Repaso Global Geometría

1 $U = \left\{ p(x) \in \mathbb{R}_2[x] \mid \int_0^1 p(x)\,dx = 0 \right\}$ $V = \mathbb{R}_2[x]$

Calcular base, dimensión y subespacio complementario

$$p(x) = a_0x^2 + a_1x + a_2$$

$$\int p(x) = \frac{3a_0x^3}{3} + \frac{a_1x^2}{2} + a_2x$$

$$\int_0^1 p(x) = 0 \Rightarrow \frac{a_0}{3} + \frac{a_1}{2} + a_2 = 0 \Rightarrow \underbrace{2a_0 + 3a_1 + 6a_2 = 0}_{\text{Una ecuación}}$$

cartesiana $\Rightarrow$ Dimensión 2

$$B_U = \{3x^2 - 1, 2x - 1\} \qquad \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 6 \neq 0$$

Subespacio complementario: $W = \mathcal{L}\{1\}$

2. $V = \mathbb{R}_n[x]$ $U_1 = \{p(x) \in \mathbb{R}_n[x] / p(1) + p'(1) = 0\}$

$U_2 = \{p(x) \in \mathbb{R}_n[x] \mid p(0) + p''(0) = 0\}$

¿Son subespacios? ¿$U_1 \oplus U_2$?

$\boxed{U_1}$ $p(x) = a_nx^n + \dots + a_0$

$$\left.\begin{array}{l} p(1) = a_n + \dots + a_0 \\ p'(1) = na_n + \dots + a_1 \end{array}\right\} p(1) + p'(1) = 0 \Rightarrow \underbrace{(n+1)a_n + \dots + 2a_1 + a_0 = 0}_{\text{Ec. cartesiana}}$$

$\boxed{U_2}$

$$\left.\begin{array}{l} p(0) = a_0 \\ p''(0) = 2a_2 \end{array}\right\} p(0) + p''(0) = 0 \Rightarrow \underbrace{2a_2 + a_0 = 0}_{\text{Ec. cartesiana}}$$

¿Es U_1 subespacio?

$p(x) \in U_1$ $q(x) \in U_1$ $a, b \in \mathbb{R}$ ~~$q(x) = b_nx^n + \dots + b_0$~~

¿$ap(x) + bq(x) \in U_1$?

~~$ap(x) + bq(x) = (aa_n + bb_n)x^n + \dots + aa_0 + bb_0$~~

~~$ap(1) + bq(1)$~~ ¿$(n+1)(aa_n + bb_n) + \dots +$

$$ap(x) + bq(x) = r(x)$$

¿$r(1) + r'(1) = 0$?

$$r(1) + r'(1) = ap(1) + bq(1) + ap'(1) + b \cdot q'(1) = 0$$

$$a(\underbrace{p(1) + p'(1)}_{0 \Rightarrow p(x) \in U_1}) + b(\underbrace{q(1) + q'(1)}_{0 \Rightarrow q(x) \in U_1}) = 0$$

¿Es U_2 subespacio?

$p(x), q(x) \in U_2 \quad a, b \in \mathbb{R}$

$ap(x) + bq(x) = r(x)$

¿$r(0) + r''(0) = 0$? $\Rightarrow ap(0) + bq(0) + ap'(0) + bq'(0) = 0$

$$a(\underbrace{p(0) + p'(0)}_{0 \Rightarrow p(x) \in U_2}) + b(\underbrace{q(0) + q'(0)}_{0 \Rightarrow q(x) \in U_2}) = 0$$

Como U_1 y U_2 tienen una ecuación cartesiana, su dimensión será n ~~1~~. Por la fórmula de las dimensiones:

$$\dim_{\mathbb{R}}(U_1 + U_2) + \dim_{\mathbb{R}}(U_1 \cap U_2) = \dim_{\mathbb{R}}(U_1) + \dim_{\mathbb{R}}(U_2)$$

$$= 2n$$

$\dim_{\mathbb{R}}(U_1 + U_2) \leq n+1 \Rightarrow$ Siempre se cumple

$\Rightarrow \dim_{\mathbb{R}}(U_1 \cap U_2) \geq 2n - n - 1 = n - 1$

Si $n = 1$, $\Rightarrow$ Suma directa (Se puede dar)

Si $n \geq 2$, no sería suma directa.

3. $u_1 = 1 - x + x^2$ Probar que $B = \{v_1, v_2, v_3\}$ es base

$u_2 = 1 + x^2$ Hallar $M_{B \leftarrow B_u}$.

$u_3 = 1 + x$

$u_1 = (1, -1, 1)_{B_u} \quad u_2 = (1, 0, 1)_{B_u} \quad u_3 = (1, 1, 0)_{B_u}$

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 - 1 - 1 = -1 \neq 0 \Rightarrow \text{Son L.I.}$$

$$M_{B_u \leftarrow B} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{B \leftarrow B_u} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = (-1)\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

4. $U_\lambda = \mathcal{L}\left\{\begin{pmatrix} 1-\lambda & -2 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & \lambda+1 \end{pmatrix}\right\}$

$W_\mu = \left\{\begin{pmatrix} x & y \\ z & t \end{pmatrix} : \mu x + y + z = 0, x + \mu y + z = 0, x + y + \mu z = 0\right\}$

a) Dimensiones de U_λ y W_μ según los valores de λ y μ.

$\boxed{U_\lambda}$ $\begin{pmatrix} 1-\lambda & 2 \\ -2 & -1 \\ 0 & 0 \\ -4 & \lambda+1 \end{pmatrix}$ $\begin{vmatrix} 1-\lambda & 2 \\ -2 & -1 \end{vmatrix} = -1+\lambda+4 = \lambda+3 \Rightarrow \lambda+3=0 \Rightarrow \lambda=-3$

$\begin{vmatrix} 1-\lambda & 2 \\ -4 & \lambda+1 \end{vmatrix} = -\lambda^2+1+8 \Rightarrow \lambda^2-9=0 \Rightarrow \lambda=\pm 3$

Si $\lambda=3$, $\dim_{\mathbb{R}} U_\lambda = 1$. Si $\lambda \neq 3$, $\dim_{\mathbb{R}} U_\lambda = 2$

$\boxed{W_\mu}$ $\begin{cases} \mu x + y + z = 0 \\ x + \mu y + z = 0 \\ x + y + \mu z = 0 \end{cases}$ $\left(\begin{array}{ccc|c} \mu & 1 & 1 & 0 \\ 1 & \mu & 1 & 0 \\ 1 & 1 & \mu & 0 \end{array}\right)$

$\begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = \mu^3+1+1-\mu-\mu-\mu \Rightarrow \mu^3-3\mu+2=0$

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$\mu^2+\mu-2=0$

$\mu = \dfrac{-1\pm\sqrt{9}}{2} = \begin{cases} -2 \\ 1 \end{cases}$

Si $\mu=-2 \Rightarrow \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = -5 \neq 0 \Rightarrow \dim_{\mathbb{R}} W_\mu = 2$

Si $\mu = 1 \Rightarrow \dim_{\mathbb{R}} W_\mu = 3$

Si $\mu \neq -2$ y $\mu \neq 1 \Rightarrow \dim_{\mathbb{R}} W_\mu = 1$

b) 1) $U_\lambda \oplus W_\mu$ 2) $M_2(\mathbb{R}) = U_\lambda + W_\mu$

~~Para que~~ $\dim_{\mathbb{R}} U_\lambda + \dim_{\mathbb{R}} W_\mu = \dim_{\mathbb{R}}(U_\lambda \cap W_\mu) + \dim(U_\lambda + W_\mu)$

Por esta fórmula, sabemos que $\mu \neq 2$ y $\mu \neq 1$. Es decir, si $\mu \neq 2$ o $\mu \neq 1$, 1) y 2) no se cumplen, sea cual sea el valor de λ.

Si $\lambda=3, \mu=-2$, tampoco 1) ni 2).

Si $\lambda = 3$ y $\mu = 1$:

$U_\lambda + W_\mu = U_3 + W_1 = \mathcal{L}\left\{ \begin{pmatrix} -2 & 1 \\ 1, & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ 0 & 4 \end{pmatrix} \right\}$

$\{ x+y+z=0 \Rightarrow$ Sol.: $(-y-z, y, z, t)$

$\begin{pmatrix} -2 & -1 & -1 & -2 \\ 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$ $\begin{vmatrix} -1 & -1 & -2 \\ 1 & 1 & -2 \\ 0 & 1 & 4 \end{vmatrix} = -4-2+4-2 = -4 \neq 0$

$\Downarrow$

$\dim_{\mathbb{R}}(U_\lambda + W_\mu)$

(Se verifican 1) y 2))

Si $\lambda \neq 3$ y $\mu = 1$:

$$W_1 = \mathcal{L}\left\{\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$$

$$U_\lambda = \mathcal{L}\left\{\begin{pmatrix} 1-\lambda & -2 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & \lambda+1 \end{pmatrix}\right\}$$

$$\begin{pmatrix} -2 & -1 & -1 & 1-\lambda & 2 \\ 1 & 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & \lambda+1 \end{pmatrix} \quad \begin{vmatrix} -2 & -1 & -1 & 1-\lambda \\ 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1-\lambda \\ 1 & 1 & -2 \\ 0 & 1 & -4 \end{vmatrix} =$$

$$= 4 + 1 - \lambda - 4 - 2 = -\lambda - 1$$

$$-\lambda - 1 = 0 \Rightarrow \lambda = -1$$

$$\begin{vmatrix} -2 & -1 & -1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \lambda+1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & \lambda+1 \end{vmatrix} = -\lambda - 1 + 2 + \lambda + 1 - 1 = 1 \neq 0$$

Sea cual sea el valor de λ, $\dim_{\mathbb{R}}(W_1 + U_\lambda) = 4$, pero no es suma directa ya que $W_1 \cap U_\lambda$ tiene dimensión 1.

Si $\lambda \neq 3$ y $\mu = 2$:

$$U_\lambda = \mathcal{L}\left\{\begin{pmatrix} 1-\lambda & -2 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & \lambda+1 \end{pmatrix}\right\}$$

$$\begin{cases} 2x + y + z = 0 \Rightarrow y = -2x - z \\ x + 2y + z = 0 \end{cases} \quad x - 4x - 2z + z = 0 \Rightarrow x = -\frac{z}{3} \quad y = -\frac{z}{3}$$

Sol.: $\left(-\frac{z}{3}, -\frac{z}{3}, z, t\right)$

$$W_2 = \mathcal{L}\left\{\begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\right\}$$

$$U_\lambda + W_2 = \mathcal{L}\left\{\begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1-\lambda & -2 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & \lambda+1 \end{pmatrix}\right\}$$

$$\begin{pmatrix} -1 & -1 & 1-\lambda & 2 \\ -1 & -1 & -2 & -1 \\ 3 & 3 & 0 & 0 \\ 0 & 1 & -4 & \lambda+1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 1-\lambda & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 3 & 0 & 0 \\ -1 & 1 & -4 & \lambda+1 \end{pmatrix}$$

$$-(-1) \cdot \begin{vmatrix} -1 & 1-\lambda & 2 \\ -1 & -2 & -1 \\ 3 & 0 & 0 \end{vmatrix} = -3 + 3\lambda + 12 \Rightarrow 3\lambda = -9 \Rightarrow \lambda = -3$$

Si $\lambda = -3$, no se cumplen ni $\Rightarrow$ 1) ni 2)

$\Downarrow$

Si $\lambda \neq -3$, se cumplen 1) y 2), por la fórmula de las dimensiones.

5.

$$U_\lambda = \mathcal{L}\left\{\begin{pmatrix} \lambda & \lambda \\ \lambda & -1 \end{pmatrix}, \begin{pmatrix} 0 & -\lambda \\ -\lambda & 2 \end{pmatrix}\right\}$$

Para cada λ, hallar la dimensión de $U_\lambda \cap W_\lambda$.

$$W_\lambda = \mathcal{L}\left\{\begin{pmatrix} 2 & -2 \\ -2 & \lambda \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}\right\}$$

$S_2(\mathbb{R})$

Primero estudiemos las dimensiones de U_λ y W_λ:

$\boxed{U_\lambda}$
$$\begin{pmatrix} \lambda & 0 \\ \lambda & -\lambda \\ \lambda & -\lambda \\ -1 & 2 \end{pmatrix}$$

$\begin{vmatrix} \lambda & 0 \\ \lambda & -\lambda \end{vmatrix} = -\lambda^2 - \lambda \Rightarrow \lambda^2 + \lambda = 0 \Rightarrow \lambda(1+\lambda) = 0$ $\boxed{\lambda = 0 \quad \lambda = -1}$

$\begin{vmatrix} \lambda & 0 \\ -1 & 2 \end{vmatrix} = 2\lambda \Rightarrow 2\lambda = 0 \Rightarrow \boxed{\lambda = 0}$

Para $\lambda = 0$, $\dim_{\mathbb{R}} U_\lambda = 1$

Para $\lambda \neq 0$, $\dim_{\mathbb{R}} U_\lambda = 2$

$\boxed{W_\lambda}$
$$\begin{pmatrix} 2 & 3 \\ -2 & 1 \\ -2 & 1 \\ \lambda & -3 \end{pmatrix}$$

$\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} = 2 + 6 = 8 \neq 0 \Rightarrow \dim_{\mathbb{R}} W_\lambda = 2$ para $\lambda \in \mathbb{R}$

$\boxed{U_\lambda \cap W_\lambda}$ Haré uso de la fórmula de las dimensiones por lo que me centraré en $U_\lambda + W_\lambda$ para deducir la dimensión de $U_\lambda \cap W_\lambda$:

$$\dim_{\mathbb{R}}(U_\lambda + W_\lambda) + \dim_{\mathbb{R}}(U_\lambda \cap W_\lambda) = \dim_{\mathbb{R}}(U_\lambda) + \dim_{\mathbb{R}}(W_\lambda)$$

Para $\lambda = 0 \Rightarrow \dim_{\mathbb{R}} U_\lambda = 1$, $\dim_{\mathbb{R}} W_\lambda = 2$:

Juntamos las bases
$$\begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & -3 \end{pmatrix}$$

$\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} = 8 \neq 0$

$\begin{vmatrix} 0 & 2 & 3 \\ 0 & -2 & 1 \\ -1 & 0 & -3 \end{vmatrix} = -2 - 6 = -8 \neq 0 \Rightarrow$ Rango 3

$\Downarrow$

$\dim_{\mathbb{R}}(U_\lambda + W_\lambda) = 3$

$\Downarrow$

$\dim_{\mathbb{R}}(U_\lambda \cap W_\lambda) = 0$

Para $\lambda \neq 0 \Rightarrow \dim_{\mathbb{R}} U_\lambda = 2$, $\dim_{\mathbb{R}} W_\lambda = 2$

Juntamos las bases
$$\begin{pmatrix} \lambda & 0 & 2 & 3 \\ \lambda & -\lambda & -2 & 1 \\ \lambda & -\lambda & -2 & 1 \\ -1 & 2 & \lambda & -3 \end{pmatrix} \rightarrow \xcancel{\begin{pmatrix} \lambda & 0 & 8 & 3 \\ \lambda & -\lambda & 0 & 1 \\ \lambda & -\lambda & 0 & 1 \\ -1 & 2 & \lambda-6 & -3 \end{pmatrix}}$$

$$\begin{pmatrix} \lambda & 0 & 2 & 3 \\ \lambda & -\lambda & -2 & 1 \\ -1 & 2 & \lambda & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 2 & 3 \\ -\lambda & -2 & 1 \\ 2 & \lambda & -3 \end{vmatrix} = 4 - 3\lambda^2 + 12 - 6\lambda \Rightarrow 3\lambda^2 + 6\lambda - 16 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36+192}}{6} = \begin{cases} \to -1 - \frac{\sqrt{57}}{3} \\ \to -1 + \frac{\sqrt{57}}{3} \end{cases}$$

$$\begin{vmatrix} \lambda & 2 & 3 \\ \lambda & -2 & 1 \\ -1 & \lambda & -3 \end{vmatrix} = 6\lambda - 2 + 3\lambda^2 - 6 + 6\lambda - \lambda^2 \Rightarrow 2\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda^2 + 6\lambda - 4 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36+16}}{2} = \begin{cases} \to -3 - \sqrt{13} \\ \to -3 + \sqrt{13} \end{cases}$$

Como no hay un λ común, $\dim_{\mathbb{R}}(V_\lambda + W_\lambda) = 3 \Rightarrow$ $\boxed{\dim_{\mathbb{R}}(V_\lambda \cap W_\lambda) = 1}$

6. Endomorfismo f de $\mathbb{R}_2[x]$ que verifique que $f \circ f = f$ y

$Im(f) = \{p(x) \in \mathbb{R}_2[x] : p(0) + p'(0) = 0,\ p'(0) + p''(0) = 0\}$

$p(x) = ax^2 + bx + c \qquad p(0) = c \qquad b + c = 0$

$p'(x) = 2ax + b \qquad p'(0) = b \qquad 2a + b = 0$ } Sol.: $(\frac{c}{2}, -c, c)$

$p''(x) = 2a \qquad p''(0) = 2a \qquad$ 2 ec. cant. $\Rightarrow \dim_{\mathbb{R}} Im(f) = 1$

$Im(f) = \mathcal{L}\{x^2 - 2x + 2\} \Rightarrow Ker(f)$ tiene dimensión 2

$B = \{1, x, x^2\} \Rightarrow f(1) = x^2 - 2x + 2$

$f \circ f = f \Rightarrow f(f(1)) = f(1)$

$f(x^2 - 2x + 2) = x^2 - 2x + 2$

$f(x^2) - 2f(x) + 2f(1) = x^2 - 2x + 2$

$f(x^2) - 2f(x) = -x^2 + 2x - 2$

~~¿$f(x^2) = -x^2$? $f(x) = -x + 1$~~

$\bar{B} = \{1, x, x^2 - 2x + 2\} \qquad f(x^2 - 2x + 2) = x^2 - 2x + 2$

$f(1) = 0$

¿$f \circ f = f$? $\quad f(x) = 0$

$f \circ f(x^2 - 2x + 2) = f(f(x^2 - 2x + 2)) = f(x^2 - 2x + 2) = x^2 - 2x + 2$ ✓

$f \circ f(1) = f(f(1)) = f(0) = 0$ ✓ $\quad f \circ f(x) = f(f(x)) = f(0) = 0$ ✓

$$M(f; \bar{B}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\bar{B} \leftarrow B_u} = (M_{B_u \leftarrow \bar{B}})^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}^{-1}$$

$$M(f; B_u) = M_{B_u \leftarrow \bar{B}} \cdot M(f; \bar{B}) \cdot M_{\bar{B} \leftarrow B_u}$$