Distribución binomial (demostraciones)

$$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x},\quad x=0,1,\ldots,n;\qquad M_X(t)=\left(pe^t+1-p\right)^n,\quad t\in\mathbb{R}.$$

- **Media:** $E[X]=np$

  ◇ *Cálculo a partir de la definición*:

$$E[X]=\sum_{x=0}^{n}x\binom{n}{x}p^x(1-p)^{n-x}=\sum_{x=1}^{n}x\binom{n}{x}p^x(1-p)^{n-x}\overset{1}{=}np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}(1-p)^{n-x}=\left[\begin{array}{c}m=n-1\\ y=x-1\end{array}\right]$$
$$=np\sum_{y=0}^{m}\binom{m}{y}p^y(1-p)^{m-y}=np.$$

  ◇ *Cálculo a partir de la función generatriz de momentos*:

$$E[X]=\left.\frac{dM_X(t)}{dt}\right|_{t=0}=n\left(pe^t+1-p\right)^{n-1}pe^t\Big|_{t=0}=np.$$

- **Momento de orden dos:** $E[X^2]=n(n-1)p^2+np$

  ◇ *Cálculo a partir de la definición*: $E[X^2]=E[X(X-1)]+E[X]=E[X(X-1)]+np$

$$E[X(X-1)]=\sum_{x=0}^{n}x(x-1)\binom{n}{x}p^x(1-p)^{n-x}=\sum_{x=2}^{n}x(x-1)\binom{n}{x}p^x(1-p)^{n-x}\overset{2}{=}$$
$$=n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}(1-p)^{n-x}=\left[\begin{array}{c}m=n-2\\ y=x-2\end{array}\right]=n(n-1)p^2\sum_{y=0}^{m}\binom{m}{y}p^y(1-p)^{m-y}=$$
$$=n(n-1)p^2.$$

  ◇ *Cálculo a partir de la función generatriz de momentos*:

$$E[X^2]=\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}=n\Big[(n-1)\left(pe^t+1-p\right)^{n-2}p^2e^{2t}+pe^t\left(pe^t+1-p\right)^{n-1}\Big]_{t=0}=n(n-1)p^2+np.$$

- **Varianza:** $Var[X]=E[X^2]-(E[X])^2=np(1-p).$

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[1] $x\neq 0\Rightarrow x\binom{n}{x}=n\binom{n-1}{x-1}$

[2] $x\neq 0,1\Rightarrow x(x-1)\binom{n}{x}=n(n-1)\binom{n-2}{x-2}$

Distribución de Poisson (demostraciones)

$$P(X=x)=e^{-\lambda}\frac{\lambda^x}{x!},\quad x=\mathbb{N}\cup\{0\};\qquad M_X(t)=e^{\lambda\left(e^t-1\right)},\quad t\in\mathbb{R}.$$

Aproximación de las probabilidades binomiales por las de Poisson:

$$X_n\to B(n,p_n)\ \Rightarrow \lim_{\substack{n\to+\infty\\ np_n\to\lambda}} P(X_n=x)=P(Y=x),\quad x\in\mathbb{N}\cup\{0\},\ \text{con } Y\to\mathcal{P}(\lambda).$$

Demostración:

$$P(X_n=x)=\binom{n}{x}p_n^x(1-p_n)^{n-x}=\frac{n!}{x!(n-x)!}\frac{(np_n)^x}{n^x}(1-p_n)^{n-x}=\frac{(np_n)^x}{x!}\frac{n(n-1)\cdots(n-x+1)}{n^x}(1-p_n)^{n-x}$$

- $\displaystyle\lim_{\substack{n\to+\infty\\ np_n\to\lambda}}\frac{(np_n)^x}{x!}=\frac{\lambda^x}{x!}.$
- $\displaystyle\lim_{n\to+\infty}\frac{n(n-1)\cdots[n-(x-1)]}{n^x}=1$ (cociente de polinomios de grado x).
- $\displaystyle\lim_{\substack{n\to+\infty\\ np_n\to\lambda}} p_n=0\Rightarrow\lim_{\substack{n\to+\infty\\ np_n\to\lambda}}(1-p_n)^{n-x}=e^{\lim\limits_{\substack{n\to+\infty\\ np_n\to\lambda}}(n-x)(-p_n)}=e^{-\lambda}.$ $\square$

- **Media:** $E[X]=\lambda$
  - ◇ *Cálculo a partir de la definición*:

$$E[X]=\sum_{x=0}^{+\infty}x\frac{e^{-\lambda}\lambda^x}{x!}=\sum_{x=1}^{+\infty}x\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\sum_{x=1}^{+\infty}x\frac{\lambda^x}{x(x-1)!}=\lambda e^{-\lambda}\sum_{x=1}^{+\infty}\frac{\lambda^{x-1}}{(x-1)!}=[y=x-1]=\lambda e^{-\lambda}\sum_{y=0}^{+\infty}\frac{\lambda^y}{y!}=\lambda.$$

  - ◇ *Cálculo a partir de la función generatriz de momentos*:

$$E[X]=\left.\frac{dM_X(t)}{dt}\right|_{t=0}=\left.\lambda e^t e^{\lambda(e^t-1)}\right|_{t=0}=\lambda.$$

- **Momento de orden dos:** $E[X^2]=\lambda^2+\lambda$
  - ◇ *Cálculo a partir de la definición*: $E[X^2]=E[X(X-1)]+E[X]=E[X(X-1)]+\lambda$

$$E[X(X-1)]=\sum_{x=0}^{+\infty}x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\sum_{x=2}^{+\infty}x(x-1)\frac{\lambda^x}{x(x-1)(x-2)!}=\lambda^2e^{-\lambda}\sum_{x=2}^{+\infty}\frac{\lambda^{x-2}}{(x-2)!}=[y=x-2]=\lambda^2$$

  - ◇ *Cálculo a partir de la función generatriz de momentos*:

$$E[X^2]=\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}=\lambda\left.\Big(e^te^{\lambda(e^t-1)}+\lambda e^{2t}e^{\lambda(e^t-1)}\Big)\right|_{t=0}=\lambda(1+\lambda)..$$

- **Varianza:** $Var[X]=E[X^2]-(E[X])^2=\lambda.$

Distribución geométrica (demostraciones)

$$P(X=x)=p(1-p)^x, \quad x=\mathbb{N}\cup\{0\}$$

$$F_X(x)=\begin{cases}0, & x<0\\ 1-(1-p)^{[x]+1}, & x\geq 0\end{cases} ; \qquad M_X(t)=\frac{p}{1-(1-p)e^t}, \quad t<-\ln(1-p).$$

- **Media:** $E[X]=\dfrac{1-p}{p}\cdot$

  ◇ *Cálculo a partir de la definición*:

$$E[X]=\sum_{x=0}^{+\infty}xp(1-p)^x=p(1-p)\sum_{x=0}^{+\infty}x(1-p)^{x-1}={}^3=p(1-p)\frac{1}{p^2}=\frac{1-p}{p}\cdot$$

  ◇ *Cálculo a partir de la función generatriz de momentos*:

$$E[X]=\left.\frac{dM_X(t)}{dt}\right|_{t=0}=\left[\frac{p(1-p)e^t}{(1-(1-p)e^t)^2}\right]_{t=0}=\frac{1-p}{p}\cdot$$

- **Momento de orden dos:** $E[X^2]=\dfrac{(1-p)(2-p)}{p^2}$

  ◇ *Cálculo a partir de la definición*: $E[X^2]=E[X(X-1)]+E[X]=\dfrac{2(1-p)^2}{p^2}+\dfrac{1-p}{p}=\dfrac{(1-p)(2-p)}{p^2}\cdot$

$$E[X(X-1)]=\sum_{x=0}^{+\infty}x(x-1)p(1-p)^x=p(1-p)^2\sum_{x=0}^{+\infty}x(x-1)(1-p)^{x-2}={}^4=p(1-p)^2\frac{2}{p^3}=\frac{2(1-p)^2}{p^2}\cdot$$

  ◇ *Cálculo a partir de la función generatriz de momentos*:

$$E[X^2]=\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}=\left[\frac{p(1-p)(2-p)e^t}{(1-(1-p)e^t)^3}\right]_{t=0}=\frac{(1-p)(2-p)}{p^2}\cdot$$

- **Varianza:** $Var[X]=E[X^2]-(E[X])^2=\dfrac{1-p}{p^2}\cdot$

Propiedad de falta de memoria:

$$X\to G(p)\ \Rightarrow\ P\left(X\geq h+k/X\geq h\right)=P\left(X\geq k\right), \quad \forall h,k\in\mathbb{N}\cup\{0\}.$$

Demostración:

$$P(X\geq x)=1-P(X<x)=1-F_X(x-1)=1-[1-(1-p)^x]=(1-p)^x, \quad \forall x\in\mathbb{N}\cup\{0\}.$$

$$\Big\Downarrow\, k,h\in\mathbb{N}\cup\{0\}$$

$$P(X\geq h+k/X\geq h)=\frac{P(X\geq h+k, X\geq h)}{P(X\geq h)}=\frac{P(X\geq h+k)}{P(X\geq h)}=\frac{(1-p)^{h+k}}{(1-p)^h}=(1-p)^k=P(X\geq k). \qquad \square$$

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[3] $|a|<1\ \Rightarrow\ \displaystyle\sum_{x=0}^{+\infty}xa^{x-1}=\frac{1}{(1-a)^2}$

[4] $|a|<1\ \Rightarrow\ \displaystyle\sum_{x=0}^{+\infty}x(x-1)a^{x-2}=\frac{2}{(1-a)^3}$

Distribución binomial negativa (demostraciones)

$$P(X=x)=\binom{x+k-1}{x}(1-p)^x p^k,\quad x\in\mathbb{N}\cup\{0\}\,;\qquad M_X(t)=\left[\frac{p}{1-(1-p)e^t}\right]^k,\quad t<-\ln(1-p).$$

$$M_X(t)=[M_Y(t)]^k,\ \text{con } Y\to G(p).$$

- **Media:** $E[X]=k\dfrac{1-p}{p}\cdot$

$\diamond\ E[X]=\left.\dfrac{dM_X(t)}{dt}\right|_{t=0}=\left[k\,(M_Y(t))^{k-1}\,\dfrac{dM_Y(t)}{dt}\right]_{t=0}=k\,(M_Y(0))^{k-1}\,\left.\dfrac{dM_Y(t)}{dt}\right|_{t=0}=kE[Y].$

- **Varianza:** $Var[X]=k\dfrac{1-p}{p^2}\cdot$

$$\diamond\ E[X^2]=\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}=k\left[(k-1)\,(M_Y(t))^{k-2}\left(\frac{dM_Y(t)}{dt}\right)^2+(M_Y(t))^{k-1}\,\frac{d^2M_Y(t)}{dt^2}\right]_{t=0}=$$

$$=k\left[(k-1)\,(M_Y(0))^{k-2}\left(\left.\frac{dM_Y(t)}{dt}\right|_{t=0}\right)^2+(M_Y(0))^{k-1}\,\left.\frac{d^2M_Y(t)}{dt^2}\right|_{t=0}\right]=$$

$$=k\left[(k-1)(E[Y])^2+E[Y^2]\right]=k\left[k(E[Y])^2+E[Y^2]-(E[Y])^2\right]=k^2(E[Y])^2+kVar[Y]\cdot$$

$\diamond\ Var[X]=E[X^2]-(E[X])^2=kVar[Y]\cdot$

Distribución hipergeométrica (demostraciones)

$$P(X=x)=\frac{\binom{N_1}{x}\binom{N-N_1}{n-x}}{\binom{N}{n}},\quad x=0,\dots,n\ /\ x\le N_1,\ n-x\le N-N_1.$$

• **Media:** $E[X]=\sum_{x=0}^{n} x\frac{\binom{N_1}{x}\binom{N-N_1}{n-x}}{\binom{N}{n}}=n\frac{N_1}{N}\cdot$

$$\sum_{x=0}^{n} x\binom{N_1}{x}\binom{N-N_1}{n-x}={}^{5}=N_1\sum_{x=1}^{n}\binom{N_1-1}{x-1}\binom{N-N_1}{n-x}=[\ y=x-1\ ]=N_1\sum_{y=0}^{n-1}\binom{N_1-1}{y}\binom{N-N_1}{(n-1)-y}=$$

$$=N_1\binom{N-1}{n-1}\Rightarrow E[X]=\frac{N_1\binom{N-1}{n-1}}{\binom{N}{n}}=n\frac{N_1}{N}\cdot$$

• **Varianza:** $Var[X]=E[X^2]-(E[X])^2=E[X(X-1)]+E[X]-(E[X])^2=n\frac{N_1}{N}\left(1-\frac{N_1}{N}\right)\frac{N-n}{N-1}\cdot$

$\diamond$ $E[X(X-1)]=\sum_{x=0}^{n}x(x-1)\frac{\binom{N_1}{x}\binom{N-N_1}{n-x}}{\binom{N}{n}}=\frac{N_1(N_1-1)}{\binom{N}{n}}\binom{N-2}{n-2}.$

$$\sum_{x=0}^{n}x(x-1)\binom{N_1}{x}\binom{N-N_1}{n-x}={}^{6}=N_1(N_1-1)\sum_{x=2}^{n}\binom{N_1-2}{x-2}\binom{N-N_1}{n-x}=[y=x-2]$$

$$=N_1(N_1-1)\sum_{y=0}^{n-2}\binom{N_1-2}{y}\binom{N-N_1}{(n-2)-y}=N_1(N_1-1)\binom{N-2}{n-2}.$$

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[5] $x\neq 0\ \Rightarrow\ x\binom{N_1}{x}=N_1\binom{N_1-1}{x-1}$

[6] $x\neq 0,1\ \Rightarrow\ x(x-1)\binom{N_1}{x}=N_1(N_1-1)\binom{N_1-2}{x-2}$