Filter Designs and Analyses (April 2015)

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***Abstract*—Filter designs utilize their unique frequency response for a broad range of applications including noise elimination, signal separation, and signal enhancement. This report focuses on understanding the response of three basic filter types; Series RL, active, and a butterworth filters. The foremost two designs were provided and analyzed, the latter was designed, then analyzed. A generic filter analysis approach was used, applicable to nearly every filter design, to calculate theoretical results. These results were then verified through computer simulation tools and prototype construction. Results implicate that better filters can be designed by increasing the order of the filter design at the cost of complexity, hardware, and power consumption.**

***Index Terms*— Filter, Butterworth, Order, Active, Passive, RL, Series.**

I. INTRODUCTION

THERE exists a multitude of extensive, complex filter designs, each intended to accomplish specific, yet distinct, tasks. Underlying each of these designs are fundamental properties of every filter, generically grouping them into four main filter types; low-pass, high-pass, band-pass, and band-reject filters. Furthermore, filter designs have similar analysis procedures, making their implementation easier by following a generic approach for each filter design.

II. GENERIC FILTER DESIGN ANALYSIS PROCESS

When provided a filter design, the type of filter and its order should first be identified to later verify your analysis. This can be done by considering generic instances of each type of filter, where higher order filters are typically combinations of first order filters. Once this has been established, the transfer function, denoted H(s), must be obtained. H(s) is the output of the filter divided by the input of the filter. The frequency response of the system can then be obtained from the transfer function by substituting j in for s, denoted H(jω), where ω represents the frequency and j represents the imaginary number $\sqrt{-1}$. The magnitude of the frequency response is then considered with the conditions outlined below in table I.

	Band-Pass	High-Pass	Low-Pass	Band-Reject
$\omega \to 0$, $\lvert H(j\omega)\rvert \to$	0	0	H_{max}	H_{max}
$\omega \to 0$, $\lvert H(j\omega)\rvert \to$	0	0	H_{max}	H_{max}

TABLE I: As the frequency approaches a limit, the response of the system signifies the filter type.

By considering the frequency response of the circuit as the frequency approaches these limits, the filter can be classified into one of the four main categories, as shown in table I. Introducing H_{max} to be the maximum gain of the system.

Determining the cutoff frequency is the next step in the process. Since the cutoff frequency is defined to be the frequency at which the circuit produces half-power, the frequency response can be set equal 1/2 H_{max}. When considering output voltage or current, however, this ratio is increased to $\sqrt{1/2}H_{max}$ since voltage and current are proportional to the square root of power. This is accomplished by solving (1) for the cutoff frequency.

$$|H(j\omega)| = \frac{H_{max}}{\sqrt{2}} \tag{1}$$

III. FIRST ORDER RL SERIES LOW-PASS FILTER

Consider the following circuit illustrated in Fig 1. It can be observed that this circuit is a first order, low-pass filter design. Following the generic analysis approach outlined above,Appendix A demonstrates the analysis of this filter design.

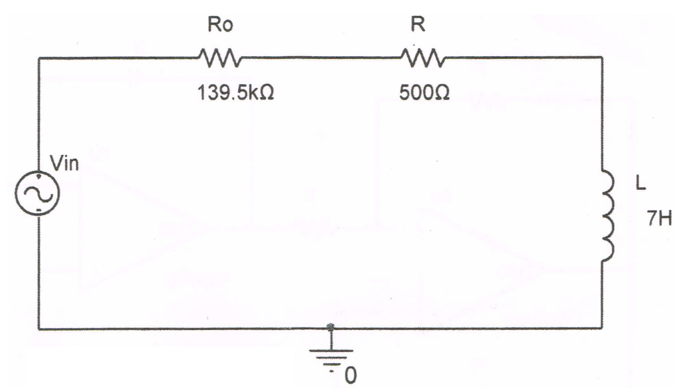


Fig. 1: Active, second order, low-pass filter design with the output taken across R_o

Notice the analysis first considers generic instances of the resistors and inductor. This greatly enhances the design by improving the efficiency at which a redesign can be made if constraints were to change.

Verification of the analysis is then done using Computer-Aided-Design (CAD) tools such as OrCAD and their simulation tools, shown in Figs. 2 to 4 below.

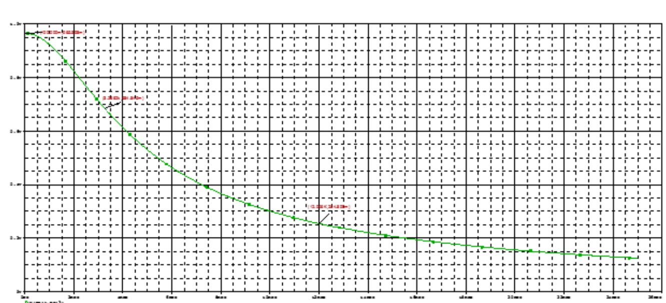


Fig. 2: Linear Plot - Output Voltage vs Frequency

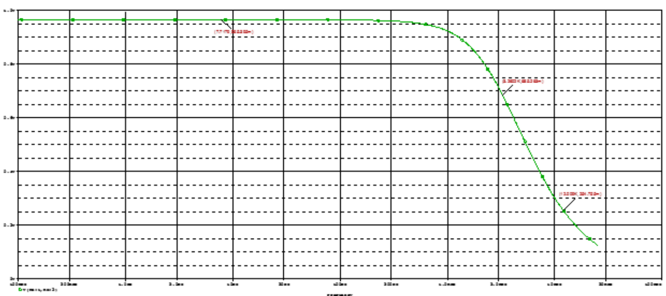


Fig. 3: Logarithmic Plot - Output Voltage vs Frequency

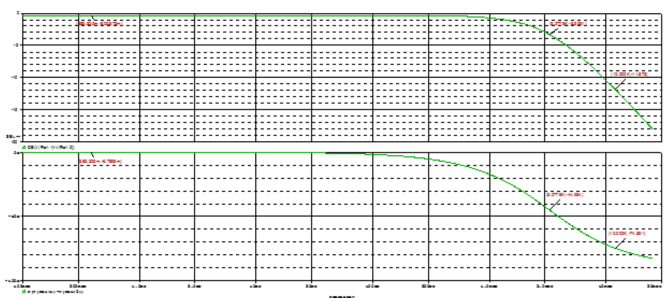


Fig. 4: Bode Plot - Output Voltage vs Frequency

The plots above all probably look familiar except for the bode plot. A bode plot is, simply put, a plot of the magnitude or phase of a system against its input frequency. In this case, both the output voltage and output power across R_o is shown in the bode plot. Further inspection done in Section VII.

IV. Active Second Order Low-Pass Filter

Now Consider an active low pass filter as shown in Fig. 5. As their name implies, active filters contain active components such as operational amplifiers. Following the generic analysis approach outlined in section I, Appendix B demonstrates the analysis of this filter design.

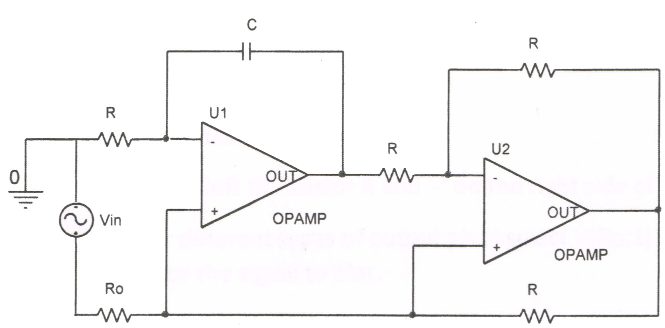


Fig. 5: Active, second order, low-pass filter design with the output taken across R_o

Notice, once again, the analysis first considers generic instances of the resistors and capacitors. This greatly enhances the design by improving the efficiency at which a redesign can be made if constraints were to change. Verification of the analysis is then done, once again, using CAD tools shown in Figs. 6 to 8 below.

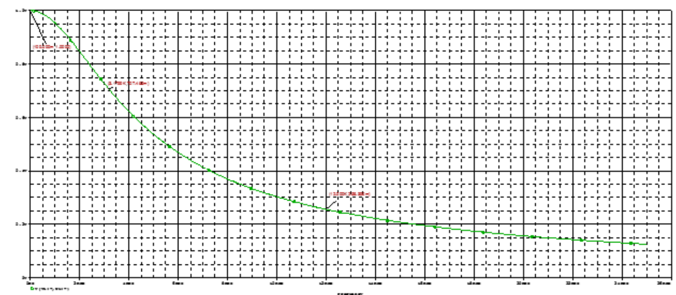


Fig. 6: Linear Plot - Output Voltage vs Frequency

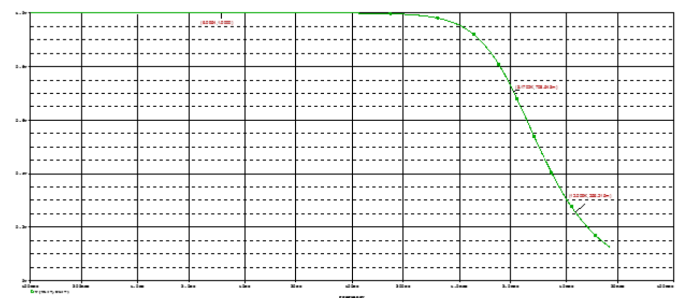


Fig. 7: Logarithmic Plot - Output Voltage vs Frequency

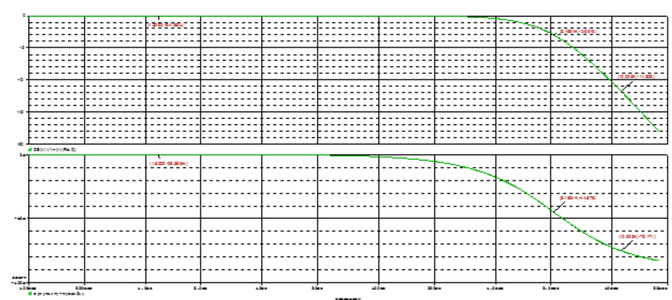


Fig. 8: Bode Plot - Output Voltage vs Frequency

V. Hardware Analysis

Further verification of the filter designs can be done by reconstructing the circuits. Table II below shows the results.

Frequency (Hz)	RL Filter V_o (V)	Active Filter V_o (V)
10	0.976	1
50	0.964	0.940
100	0.952	0.920
200	0.944	0.880
500	0.876	0.780
1k	0.776	0.620
2k	0.616	0.400
5k	0.376	0.060
10k	0.472	0.020
12k	0.520	0
15k	0.600	0
20k	0.680	0
25k	0.736	0

TABLE II: As the frequency approaches a limit, the response of the system signifies the filter type.

Figs. 9 and 10 below illustrate the frequency response of their corresponding filters with the input voltage at 1V peak-to-peak.

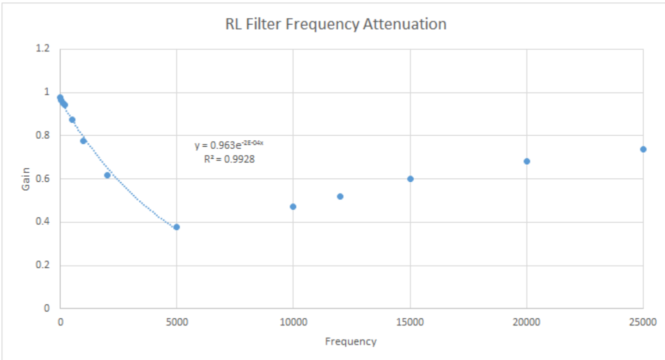


Fig. 9: Data Plot - Output Voltage vs Frequency

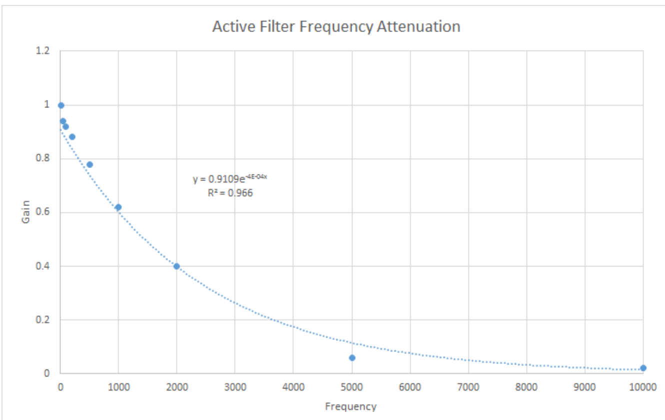


Fig. 10: Data Plot - Output Voltage vs Frequency

From these plots, expressions were added to approximate the gain and cutoff frequency. The RL filter had some discrepancies caused by an unaccounted phase shift and some noise within the circuit. The expression was then made for just the undistorted portion of the data.

Using these expressions, the gain at 12 kHz is calculated and the cutoff frequency approximated to be:

RL Filter:

$$|H(j\omega)| = 0.963e^{-0.2m*12k} = 0.0874$$

$$\frac{1}{\sqrt{2}} = e^{-0.2m*f_c}$$

$$f_c = 1733Hz \Rightarrow \omega_c = 2\pi f_c = 10888rad/sec$$

Active Filter:

$$|H(j\omega)| = 0.9109e^{-0.4m*12k} = 0.0075$$

$$\frac{1}{\sqrt{2}} = e^{-0.4m*f_c}$$

$$f_c = 866.4Hz \Rightarrow \omega_c = 2\pi f_c = 5444rad/sec$$

Figs. 11 and 12 show the actual output at R_o, where channel 1 is the input voltage and channel 2 is the other end of the output resistor R_o. Thus V_o is equal to channel 1 voltage minus channel 2 voltage.

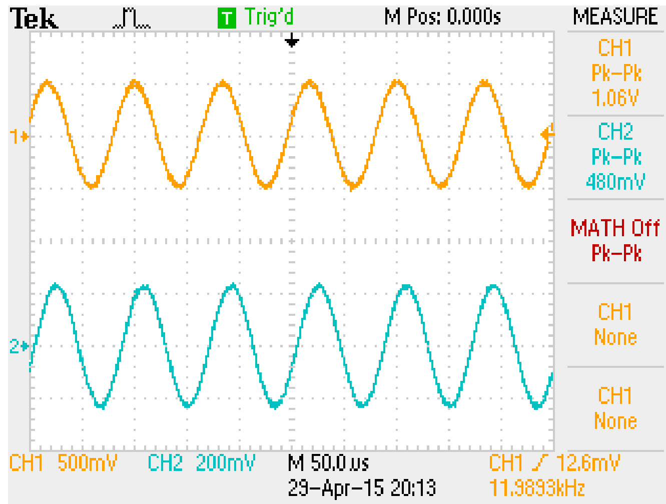


Fig. 11: hardware simulation at 12kHz output voltage - RL Filter

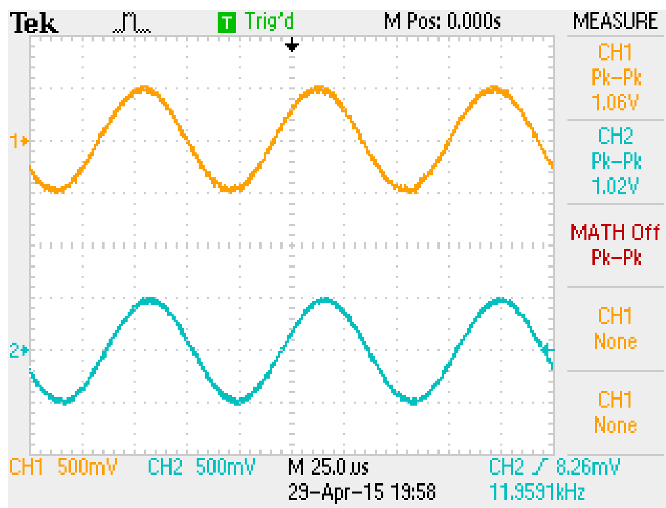


Fig. 12: hardware simulation at 12kHz output voltage - Active Filter

VI. Filter Comparison

Both the RL filter and the active filter are low-pass filters. What makes them different? Shown in Figs. 13 to 16 are the simulation outputs of each filter plotted together for comparison. The red line is the RL filter and the green line the active filter. By inspection, the active filter has a higher passband gain and approximately the same cut-off frequency. Furthermore, the active filter produces greater power at the same frequency. However, the active filter requires additional hardware to implement.

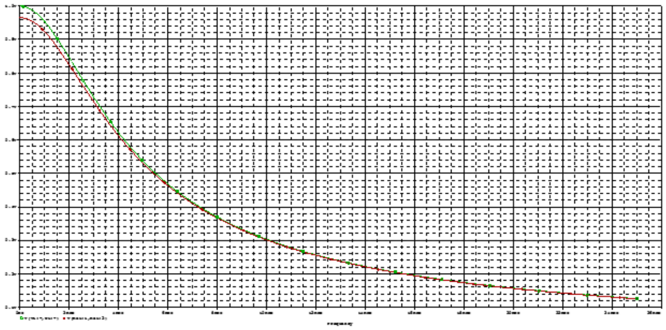


Fig. 13: Linear Plot - Output Voltage vs Frequency

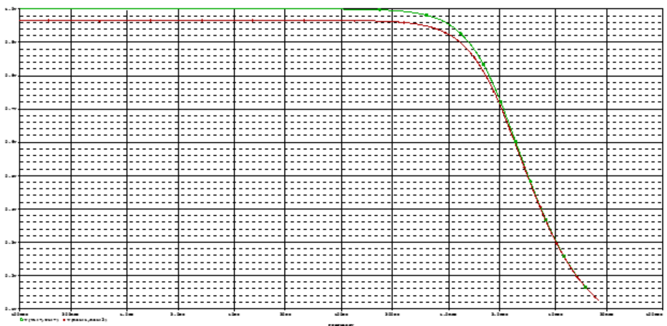


Fig. 14: Logarithmic Plot - Output Voltage vs Frequency

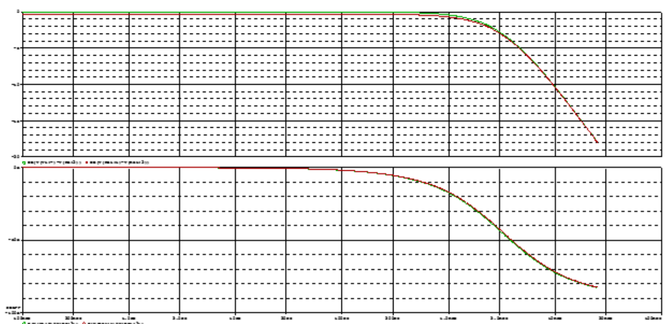


Fig. 15: Bode Plot - Output Voltage vs Frequency

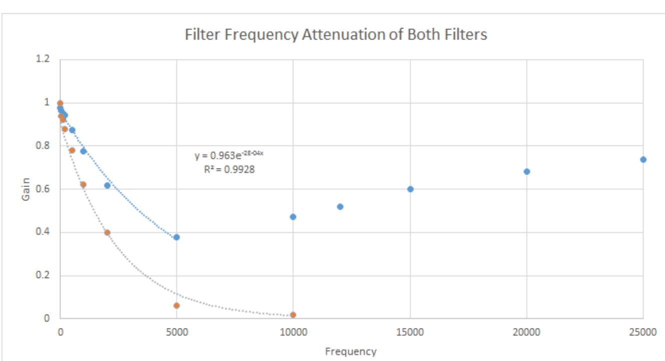


Fig. 16: Hardware Analysis Comparison Plot - Output Voltage vs Frequency

To explain this, consider the input resistance of each circuit. This is done by using a test source, V_x in place of R_o and solving for the current through the test source, I_x. Using Ohm's law, the equivalent input resistance, R_{in} is equivalent to $\frac{V_x}{I_x}$. It follows that the input resistance is:

RL Filter:

$$R_{in} = \sqrt{250k + \omega^2}$$

Active Filter:

$$R_{in} = 0.05\omega$$

The input resistance of the RL circuit is clearly much greater than the input resistance of the Active filter for all frequencies. This implies that both gain and power production across R_{out} will be larger for the active filter.

To summarize, table III below compares the theoretically calculated values against the simulation analysis and hardware results:

Results	ω_c (rad/sec)		H(jω) at $\omega = 12kHz$		H_max	
Theoretical	19858	20000	0.2555	0.2564	0.9964	1.0
Simulational	20280	19952	0.2548	0.2564	0.9654	1.0
Hardware	10888	5444	0.0874	0.0075	0.9630	0.9109

TABLE III: Results of both the RL and Active filters. The RL filter is the left columns, the active filter is the right columns

Both the theoretical and simulational results verify the design analysis, however the experimental results do not confirm the analysis. This could be due to various reasons, including component integrity, affected by the manufacturing process, the old equipment used, and the precision of the equipment used.

VII. High-Pass Butterworth Filter

Filter designs are always being improved and enhanced. Not only do these designs try to flatten the passband gain, but to create a steeper cutoff point. The drawback is these designs are usually constructed at the cost of greater power consumption and the use of additional hardware.

One type of filter that accomplishes this is the butterworth filter. An n^{th} order butterworth filter design can be derived following a similar approach outlined in section I, however, due to the widespread use and complexity of higher order butterworth filters, a generic design template is used to construct one.

Consider the need for a high pass butterworth filter with -3dB attenuation at 20 krad/sec and -27dB attenuation at 10 krad/sec. Then suppose only 0.1μF capacitors are available and the filter needs a passband gain of 3. The design of this filter is shown in Appendix C. The resulting circuit required to build this butterworth filter is shown below in Fig. 17.

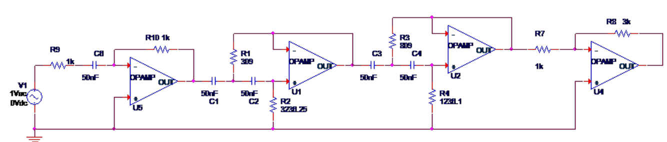


Fig. 17: 5^{th} order, high-pass butterworth filter design with the output taken at the last operational amplifier

Once again, the analysis can and should be verified using CAD tools as shown in Figs. 10 to 12 below.

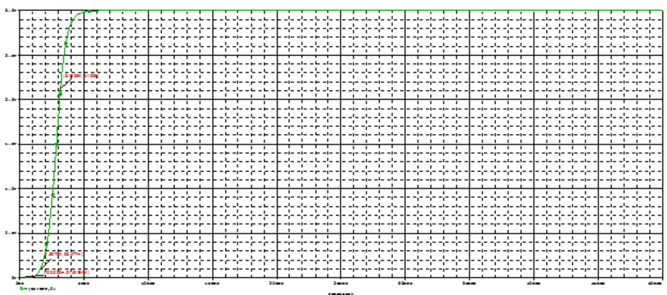


Fig. 18: Linear Plot - Output Voltage vs Frequency

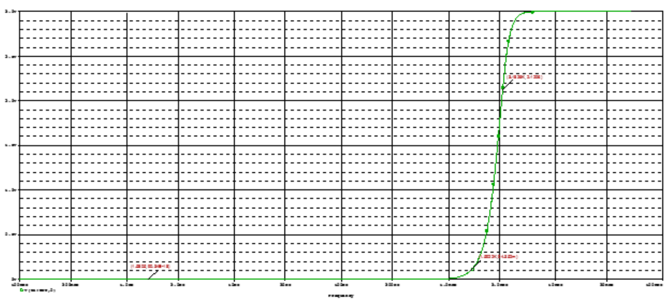


Fig. 19: Logarithmic Plot - Output Voltage vs Frequency

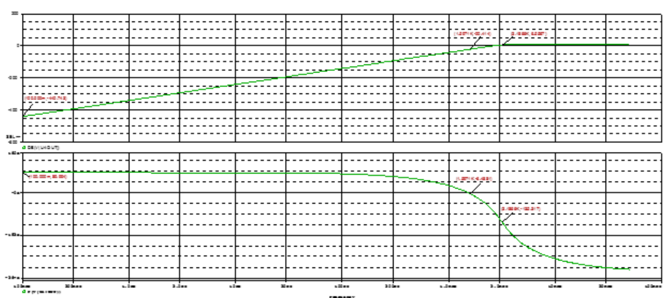


Fig. 20: Bode Plot - Output Voltage vs Frequency

To summarize, table IV below compares the theoretically calculated values against the simulation analysis results:

Results	ω_c (rad/sec)	H(jω) (dB) at $\omega = 10krad/sec$	H_max
Design	20000	-27dB	3.0
Theoretical	20000	-30.11	3.0
Simulational	200009	-20.414	3.0

TABLE IV: Results of the butterworth filter design, comparing actual design specs with simulational and theoretical results.

The entire simulational plot results are shifted up about 6.6dB. This accounts for the difference in magnitude of the frequency response at 10krad/sec. This could possibly be due to phase shift.

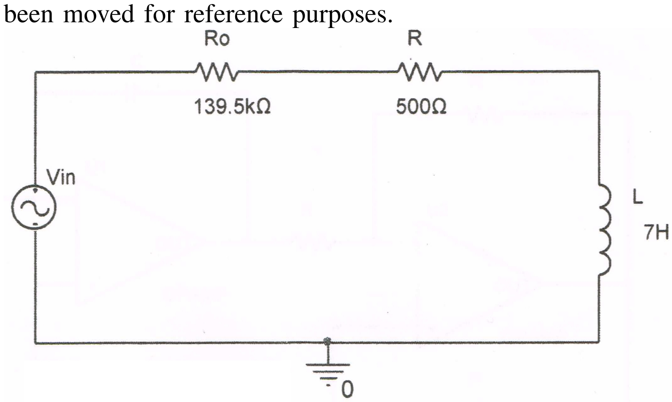
VIII. Conclusion

Despite the different class types of the outlined filters, their exist fundamental similarities between them. Each of these filters have a cutoff frequency and a passband gain. This means that each filter can be designed to allow only certain frequencies through and to attenuate other frequencies.

The fundamental differences between these filters is that the RL filter and the active filter are low-pass filters, while the butterworth filter is a high-pass filter. Furthermore, the RL filter is a 1^{st} order filter, the Active filter is a 2^{nd} order filter, and the butterworth is a 5^{th} order filter. The order, as seen throughout this report, dictates the steepness of the cutoff, the maximum passband gain, and the flatness of the passband gain. The higher the order, generally the better the filter, at the cost of additional hardware and power consumption.

Appendix A
RL Series Filter Low-Pass Analysis

The circuit shown Below is Fig. 1 from section III. It has been moved for reference purposes.



Below is the generic variable analysis of the RL series, low-pass circuit shown in the above figure:

Let $r = R_o + R$

The transfer function is then defined to be:

$$H(S) = \frac{V_o(S)}{V_{in}(S)} = \frac{R_o}{r + SL}$$

With its magnitude:

$$|H(j\omega)| = |\frac{R_o}{r + jL\omega}| = \frac{R_o}{\sqrt{r^2 + (L\omega)^2}}$$

Then consider the response at the limits of the system:

$$\omega \to \infty, |H(j\omega)| \to 0$$

$$\omega \to 0, |H(j\omega)| \to \frac{R_o}{r}$$

The maximum gain is then:

$$H_{max} = \frac{R_o}{r}$$

Finding the Cut-off Frequency:

$$\frac{H_{max}}{\sqrt{2}} = \frac{R_o}{\sqrt{r^2 + (\omega_c L)^2}} \Rightarrow H_{max}\sqrt{r^2 + (\omega_c L)^2} = \sqrt{2}R_o$$

$$\omega_c = \frac{\sqrt{(2R_o^2 - H_{max}^2 r^2)}}{H_{max}L}$$

Plugging the given values in:

$$r = R_o + R = 139.5k\Omega + 500\Omega = 140k\Omega$$

Maximum Gain:

$$H_{max} = \frac{R_o}{r} = \frac{139.5k\Omega}{140k\Omega} = .9964$$

Cutoff Frequency:

$$\omega_c = \frac{\sqrt{(2R_o^2 - H_{max}^2 r^2)}}{H_{max}L} = \frac{\sqrt{(2 * 139.5k\Omega^2 - .9964^2 * 140k\Omega^2)}}{.9964 * 7}$$

$$\omega_c = 19.858krad/sec$$

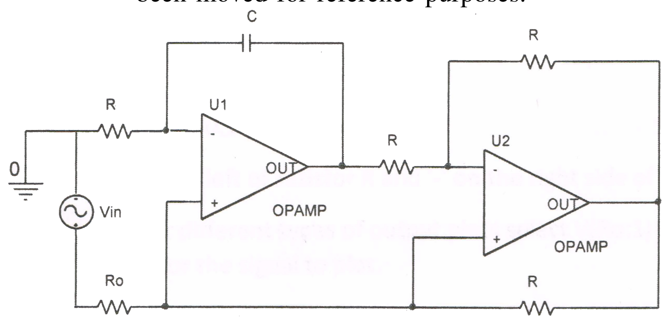
$$f_c = \frac{\omega_c}{2\pi} = \frac{19857.5}{2\pi} = 3160.4Hz$$

Output Magnitude at a 12 krad/sec Frequency:

$$|H(j\omega)| = \frac{R_o}{\sqrt{r^2 + (L\omega)^2}} = \frac{139.5k\Omega}{\sqrt{140k\Omega^2 + (7 * 12k)^2}} = 0.2555$$

Appendix B
Active Filter Low-Pass Analysis

The circuit shown Below is Fig. 5 from section IV. It has been moved for reference purposes.



Below is the generic variable analysis of the RL series, low-pass circuit shown in the above figure:

By Inspection the Node Voltages are:

$$V_{n1} = V_{n2} = V_{p1} = V_{p2} = V$$

Where V can be found by inspection across R_o:

$$V = V_{in} - V_o$$

Using Nodal Analysis at V_{n1}:

$$\frac{V_{in} - V_o}{R} + \frac{V_{in} - V_o - V_{o1}}{\frac{1}{SC}} = 0$$

$$V_{in} - V_o + RSC(V_{in} - V_o - V_{o1}) = 0 \tag{2}$$

Using Nodal Analysis at V_{n2}:

$$\frac{V_{in} - V_o - V_{o1}}{R} + \frac{V_{in} - V_o - V_{o2}}{R} = 0$$

$$V_{o2} = 2V_{in} - 2V_o - V_{o1} \tag{3}$$

Obtaining a third equation by using Ohms across bottom resistors:

$$\frac{V_o}{R_o} = \frac{V_{in} - V_o - V_{o2}}{R}$$

$$V_{o2} = \frac{-2RV_o + R_oV_{in}}{R_o} \tag{4}$$

Substituting equation 4 into equation 3:

$$2V_{in} - 2V_o - V_{o1} = \frac{-2RV_o + R_oV}{R_o}$$

$$V_{o1} = \frac{R_oV_{in} - 2R_oV_o + 2RV_o}{R_o} \tag{5}$$

Substituting equation 5 into equation 2:

$$V_{in} - V_o + RSC\left[(V_{in} - V_o) - \frac{R_oV_{in} - 2R_oV_o + 2RV_o}{R_o}\right] = 0$$

$$R_o * V_{in} = V_o * \left(2R^2SC - RR_oSC + R_o\right)$$

It follows that the transfer is:

$$H(S) = \frac{V_o(S)}{V_{in}(S)} = \frac{R_o}{2R^2SC - RR_oSC + R_o}$$

$$H(S) = \frac{R_o}{R_o + SC(2R^2 - RR_o}$$

Let $r = 2R^2 - RR_o$

$$H(S) = \frac{R_o}{R_o + rSC}$$

The magnitude can easily be calculated now:

$$|H(j\omega)| = |\frac{R_o}{R_o + j\omega rC}|$$

$$|H(j\omega)| = \frac{R_o}{\sqrt{R_o^2 + (rC\omega)^2}}$$

Then consider the response at the limits of the system:

$$\omega \to \infty, |H(j\omega)| \to 0$$

$$\omega \to 0, |H(j\omega)| \to 1$$

The maximum gain is then:

$$H_{max} = 1$$

Finding the Cut-off Frequency:

$$\frac{H_{max}}{\sqrt{2}} = \frac{R_o}{\sqrt{R_o^2 + (rC\omega)^2}}$$

$$\omega_c = \frac{R_o}{rC}$$

Plugging the given values in for R,R_o, and C:

$$r = 2R^2 - RR_o = 2 * (1k\Omega)^2 - (1k\Omega)^2 = 1M\Omega^2$$

Cutoff Frequency:

$$\omega_c = \frac{R_o}{rC} = \frac{1k\Omega}{1M\Omega * 50nF}$$

$$\omega_c = 20.0krad/sec$$

$$f_c = \frac{\omega_c}{2\pi} = \frac{20000}{2\pi} = 3183.1Hz$$

Output Magnitude at a 12 krad/sec Frequency:

$$|H(j\omega)| = \frac{R_o}{\sqrt{R_o^2 + (rC\omega)^2}} = 0.2564$$

$$V_{o1} = \frac{R_oV_{in} - 2R_oV_o + 2RV_o}{R_o} \tag{6}$$

Appendix C
Butterworth High-Pass Filter Design

Given the conditions stated in Section V, the following can be deduced:

$$A_p = -3dB, \omega_p = 20krad/sec$$

$$A_s = -27dB, \omega_p = 10krad/sec$$

The order of the filter is then given by equation 7

$$n = \frac{\log\left(\frac{\sigma_s}{\sigma_p}\right)}{\log\left(\frac{\omega_s}{\omega_p}\right)} \tag{7}$$

Where $\sigma_x = [10^{-0.1A_x} - 1]^{.5}$
It follows that $\sigma_p = 0.9976$ and $\sigma_s = 22.3649$.
Furthermore, the order is calculated to be $n = -4.487$.
Rounding up to achieve the correct filter parameters, $n = 5$ is the butterworth order needed for the design.
Given standardized polynomials of butterworth filters along with the generic prototype filter stages, shown in Fig. 21, Fig. 22, and table V below, H(S) can be easily determined as follows:

n	Factors of Polynomials
1	$S + 1$
2	$S^2 + 1.4142S + 1$
3	$(S + 1)(S^2 + S + 1)$
4	$(S^2 + 0.7654S + 1)(S^2 + 1.18478S + 1)$
5	$(S + 1)(S^2 + 0.618S + 1)(S^2 + 1.618S + 1)$

TABLE V: Butterworth filter of degree n Polynomials

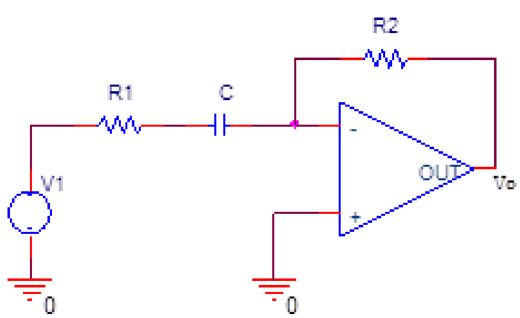


Fig. 21: 1^{st} Order, High-Pass Filter Prototype

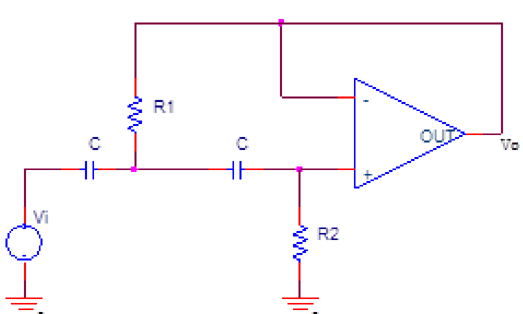
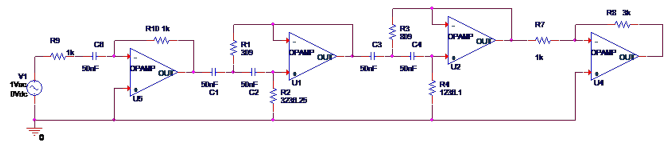


Fig. 22: 2^{nd} Order, High-Pass Filter Prototype

It follows that the prototype transfer function is:

$$H(S) = \frac{S^5}{(S+1)(S^2 + b_{11}S + 1)(S^2 + b_{12}S + 1)}$$

Where the design of the required butterworth filter can be made with one first order high-pass prototype filter in series with two second order high-pass prototype filters, as shown in the circuit below.



The corresponding prototype values for all stages are:

$$C_p = 1F, \omega_{cp} = 1rad/sec$$

The prototype values for 1^{st} Stage (1^{st} Order) are:

$$R_{p1} = R_{p2} = 1\Omega$$

The prototype values for 2^{nd} Stage (2^{nd} Order) are:

$$b_{11} = 0.618 = \frac{2}{R_{p2}} \Rightarrow R_{p2} = 3.236\Omega$$

$$1 = \frac{1}{R_{p1}R_{p2}} \Rightarrow R_{p1} = 0.309\Omega$$

The prototype values for 3^{rd} Stage (2^{nd} Order) are:

$$b_{12} = 1.618 = \frac{2}{R_{p2}} \Rightarrow R_{p2} = 1.721\Omega$$

$$1 = \frac{1}{R_{p1}R_{p2}} \Rightarrow R_{p1} = 0.5809\Omega$$

The scaling factors can then be determined and used to shift the response as desired using the following:

$$\omega_{cd} = \omega_{cd} * k_m$$

$$R_d = R_{px} * k_m$$

$$C_d = \frac{C_p}{k_f k_m}$$

To ensure the magnitude and scale of the resistors is approximately correct, table VI was formed:

				1st Order	2nd Order (1)		2nd Order (2)	
		C_D	k_m	R1=R2	R1	R2	R1	R2
Series	Four	400.00	125.00	125.00	38.63	404.53	101.13	154.51
	Three	300.00	166.67	166.67	51.50	539.37	134.83	206.02
	Two	200.00	250.00	250.00	77.25	809.06	202.25	309.02
	One	100.00	500.00	500.00	154.50	1618.12	404.50	618.05
Parallel	Two	50.00	1000.00	1000.00	309.00	3236.25	809.00	1236.09
	Three	33.30	1501.50	1501.50	463.96	4859.23	1214.71	1856.00
	Four	25.00	2000.00	2000.00	618.00	6472.49	1618.00	2472.19

TABLE VI

With the given design constraints, the capacitor used dictated the magnitudes of the resistors. With only one 0.1μF capacitor, the resulting 154.5Ω resistor is a bit small, so two capacitors in parallel were used in the design to increase this resistor's magnitude.

The scaled transfer function is then given by:

$$H(S) = \frac{\left(\frac{S}{20k}\right)^5}{\left[\frac{S}{20k}+1\right]\left[\left(\frac{S}{20k}\right)^2+\frac{0.618S}{20k}+1\right]\left[\left(\frac{S}{20k}\right)^2+\frac{1.618S}{20k}+1\right]}$$

Where $20k^5$ can be distributed through the denominator, resulting in:

$$H(S) = \frac{S^5}{[S+20k]\,[S^2+12360S+(20k)^2]\,[S^2+32360S+(20k)^2]}$$

The actual gain of the device can then be calculated by substituting jω in for S and taking the magnitude of the new expression. Alternatively, jω can be substituted in for S and then the magnitude can be found in decibels. For example, at a frequency of 10krad/sec:

$$H(j\omega) = \frac{(j10k)^5}{[j10k+20k]\,[(300M+12360(j10k)]\,[300M+32360(j10k)]}$$

$$H(j\omega) = .031057 - j.00333$$

With a gain of 3:

$$|H(j\omega)| = 3 * 0.031235 = .0937V$$

In decibels:

$$-10\log|H(j\omega)| = -30.107dB$$

References

[1] N. Riedel, Active Filter Circuits in ,Electric Circuits 9th ed. New Jersey

[2] "Butterworth Filter Design and Low Pass Butterworth Filters." Basic Electronics Tutorials. N.p., 14 Aug. 2013. Web. 01 May 2015.

[3] "How to Write an Abstract." How to Write an Abstract. N.p., n.d. Web. 01 May 2015.