1) Probar que, para $a \in \,]0,1[$, se tiene:

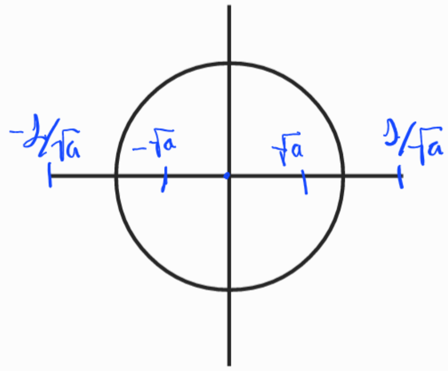
$$\int_0^{2\pi} \frac{\cos^2(3t)\,dt}{1+a^2-2a\cos(2t)} = \pi\,\frac{a^2-a+1}{1-a}$$

$$1+a^2-2a\cos(2t) = |1-ae^{2it}|^2 = (1-ae^{2it})(1-ae^{-2it})$$

$$\cos^2(3t) = \mathrm{Re}\left(\frac{1+e^{6it}}{2}\right)$$

Usando $\sigma: [0,2\pi] \to \mathbb{C}$, $t \to e^{it}$ $\Rightarrow \int_\sigma f(z)\,dz = \int_0^{2\pi} f(e^{it})\cdot ie^{it}\,dt$

$$f(z) = \frac{\frac{1+z^6}{2}}{(1-az^2)(1-az^{-2})}\cdot\frac{1}{z}$$ con polos $\begin{cases} 0 \\ \pm 1/\sqrt{a} \\ \pm\sqrt{a} \end{cases}$



Aplicamos el Teorema de los residuos:

$A = \langle \pm\sqrt{a}, 0, \pm 1/\sqrt{a} \rangle$ $\quad f \in H(\Omega \setminus A)$

$\Omega = \mathbb{C}$ $\quad \Gamma = C(0,1)$ nul-homólogo con respecto a $\mathbb{C}$ ($\mathbb{C}$ es $H\mathbb{C}$)

$A \cap \Omega = \emptyset$ $\quad \Gamma \subset \Omega \setminus A$

$$\Rightarrow \int_{C(0,1)} f(z)\,dz = 2\pi i\left[\mathrm{Res}(f,\sqrt{a}) + \mathrm{Res}(f,-\sqrt{a}) + \mathrm{Res}(f,0)\right]$$

$$\lim_{z\to} z f(z) = \lim_{z\to 0} \frac{(1+z^6)z}{2(1-az^2)(1-\frac{a}{z^2})z}\cdot\frac{z}{z} = \lim_{z\to 0}\frac{(1+z^6)z^2}{2(1-az^2)(z^2-a)} = 0 \Rightarrow \mathrm{Res}(f,0) = 0$$

$$\operatorname{Res}(f,\sqrt{a}) = \lim_{z\to\sqrt{a}} (z-\sqrt{a})\, f(z) = \lim_{z\to\sqrt{a}} \frac{\cancel{(z-\sqrt{a})}(1+z^6)\, z}{2(1-az^2)(z+\sqrt{a})\cancel{(z-\sqrt{a})}} = \frac{(1+a^3)\cancel{\sqrt{a}}}{2(1-a^2)\,2\cancel{\sqrt{a}}}$$

$$= \frac{(1+a^3)}{4(1-a^2)} = \frac{(a^3+1)}{-4(a^2-1)} = \frac{\cancel{(a+1)}(a^2-a+1)}{-4\cancel{(a+1)}(a-1)} = \frac{(a^2-a+1)}{4(1-a)}$$

$$\operatorname{Res}(f,-\sqrt{a}) = \lim_{z\to-\sqrt{a}} (z+\sqrt{a})\, f(z) = \frac{\cancel{(z+\sqrt{a})}(1+z^6)\, z}{2(1-az^2)\cancel{(z+\sqrt{a})}(z-\sqrt{a})} = \frac{-(1+a^3)\cancel{\sqrt{a}}}{-2(1-a^2)\,2\cancel{\sqrt{a}}} = \frac{(1+a^3)}{4(1-a^2)}$$

$$= \frac{(a^2-a+1)}{4(1-a)}$$

$$\Longrightarrow \int_{C(0,1)} f(z)\,dz = \cancel{2}\pi i \left(\cancel{2}\, \frac{(a^2-a+1)}{\cancel{4}(1-a)} \right) = \pi i \,\frac{(a^2-a+1)}{(1-a)}$$

Usando $$\int_{C(0,1)} f(z)\,dz = \int_0^{2\pi} f(e^{it})\, i e^{it}\,dt = i \int_0^{2\pi} \frac{\cos^2(3t)}{1+a^2-2a\cos t}\,dt$$

$$\operatorname{Re} \int_0^{2\pi} f(e^{it})\, e^{it}\,dt = \int_0^{2\pi} \operatorname{Re}\left(f(e^{it})\, e^{it}\right)dt = \int_0^{2\pi} \frac{\cos^2(3t)}{1+a^2-2a\cos t}\,dt$$

$$\Longrightarrow \int_0^{2\pi} \frac{\cos^2(3t)}{1+a^2-2a\cos t}\,dt = \pi\, \frac{(a^2-a+1)}{(1-a)}$$

Nota: $i\int_0^{2\pi} f(e^{it})\, e^{it}\,dt = i(\operatorname{Re} + \operatorname{Im}) = \pi i\, \dfrac{(a^2-a+1)}{(1-a)} \Longrightarrow \operatorname{Im} = 0$

② Probar que, para $n \in \mathbb{N}$, se tiene:

$$\int_0^{2\pi} \frac{(1+2\cos t)^n \cos(nt)}{3+2\cos t} = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

$3+2\cos t = 1+2+2\cos t =$

③ Probar que, para $n \in \mathbb{N}$, se tiene:

$$\int_0^{2\pi} e^{\cos(t)} \cos(nt - \operatorname{sen} t)\, dt = \frac{2\pi}{n!}$$

$$e^{i(nt - \operatorname{sen} t)} = \cos(nt - \operatorname{sen} t) + i \operatorname{sen}(nt - \operatorname{sen} t)$$

$$\operatorname{Re}\left(e^{\cos t + i(nt - \operatorname{sen} t)}\right) = e^{\cos(t)} \cos(nt - \operatorname{sen} t)$$

$$e^{it} = \cos(t) + i \operatorname{sen} t \Rightarrow e^{e^{it}} = e^{\cos(t)} e^{i \operatorname{sen} t} \Rightarrow e^{e^{-it}} = e^{\cos(t) - i \operatorname{sen} t}$$

$$= e^{\cos t}(\cos(\operatorname{sen}(t)) - i \operatorname{sen}(\cos t))$$

$$\cos(nt - \operatorname{sen}(t)) + i \operatorname{sen}(nt - \operatorname{sen} t) = e^{i(nt - \operatorname{sen} t)} = (e^{it})^n e^{-i \operatorname{sen} t}$$

$$\Rightarrow e^{\cos(t)}\left[\cos(nt - \operatorname{sen}(t)) + i \operatorname{sen}(nt - \operatorname{sen} t)\right] = e^{\cos t} e^{-i \operatorname{sen} t} (e^{it})^n = e^{int}(e^{\cos t - \operatorname{sen} t})$$

$$= (e^{it})^n e^{e^{-it}}$$

Si definimos $f(z)$ como $z^n e^{1/z} \cdot \frac{1}{z}$

$$\Rightarrow f(e^{it})\, i e^{it} = i(e^{it})^n e^{e^{-it}} \cdot \frac{1}{\cancel{e^{it}}} \cdot \cancel{e^{it}} = i(e^{it})^n e^{e^{-it}}$$

$$\Rightarrow \int_0^{2\pi} e^{\cos(t)} \cos(nt - \operatorname{sen} t)\, dt = \operatorname{Im} \int i(e^{it})^n e^{e^{-it}} dt = \operatorname{Re} \int e^{it} e^{e^{-it}} dt =$$

$$= \operatorname{Res}(f, 0) = \operatorname{Im} \int_{C(0,1)} f(z)\, dz$$

$f(z) = z^{n-1} e^{1/z}$ tiene una singularidad esencial en 0.

$\Rightarrow$ Tenemos que calcular el coeficiente c_{-1} del desarrollo de Laurent en 0 de f

El desarrollo en serie de Laurent de

$$e^{1/z} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!\, z^k} \quad \forall z \in \mathbb{C}^*$$

$$\Longrightarrow z^{n-1} e^{1/z} = z^{n-1} + \sum_{k=1}^{\infty} \frac{1}{k!\, z^k} z^{n-1} = z^{n-1} + \sum_{k=1}^{\infty} \frac{1}{k!} z^{n-k-1}$$

$\Longrightarrow C_{-1} = \frac{1}{n!}$ (Es el coeficiente que acompaña a $z^{-1} = z^{n-k-1} \Longrightarrow n = k$)

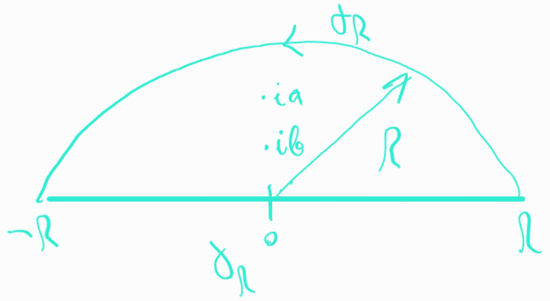
$$\Longrightarrow \int_{C(0,1)} f(z)\, dz = \frac{2\pi i}{n!} = i \operatorname{Re} \int_0^{2\pi} e^{it} e^{e^{-it}}\, dt$$

$$\Longrightarrow \int_0^{2\pi} e^{\cos(t)} \cos(nt - \operatorname{sen} t)\, dt = \frac{2\pi}{n!} \quad \blacksquare$$

④ Prueba que, para cualesquiera $a, b \in \mathbb{R}^+$, se tiene:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$



$\Gamma = \gamma_R + \sigma_R$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \sum_{a\in A} \mathrm{Ind}_{\Gamma_R}(a)\,\mathrm{Res}(f(z),a)$$

$$||$$

$$\int_{\gamma_R} f(z) + \int_{\sigma_R} f(z)\,dz$$

$\int_{\gamma_R} f(z)$ $\downarrow$ Integral que queremos

$\int_{\sigma_R} f(z)\,dz \xrightarrow{R\to\infty} 0$

IDEA GENERAL

Solución: Tomamos $\Omega = \mathbb{C}$ $\quad f: \mathbb{C}\setminus A \to \mathbb{C}$ $\quad f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)^2}$ $\quad f \in H(\Omega\setminus A)$

$A = \{\pm ia, \pm ib\}$

$$\boxed{\begin{array}{l} f \in H(\Omega)\setminus A \\ A' \cap \Omega = \emptyset \\ \Gamma_R \text{ es nulhomólogo} \\ \text{con respecto a } \Omega \\ \Gamma_R^* \subset \Omega\setminus A \end{array}}$$

Para cada $R > \max\{|a|, |b|\}$ definimos el ciclo

$\Gamma_R = \gamma_R + \sigma_R$ $\quad \gamma_R [-R, R] \to \quad \gamma_R(x) = x$

$\sigma_R : [0, \pi] \to \mathbb{C}$ $\quad \sigma_R(t) = Re^{it}$

Suponemos $a \neq b$

$\Downarrow$

Teorema de los residuos

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \Big(\underbrace{\boxed{\operatorname{Ind}_{\Gamma_R}(ia)}}_{=1} \operatorname{Res}(f(z), ia) + \underbrace{\boxed{\operatorname{Ind}_{\Gamma_R}(ib)}}_{=1} \operatorname{Res}(f(z), ib) \Big)$$ No depende de R

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \left[\operatorname{Res}(f(z), ia) + \operatorname{Res}(f(z), ib) \right]$$

$\displaystyle\int_{\gamma_R} f(z)\,dz + \int_{\sigma_R} f(z)\,dz$ Tomamos límite con $R \to \infty$ y obtenemos

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz + \lim_{R\to\infty} \int_{\sigma_R} f(z)\,dz = 2\pi i \left[\operatorname{Res}(f(z), ia) + \operatorname{Res}(f(z), ib) \right] \quad (1)$$

$$\lim_{R\to\infty} \int_{\gamma_R} f(z) = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\left| \int_{\sigma_R} f(z)\,dz \right| \leq \underbrace{l(\sigma_R)}_{=\pi R} \cdot \sup\{ |f(z)|, z \in \sigma_R^* \} \leq \frac{\pi R}{(R^2-a^2)(R^2-b^2)^2} \xrightarrow{R\to\infty} 0$$

Si $z \in \gamma_R^* \Rightarrow |z| = R$

$$\left.\begin{array}{l} |z^2+a^2| \geq |z|^2 - |a|^2 = R^2 - a^2 > 0 \\ |z^2+b^2|^2 \geq (|z|^2 - |b|^2)^2 = (R^2-b^2)^2 \end{array}\right\} \Rightarrow \sup \leq \frac{1}{(R^2-|a|^2)(R^2-|b|^2)^2}$$

Todo lo anterior nos dice que

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\operatorname{Res}(f(x), ia) + \operatorname{Res}(f(x), ib)\right)$$

Calculamos los residuos

$$\lim_{(z\to ia)} (z-ia) f(z) = \lim_{z\to ia} \frac{\cancel{(z-ia)}}{(z+ia)\cancel{(z-ia)}(z^2+b^2)^2} = \frac{1}{(2ia)(b^2-a^2)^2}$$

f tiene un polo de orden 1 en ia y el $\operatorname{Res}(f(z), ia) = \frac{1}{(2ia)(b^2-a^2)^2}$

$$\lim_{z\to ib} (z-ib)^2 f(z) \neq 0$$

$$\operatorname{Res}(f(z), ib) = \frac{1}{(2-1)!} \lim_{z\to ib} \frac{d^{2-1}}{dz^{2-1}} \left((z-ib)^2 f(z)\right)$$

$$(z-ib)^2 \frac{1}{(z^2+a^2)(z^2+b^2)^2} = \frac{\cancel{(z-ib)^2}}{(z+ib)^2\cancel{(z-ib)^2}(z^2+a^2)}$$

$$\lim_{z\to ib} \left(\frac{1}{(z^2+a^2)(z+ib)^2}\right)' = \frac{-2z(z+ib)^{\cancel{2}} - 2(z^2+a^2)\cancel{(z+ib)}}{(z^2+a^2)^2(z+ib)^{\cancel{4}3}} = \frac{-2\left(ib(2ib)+(a^2-b^2)\right)}{8(a^2-b^2)^2(ib)^3} =$$

$$= \frac{(a^2-3b^2)}{4(b^2-a^2)^2 b^3 i}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{2}\pi\cancel{i}\left(\frac{1}{(\cancel{2}\cancel{i}a)(b^2-a^2)^2} + \frac{(a^2-3b^2)}{2\cancel{4}(b^2-a^2)^2 b^3 \cancel{i}}\right) = \pi\left(\frac{1}{a(b^2-a^2)^2} + \frac{(a^2-3b^2)}{(b^2-a^2)b^3}\right)$$

5) Probar que, para $a \in \mathbb{R}^+$, se tiene:

$$\int_{-\infty}^{+\infty} \frac{x^6\,dx}{(x^4+a^4)^2} = \frac{3\pi\sqrt{2}}{8a}$$

Paso 1: Analizamos los polos de $\dfrac{x^6}{(x^4+a^4)^2}$

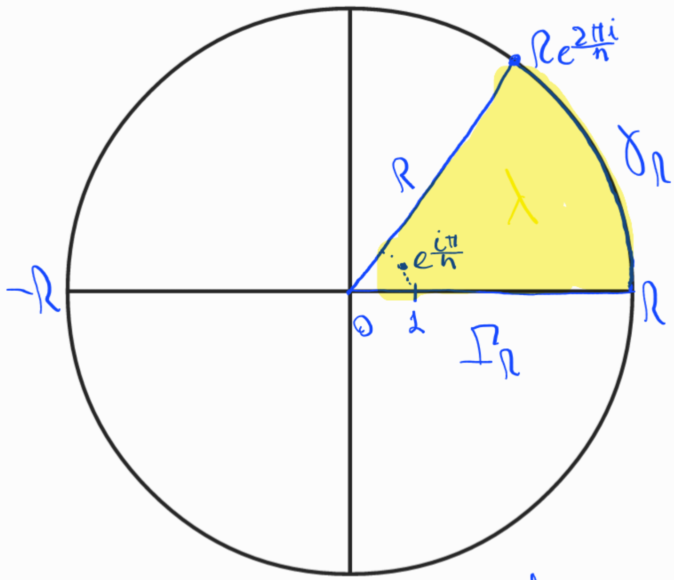
$$x^4 + a^4 = 0 \iff x^4 = -a^4 \iff x \begin{cases} \frac{\sqrt{2}}{2}(1+i)\,a \\ \frac{\sqrt{2}}{2}(1-i)\,a \\ \frac{\sqrt{2}}{2}(-1+i)\,a \\ \frac{\sqrt{2}}{2}(-1-i)\,a \end{cases}$$

6. Dado $n \in \mathbb{N}$ con $n > 1$, integrar una conveniente función sobre un camino cerrado que recorra la frontera del sector $D(0,R) \cap \{z \in \mathbb{C}^* : 0 < \arg z < 2\pi/n\}$ con $R \in \mathbb{R}^+$ para probar que:

$$\int_0^{+\infty} \frac{dx}{1+x^n} = \frac{\pi}{n} \operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Si llamamos λ a $D(0,R) \cap \{z \in \mathbb{C}^* : 0 < \arg z < 2\pi/n\}$ con $R \in \mathbb{R}^+$

Queremos integrar una cierta $f = ?$ en Γ_R



$z^n + 1 = 0 \iff z^n = -1 \iff z \in \{ e^{\frac{i\pi}{n} + \frac{2k\pi i}{n}} ; k = 0, \dots, n-1 \} = A$

Como $0 < \arg(z) < \frac{2\pi}{n}$ Γ solo rodea a un punto de A.

Aplicamos el Teorema de los residuos a

$\Omega = \mathbb{C}$

$A' \cap \Omega = \emptyset$

$f(z) = \frac{1}{1+z^n} \in \mathcal{H}(\Omega \setminus A)$

$\Gamma_R = [0,R] + \gamma_R + [Re^{\frac{2\pi i}{n}}, 0]$

Nulhomólogo en $\mathbb{C}$, $\Gamma^* \subset \mathbb{C} \setminus A$

$\Longrightarrow \begin{cases} \displaystyle\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}\left(f(z), e^{\frac{i\pi}{n}}\right) = \\ \displaystyle = \int_0^R f(z)\,dz + \int_{\gamma_R} f(z)\,dz - \int_{[0,Re^{\frac{2\pi i}{n}}]} f(z)\,dz = \end{cases}$

Tomando límite cuando $R \to \infty$

$$I = \int_0^{+\infty} \frac{1}{1+x^n} dx = \int_0^{+\infty} \frac{1}{1+z^n} dz = 2\pi i \operatorname{Res}(f, e^{\frac{\pi i}{n}}) + \lim_{R\to\infty} \left(\int_{[0, Re^{\frac{2\pi i}{n}}]} f(z)\,dz - \int_{\gamma_R} f(z)\,dz \right)$$

Analicemos por partes

$$\cdot \int_{\gamma_R} f(z)\,dz = \int_0^{2\pi/n} \frac{iRe^{it}}{1+(Re^{it})^n} dt \Rightarrow \left| \int_0^{2\pi/n} \frac{iRe^{it}}{1+(Re^{it})^n} \right| \leq 2\pi/n \, R \cdot \frac{1}{R^{n-1}} = \frac{2\pi}{n} \cdot \frac{1}{R^{n-2}}$$

$$\lim_{R\to\infty} \downarrow \text{ Porque } n > 2$$

$$0$$

$\gamma_R : [0, 2\pi/n] \to \mathbb{C}$

$t \to Re^{it}$

$\gamma_R'(t) = iRe^{it}$

$$\sup \left\{ \left| \frac{iRe^{it}}{1+(Re^{it})^n} \right| \right\} \leq \frac{R}{R^n} = \frac{1}{R^{n-1}}$$

$$\lim_{R\to\infty} \cdot \int_{\gamma_R} f(z)\,dz = 0$$

$$\cdot \int_{[0, Re^{2\pi i/n}]} f(z)\,dz = \int_0^R \frac{e^{2\pi i/n}}{1+(te^{2\pi i/n})^n} = \int_0^R \frac{e^{2\pi i/n}}{1+t^n} = e^{2\pi i/n} \underbrace{\int_0^R \frac{1}{1+t^n} dt}_{I}$$

$\varphi [0, R] \to \mathbb{C}$

$t \to te^{2\pi i/n}$

$\varphi'(t) = e^{2\pi i/n}$

$$\operatorname{Res}(f, e^{i\pi/n}) = \lim_{z\to e^{i\pi/n}} (z - e^{i\pi/n}) \frac{1}{1+z^n} = \frac{0}{0} \overset{L'H}{=} \lim_{z \to e^{i\pi/n}} \frac{1}{nz^{n-1}} =$$

$$= \frac{1}{ne^{\pi i} \cdot e^{-i\pi/n}} = \frac{-e^{i\pi/n}}{n}$$

$$\left(1-\frac{e^{2\pi i}}{n}\right)\int_0^\infty \frac{1}{1+z^6} = -\frac{e^{\pi i}}{n}2\pi i$$

$$\left(1-\cos(2\pi/n)-i\,\text{sen}(2\pi/n)\right)\int_0^\infty \frac{1}{1+x^6} = \frac{-2\pi i}{n}\left(\cos\left(\frac{\pi}{n}\right)+i\,\text{sen}(\pi/n)\right)$$

$$(1-\cos(2\pi/n))\,I = 2\pi\,\frac{\text{sen}(\pi/n)}{n}$$ Debe salir algo equivalente

$$\text{sen}(2\pi/n)\,I = \frac{2\pi}{n}\cos\left(\frac{\pi}{n}\right)$$

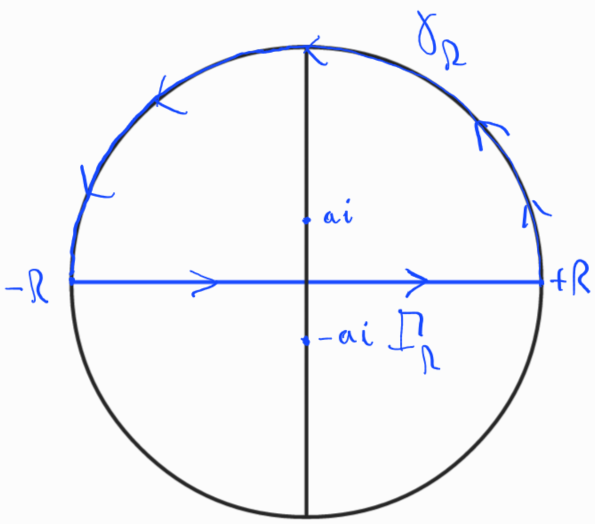
$$\cancel{2\cos(\pi/n)}\,\text{sen}\left(\frac{\pi}{n}\right) I = \frac{\pi}{n}\,\cancel{2\cos(\pi/n)} \Longrightarrow \boxed{I = \frac{\pi}{n}\cdot \text{cosec}\,\frac{\pi}{n}}$$

7 Probar que, para $a,t\in\mathbb{R}^+$, se tiene:

$$\int_{-\infty}^{+\infty}\frac{\cos(tx)\,dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}(1+at)e^{-at}$$

$$e^{itx} = \cos(tx)+i\,\text{sen}(tx) \Rightarrow \text{Re}(e^{itx}) = \cos(tx)$$

Sea $f(z) = \dfrac{e^{itz}}{(z^2+a^2)^2}$ los polos de $f(z)$ son $\{z\in\mathbb{C}: z^2=-a^2\} = \{\pm ai\} = A$



$\forall R>|a|$ Γ no se traga ninguna singularidad nueva.

Aplicamos el teorema de los residuos

$\Omega = \mathbb{C}$

$\Omega \cap A' = \phi$

Γ nulhomólogo en Ω

$\Gamma^* \subset \Omega \setminus A$

$f \in H(\Omega \setminus A)$

$\Gamma = \Gamma_R + \gamma_R$

$$\int_\Gamma f(z)\,dz = 2\pi i \, \mathrm{Ind}_\Gamma(ai)\, \mathrm{Res}(f, ai) = 2\pi i \,\mathrm{Res}(f, ai)$$

$$= \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$\lim_{R\to} \int_\Gamma f(z)\,dz = \lim_{R\to\infty} \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$\cdot \int_\gamma f(z)\,dz = \int_0^\pi \frac{e^{itRe^{i}}\, iRe^{iy}}{((Re^{iy})^2+a^2)^2}\,dy \Rightarrow \left| \int_0^\pi \frac{e^{itRe^{it}}\, iRe^{iy}}{((Re^{iy})^2+a^2)^2} \right| \leq \pi R \frac{R}{(R^2-a^2)^2} \xrightarrow{R\to\infty} 0$$

$\gamma\, [0,\pi] \to \mathbb{C}$

$y \to Re^{iy}$

$\gamma'(t) = iRe^{iy}$

$$\sup \left\{ \left| \frac{e^{itRe^{it}}\, iRe^{iy}}{((Re^{iy})^2+a^2)^2} \right| \right\} \leq \frac{R}{(R^2-a^2)^2}$$

Por lo tanto $\int_{-\infty}^{+\infty} f(z)\,dz = 2\pi i \,\mathrm{Res}(f, ai)$

Calculamos ahora $\mathrm{Res}(f, ai)$. Sabemos que f tiene un polo de orden 2

$$\Rightarrow \mathrm{Res}(f, ai) = \lim_{z\to ai} \frac{d}{dz} (z-ai)^2 \frac{e^{itz}}{(z^2+a^2)^2} = \lim_{z\to ai} \frac{d}{dz} \frac{e^{itz}}{(z+ai)^2}$$

$$= \lim_{z\to ai} \frac{it\,e^{itz}(z+ai)^2 - e^{itz}(2(z+ai))}{(z+ai)^4}$$

$$= \lim_{z \to ai} e^{itz} \left(\frac{it\,\cancel{(z+ai)^2}}{(z+ai)^{4+2}} - \frac{2\,\cancel{(z+ai)}}{(z+ai)^{4+3}} \right) = e^{-at} \frac{it}{-4a^2} - \frac{i}{4a^3}$$

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{(x^2+a^2)^2} = 2\pi i\, e^{-at} \left(\frac{-ita}{4a^3} - \frac{i}{4a^3} \right) = \frac{\pi}{2a^3} (1+at) e^{-at}$$

$$\Rightarrow \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{itx}}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} \frac{\cos(tx)}{(x^2+a^2)^2} dt = \frac{\pi}{2a^3} (1+at) e^{-at}$$

8 Probar que: $\displaystyle\int_{-\infty}^{\infty} \frac{x \operatorname{sen}(\pi x)}{x^2-5x+6} dx = -5\pi$

Para resolver el ejercicio usaremos lo siguiente:

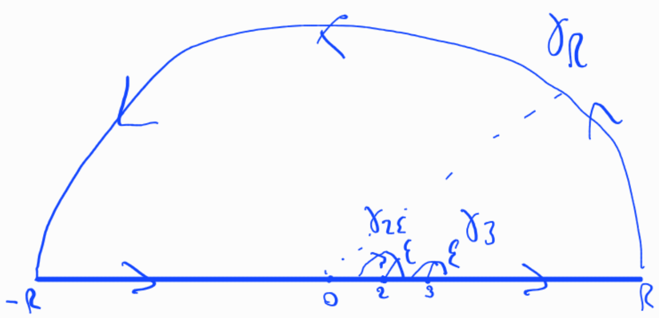
Proposición: Sea $f \in H(D(0,R) \setminus \{a\})$ con un polo de orden 1 en a. Para cada $R > \varepsilon > 0$ consideramos $\gamma_\varepsilon : [\theta_1, \theta_2] \to \mathbb{C}$ $\gamma_\varepsilon(t) = a + \varepsilon e^{it}$

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\, dz = i(\theta_2 - \theta_1) \operatorname{Res}(f(z), a)$$

$$e^{i\pi z} = \cos(\pi z) + i \operatorname{sen}(\pi z) \Rightarrow \operatorname{Im} e^{i\pi z} = \operatorname{sen} \pi z$$

Sea $f(z) = \dfrac{x e^{i\pi z}}{x^2-5x+6} = \dfrac{e^{i\pi z}}{(x-2)(x-3)}$ $\qquad A = \{2,3\}$

Sea Γ el camino



Γ_R

$\Gamma = \gamma_R + \gamma_{2\varepsilon} + \gamma_{3\varepsilon} + [-R, 2-\varepsilon] + [2+\varepsilon, 3-\varepsilon] + [3+\varepsilon, R]$

Aplicamos el Teorema de los residuos a

$$\left.\begin{array}{l} \Omega = \mathbb{C} \\ A = \{2,3\},\ A' \cap \mathbb{C} = \emptyset \\ f \in \mathcal{H}(\Omega \setminus A) \\ \Gamma \text{ es nulhomólogo con respecto a } \Omega \\ \Gamma^* \subset \Omega \setminus A \end{array}\right\} \int_\Gamma f(z)\,dz = 0$$

$$\lim_{\varepsilon \to 0} \int_\Gamma f(z)\,dz = \underbrace{\int_{[-R,R]} f(z)\,dz}_{I} + \int_{\gamma_R} f(z)\,dz + \int_{\gamma_{2\varepsilon}} f(z)\,dz + \int_{\gamma_{3\varepsilon}} f(z)\,dz = 0$$

Veamos que $\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0$

Si $z \in \gamma_R \Rightarrow z = R(\cos(t) + i\,\mathrm{sen}(t))$ $t \in [0,\pi]$

$|e^{i\pi z}| = e^{\mathrm{Re}(i\pi z)} = e^{-\pi R\,\mathrm{sen}(t)}$

$$\left|\int_{\gamma_z} f(z)\,dz\right| \leq \int_{\gamma_z} \frac{R\,|e^{i\pi z}|}{(R-2)(R-3)}\,dz = \frac{R}{(R-2)(R-3)} \int_0^\pi R\, e^{-\pi R\,\mathrm{sen}(t)}\,dt = *$$

$$\left.\begin{array}{l} \gamma: [0,\pi] \to \mathbb{C} \\ t \to Re^{it} \\ \gamma' = iRe^{it} \end{array}\right\} |e^{i\pi z}| = |e^{i\pi Re^{it}} \cdot iRe^{it}| \Rightarrow R\, e^{-\pi R\,\mathrm{sen}(t)}$$

$$* = \frac{R^2}{(R-2)(R-3)} \int_0^{\pi} e^{-\pi R \operatorname{sen}(t)} \Longrightarrow \lim_{R\to\infty} \frac{R^2}{(R-2)(R-3)} \int_0^{\pi} e^{-\pi R \operatorname{sen}(t)} = \int \lim_{R\to\infty} e^{-\pi R \operatorname{sen} t} =$$

$$\downarrow$$
$$1$$

$(\operatorname{sen}(t) > 0$ en $]0,\pi[$

$$= \int_0^{\pi} 0 = 0$$

$$\Longrightarrow \lim_{R\to\infty} \int_{\gamma} f(z)\,dz = 0$$

$$\cdot \int_{\gamma_{2\varepsilon}} f(z)dz = -\pi i \operatorname{Res}(f,2) \qquad \operatorname{Res}(f,2) = \lim_{z\to 3} \cancel{(z-2)} \frac{z e^{i\pi z}}{\cancel{(z-2)}(z-3)} = \frac{z e^{2\pi i}}{-1} = -2$$

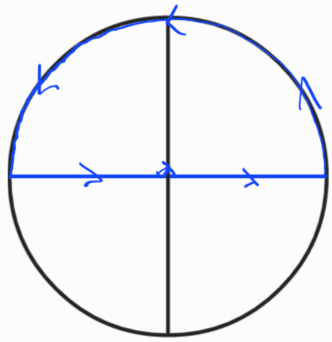
$$\cdot \int_{\gamma_{3\varepsilon}} f(z)\,dz = -\pi i \operatorname{Res}(f,3) \qquad \operatorname{Res}(f,3) = \lim_{z\to 3} \frac{\cancel{z-3}}{\cancel{z-3}} \frac{z e^{i\pi z}}{z-2} = -3$$

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\Gamma} f(z)\,dz = 0 = \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2-5x+6} + 5\pi i \cancel{\int_{\gamma_R} f(z)\,dz}^{\,0}$$

$$\Longrightarrow \operatorname{Im} \int_{-\infty}^{\infty} \frac{x\, e^{i\pi x}}{x^2-5x+6} = -5\pi = \int \frac{x \operatorname{sen}^{i\pi x}}{x^2-5x+6}$$

⑨. Integrando la función $z \to \frac{1-e^{2iz}}{z^2}$ sobre un camino cerrado que recorra la frontera de la mitad superior del anillo $A(0,\varepsilon,R)$, probar que:

$$\int_0^{+\infty} \frac{(\operatorname{sen} x)^2}{x^2}\,dx = \frac{\pi}{2}$$



$$(e^{iz})^2 = (\cos z + i \operatorname{sen} z)^2 = \cos^2(z) + 2i \operatorname{sen} z \cos z - \operatorname{sen}^2(z) = 1 - 2i \operatorname{sen} z \cos z - 2\operatorname{sen}^2(z)$$

$$\frac{1-(e^{iz})^2}{z^2} = \frac{2i \operatorname{sen} z \cos z}{z^2} + \frac{2\operatorname{sen}^2(z)}{z^2} \Longrightarrow \frac{\operatorname{sen}^2(z)}{z^2} = \operatorname{Re}\left(\frac{1-e^{2iz}}{2z^2}\right)$$

Aplicamos el Teorema de los residuos + proposición

$\Omega = \mathbb{C}$

$A = \{0\} \Rightarrow A' \cap \Omega = \emptyset$

$\Gamma^* \subset \Omega \setminus A$

Γ nulhomólogo en $\mathbb{C}$

$f \in (\Omega \setminus A)$

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Ind}(\Gamma, 0)\cdot \operatorname{Res}(f,0) = 0 =$$

$$= \int_{[-R,-\varepsilon]+[\varepsilon,R]} f(z)\,dz + \int_{\gamma_\varepsilon} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

Tomando $\lim_{\varepsilon \to 0} \int_\Gamma f(z)\,dz = \int_{[-R,R]} f(z)\,dz + -\pi i \operatorname{Res}(f,0) + \int_{\gamma_R} f(z)\,dz = 0$

Veamos que $\int_{\gamma_R} f(z)\,dz = 0$

$$\int_{\gamma_R} f(z)\,dz = \int_{\gamma_R} \frac{1-e^{2iz}}{2z^2} = \int_0^{\pi} \frac{1-e^{iRe^{it}}}{2(Re^{it})^{\cancel{2}}}\, \cancel{iRe^{it}} \Rightarrow \left|\int_0^{\pi} \frac{1-e^{iRe^{it}}\,i}{2Re^{it}}\right| \leq$$

$\gamma[0,\pi] \to \mathbb{C}$

$t \to Re^{it}$

$\gamma'(t) = iRe^{it}$

$$\leq \frac{2\pi}{2R} \xrightarrow{R\to\infty} 0$$

$$\sup\left\{ \left|\frac{1-e^{iRe^{it}}\,i}{2Re^{it}}\right| \right\} \leq \frac{2}{R}$$

$$\left|e^{iRe^{it}}\right| = \left|e^{iR(\cos t + i\,\text{sen}\,t)}\right| = e^{-R\,\text{sen}\,t} \leq 1$$

Calculemos ahora

$$\int_{\gamma_\varepsilon} f(z)\,dz = -\pi i\,\text{res}(f,0) \qquad \text{res}\,f(0) = \lim_{z\to 0} z\,\frac{1-e^{2iz}}{2z^2} = \lim_{z\to 0} \frac{1-e^{2iz}}{2z} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{z\to 0} \frac{-2ie^{2iz}}{2} = -i \Rightarrow \int_{\gamma_\varepsilon} f(z)\,dz = \pi$$

Recapitulando

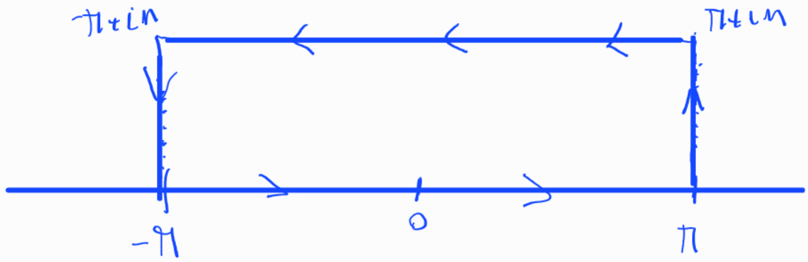
$$\int_{\Gamma} f(z)\,dz = 0 = \int_{[-\infty,\infty]} f(z)\,dz + \pi$$

$$\int_{-\infty}^{\infty} \frac{e^{2iz}}{2z^2} = \pi \Rightarrow \text{Re}\int_{-\infty}^{\infty} \frac{e^{2iz}}{2z^2} = \pi = \int_{-\infty}^{\infty} \frac{\text{sen}^2 x}{x^2}\,dx = \pi$$

Como $\frac{\text{sen}^2(x)}{x^2}$ es par $\int_0^{\infty} \frac{\text{sen}^2(x)}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\text{sen}^2 x}{x^2}\,dx = \frac{\pi}{2}$

(10) Dado $a \in \mathbb{R}$ con $a > 1$, integrar la función $z \longrightarrow \dfrac{z}{a - e^{-iz}}$ sobre la poligonal $[-\pi, \pi, \pi + in, -\pi + in, -\pi]$ con $n \in \mathbb{N}$, para probar que:

$$\int_{-\pi}^{\pi} \frac{x \operatorname{sen} x \, dx}{1 + a^2 - 2a \cos x} = \frac{2\pi}{a} \log\left(\frac{1+a}{a}\right)$$



$1 + a^2 - 2a \cos t = \cos^2 z + \operatorname{sen}^2 z + a^2 - 2a \cos z = (a - \cos z)^2 + \operatorname{sen}^2 z =$

$= |a - \cos(x) + i \operatorname{sen} x|^2 = |a - e^{-iz}|^2 = (a - e^{-iz})(a - e^{iz})$

$$\frac{z}{a - e^{iz}} = \frac{z(a - e^{iz})}{(a - e^{-iz})(a - e^{iz})} = \frac{az - z\cos(z) - iz \operatorname{sen} z}{1 + a^2 - 2a \cos z} \Longrightarrow -\operatorname{Im}\left(\frac{z}{a - e^{-iz}}\right) = \frac{z \operatorname{sen} z}{1 + a^2 - 2a \cos z}$$

Buscamos las singularidades

$a - e^{-iz} = 0 \Longleftrightarrow a = \dfrac{1}{e^{iz}} \Longleftrightarrow e^{iz} = \dfrac{1}{a} \Longleftrightarrow iz = \log\left(\dfrac{1}{a}\right) \Longleftrightarrow z = i \log(a)$

Por tanto, los punto donde se anula el denominador son $\{z \in \mathbb{C} : z \in i \operatorname{Log}(a)\}$

$\Longrightarrow z \in \{ i \log(a) + 2\pi\mathbb{Z} \} = A$

Nuestra poligonal Γ solo rodea a uno de los puntos $(i \log(a))$ $\forall n > \log(a)$

Ahora procedemos a aplicar el teorema de los residuos.

$\Omega = \mathbb{C}$

$\Omega \cap A' = \phi$

$\Gamma \subset \Omega \setminus A$

Γ es nulhomologo a Ω

$f \in H(\Omega \setminus A)$

$$\int_\Gamma f(z)\,dz = 2\pi i \cdot \mathrm{Ind}_\Gamma(i\log(a))\,\mathrm{Res}(f, i\log(a)) = 2\pi i \log$$

$$\int_\Gamma f(z)\,dz = \int_{[-\pi,\pi]} f(z)\,dz + \int_{[\pi,\pi+in]} f(z)\,dz - \int_{[-\pi+in,\pi+in]} f(z)\,dz - \int_{[\pi,-\pi+in]} f(z)\,dz$$

Analicemos por partes

- $$\int_{[\pi,\pi+in]} f(z)\,dz = \int_0^n f(\pi+it)\,i\,dt$$

$\sigma\,[0,n] \to \mathbb{C}$

$t \to \pi + it$

- $$\int_{[-\pi,-\pi+in]} f(z)\,dz = \int_0^n f(-\pi+it)\,dt$$

$\sigma\,[0,n] \to \mathbb{C}$

$t \to -\pi + it$

$$\int_{[\pi,\pi+in]} f(z)\,dz\,dt - \int_{[-\pi,-\pi+in]} f(z)\,dz = i\int_0^n \frac{\pi+it}{a-e^{-i(\pi+it)}} - \frac{-\pi+it}{a-e^{-i(-\pi+it)}} \overset{\uparrow}{=} i\int_0^n \frac{2\pi}{a-e^{\pi i + t}}$$

la exp es $2\pi i$ periodica $e^{-i(\pi+it)} = e^{-i(-\pi+it)}$

$$= 2\pi i \int_0^n \frac{1}{a+e^t} = \frac{2\pi i}{a}\int_0^n \frac{a+e^t-e^t}{a+e^t} = \frac{2\pi i}{a}\int_0^n 1 - \frac{e^t}{a+e^t}\,dt =$$

$$= \frac{2\pi i}{a}\left(t - \log(a+e^t)\right)\Big|_0^n = \frac{2\pi i}{a}\left[\log(e^n) - \log(a+e^n) + \log(a+1)\right] =$$

$$= \frac{2\pi i}{a}\left(\log\left(\frac{e^n}{a+e^n}\right) + \log(a+1)\right) \xrightarrow[\lim\, n\to\infty]{} \frac{2\pi i}{a}\log(a+1)$$

$$\int_{[-\pi+in,\pi+in]} f(z)\,dz = \int_{-\pi}^{\pi} \frac{t+in}{a+e^{-i(t+in)}} = \int_{-\pi}^{\pi} \frac{t+in}{a+e^{n-it}} \Rightarrow \left|\int_{-\pi}^{\pi} \frac{t+in}{a+e^{n-it}}\right| \leq 2\pi \frac{\pi+n}{|a|-|e^n|} \xrightarrow{n\to\infty} 0$$

$$\gamma: [-\pi,\pi] \to \mathbb{C}$$

$$t \to t+in$$

$$\sup\left\{\left|\frac{t+in}{a+e^{n-it}}\right|\right\} \leq \frac{\pi+n}{|a|-|e^n|}$$

Calculemos ahora el residuo y tendremos todas las piezas para construir la integral

$$\mathrm{Res}(f, i\log(a)) = \lim_{z\to i\log(a)} (z-i\log(a))\frac{z}{a-e^{-iz}} \overset{L'H}{=} \lim_{z\to i\log(a)} \frac{(z-i\log(a))+z}{ie^{-iz}} =$$

$$= \frac{i\log(a)}{ia} = \frac{\log(a)}{a} \in \mathbb{C}$$

Uniendo todas las piezas

$$\int_{\Gamma} f(z)\,dz = \int_{-\pi}^{\pi} f(z)\,dz + \frac{2\pi i}{a}\log(a+1) = 2\pi i \frac{\log(a)}{a}$$

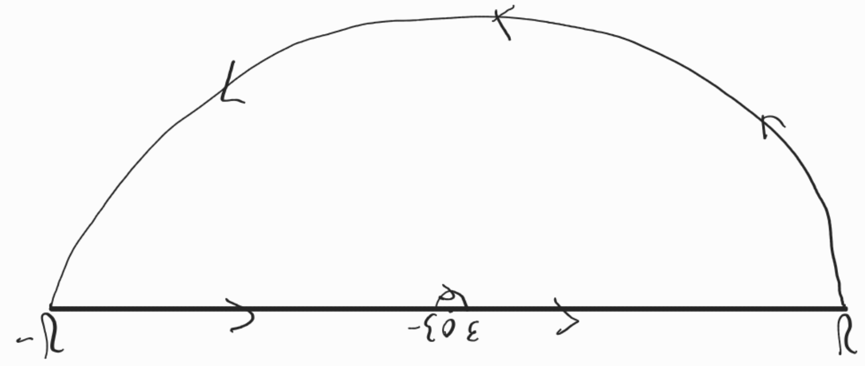
$$\Rightarrow \int_{-\pi}^{\pi} f(z)\,dz = \frac{2\pi i}{a}\log\left(\frac{a}{a+1}\right)$$

$$\mathrm{Im}\int_{-\pi}^{\pi} f(z)\,dz = -\int \frac{x\operatorname{sen}x}{1+a^2-2a\cos x} = \frac{2\pi}{a}\log\left(\frac{a}{a+1}\right)$$

$$\Rightarrow \int \frac{x\operatorname{sen}x}{1+a^2-2a\cos x} = \frac{2\pi}{a}\log\left(\frac{1+a}{a}\right)$$

(11) Integrando una conveniente función compleja a lo largo de la frontera de la mitad superior del anillo $A(0,\varepsilon,R)$, probar que, para $\alpha \in \,]-1,3[$, se tiene

$$\int_0^{+\infty} \frac{x^a\,dx}{(1+x^2)^2} = \frac{\pi}{4}(1-a)\sec\frac{\pi a}{2}$$



$z^a = e^{a\log(z)}$

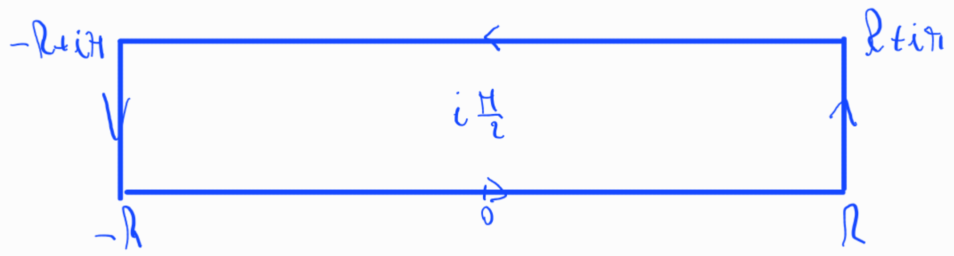
$\log(z)$ el logaritmo holomorfo en $\mathbb{C}^*\setminus\{z\in\mathbb{C};\ \arg(z) = \frac{-\pi}{2}\}$

$f(z) = \dfrac{e^{a\log(z)}}{(1+z^2)^2}$ Y usar esta.

Otra forma

$$\int_0^{+\infty} \frac{x^a}{(1+x^2)^2}\,dx = \left\{\begin{array}{l}\text{Cambio variable}\\ x = e^t \to dx = e^t dt\\ x\to 0 \Rightarrow t\to-\infty\\ x\to\infty \Rightarrow t\to+\infty\end{array}\right\} = \int_{-\infty}^{+\infty} \frac{e^{t(a+1)}}{(1+e^{2t})^2}$$

$$\int_{-\infty}^{+\infty} \frac{e^{t(a+1)}}{(1+e^{2t})^2}$$



$f:\mathbb{C}\setminus A \to \mathbb{C}$ dada por $f(z) = \dfrac{e^{z(a+1)}}{(1+e^{2z})^2}$

donde $A = \{z\in\mathbb{C};\ 1+e^{2z}=0\}$

Buscamos $z\in\mathbb{C}$ verificando $1+e^{2z}=0 \Leftrightarrow e^{2z}=-1 \Leftrightarrow 2z\in\log(-1)$
$= i\cdot\mathrm{Arg}(-1) = i(\pi+2\pi k) \Leftrightarrow z\in i\left(\frac{\pi}{2}+\pi k\right)$

$A' = \emptyset$
$\Omega = \mathbb{C}$
$f \in \mathcal{H}(\Omega \setminus A)$
$A' = \emptyset \; (\Rightarrow A' \cap \Omega = \emptyset)$
$R > 0$
$\Gamma_R = [-R, R, R+\pi i, R-\pi i, -R]$
$\Gamma_R^* \subset \Omega \setminus A$

$$\xrightarrow{\text{T. Residuos}} \int_{\Gamma_R} f(z)\,dz = 2\pi i \,\mathrm{Ind}_{\Gamma_R}(i\tfrac{\pi}{2})\,\mathrm{Res}\left(f(z), i\tfrac{\pi}{2}\right) \quad (*)$$

- $\displaystyle \int_{-R}^{R} f(z)\,dz = \int_{-R}^{R} \frac{e^{t(a+1)}}{(1+e^{2t})^2} =$ Integral que queremos

$\gamma\,[-R,R] \to \mathbb{C}$
identidad

- $\displaystyle \int_{[R+i\pi,\,-R+i\pi]} f(z)\,dz = -\int_{[-R+i\pi,\,R+i\pi]} f(z)\,dz = -\int_{-R}^{R} \frac{e^{(t+i\pi)(a+1)}}{(1+e^{2(t+i\pi)})^2}\,dt =$

$\gamma\,[-R,R] \to \mathbb{C}$
$t \mapsto t+i\pi$
$\gamma'(t) = 1$

$$= -\int \frac{e^{t(a+1)} \cdot e^{i\pi a}\, e^{i\pi}}{(1+e^{2t})^2} = e^{i\pi a} \int_{-R}^{R} \frac{e^{t(a+1)}}{(1+e^{2t})^2}$$

$e^{i\pi} = -1$

$$\int_{-R}^{R} f(z)\,dz + \int_{[-R+i\pi,\,R+i\pi]} f(z)\,dz = (e^{\pi i a}+1) \int_{-R}^{R} \frac{e^{t(a+1)}}{(1+e^{2t})^2}\,dt$$

- $\displaystyle \left| \int_{[R,\,R+i\pi]} f(z)\,dz \right| \leq \pi \frac{e^{R(a+1)}}{e^{4R} - e^{2R} + 1} \xrightarrow{R\to\infty} 0 \quad (a < 3)$

$|e^{z(a+1)}| = e^{R(a+1)}$; $|1+e^{2t}|^2 \leq (|e^{2t}| - 1)^2 \leq (e^{2R}+1)^2 = e^{4R} - e^{2\pi} + 1$

$$\left|\int_{[-R,-R+i\pi]} f(z)\,dz\right| \leq \pi \sup\{|f(z)| \,,\, z\in[-R,-R+i\pi]\} \leq \frac{\pi e^{-R(a+1)}}{(1-e^{2R})^2} \xrightarrow{R\to\infty} 0$$

Usando todo esto y tomando límite en *

$$(1+e^{i\pi a})\int_{-R}^{R} \frac{e^{t(a+1)}}{(1+e^{2t})^2}\,dt = 2\pi i \operatorname{Res}\left(f(z), i\frac{\pi}{2}\right)$$

12. Probar que para $\alpha \in]0,1[$, se tiene:

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}\, dx}{1+e^{x}+e^{2x}} = \int_{0}^{+\infty} \frac{t^{\alpha-1}\, dt}{1+t+t^2} = \frac{2\pi}{\sqrt{3}} \frac{\operatorname{sen}(\pi(1-\alpha)/3)}{\operatorname{sen}(\pi\alpha)}$$

Hacemos el cambio $e^{x}=t \Rightarrow e^{x}dx = dt \Rightarrow dx = \frac{dt}{t}$

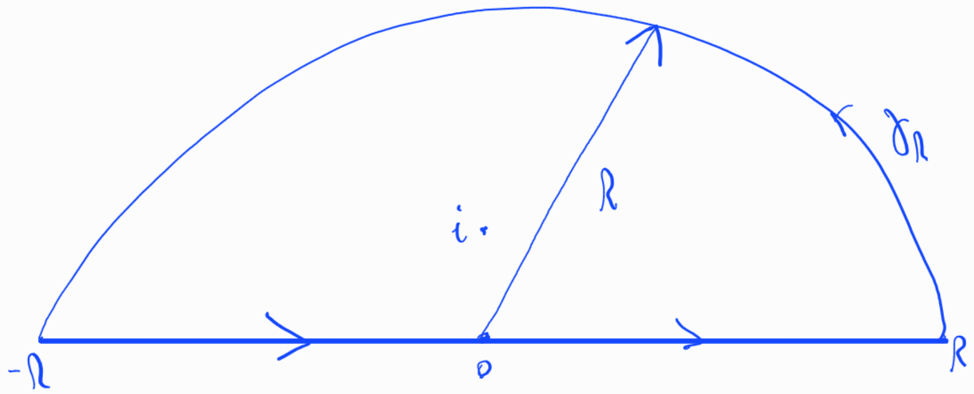
$x=\infty \quad e^{x}=\infty \Rightarrow t=\infty$

$x=-\infty \quad e^{x}=0 \Rightarrow t=0$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{e^{\alpha x}\, dx}{1+e^{x}+e^{2x}} = \int_{0}^{+\infty} \frac{t^{\alpha-1}}{1+t+t^2}$$

(13) Integrando la función $z \longrightarrow \dfrac{\log(z+i)}{1+z^2}$ sobre un camino cerrado que recorra la frontera del conjunto $\{z\in\mathbb{C} : |z|<R, \operatorname{Im} z>0\}$, con $R\in\mathbb{R}$ y $R>1$, calcular

$$\int_{-\infty}^{+\infty}\frac{\log(1+x^2)}{1+x^2}\,dx$$



Definimos $f(z)=\dfrac{\log(z+i)}{z^2+1}$ $\quad A=\{i\}$ $\quad z^2+1=0 \iff z^2=-1 \iff z=\pm i$

Aplicamos el Teorema de los residuos

$\Omega=\mathbb{C}$

$f\in H(\mathbb{C}\setminus A)$

$A'\cap\Omega=\emptyset$

Γ es nulhomólogo en Ω

$\Gamma^*\subset\Omega\setminus A$

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f,i) = \int_{-R}^{R} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$\operatorname{Res}(f,i)=\lim_{z\to i}\,\cancel{(z-i)}\,\frac{\log(z+i)}{\cancel{(z-i)}(z+i)} = \frac{\log(2i)}{2i} = \frac{\log(2)+\frac{\pi i}{2}}{2i} =$$

$$\Longrightarrow \int_\Gamma f(z)\,dz = \log(2)\,\pi + i\,\frac{\pi^2}{2}$$

$$\int_{-R}^{R}\frac{\log(z+i)}{z^2+1} = \underbrace{\int_{-R}^{0}\frac{\log(z+i)}{z^2+1}}_{①} + \underbrace{\int_{0}^{R}\frac{\log(z+i)}{z^2+1}}_{②} = \int_0^R \frac{\log(z-i)+\log(z+i)}{z^2+1} =$$

Cambio x por $-x$ en ①

$$= \int_0^R \frac{\log(z^2+1)}{z^2+1}\,dz + \int_0^R \frac{\pi i}{z^2+1}\,dz$$

$$\Rightarrow \int_{\Gamma} f(z)\,dz = \int_0^R \frac{\log(z^2+1)\,dz}{z^2+1} + \int_0^R \frac{\pi i\,dz}{z^2+1} + \int_{\gamma_R} \frac{\log(z+i)\,dz}{z^2+1} = \log(2)\pi + i\frac{\pi^2}{2}$$

Veamos que $\displaystyle\int_{\gamma_R} \frac{\log(z+i)\,dz}{z^2+1} \xrightarrow{R\to\infty} 0$ y tomando parte real nos queda

$$\int_0^R \frac{\log(z^2+1)\,dz}{z^2+1} = \log(2)\pi \Longrightarrow \int_{-R}^R \frac{\log(x^2+1)}{z^2+1} = 2\log(2)\pi$$

$$\int_{\gamma_R} \frac{\log(z+i)}{z^2+1}\,dz = \int_0^\pi \frac{\log(Re^{it}+i)\,Re^{it}\,i}{R^2e^{2it}+1} \Longrightarrow \left|\int_0^\pi \frac{\log(Re^{it}+i)}{R^2e^{2it}+1}\right| \le \pi\frac{R\cdot\log(R+1)}{R^2-1} \xrightarrow{R\to\infty} 0$$

$\gamma\,[0,\pi] \to \mathbb{C}$

$t \to Re^{it}$

$\gamma'(t) = iRe^{it}$

$$\sup\left\{\left|\frac{\log(Re^{it}+i)}{R^2e^{2it}+1}\right|\right\} \le \frac{R\cdot\log(R+1)}{R^2-1}$$

(14) Integrando una conveniente función sobre la poligonal $[-R, R, R+\pi i, -R+\pi i, -R]$. Con $R \in \mathbb{R}^+$, calcular la integral.

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\,dx$$

Consideramos la función: $f: \mathbb{C} \setminus A \to \Omega$ dada por $f(z) = \dfrac{e^{iz}}{e^z + e^{-z}}$

$A = \{z \in \mathbb{C};\ e^z + e^{-z} = 0\}$

$e^z + e^{-z} = 0 \quad e^{2z} + 1 = 0 \iff e^{2z} = -1 \iff 2z \in \operatorname{Log}(-1) = i \operatorname{Arg}(-1)$
$= i(\pi + 2\pi\mathbb{Z}) \implies z \in (i\frac{\pi}{2} + \pi\mathbb{Z})$

Solución:

$\Omega = \mathbb{C}$ (Homologicamente conexo)

$A' = \emptyset$

$f \in \mathcal{H}(\mathbb{C} \setminus A)$

$\Gamma_R = [-R, R, R+\pi i, -R+\pi i, -R]$
(nul homologo con respecto a Γ

$\xrightarrow{\text{T. Residuos}} \displaystyle\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Ind}_{\Gamma_R}\left(\frac{i\pi}{2}\right) \cdot \operatorname{Res}\left(f(z), \frac{i\pi}{2}\right) =$
$= 2\pi i \operatorname{Res}\left(f(z), \frac{i\pi}{2}\right)$

$\displaystyle\lim_{z \to \frac{i\pi}{2}} \left(z - \frac{i\pi}{2}\right) f(z) = \lim_{z \to \frac{i\pi}{2}} \frac{\left(z - i\frac{\pi}{2}\right)}{e^z + e^{-z}} e^{iz} = \frac{e^{\pi/2}}{2i} \neq 0 \implies f$ tiene un polo de orden 1 en $\frac{i\pi}{2}$ y $\operatorname{Res}\left(f(z), \frac{i\pi}{2}\right) = \frac{e^{\pi/2}}{2i}$

$\displaystyle\lim_{z \to \frac{i\pi}{2}} \frac{z - \frac{i\pi}{2}}{e^z + e^{-z}} = \lim_{z \to \frac{i\pi}{2}} \frac{1}{e^z - e^{-z}} = \frac{1}{e^{i\pi/2} - e^{-i\pi/2}} = \frac{1}{2i}$

$$\int_{\Gamma_R} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{[R,R+\pi i]} f(z)\,dz + \int_{[R+\pi i,-R+\pi i]} f(z)\,dz + \int_{[-R+\pi i,-R]} f(z)\,dz$$

$$\qquad\qquad 1 \qquad\qquad\qquad 2 \qquad\qquad\qquad 3 \qquad\qquad\qquad 4$$

$[-R,R] \to \mathbb{C}$

$\gamma(x) = x$

$$1 = \int_{-R}^{R} \frac{e^{ix}}{e^{x}+e^{-ix}}\,dx$$

$$3 = -\int_{[-R+\pi i,\, R+\pi i]} f(z)\,dz = -\int_{-R}^{R} \frac{e^{i(x+\pi i)}}{e^{x+i\pi}+e^{-x-i\pi}} = -\int_{R} \frac{e^{ix}e^{-\pi}}{-e^{x}-e^{-x}} = e^{-\pi}\int_{-R}^{R} \frac{e^{ix}}{e^{x}+e^{-x}}\,dx$$

Por tanto

$$1 + 3 = (1+e^{-\pi})\int_{-R}^{R} \frac{e^{ix}}{e^{x}+e^{-x}}\,dx$$

Probamos que

$$\lim_{R\to\infty} \begin{matrix} 2 \\ 4 \end{matrix} = 0 \qquad 4 \text{ es analogo a } 2$$

$$2 \quad \left| \int_{[R,R+\pi i]} f(z)\,dz \right| \leq \pi \cdot \sup\{|f(z)| \ z\in[R,R+\pi i]\} \leq \pi \frac{1}{e^{R}-e^{-R}} \xrightarrow{R\to\infty} 0$$

Si $z \in [R, R+\pi i] \Rightarrow z = R+ti \quad t\in[0,\pi]$

$$f(z) = \frac{e^{iz}}{e^{z}+e^{-z}}$$

$$|e^{iz}| = |e^{iR-t}| = e^{-t} \leq 1$$

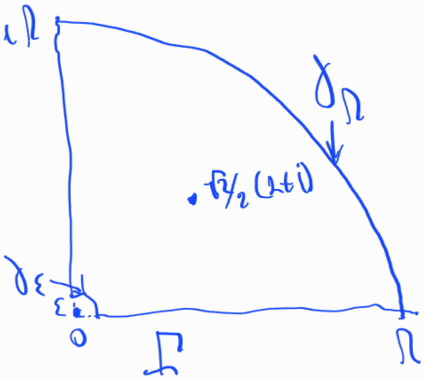
$$\left.\begin{aligned} |e^{-z}| &= |e^{-R-ti}| = e^{-R} \\ |e^{z}| &= |e^{R+ti}| = e^{R} \end{aligned}\right\} \Rightarrow |e^{z}-e^{-z}| \geq |e^{z}|-|e^{-z}| = e^{R}-e^{-R} > 0$$

Entonces tomando límite $R \to \infty$ en (*) obtenemos

$$(1+e^{-\pi})\int_{-\infty}^{\infty} \frac{e^{ix}}{e^{x}+e^{-x}}\,dx = \pi e^{-\pi/2} \underset{\text{parte real}}{\overset{\text{tomamos}}{\Longrightarrow}} \int_{-\infty}^{\infty} \frac{\cos x}{e^{x}+e^{-x}}\,dx = \frac{\pi e^{-\pi/2}}{1+e^{-\pi}} = \frac{\pi e^{\pi/2}}{e^{\pi}+1}$$

25. Integrando una conveniente función sobre un camino cerrado que recorra la frontera del conjunto $\{z \in \mathbb{C} : \varepsilon < |z| < R,\ 0 < \arg z < \pi/2\}$ con $0 < \varepsilon < 1 < R$.

$$\int_0^{+\infty} \frac{\log(x)}{1+x^4} \qquad f(z) = \frac{\log(z)}{1+x^4}$$



$$1+x^4=0 \iff x = \begin{cases} \sqrt{2}/2\,(1+i) \\ \sqrt{2}/2\,(1-i) \\ \sqrt{2}/2\,(-1+i) \\ \sqrt{2}/2\,(-1-i) \end{cases} \quad A = \{\sqrt{2}/2\,(1+i), \sqrt{2}/2\,(1-i), \sqrt{2}/2\,(-1+i), \sqrt{2}/2\,(-1-i)\}$$

Aplicamos el teorema de los residuos

$\Omega = \mathbb{C} \setminus \mathbb{R}^-$

$f \in H(\Omega \setminus A)$

$\Gamma^* \subset \Omega \setminus A$

Γ es nulhomólog en Ω

$A' \cap \Omega$

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f, \sqrt{2}/2\,(1+i))$$

$$\int_\Gamma f(z)\,dz = \int_\varepsilon^R f(z)\,dz + \int_{\gamma_R} f(z)\,dz - \int_{[i\varepsilon, iR]} f(z)\,dz - \int_{\gamma_\varepsilon} f(z)\,dz$$

Tomando lim $\varepsilon \to 0$, $R \to \infty$

$$I = \int_0^R f(z)\,dz = 2\pi i \operatorname{Res}\left(f, \frac{\sqrt{2}}{2}(1+i)\right) - \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz + \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} f(z)\,dz + \lim_{\substack{\varepsilon\to 0 \\ R\to\infty}} \int_{[i\varepsilon, iR]} f(z)\,dz$$

$$\bullet \int_{[i\varepsilon, iR]} f(z)\,dz = i\int_\varepsilon^R \frac{\log(it)}{1+t^4}\,dt = i\int_\varepsilon^R \frac{\log(t)}{1+t^4} - \frac{\pi}{2}\int_\varepsilon^R \frac{1}{1+t^4}$$

$$\boxed{\begin{array}{l} \gamma\,[\varepsilon, R] \to \mathbb{C} \\ \quad t \to it \\ \gamma'(t) = i \end{array}}$$

$$\frac{\pi}{2}\int_\varepsilon^R \frac{1}{1+t^4} \underset{\text{Ej 6}}{=} \frac{\pi}{2}\,\frac{\pi}{4}\,\frac{1}{\operatorname{sen}\left(\frac{\pi}{4}\right)} = \frac{\pi^2}{8\operatorname{sen}\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}\,\pi^2}{8}$$

$$\bullet \int_{\gamma_\varepsilon} f(z)\,dz = \int_0^{\pi/2} \frac{\log(\varepsilon e^{it})}{1+\varepsilon^4 e^{4it}}\,\varepsilon i e^{it}\,dt \Rightarrow \lim_{\varepsilon\to 0} \int_0^{\pi/2} \frac{\log(\varepsilon e^{it})}{1+\varepsilon^4 e^{4it}}\,\varepsilon i e^{it}\,dt = 0$$

$\gamma\,[0, \pi/2] \to \mathbb{C}$

$t \to \varepsilon e^{it}$

$\gamma'(t) = i\varepsilon e^{it}$

$$\bullet \int_{\gamma_R} f(z)\,dz = i\int_0^{\pi/2} \frac{\log(Re^{it})}{1+R^4 e^{4it}}\,Re^{it} \Rightarrow \left| \int_0^{\pi/2} \frac{\log(Re^{it})}{1+R^4 e^{4it}}\,Re^{it}\,dt \right| \le$$

$$\le \frac{\pi}{2}\,\frac{\log(R)R + R\frac{\pi}{2}}{1+R^4} \xrightarrow{R\to\infty} 0$$

$\gamma\,[0, \pi/2] \to \mathbb{C}$

$t \to Re^{it}$

$\gamma'(t) = iRe^{it}$

$$\operatorname{Res}\left(f, \tfrac{\sqrt{2}}{2}(1+i)\right) = \lim_{z\to\frac{\sqrt{2}}{2}(1+i)} \cancel{\left(z-\tfrac{\sqrt{2}}{2}(1+i)\right)} \frac{\log(z)}{\cancel{\left(z-\frac{\sqrt{2}}{2}(1+i)\right)}\left(z+\frac{\sqrt{2}}{2}(1+i)\right)\left(z^2-\frac{1}{2}(1-i)^2\right)} =$$

$$= \frac{\log\left(\frac{\sqrt{2}}{2}(1+i)\right)}{\sqrt{2}(1+i)\,\frac{1}{2}\left((1+i)^2-(1-i)^2\right)} = \frac{\log\left(\frac{\sqrt{2}}{2}(1+i)\right)}{\frac{\sqrt{2}}{2}(1+i)\,4i} = \frac{\log(e^{i\pi/4})}{4i\,e^{i\pi/4}} = \frac{\pi}{16\,e^{i\pi/4}}$$

$$(1+i)\int_0^\infty \frac{\log(x)}{1+x^4} = 2\pi i\,\frac{\pi}{16\,e^{i\pi/4}} - \frac{\sqrt{2}\,\pi^2}{8} \Rightarrow \int_0^\infty \frac{\log(x)}{1+x^4} = \operatorname{Re}\left(2\pi i\,\frac{\pi}{16\,e^{i\pi/4}} - \frac{\sqrt{2}\,\pi^2}{8}\right)$$

$$= \frac{\pi^2}{8}\cdot\frac{1}{\cos\left(\frac{3\pi}{4}\right)} - \frac{\sqrt{2}\,\pi^2}{8} = \frac{\pi^2}{8}(-2\sqrt{2}) = -\frac{\sqrt{2}\,\pi^2}{4}$$

16. Integrando una conveniente función sobre la poligonal $[-R, R, R+2\pi i, -R+2\pi i, -R]$ con $R \in \mathbb{R}^+$, calcular la integral.

$$\int_{-\infty}^{+\infty} \frac{e^{x/2}}{e^x+1}\,dx$$

Ya calculamos primero de forma tradicional

$$\int_{-\infty}^{+\infty} \frac{e^{x/2}\,dx}{1+(e^{x/2})^2} = 2\int_0^{\infty} \frac{1}{1+t^2} = 2\arctan(t)\Big|_0^{\infty} = 2\,\frac{\pi}{2} = \pi$$

$t = e^{x/2} \quad dt = \frac{1}{2}e^{x/2}\,dx$

Ahora con el teorema de los residuos