CS3100 Final Note Sheet

Notation

$\overline{L}$, L^c	Negation of language L.
L^R	The reversal of language L.
L^*	The Kleene-Star of L.
AB	The concatenation of language A and B.
$h(L)$	A homomorphism (a function that maps every input to a unique output) of L.
$A \setminus B$	Set difference of A and B. $A-B$ is the same thing. Equivalent to $A \cap \overline{B}$.
2^A	The power-set of set A.
$\mathcal{L}(A)$	The language of machine (CFG, PDA, DFA, TM, ...) A.

$f : x_1, x_2, \ldots, x_n \to y_1, y_2, \ldots, y_n$ denotes a function f that when given $x_1, \ldots, x_n$ as inputs yields $y_1, \ldots, y_n$ as outputs.

Regular Languages

A regular language is any language that can be recognized with a DFA. Formally a DFA is a tuple $(Q, \Sigma, \delta, q_0, F)$. Where:

Q	A finite, non-empty set of states.
Σ	A finite, non-empty alphabet.
δ	A function ($\delta : Q \times \Sigma \to Q$) that maps a state, and an input in Σ to a new state.
q_0	A state in Q that DFA starts execution from.
$F \subseteq Q$	A finite, possibly empty, set of accepting states.

Alternatively, regular languages can be defined by an NFA. Formally, NFAs are the same as DFAs, except the δ function for NFAs is defined as:

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$$

Basically, the delta function can now map an input to multiple states instead of just one state.

Closures

Where R is a regular language, L is 'not regular', and ? is Unknown.

Closed:		**Unclosed:**
$\overline{R} \to R$	$h(R) \to R$	$R \cap L \to ?$
$R^* \to R$	$R \cup R \to R$	$R \cup L \to ?$
$R^R \to R$	$R \cap R \to R$	$L \cup L \to ?$
$RR \to R$	$R \setminus R \to R$	

Pumping Lemma

$$\begin{aligned} &\exists N \in \mathbb{N} : \\ &\forall w \in L : |w| \geq N \Rightarrow \\ &\exists xyz \in \Sigma^* : \quad w = xyz \\ &\qquad\qquad \wedge |xy| \leq N \\ &\qquad\qquad \wedge y \neq \varepsilon \\ &\qquad\qquad \wedge \forall i \geq 0 : xy^i z \in L \end{aligned}$$

The goals is to find some w that you can't pump y^i into, always define w in terms of N.

DFA Operations

Negation: Mark all non-final states final and all final states non-final.

Reversal: Introduce a new initial state 'q_I'. Add ε transitions from this new state to all old final states and mark these states as non-final. Reverse all arrows in the DFA, and then mark the old initial state q_0 as final.

Concatenation (AB): Add ε transitions from all of A's final states to B's initial state. Mark A's final states as non-final.

Union ($A \cup B$): Set the Q parameter of the new DFA to A's Q crossed with B's Q: $Q_{\text{new}} = A_Q \times B_Q$, Q will now contain pairs. Now change the δ function to be in terms of both A's delta function and B's delta function: $\delta_{\text{new}} = f((q_A, q_B), s) \to (\delta_A(q_A, s),\ \delta_B(q_B, s))$. That is to say, if A had state $q0_A$ that went to state $q1_A$ on input 0, and B had state $q2_B$ that went to state $q3_B$ state on input 0, then the new state $(q0_A, q2_B)$ would go to state $(q1_A, q3_B)$ on input 0.

The state that contains both DFA's initial state becomes the new initial state. New states where **either** item in the pair were final states, become final.

Intersection ($A \cap B$): Exactly the same as DFA union except only pairs of states where **both** states in the pair are final become final states.

NFA Operations

Concatenation (AB): Concatenation for an NFA is exactly like concatenation for a DFA.

Union ($A \cup B$): Introduce a new initial state q_I and add ε transitions from q_I to the initial states of A and B. Mark old initial states as no-longer initial.

Kleene-Star (A^*): First, add ε transitions from all final states to the initial state. Next, introduce a new initial state q_I, make an ε transition from this state to the old initial state, and mark this new initial state as final.

Conversions

NFA to DFA

Assume a function ε-closure(x) that when given a set of states from the NFA as input returns the set of states that can be reached from these states via ε transition.

1. The initial state of the new DFA is ε-closure($\{q_0\}$), where q_0 is the initial state of the NFA.
2. Next, form a table where the rows are the states in the DFA (to start only write the initial state) and the columns are characters in Σ. Now proceed down rows filling in the columns for each state using the following equation. If the current state is S and the column's symbol is i, the the cell S, i is:

   $$\delta(S, i) = \varepsilon\text{-closure}\Big(\bigcup_{s \in S} \delta(s, i)\Big)$$

   In English, the ε-closure of the union of all states that can be reached from this DFA state's component NFA states on input i.

   If the output state of the above function is not yet in the table, add it as a new row and process it as normal.
3. Once this process is complete, you should be able to convert the table into a standard graph by using the columns to make the transitions. **States where any component state is final, are final.**

RE to NFA

The easiest way to build and NFA from a regular expression is incrementally. Start with an atom. In the regular expression: `(a+b)*`, start with `a`. Here's how to build from there (r^n is an NFA):

a Is an NFA with two states, an initial and a final. It transitions from initial to final on input 'a'.

$r^1 + r^2$ This is just the union of NFAs r^1 and r^2.

r^1* This is just the Kleene-Star of NFA r^1.

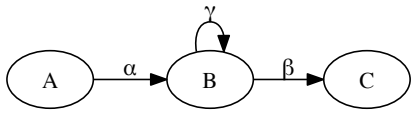
For example, the NFA for the first regular expression `(a+b)*` the union of the NFAs for `a` and `b` and then the Kleene-Star of the union. Parentheses affect order of operations just like they do in arithmetic, evaluate the inside first then the outside.

DFA to RE

To start, introduce a new initial state q_I, and make a ε transition from it, to the initial state of the DFA. Next, make a new final state q_F, make ε transitions from the DFA's final states to q_F, then mark all old final states as non-final.

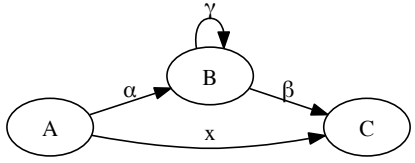
Now, the general idea, is to delete a node, and then 'fill in the gaps' with a rule. Once all nodes except the new q_I and q_F nodes have been deleted, the only edge will be between q_I and q_F, that edge will be the regular expression.

The algorithm goes like this, delete some node S, traverse all paths that go through node S, and replace them with a single edge from the origin to the destination. The rules for what the edge is called are as follows:



$\alpha\gamma^*\beta$

Which means, if we were to delete state B above, the our new edge from A to C would be $\alpha\gamma^*\beta$. Note, if γ is empty (there's no self loop) then it's just $\alpha\beta$.



$x + (\alpha\gamma^*\beta)$

Which means that if we delete state B above, and an edge from A to C exists already, then the new edge is the union of both the general equation above, and the existing edge.

DFA Minimization

Table Conversion Algorithm

Draw a table with all states of the DFA on both the x and the y axis. Consider all input strings of length $0..n$. Start from a state s on the x axis, if an input string of length n reaches a state that is final, and a state on the y axis given the same input does not reach a final state (or vice versa), then we know that those states are not equal, and their cell can be marked off. Once a round of this goes by without any cells being marked, then we know that the states corresponding to cells that are not marked are equivalent.

Brzozowski's Algorithm

This algorithm is quite simple. The crux of it is that an NFA to DFA conversion naturally results in a minimization of the NFA. So the algorithm is as-follows: Take a DFA, reverse it to get an NFA, convert that NFA to a DFA again, take this new reversed DFA and reverse it to get the original language back, then convert the NFA resulting from the reversal into a DFA.

Context Free Languages

A CFL can be represented in two ways, with a Context Free Grammar (CFG) and with a Pushdown Automata (PDA). PDA's can be both deterministic and non-deterministic, but unlike DFAs, Deterministic PDAs cannot represent the full set of CFLs. However, any language that can be expressed as a DPDA is not inherently ambiguous.

The Formalism for PDAs is $(Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$. It's basically the same as it is for DFAs with three changes. PDAs have a stack alphabet Γ which Σ is often a subset of, but not always. z_0 is the initial contents of the stack. The delta function is also changed to:

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$$

PDAs accept when either the stack becomes empty, or they reach a state in F.

Closures

Where C is a context-free language, R is a regular language, and ? is an Unknown language.

Closed:		Unclosed:
$C^R \to C$	$C \cup C \to C$	$\overline{C} \to ?$
$C^* \to C$	$C \cap R \to C$	$C \cap C \to ?$
$CC \to C$	$C \cup R \to R$	$C \setminus C \to ?$
$h(C) \to C$		

Pumping Lemma

$$\begin{aligned} \exists N \in \mathbb{N} : \\ \forall w \in L : |w| \geq N \Rightarrow \\ \exists uvxyz \in \Sigma^* : \quad & w = uvxyz \\ & \wedge |vy| > 0 \\ & \wedge |vxy| \leq N \\ & \wedge \forall i \geq 0 : uv^i xy^i z \in L \end{aligned}$$

CFG to PDA Conversion

1. Create an initial state q_0 with an initial value z_0 on the stack.
2. Create a second state q_M. Add a transition when z_0 is on the top of the stack from q_0 to q_M. Push $z_0 S$ onto the stack in this transition.
3. Create a third state q_F, mark this state as final. Add a transition from q_M to q_F when z_0 is on the top of the stack.
4. Now, populate q_M.
   - Add a transition from q_M to q_M when there is a terminal on top of the stack that matches the current input character.
   - When there exists a rule $X \to R$ where R can be a *collection* of both terminals and non-terminals, add a transition from q_M to q_M when X is on top of the stack that pushes R onto the stack.

RL and LL Grammar to NFA Conversion

CFGs where every production only contains a single non-terminal on the right (RL), or CFGs where every production only contains a single non-terminal on the left (LL) can be directly converted to NFAs. We'll with RL grammars. If a grammar is right linear, then we

know all the rules will be in the form $A \to tB$, or $A \to t$ where t is a terminal. The states of the NFA will be all the non-terminals. A rule like $A \to tB$ means that a transition from state A to state B on input t exists. A rule in the form $A \to t$ means that A goes to a new final state on input t.

To convert LL grammars to NFAs, we can convert them to RL grammars and then to NFAs. A RL grammar can be obtained from a LL grammar by reversing it (reversing the order of all productions. Bt becomes tB). This reversed RL can be converted to an NFA using the algorithm above. We can then apply the NFA reversal algorithm to obtain an NFA for the original language.

Cocke-Kasami-Younger (CKY) Parsing

CKY is an algorithm for checking if a string is accepted by a given CFG. For the CKY algorithm to be used, the CFG must be in Chomsky Normal form (All productions in the form $S \to AB$, and $A \to t$).

The CKY algorithm helps you to build a binary parse-tree backwards. To start, write out the characters of the string you're going to parse, and the non-terminals that map to them above the characters. Then, in-between each non-terminal figure out what production would generate those non-terminals. For example if you had a rule $S \to AB$, then you would write S in-between non-terminal's A and B. This should build a triangle shape. If the point of the triangle is S, then the string is accepted, if it's not S, then it didn't accept.

Ambiguous Context Free Languages

These are languages that have two separate parse-trees. To prove that a language is ambiguous, show that it actually has two separate parse-trees.

Example ambiguous grammar:

$$E \ \to \ E + E \mid E * E \mid \texttt{NUMBER}$$

Consistency and Completeness

Consistency: All strings generated by a grammar are in the language.

Completeness: The grammar generates all strings in the language.

You cannot know that a grammar defines a language until you show both. For example, if we want to define the language $\{a^n b^n | n \in \mathbb{N}\}$, the grammar:

$$S \ \to \ aabb$$

Is *consistent* because it only generates strings in the language, but not complete because it doesn't generate all strings in the language. Likewise, the grammar:

$$S \ \to \ aS \mid bS \mid \varepsilon \quad (\textit{grammar for } \{a, b\}^*)$$

Is complete, it generates all possible strings in the language, but not consistent because it generates many strings that are, in-fact, outside of the language.

Cardinality

Diagonalization

Diagonalization is a general method for showing that the cardinality of two infinite sets is not the same. So say that we have two infinite sets A and B, in this particular case, lets consider $B = 2^A$. Now, by definition B can be expressed as a set of infinite sets. We can begin writing out these sets, $s_1, s_2, \ldots, s_n$. Diagonalization tells us that no matter how many of these sets we make, there will always exists another set s that has not yet been expressed. This set s is constructed by going through the list of created sets $s_1, s_2, \ldots s_n$ and putting an element in s that does not exists in each set, this way, it cannot ever be equal to another set because it contains at least one element that is guaranteed to not be in that set.

Note, this does not apply for non-infinite mappings. A finite number of infinite sets can be mapped to a single infinite set using a pairing function.

Schröder-Bernstein Theorem

The Schröder-Bernstein theorem states that, for any two sets A and B if there exists an *injective* function from $A \to B$, and there exits an injective function from $B \to A$, then $|A| = |B|$. Note that the injective function doesn't require every item of A to map to every item of B, only that every item of A maps to *an* item of B (and vice versa).

Turing Machines

Formaly a Turing machine is represented as tuple in the form $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$. Q, Σ, q_0, and F are the same for PDAs and DFAs. Γ is the tape alphabet, and since the input string is presented on the tape, $\Sigma \subset \Gamma$ is always the case. B is the blank character. The δ is defined like so:

$$\delta : Q \times \Gamma \ \to \ Q \times \Gamma \times \{L, R\}$$

Which means the delta functions maps pairs of states an inputs, to a new state, a value to replace the current value on the tape, and *movement*. L moves the head left on the taps, and R moves the head right on the tape. Since 'staying-put' can be trivially simulated S is sometimes included as a movement as well. TMs can also be non-deterministic, in that case the δ function changes to:

$$\delta : Q \times \Gamma \ \to \ 2^{Q \times \Gamma \times \{L, R\}}$$

Which allows it to be in multiple states at the same time, since it can transition to multiple states with a single input.

Terminology and Notation

$\langle M \rangle$	String representation of Turing machine M.
halting	When a machine stops execution.
acceptance	When a machine halts in a final state.
rejection	When a machine halts and is not in a final state.
decider	A decider is a Turing machine that defines a language of Turing machines that conform to a yes or no question.

Decidability

Recognizable and Enumerable

Turing Recognizable (TR): A language L where there exists some Turing machine M that can recognize it.

Recursively Enumerable (RE): A language L where all strings in the language can be enumerated by some machine M.

Every language that is Turing Recognizable is also Recursively Enumerable since an enumerator can be built from a recognizer, and a recognizer can be built using an enumerator.

Halting Problem

The halting problem states building a Turing machine P that can detect whether any other Turing machine will halt is impossible. The proof is as follows:

Assume that we have a Turing machine P, that when given a Turing machine M and string w as input ($\langle M, w\rangle$), P will (in a finite computation time) accept in the case that M halts on input w, or reject in the case that M loops on input w. We can then define a new Turing machine Q that takes a single Turing machine M as input. Q will then ask P whether machine M halts when given itself as input (does $P(\langle M, M\rangle)$ halt?). If P accepts (says that M halts) the Q will loop. If P rejects (says M will loop) then Q halts. Now, we can supply Q as input to machine Q. Q will then run $P(\langle Q, Q\rangle)$. If P accepts, then Q will begin to loop, but P said that Q would halt. This is a contraction, a general P decider for the halting problem cannot exist.

This same proof style as above can be used to perform a proof for any decider from first principals. Just assume that the decider P exists, generate a machine Q that takes a machine as input, the runs that decider on the machine given itself as input $P(\langle M, M\rangle)$. Make machine Q return the opposite of whatever P says, and then pass Q to Q. P will never be able to decide it.

Mapping Reduction

A mapping reduction between $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$ is a function $f : \Sigma^* \to \Sigma^*$ if $\forall x \in \Sigma^*, x \in A \Leftrightarrow f(x) \in B$. More plainly, a function f such that I can pick any x in A, and $f(x)$ will also be in B. The "mapping reduction from A to B" is typically denoted as $A \leq_m B$. A mapping reduction in polynomial time is denoted $A \leq_p B$.

The general steps for a mapping reduction $A \leq_m B$ are as follows:

1. A is designated the "known undecidable" language.
2. B is designated the "unknown" language.
3. Create a function f that maps all elements of A into B.

To form f you usually assume the decider for B (D_B), then you construct a machine M that uses to decider D_B to become a decider for A (D_A). For example, we can map A_{TM} onto $Halt_{TM}$ using the following method:

Assume that a decider R for A_{TM} exists. We will now construct a decider S for $Halt_{TM}$ from R. S has two inputs a machine M and an input string w. First, S will run decider R on $\langle M, w\rangle$, if R accepts, then S accepts. If R rejects, then S accepts. We now have a decider for $Halt_{TM}$ which is undecidable, a decider for A_{TM} cannot exist.

Rice's Theorem

"Every non-trivial partitioning of the space of Turing machine codes based on the languages recognized by these Turing machines is undecidable."

More formally, given a property $\mathcal{P}$, where $\mathcal{P}$ is non-trivial (not $\emptyset$ or Σ^*) the language below is undecidable.

$$\{\langle M\rangle \mid M \text{ is a Turing machine and } \mathcal{P}(\mathcal{L}(M))\}$$

Post's Correspondence Problem

Post's Correspondence Problem (PCP) is as follows. Say we have a finite set of tuples in the form: $\{\langle x, y\rangle \mid x, y \in \Sigma^*\}$. We can think of these tuples like a set of blocks with a top half and a bottom half, where the x item is the top half and the y item is the bottom half. Now to solve the PCP, find an arrangement (with possible repetitions) where the string formed by the top blocks read left to right, is the same as the string formed by the bottom blocks read left to right.

Enumeration

The set of solvable instances of this problem can be generated quite easily. Make a machine that constantly generates two strings w_x and w_y. A split of w_x (or w_y) is a set of strings that when re-ordered and concatenated forms w_x (or w_y). Now, make a set of all possible split of w_x (S_{w_x}), and a set of splits for w_y (S_{w_y}). Then, cross every split in S_{w_x} with every split in S_{w_y}. Continue doing this for all possible strings $w_x \in \Sigma^*$, $w_y \in \Sigma^*$ where $|w_x| = |w_y|$.

Decidability

In the case of $|\Sigma| > 2$, the PCP is undecidable. This can be shown via a mapping reduction that maps PCP into A_{TM}. The proof is rather involved, but at a high-level, possible states of a Turing machine are expressed as PCP pairs. A solution to the PCP problem containing these pairs is the *execution history* of the TM that accepted the string.

NP-Completeness

Problem Classes

P: The set of problems that can be solved in polynomial time. Contained in NP.

NP: The set of problems that can be solved in non-deterministic polynomial time.

NP-hard: The set of problems that can be polynomial time reduced to every other problem in NP.

NP-complete: The set of problems that are in both NP, and NP-hard.

Proving NP-Completeness

There are two steps to proving that a languages is in NP-complete. First you have to show that it is NP, and then you have to show that it is in NP-hard.

Verifiers

One way to show that a problem is in NP is by using a verifier. A verifier is a Turing machine V_L such that for all $w \in \Sigma^*$, their exists some c such that $w \in L$ when $V_L(w, c)$ accepts. Intuitively this can be understood as "There is a machine that can check the answers to problems quickly".

To give a concrete example, consider k-Clique. A verifier for k-Clique assumes that w is some graph, and c is a set of nodes in graph w that forms a k-Clique. A machine that checks whether or not the set of nodes c actually forms a k-Clique is trivial and can easily be performed in a finite amount of time.

NP-hard Reduction

The easiest way to show that a problem is in NP-hard, is by reducing an existing NP-hard problem to the problem you want to prove is NP-hard using a polynomial mapping reduction. The classic problem to reduce from is 3-SAT.

Chomsky Hierarchy

Recursively Enumerable $\leftarrow$ Context-Sensitive $\leftarrow$ Context-Free $\leftarrow$ Regular.