Ch - 1 Introduction to Complex numbers

$\mathbb{N}$ = Set of normal num = $\{1, 2, 3, 4, \dots\}$
$\mathbb{W}$ = Set of whole num = $\{0, 1, 2, 3, 4, \dots\}$
$\mathbb{Z}$ = Set of integers = $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{Q}$ = Set of rational nos. = $\{\frac{p}{q}, q \neq 0, p, q \in \mathbb{Z}\}$
$\mathbb{R}$ = Set of ~~real~~ real nos.
$\mathbb{C}$ = Set of Complex nos.

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

⇒ Definition :-

Any number of the form $z = a + ib$ where $a, b \in \mathbb{R}$ & $i^2 = -1$ is called Complex nos.

for $z = 3 + 2i$
$z = \sqrt{3}\, i$

→ Note :-

1) for $a + ib$
real part of z, $Re(z) = R(z) = a$
Imaginary part of z, $Im(z) = b$

2) For any Complex no. $z = a + ib$ which also be mentioned as an ordered pair as (a, b).

⇒ Definition :- (Conjugate of Complex no.)

let $z = a + ib$ be any Complex no. then the Conjugate of z is denoted by $\bar{z}$. (read as z bar) & as $\bar{z} = a - ib$

for 1) $z = 3+2i \Rightarrow \bar{z} = 3-2i$

2) $z = 5i \Rightarrow \bar{z} = -5i$

3) $z = 3 \Rightarrow \bar{z} = 3$

4) $z = 1-i \Rightarrow \bar{z} = 1+i$

$\Rightarrow$ Operation of Complex numbers :-

1) Addition :-

let $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$

then

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$
$$= (a_1 + a_2) + i(b_1 + b_2)$$

$\rightarrow$ eg:

1) $z_1 = 3+2i$ & $z_2 = 1+4i$

$z_1 + z_2 = 4 + 6i$

2) Subtraction :

let $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$

then

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

$\rightarrow$ example :

1) $z_1 = 1+3i$ & $z_2 = 2-4i$

$$(z_1 - z_2) = (1+3i) - (2-4i)$$
$$= -1 + 7i$$

3) Multification :

let $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$

then

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= a_1 a_2 + ia_1 b_2 + ia_2 b_1 + i^2 b_1 b_2$$
$$= a_1 a_2 + i(a_1 b_2 + a_2 b_1) - b_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

→ example :

1) $z_1 = 1+3i$ & $z_2 = 2-4i$

$$z_1 z_2 = (1+3i)(2-4i)$$
$$= 2-4i+6i-12i^2$$
$$= (2+12)+i(-4+6)$$
$$= 14+2i$$

4) Divison :

let $z_1 = a_1+ib_1$ & $z_2 = a_2+ib_2$, $z_2 \neq 0$

then

$$\frac{z_1}{z_2} = \frac{(a_1+ib_1)}{(a_2+ib_2)} \times \frac{(a_2-ib_2)}{(a_2-ib_2)}$$
$$= \frac{a_1a_2 - ia_1b_2 + ia_2b_1 - i^2b_1b_2}{a_2^2 - i^2b_2^2}$$
$$= \frac{(a_1a_2+b_1b_2)+i(a_2b_1-a_1b_2)}{a_2^2+b_2^2}$$
$$= \frac{(a_1a_2+b_1b_2)}{a_2^2+b_2^2} + i\frac{(a_2b_1-a_1b_2)}{a_2^2+b_2^2}$$

→ example :-

1) $z_1 = 1+3i$ $z_2 = 1-i$

$$\frac{z_1}{z_2} = \frac{1+3i}{1-i} \times \frac{1+i}{1+i}$$
$$= \frac{1+i+3i-3}{1+1}$$
$$= \frac{-2+4i}{2}$$
$$= -1+2i$$

⇒ Some Properties of Conjugates :

1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

2) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

3) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

4) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, where $\bar{z}_2 \neq 0$

→ let $z_1 = 1 + 2i$ & $z_2 = 1 - 4i$

⇒ $z_1 + z_2 = 2 - 2i$

$\overline{z_1 + z_2} = 2 + 2i$

$\bar{z}_1 = 1 - 2i$

$\bar{z}_2 = 1 + 4i$

$\bar{z}_1 + \bar{z}_2 = 2 + 2i$

so, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

⇒ $z_1 - z_2 = 0 + 6i = 6i$

$\overline{z_1 - z_2} = -6i$

$\bar{z}_1 = 1 - 2i$

$\bar{z}_2 = 1 + 4i$

$\bar{z}_1 - \bar{z}_2 = 0 - 6i = -6i$

so, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

⇒ $\dfrac{z_1}{z_2} = \dfrac{1+2i}{1-4i} \times \dfrac{1+4i}{1+4i}$

$= \dfrac{1 + 4i + 2i - 8}{1 + 16}$

$= \dfrac{-7 + 6i}{17} = \dfrac{-7}{17} + \dfrac{6i}{17}$

$\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{-7}{17} - \dfrac{6i}{17}$

$\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{1-2i}{1+4i} \times \dfrac{1-4i}{1-4i}$

$= \dfrac{1 - 4i - 2i - 8}{1 + 16}$

$= \dfrac{-7 - 6i}{17}$

$\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{-7}{17} - \dfrac{6i}{17}$

$\Rightarrow \quad z_1 \cdot z_2 = (1+2i)(1-4i)$
$= 1 - 4i + 2i + 8$
$= 9 - 2i$

$\bar{z}_1 \cdot \bar{z}_2 = (1-2i)\cdot(1+4i)$
$= 1 + 4i - 2i + 8$
$= 9 + 2i$

$(\overline{z_1 \cdot z_2}) = 9 + 2i.$

So, $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$\Rightarrow$ Find conjugutes of the following:

1) $\dfrac{3+2i}{1-i}$

$\rightarrow \dfrac{3+2i}{1-i} \times \dfrac{1+i}{1+i}$

$\dfrac{3+3i+2i-2}{1+1}$

$\dfrac{1+5i}{2} = \dfrac{1}{2} + \dfrac{5}{2}i$

$\rightarrow$ So, Conjugute is $= \dfrac{1}{2} - \dfrac{5}{2}i$

2) $z = \dfrac{(2+i)(1+2i)}{3+4i}$

$z = \dfrac{2+4i+i-2}{3+4i} = \dfrac{5i}{3+4i}$

$z = \dfrac{5i}{3+4i} \times \dfrac{(3-4i)}{(3-4i)}$

$= \dfrac{15i+20}{9+16} = \dfrac{20+15i}{25} = \dfrac{4}{5} + \dfrac{3}{5}i$

$\cancel{= \dfrac{-5i}{25} = \dfrac{-i}{5}}$

So, $\bar{z} = \dfrac{4}{5} - \dfrac{3}{5}i$

$\bar{z} = \dfrac{i}{5} = \dfrac{0+i}{5}$

3) $\dfrac{2+3i}{1-i} = z$

$\rightarrow \quad z = \dfrac{2+3i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{2+2i+3i-3}{1+1} = \dfrac{-1+5i}{2} = -\dfrac{1}{2} + \dfrac{5}{2}i$

$\bar{z} = -\dfrac{1}{2} - \dfrac{5}{2}i$

4) $\dfrac{6+7i}{8i-3} = z$

$\cancel{z = \dfrac{6+7i}{8i+3} \times \dfrac{8i+3}{8i+3} = \dfrac{48i+18-56+21i}{-64-9}}$

$z = \dfrac{6+7i}{-3+8i} \times \dfrac{-3-8i}{-3-8i}$

$= \dfrac{-18-48i-21i+56}{9+64}$

$= \dfrac{38-69i}{73} = \dfrac{38}{73} - \dfrac{69i}{73}$

$\bar{z} = \dfrac{38}{73} + \dfrac{69i}{73}$

5) $\left(\dfrac{-6+2i}{-8i+1}\right)^2 = z$

$z = \left(\dfrac{-6+2i}{+1-8i} \times \dfrac{1+8i}{1+8i}\right)^2$

$= \left(\dfrac{-6-48i+2i-16}{1+64}\right)^2$

$= \left(\dfrac{-22-46i}{65}\right)^2 = \left(-\dfrac{22}{65} - \dfrac{46i}{65}\right)^2$

$= \left(\dfrac{-22}{65}\right)^2 - 2\left(\dfrac{-22}{65}\right)\left(\dfrac{-46i}{65}\right) + \left(\dfrac{-46i}{65}\right)^2$

$= \dfrac{484}{4225} - \dfrac{2024i}{4225} + \dfrac{2116}{4225}$

$= -\dfrac{1632}{4225} - \dfrac{2024i}{4225}$

Ans:- $-\dfrac{1632}{4225} + \dfrac{2024i}{4225}$

6) $\frac{1}{2-3i} + \frac{5-i}{6+2i} = z$

$\to z = \frac{1}{2-3i} \times \frac{2+3i}{2+3i} + \frac{5-i}{6+2i} \times \frac{6-2i}{6-2i}$

$= \frac{2+3i}{4+9} + \frac{30-10i-6i-2}{36+4}$

$= \frac{2+3i}{13} + \frac{28-16i}{40}$

$= \frac{2}{13} + \frac{28}{40} + \frac{3i}{13} - \frac{16i}{40}$

$= \frac{80+364}{520} + \frac{120i-208i}{520}$

$= \frac{444}{520} - \frac{88i}{520}$

$= \frac{111}{130} - \frac{22i}{130}$ $\quad \bar{z} = \frac{111}{130} + \frac{22i}{130}$

$\Rightarrow$ Modulus and its Properties:

$\to$ let $z = x+iy$ be a complex number, then modulus or absolute value of z is denoted by $|z|$ and defined as $|z| = \sqrt{x^2+y^2}$

Thus, $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2+y^2}$

$\to$ for ex., $z = 3+4i$

$|z| = \sqrt{(3)^2+(4)^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$= 5$

$\to$ If $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ then the distance between them is denoted by $|z_1-z_2|$ and is defined as

$$|z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

→ Note : 1) $|z| = 0 \iff x = 0$ and $y = 0$

2) z and $\bar{z}$ have same modulus.

→ Properties :

1) $|z_1 + z_2| \leq |z_1| + |z_2|$

2) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

3) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

4) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $|z_2| \neq 0$

5) $|z|^2 = z \cdot \bar{z}$

6) $|z^2| = |z|^2$

7) $Re(z) \leq |Re(z)| \leq |z|$

8) $Im(z) \leq |Im(z)| \leq |z|$

⇒ Polar form of complex number :-

→ let $z = x + iy$ be a complex number

taking $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

$\therefore r = \sqrt{x^2 + y^2}$

→ Now, $z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta) = r\,cis\,\theta$

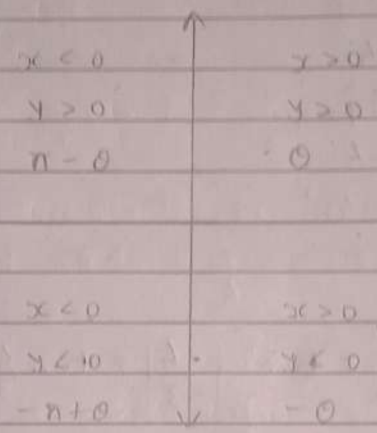
$= re^{i\theta}$

→ where r is called modulus of z and θ is the argument (amplitude) of z.

⇒ Argument or amplitude of a complex number:-

→ Suppose $z = x+iy$ is a complex number then the argument (amplitude) of z is denoted by Arg z or Amp z and defined as

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$



→ if $-\pi < \theta \leq \pi$, then θ is called principal argument of z, denoted by Arg z or Amp z and $\theta + 2k\pi$, $k \in \pi$ is called general argument of z, denoted by arg z or amp z

Where, $\theta = \tan^{-1}\left|\frac{y}{x}\right|$

→ Note:-

1) If $z = x+iy$ lies in 1^{st} and or 2^{nd} quadrant. then $0 < \theta < \pi$.

2) If $z = x+iy$ lies in 3^{rd} or 4^{th} quadrant then $-\pi < \theta < 0$.

3) As $\tan\theta$ is Perodic function with period π, $\theta = \text{Arg} z = \tan^{-1}\left(\frac{y}{x}\right)$ is multivalued function

i.e value of θ is not unique

4) Argument of origin (i.e for $z=0$) is not defined.

⇒ Express each of the following in polar form:

1) $-3-4i$

let $z = -3-4i$

$= x+iy$

⇒ $x=-3,\ y=-4$

$r = \sqrt{(-3)^2+(-4)^2}$

$= \sqrt{9+16}$

$r = 5$

⇒ $\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-4}{-3}\right| = \tan^{-1}\frac{4}{3}$

→ As $(-3,-4)$ lies in third quadrant so,

$\theta = -\pi + \tan^{-1}(4/3) = Arg(z) = Amp(z)$

→ Polar form :

$z = re^{i\theta}$

$z = 5e^{i(\tan^{-1}(4/3)-\pi)}$

2) $\left(\frac{6+8i}{4-3i}\right)^2$

→ $z = \left(\frac{6+8i}{4-3i} \times \frac{4+3i}{4+3i}\right)^2$

$= \left(\frac{24+18i+32i-24}{16+9}\right)^2$

$= \left(\frac{50i}{25}\right)^2 = (2i)^2 = (0+2i)^2 = -4$

→ $x=0,\ y=2$ → $x=-4,\ y=0$

$r = \sqrt{(0)^2+(2)^2} = 2$ $r = \sqrt{(-4)^2+(0)^2} = 4$

$\rightarrow$ $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left|\frac{0}{-4}\right| = \tan^{-1}(0) = 0$

$\rightarrow$ Polar form :- $\Rightarrow$ As $(-4, 0)$ lies in Second quadrant:

$z = re^{i\theta}$ $\qquad \theta = \pi - 0 = \pi$

$= 4e^{i\pi}$

$= 4e^{i\pi}$

$\Rightarrow$ Express the following in Cartesian (rectangular) form:

1) $2, \angle 30^\circ$

$\rightarrow$ here,

$r = 2$ & $\theta = 30^\circ = \pi/6$

$\rightarrow$ $x = r\cos\theta$ | $y = r\sin\theta$

$= 2\cos(\pi/6)$ $\qquad = 2\sin(\pi/6)$

$= \sqrt{3}$ $\qquad = 1$

$\rightarrow$ Cartesian form :-

$z = x + iy$

$= \sqrt{3} + 1i$

2) $4, \angle 300^\circ$

$\rightarrow$ here

$r = 4$ & $\theta = 300^\circ \times \frac{\pi}{180} = \frac{5\pi}{3}$

$\rightarrow$ $x = r\cos\theta$ $\qquad y = r\sin\theta$

$= 4\cos(5\pi/3)$ $\qquad = 4\sin(5\pi/3)$

$= 4\cos(2\pi - \pi/3)$ $\qquad = 4\sin(2\pi - \pi/3)$

$= 2$ $\qquad = -2\sqrt{3}$

$\rightarrow$ Cartesian form :

$z = x + iy$

$= 2 - 2\sqrt{3}i$

⇒ Find the modulus and argument of the following complex number :-

1) $1-i$

→ $z = 1-i$

$= x+iy$

$x=1, \ y=-1$

→ $|z| = \sqrt{(1)^2+(-1)^2}$

$= \sqrt{2}$

→ Now, $\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{1}\right| = \tan^{-1} 1 = \frac{\pi}{4}$

→ As $(1,-1)$ lies in forth quadurant so,

$\theta = \arg(z) = -\frac{\pi}{4} + 2K\pi, \ K \in Z$

$Arg(z) = -\theta$

$= -\frac{\pi}{4}$

2) $\frac{1-i}{1+i}$

→ let $z = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1-i-i-1}{1+1} = -\frac{2i}{2} = -i$

$z = 0-i$

$= x+iy$

So, $x=0, \ y=-1$

→ $|z| = \sqrt{(0)^2+(-1)^2} = 1$

→ Now,

$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|-\frac{1}{0}\right| = \tan^{-1}\infty = \frac{\pi}{2}$

→ As $(0,-1)$ lies between 3^{rd} and 4^{th} quadurant

So, $\theta = -\frac{\pi}{2} = Arg(z)$

$\arg(z) = -\frac{\pi}{2} + 2K\pi, \ K \in Z$

3) $(1+i)\, e^{i\pi/6}$

→ $\because\ e^{i\theta} = (\cos\theta + i\sin\theta)$

→ $z = (1+i)\left[\cos(\pi/6) + i\sin(\pi/6)\right]$

$= (1+i)\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2} + i\frac{(\sqrt{3}+1)}{2}$

→ So, $x = \frac{\sqrt{3}-1}{2}$, $y = \left(\frac{\sqrt{3}+1}{2}\right)$

$z = \sqrt{x^2+y^2}$

$= \sqrt{\frac{(\sqrt{3}-1)^2}{4} + \frac{(\sqrt{3}+1)^2}{4}}$

$= \frac{1}{2}\sqrt{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1}$

$= \frac{1}{2}\sqrt{8} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

→ $\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}+1}{2} \times \frac{\sqrt{3}-1}{2}\right|$

$= \tan^{-1}\left(\frac{\sqrt{3}+1}{2} \times \frac{2}{\sqrt{3}-1}\right)$

$= \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

→ As (x, y) lies in first quadrant so,

$\theta = \operatorname{Arg}(z) = \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

$\theta = \arg(z) = \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + 2k\pi,\ k \in \pi$

Example 1: Find z, if $\operatorname{Arg}(z+2i) = \frac{\pi}{4}$ and $\operatorname{Arg}(z-2i) = \frac{3\pi}{4}$.

→ let, $z = x+iy$

$\operatorname{Arg}(z+2i) = \frac{\pi}{4}$

$\operatorname{Arg}(x+iy+2i) = \frac{\pi}{4}$

$\operatorname{Arg}(x+i(2+y)) = \frac{\pi}{4}$

$\tan^{-1}\left|\frac{y+2}{x}\right| = \frac{\pi}{4}$

$\frac{y+2}{x} = \tan\left(\frac{\pi}{4}\right)$

$\frac{y+2}{x} = 1$

$y+2 = x$ — (i)

$\operatorname{Arg}(z-2i) = \frac{3\pi}{4}$

$\operatorname{Arg}(x+yi-2i) = \frac{3\pi}{4}$

$\operatorname{Arg}(x+i(y-2)) = \frac{3\pi}{4}$

$\tan^{-1}\left(\frac{y-2}{x}\right) = \frac{3\pi}{4}$

$\frac{y-2}{x} = \tan\left(\pi - \frac{\pi}{4}\right)$

$\frac{y-2}{x} = -1$

$y-2 = -x$ — (ii)

→ Adding eq. (i) & (ii)

$y+2+y-2 = x-x$

$2y = 0$

$y = 0$

→ from eq. (i)

$0+2 = x$

$x = 2$

→ So,

$z = x+iy$

$= 2+0i$

$= 2$

⇒ Example 2 : Find the locus of point z such that $|z+i| = |z|$ & $Arg\left(\frac{z+1}{z}\right) = \frac{\pi}{4}$

→ let, $z = x+iy$

$|z+i| = |z|$

$|x+iy+i| = |x+iy|$

$\sqrt{(x+1)^2 + y^2} = \sqrt{x^2+y^2}$

$(x+1)^2 + y^2 = x^2+y^2$

$(x+1)^2 = x^2$

$\sqrt{x^2+(y+1)^2} = \sqrt{x^2+y^2}$

$x^2+(y+1)^2 = x^2+y^2$

$(y+1)^2 = y^2$

$y+1 = y$

$y = -\frac{1}{2}$

$Arg\left(\frac{x+iy+i}{x+iy}\right) = \frac{\pi}{4}$

$Arg\left(\frac{x+i(y+1)}{x+iy} \times \frac{x-iy}{x-iy}\right) = \frac{\pi}{4}$

$Arg\left(\frac{x^2 - iyx + ix(y+1) + y(y+1)}{x^2+y^2}\right) = \frac{\pi}{4}$

$Arg\left(\frac{x^2+y^2+y + i(\cancel{xy}+x-\cancel{xy})}{x^2+y^2}\right) = \frac{\pi}{4}$

$\tan^{-1}\left(\frac{\cancel{xy}+x-\cancel{xy}}{x^2+y^2} \times \frac{x^2+y^2}{x^2+y^2+y}\right) = \frac{\pi}{4}$

$\frac{\cancel{xy}+x-\cancel{xy}}{x^2+y^2+y} = 1$

$\cancel{xy}+x-\cancel{xy} = x^2+y^2+y$

$x^2+y^2+y-x = 0$

$x^2+\frac{1}{4}-\frac{1}{2}-x = 0 \quad \left(\because y = -\frac{1}{2}\right)$

$x^2-x = 1/4$

$\cancel{x(x-1) = 1/4}$

$4x^2-4x-1 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-(-4) \pm \sqrt{16+16}}{8}$

$= \frac{4 \pm \sqrt{32}}{8} = \frac{1}{2} \pm \frac{\sqrt{2}}{\sqrt{2}\,2}$

$= \frac{1 \pm \sqrt{2}}{2}$

⇒ Example 3: If $z_1 = -1 + \sqrt{3}i$ and $z_2 = 2 + i2\sqrt{3}i$.
find $Arg(z_1 + z_2)$.

→ So, $z_1 + z_2 = -1 + \sqrt{3}i + 2 + 2\sqrt{3}i$
$= 1 + i(\sqrt{3} + 2\sqrt{3})$
$= 1 + i(3\sqrt{3})$

→ $Arg(z) = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{3\sqrt{3}}{1}\right|$

→ As (x, y) lies in 1^{st} quadrant.
$\theta = \tan^{-1}(3\sqrt{3})$

⇒ Properties of Argument :- (only for general argument)

1) $\arg z = -\arg \bar{z}$.

2) $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

3) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

4) $\arg\left(\frac{1}{z}\right) = -\arg z$

5) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z = \arg z^2$

6) $\arg(z^n) = n \arg z$

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$\Rightarrow$ De moivre's Theorem :

$\rightarrow$ If n is any rational number, then the value of one of its values of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

$\Rightarrow$ Results from De moivre's theorem :-

1) $$(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

2) $$(\cos\theta - i\sin\theta)^{-n} = \cos(-n\theta) - i\sin(-n\theta)$$
$$= \cos n\theta + i\sin n\theta$$

3) $$(\cos\theta - i\sin\theta)^{n} = \cos(n\theta) - i\sin(n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

4) $$(\sin\theta + i\cos\theta)^{n} = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$$
$$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$$

5) $$\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= \cos\theta - i\sin\theta$$