Ejercicio Examen Convocatoria Extraordinaria 2020

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad f_A : \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}^2(\mathbb{R})$ matriz en la base usual es A

¿Son A y C equivalentes?
Si lo son, encontrar matrices regulares P y Q de orden 2×2 tales que $C = Q^{-1}AP$. Obtener bases $\mathcal{B}$ y $\bar{\mathcal{B}}$ de $\mathbb{R}^2$ tales que C sea la matriz asociada a f_A.

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0 \Rightarrow$ Son eq. ya que A tiene rango 1 también.

$$A \sim C \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = Q_1^{-1} A P_1 = Q_2^{-1} C P_2$$

$$C = \underbrace{Q_2 Q_1^{-1}}_{Q^{-1}} A \underbrace{P_1 P_2^{-1}}_{P}$$

$A = M(f_A : B_u)$ ~~Hallemos un~~

$\hookrightarrow$ Debido a esto, sabemos que $(0,1) \in \text{Ker}(f_A)$

Si como base de llegada tomamos $B' = \{(1,1),(0,1)\}$:

$$M(f_A ; B' \leftarrow B_u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Entonces, } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \underset{Q_1^{-1}}{\underset{\shortparallel}{M_{B' \leftarrow B_u}}} \cdot A \cdot \underset{P_1}{\underset{\shortparallel}{I_2}}$$

$$\left(M_{B_u \leftarrow B'}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = Q_1^{-1}$$

$C = M(g : B_u) \Rightarrow$ Hallemos un vector del $\text{Ker}(g)$:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases} \quad \text{Sol.: } (-2y, y) \quad (-2,1) \in \text{Ker}(g)$$

~~Debemos hallar también~~

Como $f(1,0) = (1,2)$ y $f(-2,1) = (0,0)$, si tomamos $\overline{B} = \{(1,0),(-2,1)\}$ y $\overline{B'} = \{(1,2),(0,1)\}$:

$$M(g ; \overline{B'} \leftarrow \overline{B}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \underset{Q_2^{-1}}{\underset{\shortparallel}{M_{\overline{B'} \leftarrow B_u}}} \cdot \underset{C}{\underset{\shortparallel}{M(g ; B_u)}} \cdot \underset{P_2}{\underset{\shortparallel}{M_{B_u \leftarrow \overline{B}}}}$$

$$Q_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$P_2^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Y ahora volvemos a que:

$$C = \underbrace{Q_2 Q_1^{-1}} A \underbrace{P_1 P_2^{-1}} \longrightarrow P = P_2^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = Q^{-1}$$

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Comprobación

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \checkmark$$

Para el segundo apartado:

$$C = \underbrace{Q^{-1}}_{B' \leftarrow B_u} \underbrace{A}_{B_u} \underbrace{P}_{B_u \leftarrow B}$$

$$P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow B = \{(1,0),(2,1)\}$$

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow B' = \{(1,1),(0,1)\}$$

¿Son semejantes A y C?

$$C = P^{-1}AP \Rightarrow PC = AP$$

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} x+2y & 2x+4y \\ z+2t & 2z+4t \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$x + 2y = x \Rightarrow y = 0$

$2x + 4y = y \Rightarrow x = 0$

$z + 2t = z \Rightarrow t = 0$

$2z + 4t = t \Rightarrow z = 0$

No son semejantes

Ejercicio Convocatoria Extraordinaria

En $M_2(\mathbb{R})$ se considera el subespacio $S_2(\mathbb{R})$ formado por las matrices simétricas y

$$U_\lambda = \mathcal{L}\left\{\begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix}, \begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix}\right\}$$

a) Calcular, para cada $\lambda \in \mathbb{R}$, una base de U_λ, $U_\lambda \cap S_2(\mathbb{R})$, $U_\lambda + S_2(\mathbb{R})$.

$$S_2(\mathbb{R}) = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$$M_2(\mathbb{R}) = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$U_\lambda = \mathcal{L}\{(1,-1,\lambda+1,2),(\lambda,-\lambda,2,\lambda-2)\}$$

$$\begin{pmatrix} 1 & -1 & \lambda+1 & 2 \\ \lambda & -\lambda & 2 & \lambda-2 \end{pmatrix} \qquad \begin{vmatrix} 1 & -1 \\ \lambda & -\lambda \end{vmatrix} = -\lambda+\lambda = 0$$

$$\begin{vmatrix} 1 & \lambda+1 \\ \lambda & 2 \end{vmatrix} = 2-\lambda^2-\lambda$$

$$\lambda^2+\lambda-2=0$$

$$\lambda = \frac{-1\pm\sqrt{1+8}}{2} = \begin{cases} -2 \\ 1 \end{cases}$$

$$\begin{vmatrix} 1 & 2 \\ \lambda & \lambda-2 \end{vmatrix} = \lambda-2-2\lambda \Rightarrow \lambda+2=0 \Rightarrow \lambda=-2$$

Rango $=1 \Leftrightarrow \lambda = 2 \Rightarrow \dim_K(U)=1$

Para $\lambda=-2$:

$$U_\lambda = \mathcal{L}\left\{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\right\}$$

$\boxed{U_\lambda + S_2(\mathbb{R})}$

Se unen las bases

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -1+1=0 \Rightarrow U_\lambda + S_2(\mathbb{R}) = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$\boxed{U_\lambda \cap S_2(\mathbb{R})}$

$$\underbrace{\dim_{\mathbb{R}} U_\lambda}_{1} + \underbrace{\dim_{\mathbb{R}} S_2(\mathbb{R})}_{3} = \underbrace{\dim_{\mathbb{R}}(U_\lambda + S_2(\mathbb{R}))}_{3} + \underbrace{\dim_{\mathbb{R}}(U_\lambda \cap S_2(\mathbb{R}))}_{1}$$

3 ec. cartesianas

$\boxed{U_\lambda}$

$$\begin{pmatrix} 1 & x \\ -1 & y \\ -1 & z \\ 2 & t \end{pmatrix} \text{ Rango } 1 \qquad \begin{vmatrix} 1 & x \\ -1 & y \end{vmatrix} = x+y=0$$

$$\begin{vmatrix} 1 & x \\ -1 & z \end{vmatrix} = x+z=0$$

$$\begin{vmatrix} 1 & x \\ 2 & t \end{vmatrix} = t-2x=0 \qquad \begin{cases} x+y=0 \\ x+z=0 \\ 2x-t=0 \end{cases}$$

$\boxed{S_2(\mathbb{R})}$

$$\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 1 & z \\ 0 & 1 & 0 & t \end{pmatrix} \text{ Rango } 3 \qquad \begin{vmatrix} 0 & 1 & y \\ 0 & 1 & z \\ 1 & 0 & t \end{vmatrix} = 0 \Rightarrow y-z=0$$

$$\begin{cases} x+y=0 \\ x+z=0 \\ 2x-t=0 \\ y-z=0 \Rightarrow 1^a - 2^a \text{ (La eliminamos} \end{cases} \qquad \text{Sol.: } \left(\frac{t}{2}, -\frac{t}{2}, -\frac{t}{2}, t\right)$$

$$U_\lambda \cap S_2(\mathbb{R}) = \mathcal{L}\left\{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\right.$$

Para $\lambda \neq -2$:

$$U_\lambda = \mathcal{L}\left\{\begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix}, \begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix}\right\}$$

$\boxed{U + S_2(\mathbb{R})}$

$$\begin{pmatrix} 1 & \lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 & 1 \\ \lambda+1 & 2 & 0 & 0 & 1 \\ 2 & \lambda-2 & 0 & 1 & 0 \end{pmatrix} \quad \text{Rango} \leq 4$$

$$\begin{vmatrix} \lambda & 1 & 0 & 0 \\ -\lambda & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ \lambda-2 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} -\lambda & 0 & 1 \\ 2 & 0 & 1 \\ \lambda-2 & 1 & 0 \end{vmatrix} = -2-\lambda$$

$$-2-\lambda=0 \Rightarrow \lambda=-2$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ \lambda+1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 0 & 1 \\ \lambda+1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\lambda-1-1 = -\lambda-2=0 \Rightarrow \lambda=-2$$

NO PUES HEMOS SUPUESTO $\lambda \neq -2$

Por lo tanto, $\dim_{\mathbb{K}}(U_\lambda + S_2(\mathbb{R})) = 4$ para $\lambda \neq -2$:

$$U_\lambda + S_2(\mathbb{R}) = \mathcal{L}\{(1,-1,\lambda+1,2),(1,0,0,0),(0,0,0,1),(0,1,1,0)\}$$

$\boxed{U_\lambda \cap S_2(\mathbb{R})} \Rightarrow \dim_{\mathbb{K}}(U_\lambda \cap S_2(\mathbb{R})) = 1$

$\boxed{S_2(\mathbb{R})}$ $y - z = 0$

$\boxed{U_\lambda}$ $\begin{pmatrix} 1 & \lambda & x \\ -1 & -\lambda & y \\ \lambda+1 & 2 & z \\ 2 & \lambda-2 & t \end{pmatrix}$ Rango 2

$$\begin{vmatrix} 1 & \lambda & x \\ 2 & \lambda-2 & t \\ -1 & -\lambda & y \end{vmatrix} = 0$$

$$\Downarrow$$

$$(\lambda-2)y - \lambda t - 2\lambda x + (\lambda-2)x - 2\lambda y + t\lambda = 0$$

$$\Rightarrow (-\lambda-2)x + (-\lambda-2)y = 0$$

$$\begin{vmatrix} 1 & \lambda & x \\ 2 & \lambda-2 & t \\ \lambda+1 & 2 & z \end{vmatrix} = 0 \Rightarrow (\lambda-2)z + (\lambda^2+\lambda)t + 4x - (\lambda^2-\lambda-2)x - 2\lambda z - 2t = 0$$

$$\Rightarrow (-\lambda^2+\lambda+6)x + (-\lambda-2)z + (\lambda^2+\lambda-2)t = 0$$

Ec. cartesianas de $U_\lambda \cap S_2(\mathbb{R})$:

$$\begin{cases} y - z = 0 \Rightarrow y = z \\ (-\lambda-2)x + (-\lambda-2)y = 0 \Rightarrow x = -z \\ (-\lambda^2+\lambda+6)x + (-\lambda-2)z + (\lambda^2+\lambda-2)t = 0 \end{cases}$$

$$\Rightarrow (\lambda^2-2\lambda-8)z + (\lambda^2+\lambda-2)t = 0$$

Sol.: $\left(-z, z, z, \left(\frac{4-\lambda}{\lambda-1}\right)z\right)$ con $\lambda \neq 1$

$$U_\lambda \cap S_2(\mathbb{R}) = \mathcal{L}\left\{\begin{pmatrix} -1 & 1 \\ 1 & \frac{4-\lambda}{\lambda-1} \end{pmatrix}\right\}$$

b) Para $\lambda = 1$, ampliar la base de U_1 obtenida en el apartado anterior hasta una base B de $\mathcal{M}_2(\mathbb{R})$. Obtener la base dual de B. Calcular el anulador de U_1.

$$B = \left\{\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 2 & -1 & 0 \end{vmatrix} = 1-4 = -3 \neq 0 \Rightarrow \text{Forman base}$$

$B_u^* = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ $\quad B^* = \{\psi_1, \psi_2, \psi_3, \psi_4\}$

$\varphi = a\varphi_1 + b\varphi_2 + c\varphi_3 + d\varphi_4$

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & -1 & 1 & 1 \end{pmatrix}} = I_3$$

$C(\ )^{-1}?$
$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$B^* = \left\{(0,0,1,-1), \left(\frac{1}{3}, \frac{2}{3}, -1, 1\right), \left(\frac{2}{3}, \frac{1}{3}, -1, 0\right), (0,0,0,1)\right\}$$

$$\underset{2}{\underset{\shortparallel}{\dim_K \mathrm{an}(V_\lambda)}} = \underset{4}{\underset{\shortparallel}{\dim_K M_2(\mathbb{R})}} - \underset{2}{\underset{\shortparallel}{\dim_K (V_\lambda)}}$$

$$\left.\begin{aligned} &\varphi\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} = 0 \Rightarrow a-b+2c+2d=0 \\ &\varphi\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = 0 \Rightarrow a-b+2c-d=0 \end{aligned}\right\} d=0$$

$$a-b+2c=0$$

Sol.: $(b-2c, b, c, 0)$

$$an(V) = \mathcal{L}\{-\varphi_1+\varphi_2+\varphi_3, -2\varphi_1+\varphi_3\}$$