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The Geometric Representation of Signal Waveforms

The Problem

Digital communication systems generally implement a sequence of binary bits of information as the modulator input. The modulator may then transmit this information k bits at a time by employing $M = 2^k$ distinct signal waveforms. The problem exists in the complexity of straightforward analysis without some form of simplification. Using a vector representation of these signal waveforms, communication channels can be represented by vector channels, often significantly reducing the analytical complexity.

Context

Suppose we have a set of M signal waveforms, denoted $s_m(t)$, that are to be sent over a communication channel. To implement the vector representation of these waveforms, a set of $N \leq M$ orthonormal waveforms, called basis functions, must first be constructed. The dimension of the signal space (N) is defined by the number of basis functions required to represent all signal waveforms. Due to the nature of these functions, N does not necessarily need to be the same as the number of signal waveforms (M) and certainly cannot be greater than M.

It should be apparent that all signal waveforms can be represented as a linear combination of these basis functions. If a signal waveform can be represented by previously constructed basis functions, then a new basis function is not needed. This is how the dimension of the signal space can sometimes be less than the number of signal waveforms. There are two main methods used to determine these basis functions; the Gram-Schmidt Orthogonalization Procedure and a more intuitive method based on inspection.

The Solution: Gram-Schmidt Procedure

A signal waveform is chosen and normalized, forming the first basis function $\psi_1(t)$. To normalize the signal, refer to (4), but replace $\mathrm{d}_k(t)$ with the chosen signal waveform. Note that it does not matter which signal waveform is chosen first, since this function forms the first basis function. Each of the remaining basis functions, found by examining one signal at a time, are then calculated by first determining the signal waveform's depende nce on previously calculated basis functions. This dependence is shown below for the k^{th} signal waveform:

$$C_{ki} = \int_{-\infty}^{\infty} \mathrm{s}_k(t)\psi_i(t)dt\,, \qquad i = 1,2,\ldots,k-1 \tag{1}$$

The waveform's dependence upon other basis functions is removed:

$$\mathrm{d}_k(t) = \mathrm{s}_k(t) - \sum_{i=1}^{k-1} C_{ki}\,\psi_i(t) \tag{2}$$

It should be noted that whenever $\mathrm{d}_k(t) = 0\ \forall\ t$, the signal $\mathrm{s}_k(t)$ can already be represented by a linear combination of the previously determined basis functions in terms of C_{ki}. The energy content of this signal is then determined for use in the normalization process:

$$\mathcal{E}_k = \int_{-\infty}^{\infty} \mathrm{d}_k^2(t)dt \tag{3}$$

Finally, the k^{th} basis function is the found by normalizing $\mathrm{d}_k(t)$:

$$\psi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}} \tag{4}$$

Once this process has been completed for every signal waveform, each signal can then be written as an exact linear combination of the basis functions:

$$\mathrm{s}_m(t) = \sum_{i=1}^{N} s_{mn}\,\psi_n(t), \qquad m = 1,2,\ldots,M \tag{5}$$

Where the coefficients s_{mn} are the same as the C_{ki} terms found for that signal waveform. Note the last coefficient s_{nn} must still be determined using (1) but with the k^{th} signal waveform. From there, every signal waveform can be represented in vector form:

$$\mathrm{s}_m = (s_{m1}, s_{m2}, \dots, s_{mN}) \tag{6}$$

An important outcome of the process is that the energy content of the m^{th} signal waveform is simply the square of the length of the vector:

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^2(t)dt = \sum_{n=1}^{N} s_{mn}^2 = \|s_m\|^2 \tag{7}$$

Another aspect that arises from this process is the simplicity of calculating distances between signals. This is especially important in communication systems because it will illustrate which signals contribute the most amount of error in a system. The distance between signals s_m and s_n can be determined as follows:

$$d_{mn} = \sqrt{(s_{m1} - s_{n1})^2 + (s_{m2} - s_{n2})^2 + \cdots + (s_{mN} - s_{nN})^2} \tag{8}$$

To clarify and illustrate this process, consider the signal waveforms of problem 8.4 in Proakis and Salehi [1] as shown in the figure below:

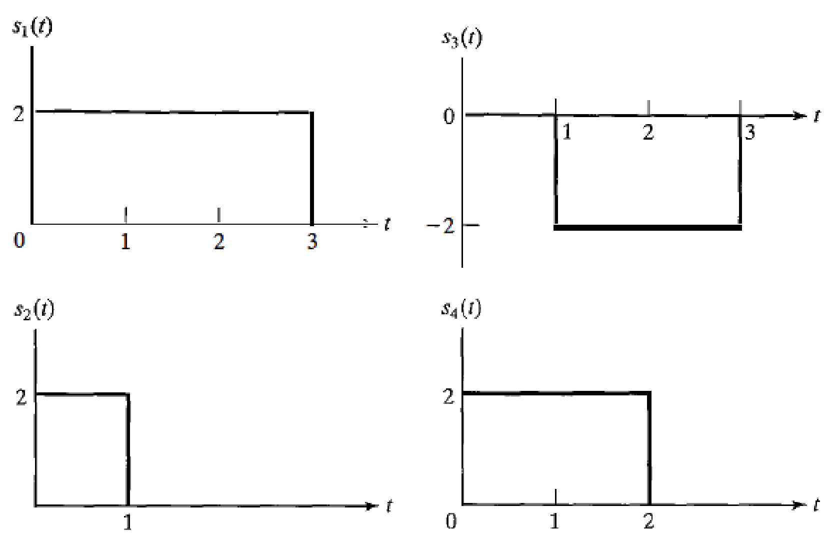
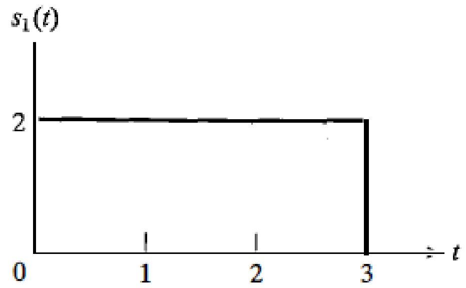


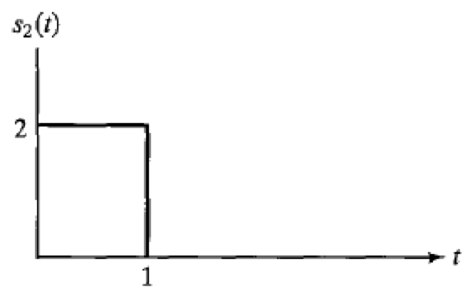
Figure 1: Signal waveforms to be orthonormalized

Applying the Gram-Schmidt process to each signal waveform in order:



$$\mathcal{E}_1 = \int_{-\infty}^{\infty} s_1^2(t)dt = \int_0^3 2^2 dt = 12 \neq 1$$

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \frac{2}{\sqrt{12}} = \frac{\sqrt{3}}{3}, \qquad 0 \leq t \leq 3$$

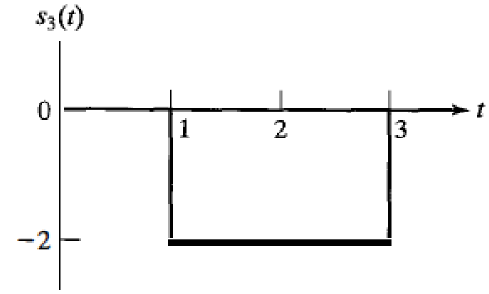


$$C_{21} = \int_{-\infty}^{\infty} s_2(t)\psi_1(t)dt = \int_0^1 2 * \frac{\sqrt{3}}{3} dt = \frac{2\sqrt{3}}{3}$$

$$\mathrm{d}_2(t) = \mathrm{s}_2(t) - C_{21}\psi_1(t) = \begin{cases} \frac{4}{3}, & 0 \leq t \leq 1 \\ -\frac{2}{3}, & 1 \leq t \leq 3 \end{cases}$$

$$\mathcal{E}_2 = \int_{-\infty}^{\infty} d_2^2(t)dt = \int_0^1 \left(\frac{4}{3}\right)^2 dt + \int_1^3 \left(\frac{-2}{3}\right)^2 dt = \frac{8}{3} \neq 1$$

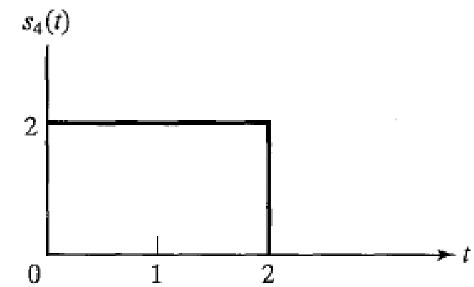
$$\psi_2(t) = \frac{d_2(t)}{\sqrt{\mathcal{E}_2}} = \begin{cases} \frac{\sqrt{6}}{3}, & 0 \leq t \leq 1 \\ -\frac{\sqrt{6}}{6}, & 1 \leq t \leq 3 \end{cases}$$



$$C_{31} = \int_{-\infty}^{\infty} s_3(t)\psi_1(t)dt = \int_1^3 -2 * \frac{\sqrt{3}}{3} dt = \frac{-4\sqrt{3}}{3}$$

$$C_{32} = \int_{-\infty}^{\infty} s_3(t)\psi_2(t)dt = \int_1^3 -2 * \frac{\sqrt{6}}{6} dt = \frac{2\sqrt{6}}{3}$$

$$\mathrm{d}_3(t) = \mathrm{s}_3(t) - C_{31}\psi_1(t) - C_{32}\psi_2(t) = 0$$



$$C_{31} = \int_{-\infty}^{\infty} s_4(t)\psi_1(t)dt = \int_0^2 2 * \frac{\sqrt{3}}{3} dt = \frac{4\sqrt{3}}{3}$$

$$C_{32} = \int_{-\infty}^{\infty} s_4(t)\psi_2(t)dt = \int_0^1 2 * \frac{\sqrt{6}}{3} dt + \int_1^2 2 * \frac{-\sqrt{6}}{6} dt = \frac{2\sqrt{6}}{6}$$

$$\mathrm{d}_3(t) = \mathrm{s}_4(t) - C_{31}\psi_1(t) - C_{32}\psi_2(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ 1, & 1 \leq t \leq 2 \\ -1, & 2 \leq t \leq 3 \end{cases}$$

$$\mathcal{E}_3 = \int_{-\infty}^{\infty} d_3^2(t)dt = \int_1^2 1^2 dt + \int_1^2 (-1)^2 dt = 2 \neq 1$$

$$\psi_3(t) = \frac{d_3(t)}{\sqrt{\mathcal{E}_3}} = \begin{cases} 0, & 0 \leq t \leq 1 \\ \frac{\sqrt{2}}{2}, & 1 \leq t \leq 2 \\ \frac{-\sqrt{2}}{2}, & 2 \leq t \leq 3 \end{cases}$$

From which these signals can be represented in vector form, their lengths and energy contents found, and the distances between signals considered:

Signal Representation

$$s_1(t) = 2\sqrt{3}\psi_1(t) = (\mathbf{2\sqrt{3}, 0, 0})$$

$$s_2(t) = \frac{2\sqrt{3}}{3}\psi_1(t) + \frac{2\sqrt{6}}{3}\psi_2(t) = \left(\frac{\mathbf{2\sqrt{3}}}{\mathbf{3}}, \frac{\mathbf{2\sqrt{6}}}{\mathbf{3}}, \mathbf{0}\right)$$

$$s_3(t) = \frac{-4\sqrt{3}}{3}\psi_1(t) + \frac{2\sqrt{6}}{3}\psi_2(t) = \left(\frac{\mathbf{-4\sqrt{3}}}{\mathbf{3}}, \frac{\mathbf{2\sqrt{6}}}{\mathbf{3}}, \mathbf{0}\right)$$

$$s_4(t) = \frac{4\sqrt{3}}{3}\psi_1(t) + \frac{2\sqrt{6}}{6}\psi_2(t) + \sqrt{2}\psi_3(t) = \left(\frac{\mathbf{4\sqrt{3}}}{\mathbf{3}}, \frac{\mathbf{\sqrt{6}}}{\mathbf{3}}, \sqrt{\mathbf{2}}\right)$$

Signal Lengths

$$\|s_1\| = \sqrt{(\mathbf{2\sqrt{3}})^2 + 0^2 + 0^2} = \mathbf{2\sqrt{3}}$$

$$\|s_2\| = \sqrt{\left(\frac{2\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2 + 0^2} = \mathbf{2}$$

$$\|s_3\| = \sqrt{\left(\frac{-4\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2 + 0^2} = \mathbf{2\sqrt{2}}$$

$$\|s_4\| = \sqrt{\left(\frac{4\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 + \sqrt{2}^2} = \mathbf{2\sqrt{2}}$$

Signal Energy Content

$$\mathcal{E}_1 = \|s_1\|^2 = \mathbf{12} \qquad \mathcal{E}_2 = \|s_2\|^2 = \mathbf{4} \qquad \mathcal{E}_3 = \|s_3\|^2 = \mathbf{8} \qquad \mathcal{E}_4 = \|s_4\|^2 = \mathbf{8}$$

Distances Between Signals

$$d_{12} = \sqrt{\left(2\sqrt{3} - \frac{2\sqrt{3}}{3}\right)^2 + \left(0 - \frac{2\sqrt{6}}{3}\right)^2 + 0^2} \cong \mathbf{2.83}$$

$$d_{13} = \sqrt{\left(2\sqrt{3} + \frac{4\sqrt{3}}{3}\right)^2 + \left(0 - \frac{2\sqrt{6}}{3}\right)^2 + 0^2} \cong \mathbf{1.41}$$

$$d_{14} = \sqrt{\left(2\sqrt{3} - \frac{4\sqrt{3}}{3}\right)^2 + \left(0 - \frac{\sqrt{6}}{3}\right)^2 + \sqrt{2}^2} = \mathbf{1.41}$$

$$d_{23} = \sqrt{\left(\frac{2\sqrt{3}}{3} + \frac{4\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3} - \frac{2\sqrt{6}}{3}\right)^2 + 0^2} \cong \mathbf{3.46}$$

$$d_{24} = \sqrt{\left(\frac{2\sqrt{3}}{3} - \frac{4\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3} - \frac{\sqrt{6}}{3}\right)^2 + 2} \cong \mathbf{1.41}$$

$$d_{34} = \sqrt{\left(-\frac{4\sqrt{3}}{3} - \frac{4\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3} - \frac{\sqrt{6}}{3}\right)^2 + 2} \cong \mathbf{4.69}$$

Outcome

The vector representation of any set of signal waveforms is not unique since the selection of the basis waveforms is not unique. In fact, if the above example is considered with corresponding basis functions shown in figure 2 below, the four signal waveforms can clearly be represented by linear combinations of these basis functions.

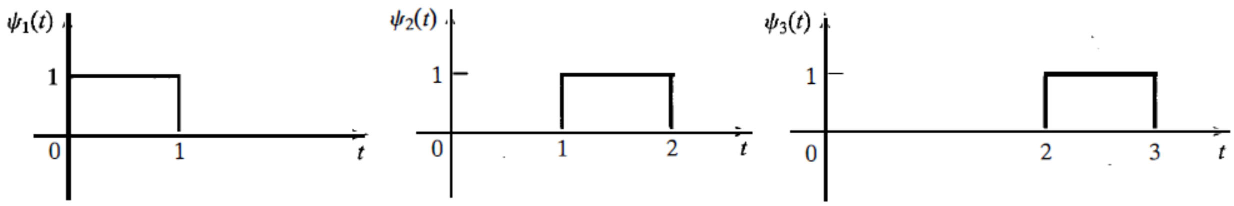


Figure 2: Intuitive approach basis functions for example problem

This is the other method of orthonormalizing a set of signal waveforms. Simply by inspection, a set of orthonormal functions can often be created to represent all signal waveforms in a set. Note that the signal lengths, energy contents, the dimension of the signal space, and the distances between signals does not change; it is only the way in which the signals are represented that changes. Changing basis functions essentially just rotates the signal vectors about the origin of the signal space. In the end, the method of generating basis functions based on inspection is almost always the easier method to implement, but the Gram-Schmidt Process will guarantee an outcome.

References

[1] J. G. Proakis and M. Salehi, *Fundamentals of Communication Systems*, 2nd ed. Upper Saddle River, New Jersey: Pearson Prentice Hall, 2014.