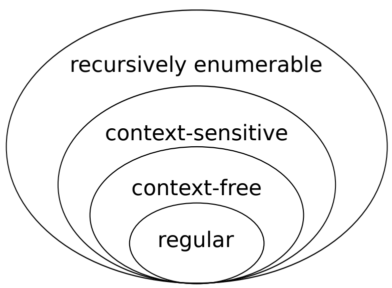
CS3100 Final Note Sheet

Notation

$\overline{L}$	Negation of language L.
L^R	The reversal of language L.
L^*	The Kleene-Star of L.
AB	The concatenation of language A and B.
$h(L)$	A homomorphism (a function that maps every input to a unique output) of L.
$A \setminus B$	Set difference of A and B. $A - B$ is the same thing.
2^A	The power-set of set A.

$f : x_1, x_2, \ldots, x_n \to y_1, y_2, \ldots, y_n$ denotes a function f that when given $x_1, \ldots, x_n$ as inputs yields $y_1, \ldots, y_n$ as outputs.

Chomsky Hierarchy



Regular Languages

A regular language is any language that can be recognized with a DFA. Formally a DFA is a tuple $(Q, \Sigma, \delta, q_0, F)$. Where:

Q	A finite, non-empty set of states.
Σ	A finite, non-empty alphabet.
δ	A function ($\delta : Q \times \Sigma \to Q$) that maps a state, and an input in Σ to a new state.
q_0	A state in Q that DFA starts execution from.
$F \subseteq Q$	A finite, possibly empty, set of accepting states.

Alternatively, regular languages can be defined by an NFA. Formally, NFAs are the same as DFAs, except the δ function for NFAs is defined as:

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q$$

Basically, the delta function can now map an input to multiple states instead of just one state.

Closures

Where R is a regular language, L is 'not regular', and ? is Unknown.

Closed:		**Unclosed:**
$\overline{R} \rightarrow R$	$h(R) \rightarrow R$	$R \cap L \rightarrow ?$
$R^* \rightarrow R$	$R \cup R \rightarrow R$	$R \cup L \rightarrow ?$
$R^R \rightarrow R$	$R \cap R \rightarrow R$	$L \cup L \rightarrow ?$
$RR \rightarrow R$	$R \setminus R \rightarrow R$	

Pumping Lemma

$$\begin{aligned}
&\exists N \in \mathbb{N} : \\
&\quad \forall w \in L : |w| \geq N \Rightarrow \\
&\exists xyz \in \Sigma^* : \quad w = xyz \\
&\qquad \wedge |xy| \leq N \\
&\qquad \wedge y \neq \varepsilon \\
&\qquad \wedge \forall i \geq 0 : xy^i z \in L
\end{aligned}$$

DFA Operations

Negation: Mark all non-final states final and all final states non-final.

Reversal: Introduce a new initial state 'q_I'. Add ε transitions from this new state to all old final states and mark these states as non-final. Reverse all arrows in the DFA, and then mark the old initial state q_0 as final.

Concatenation (AB): Add ε transitions from all of A's final states to B's initial state. Mark A's final states as non-final.

Union ($A \cup B$): Set the Q parameter of the new DFA to A's Q crossed with B's Q: $Q_{\text{new}} = A_Q \times B_Q$, Q will now contain pairs. Now change the δ function to be in terms of both A's delta function and B's delta function: $\delta_{\text{new}} = f((q_A, q_B), s) \rightarrow (\delta_A(q_A, s),\ \delta_B(q_B, s))$. That is to say, if A had state $q0_A$ that went to state $q1_A$ on input 0, and B had state $q2_B$ that went to state $q3_B$ state on input 0, then the new state $(q0_A, q2_B)$ would go to state $(q1_A, q3_B)$ on input 0.

The state that contains both DFA's initial state becomes the new initial state. New states where **either** item in the pair were final states, become final.

Intersection ($A \cap B$): Exactly the same as DFA union except only pairs of states where **both** states in the pair are final become final states.

NFA Operations

Concatenation (AB): Concatenation for an NFA is exactly like concatenation for a DFA.

Union ($A \cup B$): Introduce a new initial state q_I and add ε transitions from q_I to the initial states of A and B. Mark old initial states as no-longer initial.

Kleene-Star (A^*): First, add ε transitions from all final states to the initial state. Next, introduce a new initial state q_I, make an ε transition from this state to the old initial state, and mark this new initial state as final.

Conversions

NFA to DFA

Assume a function ε-closure(x) that when given a set of states from the NFA as input returns the set of states that can be reached from these states via ε transition.

1. The initial state of the new DFA is ε-closure$(\{q_0\})$, where q_0 is the initial state of the NFA.
2. Next, form a table where the rows are the states in the DFA (to start only write the initial state) and the columns are characters in Σ. Now proceed down rows filling in the columns for each state using the following equation. If the current state is S and the column's symbol is i, the the cell S, i is:

   $$\delta(S, i) = \varepsilon\text{-closure}\Big(\bigcup_{s \in S} \delta(s, i)\Big)$$

   In English, the ε-closure of the union of all states that can be reached from this DFA state's component NFA states on input i.

   If the output state of the above function is not yet in the table, add it as a new row and process it as normal.
3. Once this process is complete, you should be able to convert the table into a standard graph by using the columns to make the transitions. **States where any component state is final, are final.**

RE to NFA

The easiest way to build and NFA from a regular expression is incrementally. Start with an atom. In the regular expression: `(a+b)*`, start with `a`. An NFA for a single atom is simply an initial state and `a` transition to a single final state. Once you have the initial NFAs you can simply apply the NFA operations on them. Expression `ab` is the concatenation of the NFAs for `a` and `b`. Expression `a+b` is the union of the NFAs for `a` and `b`, and `a*` is the Kleene-Star operation on the NFA for `a`. These can be chained together. For example, the NFA for the first regular expression `(a+b)*` the union of the NFAs for `a` and `b` and then the Kleene-Star of the union. Parentheses affect order of operations just like they do in arithmetic, evaluate the inside first then the outside.

DFA to RE

DFA Minimization

Table Conversion Algorithm

Brzozowski's Algorithm

This algorithm is quite simple. The crux of it is that an NFA to DFA conversion naturally results in a minimization of the NFA. So the algorithm is as-follows: Take a DFA, reverse it to get an NFA, convert that NFA to a DFA again, take this new reversed DFA and reverse it to get the original language back, then convert the NFA resulting from the reversal into a DFA.

Context Free Languages

Closures

Where C is a context-free language, R is a regular language, and ? is an Unknown language.

Closed:		**Unclosed:**
$C^R \rightarrow C$	$C \cup C \rightarrow C$	$\overline{C} \rightarrow ?$
$C^* \rightarrow C$	$C \cap R \rightarrow C$	$C \cap C \rightarrow ?$
$CC \rightarrow C$	$C \cup R \rightarrow R$	$C \setminus C \rightarrow ?$
$h(C) \rightarrow C$		

Pumping Lemma

$$
\begin{aligned}
&\exists N \in \mathbb{N} : \\
&\quad \forall w \in L : |w| \geq N \Rightarrow \\
&\exists uvxyz \in \Sigma^* : \quad w = uvxyz \\
&\qquad \wedge |vy| > 0 \\
&\qquad \wedge |vxy| \leq N \\
&\qquad \wedge \forall i \geq 0 : uv^ixy^iz \in L
\end{aligned}
$$

CFG to PDA Conversion

Chomsky Normal Form

Cocke-Kasami-Younger (CKY) Parsing

Ambiguous Context Free Languages

These are languages that have two separate parse-trees. To prove that a language is ambiguous, show that it actually has two separate parse-trees.

Example ambiguous grammar:

$$E \rightarrow E + E \mid E * E \mid \texttt{NUMBER}$$

Consistency and Completeness

Consistency: All strings generated by a grammar are in the language.

Completeness: The grammar generates all strings in the language.

You cannot know that a grammar defines a language until you show both. For example, if we want to define the language $\{a^nb^n | n \in \mathbb{N}\}$, the grammar:

$$S \rightarrow aabb$$

Is *consistent* because it only generates strings in the language, but not complete because it doesn't generate all strings in the language. Likewise, the grammar:

$$S \rightarrow aS \mid bS \mid \varepsilon \qquad (\textit{grammar for } \{a, b\}^*)$$

Is complete, it generates all possible strings in the language, but not consistent because it generates many strings that are, in-fact, outside of the language.

Cardinality

Diagonalization

Schröder-Bernstein Theorem

The Schröder-Bernstein theorem states that, for any two sets A and B if there exists an *injective* function from $A \rightarrow B$, and there exits an injective function from $B \rightarrow A$, then $|A| = |B|$. Note that the injective function doesn't require every item of A to map to every item of B, only that every item of A maps to *an* item of B (and vice versa).

Pairing Functions

Turing Machines

Terminology and Notation

$\langle M \rangle$	String representation of Turing machine M.
halting	When a machine stops execution.
acceptance	When a machine halts in a final state.
rejection	When a machine halts and is not in a final state.
decider	A decider is a Turing machine that defines a language of Turing machines that conform to a yes or no question.

Post's Correspondence Problem

Decidability

Turing Recognizable & Recursively Enumerable

Bang.

Halting Problem

The halting problem states building a Turing machine P that can detect whether any other Turing machine will halt is impossible. The proof is as follows:

Assume that we have a Turing machine P, that when given a Turing machine M and string w as input ($\langle M, w \rangle$), P will (in a finite computation time) accept in the case that M halts on input w, or reject in the case that M loops on input w. We can then define a new Turing machine Q that takes a single Turing machine M as input. Q will then ask P whether machine M halts when given itself as input (does $P(\langle M, M \rangle)$ halt?). If P accepts (says that M halts) the Q will loop. If P rejects (says M will loop) then Q halts. Now, we can supply Q as input to machine Q. Q will then run $P(\langle Q, Q \rangle)$. If P accepts, then Q will begin to loop, but P said that Q would halt. This is a contraction, a general P decider for the halting problem cannot exist.

Mapping Reduction

A mapping reduction between $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$ is a function $f : \Sigma^* \to \Sigma^*$ if $\forall x \in \Sigma^*, x \in A \Leftrightarrow f(x) \in B$. More plainly, a function f such that I can pick any x in A, and $f(x)$ will also be in B. The "mapping reduction from A to B" is typically denoted as $A \leq_m B$. A mapping reduction in polynomial time is denoted $A \leq_p B$.

The general steps for a mapping reduction $A \leq_m B$ are as follows:

1. A is designated the "known undecidable" language.
2. B is designated the "unknown" language.
3. Create a function f that maps all elements of A into B.

To form f you usually assume the decider for B (D_B), then you construct a machine M that uses to decider D_B to become a decider for A (D_A). For example, we can map A_{TM} onto $Halt_{TM}$ using the following method:

Assume that a decider R for A_{TM} exists. We will now construct a decider S for $Halt_{TM}$ from R. S has two inputs a machine M and an input string w. First, S will run decider R on $\langle M, w$, if R accepts, then S accepts. If R rejects, then S accepts. We now have a decider for $Halt_{TM}$ which is undecidable, a decider for A_{TM} cannot exist.

Rice's Theorem

"Every non-trivial partitioning of the space of Turing machine codes based on the languages recognized by these Turing machines is undecidable."

More formally, given a property $\mathcal{P}$, where $\mathcal{P}$ is non-trivial (not $\emptyset$ or Σ^*) the language below is undecidable.

$$\langle M \rangle \mid M \text{ is a Turing machine and } \mathcal{P}(Lang(M))$$

NP-Completeness

Problem Classes

P: The set of problems that can be solved in polynomial time. Contained in NP.

NP: The set of problems that can be solved in non-deterministic polynomial time.

NP-hard: The set of problems that can be polynomial time reduced to every other problem in NP.

NP-complete: The set of problems that are in both NP, and NP-hard.

Proving NP-Completeness

There are two steps to proving that a languages is in NP-complete. First you have to show that it is NP, and then you have to show that it is in NP-hard.

Verifiers

One way to show that a problem is in NP is by using a verifier. A verifier is a Turing machine V_L such that for all $w \in \Sigma^*$, their exists some c such that $w \in L$ when $V_L(w, c)$ accepts. Intuitively this can be understood as "There is a machine that can check the answers to problems quickly".

[**NEEDS MORE**]

Deciders

NP-hard Reduction