Ejercicios Propuestos

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Ejercicio 1

$$\bar{x} = \sum_{j=1}^{p} f_{.j}\bar{x}_j = \sum_{j=1}^{p} f_{.j} \sum_{i=1}^{k} f_i^j x_i = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{.j} f_i^j x_i = \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{n_{.j}}{n}\frac{n_{ij}}{n_{.j}} x_i = \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{n_{ij}}{n} x_i =$$

$$= \frac{1}{n}\sum_{i=1}^{k} x_i \sum_{j=1}^{p} n_{ij} = \frac{1}{n}\sum_{i=1}^{k} n_{i.} x_i = \sum_{i=1}^{k} f_{i.} x_i = \bar{x}$$

$$\bar{y} = \sum_{i=1}^{k} f_{i.}\bar{y}_i = \sum_{i=1}^{k} f_{i.} \sum_{j=1}^{p} f_j^i y_j = \sum_{j=1}^{p}\sum_{i=1}^{k} f_{i.} f_j^i y_j = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{i.}}{n}\frac{n_{ij}}{n_{i.}} y_j = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{ij}}{n} y_j =$$

$$= \frac{1}{n}\sum_{j=1}^{p} y_j \sum_{i=1}^{k} n_{ij} = \frac{1}{n}\sum_{j=1}^{p} n_{.j} y_j = \sum_{j=1}^{p} f_{.j} y_j = \bar{y}$$

Ejercicio 2

$$\sigma_x^2 = \sum_{i=1}^{k} f_{i.}(x_i-\bar{x})^2 = \sum_{i=1}^{k} \frac{n_{i.}}{n}(x_i-\bar{x})^2 = \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{n_{ij}}{n}(x_i-\bar{x})^2 = \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{n_{.j}}{n}\frac{n_{ij}}{n_{.j}}(x_i-\bar{x})^2 =$$

$$\sum_{j=1}^{p}\sum_{i=1}^{k} f_{.j} f_i^j (x_i-\bar{x})^2 = \sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i-\bar{x})^2\right] = \sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i-\bar{x}_j+\bar{x}_j-\bar{x})^2\right] =$$

$$= \sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i-\bar{x}_j)^2 + \sum_{i=1}^{k} f_i^j (\bar{x}_j-\bar{x})^2 + 2\sum_{i=1}^{k} f_i^j (x_i-\bar{x}_j)(\bar{x}_j-\bar{x})\right] =^*$$

$$= \sum_{j=1}^{p} f_{.j} \left[\sum_{i=1}^{k} f_i^j (x_i - \bar{x}_j)^2 + \sum_{i=1}^{k} f_i^j (\bar{x}_j - \bar{x})^2 \right] = \sum_{j=1}^{p} f_{.j} \left[\sigma_{x,j}^2 + (\bar{x}_j - \bar{x})^2 \sum_{i=1}^{k} f_i^j \right] =$$

$$= \sum_{j=1}^{p} f_{.j} \left[\sigma_{x,j}^2 + (\bar{x}_j - \bar{x})^2 \right] = \sum_{j=1}^{p} f_{.j} \sigma_{x,j}^2 + \sum_{j=1}^{p} f_{.j} (\bar{x}_j - \bar{x})^2$$

*Donde se ha usado que:

$$2 \sum_{i=1}^{k} f_i^j (x_i - \bar{x}_j)(\bar{x}_j - \bar{x}) = 2(\bar{x}_j - \bar{x}) \sum_{i=1}^{k} f_i^j (x_i - \bar{x}_j) = 2(\bar{x}_j - \bar{x}) \left[\sum_{i=1}^{k} f_i^j x_i - \sum_{i=1}^{k} f_i^j \bar{x}_j \right] =$$

$$= 2(\bar{x}_j - \bar{x}) \left[\bar{x}_j - \bar{x}_j \sum_{i=1}^{k} f_i^j \right] = 2(\bar{x}_j - \bar{x}) \left[\bar{x}_j - \bar{x}_j \frac{1}{n_{.j}} \sum_{i=1}^{k} n_{ij} \right] = 2(\bar{x}_j - \bar{x}) \left[\bar{x}_j - \bar{x}_j \frac{n_{.j}}{n_{.j}} \right] = 0$$

$$\sigma_y^2 = \sum_{j=1}^{p} f_{.j} (y_j - \bar{y})^2 = \sum_{j=1}^{p} \frac{n_{.j}}{n} (y_j - \bar{y})^2 = \sum_{j=1}^{p} \sum_{i=1}^{k} \frac{n_{ij}}{n} (y_j - \bar{y})^2 = \sum_{j=1}^{p} \sum_{i=1}^{k} \frac{n_{i.}}{n} \frac{n_{ij}}{n_{i.}} (y_j - \bar{y})^2 =$$

$$\sum_{i=1}^{k} \sum_{j=1}^{p} f_{i.} f_j^i (y_j - \bar{y})^2 = \sum_{i=1}^{k} f_{i.} \left[\sum_{j=1}^{p} f_j^i (y_j - \bar{y})^2 \right] = \sum_{i=1}^{k} f_{i.} \left[\sum_{j=1}^{p} f_j^i (y_j - \bar{y}_i + \bar{y}_i - \bar{y})^2 \right] =$$

$$= \sum_{i=1}^{k} f_{i.} \left[\sum_{j=1}^{p} f_j^i (y_j - \bar{y}_i)^2 + \sum_{j=1}^{p} f_j^i (\bar{y}_i - \bar{y})^2 + 2 \sum_{j=1}^{p} f_j^i (y_j - \bar{y}_i)(\bar{y}_i - \bar{y}) \right] =^*$$

$$= \sum_{i=1}^{k} f_{i.} \left[\sum_{j=1}^{p} f_j^i (y_j - \bar{y}_i)^2 + \sum_{j=1}^{p} f_j^i (\bar{y}_i - \bar{y})^2 \right] = \sum_{i=1}^{k} f_{i.} \left[\sigma_{y,i}^2 + (\bar{y}_i - \bar{y})^2 \sum_{j=1}^{p} f_j^i \right] =$$

$$= \sum_{i=1}^{k} f_{i.} \left[\sigma_{y,i}^2 + (\bar{y}_i - \bar{y})^2 \right] = \sum_{i=1}^{k} f_{i.} \sigma_{y,i}^2 + \sum_{i=1}^{k} f_{i.} (\bar{y}_i - \bar{y})^2$$

*Donde se ha usado que:

$$2\sum_{j=1}^{p} f_j^i(y_j-\bar{y}_i)(\bar{y}_i-\bar{y}) = 2(\bar{y}_i-\bar{y})\sum_{j=1}^{p} f_j^i(y_j-\bar{y}_i) = 2(\bar{y}_i-\bar{y})\left[\sum_{j=1}^{p} f_j^i y_j - \sum_{j=1}^{p} f_j^i \bar{y}_i\right] =$$

$$= 2(\bar{y}_i-\bar{y})\left[\bar{y}_i - \bar{y}_i\sum_{j=1}^{p} f_j^i\right] = 2(\bar{y}_i-\bar{y})\left[\bar{y}_i - \bar{y}_i\frac{1}{n_{i.}}\sum_{j=1}^{p} n_{ij}\right] = 2(\bar{y}_i-\bar{y})\left[\bar{y}_i - \bar{y}_i\frac{n_{i.}}{n_{i.}}\right] = 0$$

Ejercicio 3

Demuestra que $\mu_{11} = m_{11} - m_{10}m_{01}$, siendo:

$$\mu_{11} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i-\bar{x})(y_j-\bar{y}) \qquad m_{11} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i y_j$$

$$m_{10} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i \qquad m_{01} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j$$

$$\mu_{11} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i-\bar{x})(y_j-\bar{y}) = \sum_{i=1}^{k}\sum_{j=1}^{p}(f_{ij}x_i y_j + f_{ij}\bar{x}\bar{y} - f_{ij}y_j\bar{x} - f_{ij}x_i\bar{y}) =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i y_j + \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}\bar{x}\bar{y} - \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j\bar{x} - \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i\bar{y} =$$

$$= m_{11} + \bar{x}\bar{y}\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij} - \bar{x}\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j - \bar{y}\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i =$$

$$= m_{11}+\bar{x}\bar{y}-\bar{x}\sum_{j=1}^{p}\sum_{i=1}^{k} f_{ij}y_j - \bar{y}\sum_{i=1}^{k} x_i\sum_{j=1}^{p} f_{ij} = m_{11}+\bar{x}\bar{y}-\bar{x}\sum_{j=1}^{p} y_j\sum_{i=1}^{k} f_{ij} - \bar{y}\sum_{i=1}^{k} x_i\sum_{j=1}^{p} f_{ij} =$$

$$= m_{11} + \bar{x}\bar{y} - \bar{x}\sum_{j=1}^{p} f_{.j}y_j - \bar{y}\sum_{i=1}^{k} f_{i.}x_i = m_{11} + \bar{x}\bar{y} - \bar{x}\bar{y} - \bar{x}\bar{y} = m_{11} - \bar{x}\bar{y} =$$

$$= m_{11} - \left[\sum_{i=1}^{k} f_{i.}x_i\right]\left[\sum_{j=1}^{p} f_{.j}y_j\right] = m_{11} - \left[\sum_{i=1}^{k} x_i \sum_{j=1}^{p} f_{ij}\right]\left[\sum_{j=1}^{p} y_j \sum_{i=1}^{k} f_{ij}\right] =$$

$$= m_{11} - \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i\right]\left[\sum_{j=1}^{p}\sum_{i=1}^{k} f_{ij}y_j\right] = m_{11} - \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j\right] =$$

$$= m_{11} - m_{10}m_{01}$$

Si X e Y son independientes, demostrar que:

i) $m_{rs} = m_{r0}m_{0s}$, $\mu_{rs} = \mu_{r0}\mu_{0s}$
ii) $\mu_{11} = 0$

*En este ejercicio aplicaremos lo visto en clase: cuando X e Y son independientes, entonces $f_{ij} = f_{i.}f_{.j}$.

Empecemos demostrando i):

$$m_{rs} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i^r y_j^s = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{i.}f_{.j}x_i^r y_j^s = \left[\sum_{i=1}^{k} f_{i.}x_i^r\right]\left[\sum_{j=1}^{p} f_{.j}y_j^s\right] =$$

$$= \left[\sum_{i=1}^{k} x_i^r \sum_{j=1}^{p} f_{ij}\right]\left[\sum_{j=1}^{p} y_j^s \sum_{i=1}^{k} f_{ij}\right] = \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i^r\right]\left[\sum_{j=1}^{p}\sum_{i=1}^{k} f_{ij}y_j^s\right] =$$

$$= \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i^r\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j^s\right] = m_{r0}m_{0s}$$

$$\mu_{rs} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \bar{x})^r (y_j - \bar{y})^s = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{i.}f_{.j}(x_i - \bar{x})^r (y_j - \bar{y})^s =$$

$$= \left[\sum_{i=1}^{k} f_{i.}(x_i - \bar{x})^r\right] \left[\sum_{j=1}^{p} f_{.j}(y_j - \bar{y})^s\right] = \left[\sum_{i=1}^{k}(x_i - \bar{x})^r \sum_{j=1}^{p} f_{ij}\right] \left[\sum_{j=1}^{p}(y_j - \bar{y})^s \sum_{i=1}^{k} f_{ij}\right] =$$

$$= \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \bar{x})^r\right] \left[\sum_{j=1}^{p}\sum_{i=1}^{k} f_{ij}(y_j - \bar{y})^s\right] =$$

$$= \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \bar{x})^r\right] \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(y_j - \bar{y})^s\right] = \mu_{r0}\mu_{0s}$$

La demostración de ii) es inmediata ya que es consecuencia de lo ya demostrado:

Antes de i) se ha demostrado que $\mu_{11} = m_{11} - m_{10}m_{01}$, y como X e Y son independientes y por lo último demostrado en i), esto es equivalente a $\mu_{11} = m_{11} - m_{11} = 0$.