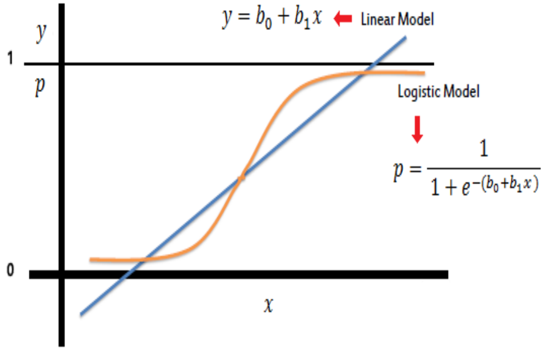
Linear Regression &

Logistic Regression



Fast Campus
Start Deep Learning with TensorFlow

Let’s Go to the Deep Learning World!!

쉬운 것부터 시작해봅시다

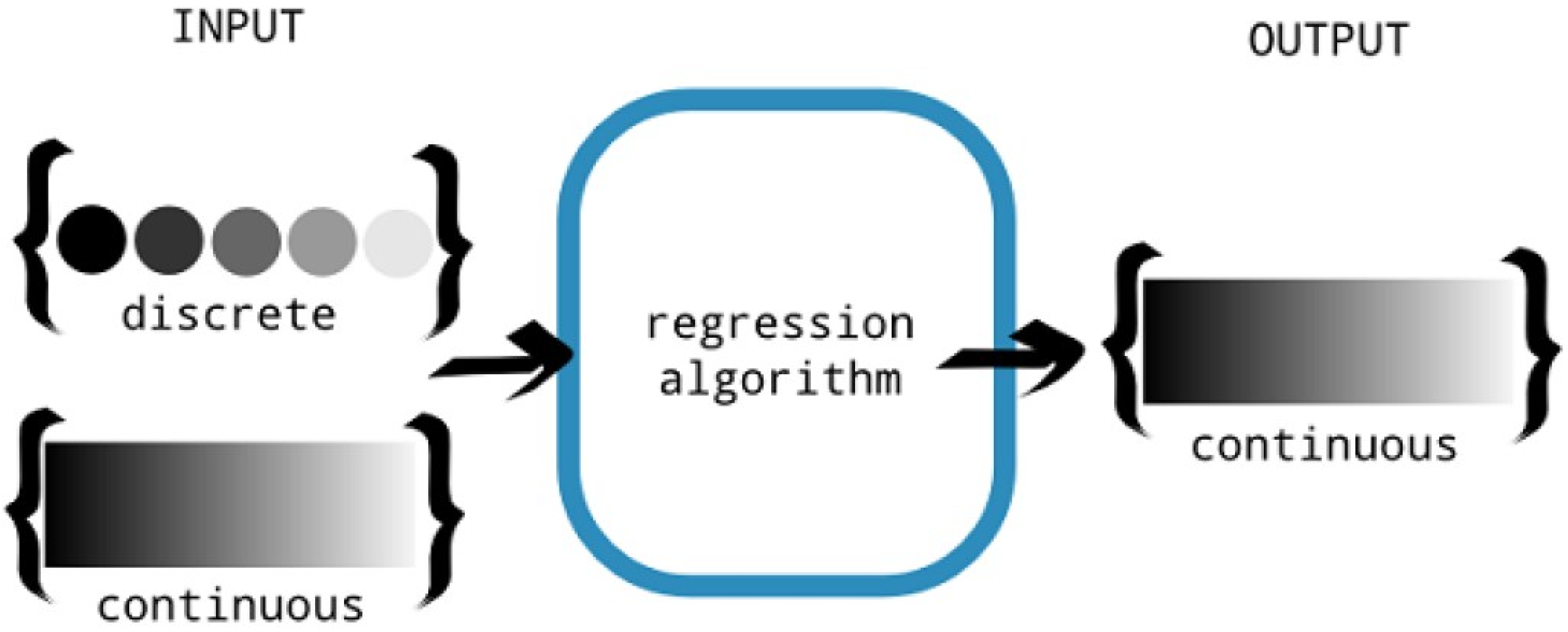
초등학교 6학년 수학 – 정비례

2배 3배 4배

닭 수 (마리)	1	2	3	4	5	……
다리 수 (개)	2	4	6	8	10	……

2배 3배 4배

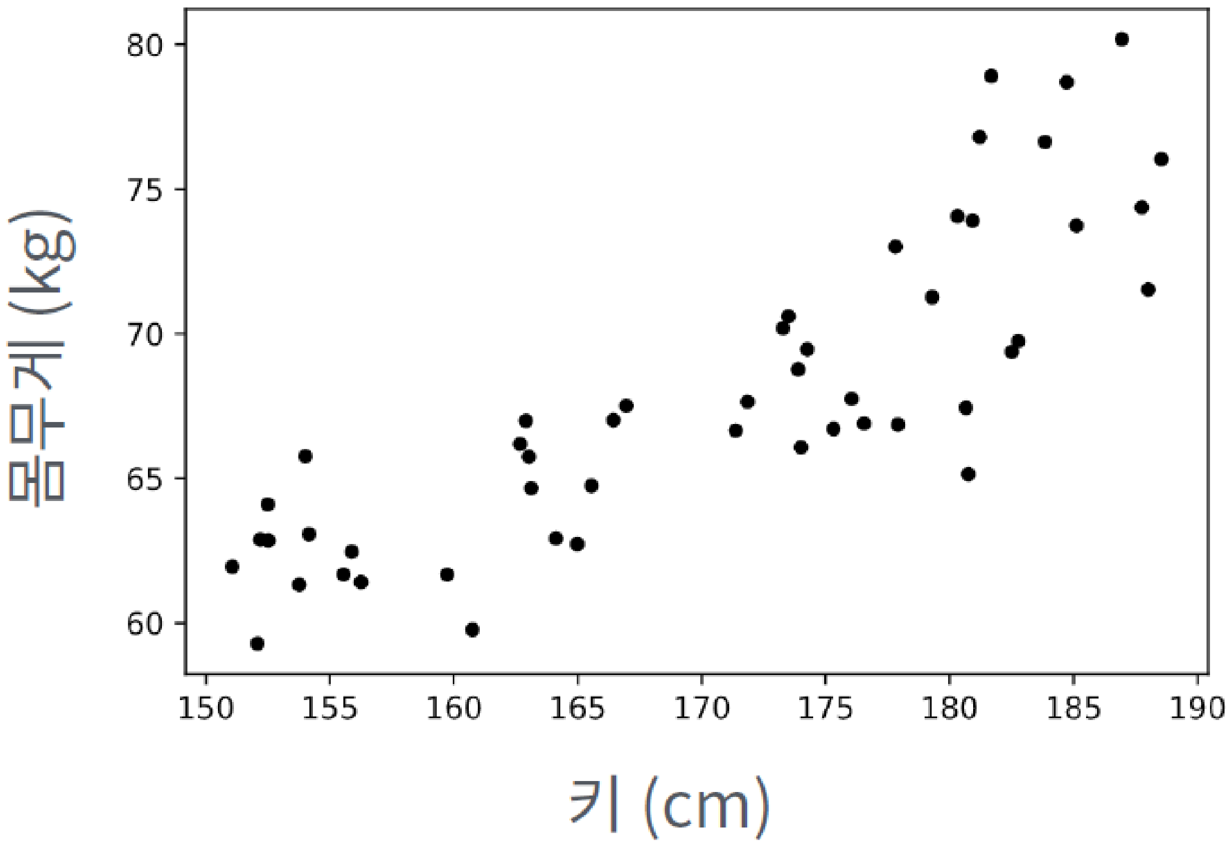
Regression?



INPUT
OUTPUT
discrete
continuous
regression
algorithm
continuous

Linear Regression

- 어느 학교 학생들의 신체검사 자료
- 새로 전학온 학생 A의 키가 175cm일 때 예상 몸무게는?



Linear Regression

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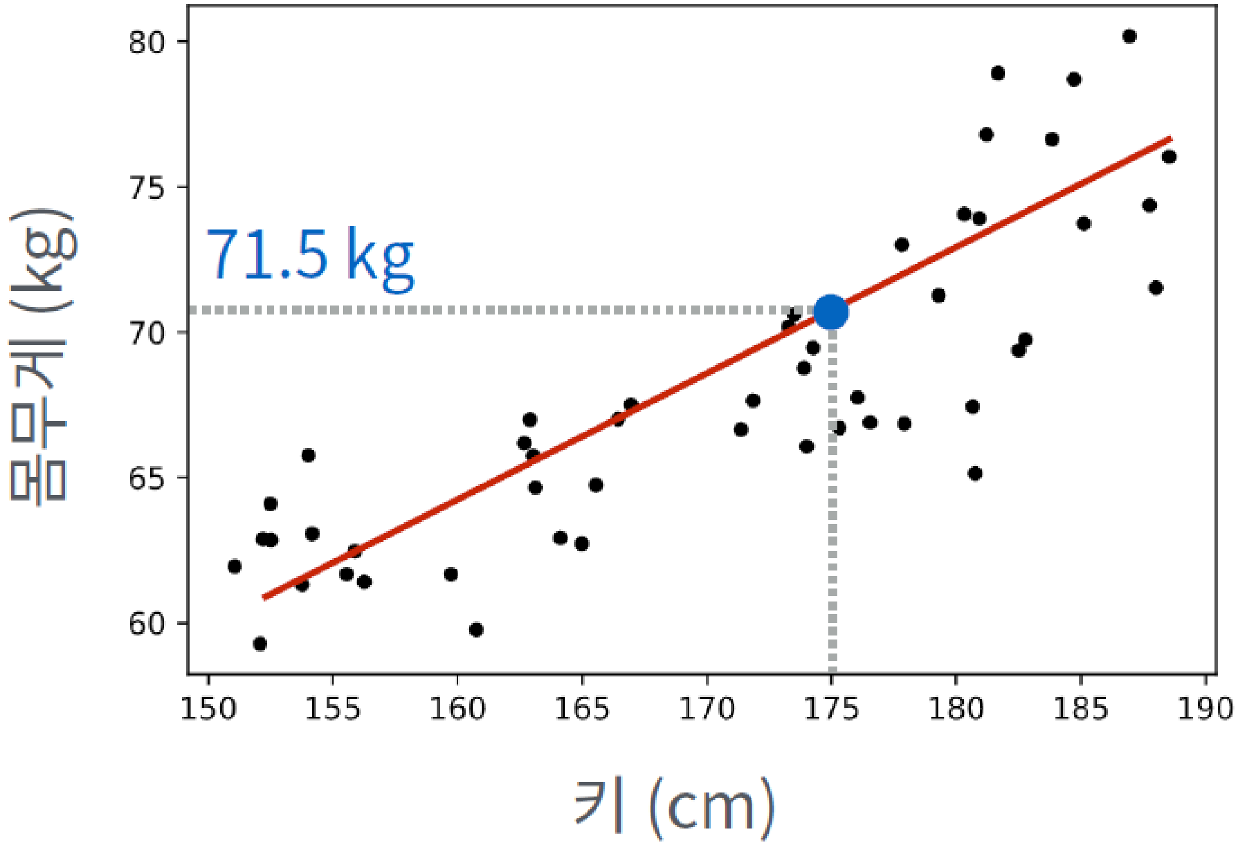
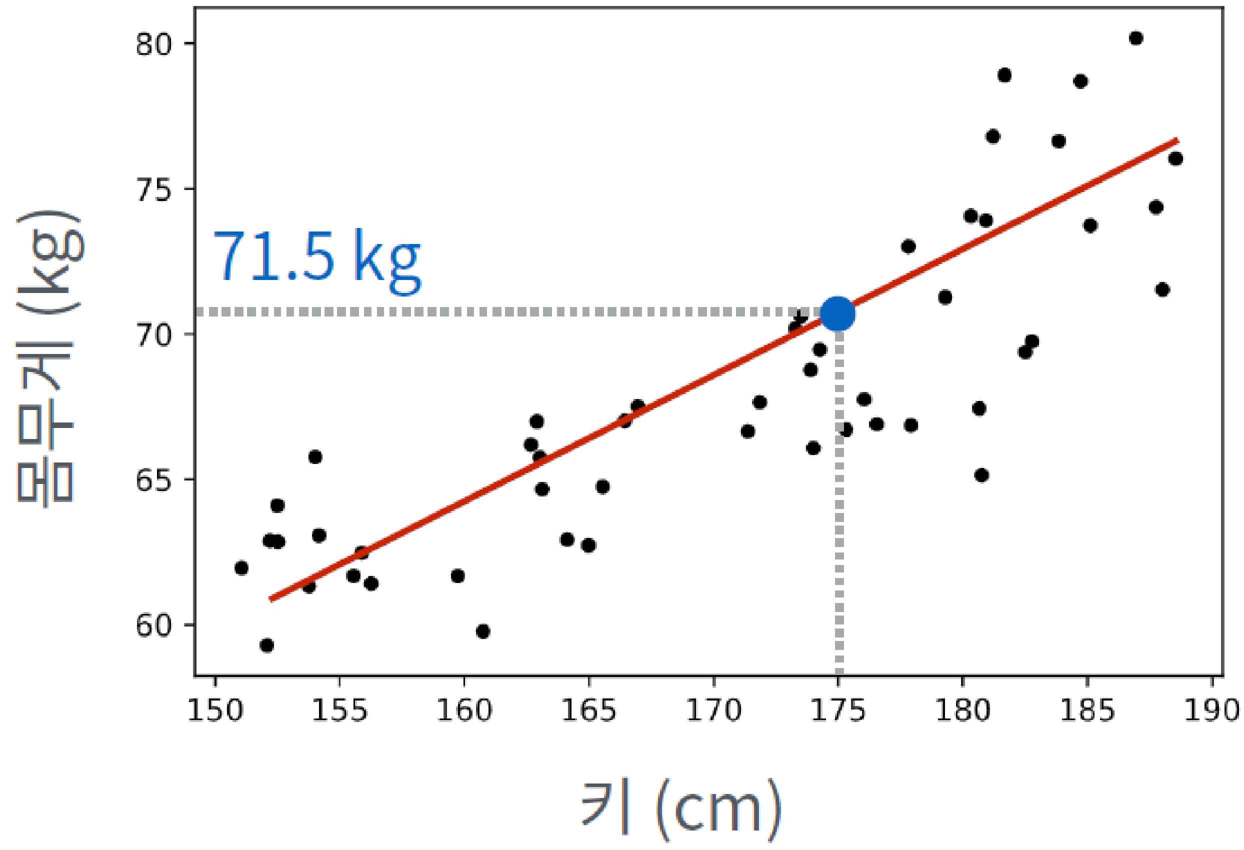
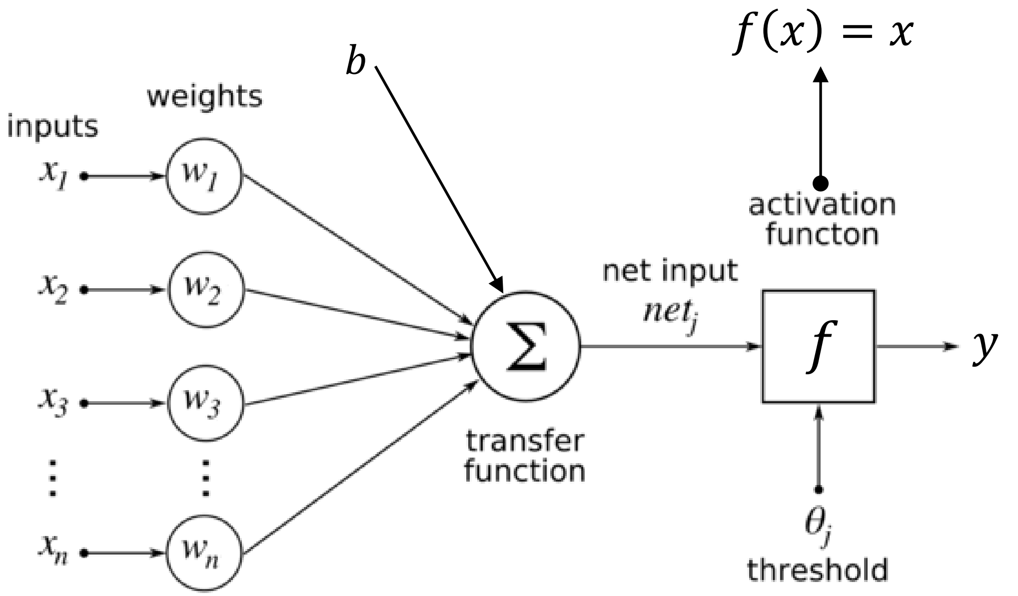


Linear Regression

- 선형함수(예 : 1차함수)로 주어진 data를 근사한다
- $y = wx + b$





<Perceptron>

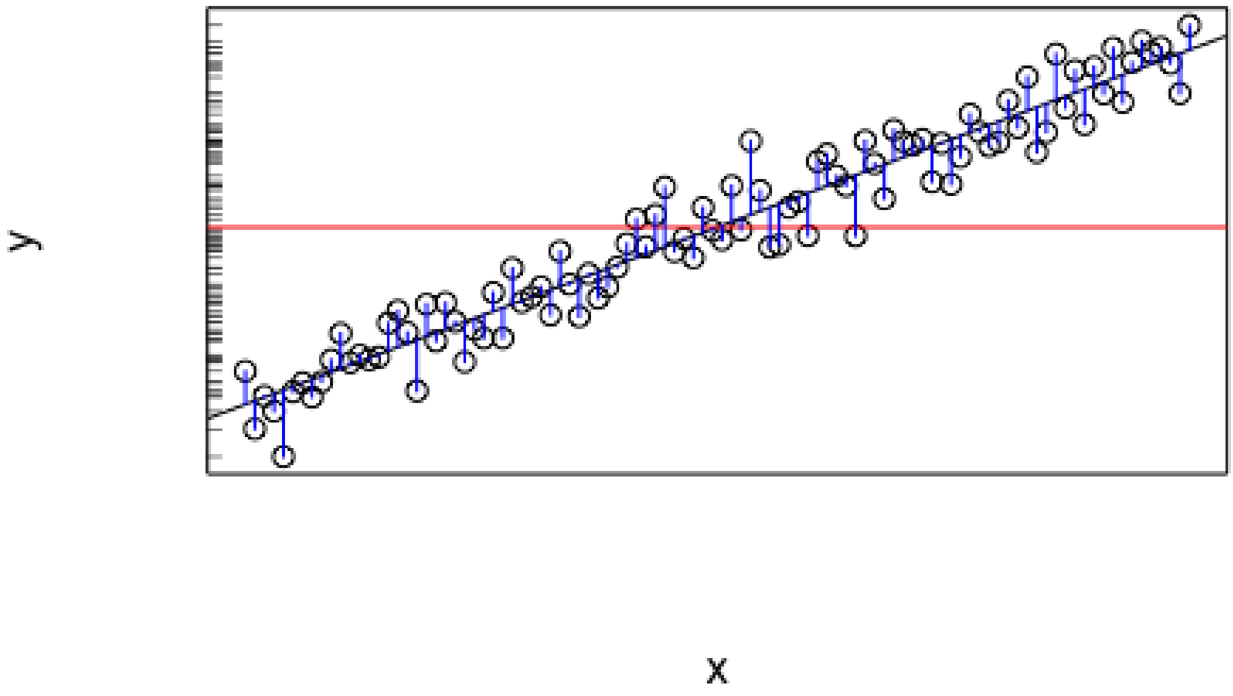
$$y = f(\mathbf{wx} + \mathrm{b})$$

$$\boldsymbol{w} = [w_1\ w_2\ w_3\ \ldots\ w_n]$$

$$\boldsymbol{x} = [x_1\ x_2\ x_3\ \ldots\ x_n]^T$$

Linear Regression

- 잘 예측했는지 측정할 척도(metric)가 필요함



$$y^* = wx + b \text{ (예측값)}$$

$$Cost(Loss) = \sum_i (y_i - y_i^*)^2$$

$$= \sum_i (y_i - wx_i - b)^2$$

Linear Regression

- Cost(Loss) 값을 minimize하는 w와 b를 구하면 될텐데.... 어떻게?
  - Random Search – 가능????
  - Cost function을 미분해서 최솟값(미분=0이되는 점)을 찾자!

약간의 수학을(미분을...) 조금 해야겠습니다

b 구하기

$$L = \sum_i (y_i - wx_i - b)^2$$

$$\frac{\delta L}{\delta b} = \frac{\delta \sum_i (y_i - wx_i - b)^2}{\delta b}$$

$$= -2\sum_i (y_i - wx_i - b) = ny_{avg} - nwx_{avg} - nb = 0$$

$$\therefore \boldsymbol{b = y_{avg} - wx_{avg}}$$

w 구하기

$$L = \sum_i (y_i - wx_i - b)^2$$

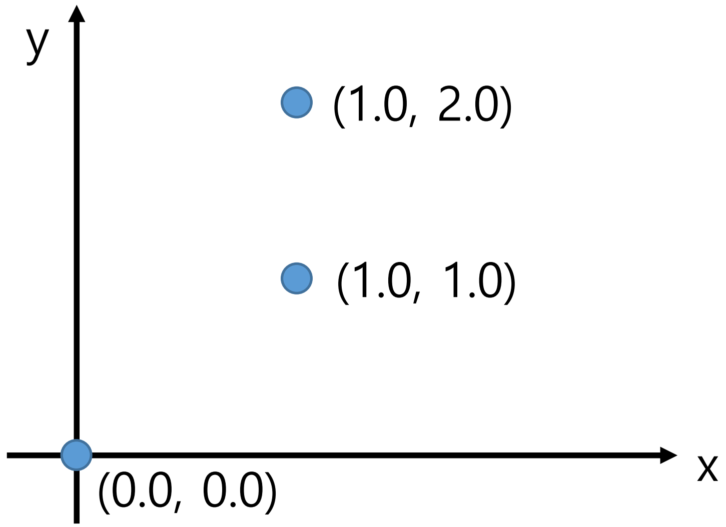
$$\frac{\delta L}{\delta w} = \frac{\delta \sum_i (y_i - wx_i - b)^2}{\delta w}$$

$$= -2\sum_i x_i(y_i - wx_i - b) = -2\sum_i x_i\left(y_i - wx_i - y_{avg} + wx_{avg}\right)$$

$$= 0$$

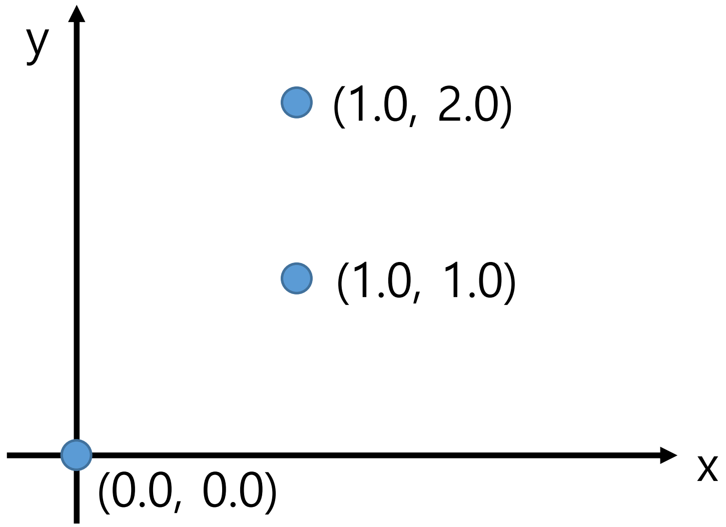
Example

- Find the linear function(f) that best describes the given data
  - $H(x, w_0, w_1) = w_1 x + w_0$



Example

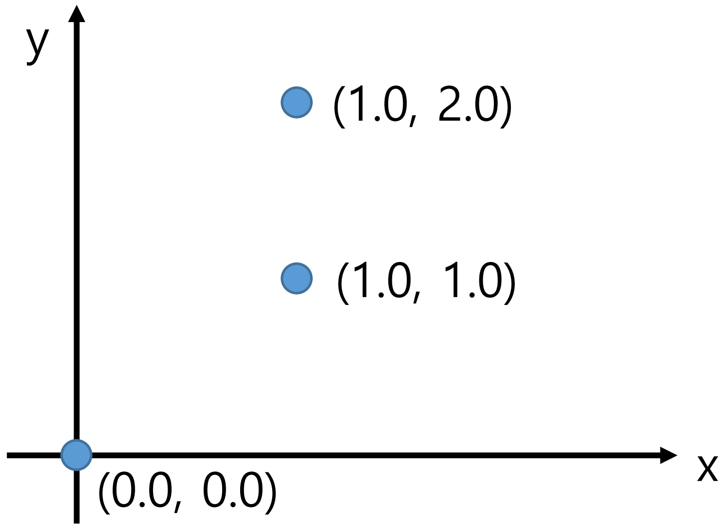
- $H(0, w_0, w_1) \approx 0.0$
- $H(1, w_0, w_1) \approx 1.0$
- $H(1, w_0, w_1) \approx 1.0$



Example

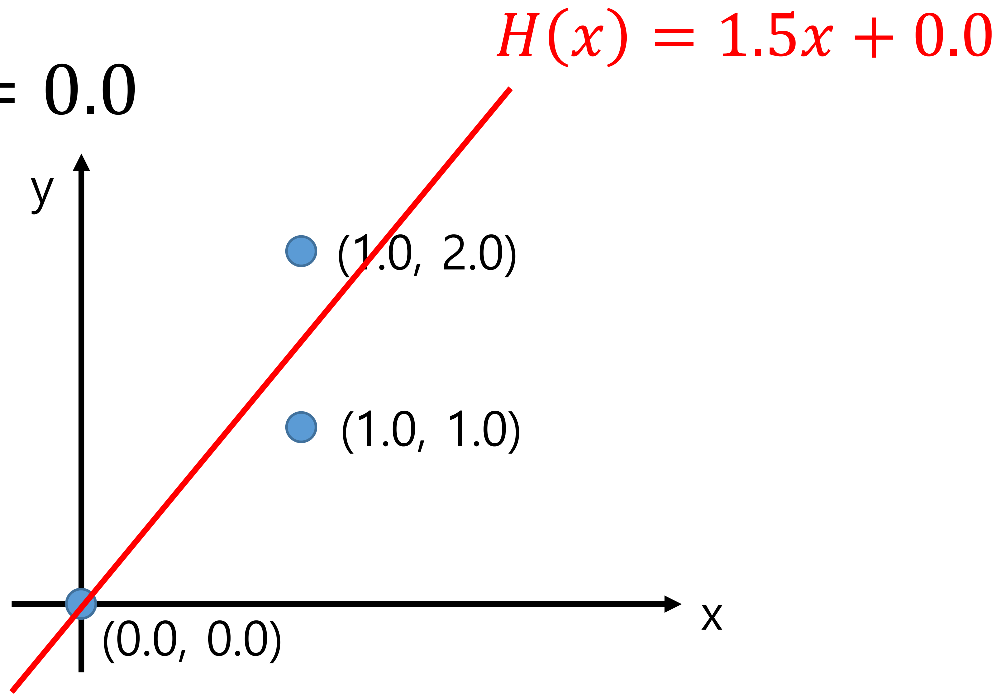
- $L = \sum_i (y_i - w_1 x_i - w_0)^2$

$$= (0.0 - w_1 \cdot 0.0 - w_0)^2 + (1.0 - w_1 \cdot 1.0 - w_0)^2 + (2.0 - w_1 \cdot 1.0 - w_0)^2$$
$$= 2{w_1}^2 + 3{w_0}^2 - 6w_1 - 6w_0 + 4w_1 w_0 + 5$$



Example

- $\frac{\partial L}{\partial w_1} = 4w_1 + 4w_0 - 6 = 0$
- $\frac{\partial L}{\partial w_0} = 4w_1 + 6w_0 - 6 = 0$
- $\therefore\ w_1 = 1.5, w_0 = 0.0$



Multi Variable Linear Regression

- x가 scalar값(1개)가 아니라 vector가 된다면??

- Input
  - X1 : Facebook 광고료
  - X2 : TV 광고료
  - X3 : 신문 광고료
- Output
  - 판매량

FB	TV	신문	판매량
X_1	X_2	X_3	Y
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	9.3
151.5	41.3	58.5	18.5
180.8	10.8	58.4	12.9
8.7	48.9	75	7.2
57.5	32.8	23.5	11.8
⋮	⋮	⋮	⋮

Multi Variable Linear Regression

$$\begin{aligned} \mathbf{w}_{\text{lin}} &= \underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \, \mathrm{E}_{\text{in}}(\mathbf{w}) \\ &= \underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \, \frac{1}{N}||X\mathbf{w} - \mathbf{y}||^2 \\ &= \underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \, \frac{1}{N}(\mathbf{w}^\top X^\top X\mathbf{w} - 2\mathbf{w}^\top X^\top \mathbf{y} + \mathbf{y}^\top \mathbf{y}) \end{aligned}$$

Multi Variable Linear Regression

- Find w that minimize $E_{in}(w)$ by requiring

$$\nabla \mathrm{E}_{\mathrm{in}}(\mathbf{w}) = \mathbf{0}$$

- From previous equation

$$\nabla \mathrm{E}_{\mathrm{in}}(\mathbf{w}) = \frac{2}{N}(X^\top X\mathbf{w} - X^\top \mathbf{y})$$

$$\begin{aligned} \mathbf{w}_{\mathrm{lin}} &= \underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \mathrm{E}_{\mathrm{in}}(\mathbf{w}) \\ &= \underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \frac{1}{N}||X\mathbf{w} - \mathbf{y}||^2 \\ &= \underset{\mathbf{w}\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \frac{1}{N}(\mathbf{w}^\top X^\top X\mathbf{w} - 2\mathbf{w}^\top X^\top \mathbf{y} + \mathbf{y}^\top \mathbf{y}) \end{aligned}$$

Multi Variable Linear Regression

- Two scenarios
  - If $X^T X$ is invertible

    $$\boldsymbol{w} = (X^T X)^{-1} X^T$$

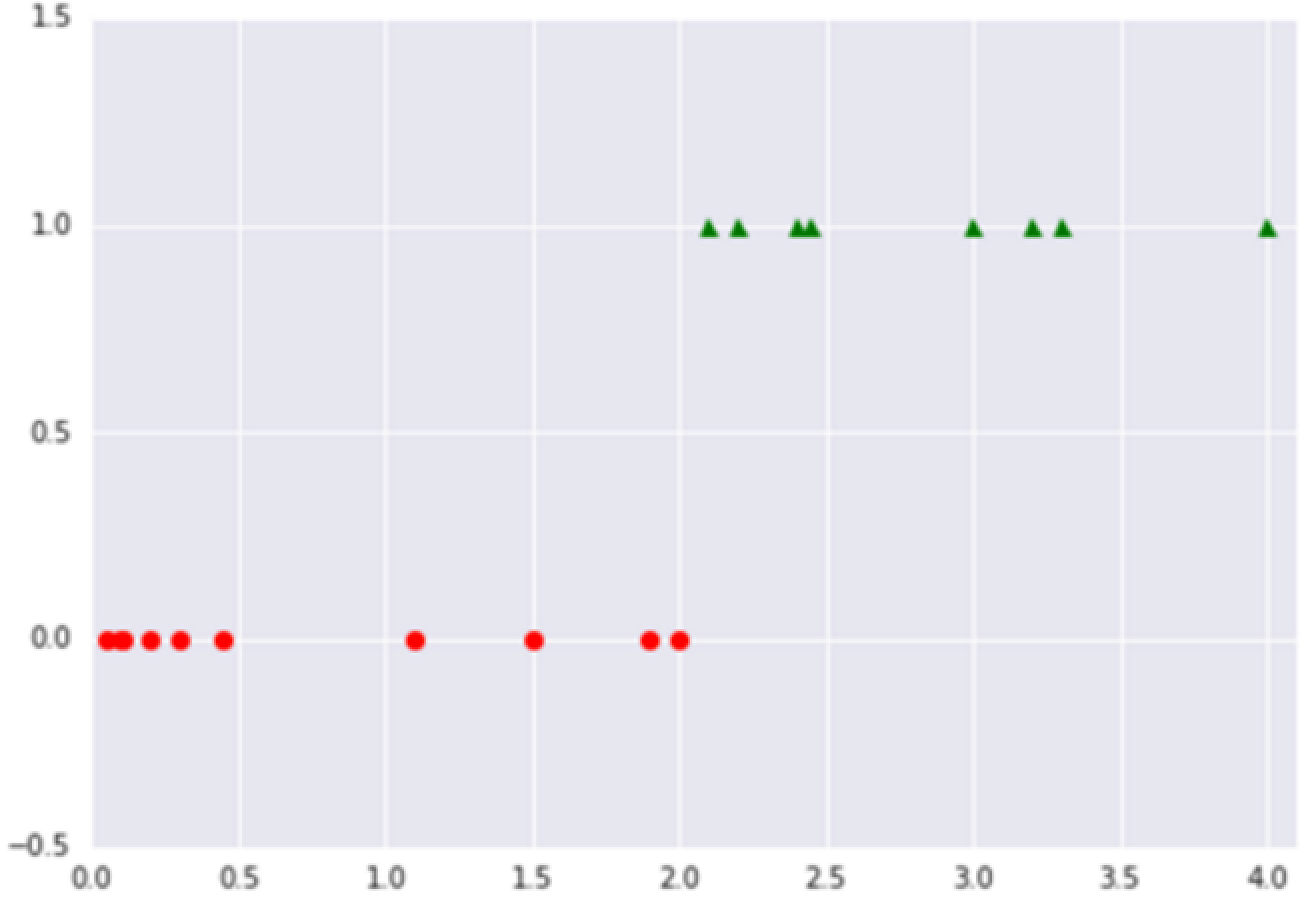
  - If $X^T X$ is not invertible

    Pseudo-inverse defined, but no unique solution

Classification도 할 수 있지 않을까요?

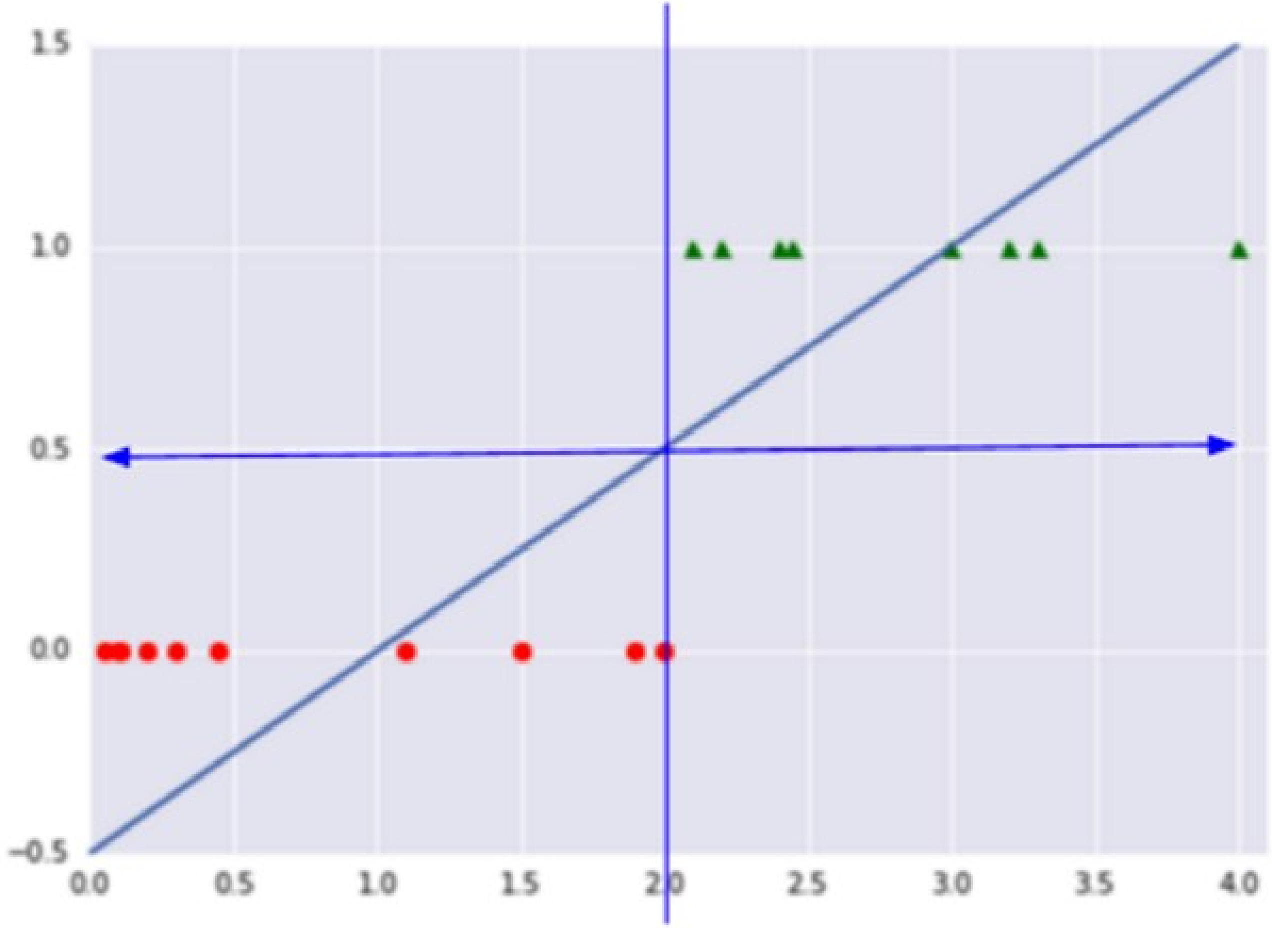
Binary Classification

- 종양의 크기에 따른 양성/음성 판별 문제
  - 1 : 양성(암), 0: 음성(정상)



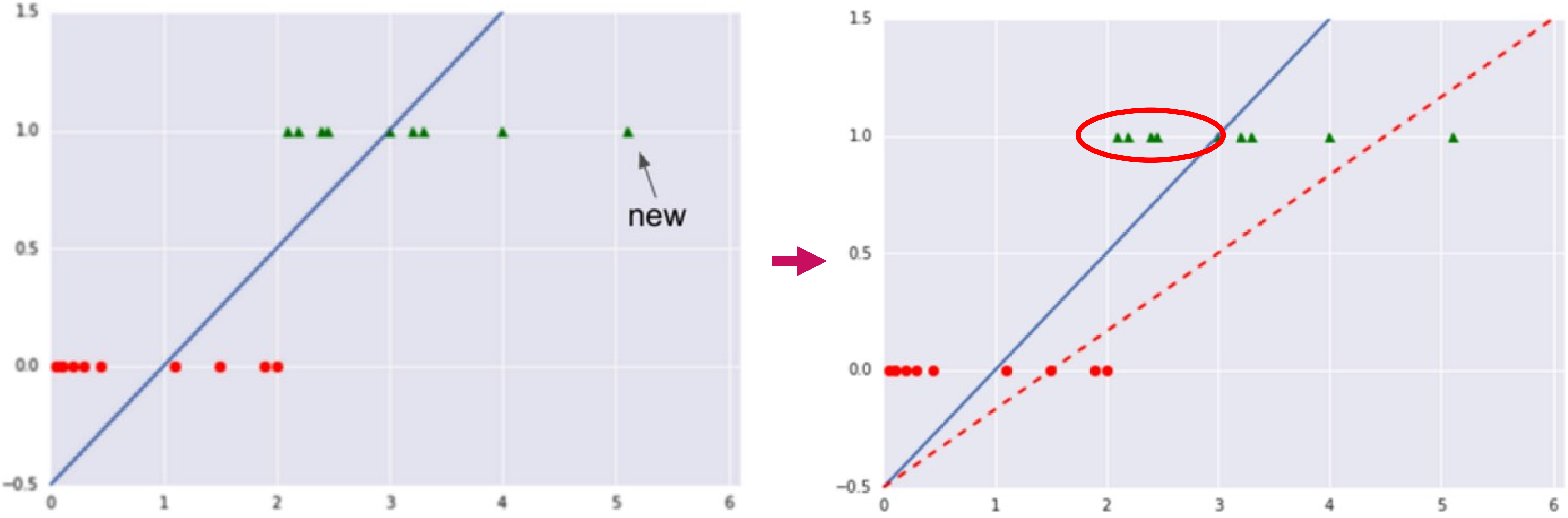
Binary Classification

- Linear Regression으로 해봅시다
  - Regression 예측값이 0.5 이상이면 양성, 0.5 이하면 음성으로 판별



Binary Classification

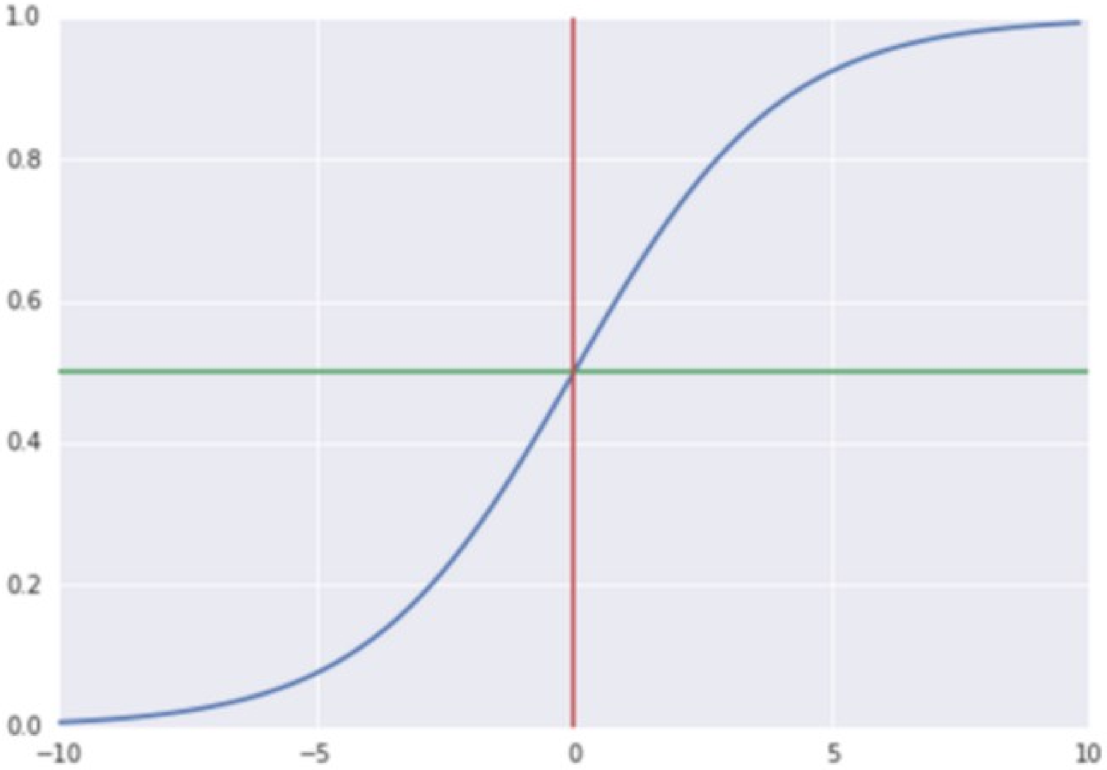
- 종양의 크기가 매우 큰 data(outlier)가 추가된 경우



Binary Classification

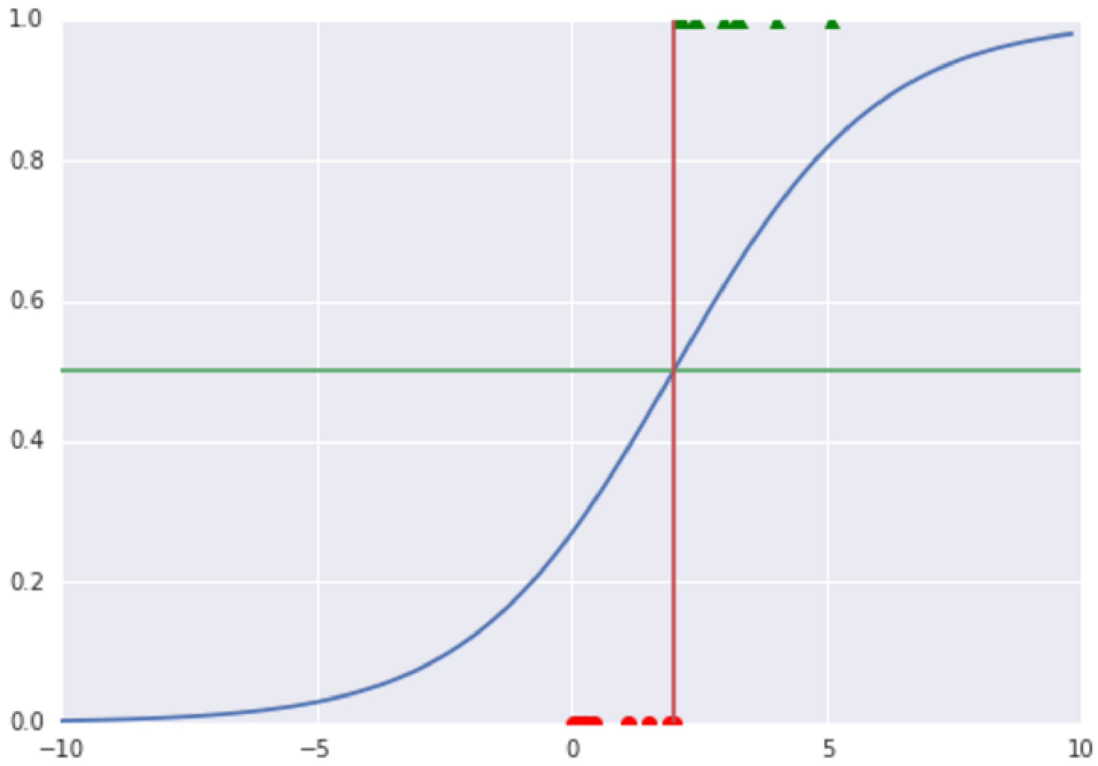
- 아주 크거나 아주 작은 data에 영향을 많이 받지 않았으면 좋겠다
- Binary classification에 맞게 0에서 1사이 값으로 나오면 좋겠다

→ Sigmoid 를 써보자



Logistic Regression

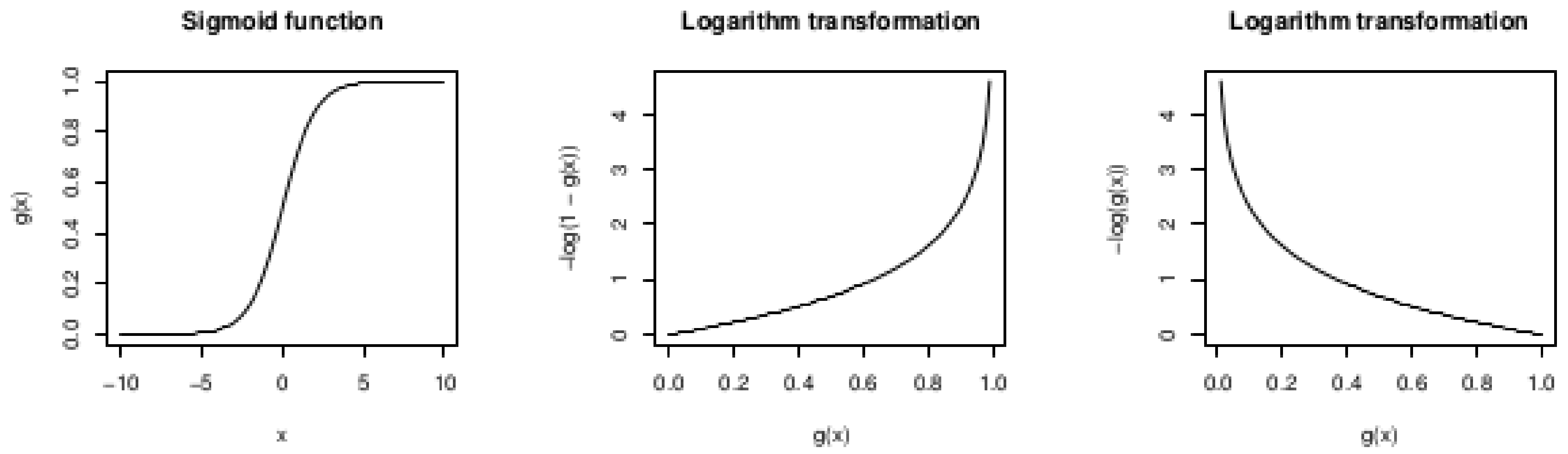
- Linear Regression 식에 Sigmoid 함수를 통과시킨 것
  - $H(x) = \sigma(\boldsymbol{w}^T\boldsymbol{x} + b) = \dfrac{1}{1+ e^{-(\boldsymbol{w}^T\boldsymbol{x}+b)}} = P(y|\boldsymbol{x})$



Logistic Regression

- 새로운 Cost(Loss) function을 정의(maximum likelihood estimation)

$$cost(W) = -\frac{1}{m}\sum ylog(H(x)) + (1-y)log(1-H(x))$$



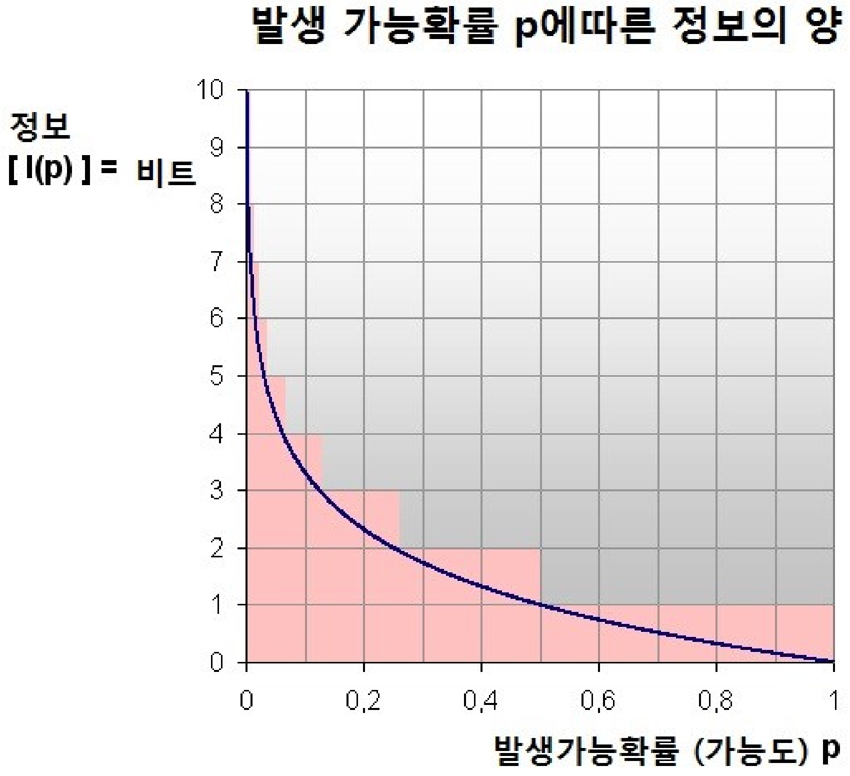
(a) *Sigmoid function.* (b) *Cost for $y = 0$.* (c) *Cost for $y = 1$.*

Figure B.1: *Logarithmic transformation of the sigmoid function.*

entropy

- Information Theory에서 Entropy란 정보량의 기댓값(평균)이다

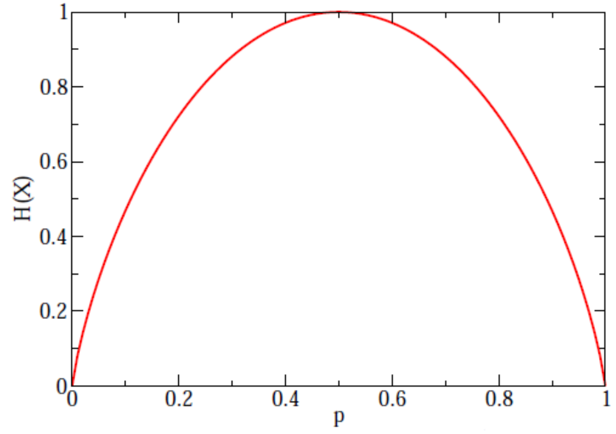
$$\mathrm{H}(X) = \mathrm{E}[\mathrm{I}(X)] = \mathrm{E}[-\ln(\mathrm{P}(X))].$$



- 항상 일어나는 사건이라면 p=1 이고 이것이 가지고 있는 정보량은 0이다.
- 일어날 확률이 적을 수록 많은 정보를 담고 있다.

Entropy

$$
\begin{aligned}
\mathrm{H}(X) &= \mathrm{E}[\mathrm{I}(X)] = \mathrm{E}[-\ln(\mathrm{P}(X))]. \\
&= \sum_{i=1}^{n} \mathrm{P}(x_i)\,\mathrm{I}(x_i) = -\sum_{i=1}^{n} \mathrm{P}(x_i) \log_b \mathrm{P}(x_i),
\end{aligned}
$$



Ex) 동전던지기

P(앞) = ½
P(뒤) = ½

$$H(X) = -(0.5\ * \log_2 0.5 + 0.5\ * \log_2 0.5) = 0.5 + 0.5 = 1$$

이 확률 분포를 표현하기 위하여 평균 1 bit가 필요하다는 의미!

Cross Entropy

True distribution p,
Model이 예측한 distribution을 q라고 하면

Cross entropy를 minimize 하는 것은 p와 q의 분포가 가까워지도록 만드는 것과 같다

$$
\begin{aligned}
H(p,q) &= -\sum_i p_i \log q_i \\
&= -\sum_i p_i \log q_i \boxed{-\sum_i p_i \log p_i} + \sum_i p_i \log p_i \\
&= \boxed{H(p)} + \sum_i p_i \log p_i - \sum_i p_i \log q_i \\
&= H(p) + \sum_i p_i \log \frac{p_i}{q_i} \\
&= H(p) + D_{KL}(p \parallel q)
\end{aligned}
$$

P 자체의 entropy(불변)

p를 기준으로 q가 얼마나 다른가
(p의 distribution과 q의 distribution의 거리)

Kullback–Leibler(KL) Divergence

p의 entropy　　p, q의 cross entropy

- 두 확률 분포간의 거리(?)를 나타냄
  - $D_{KL}(p \parallel q) = p \log \frac{p}{q} = p \log p + (-p \log q) = -H(p) + H(p, q) = H(p, q)$
  - p는 label, q는 network output 이라고 할 경우, network의 목표는 label이 나타내는 확률 분포 p와 최대한 가까운 q를 학습하고자 하는 것임
  - 일반적으로 label은 one-hot encoding 하기 때문에, 각 label의 값이 확률이라고 생각하면 p의 entropy, $H(p) = 0$
  - $D_{KL}(p \parallel q) > 0$

Likelihood

- Likelihood – 2개의 class(y = +1, y = -1)의 경우,

$$P(y|\mathbf{x}) = \begin{cases} h(\mathbf{x}) & \text{for } y = +1 \\ 1 - h(\mathbf{x}) & \text{for } y = -1 \end{cases}$$

$$h(\boldsymbol{x}) = \sigma(\boldsymbol{w}^T \boldsymbol{x} + b)$$

$$P(y|\mathbf{x}) = h(\mathbf{x})^{[\![y=+1]\!]}(1 - h(\mathbf{x}))^{[\![y=-1]\!]}$$

- Likelihood of data $(\boldsymbol{x}_1, y_1), \dots, (\boldsymbol{x}_N, y_N)$

$$\prod_{n=1}^{N} P(y_n|\mathbf{x}_n) = \prod_{n=1}^{N} h(\mathbf{x}_n)^{[\![y_n=+1]\!]}(1 - h(\mathbf{x}_n))^{[\![y_n=-1]\!]}$$

Negative Log-Likelihood(NLL)

- Maximize likelihood = Minimize negative log-likelihood(NLL)

$$NLL(\mathbf{w}) \propto -\frac{1}{N}\log\left\{\prod_{n=1}^{N} P(y_n|\mathbf{x}_n)\right\}$$

$$= -\frac{1}{N}\log\left\{\prod_{n=1}^{N} h(\mathbf{x}_n)^{[\![y_n=+1]\!]}(1-h(\mathbf{x}_n))^{[\![y_n=-1]\!]}\right\}$$

$$= \boxed{\frac{1}{N}\sum_{n=1}^{N}\left\{[\![y_n=+1]\!]\log\frac{1}{h(\mathbf{x}_n)} + [\![y_n=-1]\!]\log\frac{1}{1-h(\mathbf{x}_n)}\right\}}$$

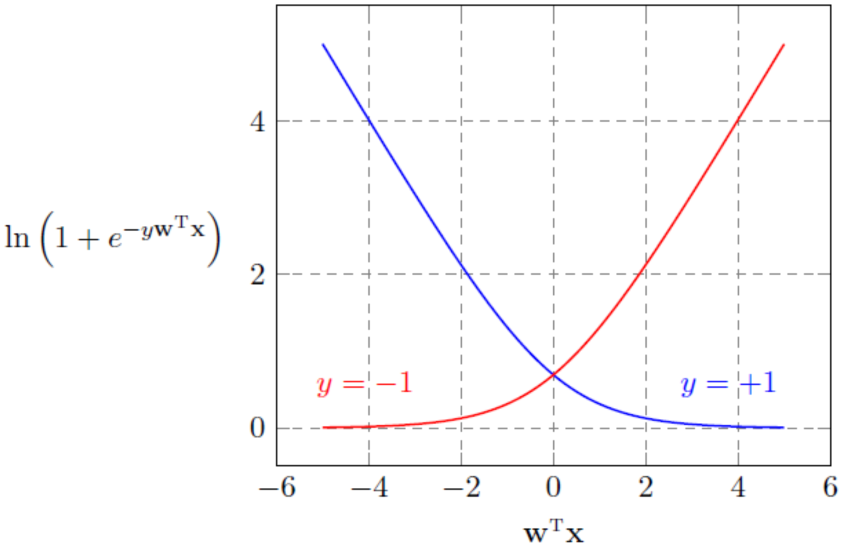
Cross-Entropy!

Minimizing NLL

- Minimize $$-\frac{1}{N}\log\left\{\prod_{n=1}^{N}P(y_n|\mathbf{x}_n)\right\}=\frac{1}{N}\sum_{n=1}^{N}\log\frac{1}{P(y_n\,|\,\mathbf{x}_n)}$$
- We can define loss(error) function as below

$$\mathrm{E}_{\mathrm{in}}(\mathbf{w})=\frac{1}{N}\sum_{n=1}^{N}\log\left(1+e^{-y_n\mathbf{w}^\top\mathbf{x}_n}\right)$$

$$\mathbf{e}(h(\mathbf{x}_n),y_n)=\ln\left(1+e^{-y_n\mathbf{w}^\top\mathbf{x}_n}\right)$$



Minimizing NLL

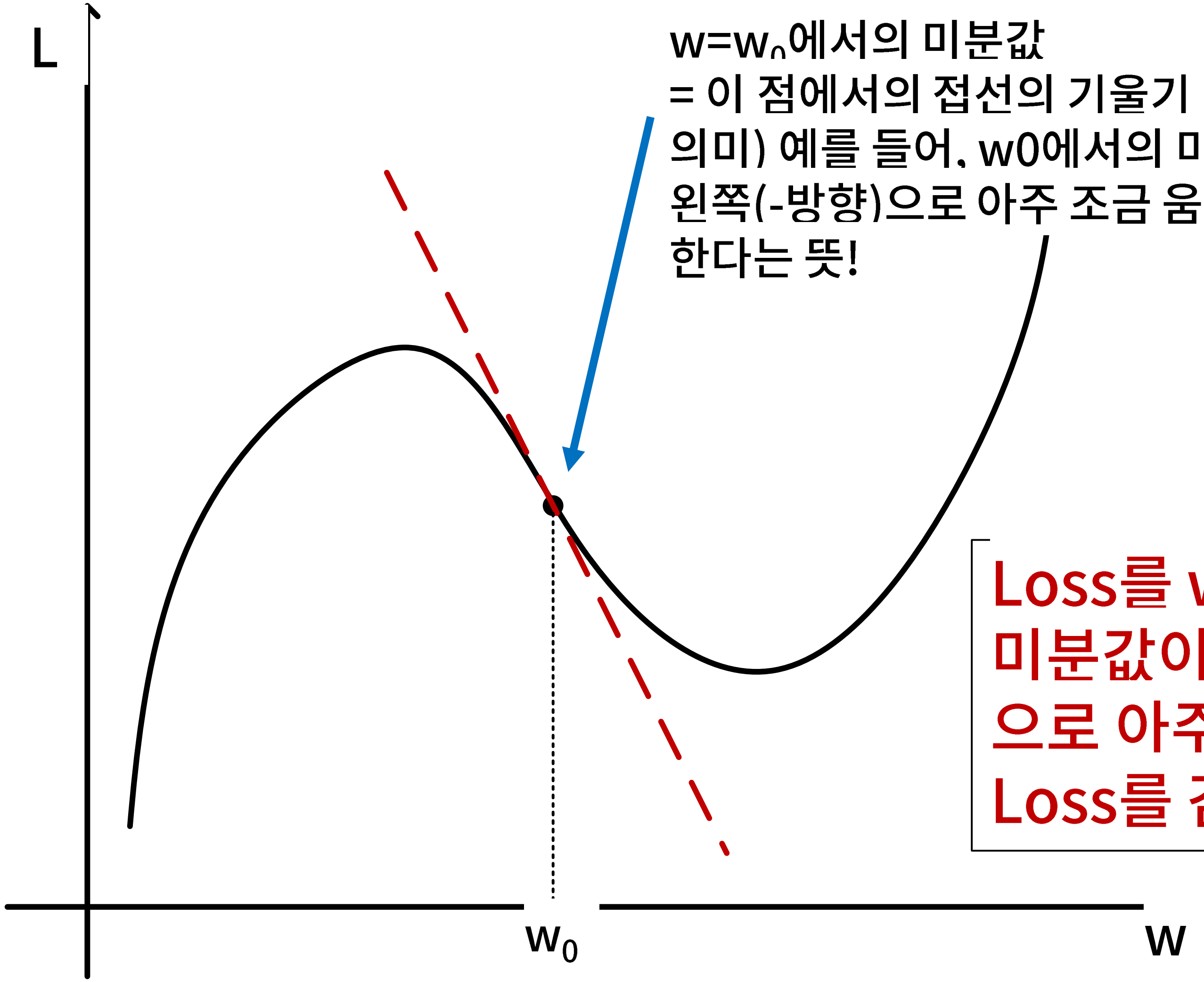
- Unfortunately, not easy to manipulate analytically

$$\nabla \mathrm{E}_{\mathrm{in}}(\mathbf{w}) = -\frac{1}{N}\sum_{n=1}^{N}\frac{y_n\mathbf{x}_n}{1+e^{y_n\mathbf{w}^{\mathrm{T}}\mathbf{x}_n}}$$

$$= \frac{1}{N}\sum_{n=1}^{N} -y_n\mathbf{x}_n\theta(-y_n\mathbf{w}^{\mathrm{T}}\mathbf{x}_n)$$

- We need iterative optimization
- Use 미분!

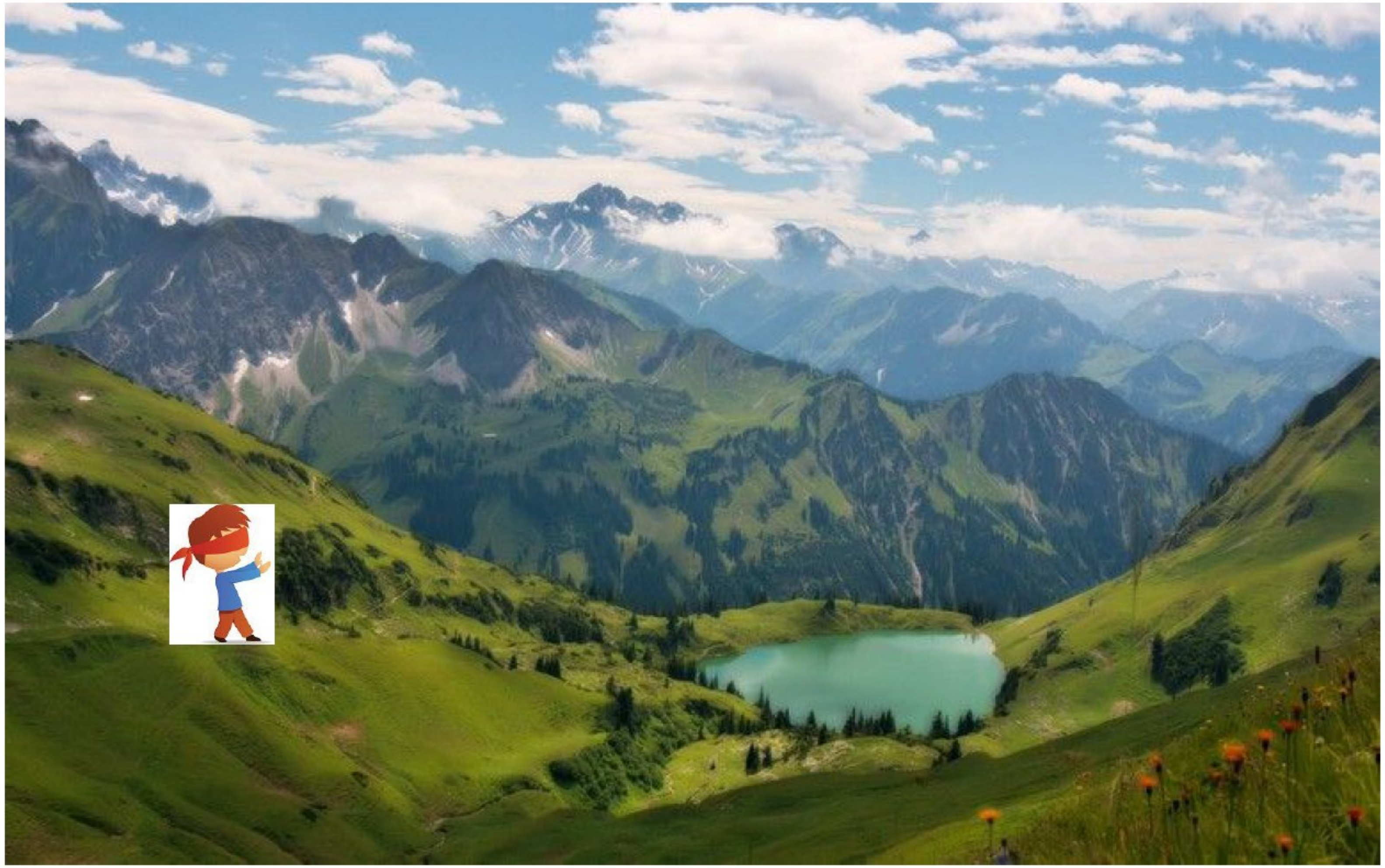
미분??



$w=w_0$에서의 미분값
= 이 점에서의 접선의 기울기
의미) 예를 들어, w0에서의 미분값이 -2라면, w를 w0에서 왼쪽(-방향)으로 아주 조금 움직이면 그 2배만큼 L 값이 증가한다는 뜻!

Loss를 w로 미분하고,
미분값이 가리키는 방향의 반대방향으로 아주 조금씩 w를 바꿔나가면
Loss를 감소시킬 수 있다!!

Gradient Descent



Gradient Descent

Loss Function의 미분(Gradient)를 이용하여 weight를 update하는 방법

$$w_{new} = w_{old} - \eta \nabla_w L$$

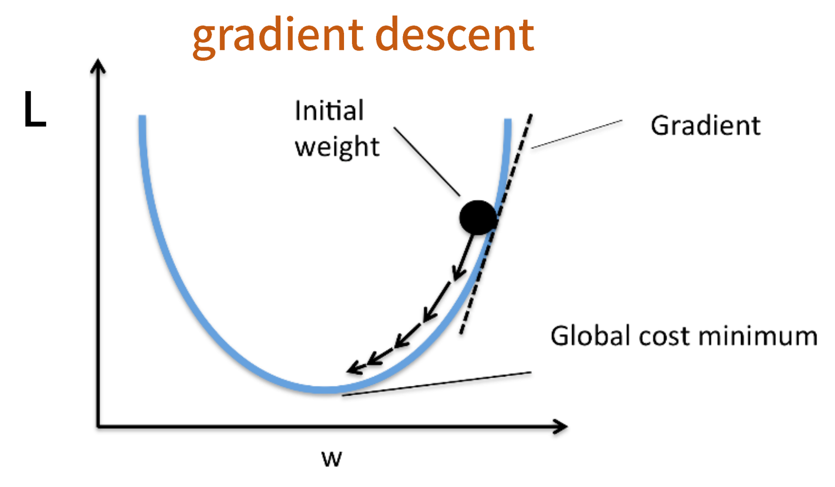
Weight update $= w_{new} - w_{old}$

$$= -\eta \nabla_w L$$

Loss를 감소시키는 방향 (Descent)

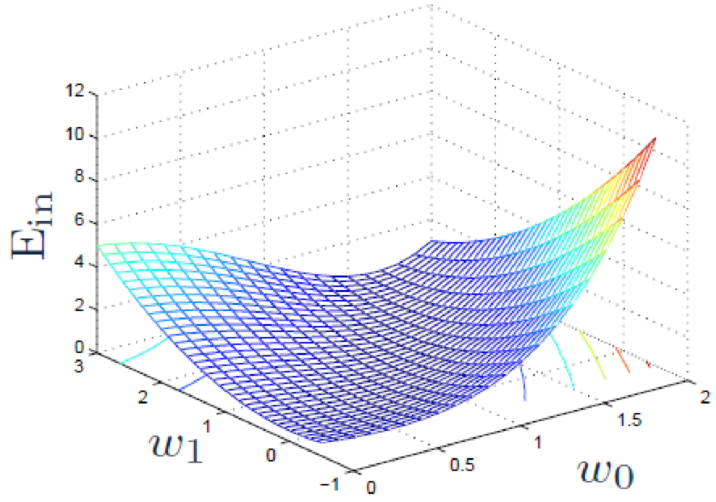
아주 조금씩 이동 (Learning Rate)

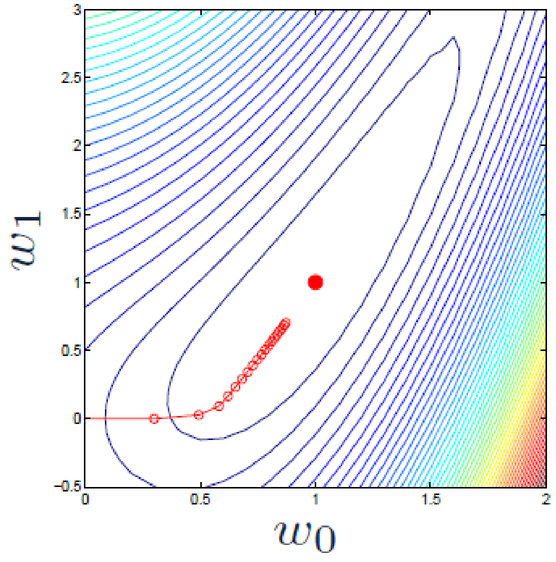
미분값 (Gradient)



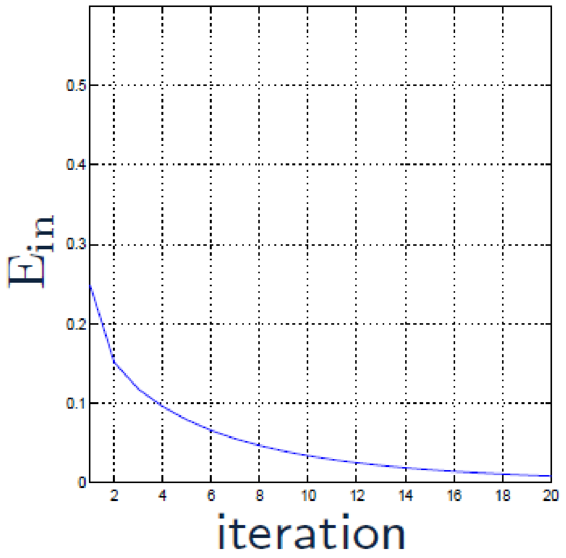
Gradient Decent Example

- **Global minimum 0 at (1,1)**

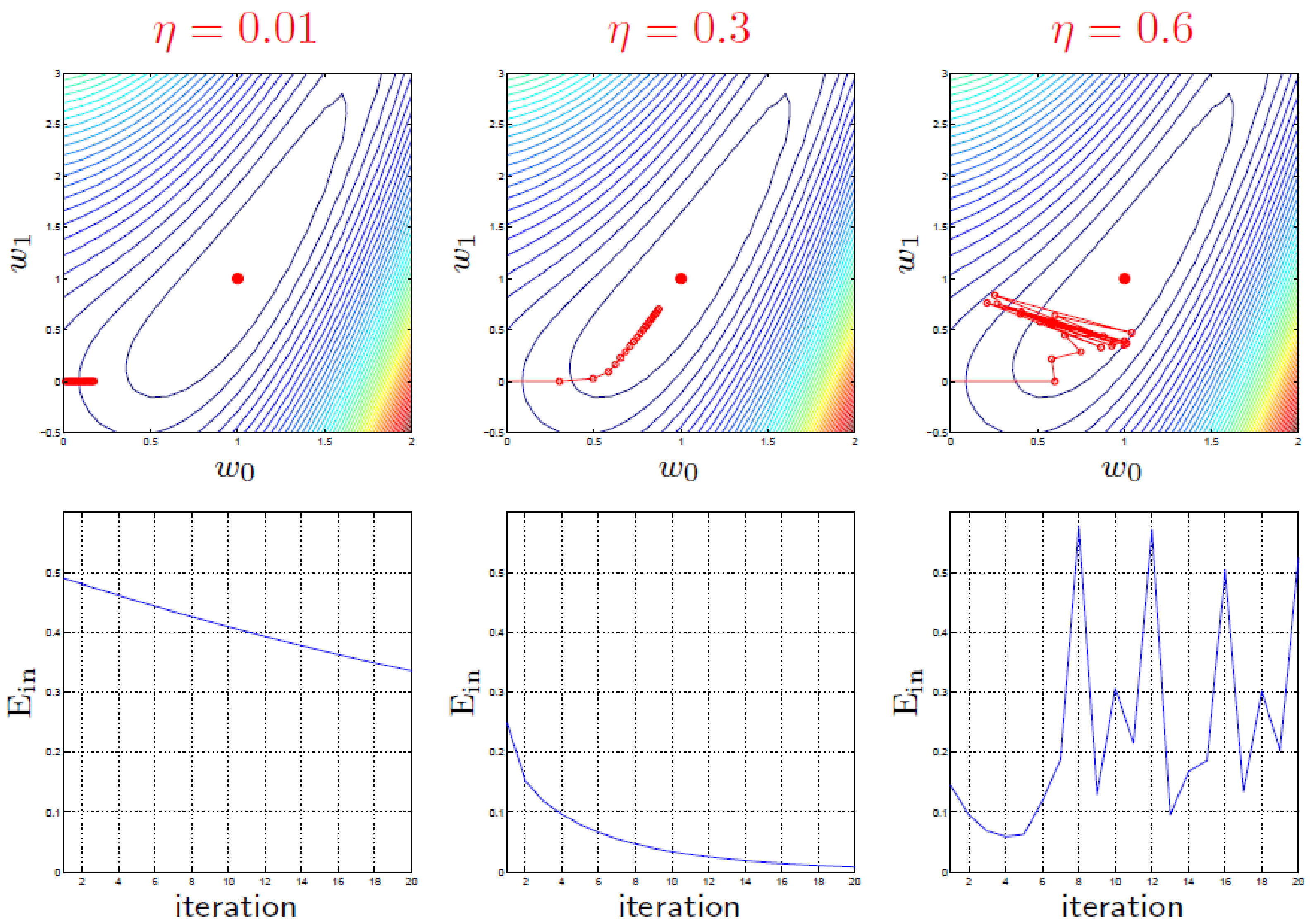




- **Start at (0,0)**
  - # iterations (steps) = 20
  - step size $\eta = 0.3$

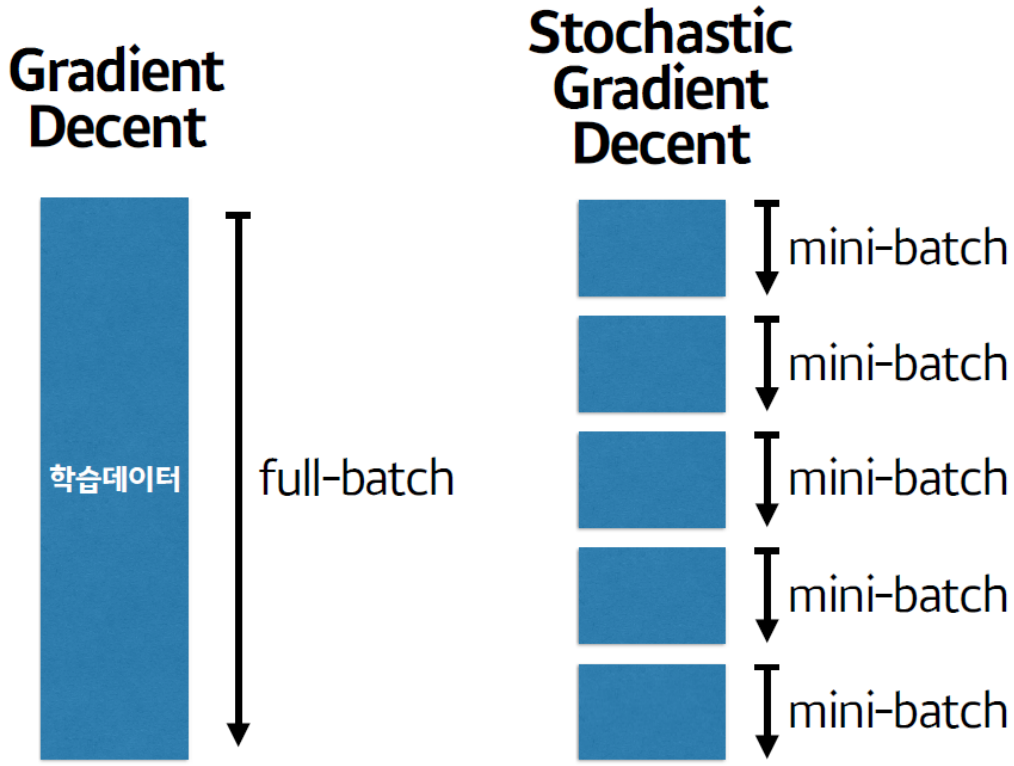


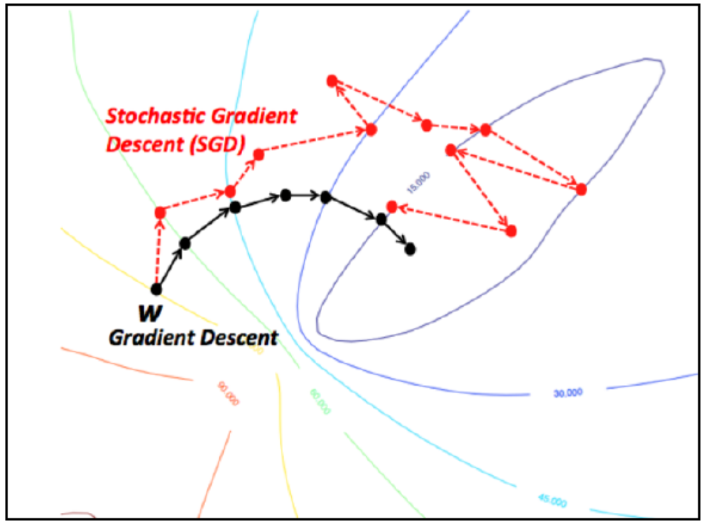
Learning Rate



Stochastic Gradient Descent, Mini-batch Training

- Data가 너무 많아서 한번에 다 넣고 학습하면 시간도 오래걸리고, memory도 부족하게 됨





전부다 읽고나서
최적의 1스텝 간다.

작은 토막마다
일단 1스텝간다.