1.

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ 3 & 0 & 1 & -1 \\ 1 & 3 & 0 & 4 \\ 3 & -9 & 4 & 5 \end{bmatrix} \quad b_1 = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix} \quad b_2 = \begin{pmatrix} -1.1 \\ -1.9 \\ 3.9 \\ 0.01 \end{pmatrix}$$

$$A_{x_1} = b_1 \qquad A_{x_2} = b_2$$

$$\left[\begin{array}{cccc|c|c} 1 & 3 & 0 & 4 & 4 & 3.9 \\ 3 & 0 & 1 & -1 & 2 & 1.9 \\ 0 & -1 & 3 & 0 & -1 & -1.1 \\ 3 & -9 & 4 & 5 & 0 & 0.01 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 - 3R_1 \\ \\ R_4 = R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c|c} 1 & 3 & 0 & 4 & 4 & 3.9 \\ \overset{0}{0} & \overset{1}{-9} & \overset{-1/9}{1} & \overset{13/9}{-13} & \overset{10/9}{-10} & \overset{9.8/9}{-9.8} \\ 0 & -1 & 3 & 0 & -1 & -1.1 \\ 0 & -18 & 4 & -7 & -12 & -11.69 \end{array}\right] \begin{array}{l} \\ R2 = R2/-9 \\ R_3 = R_3 + R_2 \\ R_4 = R_4 + 18R_2 \end{array}$$

$$\left[\begin{array}{cccc|c|c} 1 & 3 & 0 & 4 & 4 & 3.9 \\ 0 & 1 & -1/9 & 13/9 & 10/9 & 9.8/9 \\ 0 & 0 & \overset{1}{2.88} & \overset{0.502}{13/9} & \overset{0.038}{0.11} & \overset{-0.038}{-0.11} \\ 0 & 0 & 2 & 19 & 8 & 7.91 \end{array}\right] \begin{array}{l} \\ \\ R_3 = R_3/2.88 \\ R_4 = R_4 - 2R_3 \end{array}$$

$$\left[\begin{array}{cccc|c|c} 1 & 3 & 0 & 4 & 4 & 3.9 \\ 0 & 1 & -1/9 & 13/9 & 10/9 & 9.8/9 \\ 0 & 0 & 1 & 0.502 & 0.038 & -0.038 \\ 0 & 0 & 0 & 17.996 & 7.924 & 7.926 \end{array}\right]$$

$$\left[\begin{array}{cccc|c|c} & & & & b_1 & b_2 \\ 1 & 3 & 0 & 4 & 4 & 3.9 \\ 0 & 1 & -1/9 & 13/9 & 10/9 & 9.8/9 \\ 0 & 0 & 1 & 0.502 & 0.038 & -0.038 \\ 0 & 0 & 0 & 1 & 0.44 & 0.44 \end{array}\right]$$

X_1

$b_4 = 0.44$

$b_3 + 0.502 b_4 = 0.038 \qquad b_3 = -0.183$

$b_2 - \frac{b_3}{9} + \frac{13 b_4}{9} = 10/9 \qquad b_2 = 0.456$

$b_1 + 3b_2 + 4b_4 = 4 \qquad b_1 = 0.872$

$$x_1 = \begin{pmatrix} 0.872 \\ 0.456 \\ -0.183 \\ 0.44 \end{pmatrix}$$

X_2

$b_4 = 0.44$

$b_3 + 0.502 b_4 = -0.038 \qquad b_3 = -0.259$

$b_2 - \frac{b_3}{9} + \frac{13 b_4}{9} = 9.8/9 \qquad b_2 = 0.425$

$b_1 + 3b_2 + 4b_4 = 3.9 \qquad b_1 = 0.865$

$$x_2 = \begin{pmatrix} 0.865 \\ 0.425 \\ -0.259 \\ 0.44 \end{pmatrix}$$

b) $\text{cond}(A) = |A| \cdot |A^{-1}|$

Norm 1

$|A|_1 = 13 \qquad |A^{-1}|_1 = 0.6347$

$\text{cond}_1(A) = 13 \cdot 0.6347 = 8.2511$

Norm ∞

$|A|_1 = 21 \quad |A^{-1}|_1 = 0.4936$

$\text{cond}_\infty(A) = 21 \cdot 0.4936 = 10.3656$

c) $\Delta x = x_1 - x_2$ $\quad \dfrac{|\Delta x|}{|x_1|} = \dfrac{0.338}{1.26} = 0.268$

$= 0.338$

$\Delta b = b_1 - b_2$ $\quad \dfrac{0.54}{2.24} = 0.241$

$= 0.54$

$= 0.268 \leq 10.3656 \cdot 0.241$

$= 0.268 \leq 2.498$

d) $A = LU$

*From a

$$U = \left[\begin{array}{cccc|c|c} & & & & b_1 & b_2 \\ 1 & 3 & 0 & 4 & 4 & 3.9 \\ 0 & 1 & -1/9 & 13/9 & 10/9 & 9.8/9 \\ 0 & 0 & 1 & 0.502 & 0.038 & -0.038 \\ 0 & 0 & 0 & 1 & 0.44 & 0.44 \end{array}\right]$$

*From the transformations in part a

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 18 & 2 & 1 \end{bmatrix} \qquad b_1 = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix}$$

$Ux = y$

$x_4 = -85$

$x_3 + 0.502x_4 = -1 \qquad x_3 = 41.67$

$x_2 - \frac{1}{9}x_3 + \frac{13}{9}x_4 = 5 \qquad x_2 = -113.15$

$x_1 + 3x_2 + 4x_4 = -1 \qquad x_1 = 678.45$

$$x = \begin{pmatrix} 678.45 \\ -113.15 \\ 41.67 \\ -85 \end{pmatrix}$$

$Ly = b$

$y_1 = -1$

$3y_1 + y_2 = 2 \qquad y_2 = 5$

$y_2 + y_3 = 4 \qquad y_3 = -1$

$3y_1 + 18y_2 + 2y_3 + y_4 = 0 \qquad y_4 = -85$

$$y = \begin{pmatrix} -1 \\ 5 \\ -1 \\ -85 \end{pmatrix}$$

e) Reordering so diagonal is all non-zero

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ 3 & 0 & 1 & -1 \\ 1 & 3 & 0 & 4 \\ 3 & -9 & 4 & 5 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 3 & 0 & 1 & -1 \\ 3 & -9 & 4 & 5 \\ 0 & -1 & 3 & 0 \\ 1 & 3 & 0 & 4 \end{bmatrix}$$

$|3| > |1| + |-1|$

$|9| \not> |3| + |4| + |5|$ $\times$

$|3| > |-1|$

$|4| \not> |3| + |1|$ $\times$

$\therefore$ sufficent condition fails

$$b_1 = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 0 \\ -1 \\ 4 \end{pmatrix}$$

$$x_1 = \frac{2 - x_3 + x_4}{3}$$

$$x_2 = \frac{-3x_1 - 4x_3 - 5x_4}{-9}$$

$$x_3 = \frac{-1 + x_2}{3}$$

$$x_4 = \frac{4 - x_1 - 3x_2}{4}$$

Assume x's $= 0$ for first iteration

iteration	x_1	x_2	x_3	x_4
0	0	0	0	0
1	0.66	0	-0.33	1
2	1.11	0.63	-0.12	0.83
3	1.06	0.69	-0.18	0.25
4	0.79	0.44	-0.1	0.22
5	0.78	0.34	-0.19	0.48
6	0.89	0.44	-0.22	0.55
7	0.92	0.50	-0.19	0.42
8	0.88	0.47	-0.17	0.39
9	0.85	0.44	-0.18	0.42
10	0.87	0.44	-0.19	0.46

$$x = \begin{pmatrix} 0.87 \\ 0.44 \\ -0.19 \\ 0.46 \end{pmatrix}$$

2, a) $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 3^2 = \lambda^2 - 10\lambda + 16 = 0$$

$$\lambda_1 = 8 \quad \lambda_2 = 2$$

$$(A - 8I)x = 0 \rightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} x = 0 \qquad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2$$

$$(A - 2I)x = 0 \rightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} x = 0 \qquad x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

b) $A = Q D_\lambda Q^T$ where $Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ $\quad D_\lambda = \begin{pmatrix} \overset{\lambda_1}{8} & 0 \\ 0 & \underset{\lambda_2}{2} \end{pmatrix}$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ \frac{-1}{\sqrt{2}} & 1/\sqrt{2} \end{bmatrix}$$

c) $A = U \Sigma V^T$ $\qquad$ Symm matrix → $U = V = Q$ $\quad \Sigma = \sqrt{A}$

$$\Sigma = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

d) Since U, V equal Q for symmetric matrices, the difference lies within the SVD, which represents the square roots of the eigenvalues, which emphasis the strength of each values impact, wheres the symmetric eigenvalue decomp. directly relates to the eigenvalues directly.

3.

a) # of linear independant col. vectors

max = 3 (three cols.)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

b) $$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\det|A^T A - \lambda I| = \det\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 8\lambda^2 - 17\lambda + 10$$

$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 5$

$\lambda_1 = 1$ example:

$$A^TA - 1I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 \qquad x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \lambda_3 = 5 \qquad x_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

SVD of A

$$V = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \qquad U = AV\Sigma^{-1}$$

$\uparrow$ $\uparrow$ $\nwarrow$

$\sqrt{1^2+1^2}$ $\sqrt{1^2+1^2+1^2}$ $\sqrt{1^2+2^2+1^2}$

$$U = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & .82 & .37 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0 & -.41 & .37 \\ .71 & 0 & .55 \\ 0 & -.41 & .37 \\ -.71 & 0 & .55 \end{bmatrix}$$

$$A = U \Sigma V^T = \begin{bmatrix} 0 & .82 & .37 \\ 0 & -.41 & .37 \\ .71 & 0 & .55 \\ 0 & -.41 & .37 \\ -.71 & 0 & .55 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \end{bmatrix}$$

c)

d) 3, three singular values from part A

e)

f)

4. Show singular value $\sigma_i = u_i^T A v_i$

Using SVD $A = U \Sigma V^T = \sum_{i=1}^{n} \sigma_k u_k v_k^T$

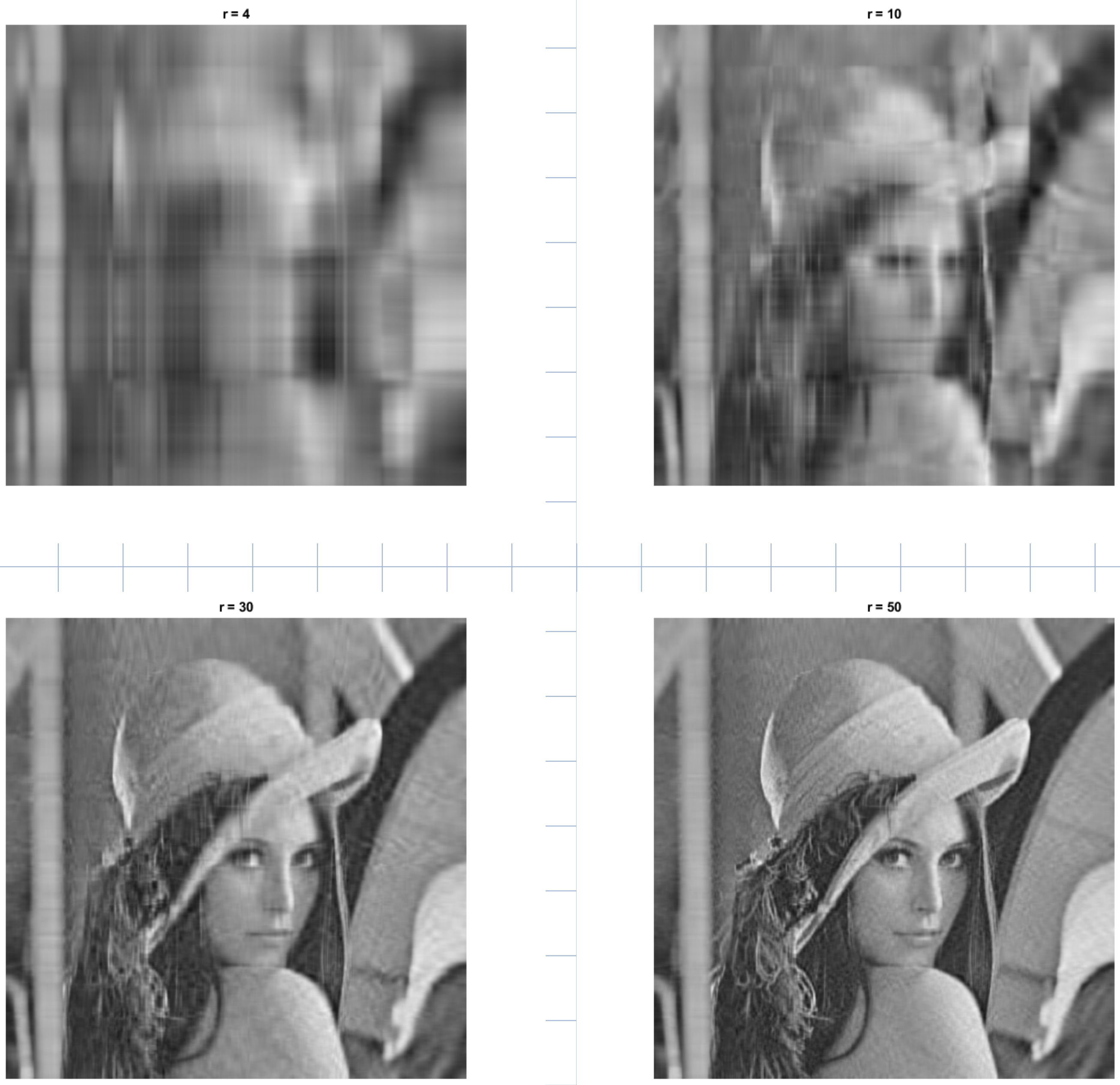
Isolate σ_i from sum

$$u_i^T A v_i = u_i^T \left(\sum_1^n \sigma_i u_i v_i^T \right) \cdot v_i$$

$$u_i^T A v_i = \sum_1^n \sigma_i \underbrace{(u_i^T u_i)}_{\delta_{ki}} \underbrace{(v_i^T \cdot v_i)}_{\delta_{ki}} = \sigma_i$$

$$\therefore u_i^T A v_i = \sigma_i$$

c)



d) Minimum rank ≈ 15 which resaults in the image being recognizable.

e)

Exclude 1st singular value

Excluding the first singular value resaults in a much darker image.

c)

This approach is only useful if the noise doesn't effect the base image.