Contact Surgery

Joshua Tomlin

October 2019

A thesis submitted for the degree of Bachelor of Philosophy (Honours)
of the Australian National University

Australian National University

Declaration

The work in this thesis is my own except where otherwise stated.

Joshua Tomlin

Acknowledgements

Firstly, I'd like to thank my supervisor Joan Licata for always making time for me and constantly being so positive and insightful. Joan, I would always leave every meeting with you filled with confidence, and feel as if anything confusing was becoming so clear. Thank you in general to all the amazing lectures at MSI, I'm so lucky to have had so many incredible and inspiring lectures over these years.

Thank you to all my friends, including the other maths students and the honours corner for making the whole year more enjoyable. Especially thank you to my closest friends, the endless support and thoughtfulness you showed me during these busy stressful times was invaluable to me. We got there.

Abstract

In this thesis, we investigate different techniques and constructions of contact surgery on contact 3-manifolds. We begin by examining admissible transverse surgery, a widely applicable contact surgery technique that can be performed on any transverse knot in any contact manifold. Then, we ask what restrictions we need to perform tight surgery on a tight manifold. We describe a construction of tight surgery on the unique tight contact structure ξ_{tight} on S^3 that relies on Reeb dynamical properties of a contact form representing ξ_{tight}. Finally, we explore how to extend these dynamical tight surgery techniques to open books. For certain open books, we improve the dynamics of a supported contact form near the binding in a effort to make these tight surgery techniques more applicable to general open books.

Contents

Notation

Notation

M	A smooth, closed, oriented 3-manifold.
ξ	A (positive) contact structure on M.
α	A contact form for ξ.
K	A knot in (M, ξ), usually transverse.
N(K)	A regular neighbourhood of K, diffeomorphic to a solid torus $D^2 \times S^1$ with K identified as the core.
D^n	The closed unit disk in $\mathbf{R}^n$.
S^n	The unit sphere in $\mathbf{R}^{n+1}$.
S	A solid torus $S = D^2 \times S^1$.
T	A torus $T = S^1 \times S^1$.
μ, λ	A meridional and longitudinal curve respectively on a torus T.
Σ	A surface, possibly with boundary.
(B, π)	An open book with binding B and fibration π.
(Σ, h)	An abstract open book with page Σ and monodromy h.

Chapter 1

Introduction

In this thesis, we investigate different techniques and constructions of contact surgery on contact 3-manifolds. Specifically, we examine how generally applicable or restrictive these surgery methods are, what techniques can be used to perform tight surgery, and how these tight surgery techniques can be extend to more general settings such as open books.

Chapter 2 is a brief overview of essential definitions and theorems in contact geometry, and a short description of topological surgery. For Sections 2.1-2.4, we follow [Gei08], [Mas14] and [Etn03], however the definitions and results presented in these sections are all well documented contact geometry literature. The book [Gei08] is a particularly thorough and extensive source containing all of this material. For Section 2.5, we are also just reiterating classical results in low dimensional topology. A standard resource for the content in this section is [Rol90] by Roflsen.

In Chapter 3, we describe two equivalent methods of contact surgery on transverse knots, known as admissible (transverse) surgery. These surgery methods are described briefly in [BE13], and we take the time to explain the details and prove the stated properties. In Section 3.1, we develop a model contact structure on a solid torus to model so-called 'standard neighbourhoods' of transverse knots, noting that every transverse knot admits a standard neighbourhood. All of these definitions are made in [BE13] and we give insight to the ideas behind them. In Section 3.3, we give a detailed description of two diffeomorphic methods of topological surgery which are the basis for the two methods of admissible surgery. We prove they are diffeomorphic by following a construction outlined in [BE13], while filling in the details. In Section 3.4, we describe the admissible surgery constructions given in [BE13] in detail. To help fill in the smaller details in these constructions, we also refer to papers [Ler01] and [Gei97] by Lerman and Geiges respectively, which are also used as references in [BE13].

In Chapter 4, we aim to refine contact surgery on transverse knots to preserve tightness, asking what restrictions we need to do this. We show that tight surgery can be performed on S^3 with its unique tight contact structure. The tight surgery construction we describe is presented in [EG99], which utilises deep results connecting Reeb orbits of contact forms to tightness from [HK99] and [HWZ95]. In Section 4.2, we construct a family of model contact forms on S^3 and examine their Reeb dynamics. The same forms are constructed in [EG99], however we provide calculations of their important properties such as their Reeb orbits and their characteristic foliations. Further, we give insight to how these contact forms are being constructed and why. In Section 4.3, we describe the construction given in [EG99] and fill in all the important details with insight into the ideas governing the proof.

Finally, in Chapter 5, we aim to apply the surgery techniques from Chapter 4 to the more general setting of open books. We begin the chapter by giving a basic overview of open books in Section 5.1, making the fundamental definitions, proving the basic properties and demonstrating the relevance of open books to contact geometry. For Section 5.1, we follow the works [Etn06] and [Gei08] of Etnyre and Geiges respectively, however these definitions, examples and results are commonly known and standard to the field. The majority of Section 5.2 is original work that is built upon ideas and techniques from Chapter 4. We take the ideas of the tight surgery construction in Section 4.3 to improve the dynamics of a supported contact form near the binding of certain open books. We described a construction of a family of contact forms near the binding with nice dynamics that we already know how to perform surgery on from Section 4.3. A corollary of this is that tight surgery can be performed on these open books so long as the dynamics of a supported contact form (extending one of our constructed contact forms) can be controlled on the interior of the pages.

Chapter 2

Background

In this chapter, we give a brief overview of the basic definitions and properties in the field of contact geometry. Contact geometry is an area of mathematics for studying odd dimensional manifolds, consider the odd dimensional analogue of symplectic geometry. In this thesis, we restrict our attention to 3-manifolds. In particular, let M be a closed, oriented 3-manifold.

The following definitions and results are all standard in the field, and there are many great resources providing insightful commentary and detailed proofs of these results. These works include lecture notes of Etnyre ([Etn03]) and Massot ([Mas14]), and a comprehensive and thorough book by Geiges ([Gei08]). Further, the lecture notes [Mas14] are illustrated with computer generated graphics to help visualise some of the fundamental examples of contact structures. For these reasons, we keep the explanations and proofs in this section quite brief, referring the reader to these resources.

2.1 Contact Structures

A contact manifold is a 3-manifold endowed with some extra structure. This extra structure is a certain choice of tangent plane at every point, known as a plane field.

Definition 2.1. A (positive) *plane field* ξ on M is a (positively cooriented) two dimensional sub-bundle of the tangent bundle TM.

A plane field ξ on M is a (smooth) association of a tangent plane ξ_x to every point in $x \in M$. A plane field is cooriented if there is a consistent, continuous choice of positive normal vector to each of the planes. For a more detailed definition, see [Mas14]. A coorientation of ξ combined with an orientation of M defines an orientation on ξ.

Definition 2.2. A (positive) *contact structure* ξ is a (positive) plane field on M that is nowhere integrable. The pair (M, ξ) is called a contact manifold.

For a plane field ξ to be a contact structure on M, we require that ξ is nowhere integrable. This means that ξ is not tangent to any surface $\Sigma \subset M$. This enforces that the contact planes are constantly twisting about curves that are tangent to ξ. This property of contact structures is formalised and nicely illustrated with graphics in [Mas14].

Remark 2.3. We have assumed M to be oriented, so we will only work with positive contact structures on M. Thus in this thesis, a contact structure on M implicitly means a positive contact structure on M.

A plane in a vector space can always be represented as the kernel of a linear functional on that vector space. That is, if $\alpha_x : T_xM \to \mathbf{R}$ is a linear map and $\alpha_x \neq 0$, then $\ker \alpha_x \subset T_xM$ is a plane and all planes in T_xM arise in this way. With that as motivation, we make the following definition.

Definition 2.4. A *contact form* representing a contact structure ξ is a (positive) 1-form $\alpha \in T^*M$ such that $\ker \alpha = \xi$.

The restriction that α is positive is so that the orientation of $\ker \alpha$ agrees with the orientation of ξ. From Definition 2.4, two contact forms α and α' represent the same contact structure if and only if they have the same kernel. This is true if and only if $\alpha = f \cdot \alpha'$ for some positive function $f : M \to \mathbf{R}$. In general, any contact structure can be represented locally by a contact form. Since M is oriented, we are able to represent any contact structure globally by a contact form (see [Etn03]). This means that defining a contact structure on M is equivalent to defining a contact form on M up to positive scaling.

Let $\alpha \in T^*M$ be a positive 1-form. Then $\ker \alpha$ is a plane field, as described above. A natural question to ask is "when is this plane field a contact structure?". By a standard result in the field of differential geometry know as Frobenius integrability (see [Lee13]), it can be shown that

$$\ker \alpha \text{ is a contact structure} \Leftrightarrow \alpha \wedge d\alpha > 0. \tag{2.1}$$

Equation 2.1 is known as the contact condition. For a proof of the contact condition, see [Gei08].

Remark 2.5. In general, a $(2n+1)$-dimensional contact form on a $(2n+1)$-dimensional manifold M is a 1-form α_M satisfying the higher dimensional contact condition $\alpha_M \wedge (d\alpha_M)^n > 0$.

Given two contact manifolds (M, ξ) and (M', ξ'), we define a notion of equivalence between them called a contactomorphism.

Definition 2.6. A diffeomorphism $F : M \to M'$ is a *contactomorphism* between contact manifolds (M, ξ) and (M, ξ') if $F_*(\xi) = \xi'$.

A contactomorphism is diffeomorphism that sends contact planes to contact planes. This is a natural notion of equivalence to expect because it transports the structure of (M, ξ) to (M', ξ') and vice versa. We will say two lower dimensional submanifolds $N \subset M$ and $N' \subset M'$ are locally contactomorphic if they have contactomorphic neighbourhoods.

If α and α' are contact forms representing ξ and ξ' respectively, then $F : M \to M'$ is a contactomorphism if $F^*\alpha' = f \cdot \alpha$ for some positive function $f : M \to \mathbf{R}$. Sometimes, we study the actual contact forms α and α', rather than the contact structures $\ker \alpha$ and $\ker \alpha'$. In these situations, we require a more strict definition of a contactomorphism

Definition 2.7. A diffeomorphism $F : M \to M'$ is a *strict contactomorphism* between contact manifolds (M, α) and (M, α') if $F^*\alpha' = \alpha$.

Example 2.8. Consider $\mathbf{R}^3$ with standard coordinates (x, y, z). Define a 1-form α_{std} and plane field ξ_{std} on $\mathbf{R}^3$ by

$$\begin{aligned} \alpha_{\text{std}} &= dz + x dy \\ \xi_{\text{std}} &= \ker \alpha_{\text{std}}. \end{aligned} \tag{2.2}$$

Then $d\alpha_{\text{std}} = dx \wedge dy$ and $\alpha_{\text{std}} \wedge d\alpha_{\text{std}} = dx \wedge dy \wedge dz > 0$ which is the standard volume form on $\mathbf{R}^3$. This shows that α_{std} is a contact form and that $(\mathbf{R}^3, \xi_{\text{std}})$ is a contact manifold. The contact structure ξ_{std} is know as the standard contact structure on $\mathbf{R}^3$. See [Mas14] for illustrations of this contact structure and the next examples we describe.

To understand $(\mathbf{R}^3, \xi_{\text{std}})$, first notice that α_{std} is invariant under translations in the z-coordinate. This means that the contact planes ξ_{std} are vertically invariant, and we can understand ξ_{std} by understanding it in the x-y plane. Let $v = v_x\partial_x + v_y\partial_y + v_z\partial_z \in T\mathbf{R}^3$ be a vector in the tangent space of $\mathbf{R}^3$ represented in the standard coordinate basis $\{\partial_x, \partial_y, \partial_z\}$. Then $v \in \xi_{\text{std}}$ if and only if $\alpha_{\text{std}}(v) = 0$, which gives the following relation.

$$\begin{aligned} \alpha(v) &= 0 \\ (dz + xdy)(v_x\partial_x + v_y\partial_y + v_z\partial_z) &= 0 \\ v_z + xv_y &= 0. \end{aligned} \tag{2.3}$$

This means that $(\xi_{\text{std}})_{(x,y,z)}$ is spanned by the vectors ∂_x and $\partial_z + x\partial_y$. This describes planes tangent to lines in the x-y plane with a constant y-coordinate, that have a $\frac{dz}{dy}$-slope of $-x$. That is, the planes are invariant under translations in the x-coordinate, and perpendicular to the z-axis at $x = 0$. Then, looking down the positive x-axis from the origin, the planes are twisting in an anti-clockwise direction.

Example 2.9. Consider $\mathbf{R}^3$ with cylindrical coordinates (ρ, θ, z) so that $\rho \in [0, \infty), \theta \in \mathbf{R}/2\pi\mathbf{Z}$ and $z \in (-\infty, \infty)$. Define a contact structure ξ_{radial} on $\mathbf{R}^3$ by

$$\xi_{\text{radial}} = \ker(dz + \rho^2 d\theta). \tag{2.4}$$

The plane field ξ_{radial} can be shown to be contact by checking the contact condition for $\alpha_{\text{radial}} = dz + \rho^2 d\theta$. Similar to Example 2.8, this contact structure is invariant under translations in the z direction and perpendicular to the z-axis when $\rho = 0$. This time, ξ_{radial} is radially symmetric, with the contact planes having a $\frac{dz}{d\theta}$-slope of $-\rho^2$. However, $(\mathbf{R}^3, \xi_{\text{radial}})$ is contactomorphic to $(\mathbf{R}^3, \xi_{\text{std}})$ and a contactomorphism is exhibited in [Mas14].

Example 2.10. Consider $\mathbf{R}^3$ with cylindrical coordinates (r, θ, z) so that $r \in [0, \infty), \theta \in \mathbf{R}/2\pi\mathbf{Z}$ and $z \in (-\infty, \infty)$. Define a contact structure ξ_{OT} on $\mathbf{R}^3$ by

$$\xi_{\text{OT}} = \ker(r \sin r d\theta + \cos r dz). \tag{2.5}$$

The plane field ξ_{OT} can be shown to be a contact structure checking that $\alpha_{\text{OT}} = r \sin r d\theta + \cos r dz$ satisfies the contact condition. This example is similar to Example 2.9 because it is invariant under translations in the z direction and radially symmetric. However, the $\frac{dz}{d\theta}$-slope of the tangent planes is $r \tan r$, which is zero whenever r is a multiple of π. That is, the planes twist infinitely many times around radial lines perpendicular to the z-axis. This is different to both Examples 2.8 and 2.9 where the twisting of the contact planes only approaches a half turn away from the z-axis.

The contact manifold $(\mathbf{R}^3, \xi_{\mathrm{OT}})$ is not contactomorphic to $(\mathbf{R}^3, \xi_{\mathrm{std}})$ because $(\mathbf{R}^3, \xi_{\mathrm{OT}})$ is *overtwisted* while $(\mathbf{R}^3, \xi_{\mathrm{std}})$ is *tight*, which will be explained in Section 2.3. However, let $V = \mathrm{Int}(D^2_\pi) \times \mathbf{R}$ be the submanifold of $\mathbf{R}^3$ defined by

$$V = \{(r, \theta, z) \in \mathbf{R}^3 \mid r < \pi\}. \tag{2.6}$$

Then $(V, \xi_{\mathrm{OT}}|_V)$ is contactomorphic to $(\mathbf{R}^3, \xi_{\mathrm{radial}})$. A contactomorphism between them can be exhibited by reparameterising the radial coordinate $0 \leq r < \pi$ in V to $[0, \infty)$.

Examples 2.8 and 2.9 at first look different, but turn out to be contactomorphic. This is reasonable because the contact structures have similar symmetries and the contact planes approach a half twist as you move along a line away from the z-axis. However, Example 2.10 looks different because the contact planes twist infinitely many times as you move along a radial line away from the z-axis. This raises the question of "how do we tell when contact structures are not contactomorphic?". Specifically, what invariants are there to distinguish between contact structures?

2.2 Local Behaviour of Contact Manifolds

2.2.1 The Darboux Theorem

One of the most fundamental theorems in contact geometry is the Darboux theorem, which describes the local behaviour of contact manifolds.

Theorem 2.11 (Darboux)**.** *Let (M, ξ) be a contact manifold and $p \in M$. Then there exists a neighbourhood $N(p)$ of p and a neighbourhood U of the origin in $\boldsymbol{R}^3$ such that $(N(p), \xi)$ is contactomorphic to (U, ξ_{std}).*

Such a contactomorphism can be assumed to send p to the origin in $\mathbf{R}^3$. Further, if a contact form α is chosen for ξ, then there exists a strict contactomorphism from $(N(p), \alpha)$ to $(U, \alpha_{\mathrm{std}})$ for suitably chosen $N(p)$ and U. For a detailed proof of this theorem, see [Gei08].

A consequence of the Darboux theorem is that contact manifolds have no useful local invariants because near a point, all contact manifolds are the same. This means to distinguish between contact manifolds, we need to examine information more global than a neighbourhood of point.

2.2.2 Knots in Contact Manifolds

To study contact surgery, we will need to study some knots. Let K be a knot in a contact manifold (M, ξ). Then at any point $x \in K$, K is either transverse or tangent to ξ_x

Definition 2.12. A knot K in a contact manifold (M, ξ) is *Legendrian* if at every point $x \in K$, K is (positively) tangent to ξ_x. Similarly, K is a *transverse* knot if at every point $x \in K$, K is (positively) transverse to ξ_x.

It is a standard result that every knot in a contact manifold $(M,\ \xi)$ is both isotopic to a Legendrian knot and isotopic to a transverse knot. In this thesis, we work only with transverse knots. The main theorem about transverse knots that we will use is the following from [Gei08].

Theorem 2.13. *If $K_i \subset (M_i, \xi_i)$, $i = 0, 1$ are transverse knots, then there exists neighbourhoods $N(K_i)$ of K_i which are contactomorphic. Such a contactomorphism $\Phi : N(K_0) \to N(K_1)$ can be assumed to send K_0 to K_1 and a preferred longitude on $\partial N(K_0)$ to a preferred longitude on $\partial N(K_1)$.*

An identical statement holds for Legendrian knots. This theorem is similar in style to the Darboux theorem in that all transverse knots are locally contactomorphic. We will use this property to perform surgery on transverse knots by defining a model contact structure for neighbourhoods of transverse knots in Section 3.1. This is similar to how we defined $(\mathbf{R}^3, \xi_{\text{std}})$ as a model contact structure for neighbourhoods of points in contact manifolds.

2.2.3 Characteristic Foliation

One way to study a contact manifold (M, ξ) is to study the surfaces Σ in M. Specifically, how the contact structure behaves on or near a surface Σ. The contact structure ξ is defined to be nowhere integrable and therefore ξ is not tangent Σ on any open set $U \subset \Sigma$. This means that $l_x = \xi_x \cap T_x\Sigma$ is a one dimensional subspace of T_xM for almost every $x \in \Sigma$ and that we are able to integrate l_x to a singular foliation Σ_ξ of Σ ([Etn03]).

Definition 2.14. The singular foliation Σ_ξ is the *characteristic foliation* of ξ on Σ.

The characteristic foliation Σ_ξ determines the germ of ξ about Σ. This idea is made precise by the following theorem of Giroux, proved in [Gei08].

Theorem 2.15 (Giroux)**.** *Let Σ_i be closed surfaces in contact manifolds (M_i, ξ_i), $i = 0, 1$ and $\Phi : \Sigma_0 \to \Sigma_1$ a diffeomorphism with $\Phi((\Sigma_0)_{\xi_0}) = (\Sigma_1)_{\xi_1}$. Then there is a contactomorphism $\Psi : N(\Sigma_0) \to N(\Sigma_1)$ of suitable neighbourhoods $N(\Sigma_i)$ of Σ_i with $\Psi(\Sigma_0) = \Sigma_1$ and $\Psi|_{\Sigma_0} = \Phi$.*

This theorem can be thought of as a higher dimensional analogue of the Darboux theorem and Theorem 2.13. The Darboux theorem is the statement that 0-dimensional submanifolds of contact manifolds are locally contactomorphic, Theorem 2.13 is the statement that transverse knots in contact manifolds (which are 1-dimensional submanifolds) are locally contactomorphic, and Theorem 2.15 is the statement that diffeomorphic 2-dimensional (closed) submanifolds of contact manifolds with matching characteristic foliations are locally contactomorphic. Further, Theorem 2.15 states that given a diffeomorphism between two surfaces that sends the characteristic foliation on one surface to the characteristic foliation on the other, these surfaces are locally contactomorphic by a contactomorphism that extends this diffeomorphism. This emphasises that the local information of a contact structure near a surface is contained in its characteristic foliation on the surface.

Theorem 2.15 is a very useful result for our purposes because it reduces the information of the contact structure about Σ to a family of curves foliating Σ. We will use this extensively to glue contact manifolds together along their boundaries when performing surgery. That is, we will ensure that the characteristic foliations match up when we do the gluing. Then this theorem guarantees that this gluing extends to a contactomorphism on a neighbourhood of the surfaces. This makes extending forms smoothly to the resulting manifold easier because we have a neighbourhood where we can make the two contact structures agree.

2.3 Tight and Overtwisted

We saw in Section 2.2.1 that contact manifolds have no interesting local invariants. So we ask how globally we need to look to find a useful invariant. This leads us to a fundamental dichotomy in contact geometry, that being between tight and overtwisted contact structures.

Definition 2.16. A disk $D \subset M$ in a contact manifold (M, ξ) is *overtwisted* if for all $x \in \partial D$, $T_x D = \xi_x$.

A contactomorphism between contact manifolds sends an overtwisted disk to an overtwisted disk. Therefore the existence of an overtwisted disk in a contact manifold is invariant under contactomorphism. Hence the following definition.

Definition 2.17. A contact manifold is *overtwisted* if it admits an overtwisted disk. A contact manifold which is not overtwisted is called *tight*.

Example 2.18. Recall from Example 2.10 the contact manifold $(\mathbf{R}^3, \xi_{\mathrm{OT}})$ defined in cylindrical coordinates by

$$\xi_{\mathrm{OT}} = \ker(r \sin r d\theta + \cos r dz). \tag{2.7}$$

Let P denote the $z = 0$ plane in $\mathbf{R}^3$ and let $D \subset P$ be the disk of radius π centred at the origin. Restricting attention to P, we noted that ξ_{OT} is tangent to P whenever r is a multiple of π. This means that ξ_{OT} is tangent to ∂D and that D is an overtwisted disk in $(\mathbf{R}^3, \xi_{\mathrm{OT}})$. Thus $(\mathbf{R}^3, \xi_{\mathrm{OT}})$ is overtwisted.

Example 2.19. Consider $S^3 = \{(re^{i\theta}, \sqrt{1-r^2}e^{i\phi})\} \subset \mathbf{C}^2$ with $r \in [0,1]$ and $\theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. Define a contact form α_{tight} and contact structure ξ_{tight} on S^3 by

$$\begin{aligned} \alpha_{\mathrm{tight}} &= r^2 d\theta + (1-r^2) d\phi \\ \xi_{\mathrm{tight}} &= \ker \alpha_{\mathrm{tight}}. \end{aligned} \tag{2.8}$$

Then ξ_{tight} is tight, as stated in Theorem 2.20.

This dichotomy between tight and overtwisted contact structures is fundamental to the field of contact geometry. One reason is because of the following two theorems due to Bennequin and Eliashberg respectively.

Theorem 2.20 (Bennequin [Ben83])**.** *The contact manifolds $(\boldsymbol{R}^3, \xi_{std})$ and (S^3, ξ_{tight}) are tight.*

Theorem 2.21 (Eliashberg [Eli92])**.** *All tight contact structures on $\boldsymbol{R}^3$ are contactomorphic to ξ_{std}. Similar, all tight contact structures on S^3 are contactomorphic to ξ_{tight}.*

Thus both $\mathbf{R}^3$ and S^3 have unique tight contact structures (up to contactomorphism). Overtwisted contact structures have been classified by Eliashberg in [Eli89]. In particular, Eliashberg showed that every homotopy class of a plane field on a closed 3-manifold has a unique overtwisted representative. The classification of tight contact structures on the other hand is at the forefront of current contact geometry research. One reason that classifying tight contact structures is more difficult than classifying overtwisted contact structures is that proving a contact manifold is tight involves proving the non-existence of an overtwisted disk, and methods for doing so are currently scarce. In this thesis, we will explore some ideas and techniques for constructing tight manifolds through surgery.

2.4 Reeb Dynamics

As mentioned before, a choice of contact form for a contact manifold (M, ξ) is strictly more information than the contact structure itself. A consequence of this is that a contact form carries more geometry than a contact structure. In particular, a contact form has an associated Reeb vector field, which we define now.

Definition 2.22. Let (M, ξ) be a contact manifold and α a contact form representing ξ. The *Reeb vector field* of α is the unique vector field X_α satisfying $i_{X_\alpha} d\alpha = 0$ and $i_{X_\alpha}\alpha = 1$.

A straightforward calculation applying the Cartan formula (see [Lee13]) shows that $L_{X_\alpha}\alpha = 0$ where L_{X_α} is the Lie derivative of X_α. Geometrically, this means that X_α is a vector field transverse to ξ with flow that preserves ξ, and is scaled so that $\alpha(X_\alpha) = 1$.

Lemma 2.23. *Let X and X' be the Reeb vector fields of the contact manifolds (M, α) and (M, α') respectively. Suppose that $F : (M, \alpha) \to (M', \alpha')$ is a surjective submersion such that $\alpha = F^*\alpha'$. Then $X' = F_*(X)$.*

Proof. Let X be Reeb vector field of α and let X' be the Reeb vector field for α'. Then $\alpha(X) = 1$ and $d\alpha(X, Y) = 0$ for all $Y \in TM$. We show that $F_*(X)$ satisfies the conditions $\alpha'(F_*(X)) = 1$ and $d\alpha'(F_*(X), Y') = 0$ for all $Y' \in TM'$. Then by the uniqueness of the Reeb vector, we must have $F_*(X) = X'$. To check that $\alpha'(F_*(X)) = 1$, we have

$$\begin{aligned} \alpha'(F_*(X_\alpha)) &= (F^*\alpha')(X) \\ &= \alpha(X) \\ &= 1. \end{aligned}$$

Let $Y' \in TM'$ be some vector field. Since F is a surjective submersion, there exists $Y \in TM$ such that $F_*Y = Y'$. Note that pull backs of smooth functions commute with exterior derivative.

$$\begin{aligned} d\alpha'(F_*(X), Y') &= (F^*d\alpha')(X, Y) \\ &= (dF^*\alpha')(X, Y) \\ &= d\alpha(X, Y) \\ &= 0. \end{aligned}$$

Thus $F_*(X) = X'$. □

In particular, Lemma 2.23 applies to strict contactomorphisms $F : (M, \alpha) \to (M', \alpha')$.

2.4.1 Reeb Orbits

In Chapter 4, we are interested in the dynamics of the integral curves of X_α.

Definition 2.24. Let (M, α) be a contact manifold with Reeb vector field X_α. The Reeb orbits of α are the integral curves of X_α.

A consequence of Lemma 2.23 is that if $F : (M, \alpha) \to (M', \alpha')$ is a strict contactomorphism, then F takes the Reeb orbits of α to the Reeb orbits of α'. Proving this is a direct application of a property known as the 'naturality of integral curves' in Lee's differential geometry book [Lee13].

In Chapter 4, we specifically study Reeb orbits which are periodic. To study periodic Reeb orbits, we briefly describe some standard dynamical definitions and properties, see [Gei08] and [Vil01].

Let γ_x be a periodic Reeb orbit of α through the point $x \in M$. Let $N(\gamma_x) = D^2 \times S^1$ be a neighbourhood of γ_x with the core of $N(\gamma_x)$ equal to γ_x. Fix a point $p \in S^1$ such that $x \in D^2 \times \{p\}$. Consider another point $x' \in D^2 \times \{p\}$ and let $\gamma_{x'} : \mathbf{R} \to M$ denote the Reeb orbit of α through x'. That is, $\gamma_{x'}(0) = x'$ and $\gamma'_{x'}(t) = (X_\alpha)_{\gamma_{x'}(t)}$. Then for x' sufficiently close to the core of $N(\gamma_x)$, $\gamma_{x'}(t) \in D^2 \times \{p\}$ for some $t > 0$. Define $h(x') = \gamma_{x'}(t_1)$ where $t_1 > 0$ is the smallest value such that $\gamma_{x'}(t_1) \in D^2 \times \{p\}$. Then h is well defined on a sufficiently small open neighbourhood $U \subset D^2 \times \{p\}$ containing x.

Definition 2.25. The map $h : U \to D^2 \times \{p\}$ is called the *(Poincaré) first return map* of γ at x. We call $Dh|_x : TU_x \to T(D^2 \times \{p\})|_x$ the linearised first return map of γ at x.

For a detailed description of the first return map and a proof that it is well defined, see [Vil01]. To distinguish between different types of periodic Reeb orbits, we make the following definitions.

Definition 2.26. Let (M, α) be a contact manifold and let γ_x be a periodic Reeb orbit through $x \in M$. Let h be the first return map of γ at x. Then γ is called *degenerate* if 1 is an eigenvalue of the linearised return map $Dh|_x$. Otherwise, γ is called non-degenerate if 1 is not an eigenvalue of $Dh|_x$.

The idea is that if 1 is not an eigenvalue of Dh, then there cannot be periodic orbits accumulating at γ (see [Gei08]). Thus a Reeb orbit being non-degenerate obstructs this behaviour of clustering periodic orbits. We can further classify non-degenerate periodic orbits by examining the spectrum of the linearised return map. The fact that Definitions 2.26 and 2.27 do not depend on the chosen point x is explained in [Vil01].

Definition 2.27. Let (M, α) be a contact manifold and let γ_x be a non-degenerate periodic Reeb orbit through $x \in M$. Let h be the first return map of γ at x. Then γ is called *elliptic* if the (complex) eigenvalues of the linearised return map $Dh|_x$ have norm 1. Otherwise, γ is called hyperbolic if the eigenvalues of $Dh|_x$ are both not norm 1.

Notice that Dh is a real invertible matrix and therefore its (possibly complex) eigenvalues are conjugate. So this definition covers all cases. The first return map having only eigenvalues of norm 1 means that there is a neighbourhood $N(\gamma_x) = D^2 \times S^1$ such that the every Reeb orbit near γ_x is tangent to a torus $T_r \subset N(\gamma_x)$ of fixed radius $r \in [0, 1]$. That is, the Reeb orbits are neither being attracted to nor repelled from γ_x. The Reeb orbit γ_x is hyperbolic if the Reeb orbits near γ_x are either repelled away or attracted towards γ_x. For a more rigorous description and proof of the claimed properties, we refer the reader to [Vil01].

2.5 Surgery Basics

Dehn surgery is a classical tool used in 3-manifold topology to modify a 3-manifold M into a new 3-manifold. This is accomplished by removing a neighbourhood $N(K)$ of a knot $K \subset M$, and then filling $\overline{M - N(K)}$ with a new solid torus $S = D^2 \times S^1$ glued to $\partial(\overline{M - N(K)})$. For detailed explanations and proofs of the following definitions and properties, we refer the reader to [Rol90].

Let $S = D^2 \times S^1$ be a solid torus with boundary torus $T = \partial S$. Since T is the boundary of a solid torus, there is a canonical choice of (isotopy class of) meridian μ on T, namely a curve that is null homologous in S. Let λ be some choice of longitude on T, a choice which is not canonical. The first homology group $H_1(T)$ of T is isomorphic to $\mathbf{Z} \times \mathbf{Z}$ and the curves μ and λ form a $\mathbf{Z}$-linear homology basis of $H_1(T)$. We define the slope of a curve $\gamma = a\mu + b\lambda$ on T to be $\frac{a}{b}$.

Any map $\Phi : T \to T$ can be described up to isotopy by how it acts on curves in T. Since μ and λ is a homology basis for $H_1(T)$, this means that we can represent Φ as a matrix with integer coefficients acting on the ordered basis (μ, λ) of $H_1(T)$.

$$\Phi = \begin{pmatrix} p & m \\ q & n \end{pmatrix}. \tag{2.9}$$

The above example describes a map $\Phi : T \to T$ that sends μ to a slope-$\frac{p}{q}$ curve and λ to a slope-$\frac{m}{n}$ curve. If Φ is an orientation preserving diffeomorphism, then this forces the determinant of Φ to be equal to 1. That is, Φ is an orientation preserving diffeomorphism if and only if $pn - qm = 1$.

Let $r = \frac{p}{q}$ be a fixed rational number with p and q coprime integers.

Definition 2.28. An *r-surgery map* $\Phi : T \to T$ is an orientation preserving diffeomorphism of the form

$$\Phi = \begin{pmatrix} p & m \\ q & n \end{pmatrix}, \tag{2.10}$$

for some $m, n \in \mathbf{Z}$ such that $pn - qm = 1$.

This means that an r-surgery map Φ sends the meridian on T to a slope-r curve and a longitude on T to some slope-$\frac{m}{n}$ curve. When choosing an r-surgery map, the choice of $m, n \in \mathbf{Z}$ such that $pn - pm = 1$ is not unique, but such integers always exist.

The process of r-surgery on K is to first remove a neighbourhood $N(K)$ of K. Then the boundary of $\overline{M - N(K)}$ is a torus, so we glue a solid torus S to $\partial(\overline{M - N(K)})$ by an r-surgery map.

Definition 2.29. Let K be a knot in a closed orientable manifold M. Let $N(K) = D^2 \times S^1$ be a neighbourhood of K and let $S = D^2 \times S^1$ be a solid torus. Choose $\Phi : \partial S \to \partial(\overline{M - N(K)})$ to be some r-surgery map. Then the manifold $M_K(r)$ obtained from r-surgery on K is defined to be

$$M_K(r) = \overline{M - N(K)} \cup_\Phi S. \tag{2.11}$$

The fact that the homeomorphism type of $M_K(r)$ is independent of choice of r-surgery map is a classical result due to Alexander's Theorem (see [Rol90]). This is the statement that r-surgery on K is completely determined up to homeomorphism by requiring that a slope-r curve on $\overline{M - N(K)}$ bounds a disk in $M_K(r)$.

In this thesis, we will extend the techniques of classical surgery to contact manifolds. That is, given a knot K in a contact manifold (M, ξ), we will construct a contact structure $\xi_K(r)$ on the surgered manifold $M_K(r)$ such that $\xi_K(r)$ extends ξ. We say that the contact manifold $(M_K(r), \xi_K(r))$ is the result of contact r-surgery on (M, ξ).

Definition 2.30. A *contact r-surgery* on a knot K in a contact manifold (M, ξ) is a contact structure $\xi_K(r)$ on the surgered manifold $M_K(r)$ such that $\xi_K(r)$ extends ξ. We call the contact manifold $(M_K(r), \xi_K(r))$ the result of contact r-surgery on K.

Chapter 3

Admissible Transverse Surgery

Surgery is a classical tool in 3-manifold topology for constructing new 3-manifolds out of existing ones. Naturally, we would like to extend these techniques to contact manifolds. In a contact manifold, there are two obvious choices of families of knots to perform surgery on: transverse knots and Legendrian knots. In this thesis, we will explore surgery constructions on transverse knots. Specifically, in this chapter we describe two equivalent ways to perform contact surgery on any transverse knot in any contact manifold. These constructions are known as admissible transverse surgery and are described by Baldwin and Etnyre in [BE13]. Often we will refer to admissible transverse surgery as just admissible surgery for convenience, although it should be noted that Baldwin and Etnyre also describe an admissible surgery construction for Legendrian knots in [BE13].

Transverse knots are a convenient family of knots to perform surgery on because, as we saw in Theorem 2.13, all transverse knots are locally contactomorphic. In this chapter, we will construct a contact structure ξ_U on a solid torus U to model neighbourhoods of transverse knots on through Theorem 2.13. This means that any transverse knot K in (M, ξ) will have a neighbourhood $N(K)$ contactomorphic to some solid torus contained in $(U, \xi|_U)$. Then we will develop admissible surgery techniques by studying the contact structure $\xi|_U$ on U.

We follow [BE13], first constructing a model contact structure on a solid torus, then describing two methods of classical surgery and adapting these methods to contact manifolds. The constructions in [BE13] are described quite briefly, so we take the time to fill in the missing details and focus attention on explaining the ideas behind the constructions.

3.1 A Model Contact Structure on $D^2 \times S^1$

Let (M, ξ) be a contact manifold and K a transverse knot in M with $N(K)$ a solid torus neighbourhood of K. The process of surgery on K involves removing $N(K)$ from M and gluing in a new solid torus in some possibly different way. In order to perform contact surgery on K, it is useful to understand ξ restricted to $N(K)$. The fundamental idea behind admissible surgery is Theorem 2.13, that all transverse knots have contactomorphic neighbourhoods. This is similar to the Darboux theorem that every point in a contact manifolds has a neighbourhood contactomorphic to the origin in $(\mathbf{R}^3, \xi_{std})$ (see Theorem 2.11). With this in mind, we develop a standard contact structure on a solid torus to model neighbourhoods of knots on. The following model solid torus construction was introduced in [BE13]. We expand the details of this construction, giving insight to the ideas behind it and proving some of the claimed properties.

Let $U = \mathbf{R}^2 \times S^1$ be an open solid torus with polar coordinates $((r, \theta), \phi)$ such that $r \in [0, \infty)$, $\theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}$ and let $f : [0, \infty) \to [0, \pi)$ be some strictly increasing smooth surjection. In these coordinates, we think of $\mathbf{R}^2$ as a disk of infinite radius. Define a contact structure ξ_U on U by

$$\xi_U = \ker\left(f(r) \sin f(r) d\theta + \cos f(r) d\phi\right). \tag{3.1}$$

Lemma 3.1. *The contact manifold (U, ξ_U) is tight.*

Proof. Recall from Example 2.10 that there is an overtwisted contact structure ξ_{OT} on $\mathbf{R}^3$ given by

$$\xi_{\mathrm{OT}} = \ker\left(r \sin r d\theta + \cos r dz\right). \tag{3.2}$$

Further, we showed in Example 2.10 that ξ_{OT} restricts to a tight contact structure on $V = \mathrm{Int}(D^2_\pi) \times \mathbf{R} \subset \mathbf{R}^3$. Let $\pi : V \to S_\pi$ be the map that quotients the z-axis by $2\pi\mathbf{Z}$. Then S_π is a solid torus of radius π with coordinates $((r, \theta), \phi)$, and $\pi_*(\xi_{\mathrm{OT}})$ is a contact structure on S_π described in these coordinates by

$$\pi_*(\xi_{\mathrm{OT}}) = \ker(r \sin r d\theta + \cos r d\phi). \tag{3.3}$$

The contact manifold (V, ξ_{OT}) is the universal cover for $(S_\pi, \pi_*(\xi_{\mathrm{OT}}))$ and therefore $(S_\pi, \pi_*(\xi_{\mathrm{OT}}))$ is universally tight (see [Hon01] for the appropriate definitions). In particular, $(V_\pi, \pi_*(\xi_{\mathrm{OT}}))$ is tight.

Let $F : U \to S_\pi$ be the diffeomorphism that reparameterises the radial coordinate of U by the function $f : [0, \infty) \to [0, \pi)$. Then $F_*(\xi_U) = \pi_*(\xi_{\mathrm{OT}})$ and F is a contactomorphism. Therefore (U, ξ_U) is tight. In particular, (U, ξ_U) is contactomorphic to the largest radially symmetric open neighbourhood of the z-axis in $(\mathbf{R}^3, \xi_{\mathrm{OT}})$ that does not contain an overtwisted disk. $\square$

To understand the contact structure ξ_U on U, we consider a family of tori U_a of radius $a > 0$.

$$U_a = \{(r, \theta, \phi) \mid r \leq a\} \subset U \tag{3.4}$$

$$\partial U_a = \{(r, \theta, \phi) \mid r = a\} \subset U. \tag{3.5}$$

Denote the core of U by K_U. Since we have defined coordinates on ∂U_a, there is a natural choice of representative meridional and longitudinal curves μ and λ on ∂U.

$$\mu = \{(a, \theta, 0) \mid \theta \in \mathbf{R}/2\pi\mathbf{Z}\} \subset \partial U_a \tag{3.6}$$

$$\lambda = \{(a, 0, \phi) \mid \phi \in \mathbf{R}/2\pi\mathbf{Z}\} \subset \partial U_a \tag{3.7}$$

The choice of (isotopy class of) the meridian μ is canonical since ∂U_a is the boundary of a solid torus. That is, μ can be chosen to be a curve that is null homologous in U_a. The choice of longitude λ is not canonical; however, with coordinates already chosen there is a natural choice. The curves μ and λ form a $\mathbf{Z}$-linear homology basis $H_1(\partial U_a) \cong \mathbf{Z} \times \mathbf{Z}$. They will be used to describe the slope of curves on ∂U_a and surgeries on K_U.

An important property of the solid torus U_a is the slope of the characteristic foliation of ξ_U on the boundary torus ∂U_a. This is because it determines the germ of ξ near ∂U_a, as explained in Section 2.2.3. In [BE13], it is stated that the slope of the characteristic function on ∂U_a is a monotonically increasing function of a; here we prove this assertion.

Proposition 3.2. *The characteristic foliation of ξ on ∂U_a has a $\frac{d\theta}{d\phi}$-slope of $-\frac{1}{f(a)} \cot f(a)$.*

Proof. Let $\{\partial_\theta, \partial_\phi\}$ be the standard coordinate basis for the tangent space of ∂U_a. Suppose $v = v_\theta \partial_\theta + v_\phi \partial_\phi$ is tangent to ξ_U. This gives the following relation between v_θ and v_ϕ:

$$(f(a) \sin f(a) d\theta + \cos f(a) d\phi)(v_\theta \partial_\theta + v_\phi \partial_\phi) = 0$$

$$f(a) \sin f(a) v_\theta + \cos f(a) v_\phi = 0$$

$$v_\theta = -\frac{1}{f(a)} \cot f(a) v_\phi. \tag{3.8}$$

Therefore the characteristic foliation of ∂U_a is linear with $\frac{d\theta}{d\phi}$-slope equal to $-\frac{1}{f(a)}\cot f(a)$. $\square$

Remark 3.3. The normal convention for denoting the slope of a curve on a torus is the $\frac{d\phi}{d\theta}$-slope. That is, the number of copies of the longitude over the number of copies of the meridian. In this section only, when we refer to the slope of a curve we mean the $\frac{d\theta}{d\phi}$-slope, which is the inverse of the normal slope convention. This is only because it is easier to talk about the $\frac{d\theta}{d\phi}$-slope changing from $-\infty$ up to ∞ rather than the $\frac{d\phi}{d\theta}$-slope changing from -0 down to $-\infty = \infty$ and back down to 0; this follows the approach taken in [BE13].

In particular, the slope of the characteristic foliation of ∂U_a is strictly increasing from $-\infty$ to ∞ as a increases from 0 to ∞. When $r = 0$, the contact planes are tangent to the r-θ plane (thought of as perpendicular to the z-axis under the standard inner product on $\mathbf{R}^3$). Thus the core K_U of U is a transverse knot. The contact planes turn anticlockwise in the θ-ϕ plane as we move away from the z-axis on a radial line. They become vertical when $f(r) = \frac{\pi}{2}$, and they tend to flat again as $r \to \infty$. This is similar to the twisting of the contact planes along radial lines exhibited in the contact structure $(\mathbf{R}^3, \xi_{\text{radial}})$, defined in Example 2.9.

The characteristic foliation on the boundary of the solid torus U_a plays a significant role in both admissible surgery constructions. Thus, we parametrise the solid tori by the slopes of the characteristic foliations on their boundaries instead of parameterising them by their radii. For $s \in (-\infty, \infty)$, let S_s be the solid torus such that its boundary torus $T_s = \partial S_s$ has characteristic foliation of slope s. Additionally, let $S_{s,s'}$ denote the thickened torus $\overline{S_{s'} - S_s}$ for $s < s'$.

Now we have a model contact structure ξ_U on U with the necessary notation and properties described. Next we show how to model neighbourhoods of transverse knots on (U, ξ_U).

3.2 Standard Neighbourhoods

The following result is one of the main tools used in [BE13] for both admissible surgery constructions. Definition 3.5 is also terminology used in [BE13].

Theorem 3.4 (Standard Neighbourhood Theorem, [Gei08])**.** *Let K be a transverse knot in a contact manifold (M, ξ). Then there exists a neighbourhood $N(K)$ of K contactomorphic to S_s for some $s > 0$. This contactomorphism can be chosen such that K is sent to the core of S_s and a preferred longitude on $\partial N(K)$ is sent to the preferred longitude λ on ∂S_s.*

Proof. Let K be a transverse knot in (M, ξ). The core K_U of the solid torus U is a transverse knot in the contact manifold (U, ξ_U). By Theorem 2.13, there exists a neighbourhood $N(K)$ of K contactomorphic to a neighbourhood $N(K_U)$ of K_U. By restricting to a smaller neighbourhood if necessary, we can assume that $N(K_U) = S_s$ for some $s > 0$. Also from Theorem 2.13, we can assume this contactomorphism sends K to K_U and a preferred longitude on $\partial N(K)$ to the preferred longitude λ on ∂S_s. $\square$

Definition 3.5. Such a neighbourhood $N(K)$ contactomorphic to S_s with the properties described above is called a *standard neighbourhood of K of thickness s*.

Theorem 3.4 is similar in style to the Darboux theorem that every point in a contact manifold has a neighbourhood contactomorphic to a neighbourhood of the origin in $(\mathbf{R}^3, \xi_{std})$. Instead, Theorem 3.4 states that every transverse knot has a neighbourhood contactomorphic to a neighbourhood of K_U in (U, ξ_U). Further, we can identify $N(K)$ with S_s, with K identified as the core of S_s and a longitude on $\partial N(K)$ identified with λ. Thus we know exactly what the contact structure is inside $N(K)$. This is useful for extending a contact structure to a surgered manifold because while the homeomorphism-type of a surgered manifold depends only on the boundary identifications, extending a (smooth) contact structure requires control of a neighbourhood of the identification boundary.

The following lemma is another tool helpful for extending a contact structure to a surgered manifold. This result is stated in [BE13], but is not proved. Lemma 3.6 is a special case of Theorem 2.15 from [Gei08], as explained in Chapter 2.

Lemma 3.6. *Let T be a torus in a contact manifold (M, ξ) and $\Phi : T_s \to T$ a diffeomorphism that sends the characteristic foliation on T_s to the characteristic foliation on T. Then Φ extends to a contactomorphism from a neighbourhood of $N(T_s)$ of T_s to a neighbourhood $N(T)$ of T.*

We will apply Lemma 3.6 through the following corollary.

Corollary 3.7. *Suppose (M, ξ) is a contact manifold where M has torus boundary T. Let $\Phi : \partial S_s \to T$ be a diffeomorphism that sends the characteristic foliation on ∂S_s to the characteristic foliation on T. Then $\xi_\Phi = \xi \cup_\Phi \xi_U$ is a smooth contact structure on $M \cup_\Phi S_s$.*

Proof. By Lemma 3.6, Φ extends to a contactomorphism $\Phi : N(\partial S_s) \to N(T)$. This means that ξ_Φ is smooth because ξ and $\Phi_*(\xi_U)$ agree on the neighbourhood

$N(T)$. The fact that ξ_Φ is contact is immediate because both ξ and ξ_U are contact. □

Corollary 3.7 is most useful for our purposes because it shows that to glue contact manifolds by a diffeomorphism Φ between their boundaries and extend their contact structures smoothly, it is enough to ensure that Φ sends the characteristic foliation on one boundary to the characteristic foliation on the other.

3.3 Two Methods of Surgery

We will perform contact surgery on a transverse knot K in the contact manifold (M, ξ) in two equivalent ways by first defining two equivalent methods of classical surgery on M. Let $N(K)$ be a neighbourhood of K and consider $\overline{M - N(K)}$ with $T = \partial(\overline{M - N(K)})$. Let $r = \frac{p}{q}$ be a rational surgery coefficient with p and q coprime integers. Let $S = D^2 \times S^1$ be a solid torus and fix preferred longitudes on both T and ∂S. Recall from Section 2.5 that an r-surgery map $\Phi : \partial S \to T$ can be defined by the matrix

$$\Phi = \begin{pmatrix} p & m \\ q & n \end{pmatrix}, \tag{3.9}$$

for any choice of integers m and n such that $pn - qm = 1$. Also recall that the surgured manifold $M_K(r)$ is constructed as

$$M_K(r) = \overline{M - N(K)} \cup_\Phi S.$$

We will take this as the standard definition of topological surgery and call this the 'normal' or 'filling' method. However, there is another equivalent approach where instead of "filling a torus shaped boundary", the torus boundary T in $\overline{M - N(K)}$ is collapsed to an S^1. In this section we define this alternative 'collapsing' method and prove that both methods give diffeomorphic manifolds.

Example 3.8 ('Dimensionally Reduced Surgery')**.** Here we will consider a toy, dimensionally reduced example to illustrate the ideas behind the two equivalent methods of surgery. Consider capping off a boundary component of a surface. That is, let Σ be a surface with non-empty boundary and let T be an S^1-boundary component in Σ. We describe two methods of this, one 'filling' the boundary and one 'collapsing' the boundary. The filling method is to glue in a disk D^2 by some homeomorphism of S^1. By Alexander's Trick (see [Rol90]) there is only one way to do this up to homeomorphism.

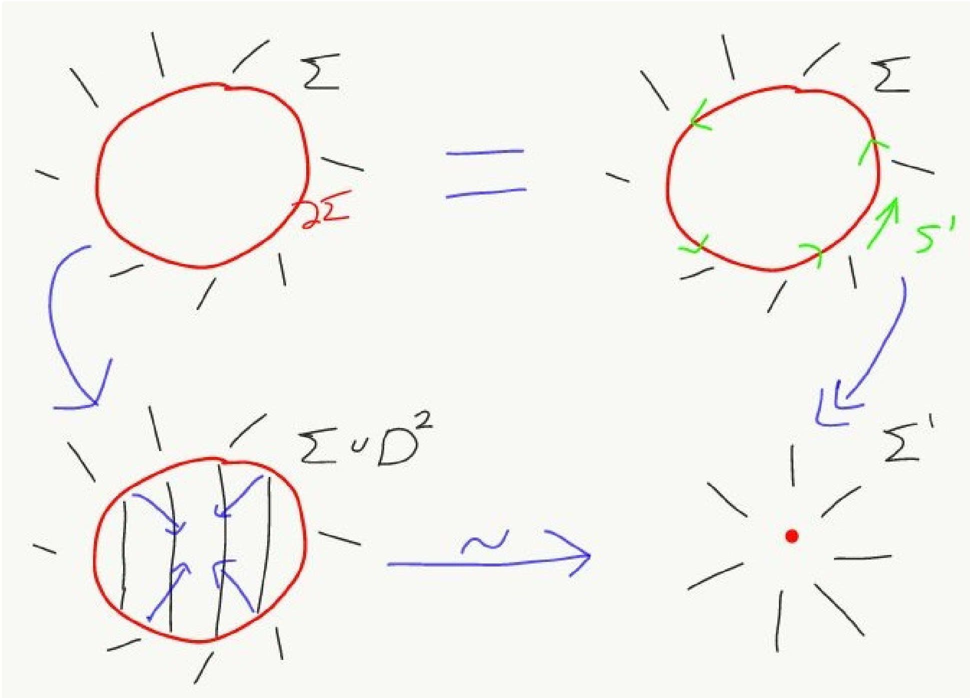

Figure 3.1: A homeomorphism between the filling and the collapsing of $\partial\Sigma = T$

Alternatively, we can collapse the boundary component T to a point. To do this, define an S^1-action on T by rotation. That is, T is just a copy of S^1, and S^1 acts on itself by rotation. Since we are going to quotient by this action, the only important information is what the orbits are. The orbits define an equivalence relation $\sim$ on T by $x \sim x'$ if and only if x and x' are in the same orbit. This naturally extends to an equivalence relation on Σ by defining $\sim$ to be the trivial equivalence relation on the interior of Σ. Now form the quotient manifold Σ' by

$$\Sigma' = \Sigma / \sim . \tag{3.10}$$

That is, any points on the boundary component T that are in the same orbit have been identified. In this case, there is only one orbit, so the whole boundary T has been collapsed to a point. This is homeomorphic to the filling of T, which can be seen by shrinking the filled-in interior disk to a point as illustrated in Figure 3.1.

Example 3.8 is the idea behind the collapsing method of surgery. Instead of filling in the torus boundary so that a slope-r curve bounds a disk, we collapse the torus boundary to an S^1 along an action which has slope-r orbits.

An S^1-Action on a Torus

Let $T = S^1 \times S^1$ be a torus with coordinates $(\theta, \phi) \in \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$. Define a free (proper) S^1-action on T by

$$\begin{aligned} \Psi : S^1 \times T &\to T \\ \Psi_\psi(\theta, \phi) &= (\theta + q\psi, \phi + p\psi). \end{aligned} \tag{3.11}$$

Here, $\psi \in S^1 = \mathbf{R}/2\pi\mathbf{Z}$. The orbits of this action are the slope $\frac{p}{q} = r$ curves on T. This can be seen by fixing (θ, ϕ) and noticing that the line $x = \theta + q\psi, y = \phi + p\psi$ for varying ψ has slope equal to r. The explicit action is not important for the surgery we are about to describe, it is only necessary that the orbits are the slope-r curves on the torus T. However, having an explicitly described action is useful later when we extend a contact structure to the surgered manifold.

Suppose now that T is a torus boundary component of M. Define an equivalence relation $\sim$ on T by $x \sim x'$ if and only if x and x' are in the same orbit of Ψ. As mentioned before, this extends naturally to an equivalence relation on M. We obtain a new space M' by collapsing T along this action.

$$M' := M/\sim \tag{3.12}$$

Let $\pi : M \to M'$ be the corresponding quotient map and for any subset $U \subset M$, we will denote $\pi(U)$ as U'. This gives a new space M' where each slope-r curve on T has been identified to a point, collapsing all of the boundary torus T to an S^1.

3.3.1 Defining a Smooth Structure on M'

Collapsing the boundary T of M to an S^1 is a dramatic change to M, and at first it is not clear that M' is smooth. In this section, we will construct a smooth structure on M', and show that M' is diffeomorphic to $M_K(r)$. The following proof is from [BE13], although the explanations there are quite brief. We take the care to provide the details and ideas.

Proposition 3.9. *The space $M' = M/\sim$ admits a smooth structure.*

Proof. To construct a smooth structure on M' it is enough to show that there is a neighbourhood of $T' = \pi(T)$ that admits a smooth structure compatible with the smooth structure on $M' - T' = M - T$. Let $N = T \times [0, 1]$ be a collar neighbourhood of the boundary torus T with $T \times \{0\} = T$. Our idea is to construct a new space P with a smooth, free S^1-action such that P/S^1 is homeomorphic to $N' = \pi(N)$. Then P/S^1 has a smooth structure by the general theory of smooth Lie group actions on manifolds (see [Lee13]). This allows us to transport the smooth structure on P/S^1 to N'.

Define a smooth map $f : N \times \mathbf{C} \to \mathbf{R}$ by

$$\begin{aligned} &f : N \times \mathbf{C} \to \mathbf{R} \\ &f((x,t),z) = t - |z|^2, \end{aligned} \tag{3.13}$$

where $(x,t) \in N = T \times [0,1]$. This map f has surjective derivative at all points in $N \times \mathbf{C}$, and $0 \in \mathbf{R}$ is in the image of f. In particular, zero is a regular value of f. Hence $f^{-1}(0)$ is a smooth submanifold of $N \times \mathbf{C}$ (see [Lee13] for definitions of this terminology and proof of this standard fact about submanifolds). Let P be the submanifold of $N \times \mathbf{C}$ defined by

$$P = f^{-1}(0).$$

Then in coordinates, P has the following description.

$$\begin{aligned} P &= \{((x,t),z) \in N \times \mathbf{C} \mid t = |z|^2\} \\ &= \{(x, r^2, re^{i\theta}) \mid x \in T, re^{i\theta} \in D^2\}. \end{aligned} \tag{3.14}$$

In particular, P is homeomorphic to $T \times D^2$ by forgetting the second coordinate in Equation 3.14. There is a natural inclusion $i : N \to P$ of N into P by

$$\begin{aligned} &i : N \to P \\ &i(x,t) = (x,t,\sqrt{t}). \end{aligned} \tag{3.15}$$

Additionally, define an extension of the free S^1-action on T to a free S^1-action on P by

$$e^{i\phi} \cdot (x, re^{i\theta}) = (e^{i\phi} \cdot x, re^{i(\theta-\phi)}). \tag{3.16}$$

Equation 3.16 defines the S^1-action on the T component of P to be the given S^1-action on T and the S^1-action on the D^2 component of P to be a clockwise rotation of D^2. When $r = 0$ is fixed, this S^1-action on P is the given S^1-action on $T = T \times \{0\}$. Consider P/S^1 and let $\rho : P \to P/S^1$ denote the quotient map.

Lemma 3.10. *The space P/S^1 is homeomorphic to N'.*

Proof. Define a map $F : N' \to P/S^1$ by

$$\begin{aligned} F &: N' \to P/S^1 \\ F([x,t]) &= [x, t, \sqrt{t}]. \end{aligned} \tag{3.17}$$

The map F is well defined on N' because the S^1-action on $T \times \{0\} \subset P$ is the S^1-action on T. That is, F is the descent of $i : N \to P$ to a map between the quotient spaces $F : N' \to P/S^1$.

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"] & P \arrow[d, two heads, "\rho"] \\ N' \arrow[r, dashed, "F"'] & P/S^1. \end{tikzcd} \tag{3.18}$$

This map F has an inverse $F^{-1} : P/S^1 \to N'$ given by

$$\begin{aligned} F^{-1} &: P/S^1 \to N' \\ F^{-1}([x, r^2, re^{i\theta}]) &= (e^{i\theta} \cdot x, r^2). \end{aligned} \tag{3.19}$$

We check that the map F^{-1} is well defined.

$$\begin{aligned} F^{-1}([e^{i\phi} \cdot (x, r^2, re^{i\theta})]) &= F^{-1}([e^{i\phi}x, r^2, re^{i(\theta-\phi)}]) \\ &= (e^{i(\theta-\phi)} \cdot (e^{i\phi}x), r^2) \\ &= (e^{i\theta} \cdot x, r^2) \\ &= F^{-1}([x, r^2, re^{i\theta}]). \end{aligned}$$

The maps F and F^{-1} are continuous inverses of each other, hence we have shown that P/S^1 is homeomorphic to N'. $\square$

We showed above that P is a smooth manifold. The S^1-action on P is smooth, free, and proper. Therefore P/S^1 admits a smooth structure (see [Lee13]). We define a smooth structure on N' through the homeomorphism $F : N' \to P/S^1$. That is, given a set of charts

$$\{\varphi_\alpha : U_\alpha \subset P/S^1 \to \mathbf{R}^3\} \tag{3.20}$$

on P/S^1, we define a set of charts

$$\{\varphi_\alpha \circ F : F^{-1}(U_\alpha) \subset N' \to \mathbf{R}^3\} \tag{3.21}$$

on N'. Since $N' - T' = T \times (0,1]$ and $F|_{N'-T'} : N' - T' \to P/S^1$ is smooth with respect to the smooth structure on $T \times (0,1]$, these charts are compatible with the smooth structure on M. $\square$

Remark 3.11. In the proof of Proposition 3.9, we have not only shown that N' has a smooth structure, but we have given it a particular smooth structure. That is, we have identified N' with P/S^1 under the homeomorphism F. So to show, for example, that a function $f : N' \to \mathbf{R}$ is smooth, we show that $f \circ F^{-1} : P/S^1 \to \mathbf{R}$ is smooth. Similarly, the differential forms on N' are described as the differential forms on P/S^1 pulled back by F.

Now that we have a smooth structure on M', we aim to show that M' is diffeomorphic to $M_K(r)$. Firstly, we will show that M' and $M_K(r)$ are homeomorphic. The idea behind this is that filling in a boundary curve with a disk is homeomorphic to collapsing that curve to a point, as previously mentioned in Example 3.8.

Proposition 3.12. *The topological space M' is homeomorphic to $M_K(r)$.*

Proof. Let U be a neighbourhood of the torus boundary component $T = \partial(\overline{M - N(K)})$ in $\overline{M - N(K)}$. Notice that in the construction of M' we have $\overline{M' - U'} = \overline{M - U}$. Further, U' is a solid torus in M'.

Consider a slope-r curve γ on $\partial U = \partial U'$. We can isotope γ inside U to a slope-r curve on T. This means that γ is null homotopic in U' because the slope-r curves on T have been collapsed to a point. Thus we have shown that M' is equal to $\overline{M - U}$ filled with a solid torus U' such that a slope-r curve on $\partial(\overline{M - U})$ now bounds a disk. Surgery on M is determined up to homeomorphism by which curve on $\partial(\overline{M - U})$ is chosen to bound a disk, therefore we have shown that M' and $M_K(r)$ are homeomorphic. □

Corollary 3.13. *The smooth manifolds M' and $M_K(r)$ are diffeomorphic.*

Proof. It is a fact that any homeomorphic smooth 3-manifolds are diffeomorphic. This is because of the uniqueness of smooth structures on 3-manifolds, which is known as Moise's Theorem (see [Moi77]). Proposition 3.9 shows that M' is a smooth manifold and Proposition 3.12 shows M' and $M_K(r)$ are homeomorphic. □

Thus we have shown that the filling and collapsing methods of surgery give diffeomorphic manifolds. Notice that we could have used Proposition 3.12 to define the smooth structure on M', and then we get Corollary 3.13 without having to do the work of Proposition 3.9. However, we having an explicit description of the smooth structure on M' is necessary work for the admissible surgery constructions.

3.4 Admissible Surgery Constructions

Following the filling and collapsing methods of topological surgery, we describe two approaches to admissible transverse surgery on contact manifolds. The goal of this section is to prove the below theorem via two different constructions.

Theorem 3.14 (Baldwin, Etnyre)**.** *Let K be a transverse knot in a contact manifold (M, ξ) with a standard neighbourhood $N(K)$ of thickness $s > 0$. Then for any (reduced) rational number $r = \frac{p}{q} \in (-\infty, s)$, there exists a contact structure $\xi_K(r)$ on $M_K(r)$ that extends ξ.*

Such a contact surgery on K in (M, ξ) performed by these methods is called an admissible (transverse) r-surgery on K in (M, ξ). Both constructions we describe are outlined in [BE13]. We expand on the ideas behind the proofs, and provide detailed proofs of any claims.

As stated in Theorem 3.14 let $N(K)$ be a standard neighbourhood of K of thickness $s > 0$ and let $r = \frac{p}{q} \in (-\infty, s)$ be a rational surgery coefficient with p and q coprime. Note that the surgeries Theorem 3.14 restricts to are bounded by the thickness of this standard neighbourhood.

3.4.1 The 'Filling' Construction

In this 'filling' construction, we perform admissible surgery on a transverse knot K in a contact manifold (M, ξ) using the standard filling method of topological surgery on K in M. That is, we glue a solid torus to $\partial(\overline{M - N(K)})$ by an r-surgery map Φ. In order to extend ξ to the newly glued in solid torus, we will glue the model solid torus (S_b, ξ_U) to $\partial(\overline{M - N(K)})$ for some suitably chosen $b \in (-\infty, s)$. The parameter b is chosen so that the characteristic foliation of ξ_U on T_b is sent to the characteristic foliation of ξ on $\partial\overline{M - N(K)}$ under Φ. Then by Corollary 3.7, we conclude that the contact structure $\xi_K(r) = \xi \cup_\Phi \xi_U$ is a smooth extension of ξ.

Proof of Theorem 3.14. (Filling Construction). Let $N(K)$ be a standard neighbourhood of thickness s of a transverse knot K in a contact manifold (M, ξ). We identify $N(K)$ with S_s so that to perform surgery on K, we perform surgery on the core K_U of S_s.

Notice that the surgery coefficient r is chosen to be bounded above by s, the thickness of $N(K)$. This means that the solid torus S_r is contained in the solid torus S_s. Let $b \in (r, s)$ so that $S_r \subset S_b \subset S_s$. Note that this is the part of the proof that requires $r < s$, limiting the surgeries we can do with this construction.

Fix an $a \in (-\infty, \infty)$ and let $\Phi : \partial S_a \to \partial S_b$ be an $r-$surgery gluing map as

$$\Phi = \begin{bmatrix} p & m \\ q & n \end{bmatrix}. \tag{3.22}$$

That is, m and n are integers such that $pn - qm = 1$. We will perform r-surgery on M by removing the solid torus S_b and gluing the solid torus S_a to $\overline{M - S_b}$ by the map Φ for some appropriately chosen parameter a.

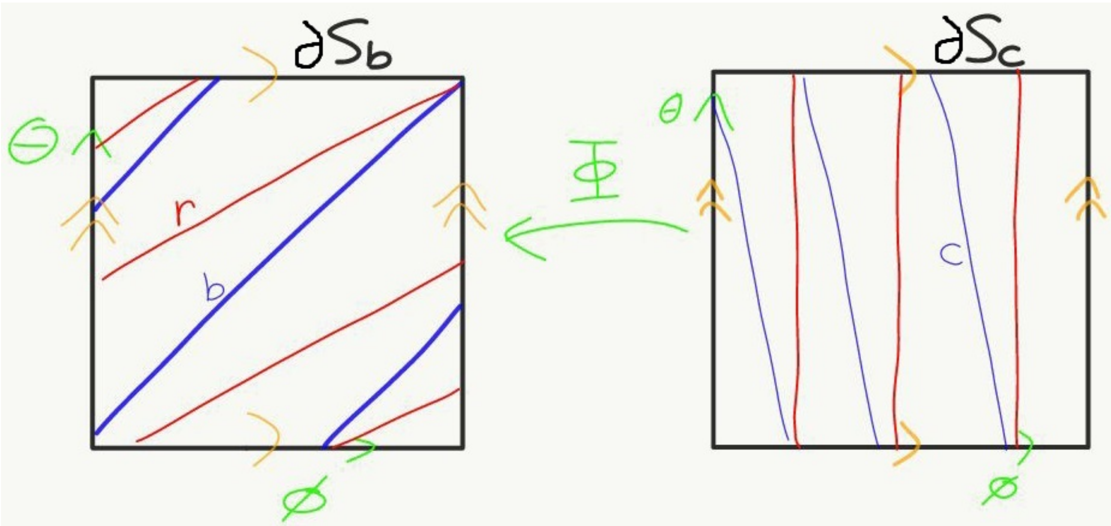

Figure 3.2: This diagram is the essential idea of the proof of the filling construction. There are two curves important curves on ∂S_b, the surgery slope-r curve (red) and the characteristic foliation slope-b curve (blue). To perform the surgery topologically, we need the surgery slope-r curve to bound a disk in the surgered manifold. This means we need the red curve on ∂S_b to be sent to a slope ∞ curve on ∂S_a by Φ^{-1}. This defines so the surgery, so now we examine where the blue curve on ∂S_b is sent by Φ^{-1}. We choose a to be the slope of the image of the blue curve on ∂S_b under Φ^{-1}. This ensures that the characteristic foliation on ∂S_a is sent to the characteristic foliation on ∂S_b under Φ. Then we can apply Corollary 3.7 to extend the contact structure ξ to $M_K(r)$.

The remainder of this proof will formalise the argument described in Figure 3.4.1. The following lemma is stated in [BE13] and we provide a proof.

Lemma 3.15. *There exists $a \in (-\infty, \infty)$ such that the characteristic foliation of ∂S_a is mapped under Φ to the characteristic foliation of ∂S_b.*

Proof. The characteristic foliation of ∂S_b is by slope-b curves on ∂S_b, which are the integral curves of the vector field $X_b = b\partial_\theta + \partial_\phi$. We calculate the slope of X_b after applying Φ_*^{-1} in the standard coordinate basis $(\partial_\theta, \partial_\phi)$.

$$\begin{aligned} \Phi_*^{-1} X_b &= \begin{bmatrix} n & -m \\ -q & p \end{bmatrix} \begin{bmatrix} b \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} nb - m \\ -qb + p \end{bmatrix} \end{aligned}$$

Let $a = \frac{nb-m}{-qb+p}$. This is chosen so that the characteristic foliation of ∂S_a is the foliation of the integral curves of $X_a = \Phi_*^{-1} X_b$. Note importantly that $-qb+p \neq 0$ because $b > r$ (in particular, $b \neq \frac{p}{q}$). So $a \in (-\infty, \infty)$. Since $\Phi_* X_a = X_b$, we have that the characteristic foliation of ∂S_a is mapped by Φ to the characteristic foliation of ∂S_b. $\square$

Choose an $a \in (-\infty, \infty)$ according to Lemma 3.15 so that $\Phi : \partial S_a \to \partial S_b$ sends the characteristic foliation on ∂S_a to the characteristic foliation on ∂S_b. Construct the topologically surgered manifold $M_K(r)$ by

$$M_K(r) = \overline{M - S_b} \cup_\Phi S_a. \tag{3.23}$$

Define a contact structure $\xi_K(r)$ on $M_K(r)$ by

$$\xi_K(r) = \xi \cup_\Phi \xi_U. \tag{3.24}$$

Since Φ sends the characteristic foliation of ξ_U on ∂S_a to the characteristic foliation of ξ on $\partial S_b = \partial(\overline{M - S_b})$, we have by Corollary 3.7 that $\xi_K(r)$ is a smooth contact structure on $M_K(r)$. $\square$

3.4.2 The 'Collapsing' Construction

Now we explore the alternate, 'collapsing' construction of admissible surgery. This construction is given in [BE13], and we prove some of the details referring to [Ler01] and [Gei97].

Proof of Theorem 3.14 (Collapsing Construction). Let K be a transverse knot in a contact manifold (M, ξ). Let $N(K)$ be a standard neighbourhood of K

of thickness $s > 0$ and let $r \in (-\infty, s)$ be rational. Identifying $(N(K), \xi)$ with (S_s, ξ_U), consider the solid torus torus S_r contained in the solid torus $S_s = N(K)$. Remove S_r from M to form a manifold with torus boundary $\overline{M - S_r}$. Then $\partial(\overline{M - S_r}) = T_r$. Define an S^1-action $\Psi : S^1 \times T_r \to T_r$ on T_r as in Equation 3.11.

$$\begin{aligned} \Psi : S^1 \times T_r \to T_r \\ \Psi_\psi(\theta, \phi) = (\theta + q\psi, \phi + p\psi) \end{aligned} \tag{3.25}$$

Recall that the orbits of Ψ are the slope-r curves on T_r. That is, the orbits of Ψ are the leaves of the characteristic foliation of ξ on T_r. Let $N = T_r \times I$ be a neighbourhood of T_r contained in $\overline{S_s - S_r}$ such that $T_r \times \{0\} = T_r$ and the characteristic foliation of ξ on $T_r \times \{t\}$ is linear. We will perform surgery on K by collapsing T_r by the S^1-action Ψ, then we will define a contact form on this quotient.

Extend Ψ on T_r to a free S^1-action on N by

$$\begin{aligned} \psi : N \to N \\ \psi \cdot (x, t) = (\Psi_\psi(x), t). \end{aligned}$$

Here, $\psi \in S^1$ and the map $\psi : N \to N$ represents the action of ψ on N.

Lemma 3.16. *The contact structure ξ on N is invariant under the action of ψ for all $\psi \in S^1$.*

Proof. In coordinates $((\theta, \phi), t)$ on $N = T_r \times I$, we have

$$\psi((\theta, \phi), t) = ((\theta, \phi), t) + ((q\psi, p\psi), 0). \tag{3.26}$$

That is, ψ is just the addition of a constant, so its derivative ψ_* is the identity. Therefore, a slope c curve through x is sent under ψ_* to a slope c curve through $\psi(x)$. This means that the characteristic foliation at $x \in T \times \{t\}$ is sent to the characteristic foliation at $\psi(x) \in T \times \{t\}$.

Consider the two contact structures ξ and $\psi_*(\xi)$ on N. They have the same characteristic foliation on $T \times \{t\}$ for all $t \in [0, 1]$. The characteristic foliation on a surface determines the germ of the contact structure about that surface (see Theorem 2.15). Therefore $\psi_*(\xi) = \xi$ on N and ξ is invariant under the action of ψ. $\square$

Lemma 3.17. *Suppose ξ is a contact structure on N invariant under some S^1-action. Then there exists an S^1-invariant contact form α for (N, ξ).*

Lemma 3.17 is stated in [BE13], where they refer to [Ler01] for a proof. The proof given in [Ler01] is fairly standard. The idea is to average a contact form by integrating it over the action of ψ with respect to a chosen Haar measure.

Applying Lemma 3.17, choose an S^1-invariant contact form α for the contact manifold (N, ξ). Let N' be the space N with the orbits of the S^1-action on T_r collapsed. Additionally, let T_r' be the image of T_r under this quotient. To extend α to N', recall that in Proposition 3.9, we defined a submanifold $P \subset N \times \mathbf{C}$ with an S^1-action and a homeomorphism $F : N' \to P/S^1$ by

$$P = \{(x, \rho^2, \rho e^{i\varphi}) \mid (x, \rho^2) \in T_r \times I, \rho e^{i\varphi} \in D^2\} \tag{3.27}$$

$$F : N' \to P/S^1$$

$$F([x, t]) = [x, t, \sqrt{t}]. \tag{3.28}$$

Further, F is a diffeomorphism by the definition of the smooth structure on N'.

Notice that $N' - T' = (N - T)' = N \times (0, 1]$. This means that α is already defined on $N' - T'$, and we want to show that α extends to N' by defining it smoothly on T'. We will do this by defining a contact form β' on P/S^1, and then show that $\alpha' = F^*\beta'$ agrees with α on $N' - T'$. When this is accomplished, we will have that α' is a smooth extension of α to N' because α and α' agree on $N' - T' = N \times (0, 1]$.

Consider the (real valued) 1-form $\omega = \rho^2 d\varphi$ on $\mathbf{C}$, defined in polar coordinates $z = \rho e^{i\varphi}$. Note that this form is rotationally invariant. Define a 1-form β on $N \times \mathbf{C}$ by

$$\beta = \alpha + \omega. \tag{3.29}$$

Here, we slightly abuse notation by writing $\alpha \in T^*(N \times \mathbf{C})$ to mean the extension of the 1-form $\alpha \in T^*N$ to $T^*(N \times \mathbf{C})$ and similarly for ω. The 1-form β restricts to an S^1-invariant form on P because both α and ω are invariant under the S^1-action on P (see Equation 3.16). Therefore, β descends uniquely to a 1-form β' on P/S^1.

We now prove that β is a contact form on $N \times \mathbf{C}$ and that β' is a contact form on P/S^1 in the following two lemmas. Both of these lemmas are stated without proof in [BE13]. We prove the details of Lemma 3.18 independently. A much more general version of Lemma 3.19 is proved in [Ler01] and [Gei97]. We follow these proofs, however we have simplified them significantly to apply to our specific situation.

Lemma 3.18. *The 1-form β is a (5-dimensional) contact form on $N \times \boldsymbol{C}$.*

Proof. To check that β is a contact form on $N \times \mathbf{C}$, we show that $\beta \wedge (d\beta)^2$ is a positive 5-form (see Remark 2.5). Note that this is the only time in this thesis where we consider a contact manifold of degree higher than 3.

$$\beta \wedge (d\beta)^2 = (\alpha + \omega) \wedge (d\alpha + d\omega) \wedge (d\alpha + d\omega)$$

Notice that $d\alpha \wedge d\alpha$ is a 4-form on N, a 3-manifold. Therefore $d\alpha \wedge d\alpha = 0$. Similarly, $\omega \wedge d\omega = d\omega \wedge d\omega = 0$. Expanding while taking this into account, we obtain the following:

$$\begin{aligned}\beta \wedge (d\beta)^2 &= \alpha \wedge d\alpha \wedge d\omega + \alpha \wedge d\omega \wedge d\alpha \\ &= 2\alpha \wedge d\alpha \wedge d\omega \\ &> 0.\end{aligned}$$

The 5-form $\alpha \wedge d\alpha \wedge d\omega$ is positive on $N \times \mathbf{C}$ because both $\alpha \wedge d\alpha$ and $d\omega = 2\rho d\rho \wedge d\varphi$ are. Thus β is a contact form. □

Lemma 3.19. *The 1-form β' is a contact form on P/S^1.*

Proof. The standard coordinate basis of the tangent space $T(N \times \mathbf{C})_x$ is $((\partial_\theta, \partial_\phi, \partial_t), (\partial_\rho, \partial_\varphi))$ for $x = ((\theta, \phi, t), (\rho, \varphi)) \in N \times \mathbf{C}$. When we restrict to P, the vectors ∂_t and ∂_ρ become linearly dependent (see Equation 3.27). When we quotient by the S^1-action on P, ∂_φ is sent to zero. Therefore the vectors $\{\overline{\partial_\theta}, \overline{\partial_\phi}, \overline{\partial_t}\}$ span the tangent space of P/S^1, where $\overline{\partial_\theta}, \overline{\partial_\phi}$ and $\overline{\partial_t}$ are the images of $\partial_\theta, \partial_\phi$ and ∂_t in the quotient $T(P/S^1)$. The tangent space of P/S^1 is three dimensional, therefore $(\overline{\partial_\theta}, \overline{\partial_\phi}, \overline{\partial_t})$ is a (positively oriented) basis of $T(P/S^1)$. We check that $\beta' \wedge d\beta'$ is positive on this basis.

$$\begin{aligned}(\beta' \wedge d\beta')(\overline{\partial_\theta}, \overline{\partial_\phi}, \overline{\partial_t}) &= (\beta \wedge d\beta)(\partial_\theta, \partial_\phi, \partial_t) \\ &= (\alpha + \omega) \wedge (d\alpha + d\omega)(\partial_\theta, \partial_\phi, \partial_t) \\ &= (\alpha \wedge d\alpha)(\partial_\theta, \partial_\phi, \partial_t) \\ &> 0\end{aligned}$$

The form $\alpha \wedge d\alpha$ is positive by the contact condition for the contact manifold (N, α). Therefore β' is a contact form on P/S^1. □

Thus we have constructed a contact form β' on P/S^1. Define a contact form α' on N' by

$$\alpha' = F^* \beta'. \tag{3.30}$$

We will now show that α' and α agree on $N \times (0,1]$. Recall the following diagram (Equation 3.18) defined in Proposition 3.9, where $i : N \to P$ is defined by $i(x,t) = (x, t, \sqrt{t})$ (Equation 3.15).

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"] & P \arrow[d, two heads, "\rho"] \\ N' \arrow[r, dashed, "F"'] & P/S^1 \end{tikzcd} \tag{3.31}$$

Let v be a vector in the tangent space of $N' - T' = T \times (0,1]$. Then v can be viewed both as a tangent vector in TN' and as a tangent vector in TN. From the diagram in Equation 3.31, this means that we have that $\beta'(F_*v) = \beta(i_*v)$. We apply this in the following calculation:

$$\begin{aligned} \alpha'(v) &= \beta'(F_*v) \\ &= (\alpha + \omega)(i_*v) \\ &= \alpha(i_*v) + \omega(i_*v) \\ &= \alpha(v). \end{aligned}$$

The last line follows because $i : N \to P \subset N \times \mathbf{C}$ is the identity on the N component of P, and has a constant φ-coordinate on the $\mathbf{C}$ component of P. Thus i_*v has no ∂_φ component and $\omega(i_*v) = 0$. This shows that $\alpha' = \alpha$ on $N' - T' = T \times (0,1]$ and thus α' is a smooth extension of α to N'. This defines the required extension (M', ξ') of (M, ξ). □

Remark 3.20. Notice that in this proof, the form $\omega = \rho^2 d\varphi$ on P is sent to zero in the quotient $T^*(P/S^1)$ because the φ coordinate of P is collapsed in P/S^1. So it is superfluous to define β' as the descent of $\beta = \alpha + \omega$ when we could have defined it to be the descent of α. However the reason this is done in [BE13] is to emphasis that β is a contact form on $N \times \mathbf{C}$ and β' is the reduction of this form by the S^1-action. This whole construction is a specific example of a more general construction called a contact cut, described in [Ler01]. This is the contact geometry analogue of a symplectic reduction in symplectic manifold theory (see [KZ19]).

Thus we have exhibited two constructions of admissible surgery. We now prove that both constructions are contactomorphic.

Proposition 3.21. *The contact manifolds $(M_K(r), \xi_K(r))$ and (M', ξ') obtained from the filling and collapsing constructions of admissible r-surgery on K are contactomorphic.*

Proof. As before, let $N(K) = S_s$ be a standard neighbourhood of K of thickness $s > 0$. In both surgery constructions, there is a solid torus S_b such that $S_r \subset S_b \subset S_s$ and

$$\left(\overline{M_K(r) - S_b}, \xi_K(r)\right) = \left(\overline{M - S_b}, \xi\right) = \left((\overline{M - S_b})', \xi'\right). \tag{3.32}$$

Let K_U be the core of S_b and let $K' = T'_r$ be the core of the collapsed torus T_r in M'. Then K_U is transverse to $\xi_K(r)$ and the slope of the characteristic foliation on the torus T_ρ is monotonically increasing from $-\infty$ to b as ρ increases from $-\infty$ to b. The same is true for the neighbourhood S'_b of K' in (M', ξ'); the knot K' is transverse to ξ', and S'_b is exhausted by nested tori whose characteristic foliation slope increase monotonically from $-\infty$ up to b. Let $T'_\rho \subset S'_b$ denote the torus with slope-ρ characteristic foliation.

For each torus $T_\rho \subset S_s$ and $T'_\rho \subset S'_s$, we can find a diffeomorphism $\Phi_\rho : T_\rho \to T'_\rho$ that sends the characteristic foliation on T_ρ to the characteristic foliation on T'_ρ. Theorem 2.15 gives that $\Phi_\rho : T_\rho \to T'_\rho$ extends to a contactomorphism $\Phi_\rho : N(T_\rho) \to N(T'_\rho)$ for suitable open neighbourhoods $N(T_\rho)$ of T_ρ and $N(T'_\rho)$ of T'_ρ. Then the set $\{N(T_\rho)\}$ for $\rho \in (-\infty, b]$ is an open cover of S_b and similarly $\{N(T'_\rho)\}$ is an open cover for S'_b. By compactness of S_b and S'_b, choose finitely many $\rho_i, i = 1, ..., n$ such that $\{N(T_{\rho_i})\}$ covers S_b and $\{N(T'_{\rho_i})\}$ covers S'_b. Then a contactomorphism $\Phi : (S_b, \xi_K(r)) \to (S'_b, \xi')$ can be constructed by patching together the Φ_{ρ_i}'s for $i = 1, ..., n$. This shows that $(M_K(r), \xi_K(r))$ and (M', ξ') are contactomorphic because they are equal outside of S_b. □

In summary, admissible surgery is a very general and widely applicable answer to the question of "when is contact surgery possible?". We have just shown that for any transverse knot K in any contact manifold (M, ξ), there is a family of contact surgeries that can be performed on K, with the surgeries possible depending on the size of a given standard neighbourhood. This is a great first answer and allows us to build many examples of contact manifolds. However there is no guarantee that this surgery preserves tightness. Moving forward, we ask what restrictions and alternative methods of surgery we can perform to ensure the surgery preserves tightness. This may make it necessary to have more restrictions and will certainly be less generally applicable (tight manifolds are somewhat sporadic), but has the benefit of producing tight manifolds.

Chapter 4

Tight Surgery on S^3

We saw in Chapter 3 that given a (closed, oriented) contact manifold (M, ξ) and a transverse knot K in M, we are able to perform a range of contact surgeries on K, with this range being determined by what standard neighbourhood of K we can find. This is a very general statement about contact surgery because it applies to a large class of knots in any contact manifold. However, there is no guarantee that an admissible surgered contact manifold is tight, even if the contact manifold being surgered is tight. A natural question to ask then is how can we perform contact surgery in such a way that preserves tightness? Further, what restrictions do we need on (M, ξ) and possibly K in order to be able to perform tight surgery? Attempting to construct and classify tight manifolds is a very relevant area of contact geometry research because techniques for constructing examples of tight manifolds are currently limited. Developing methods of contact surgery that preserve tightness would be new progress towards this area.

In this chapter, we explore a construction of tight surgery on an unknot in $(S^3, \xi_{\text{tight}})$. That is, we show that for a specifically chosen transverse unknot γ in S^3 with its unique tight contact structure, we can perform any rational surgery on γ to obtain a new tight manifold (specifically, this will be a tight lens space). Further, the rational surgeries we can do using this method are unrestricted, unlike admissible surgery. Justifying that a surgered manifold is tight is not straightforward, because after making such a dramatic change to the manifold through surgery, it is quite plausible that this process may have introduced an overtwisted disk. This difficulty is illustrated in Example 4.14. The construction we will describe circumvents this issues by studying the Reeb dynamics of a chosen contact form for ξ_{tight} and utilising deep results between Reeb orbits and tightness due to Hofer, Wyzocki and Zehnder.

The method of tight surgery on $(S^3, \xi_{\text{tight}})$ we present in this chapter is from a paper called Tight Contact Structures via Dynamics by Etnyre and Ghrist ([EG99]). Their central idea in this paper is to use a sourced theorem that relates the Reeb dynamics of a contact form to the tightness of the associated contact structure. During the contact surgery, they keep track of the relevant Reeb dynamics information, and in the end use this theorem to justify that their surgered contact form is tight. In this chapter, we follow their construction, giving insight to the ideas behind their methods and providing details and proofs of any claims.

4.1 Reeb Dynamics and Tightness

Justifying that a contact manifold is tight is usually not easy, because proving the non-existence of an overtwisted disk can be difficult. In particular, knowing only local information about the manifold, for instance a neighbourhood of a knot as in the case of surgery, is not enough to know whether the manifold is tight. Some sort of global information is required.

The way Etnyre and Ghrist avoid this issue in [EG99] is through studying the Reeb dynamics of a carefully chosen contact form associated to a contact structure. A contact form α inherently carries more geometry than a contact structure $\xi = \ker \alpha$ because there is a Reeb vector field associated to α. Recall from Section 2.4 that the Reeb vector field X_α of a contact form α is the vector field transverse to the contact structure ξ whose flow preserves ξ, and is normalised so that $\alpha(X_\alpha) = 1$. This is equivalent to X_α satisfying the two conditions $i_{X_\alpha} d\alpha = 0$ and $i_{X_\alpha} \alpha = 1$. In [HWZ95], Hofer proved Theorem 4.1, which illustrates a deep connection between the Reeb dynamics of a contact form and the tightness of its associated contact structure. This result comes from his work towards proving the Weinstein Conjecture, which was the conjecture that every contact form has a closed Reeb orbit. This is no longer conjecture, proved to be true by Taubes in 2007 ([Tau09]).

Theorem 4.1 (Hofer [HWZ95]). *Let ξ be an overtwisted contact structure on a closed 3-manifold M and α a contact form for ξ. Then α has at least one periodic Reeb orbit of finite order in $\pi_1(M)$.*

Recall from Section 2.4.1 that the Reeb orbits of a contact form α are the integral curves of its Reeb vector field. Theorem 4.1 illustrates a connection between tight manifolds and Reeb orbits because it implies that if a contact manifold has no periodic Reeb orbits of finite order, then it must be tight. This result was refined in [HK99] by studying the dynamics of these periodic Reeb orbits.

Theorem 4.2 (Hofer, Wyzocki, and Zehnder [HK99]). *Let ξ be an overtwisted contact structure on a closed 3-manifold M and α a contact form for ξ. If α has no degenerate periodic Reeb orbits, then α has at least one closed hyperbolic Reeb orbit of finite order in $\pi_1(M)$.*

We refer the reader back to Section 2.4.1 for definitions of degenerate, elliptic and hyperbolic reeb orbits. Theorems 4.1 and 4.2 are both highly technical results which rely on the study of symplectic manifolds, specifically Gromov's theory of pseudoholomorphic curves (see [Gro85]). Additionally, these theorems are only used as sourced results in [EG99] with no proof given for either of them. For these reasons, we also do not provide proofs of these theorems, but refer the reader to [HWZ95] and [HK99].

In [EG99], Etnyre and Ghrist use Theorem 4.2 to accomplish tight surgery on S^3 through the following corollary.

Corollary 4.3. *Let ξ be a contact structure on a closed 3-manifold M and α a contact form for ξ. If α has no degenerate or hyperbolic periodic Reeb orbits, then ξ is tight.*

Thus (M, α) is tight if α is obstructed from having degenerate of hyperbolic periodic Reeb orbits. This motivates the following definitions to describe the types of contact forms we are interested in constructing for this method of tight surgery. These definitions are not standard or used in [EG99], however they highlight the type of Reeb dynamics that allow us to apply these surgery techniques.

Definition 4.4. Let α be a contact form for (M, ξ). A Reeb orbit of α is *bad* if it periodic and degenerate or hyperbolic. A contact form is said to have *globally nice dynamics* if it has no bad periodic Reeb orbits.

Definition 4.5. Let α be a contact form for (M, ξ) and S a solid torus in M. The contact form α has *nice dynamics inside* S if α has no bad Reeb orbits in S and every Reeb orbit of α is tangent to some torus T contained in S.

The idea of Etnyre and Ghrist's construction is to begin with a standard contact form α_{tight} representing ξ_{tight} on S^3 (see Section 2.3), which has infinitely many bad Reeb orbits. This contact form is modified to a contact form α_a

representing ξ_{tight} that has globally nice dynamics. Then, contact surgery is performed on a specific transverse unknot $\gamma \subset S^3$ in such a way that the resulting surgered contact form has nice dynamics. This is accomplished by showing that α_a has nice dynamics inside a neighbourhood $N(\gamma)$ of γ. When we have such a nice neighbourhood $N(\gamma)$, it can be removed without modifying the Reeb orbits outside of $N(\gamma)$. This means for the new surgered manifold to be tight, it is enough to shown that it has no bad orbits inside the surgery neighbourhood. This idea will be made more precise in Section 4.3.

4.2 A Family of Model Contact Forms on S^3

Corollary 4.3 gives a condition on the periodic Reeb orbits of a contact form to ensure tightness of a contact structure. With that as motivation, we consider the Reeb orbits of a family of particular model contact forms on S^3. To construct this family of contact forms on S^3, we follow the construction outlined in [EG99]. Each proposition and lemma in this section is stated without proof in [EG99]; we prove these claims and provide insight to the construction.

Consider $\mathbf{C}^2$ with coordinates $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$ for $r_1, r_2 \geq 0$ and $\theta_1, \theta_2 \in \mathbf{R}/2\pi\mathbf{Z}$. Define a 1-form $\widetilde{\alpha}_1$ on $\mathbf{C}^2$ by

$$\widetilde{\alpha}_1 = r_1^2 d\theta_1 + r_2^2 d\theta_2. \tag{4.1}$$

Remark 4.6. In [EG99], the definition $\widetilde{\alpha}_1 = \frac{1}{2}(r_1^2 d\theta_1 + r_2^2 d\theta_2)$ is made instead. This is most likely to emphasise that $d\widetilde{\alpha}_1$ is the standard symplectic form on $\mathbf{C}^2$. We remove this factor of a half for convenience, noting that it makes no significant differences to the tight surgery construction.

We view S^3 as a subset of $\mathbf{C}^2$, with coordinates $(\sin\rho e^{i\theta}, \cos\rho e^{i\phi})$ for $\rho \in [0, \frac{\pi}{2}]$ and $\theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. To understand this geometrically as S^3, define the following sets.

$$T_\rho = \{(\sin\rho e^{i\theta}, \cos\rho e^{i\phi}) \mid \theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}\} \tag{4.2}$$
$$S_\rho = \{(\sin r e^{i\theta}, \cos r e^{i\phi}) \mid \theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}, 0 \leq r \leq \rho\}. \tag{4.3}$$

For any $\rho \in (0, \frac{\pi}{2})$, T_ρ is a torus parametrised by θ and ϕ. As seen in Section 3.1, these coordinates on T_ρ define a representative meridian μ and preferred longitude λ. For $\rho = 0$ or $\frac{\pi}{2}$, notice that T_ρ is just S^1 parametrised by ϕ or θ, respectively. We label these circles B_1 and B_2 respectively, but will sometimes refer to them as T_0 and $T_{\frac{\pi}{2}}$ for ease of notation.

Notice that for any fixed $\rho_0 \in (0, \frac{\pi}{2})$, the torus T_{ρ_0} separates S^3 into the solid torus S_{ρ_0} and the solid torus $S^3 - \overline{S_{\rho_0}}$. This presents S^3 as two solid tori glued along their boundary T_{ρ_0}. Further, B_1 is the core of S_{ρ_0} and B_2 is the core of the complement of S_{ρ_0}. Thus the circles B_1 and B_2 are linked unknots and $B = B_1 \cup B_2$ is a Hopf link.

Define $\alpha_1 = \widetilde{\alpha}_1|_{S^3}$ to be the restriction of $\widetilde{\alpha}_1$ to S^3, given in coordinates as

$$\alpha_1 = \sin^2 \rho d\theta + \cos^2 \rho d\phi. \tag{4.4}$$

Recalling Section 2.1, we recognise the kernel of α_1 as the (unique) tight contact structure ξ_{tight} on S^3.

To understand the contact structure $(S^3, \xi_{\text{tight}})$, it is useful to understand the characteristic foliation it prints on some family of surfaces. So we calculate the characteristic foliation of ξ_{tight} on the family of tori T_ρ for $\rho \in (0, \frac{\pi}{2})$.

Proposition 4.7. *Let $\rho \in (0, \frac{\pi}{2})$. The characteristic foliation of ξ_{tight} on the torus T_ρ is the foliation of slope $-\tan^2 \rho$ curves.*

Proof. The standard coordinate basis for the tangent space of T_ρ is $\{\partial_\theta, \partial_\phi\}$. Suppose $v = v_\theta \partial_\theta + v_\phi \partial_\phi$ is a tangent vector in $\xi_{\text{tight}} = \ker \alpha_1$. Then we obtain the following relation.

$$\begin{aligned} (\sin^2 \rho d\theta + \cos^2 \rho d\phi)(v) &= 0 \\ \sin^2 \rho v_\theta + \cos^2 \rho v_\phi &= 0 \\ v_\phi &= -\tan^2 \rho v_\theta. \end{aligned} \tag{4.5}$$

That is, the space $\xi_{\text{tight}} \cap T(T_\rho) \subset T(S^3)$ is spanned by the vector field $V = \partial_\theta - \tan^2 \rho \partial_\phi$. The characteristic foliation of ξ_{tight} on T_ρ is the foliation of the integral curves V, which are the slope $-\tan^2 \rho$ curves on T_ρ. $\square$

To understand the contact form α_1 representing ξ_{tight}, we calculate the Reeb vector field X_1 associated to α_1.

Proposition 4.8. *The Reeb vector field of the contact form α_1 on S^3 is $X_1 = \partial_\theta + \partial_\phi$.*

Proof. We calculate $d\alpha_1$ and $\alpha_1(X_1)$.

$$\begin{aligned} d\alpha_1 &= \sin(2\rho) d\rho \wedge d\theta - \sin(2\rho) d\rho \wedge d\phi \\ &= \sin(2\rho) d\rho \wedge (d\theta - d\phi) \\ \alpha_1(X_1) &= (\sin^2 \rho d\theta + \cos^2 \rho d\phi)(\partial_\theta + \partial_\phi) \\ &= \sin^2 \rho + \cos^2 \rho \\ &= 1. \end{aligned}$$

Notice that X_1 is in the kernel of $\rho d\rho$ and $d\theta - d\phi$, hence $d\alpha_1(X_1, -) = 0$. Thus X_1 is the unique vector field satisfying $i_{X_1} d\alpha_1 = 0$ and $i_{X_1}\alpha_1 = 1$. □

From Proposition 4.8, the Reeb vector field $X_1 = \partial_\theta + \partial_\phi$ of α_1 has integral curves tangent to T_ρ with slope 1. That is, the Reeb orbits of this contact form are the slope 1 curves on T_ρ for $\rho \in (0, \frac{\pi}{2})$, and are tangent to T_0 and $T_{\frac{\pi}{2}}$. The Reeb orbits of α_1 having rational slope means that T_ρ is foliated by bad Reeb orbits. This is a situation we need to avoid in order to apply Corollary 4.3. Instead, we would like control over the slope of the Reeb orbits, particularly so we can choose this slope to be irrational. Doing this will remove all of the closed orbits on T_ρ for $\rho \in (0, \frac{\pi}{2})$, making it more feasible to apply Corollary 4.3.

Fix a real number $a > 0$ and consider the shear map $K_a : \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$K_a(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = (r_1 e^{i\theta_1}, a r_2 e^{i\theta_2}). \tag{4.6}$$

That is, we are scaling the second coordinate of $\mathbf{C}^2$ by a factor of a. Viewing S^3 as embedded in $\mathbf{C}^2$, we can think of this as a deformation of S^3 to $\widehat{S} = K_a(S^3)$. This deformed S^3 can be described in coordinates as below.

$$\begin{aligned}\widehat{S} &= K_a(S^3)\\ &= \left\{(\sin\rho e^{i\theta}, a\cos\rho e^{i\phi})\right\}\\ &= \left\{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \mid r_1^2 + \frac{1}{a^2} r_2^2 = 1\right\} \subset \mathbf{C}^2. \end{aligned} \tag{4.7}$$

Define a contact form on $\widehat{S}$ by

$$\widehat{\alpha} = \widetilde{\alpha}_1|_{\widehat{S}}. \tag{4.8}$$

That is, we have stretched S^3 into $\widehat{S}$ and now we are considering the contact form $\widetilde{\alpha}_1$ restricted to $\widehat{S}$. Similarly to before, we label the following distinguished tori in $\widehat{S}$.

$$\widehat{T}_\rho = \{(\sin\rho e^{i\theta}, a\cos\rho e^{i\phi}) \mid \theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}\} \tag{4.9}$$

$$\widehat{S}_\rho = \{(\sin r e^{i\theta}, a\cos r e^{i\phi}) \mid \theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}, 0 \le r \le \rho\} \tag{4.10}$$

That is, $\widehat{T}_\rho$ is the image of T_ρ under K_a and similarly for $\widehat{S}_\rho$. The tori $\widehat{T}_\rho$ naturally inherits a preferred longitude $K_a(\lambda)$.

Proposition 4.9. *The Reeb vector field of $\widehat{\alpha}$ on $\widehat{S}$ is $\widehat{X} = \partial_\theta + \frac{1}{a^2}\partial_\phi$.*

Proof. We have $\widehat{\alpha} = \sin^2 \rho d\theta + a^2 \cos^2 \rho d\phi$. By an analogous calculation to Proposition 4.8, $d\widehat{\alpha} = \sin 2\rho d\rho \wedge (d\theta - a^2 d\phi)$. Thus $d\widehat{\alpha}(\widehat{X}, -) = 0$ and $\widehat{\alpha}(\widehat{X}) = 1$, so $\widehat{X}$ is the Reeb vector field of $\widehat{\alpha}$. $\square$

From Proposition 4.9, the Reeb orbits of $\widehat{\alpha}$ are tangent to $\widehat{T}_\rho$ with slope $\frac{1}{a^2}$. Thus by varying a, we are now able to choose the slope of the Reeb orbits of $\widehat{\alpha}$. Finally, we realise $\widehat{\alpha}$ as a contact form on S^3. Define a function $H_a : \mathbf{C}^2 \to \mathbf{R}$ by

$$\begin{aligned} H_a &: \mathbf{C}^2 \to \mathbf{R} \\ H_a(z) &= r_1^2 + \frac{1}{a^2} r_2^2. \end{aligned} \tag{4.11}$$

Thus $\widehat{S} = H_a^{-1}(1)$. Now define a map $\psi : S^3 \to \mathbf{C}^2$ by

$$\begin{aligned} \psi &: S^3 \to \mathbf{C}^2 \\ z &\mapsto \frac{1}{\sqrt{H_a(z)}} z. \end{aligned} \tag{4.12}$$

Here, $z = (z_1, z_2) = (\sin \rho e^{i\theta}, \cos \rho e^{i\phi}) \in \mathbf{C}^2$.

Proposition 4.10. *The map $\psi : S^3 \to \mathbf{C}^2$ is a diffeomorphism onto $\widehat{S} \subset \mathbf{C}^2$.*

Proof. First we check that the image of ψ is contained in $H_a^{-1}(1) = \widehat{S}$.

$$\begin{aligned} H_a(\psi(z)) &= H_a \left(\frac{1}{\sqrt{H_a(z)}} z_1, \frac{1}{\sqrt{H_a(z)}} z_2 \right) \\ &= \left(\frac{\sin \rho}{\sqrt{H_a(z)}} \right)^2 + \frac{1}{a^2} \left(\frac{\cos \rho}{\sqrt{H_a(z)}} \right)^2 \\ &= \frac{\sin^2 \rho + \frac{1}{a^2} \cos^2 \rho}{H(z)} \\ &= \frac{\sin^2 \rho + \frac{1}{a^2} \cos^2 \rho}{\sin^2 \rho + \frac{1}{a^2} \cos^2 \rho} \\ &= 1. \end{aligned}$$

The map ψ is just a resizing of the radii (r_1, r_2) for each $z \in S^3$, rescaling S^3 to $\widehat{S}$. Thus ψ is a diffeomorphism onto $\widehat{S}$, with inverse given by undoing this resizing. Specifically, the inverse of ψ on $H_a^{-1}(1)$ is given by

$$\psi^{-1} \left(\rho_1 e^{i\theta}, a\sqrt{1 - \rho_1^2} e^{i\phi} \right) = \left(\frac{\rho_1^2}{a^2(1 - \rho_1^2) + \rho_1^2} e^{i\theta}, \frac{\rho_1^2}{a^2(1 - \rho_1^2) + \rho_1^2} e^{i\phi} \right). \tag{4.13}$$

$\square$

Define a contact form α_a on S^3 by

$$\alpha_a = \psi^*(\widehat{\alpha}). \tag{4.14}$$

By construction, ψ is a strict contactomorphism from (S^3, α_a) to $(\widehat{S}, \widehat{\alpha})$.

Proposition 4.11. *The Reeb vector field of the contact form α_a on S^3 is $X_a = \partial_\theta + \frac{1}{a^2}\partial_\phi$.*

Proof. Since $\alpha_a = \psi^*(\widehat{\alpha})$ and ψ is a diffeomorphism, it is enough to check that $D\psi(X_a) = \widehat{X}$ by Lemma 2.23. But ψ is the identity on the θ and ϕ coordinates of $\mathbf{C}^2$, and therefore $D\psi$ acts as the identity on ∂_θ and ∂_ϕ.

$$\begin{aligned}
(D\psi(X_a))_{\psi(z)} &= D\psi|_z \cdot (X_a)_z \\
&= D\psi|_z \left(\partial_\theta + \frac{1}{a^2}\partial_\phi\right)_z \\
&= \left(\partial_\theta + \frac{1}{a^2}\partial_\phi\right)_{\psi(z)} \\
&= \widehat{X}_{\psi(z)}.
\end{aligned}$$

That is, $D\psi(X_a) = \widehat{X}$. □

Similar to before, the Reeb orbits of α_a on S^3 are tangent to the tori T_ρ, but now have slope $\frac{1}{a^2}$. We thus have the desired control of the slope of the Reeb orbits by changing the parameter a. In particular, a can be chosen to be irrational, to remove most of the periodic orbits on T_ρ as discussed earlier.

Proposition 4.12. *The forms α_a and α_1 on S^3 are related by*

$$\alpha_a = \frac{1}{H_a(z)}\alpha_1. \tag{4.15}$$

Proof. Let $z = (\sin\rho e^{i\theta}, \cos\rho e^{i\phi})$. The map $\psi : S^3 \to \widehat{S}$ is the identity on the θ and ϕ coordinates, therefore $\psi^*(d\theta_{\psi(z)}) = d\theta_z$ and $\psi^*(d\phi_{\psi(z)}) = d\phi_z$. We calculate

α_a at z.

$$\begin{aligned}
(\alpha_a)_z &= (\psi^*\widehat{\alpha})_z \\
&= \widehat{\alpha}_{\psi(z)} \circ D\psi|_z \\
&= \left(\left(\frac{\sin\rho}{\sqrt{H(z)}}\right)^2 d\theta + \left(\frac{\cos\rho}{\sqrt{H(z)}}\right)^2 d\phi\right)_{\psi(z)} \circ D\psi|_z \\
&= \frac{1}{H(z)}\left(\sin^2\rho d\theta + \cos^2\rho d\phi\right)_{\psi(z)} \circ D\psi|_z \\
&= \frac{1}{H(z)}\left(\sin^2\rho(d\theta\circ D\psi) + \cos^2\rho(d\phi \circ D\psi)\right)_z \\
&= \frac{1}{H(z)}\left(\sin^2\rho d\theta + \cos^2\rho d\phi\right)_z \\
&= \left(\frac{1}{H(z)}\alpha_1\right)_z
\end{aligned}$$

Thus $\alpha_a = \frac{1}{H_a(z)}\alpha_1$. □

In particular, Proposition 4.12 implies that α_1 and α_a are both forms representing the same contact structure $\ker\alpha_a = \ker\alpha_1 = \xi_{\text{tight}}$. This also gives a description of α_a in coordinates.

$$\alpha_a = \frac{1}{\sin^2\rho + \frac{1}{a^2}\cos^2\rho}(\sin^2\rho d\theta + \cos^2\rho d\phi) \tag{4.16}$$

Thus we have constructed a contact form α_a for the contact manifold $(S^3, \xi_{\text{tight}})$ where we are able to choose the slope of the Reeb orbits of α_a by choosing the parameter a appropriately. Having this control over the dynamics of (S^3, α_a) allows us to usefully apply Corollary 4.3 later in the tight surgery construction. The following diagram is a summary of the constructed contact forms and how they are related.

$$\begin{tikzcd}
\mathbf{C}^2 \arrow[r, "K_a"] & \mathbf{C}^2 & \\
(S^3, \alpha_1) \arrow[u, hook] \arrow[r, "K_a"'] & (\widehat{S}, \widehat{\alpha}) \arrow[u, hook] & (S^3, \alpha_a) \arrow[l, "\sim", "\psi"']
\end{tikzcd} \tag{4.17}$$

4.3 Tight Surgery Construction

We began Section 4.2 with the tight contact structure $\xi_{\text{tight}} = \ker\alpha_1$ on S^3, represented by the contact form α_1. However, α_1 does not have nice dynamics. So, we constructed a new contact form α_a representing the same contact structure

ξ_{tight}, but with the property that the Reeb orbits are tangent to the tori T_ρ and have slope $\frac{1}{a^2}$. For the rest of this chapter, $a > 0$ is a fixed irrational number so that the Reeb orbits of α_a have irrational slope. Thus the Reeb orbits of α_a tangent to T_ρ for $\rho \in (0, \frac{\pi}{2})$ are not periodic. This means that the only periodic Reeb orbits of α_a are the two Reeb orbits tangent to the circles T_0 and $T_{\frac{\pi}{2}}$.

Proposition 4.13. *The contact form α_a on S^3 has two closed Reeb orbits, both of which are elliptic.*

Proof. Proposition 4.11 states that the Reeb vector field of α_a is $X_a = \partial_\theta + \frac{1}{a^2}\partial_\phi$. The periodic orbits of this vector field are $\gamma_1 = B_1 = T_0$ and $\gamma_2 = B_2 = T_{\frac{\pi}{2}}$. These Reeb orbits correspond to the integral curves of X_a restricted to $\rho = 0$ and $\rho = \frac{\pi}{2}$ respectively. Otherwise, the integral curves of X_a are tangent to the tori T_ρ of slope $\frac{1}{r^2}$. Since r is irrational, there are no periodic orbits tangent to any of these tori and therefore γ_1 and γ_2 are the only two periodic orbits of X_a.

The periodic Reeb orbit γ_i for $i = 0, 1$, is isolated from other periodic Reeb orbits, hence it is non-degenerate because there is no clustering of periodic Reeb orbits near γ_i (see Section 2.4.1). Further, the Reeb orbits near γ_i lie on the tori T_ρ for $\rho \in (0, \frac{\pi}{2})$. This means that the (complex) eigenvalues of the linearised first return maps must have norm 1 because these Reeb orbits are not being attracted to or repelled from γ_i. Hence both γ_0 and γ_1 are elliptic, as explained in Section 2.4.1. □

Now everything is in place for the tight surgery construction. The idea of the proof from [EG99] is as follows. First, a toroidal neighbourhood $N(\gamma)$ of the periodic Reeb orbit $\gamma = \gamma_1$ is removed from (S^3, α_a). The neighbourhood $N(\gamma)$ is chosen that they α_a has nice dynamics inside $N(\gamma)$. Then a solid torus $S = D^2 \times S^1$ is glued to $\overline{S^3 - N(\gamma)}$ via an appropriate surgery map. Pulling back α_a by this surgery map yields a contact form near ∂S. A particular model structure $\overline{\alpha}$ on a solid torus N is chosen so that $\overline{\alpha}$ is strictly contactomorphic to a neighbourhood of ∂S. We then glue $(N, \overline{\alpha})$ to $\overline{S^3 - N(\gamma)}$ and extend α_a smoothly. The fact that $\overline{\alpha}$ is one of the previously defined model forms means that this extension has nice dynamics inside N in the surgered S^3, allowing us to conclude that this surgered manifold is tight.

One precaution we need to take when attempting tight surgery is to ensure we are not introducing any overtwisted disks in the surgered manifold. When performing topological surgery on a knot γ in S^3, we are changing what curves on $\overline{S^3 - N(\gamma)}$ bound disks. Thus, it is a priori plausible that this could result introducing a new disk that happens to be overtwisted. The following example from [EG99] illustrates the difficulty of tight surgery and the necessity for $N(\gamma)$ to be sufficiently 'large' to avoid the issue of introducing an overtwisted disk. In the following construction, we will highlight where this issue is accounted for.

Example 4.14 (Surgery on a 'Small' Torus)**.** Let $\rho_0 = \tan^{-1}(\sqrt{2})$ and consider the torus T_{ρ_0} in $(S^3, \xi_{\text{tight}})$. We showed in Proposition 4.7 that the characteristic foliation of ξ_{tight} on T_{ρ_0} is the foliation of slope $-\tan^2 \rho_0$ curves. That is, slope -2 curves. Suppose we were to perform -2 surgery on γ_1 with $N(\gamma_1) = S_\rho$ for some fixed $\rho < \rho_0$. We glue a solid torus to $\overline{S^3 - N(\gamma_1)}$ such that any slope -2 curve on $\partial(\overline{S^3 - N(\gamma_1)}) = T_\rho$ bounds a disk.

Therefore any slope -2 curve on T_{ρ_0} also bounds a disk because such a curve can be isotoped to a slope -2 curve on $T_\rho \subset S_{\rho_0}$.

Let $N(T_{\rho_0}) = \overline{S_{\rho_0} - S_{\rho_0 - \varepsilon}}$ be an $\varepsilon > 0$ neighbourhood of T_{ρ_0} contained in S_{ρ_0} so that $\rho < \rho_0 - \varepsilon < \rho_0$. Then we can embed an annulus $A = S^1 \times I$ in $N(T_{\rho_0})$ so that $S^1 \times \{1\}$ is a slope -2 curve on T_{ρ_0} and tangent to ξ_{tight}. Then $S^1 \times \{0\}$ is isotopic to a -2 curve on T_ρ. Thus the curve $S^1 \times \{0\}$ bounds a disk in the surgered manifold, and we can cap off this boundary component of A. This capped off annulus is a disk whose boundary is tangent to the contact structure ξ_{tight}. Hence the surgered contact manifold is overtwisted because we have exhibited an overtwisted disk. This illustrates that in any tight surgery construction, it is necessary the surgery neighbourhood $N(\gamma_1)$ is chosen to be sufficiently 'large'.

Theorem 4.15 (Etnyre, Ghrist [EG99])**.** *There exists tight rational contact r-surgeries on the unknot in (S^3, ξ_{tight}).*

Proof. Let $r = \frac{p}{q}$ for coprime integers p and q and let $\gamma = \gamma_1 = B_1$ be one of the periodic orbits of (S^3, α_a) mentioned in Proposition 4.13. Let S_R be the solid torus neighbourhood of γ of radius $R > 0$ (see Equation 4.3). Recall that the characteristic foliation of ξ_{tight} on $T_\rho \subset S_R$ is linear of slope $-\tan^2 \rho$, and the characteristic foliation on ∂S_R has slope $-\tan^2 R$. Also recall that the Reeb orbits of α_a are tangent to T_ρ of slope $\frac{1}{a^2}$.

Let $S = D^2_R \times S^1$ be a solid torus of radius R with some fixed preferred longitude on ∂S and consider the r-surgery map (Definition 2.28) $\Phi : \partial S \to \partial S_R$ defined by

$$\Phi = \begin{pmatrix} p & m \\ q & n \end{pmatrix}. \tag{4.18}$$

Note that we are free to choose the integers m and n so long as $pn - qm = 1$.

Let $N(\partial S_R) = \partial S_R \times (R - \varepsilon, R]$ be an ε-neighbourhood of ∂S_R such that $\partial S_R \times \{\rho\} = T_\rho$ for $\rho \in (R - \varepsilon, R]$. Similarly, let $N(\partial S) = \partial S \times (R - \varepsilon, R]$ be a neighbourhood of ∂S with $\partial S \times \{R\} = \partial S$. Then smoothly extend $\Phi : \partial S \to \partial S_R$ to a map $\Phi : N(\partial S) \to N(\partial S_R)$ by

$$\begin{aligned} \Phi : N(\partial S) &\to N(\partial S_R) \\ \Phi(x, t) &= (\Phi(x), t). \end{aligned} \tag{4.19}$$

For ease of notation, we also denote this extension Φ. Now pull back the form α_a on $N(\partial S_R)$ to obtain a contact form $\Phi^*\alpha_a$ on $N(\partial S)$.

Let $\tau_\rho = -\tan^2 \rho$ and note that the slope of the characteristic foliation of $\ker \alpha_a$ on ∂S_R is τ_R. We will now calculate the characteristic foliations and Reeb orbits of $(N(\partial S), \Phi^*\alpha_a)$ for the chosen parameters m, n and R. Note that if different values of m, n or R are chosen, then the form $\Phi^*\alpha_a$ changes which potentially alters its Reeb dynamics and characteristic foliation. Hence these parameters will need to be chosen carefully.

Lemma 4.16. *Let $\rho \in (R - \varepsilon, R]$ so that $\partial S \times \{\rho\} \subset N(\partial S)$. Then the slope of the characteristic foliation of $(N(\partial S), \Phi^*\alpha_a)$ on the torus $\partial S \times \{\rho\}$ is ν_ρ where*

$$\nu_\rho = -\left(\frac{p\tau_\rho - q}{m\tau_\rho - n}\right). \tag{4.20}$$

Additionally, the Reeb vector field of $(N(\partial S), \Phi^*\alpha_a)$ is tangent to the torus $\partial S \times \{\rho\}$ with slope $\overline{a}$ (independent of ρ) where

$$\overline{a} = -\left(\frac{p - a^2 q}{m - a^2 n}\right). \tag{4.21}$$

Proof. The slope ν_ρ curves on $\partial S \times \{\rho\}$ are the integral curves of the constant vector field V where,

$$V = -\begin{bmatrix} m\tau_\rho - n \\ -p\tau_\rho + q \end{bmatrix}.$$

This is defined with respect to the standard coordinate basis $\{\Phi_*^{-1}\partial_\theta, \Phi_*^{-1}\partial_\phi\}$ for the tangent space of $\Phi^{-1}T_\rho$. The characteristic foliation on the torus $\Phi^{-1}T_\rho$ is the image of the characteristic foliation on T_ρ under Φ^{-1}. Therefore, we need to show that the integral curves of $\Phi_* V$ have slope τ_ρ.

$$\begin{aligned}
\Phi_* V &= \begin{bmatrix} p & m \\ q & n \end{bmatrix} \begin{bmatrix} m\tau_\rho - n \\ -p\tau_\rho + q \end{bmatrix} \\
&= -\begin{bmatrix} pm\tau_\rho - pn - mp\tau_\rho + qm \\ qm\tau_\rho - qn - np\tau_\rho + qn \end{bmatrix} \\
&= \begin{bmatrix} -(pn - qm) \\ -(pn - qm)\tau_\rho \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ -\tau_\rho \end{bmatrix}.
\end{aligned}$$

The slope of $\Phi_* V$ is $\frac{-\tau_\rho}{-1} = \tau_\rho$, which is the slope of the characteristic foliation on the torus T_ρ as required.

Recall that the Reeb vector field of α_a is $X_a = \partial_\theta + \frac{1}{a^2}\partial_\phi$ (see Proposition 4.11). By Lemma 2.23, the Reeb vector field of $(N(\partial S), \Phi^* \alpha_r)$ is $\Phi_*^{-1} X_a$, which is tangent to the tori $\Phi^{-1}(T_\rho)$.

$$\begin{aligned}
\Phi_*^{-1} X_a &= \begin{bmatrix} n & -m \\ -q & p \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{a^2} \end{bmatrix} \\
&= \begin{bmatrix} n - m\frac{1}{a^2} \\ -q + p\frac{1}{a^2} \end{bmatrix} \\
&= \frac{1}{a^2} \begin{bmatrix} -m + a^2 n \\ p - a^2 q \end{bmatrix}.
\end{aligned}$$

Thus we see that the slope of $\Phi_*^{-1} X_a$ is $\overline{a} = -\left(\frac{p - a^2 q}{m - a^2 n}\right)$, as required. $\square$

Now we will choose a model contact form on a solid torus $(N, \overline{\alpha})$ that is strictly contactomorphic to $(N(\partial S), \Phi^* \alpha_a)$ near ∂N. Notice that the model contact forms have positive Reeb slope and negative characteristic foliation slope. Therefore, we need to choose the parameters m, n and R such that $\overline{a} > 0$ and $\nu_\rho < 0$. The following lemma states that this is always possible.

Lemma 4.17. *For fixed $p, q \in \boldsymbol{Z}$, there exists $m, n \in \boldsymbol{Z}$ and $R \in (0, \frac{\pi}{2})$ such that $pn - qm = 1$, $\overline{a} > 0$ and $\nu_R < 0$ where $\overline{a}$ and ν_R are defined as in Lemma 4.16.*

Remark 4.18. Lemma 4.17 is stated without proof in [EG99]. It is rather straightforward and proving it does not provide much insight to the overall construction. Hence we also leave it as an exercise. However, Lemma 4.17 does addresses the issue illustrated in Example 4.14 of attempting to perform tight surgery on too small of a torus. That is, being forced to choose the parameters m, n and R in this way ensures that R is large enough to avoid introducing an overtwisted disk to the surgered manifold.

Fix parameters m, n and R given by Lemma 4.17 so that $\overline{a} > 0$ and $\nu_R < 0$. Let $N = D^2_\eta \times S^1$ be a solid torus of radius $\eta = \tan^{-1}\sqrt{|\nu_R|}$. That is, give N coordinates (ρ, θ, ϕ) where $\rho \in [0, \eta]$ and $\theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. Define a contact form $\overline{\alpha}$ on N by

$$\begin{aligned}\overline{\alpha} &= \alpha_{\frac{1}{\sqrt{\overline{a}}}}\\ &= \frac{1}{\sin^2\rho + \overline{a}\cos^2\rho}(\sin^2\rho d\theta + \cos^2\rho d\phi).\end{aligned} \tag{4.22}$$

This is a model contact form on N with all the properties discussed in the previous section, (see Equation 4.16). That is, this form is chosen so that the Reeb orbits of $(N, \overline{\alpha})$ have slope $\overline{a}$ and the slope of the characteristic foliation on ∂N is ν_R.

Let $N(\partial N) = \partial N \times (\eta - \varepsilon', \eta]$ be an ε'-neighbourhood of ∂N such that $\partial N \times \{\eta\} = \partial N$. Our goal is to define a strict contactomorphism Ψ from $(N(\partial N), \overline{\alpha})$ to $(N(\partial S), \Phi^*\alpha_a)$. We accomplish this by rescaling the radius of $N(\partial N)$ such that the characteristic foliation on $\partial N \times \{b\}$ is sent to the characteristic foliation on $\Psi(\partial N \times \{b\}) = \partial S \times \{f(b)\}$ for some function f. We also require that Ψ sends the distinguished meridian and longitude of $\partial N \times \{b\}$ to the distinguished meridian and longitude of $\partial S \times \{f(b)\}$ so that Ψ does not change the slope of any curves.

Define a function $f : (\eta - \varepsilon', \eta] \to (R - \varepsilon, R]$ by

$$\begin{aligned} f &: (\eta - \varepsilon', \eta] \to (R - \varepsilon, R]\\ f(b) &= \tan^{-1}\sqrt{-\frac{n\tan^2 b - q}{m\tan^2 b - p}}.\end{aligned} \tag{4.23}$$

The inside of the square root of $f(\eta)$ is $\tan^2 R > 0$, so this is well defined and smooth for sufficiently small ε'. Further it is a bijection for appropriately chosen ε. A straightforward calculation shows that $\nu_{f(b)} = -\tan^2 b$ for $b \in (\eta - \varepsilon', \eta]$.

Let $F : \partial N \to \partial S$ be a diffeomorphism that sends the distinguished meridian and longitude on ∂N to the distinguished meridian and longitude on ∂S. Define a diffeomorphism $\Psi : N(\partial N) \to N(\partial S)$ by

$$\Psi(x, b) = (F(x), f(b)). \tag{4.24}$$

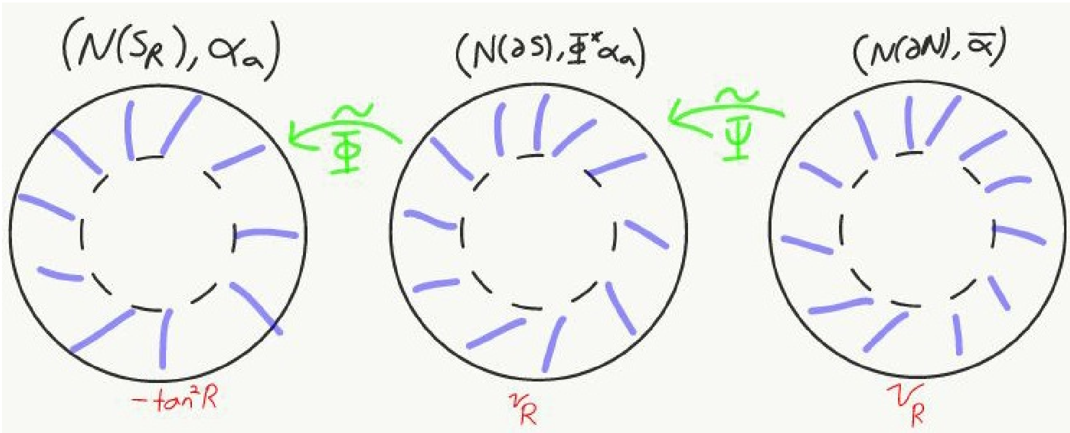

Figure 4.1: A diagram of the neighbourhoods of boundaries of the constructed solid tori and the contactomorphisms between them. The slope of the characteristic foliations on their boundaries is indicated in red.

That is, Ψ is mapping the torus $\partial N \times \{b\}$ to the torus $\partial S \times \{f(b)\}$ and preserves the slopes of curves on $\partial N \times \{b\}$.

The characteristic foliation of the torus $\partial N \times \{b\}$ has slope $-\tan^2 b$ and the characteristic foliation on the torus $\Psi(\partial N \times \{b\}) = \partial S \times \{f(b)\}$ has slope $\nu_{f(b)} = -\tan^2 b$. Therefore, Ψ maps the characteristic foliation of $\partial N \times \{b\}$ to the characteristic foliation of $\partial S \times \{f(b)\}$. Further, $(N, \overline{\alpha})$ and $(S, \Phi^*\alpha_a)$ both have slope $\overline{a}$ Reeb orbits. Therefore Ψ sends the Reeb orbits of $\overline{\alpha}$ to the Reeb orbits of α_a.

Lemma 4.19. *Let Σ be a surface and α, α' contact forms on $\Sigma \times I$. Suppose that α and α' print the same characteristic foliation on $S \times \{t\}$ for all $t \in [0, 1]$. Then $\ker \alpha = \ker \alpha'$. Further, if the Reeb vector fields X_α and $X_{\alpha'}$ are equal, then $\alpha = \alpha'$.*

Proof. The fact that $\ker \alpha = \ker \alpha'$ follows from Theorem 2.15 that the characteristic foliation of $\ker \alpha$ on the surface $S \times \{t\}$ determines the germ of that $\ker \alpha$ near $S \times \{t\}$. Since the characteristic foliations for $\ker \alpha$ and $\ker \alpha'$ agree on $S \times \{t\}$ for all $t \in [0, 1]$, we must have that $\ker \alpha = \ker \alpha'$ on $\Sigma \times I$.

Suppose that the Reeb vector fields X_α and $X_{\alpha'}$ are equal on $\Sigma \times I$. Let $v \in T(\Sigma \times I)$. Then v can be written uniquely as $w + cX_\alpha$ for some $w \in \ker \alpha$ and $c \in \mathbf{R}$. Hence $\alpha(v) = c\alpha(X_\alpha) = c$. But we have already shown that $\ker \alpha = \ker \alpha'$. Hence $\alpha'(v) = c\alpha'(X_\alpha) = c\alpha'(X_{\alpha'}) = c$. Therefore $\alpha = \alpha'$. $\square$

Thus by Lemma 4.19, $\Psi^*(\Phi^*\alpha_a) = \overline{\alpha}$ and $(\Phi \circ \Psi)$ is a strict contactomorphism from $(N(\partial N), \overline{\alpha})$ to $(N(\partial S_R), \alpha_a)$. Now glue the solid torus N to $\overline{S^3 - S_R}$ by the map $\Phi \circ \Psi$. That is, construct the surgered manifold

$$S^3_\gamma(r) = \overline{S^3 - S_R} \cup_{\Phi \circ \Psi} N. \tag{4.25}$$

Extend the contact form α_a on S^3 to a contact form $\alpha_\gamma(r)$ on $S^3_\gamma(r)$ by

$$\alpha_\gamma(r) = \alpha_a \cup_{\Phi \circ \Psi} \overline{\alpha}. \tag{4.26}$$

Then $\alpha_\gamma(r)$ is a smooth contact form because the contact forms $(\Phi \circ \Phi)^* \alpha_a$ and $\overline{\alpha}$ agree on a neighbourhood of ∂N. Thus $(S^3_\gamma(r), \ker \alpha_\gamma(r))$ is the required surgered contact manifold with $\alpha_\gamma(r)$ extending α_a. It remains to show that $\ker \alpha_\gamma(r)$ is tight.

The surgered manifold $S^3_\gamma(r)$ decomposes into two pieces: $\overline{S^3 - S_R}$ and N. Since the Reeb orbits of (S^3, α_a) are tangent to the tori T_ρ, none of the Reeb orbits in $\overline{S^3 - S_R}$ were altered during the surgery. Further, there is only one periodic Reeb orbit in $(N, \overline{\alpha})$, namely the core of N. This periodic Reeb orbit is non-degenerate and elliptic by Proposition 4.13. Thus $\alpha_\gamma(r)$ has no degenerate or hyperbolic periodic Reeb orbits and $(S^3_\gamma(r), \ker \alpha_\gamma(r))$ is tight by Corollary 4.3. □

Thus we have successfully completed tight r-surgery on a transverse unknot in $(S^3, \xi_{\text{tight}})$ (specifically, a periodic Reeb orbit in (S^3, α_a)). This is the beginning of an answer to when tight surgery is possible. In the previous chapter, we showed that certain contact surgeries are always possible on any transverse knot in any contact manifold. This is a very general statement, but is almost too general to have any hope of preserving tightness. In this chapter, we showed that by restricting to a specific transverse knot inside $(S^3, \xi_{\text{tight}})$, then tight surgery on this knot is possible so long as we are careful with the dynamics of a representative contact form for ξ_{tight}. This is a much more restrictive statement because it only applies to $(S^3, \xi_{\text{tight}})$, and is very sensitive to the chosen contact form, but it is good to have a starting point. It illustrates that at least in one simple case, tight surgery is possible. Our goal moving further is to see how much we can generalise the ideas in this construction to potentially perform tight surgery on tight contact manifolds other than $(S^3, \xi_{\text{tight}})$.

Chapter 5

Surgery on Open Books

So far in this thesis, we have explored different types of contact surgery on transverse knots in contact manifolds. In Chapter 3 we defined admissible transverse surgery, a general approach to contact surgery that applies to any transverse knot. This is a widely applicable result because every knot in a contact manifold is isotopic to a transverse knot (see [Etn03]). Thus admissible transverse surgery provides a range of contact surgeries on a large family of knots in any contact manifold.

In Chapter 4, we restricted our attention to a transverse unknot γ in $(S^3, \xi_{\text{tight}})$. In particular, γ was a periodic Reeb orbit of a contact form α_a representing the contact structure ξ_{tight}. With these restrictions and control over the Reeb dynamics of α_a, we were able to perform contact surgery on γ while preserving tightness. This raises the question of how relevant periodic Reeb orbits are to contact surgery, and whether there are periodic Reeb orbits in other manifolds that we can do surgery on. The question of whether periodic Reeb orbits of contact forms for contact 3-manifolds always exist is known as the Weinstein Conjecture. It was first conjectured by Weinstein in 1979 ([Wei79]) and has since been proven to be true by Taubes in 2007 ([Tau09]). This means that in any contact manifold, we are always able to find a periodic Reeb orbit to potentially do surgery on.

A natural place to find examples of transverse knots that we are interested in performing surgery on is in a so-called open book. An open book is a topological decomposition of a manifold that determines a class of compatible contact structures on this manifold. In fact, the "binding" of an open book is a periodic Reeb orbit for a compatible contact form. Hence we define and study open books now.

5.1 Open Books

The general study of 3-manifolds is a rich and complicated area of mathematics. One approach that aids visualisation is to break them down into pieces that we can visualise separately. An open book decomposition of a 3-manifold is exactly this, a division of a 3-manifold into two simpler pieces. The following definitions, theorems and examples in this section are based mostly on expositions in Etnyre [Etn06] and Geiges [Gei08], but are standard in the field.

Definition 5.1. An *open book decomposition* of a closed, oriented 3-manifold M is an oriented link B in M and a fibration $\pi : (M - B) \to S^1$ such that the boundary of the fibre $\pi^{-1}(\psi)$ is equal to B for all $\psi \in S^1$. The link B is called the binding and the closure of the fibres $\Sigma_\psi = \overline{\pi^{-1}(\psi)}$ are called the pages. We say that (π, B) is an open book of M with page $\Sigma \cong \Sigma_p si$.

This definition breaks M into two pieces: a tubular neighbourhood of the binding B and a fibre bundle over S^1. Fibre bundles over S^1 are easier to understand and visualise than general 3-manifolds because an S^1-fibre bundle is just a twisted product of Σ and S^1. However, an important aspect of this definition is the condition that $\partial\Sigma_\psi = B$. This means that the pages cannot be fibred over S^1 in just any way; we require that the boundary of the pages be equal to the binding. In this way, we can think of this fibre bundle as a book of Σ pages glued to B the binding, as shown in Figure 5.1.

The following examples are described by Etnyre in [Etn06].

Example 5.2 (A D^2-open book of S^3)**.** Consider $S^3 \subset \mathbf{C}^2$ and let $B = \{(z_1, 0) \mid |z_1| = 1\}$ be an unknot in S^3. Define $\pi : (S^3 - B) \to S^1$ by

$$\pi(z_1, z_2) = \frac{z_2}{|z_2|}. \tag{5.1}$$

This map π is well defined on $S^3 - B$ since $|z_2|$ is never zero. The map π sends (z_1, z_2) to the angular coordinate of z_2. This open book is illustrated in Figure 5.1.

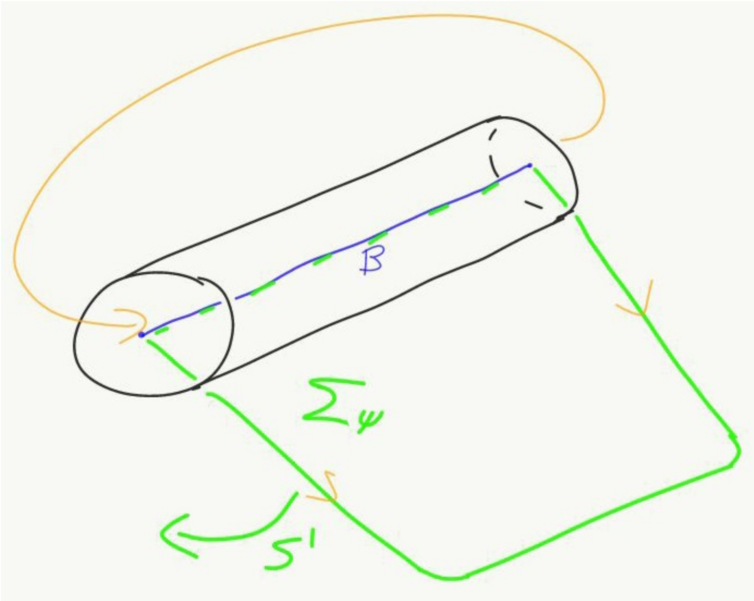

Figure 5.1: The standard depiction of an open book with a connected binding. We think of the surfaces Σ_ψ as pages of a book fibred over S^1, glued to the binding B.

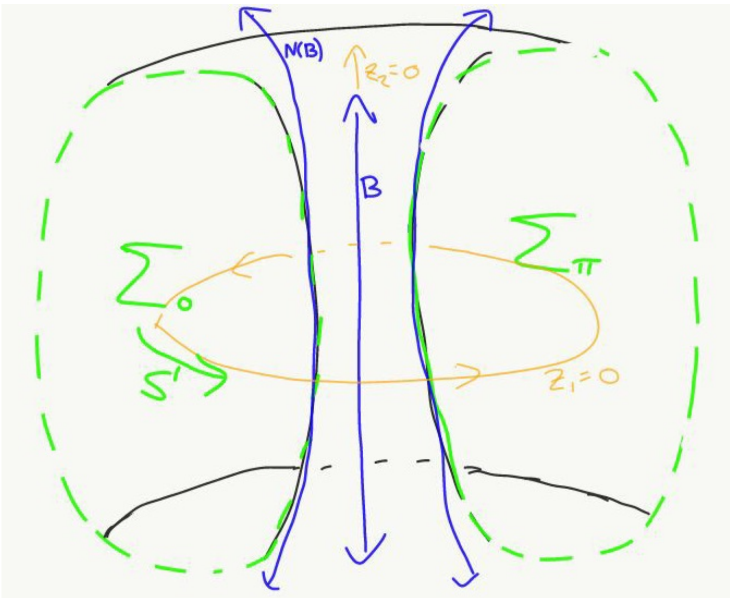

Figure 5.2: The D^2 open book of S^3 described in Example 5.2. The pages are the green disks glued to the binding B. The fibration $\pi : (S^3 - B) \to S^1$ is given by projecting onto the angular coordinate of $z_2 \neq 0$. Here we see S^3 presented as two tori, one $N(B)$ in blue, and the other the fibre bundle $\text{Int}(\Sigma) \times S^1$ in green.

Example 5.3. (An annular open book of S^3) Consider $S^3 \subset \mathbf{C}^2$ and let $H = \{(z_1, z_2) \mid z_1 z_2 = 0\} \subset S^3$ be a (positive) Hopf link. Define a fibration $\pi : (S^3 - H) \to S^1$ by

$$\pi(z_1, z_2) = \frac{z_1 z_2}{|z_1 z_2|}. \tag{5.2}$$

This map is well defined on $S^3 - H$ because $z_1 z_2 \neq 0$. In polar coordinates, the projection map π is

$$\pi(re^{i\theta}, \sqrt{1-r^2}e^{i\phi}) = e^{i(\theta+\phi)}. \tag{5.3}$$

This means a fibre $\pi^{-1}(\psi)$ is given by

$$\pi^{-1}(\psi) = \{(re^{i\theta}, \sqrt{1-r^2}e^{i\phi}) \mid \phi = \psi - \theta, r \neq 0, 1\}. \tag{5.4}$$

The set $\pi^{-1}(\psi)$ is parametrised by $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ and $r \in (0, 1)$. Thus the page $\Sigma = \overline{\pi^{-1}(\psi)}$ is an annulus $S^1 \times [0, 1]$.

In an open book, the data of the binding B is contained in the page Σ because $\partial\Sigma = B$. Further, the S^1-fibration $\pi : (M - B) \to S^1$ can be obtained from $\Sigma \times [0, 1]$ if we know how $\Sigma \times \{1\}$ is glued to $\Sigma \times \{0\}$. With that in mind, we make the following definition.

Definition 5.4. An *abstract open book* is a pair (Σ, h) where Σ is a compact, oriented surface and $h \in \mathrm{Diff}^+(\Sigma)$ is an orientation preserving diffeomorphism such that $h|_{\partial\Sigma} = \mathrm{id}$. The surface Σ is called the page and h the monodromy.

Remark 5.5. Although (Σ, h) specifies an orientation preserving diffeomorphism, much of the theory of open books depends only on the isotopy class $[h] \in \mathrm{Mod}(\Sigma)$.

As suggested earlier, an abstract open book contains the data needed to construct an open book. That is, from an abstract open book (Σ, h), we are able to construct a 3-manifold M_h with an open book decomposition. To describe this, we follow [Etn06] and [Gei08]. Let $\Sigma(h)$ be the mapping torus of (Σ, h), given by

$$\Sigma(h) = \Sigma \times [0, 2\pi] / \left((x, 2\pi) \sim (h(x), 0)\right). \tag{5.5}$$

We think of $\Sigma(h)$ as a cylinder $\Sigma \times I$, with the ends glued together by the map h. This is naturally an S^1-fibre bundle by projection onto the I component, which becomes an S^1 component in the quotient $\Sigma(h)$.

The boundary of the mapping torus is $\partial\Sigma(h) = \partial\Sigma \times S^1$, which is a disjoint union of tori $\partial\Sigma(h) = \coprod_{|\partial\Sigma|} (\partial\Sigma)_i \times S^1$. To construct M_h, we glue a solid torus

$S_i = D^2 \times S^1$ to each torus boundary component $(\partial\Sigma)_i \times S^1$ such that the curve $\{p_i\} \times S^1$ bounds a disk, where $p_i \in (\partial\Sigma)_i$. That is, let $\Phi : \coprod_{|\partial\Sigma|} \partial S_i \to \coprod_{|\partial\Sigma|} (\partial\Sigma)_i \times S^1$ be the gluing map that sends the meridian on ∂S_i to the curve $\{p_i\} \times S^1$ on $(\partial\Sigma)_i \times S^1$. Construct M_h as

$$M_h := \Sigma(h) \cup_\Phi \coprod_{|\partial\Sigma|} S_i. \tag{5.6}$$

Alternatively, this filling is diffeomorphic to collapsing each boundary component $(\partial\Sigma)_i \times S^1$ along the curve $\{p_i\} \times S^1$, as shown in Proposition 3.13. A consequence of this gluing is that the curve $(\partial\Sigma)_i \times \{t\} \subset \partial\Sigma(h)$ is now isotopic to the core of S_i.

The manifold M_h has a natural open book decomposition: the pages are copies of Σ, the binding is the union of the core of the tori S_i, and the S^1-fibration comes naturally from projection onto the S^1-factor of the mapping torus $\Sigma(h) = \overline{M_h - \coprod_{|\partial\Sigma|} S_i} \cong M_h - B$. Thus we have shown how to construct an open book from an abstract open book. Similarly, it is possible to reverse this and construct an abstract open book from an open book. The idea is to determine how the pages are glued up in the S^1-fibration to recover the monodromy. For an explicit construction of this, see [Gei08]. In light of this equivalence, we will move freely between these definitions and use whichever is most convenient for the given situation.

The following are examples of abstract open books and their associated open books. These examples are explained in [Etn06] and [Gei08], but are standard in the literature.

Example 5.6. Consider the abstract open book (Σ, h) where $\Sigma = D^2$ is a disk and h is the identity. The mapping torus $\Sigma(h)$ is a solid torus $\Sigma(h) = D^2 \times S^1$ because h is the identity. Let $((\rho, \theta), \phi)$ be coordinates on $\Sigma(h) = D^2 \times S^1$ and let $\lambda = \{(1, 0, \phi)\}$ be a preferred longitude. To construct M_h, we glue a solid torus to $\Sigma(h)$ such that the longitude λ now bounds a disk in M_h. That is, M_h is S^3. The associated S^1-fibration is the projection map onto the ϕ coordinate, which is the same S^1-fibration described in Example 5.2. That is, the open book corresponding to this abstract open book is the D^2-open book of S^3 defined in Example 5.2.

Example 5.7. Consider the abstract open book (A, id) where $A = S^1 \times I$ is an annulus. The mapping torus is $(S^1 \times I) \times S^1$, which we depict as an annulus times an interval with top and bottom identified.

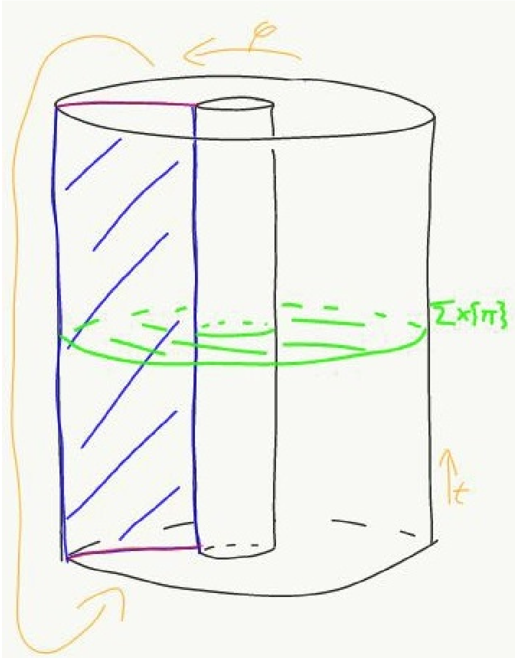

Figure 5.3: The mapping torus $A(\mathrm{id})$ for the abstract open book (A, id). We depict this as the cylinder $A \times I$ with top and bottom identified by the identity. After filling in a neighbourhood of the binding, a page A_t in the open book M_{id} is given by take a horizontal slice (which is a fixed t-coordinate).

In Figure 5.7, the vertical axis is the S^1-component of the mapping torus $A(\mathrm{id})$, so a page is a horizontal slice (which we can clearly see is an annulus). We claim that M_{id} is an open book of $S^2 \times S^1$. To realise this with the visual aid of Figure 5.7, consider the highlighted blue rectangle. In the mapping torus $\Sigma(h)$, it has its top and bottom identified, so it is an open cylinder. We can think of filling in a neighbourhood of the binding as collapsing the two vertical sides of this blue rectangle to a point. Thus, when a neighbourhood of the binding is filled in, the ends of the blue open cylinder are collapsed to a point. Therefore this blue rectangle is an S^2 in M_{id}. These blue rectangles are parametrised by S^1, hence $M_h = S^2 \times S^1$.

These examples are relatively simple because they each take the monodromy to be the identity. In Section 5.1.2, we compute a more complicated example where the monodromy is instead a Dehn twist.

Finally, it is a theorem of Alexander that every 3-manifold admits an open book decomposition. Hence open books are abundant enough to be a useful tool for studying 3-manifolds because any 3-manifold we are studying is guaranteed to have an open book decomposition. To focus on the relationship of open books to contact geometry, we do not prove this theorem but refer the reader to [Etn06] and [Rol90].

5.1.1 Supported Contact Structures

Open books are useful tools for studying 3-manifolds, but they are also useful tools for studying contact manifolds. One reason for this is the Giroux correspondence between open books and supported contact structures, stated in Theorem 5.11. In this section, we develop what it means for a contact structure to be supported by an open book and prove that every open book admits a supported contact structure.

Definition 5.8. A contact form α is supported by an open book (π, B) if $d\alpha$ restricts to a positive area from on each page Σ_ψ and $\alpha > 0$ on the binding B. That is, if $d\alpha|_{T\Sigma_\psi} > 0$ and $\alpha|_{TB} > 0$.

The condition that $\alpha > 0$ on the binding means that the binding intersects the contact planes $\xi = \ker \alpha$ (positively) transversely. The condition $d\alpha|_{T\Sigma_\psi} > 0$ means that the twisting of the planes away from the binding is bounded by the pages of the open book. That is, the majority of the twisting of the contact planes is localised to the binding.

Remark 5.9. There is a subtlety when defining supported contact structures as opposed to supported contact forms. It would be natural to define a supported contact structure as one that admits a supported contact form, however this property is not preserved under isotopy. When working with contact manifolds and open books topologically, we would like the freedom to work with them up to isotopy. Hence we define a supported contact structure to be one that is isotopic (through contact structures) to a contact structure that admits a supported contact form.

The following lemma demonstrates that a contact form being supported is equivalent to a property of its Reeb dynamics. The proof is presented in [Etn06] and we include it for insight into the relation between Reeb dynamics and supported contact forms.

Lemma 5.10. *Let X be the Reeb vector field for a contact form α. Then α is supported by an open book (B, π) if and only if X is (positively) tangent to B and (positively) transverse to the pages Σ_ψ.*

Proof. Recall that X has the properties $i_X\alpha = 1$ and $i_X d\alpha = 0$. Suppose X is tangent to B and transverse to Σ_ψ. The assumption that X is (positively) tangent to the binding implies that $\alpha(X) > 0$ on B because $X \notin \ker\alpha$. The tangent space of the binding is one dimensional, therefore $\alpha|_{TB} > 0$. We use the two Reeb vector field properties of X in the following calculation.

$$\begin{aligned} i_X(\alpha \wedge d\alpha) &= (i_X\alpha) \wedge d\alpha - \alpha \wedge (i_X d\alpha) \\ &= d\alpha. \end{aligned}$$

Let $\{Y_p, Z_p\} \in T_p\Sigma_\psi$ be a positively oriented basis for $T_p\Sigma_\psi$ at a point $p \in \Sigma_\psi$. Then by convention, $\{X_p, Y_p, Z_p\}$ is a positively oriented basis for T_pM. Applying the above calculation, we have the following.

$$\begin{aligned} d\alpha_p(Y_p, Z_p) &= i_{X_p}(\alpha_p \wedge d\alpha_p)(Y_p, Z_p) \\ &= (\alpha \wedge d\alpha)_p(X_p, Y_p, Z_p) \\ &> 0. \end{aligned}$$

The last line follows because $\alpha \wedge d\alpha > 0$ by the contact condition. Therefore $d\alpha$ restricts to a positive area form on Σ_ψ and we have shown that α is supported by (B, π).

Now suppose that α is supported by (B, π). Then $d\alpha > 0$ on Σ_ψ and $\alpha > 0$ on B. Since $i_X d\alpha = 0$ and $X \neq 0$, we must have $X \notin T\Sigma_\psi$. That is, X is transverse to the pages Σ_ψ.

Now we want to show that X is tangent to B. Let $N(B)$ be a neighbourhood of the binding and choose coordinates $((\rho, \theta), \phi)$ on $N(B) = D^2 \times S^1$ so that the intersection of the page Σ_ψ with $N(B)$ is given by $\{\theta = \psi\}$ and $\{0\} \times S^1 = B$. On $N(B)$, we can write $X = f_\rho\partial_\rho + f_\theta\partial_\theta + f_\phi\partial_\phi$ for some functions $f_\rho, f_\theta, f_\phi : N(B) \to \mathbf{R}$. At $\rho = 0$, we have $X = f_\rho(0, \theta, \phi)\partial_\rho + f_\phi(0, \theta, \phi)\partial_\phi$, noting that the functions f_ρ and f_ϕ are independent of θ when $\rho = 0$. Thus to show that X is tangent to the binding, it is enough to show that $f_\rho(0, \theta, \phi) = 0$.

Suppose (without loss of generality) that $f_\rho(0, \theta, \phi) > 0$. Then there exists $\varepsilon > 0$ such that $f_\rho(\rho, \theta, \phi) > 0$ for all $\rho < \varepsilon$. Fix $0 < \rho_0 < \varepsilon$ and some $\psi \in \mathbf{R}/2\pi\mathbf{Z}$. Then $f_\rho(\rho_0, \psi, \phi) > 0$ and $f_\rho(\rho_0, \psi + \pi, \phi) > 0$. But X is positively tangent to both pages Σ_ψ and $\Sigma_{\psi+\pi}$. This means the ∂_θ component of X at these two pages must be opposite in sign. This is a contradiction, hence $f_\rho(0, \theta, \phi) = 0$. Thus X is tangent to B as required. □

Thus Lemma 5.10 gives an alternate definition for a supported contact form based on its Reeb dynamics. This will be convenient for us later when we study the Reeb dynamics of a contact form on an open book. In particular, notice that if the contact form α is supported by the open book (B, π), then the binding B is a periodic Reeb orbit of α. Thus open books with supported contact structures come with identified periodic Reeb orbits (that is, the components of the binding) that we will apply our surgery techniques to.

One of the main reasons why open books are great tools for studying contact structures is the following Giroux correspondence.

Theorem 5.11 (Giroux [Gir02]). *Let M be a closed, oriented 3-manifold. Then there is a 1-1 correspondence between*

$$\{\textit{Positive contact structures on } M \textit{ up to isotopy}\}$$

and

$$\{\textit{Open book decompositions of } M \textit{ up to positive stabilisation}\}.$$

We have not defined positive stabilisation and this is not important for our purposes, hence this definition is suppressed. We refer the reader to [Etn06] for the associated definitions and a complete proof of this theorem.

One direction of the Giroux correspondence is given by showing that every open book supports a contact structure which is unique up to isotopy. We are interested in the existence statement of this direction of the correspondence, which we will now prove. To do this, we follow a construction of a supported contact form on an open book given in [Gei08].

Theorem 5.12. *For any abstract open book (Σ, h), there exists a contact form α on M_h supporting (Σ, h).*

Proof. For notational simplicity, we assume Σ has only one boundary component. The case of multiple boundary components is an easy extension of this.

Let $N(\partial\Sigma) = \partial\Sigma \times [0, \frac{1}{2}]$ be a collar neighbourhood of the boundary of Σ with $\partial\Sigma \times \{\frac{1}{2}\} = \partial\Sigma$. The boundary $\partial\Sigma$ is an S^1, so assign coordinates (φ, s) to $N(\partial\Sigma)$ with $\varphi \in \mathbf{R}/2\pi\mathbf{Z}, s \in [0, \frac{1}{2}]$. The construction in [Gei08] begins by proving the following lemma.

Lemma 5.13. *The set $\mathcal{B}$ of 1-forms β on Σ such that*

1. *$\beta = e^s d\varphi$ on $N(\partial\Sigma)$ and*
2. *$d\beta$ is an area form on Σ with total area 2π,*

is non-empty and convex.

For proof of this lemma, we refer the reader to [Gei08]. The idea behind showing $\mathcal{B}$ is non-empty is to use a De Rahm cohomology argument, and the convexity statement is a straightforward exercise.

The idea of the proof of Theorem 5.12 is to first choose a 1-form $\beta_0 \in \mathcal{B}$ on the page Σ by Lemma 5.13. Then $d\beta_0$ is already an area form on Σ, and has a nice description near $\partial\Sigma$. We extend this to a form β on the mapping torus $\Sigma(h)$ and modify it so that it is contact. Then we use the description of β near $\partial\Sigma(h)$ to glue in a neighbourhood of the binding and extend the contact form to this neighbourhood.

Choose such a 1-form $\beta_0 \in \mathcal{B}$. The diffeomorphism h can be assumed to be the identity near $\partial\Sigma$ (see Remark 5.5) and thus $h^*\beta_0 \in \mathcal{B}$. Let $f : [0, 2\pi] \to [0, 1]$ be some smooth function such that $f(t) = 1$ near $t = 0$ and $f(t) = 0$ near $t = 2\pi$. Define a family of 1-forms β_t by

$$\beta_t := f(t)\beta + (1 - f(t))h^*\beta. \tag{5.7}$$

By convexity of $\mathcal{B}$, $\beta_t \in \mathcal{B}$ for all $t \in [0, 2\pi]$. Define a 1-form β on $\Sigma \times [0, 2\pi]$ by $\beta|_{\Sigma\times\{t\}} = \beta_t$. When we glue up the cylinder $\Sigma \times [0, 2\pi]$ by h to obtain the mapping torus $\Sigma(h)$, the point $(x, 2\pi)$ is glued to the point $(h(x), 0)$. Therefore, a tangent vector $(v, 2\pi)$ in $T(\Sigma \times \{2\pi\})_{(x,2\pi)} = T\Sigma_x \times \{2\pi\}$ is identified with the tangent vector $(h_*v, 0)$ in $T\Sigma_{h(x)} \times \{0\}$. Thus β descends to the mapping torus because $\beta(h_*v, 0) = h^*\beta_0(v) = \beta(v, 2\pi)$. For ease of notation, we also denote this descended 1-form on $\Sigma(h)$ by β.

Thus we have constructed a 1-form β on $\Sigma(h)$ such that $d\beta$ restricted to $\Sigma_\psi \cap \Sigma(h) = \Sigma \times \{\psi\}$ is a positive area form. Now we modify β to make it satisfy the contact condition while preserving this property. Let $\alpha = \beta + Kdt$ for some positive constant K and notice that

$$\begin{aligned} \alpha \wedge d\alpha &= \beta \wedge d\beta + Kdt \wedge d\beta \\ &= \beta \wedge d\beta + Kd\beta \wedge dt. \end{aligned} \tag{5.8}$$

From Lemma 5.13, we have that $d\beta \wedge dt$ is positive. Since $\Sigma(h)$ is compact, we can choose a large enough K such that the $Kd\beta \wedge dt$ term dominates the $\beta \wedge d\beta$ term. Thus, fix a large enough K so that α is a contact form on $\Sigma(h)$. Note that $\alpha|_{\Sigma_\psi} = \beta|_{\Sigma_\psi}$.

Finally, we glue in a neighbourhood of the binding to $\Sigma(h)$ to form M_h. Let $S = D^2 \times S^1$ be a solid torus with coordinates $((\rho, \theta), \phi)$ where $\rho \in [0, 1]$,

$\theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. Let $N(\partial S) = \{\frac{1}{2} \leq \rho \leq 1\} \subset S$ be a neighbourhood of ∂S and let $N(\partial\Sigma(h)) = N(\partial\Sigma) \times S^1$ be a neighbourhood of $\partial\Sigma(h)$ with coordinates (φ, s, t). Define a gluing map $\Phi : N(\partial S) \to N(\partial\Sigma(h))$ by

$$\begin{aligned} \Phi &: N(\partial S) \to N(\partial\Sigma(h)) \\ \Phi(\rho, \theta, \phi) &= (\phi, 1-\rho, \theta) \end{aligned} \tag{5.9}$$

The idea is to glue ∂S to $\partial\Sigma(h)$ on the neighbourhoods $N(\partial S)$ and $N(\partial\Sigma(h))$. Then, we will extend the form α on $\Sigma(h)$ to $S \subset M_h$ by making sure the extension agrees with α on $N(\partial\Sigma(h))$. The gluing described by Φ is depicted in Figure 5.1.1.

We pull back α by Φ to obtain a form on $N(\partial S)$.

$$\begin{aligned} \Phi^*\alpha &= \Phi^*(e^s d\psi + K dt) \\ &= e^{1-\rho} d\phi + K d\theta \end{aligned} \tag{5.10}$$

Let $\alpha' = f_\theta(\rho)d\theta + f_\phi(\rho)d\phi$ be a 1-form on S for some smooth functions $f_\theta, f_\phi : [0,1] \to \mathbf{R}$. Our goal is to choose f_θ and f_ϕ so that α' is contact, $\alpha' = \Phi^*\alpha$ for $\frac{1}{2} \leq \rho \leq 1$ and $\alpha' = \rho^2 d\theta + d\phi$ near $\rho = 0$. These conditions will ensure that α' extends α smoothly to $N(B)$ and that $d\alpha|_{T\Sigma_\psi}, \alpha|_{TB} > 0$. This gives the two boundary conditions $f_\theta(\rho) = K, f_\phi(\rho) = e^{1-\rho}$ for $\frac{1}{2} \leq \rho \leq 1$ and $f_\theta(\rho) = \rho^2, f_\phi(\rho) = 1$ near $\rho = 0$.

Lemma 5.14. *Suppose f_θ and f_ϕ satisfy the above boundary conditions. Define $f : [0,1) \to \boldsymbol{R}^2$ by $f(\rho) = (f_\theta(\rho), f_\phi(\rho))$. Then α' is a contact form if and only if $f(\rho)$ and $\dot{f}(\rho)$ are not parallel for all $\rho \in [0,1)$.*

The proof of this lemma is a straightforward calculation of the contact condition for α', showing that $\alpha' \wedge d\alpha' = 0$ if and only if $f(\rho)$ and $\dot{f}(\rho)$ are parallel. The fact that $\alpha' \wedge d\alpha'$ must be positive comes from the fact that $\alpha' \wedge d\alpha'$ is positive at the boundaries. It is not difficult to construct f_θ and f_ϕ satisfying the required boundary conditions such that $f(\rho)$ and $\dot{f}(\rho)$ are never parallel. A nice graph of such an f is presented in [Gei08], so we refer the reader there to see a choice of such f_θ and f_ϕ. Thus we have defined a contact form α' on M_h, which is supported by (Σ, h) by construction. $\square$

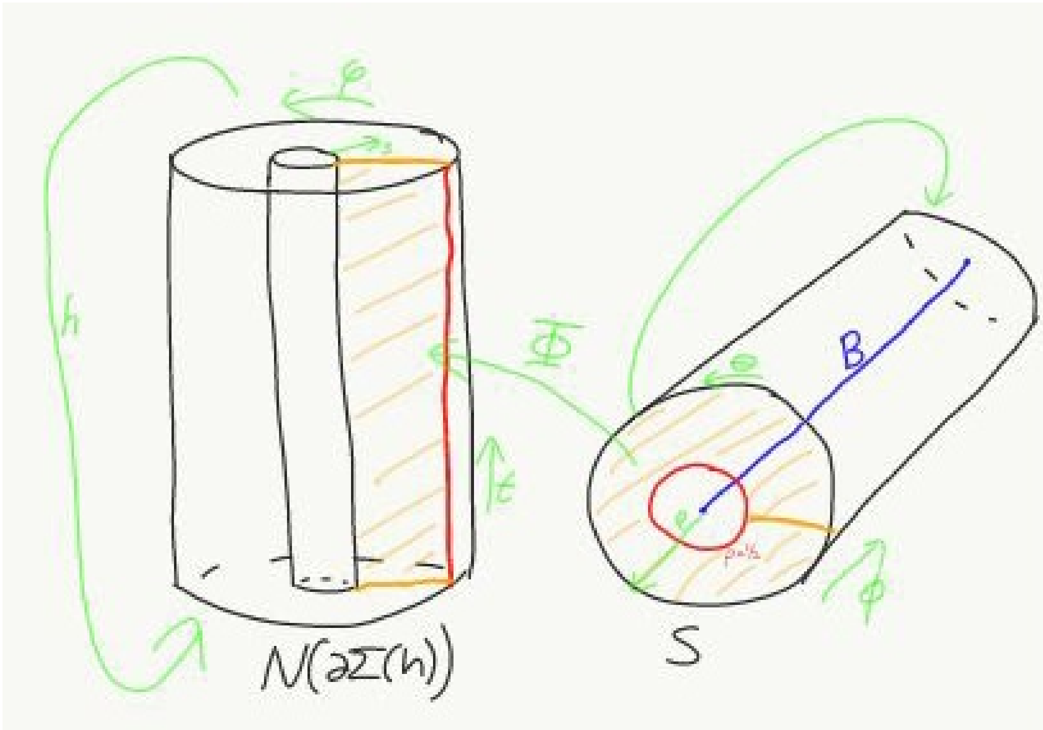

Figure 5.4: A visual representation of the gluing performed in the proof of Theorem 5.12.

Thus for any given open book (B, π), we are able to find a supported contact form α. In particular, the above construction gives an explicit description of a choice of α near the binding. That is, $\alpha = \rho^2 d\theta + d\phi$ near B. Note that $d\alpha = 2\rho d\rho \wedge d\theta$, so the Reeb vector field of α is $X_\alpha = \partial_\phi$. This means that the tori of fixed radius ρ in $N(B)$ are foliated by degenerate periodic Reeb orbits. Such Reeb orbits prevent us from using the Section 4.3 method of tight surgery. Recall that in Section 4.3, our initial contact form α_1 on S^3 had bad Reeb orbits of rational slope, which obstructed us from using Corollary 4.3. This is a similar situation, so later we aim to modify the dynamics of α near B. For now, we realise an open book that supports (S^3, α_a) from Section 4.3.

5.1.2 Annular Open Book of S^3

Let $A = S^1 \times I$ be an annulus and let $\tau : A \to A$ be a right-handed Dehn twist. The abstract open book (A, τ) is an example that has been very relevant to us. The reason is that this is an open book that supports the contact form α_a, defined in Section 4.2. This means that we can realise the tight surgery construction in

Chapter 4 as surgery on a binding component of this open book with its supported contact structure. We develop a concrete understanding of this by exploring this example now.

Recall Example 5.3 of an open book of S^3 with annular pages and binding a Hopf link. It is natural to ask what the abstract open book corresponding to this open book is. We saw in Example 5.2 that (A, id) is an abstract open book for $S^2 \times S^1$. Now we look at a similar example of an annular abstract open book where we instead take the monodromy to be a Dehn twist τ. We claim that (A, τ) is the open book corresponding to Example 5.3.

Proposition 5.15. *Let (A, τ) be an abstract open book with A an annulus and τ a right-handed Dehn twist. Then $M_\tau = S^3$ and the corresponding open book for (A, τ) is the annular open book of S^3 described in Example 5.3.*

Proof. Fix a point $p \in S^2$ and let $\gamma = \{p\} \times S^1$ be a curve in $S^2 \times S^1$. We begin with the following lemma.

Lemma 5.16. *The result of ±1-surgery on γ in $S^2 \times S^1$ is S^3.*

Proof. Let $U \subset S^2$ be a neighbourhood of p. Then $N(\gamma) = U \times S^1$ is a tubular neighbourhood of γ. Let $\lambda = \{u\} \times S^1$ be a longitude on $\partial N(\gamma)$ for some $u \in \partial U$. Notice that $S^2 - U$ is a disk and therefore $\overline{S^2 \times S^1 - N(\gamma)} = (S^2 - U) \times S^1$ is a solid torus. Further, the curve $\gamma_1 = \partial U \times \{\theta\}$ for a fixed $\theta \in S^1$ bounds a disk in the solid torus $\overline{S^2 \times S^1 - N(\gamma)}$.

Perform ±1-surgery by removing $N(\gamma)$ from $S^2 \times S^1$ and gluing in a solid torus $S = D^2 \times S^1$ to $\partial(\overline{S^2 \times S^1 - N(\gamma)})$ so that a slope ±1 curve γ_2 on $\partial\overline{S^2 \times S^1 - N(\gamma)}$ now bounds a disk in S. Then γ_1 and γ_2 intersect exactly once. Thus we have a Heegaard splitting $(\overline{S^2 \times S^1 - N(\gamma)}, \gamma_1) \cup_{\partial U \times S^1} (S, \gamma_2)$ of the surgered manifold into two solid tori where the two distinguished boundary curves intersect once and both bound disks. It is a classical theorem in 3-manifold topology (see [Rol90]) that the only 3-manifold with such a Heegaard splitting is S^3. $\square$

In Example 5.3, we showed that the manifold associated to the abstract open book (A, id) is $S^2 \times S^1$. We claim that changing the monodromy to a right-handed Dehn twist is equivalent to doing 1-surgery on γ in $S^2 \times S^1$. To see this, consider a collar neighbourhood $N(A \times \{1\}) = A \times [-\varepsilon, \varepsilon]$ of $A \times \{0\}$ in the mapping torus $A(\tau)$. This is a solid torus because it is an annulus cross an interval. Outside of $N(A \times \{0\})$, we have $\overline{A(\tau) - N(A \times \{0\})} = \overline{A(\mathrm{id}) - N(A \times \{0\})}$. For the rest of the argument, we refer to Figure 5.1.2.

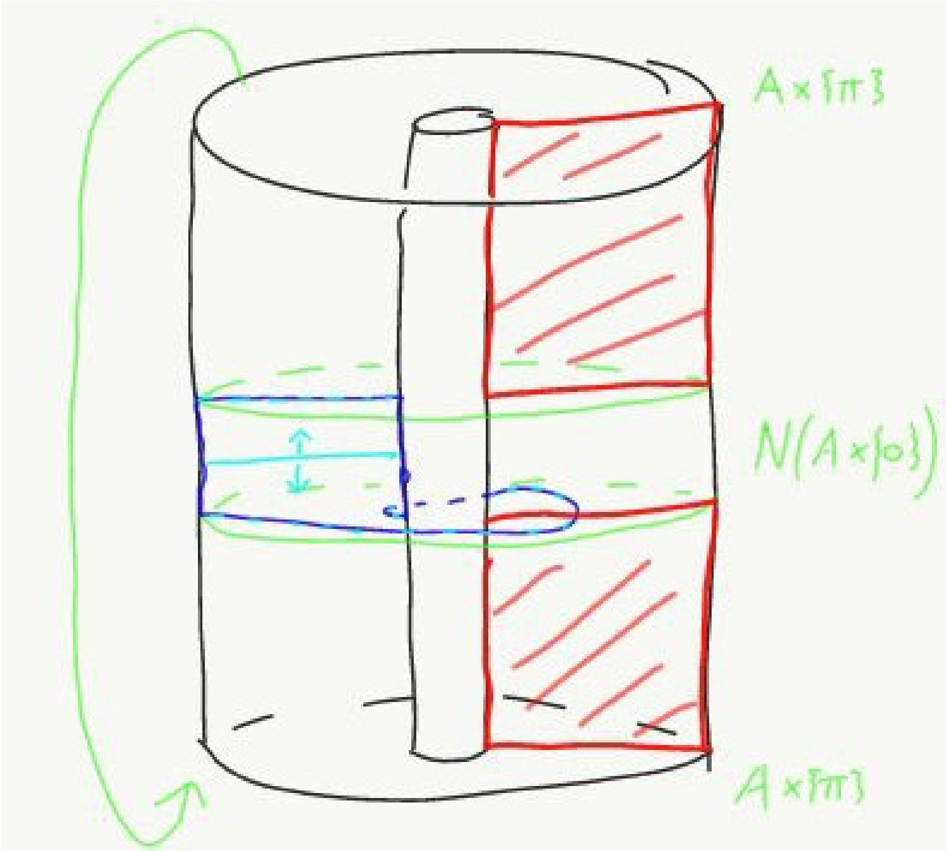

Figure 5.5: The mapping torus $A(\tau)$. To see this as the open book M_τ, we collapse the boundary circles $(S^1 \times \{0\}) \times S^1$ and $(S^1 \times \{1\}) \times S^1$ to a point. Here we have depicted the solid torus $N(A \times \{0\})$ in green and have shown that a slope-1 curve on $\partial N(A \times \{0\})$ bounds a disk. Hence M_τ is the result of 1-surgery on $S^2 \times S^1$ and therefore $M_\tau = S^3$. Further, the red and blue curves intersect once and are both meridional disks in two different solid tori. Hence we can see a Heegaard splitting of S^3 in this figure.

□

Remark 5.17. We have only shown (A, τ) corresponds to an annular open book of S^3, but we have not shown that fibration π_τ obtained from (A, τ) is the same fibration defined in Example 5.7. There are actually annular open books of S^3, one corresponding to the positive Hopf link and one to the negative Hopf link. The abstract open book (A, τ) corresponds to the positive Hopf link annular open book because we specified that τ is a right-handed Dehn twist.

Thus (A, τ) is the annular open book of S^3 described in Example 5.3. This open book has a supported contact structure, which we have already encountered.

Recall in Equation 4.16, the contact form α_a for $(S^3, \xi_{\text{tight}})$ was defined as

$$\alpha_a = \frac{1}{\sin^2\rho + \frac{1}{a^2}\cos^2\rho}(\sin^2\rho d\theta + \cos^2\rho d\phi). \tag{5.11}$$

Also recall from Proposition 4.11 that the Reeb vector field of α_a is $X_a = \partial_\theta + \frac{1}{a}\partial_\phi$.

Proposition 5.18. *Let (π, H) be the annular open book of S^3 described in Example 5.3. This open book supports the contact form α_a.*

Proof. The projection map $\pi : (S^3 - H) \to S^1$ is defined in coordinates $z = (\sin\rho e^{i\theta}, \cos\rho e^{i\phi})$ as

$$\pi(\sin\rho e^{i\theta}, \cos\rho e^{i\phi}) = e^{i(\theta+\phi)}. \tag{5.12}$$

Therefore the pages are given by $A_\psi = \pi^{-1}(\psi) = \{\theta + \phi = \psi\} \subset (S^3 - H)$. This means the tangent space TA_ψ is spanned by the vectors $\{\partial_\rho, \partial_\theta - \partial_\phi\}$. Hence we have shown that X_a intersect the pages of this open book transversely. By examining α_a when $\sin\rho = 0$ and when $\cos\rho = 0$ in Equation 4.16, we see that $\alpha_a > 0$ on H. Thus by Lemma 5.10, this open book supports α_a. □

Now we see that in Section 4.3, we performed surgery on a binding component of the open book (π, H) of S^3, endowed with a supported contact form α_a. Naturally, we ask whether we can extend these techniques to supported contact forms on other open books.

To perform tight surgery on $(S^3, \xi_{\text{tight}})$ in Section 4.3, we chose a contact form α_a representing ξ_{tight} with nice dynamics. Then we performed surgery on a binding component of H in such a way that preserved these nice dynamics, allowing us to conclude tightness of the surgered manifold. This means that the dynamics of the contact form near a binding component we would like to do surgery on are very relevant to this construction.

5.2 Dynamical Improvements near the Binding

In Theorem 5.12, we constructed a supported contact form α for an arbitrary open book (π, B). However, we saw that α does not have nice dynamics near the binding, and we do not know the dynamics on the interior of the pages. Thus we aim to modify the Reeb dynamics of α near B, working towards extending the surgery techniques presented in Chapter 4.

Our idea to improve the techniques used in Section 4.3 is to control the Reeb dynamics of a neighbourhood of the binding and the interior of the pages separately. In this section, we aim to control the Reeb dynamics near the binding. We show that for an abstract open book (Σ, h) with certain assumptions on the monodromy, we can construct a supported contact form on a neighbourhood of the binding that is strictly contactomorphic to a solid torus with a model contact form. Dynamically, this is an improvement on the supported contact form constructed on $N(B)$ in Theorem 5.12. Further, the strict contactomorphism allows us to lift the ideas of the surgery in Section 4.3 to this situation because the surgery on B can be performed completely inside $N(B)$. Thus we are able to apply the techniques of Section 4.3 to perform surgery on B while preserving the nice dynamics inside $N(B)$.

5.2.1 Parametrising a Dehn twist

In Section 5.1.2, we described an abstract open book (A, τ) where $A = S^1 \times I$ is an annulus and τ is a right-handed Dehn twist. We showed that this is an open book of $M_\tau = S^3$ with binding a (positive) Hopf link and that this open book supports the contact form α_a. Further, in Section 4.3 we performed tight surgery on a binding component of (M_τ, α_a). We are interested in lifting the ideas of Section 4.3 to an abstract open book (Σ, h) where h restricts to a single Dehn twist near a boundary component of Σ.

In order to analyse an abstract open book whose monodromy is a Dehn twist near the boundary, we first parametrise a Dehn twist on an annulus. Let (φ, s) be coordinates on $A = S^1 \times I$ with $\varphi \in \mathbf{R}/2\pi\mathbf{Z}$ and $s \in [0, 1]$. Then a parametrisation of a right-handed Dehn twist $\tau : A \to A$ in these coordinates is

$$\tau(\varphi, s) = (\varphi + 2\pi s, s). \tag{5.13}$$

This map rotates the circle $S^1 \times \{s\} \subset A$ by an angle of $2\pi s$ in an anticlockwise direction. We see that the line given by fixing some $\varphi = c$ is sent to a curve that circles around the boundary component $S^1 \times \{0\}$ as s increase. For an illustration of this, see Figure 5.2.1.

Let $S = D^2 \times S^1$ be a solid torus with coordinates $((\rho, \theta), \phi)$. In Theorem 5.12, we glued a solid torus S to a neighbourhood of the boundary $N(\partial\Sigma) = S^1 \times [0, \frac{1}{2}]$ by gluing $N(\partial\Sigma)$ to $\{\frac{1}{2} \leq \rho \leq 1\} \subset \overline{S}$. We would like to mimic this gluing procedure, so it is useful to change our Dehn twist coordinates to match this. Let $r = 1 - \frac{1}{2}s$ and notice that $r \in [\frac{1}{2}, 1]$. When we glue S in, we will match up ρ

with r on the interval $\frac{1}{2} \leq \rho \leq 1$. In the coordinates (φ, r) on A, τ is described as

$$\tau(\varphi, r) = (\varphi + 4\pi(1 - r), r). \tag{5.14}$$

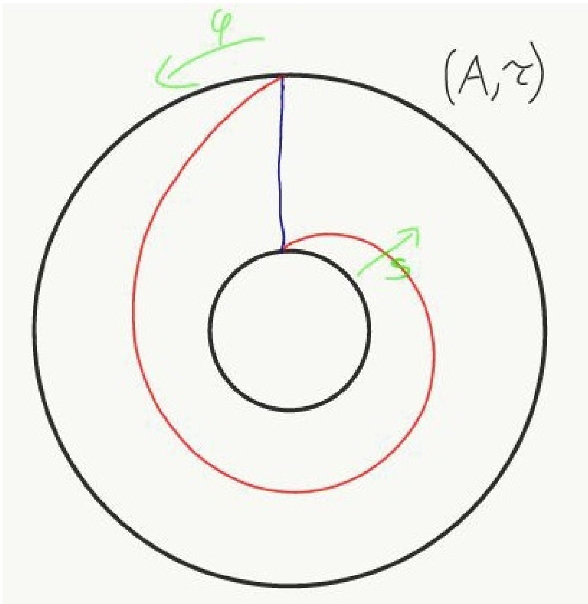

Figure 5.6: A right-handed Dehn twist.

5.2.2 The Dynamically Improved Construction

Let (Σ, h) be an abstract open book with corresponding open book decomposition (π, B) of M_h. Let Σ_ψ denote the page $\overline{\pi^{-1}(\psi)}$ for $\psi \in S^1$. Suppose that h restricts to a Dehn twist near a boundary component of Σ. For notational simplicity only, we will assume that Σ has only one boundary component, but the proof of Theorem 5.19 extends to an abstract open book with multiple boundary components. Choose a neighbourhood $N(\partial\Sigma) = \partial\Sigma \times [\frac{1}{2}, 1]$ with $\partial\Sigma \times \{\frac{1}{2}\} = \partial\Sigma$ and choose coordinates (φ, r) on $N(\partial\Sigma)$ such that

$$h(\varphi, r) = (\varphi + 4\pi(1 - r), r). \tag{5.15}$$

This is the parametrisation of h defined in Equation 5.14. Consider the cylinder $N(\partial\Sigma) \times [0, 2\pi] \subset \Sigma \times [0, 2\pi]$ with coordinates $((\varphi, r), t)$ where $t \in [0, 2\pi]$. This is a neighbourhood of the boundary of the mapping torus before it has been glued up by the monodromy h. So let $\pi : \Sigma \times [0, 2\pi] \to \Sigma(h)$ be the projection map to the mapping torus and define $N(\partial\Sigma(h)) = \pi(N(\partial\Sigma) \times [0, 2\pi])$.

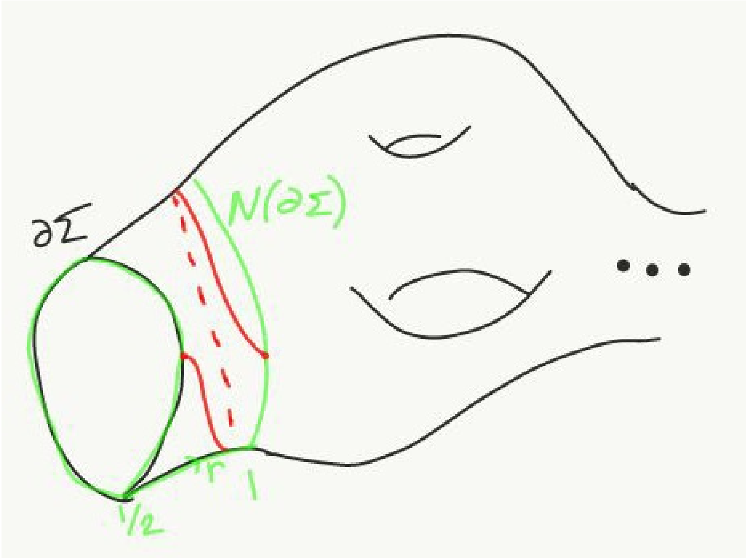

Figure 5.7: A right-handed Dehn twist near a boundary component of a surface.

Theorem 5.19. *Suppose (Σ, h) is an abstract open book such that near $\partial\Sigma$, the monodromy h restricts to a Dehn twist parametrised by Equation 5.15. Then there is a contact form α on a neighbourhood $N(\partial\Sigma(h))$ of $\partial\Sigma(h)$ with nice dynamics and Reeb orbits transverse to Σ_ψ. Further, α extends to a neighbourhood of the binding $U_h \subset M_h$ with nice dynamics inside U_h, such that its Reeb orbits are transverse to Σ_ψ and tangent to B.*

Proof. We begin by defining a contact form α on $N(\partial\Sigma(h))$ with nice dynamics and whose Reeb orbits are transverse to Σ_ψ. We then extend α to the binding and show that it still has nice dynamics and that its Reeb orbits are are transverse to Σ_ψ and tangent to B.

Let $S = D^2 \times S^1$ be a solid torus with coordinates $((\rho, \theta), \phi)$ where $\rho \in [0, 1]$ and $\theta, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. Fix an irrational number $a > 0$ and let α_a be the model contact form on S as in Equation 4.16. Let $N(B) = \{0 \leq \rho < \frac{1}{2}\} \subset S$ be a solid torus contained in S. To construct M_h, we glue $N(B)$ to $\partial\Sigma(h)$ so that the curve $\{p\} \times S^1 \subset \partial\Sigma(h)$ bounds a disk. We are going to accomplish this gluing by gluing S to $\Sigma(h)$ over the neighbourhood $N(\partial\Sigma(h))$ so that $\partial N(B)$ is identified with $\partial\Sigma(h)$. We now construct the map that achieves this gluing.

Recall that we have coordinates (φ, r, t) on $N(\partial\Sigma) \times [0, 2\pi]$ and $((\rho, \theta), \phi)$ on

S. Define a map $F : N(\partial\Sigma) \times [0, 2\pi] \to S$ by

$$\begin{aligned} F : N(\partial\Sigma) \times [0, 2\pi] &\to S \\ F(\varphi, r, t) &= (r, t, \varphi - (2r-1)t). \end{aligned} \tag{5.16}$$

The map F is defined so that it is compatible with the identifications made in the mapping torus. We show that F descends to $\widetilde{F} : N(\partial\Sigma(h)) \to S$ by showing that $F(h(\varphi, r), 0) = F(\varphi, r, 2\pi)$.

$$\begin{aligned} F(\varphi, r, 2\pi) &= (r, 2\pi, \varphi - (2r-1) \cdot 2\pi) \\ &= (r, 0, \varphi - 2\pi(2r-1) + 2\pi) \\ &= (r, 0, \varphi + 4\pi(1-r)) \\ &= F(\varphi + 4\pi(1-r), r, 0) \\ &= F(h(\varphi, r), 0). \end{aligned}$$

Let $N(\partial S) = \{\frac{1}{2} \leq \rho \leq 1\} \subset S$ be a neighbourhood of ∂S. Then the map $\widetilde{F}$ is a diffeomorphism onto $N(\partial S)$ with inverse given by

$$\begin{aligned} \widetilde{F}^{-1} : N(\partial S) &\to N(\partial\Sigma(h)) \\ (\rho, \theta, \phi) &\mapsto (\phi + (2\rho - 1)\theta, \rho, \theta). \end{aligned} \tag{5.17}$$

Checking that $\widetilde{F}^{-1}$ is the inverse of $\widetilde{F}$ is a straightforward calculation. Note that $\widetilde{F}^{-1}$ is only well defined when its codomain is the quotient $N(\partial\Sigma(h))$, not the cylinder $N(\partial\Sigma) \times [0, 2\pi]$.

The map $\widetilde{F}^{-1}$ describes a family of tori foliating $N(\partial\Sigma(h))$. Let $T_\rho, \mu_\rho, \lambda_\rho \subset S$ be the torus, meridian and longitude described in Equations 3.5, 3.6 and 3.7. This defines a family of tori $\widetilde{T}_r \subset N(\partial\Sigma(h))$ by

$$\begin{aligned} \widetilde{T}_r &= \widetilde{F}^{-1}(T_r) = \{(\varphi + (2r-1)t, r, t) \mid \varphi, t \in \mathbf{R}/2\pi\mathbf{Z}\} \\ \widetilde{\mu}_r &= \widetilde{F}^{-1}(\mu_r) = \{((2r-1)t, r, t) \mid t \in \mathbf{R}/2\pi\mathbf{Z}\} \\ \widetilde{\lambda}_r &= \widetilde{F}^{-1}(\lambda_r) = \{(\varphi + (2r-1)t, r, 0) \mid \varphi \in \mathbf{R}/2\pi\mathbf{Z}\}. \end{aligned} \tag{5.18}$$

Notice that when $\rho = \frac{1}{2}$ in Equation 5.18, the $2\rho - 1$ term vanishes. This means that $\widetilde{\mu}_{\frac{1}{2}} = \{(0, \frac{1}{2})\} \times S^1 \subset \partial\Sigma(h)$. Thus $\widetilde{F}^{-1}$ sends a meridian on $\partial N(B)$ to the curve $\{(0, \frac{1}{2})\} \times S^1$ on the boundary of $\partial\Sigma(h)$ and therefore gluing $N(B)$ to $\partial\Sigma(h)$ by $\widetilde{F}^{-1}$ will construct M_h.

Fix an irrational number $a > 0$ and define a contact form α on $N(\partial\Sigma(h))$ by

$$\alpha = \widetilde{F}^* \alpha_a. \tag{5.19}$$

We will prove that $(N(\partial\Sigma(h)), \alpha)$ has nice dynamics and that the Reeb orbits of α are transverse to Σ_ψ.

Lemma 5.20. *The Reeb vector field X of $(N(\partial\Sigma)\times[0,2\pi], F^*\alpha_a)$ is*

$$X = ((2r-1)\partial_\varphi + \partial_t) + \frac{1}{a^2}\partial_\varphi. \tag{5.20}$$

Consequently, the Reeb vector field of $(N(\partial\Sigma(h)),\alpha)$ is $\pi_*(X)$.

Proof. By Lemma 2.23, the Reeb vector field of $F^*\alpha$ is $F_*^{-1}(X_a)$. So we only need to check that $F_*(X) = X_a$.

$$\begin{aligned} F_*(X) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2t & -(2r-1) \end{bmatrix} \begin{bmatrix} (2r-1)+\frac{1}{a^2} \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ \frac{1}{a^2} \end{bmatrix} \\ &= X_a \end{aligned}$$

Also by Lemma 2.23, we have that $Y_a = \pi_*(X)$ because $\pi : \Sigma\times[0,2\pi]\to\Sigma(h)$ is a surjective submersion. $\square$

From Equation 5.18, the curves $\widetilde{\mu}_r$ and $\widetilde{\lambda}_r$ are the integral curves of the vector fields $(2r-1)\partial_\varphi + \partial_t$ and ∂_φ respectively. So we have shown that Y_a is tangent to the torus $\widetilde{T}_r$ with slope $\frac{1}{a^2}$. This shows that Y_a has no periodic orbits on $N(\partial\Sigma(h))$ because a is irrational. Thus $(N(\partial\Sigma(h)),\alpha)$ has nice dynamics. Additionally, the subset of the page Σ_ψ contained in $N(\partial\Sigma(h))$ is $\Sigma_\psi\cap N(\partial\Sigma(h))$ and has a constant t-coordinate. Since Y_a always has a non-zero ∂_t component, it is transverse to Σ_ψ.

Now we will extend α to the binding. Define $U_h\subset M_h$ by

$$U_h = N(\partial\Sigma(h)) \cup_{\widetilde{F}^{-1}} S. \tag{5.21}$$

Then the contact form α on $N(\partial\Sigma(h))$ extends to a contact form α_{U_h} on U_h by defining

$$\alpha_{U_h} = \alpha \cup_{\widetilde{F}^{-1}} \alpha_a. \tag{5.22}$$

The contact form α_{U_h} is smooth because $\alpha = \widetilde{F}^*\alpha_a$ on $N(\partial\Sigma(h))$. The form α_a has nice dynamics inside $N(B)$, as seen in Chapter 4. Further, the Reeb orbits of α_a are transverse to $\Sigma_\psi\cap N(B)$ and tangent to the core of $N(B)$. Thus α_{U_h} has nice dynamics inside U_h, with Reeb orbits transverse to Σ_ψ and tangent to the binding B. $\square$

Thus we have defined a contact form α_{U_h} on a neighbourhood of the binding $U_h \subset M_h$ with nice dynamics inside U_h and the right Reeb properties to potentially be supported by (B, π). Further, α_{U_h} has the same local model on U_h as the model contact forms exhibited in Chapter 4 because (U_h, α_{U_h}) is strictly contactomorphic to (S, α_a). This means that if α_{U_h} can be extended to M_h with globally nice dynamics, then the techniques in Section 4.3 can be directly applied to U_h and we can perform tight surgery on B.

Corollary 5.21. *Suppose that the contact form α_{U_h} on U_h can be extended to a supported contact form α_{M_h} on M_h that has nice dynamics outside of U_h. Then there exists tight rational r-surgeries on B in (M_h, α_{U_h}). Moreover, this method of tight 0-surgery performed on all but one component of B yields a tight open book.*

Proof. The first statement is a direct corollary of Theorem 5.19 and our discussion about lifting the surgery techniques from Section 4.3 to $N(B)$. For the second statement, 0-surgery on a binding component of B is just caps off a boundary component of the pages. So as long as the boundary is non-empty after a number of 0-surgeries on components of B, then (B, π) has a natural open book structure where the new page Σ' has less boundary components. If 0-surgery is performed on all binding components, then we are just left with a fibre bundle. □

Obtaining Corollary 5.21 means that we have reduced the question of "is tight surgery on an open book with monodromy a single Dehn twist near a binding component possible?" to finding a contact form on the interior of the pages with nice dynamics (that extends α_{U_h}). That is, we have taken care of all the work near the binding. We have constructed a supported contact form near the binding that has nice dynamics and that we already know how to perform tight surgery on. This achieves our goal of controlling the dynamics near the binding of these open books whose monodromy restricts to a Dehn twist near a boundary component.

Extending α_{U_h} to M_h and maintaining its nice dynamical properties is a complicated task which we do not explore. As we have already seen, the Reeb dynamics of a contact form are very sensitive to small perturbations of the manifold, so controlling them globally is a complex task. However, we already know this is possible in the case of the annular open book (A, τ) of S^3, so we should have some hope that this is possible in other cases.

Notice that in the proof of Theorem 5.19, we are able to choose the slope of the Reeb orbits of α to be any positive irrational number $\frac{1}{a^2}$. Further, the characteristic foliation of α on the tori $\widetilde{T}_r$ is linear and monotonically decreasing, just like in Chapter 4. This means that we have a lot of flexibility in our choice of α_{U_h}. So extending it to M_h should not be the limiting factor in Corollary 5.21. The more difficult task is controlling the Reeb orbits on the interior of the pages.

Bibliography

[BE13] John A. Baldwin and John B. Etnyre. Admissible transverse surgery does not preserve tightness. *Math. Ann.*, 357(2):441–468, 2013.

[Ben83] Daniel Bennequin. Entrelacements et équations de Pfaff. In *Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982)*, volume 107 of *Astérisque*, pages 87–161. Soc. Math. France, Paris, 1983.

[EG99] John Etnyre and Robert Ghrist. Tight contact structures via dynamics. *Proc. Amer. Math. Soc.*, 127(12):3697–3706, 1999.

[Eli89] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989.

[Eli92] Yakov Eliashberg. Contact 3-manifolds twenty years since J. Martinet's work. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):165–192, 1992.

[Etn03] John B. Etnyre. Introductory lectures on contact geometry. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 81–107. Amer. Math. Soc., Providence, RI, 2003.

[Etn06] John B. Etnyre. Lectures on open book decompositions and contact structures. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 103–141. Amer. Math. Soc., Providence, RI, 2006.

[Gei97] Hansjörg Geiges. Constructions of contact manifolds. *Math. Proc. Cambridge Philos. Soc.*, 121(3):455–464, 1997.

[Gei08] Hansjörg Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[Gir02] Emmanuel Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 405–414. Higher Ed. Press, Beijing, 2002.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[HK99] Helmut Hofer and Markus Kriener. Holomorphic curves in contact dynamics. In *Differential equations: La Pietra 1996 (Florence)*, volume 65 of *Proc. Sympos. Pure Math.*, pages 77–131. Amer. Math. Soc., Providence, RI, 1999.

[Hon01] Ko Honda. Factoring nonrotative $T^2 \times I$ layers. Erratum: "On the classification of tight contact structures. I" [Geom. Topol. **4** (2000), 309–368; mr1786111]. *Geom. Topol.*, 5:925–938, 2001.

[HWZ95] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants. *Geom. Funct. Anal.*, 5(2):270–328, 1995.

[KZ19] Jean-Louis Koszul and Yi Ming Zou. *Introduction to symplectic geometry.* Science Press Beijing, Beijing; Springer, Singapore, 2019. With forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2013.

[Ler01] Eugene Lerman. Contact cuts. *Israel J. Math*, 124:77-92, 2001.

[Mas14] Patrick Massot. Topological methods in 3-dimensional contact geometry. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 27–83. János Bolyai Math. Soc., Budapest, 2014.

[Moi77] Edwin E. Moise. *Geometric topology in dimensions* 2 *and* 3. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, Vol. 47.

[Rol90] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series.* Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[Tau09] Clifford Henry Taubes. The Seiberg-Witten equations and the Weinstein conjecture. II. More closed integral curves of the Reeb vector field. *Geom. Topol.*, 13(3):1337–1417, 2009.

[Vil01] Gaetano Vilasi. *Hamiltonian dynamics.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[Wei79] Alan Weinstein. On the hypotheses of Rabinowitz' periodic orbit theorems. *J. Differential Equations*, 33(3):353–358, 1979.