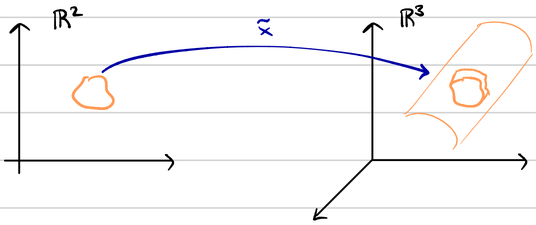
SURFACES

DEF: A subset $S \subset \mathbb{R}^3$ is a regular surface if, $\forall p \in S$, there exist open $V_p \subset \mathbb{R}^3$, $M_p \subset \mathbb{R}^3$, and $\tilde{x}: M_p \to V_p \cap S$ such that 3 conditions hold:

1. $\tilde{x}$ is differentiable
2. $\tilde{x}$ is homeomorphism
3. differential $\forall q \in M_p$ $d\tilde{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one map.



Remark: $\tilde{x}(u,v) = (x(u,v), y(u,v), z(u,v))$.
Differenciability of $\tilde{x}$ means that all maps $x: M_p \to \mathbb{R}$, $y: M_p \to \mathbb{R}$ and $z: M_p \to \mathbb{R}$ have partial derivate of all orders.

Remark: $\tilde{x}: M_p \to V_p \cap S$ is a homeomorphism $\Leftrightarrow$ $\tilde{x}: M_p \to V_p \cap S$ is continuous, injective and surjective (there will exist the inverse of $\tilde{x}$). $\exists (\tilde{x})^{-1}: V_p \cap S \to M_p$ continuous as well.

Remark: $\tilde{x}$ is differentiable $\Rightarrow$ $\tilde{x}$ is continuous

Remark: We compute matrix of $d\tilde{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ in base $e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ in $\mathbb{R}^2$ and $f_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, f_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, f_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ in $\mathbb{R}^3$.

$$d\tilde{x}_q = \begin{bmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \\ \frac{dz}{du} & \frac{dz}{dv} \end{bmatrix}$$

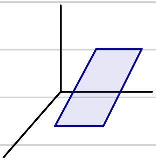
$d\tilde{x}_q$ is a linear transformation one-to-one $\Leftrightarrow$

$\Leftrightarrow$ both colums ("du", "dv") are linearly independent $\Leftrightarrow$

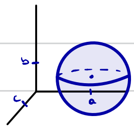
$$\Leftrightarrow \det \begin{bmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{bmatrix} \neq 0 \quad \text{or} \quad \det \begin{bmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dz}{du} & \frac{dz}{dv} \end{bmatrix} \neq 0 \quad \text{or} \quad \det \begin{bmatrix} \frac{dy}{du} & \frac{dy}{dv} \\ \frac{dz}{du} & \frac{dz}{dv} \end{bmatrix} \neq 0$$

EXAMPLE: Of regular surfaces

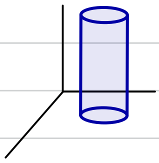
1 Plane



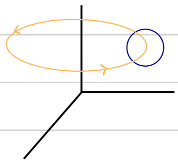
2 Sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\}$
of radius r and centre (a,b,c)

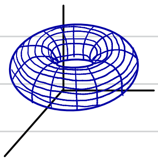


3 Cylinder $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, \ z \in \mathbb{R}\}$

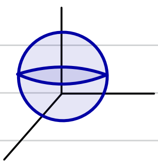


4 Torus We rotate a circle in the plane delimited by x and y around the axis Y.





EXAMPLE: We intend to prove that the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$ is a regular sphere



$z^2 = r^2 - x^2 - y^2 ; \quad \pm z = \sqrt{r^2 - x^2 - y^2}$

$f(x,y) = z = \sqrt{r^2 - x^2 - y^2}$ // Para estudiar el hemisferio superior

$D = \{(x,y,0) = x^2 + y^2 < r^2\}$ (open set in $\mathbb{R}^2$)

$f : D \to S^2_+$

The function $f(x,y)$ has partial derivates of arbitrary orders $\Rightarrow$
$\Rightarrow f$ is differentiable $\Rightarrow f$ is continuous.
1st cond

There exists $f^{-1} : S^2_+ \to D$ continuous.

Taking into account the previous conditions (f and f^{-1} cont), we can affirm that f is a homeomorphism (2nd cond)

Calculating the matrix of the differentative df:

$$\begin{bmatrix} 1 = \frac{dx}{dx} & 0 = \frac{dx}{dy} \\ 0 = \frac{dy}{dx} & 1 = \frac{dy}{dy} \\ \frac{d\sqrt{r^2-x^2-y^2}}{dx} & \frac{d\sqrt{r^2-x^2-y^2}}{dy} \end{bmatrix} \to \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

so df is one-to-one (3rd cond)

Therefore, the upper hemi-sphere is a regular surface.

The map $f_1 : D \to S^2$ $f_1(x,y) = -\sqrt{r^2 - x^2 - y^2}$, the same arguments, as in the case of f, gives that f is a regular parametrization of lower hemi-sphere.

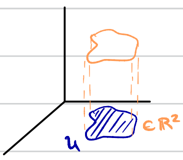
Remark: It is easy to check that $\tilde{x}(\varphi_1, \varphi_2) = (\sin\varphi_1 \cdot \cos\varphi_2, \sin\varphi_1 \sin\varphi_2, \cos\varphi_1)$

THM: If $f : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U}$ is an open set in $\mathbb{R}^2$, is a differentiable function, then the graph of f, that is:

$$Gr(f) = \{(x, y, f(x,y)) \in \mathbb{R}^3 : (x,y) \in \mathcal{U}\}$$

is a regular surface.

<u>Proof</u>: Let $\tilde{x}(u,v) = (u, v, f(u,v))$, then, $\tilde{x}$ is differentiable $\Rightarrow$ 1st
$\Rightarrow \tilde{x}$ is continuous.



$(\tilde{x}^{-1})$ exists and it is continuous

$\tilde{x}$ and $\tilde{x}^{-1}$ cont $\Rightarrow$ $\tilde{x}$ is a homeomorphism 2nd

Notice that $\frac{\partial \tilde{x}}{\partial u} = [1, 0, f_u]$, $\frac{\partial \tilde{x}}{\partial v} = [0, 1, f_v]$. Those two vectors are linearly independent, so $d\tilde{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one 3rd

+

DEF: Given a differentiable map $F : \mathcal{U} \to \mathbb{R}^m$, where $\mathcal{U} = \mathcal{U}^\circ \subset \mathbb{R}^n$, we say that a point $p \in \mathcal{U}$ is a critical point of F if $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective.

Then, $F(p)$ is called a critical value of F.

A point $q \in \mathbb{R}^m$ which is not a critical value of F is called a regular value of F.

THM: If $f : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U} = \mathcal{U}^\circ \subset \mathbb{R}^3$, is a differentiable function and a point $a \in f(\mathcal{U})$ is a regular value of f, then $f^{-1}(a)$ is a regular surface.

luce como un huevo

COROLARY: The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is a regular surface

Proof: Let $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$. f is defined in an open subset of $\mathbb{R}^3$ and f is differentive.

Partial derivative of f: $\frac{\delta f}{\delta x}=\frac{2x}{a^2}$ $\frac{\delta f}{\delta y}=\frac{2y}{b^2}$ $\frac{\delta f}{\delta z}=\frac{2z}{c^2}$

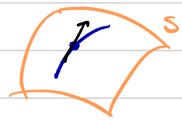
$$\left[\frac{\delta f}{\delta x},\frac{\delta f}{\delta y},\frac{\delta f}{\delta z}\right]=[0,0,0] \iff x=y=z=0.$$

But $[0,0,0]\notin f^{-1}(0)$, because $f(0,0,0)=-1$. By the theorem above, the ellipsoid is a regular surface.

+

Tangent plane

Assume that S is a regular surface and choose a point $p\in S$.



DEF: By a tangent vector to the surface S at point $p\in S$, we mean a tangent vector $\alpha'(0)$ of a diff. curve $\alpha:(-\varepsilon,\varepsilon)\to S$ such that $\alpha(0)=p$.

By T_pS we mean the space of all tangent vectors to S at point p. It is the tangent plane to S at p.

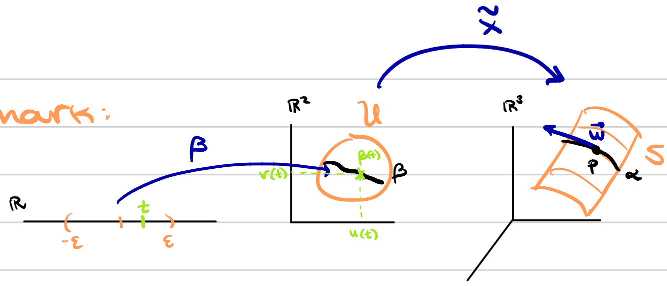
PROP: Let $x:\mathcal{U}\subset\mathbb{R}^2\to S$ be a parametrization of a regular surface S. For any point $q\in\mathcal{U}$, $dx_q(\mathbb{R}^2)=T_{x(q)}S$

Remark: Consider a parametrization $x:\mathcal{U}\subset\mathbb{R}^2\to S$ with $q\in\mathcal{U}$ such that $x(q)=p$. Then, $\tilde{x}(u,v)=(x(u,v),y(u,v),z(u,v))$, and

$$\frac{\delta\tilde{x}}{\delta u}(q),\ \frac{\delta\tilde{x}}{\delta v}(q)\in T_pS$$

Moreover, $\frac{\delta\tilde{x}}{\delta u}(q)$ and $\frac{\delta\tilde{x}}{\delta v}(q)$ are linearly independent (because $d\tilde{x}:\mathbb{R}^2\to\mathbb{R}^3$ is one to one). So those two vectors are the basis of two-dimensional tangent plane T_pS.

Remark:



$\alpha : (-\varepsilon, \varepsilon) \to S$
$\alpha(0) = p$
$\alpha'(0) = \vec{w}$

$\alpha(t) = (\tilde{x} \circ \beta)(t)$
$\beta(t) = (u(t), v(t))$

$$\alpha'(t) = \frac{d}{dt}(\tilde{x} \circ \beta)(t) = \frac{d}{dt}\tilde{x}(\beta(t)) = \frac{d}{dt}\tilde{x}(u(t), v(t)) =$$
$$= \frac{\partial \tilde{x}}{\partial u} \cdot u'(t) + \frac{\partial \tilde{x}}{\partial v} \cdot v'(t) = \alpha'(t)$$

Our tangent vector, $\vec{w} = \alpha'(0)$, is a linear combination of vectors $\frac{\partial \tilde{x}}{\partial u}$ and $\frac{\partial \tilde{x}}{\partial v}$. So $\vec{w}$ has coordinates $[\vec{u}, \vec{v}]$ in basis $B = \{\frac{\partial \tilde{x}}{\partial u}, \frac{\partial \tilde{x}}{\partial v}\}$

The first Fundamental Form

Let S be a regular surface, $p \in S$, and T_pS a tangent plane to S at p.

In $\mathbb{R}^3$ we have a scalar product $\langle \cdot , \cdot \rangle$, so we can use this scalar product for any vectors $v, w \in T_pS$

DEF: The quadratic form $I_p : T_pS \to \mathbb{R}$ defined by $I_p(w) := \langle w, w \rangle$ for any $w \in T_pS$, is called the first fundamental form of a surface S with parametrization $\tilde{x}$ at point p.

Since vector $w \in T_pS$, there exists a curve $\alpha(t) = \tilde{x}(u(t), v(t))$ such that $t \in (-\varepsilon, \varepsilon)$, $\alpha(0) = p$, $\alpha'(0) = w$.

Therefore, $I_p(w) = I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle = (*)$.

Since $\alpha'(0) = \frac{\partial}{\partial t}(\tilde{x}(u(t), v(t))) = \underbrace{\frac{\partial \tilde{x}}{\partial u}}_{\tilde{x}_u} \cdot u' + \underbrace{\frac{\partial \tilde{x}}{\partial v}}_{\tilde{x}_v} \cdot v'$, we get:

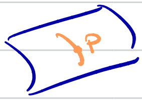
$$(*) = \langle \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v', \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v' \rangle =$$

$$= u' \langle \tilde{x}_u, \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v' \rangle + v' \langle \tilde{x}_v, \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v' \rangle =$$

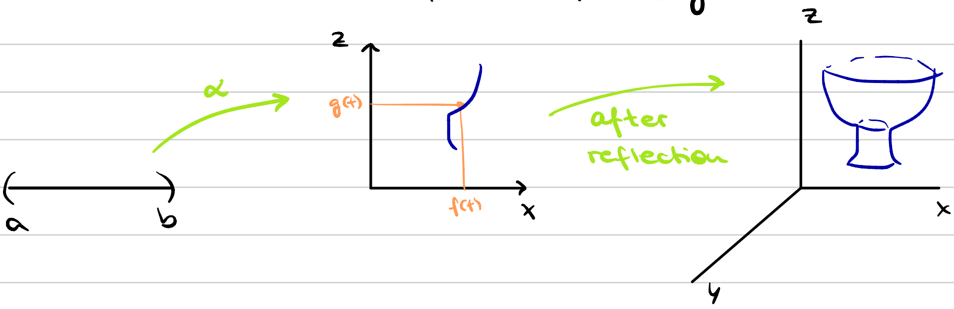
$$= (u')^2 \langle \tilde{x}_u, \tilde{x}_u \rangle + (u' \cdot v') \langle \tilde{x}_u, \tilde{x}_v \rangle + (v' \cdot u') \langle \tilde{x}_v, \tilde{x}_u \rangle + (v')^2 \langle \tilde{x}_v, \tilde{x}_v \rangle =$$

$$= (u')^2 \overbrace{\langle \tilde{x}_u, \tilde{x}_u \rangle}^{E} + 2u' \cdot v' \overbrace{\langle \tilde{x}_u, \tilde{x}_v \rangle}^{F} + (v')^2 \overbrace{\langle \tilde{x}_v, \tilde{x}_v \rangle}^{G} =$$

$$= E(u')^2 + 2F u'v' + G(v')^2$$



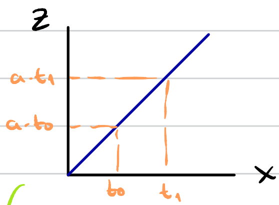
FIRST CASE: SURFACE OF REVOLUTION OBTAINED BY ROTATION OF A CURVE $\forall t \in (a,b) = (f(t), g(t))$



$\forall t \in (a,b), \quad \alpha(t) = (f(t), g(t))$

$$x(t, \varphi) = (f(t) \cdot \sin(\varphi),\ f(t) \cdot \cos(\varphi),\ g(t))$$

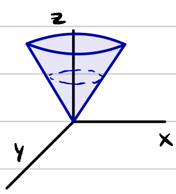
EXAMPLE: Parametrization of the half-line



$\forall t \geq 0, \quad \alpha(t) = (t, at)$

after reflection

Parametrization of a cone



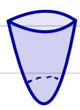
$$x(t, \varphi) = (t \sin(\varphi),\ t \cos(\varphi),\ at)$$

$\varphi \in (0, 2\pi) \quad t \geq 0.$

PROBLEM: Is a cone a regular surface?

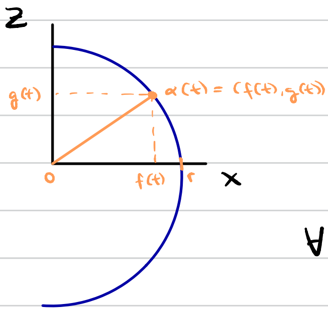
ANSWER: No, because dx at $t=0$ is not 1-to-1

OBSERVATION: Truncated cone is a regular surface



SECOND CASE: SPHERE AS A SURFACE OF REVOLUTION

Parametrization of a semi-circle



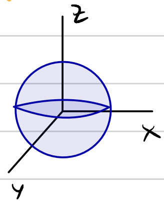
$\frac{f(t)}{r} = \cos(t)$

$\frac{g(t)}{r} = \sin(t)$

$\forall t \in (-\pi/2, \pi/2)$,

$\alpha(t) = (r\cos(t), r\sin(t))$

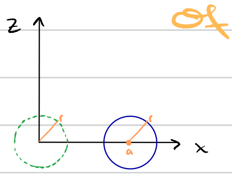
Parametrization of a sphere as a surface of revolution



$x(t,\varphi) = (r\cos(t)\sin(\varphi),$
$r\cos(t)\cos(\varphi),$
$r\sin(t))$

TORUS AS A SURFACE OF REVOLUTION

Parametrization of a circle



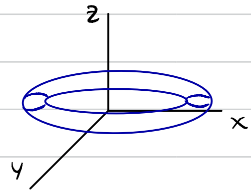
$\forall t \in (0, 2\pi)$

$\alpha(t) = (r\cos(t), r\sin(t))$

$\wr$

$\tilde{\alpha}(t) = (a + r\cos(t), r\sin(t))$

Parametrization of the torus

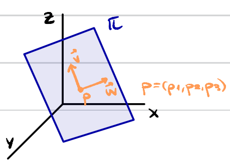


$x(t,\varphi) = ((a + r\cos(t)) \cdot \sin(\varphi),$
$(a + r\cos(t)) \cdot \cos(\varphi),$
$r\sin(t))$

DEF: The numbers E, F, G are called the coefficients of the first fundamental form I_p of the surface S with parametrization $x(u,v)$

EXAMPLE: Consider a plane π with parametrization

$\pi = \{p + s \cdot v + tw : s,t \in \mathbb{R}\} =$
$= \{(p_1 + sv_1 + tw_1,$
$p_2 + sv_2 + tw_2,$
$p_3 + sv_3 + tw_3 : t,s \in \mathbb{R}\}$



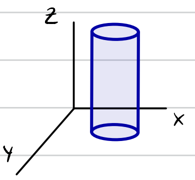
$v \perp w$
$v = [v_1, v_2, v_3]$
$w = [w_1, w_2, w_3]$

Parametrization of the plane

$x(s,t) = (p_1 + sv_1 + tw_1, p_2 + sv_2 + tw_2, p_3 + sv_3 + tw_3)$

$$\left.\begin{aligned} x_s &= [v_1, v_2, v_3] \\ x_t &= [w_1, w_2, w_3] \end{aligned}\right\} \Rightarrow \left| \begin{aligned} E &= \langle x_s, x_s \rangle = \langle v, v \rangle = \|v\|^2 \\ F &= \langle x_s, x_t \rangle = \langle v, w \rangle = 0 \\ G &= \langle x_t, x_t \rangle = \langle w, w \rangle = \|w\|^2 \end{aligned}\right.$$

EXAMPLE: Consider a cylinder $C = \{(r\cos(t), r\sin(t), s) : t \in (0, 2\pi),\ s \in \mathbb{R}\}$



Parametrization of C

$$x(t,s) = (r\cos(t), r\sin(t), s)$$

$$\left.\begin{aligned} x_t &= (-r\sin t,\ r\cos t,\ 0) \\ x_s &= (0,0,1) \end{aligned}\right\} \Rightarrow \left|\begin{aligned} E &= \langle x_t, x_t\rangle = r^2\sin^2(t) + r^2\cos^2(t) = r^2 \\ F &= \langle x_t, x_s\rangle = 0 \\ G &= \langle x_s, x_s\rangle = 1 \end{aligned}\right.$$

APPLICATIONS OF THE FIRST FUNDAMENTAL FORM

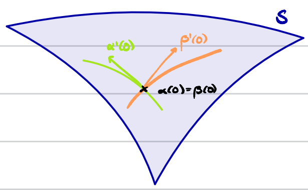
1. The lenght of a parametric curve on S

$$\alpha : (a,b) \to S$$

$$\mathcal{L}(\alpha) = \int_a^b \|\alpha'(t)\|\, dt = \int_a^b I_p(\alpha'(t))\, dt$$

2. The angle of intersection of two curves

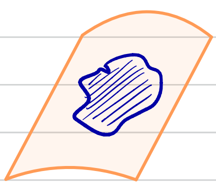
$\alpha : (a,b) \to S$ and $\beta : (a,b) \to S$, $\alpha(0) = \beta(0)$



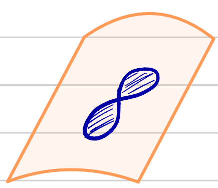
$$\cos(\varphi) = \frac{\langle \alpha'(0), \beta'(0)\rangle}{\|\alpha'(0)\| \cdot \|\beta'(0)\|} = \frac{\langle \alpha'(0), \beta'(0)\rangle}{\sqrt{I_p(\alpha'(0)) \cdot I_p(\beta'(0))}}$$

3. Area of a region on a surface

DEF: An open connected subset of a surface S, with boundary which are diffeomorph to a circle is called a region.

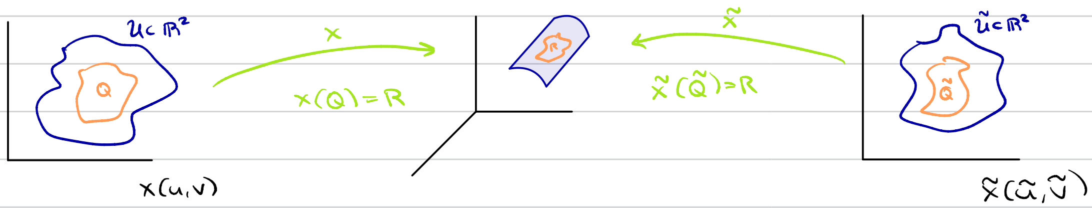


Region



Not a region

4.



Our claim is that $\iint_Q \|x_u \times x_v\| \, du dv$ does not depend on a parametrization.

$$\iint_Q \|x_u \times x_v\| \, du dv = \iint_Q \|x_u \times x_v\| \cdot \left| \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} \right| d\tilde{u} d\tilde{v} = \iint_{\tilde{Q}} \|\tilde{x}_{\tilde{u}} \times \tilde{x}_{\tilde{v}}\| \, d\tilde{u} d\tilde{v}$$

DEF: Let $R \subset S$ be a region in S and S a regular surface given by parametrization $x(u,v)$. Then, the number

$$\text{Area}(R) = \iint_Q \|x_u \times x_v\| \, du dv$$ is called the area of the region $R = x(Q)$

LEMMA: For any two vectors $v, w \in \mathbb{R}^3$, we have $\|v \times w\|^2 + \langle v, w\rangle^2 = \|v\|^2 \cdot \|w\|^2$

So $\|v \times w\| = \sqrt{\|v\|^2 \cdot \|w\|^2 - \langle v, w\rangle^2}$

COROLARY: In particular, taking vectors x_u, x_v partial derivates of parametrization $x(u,v)$, we obtain:

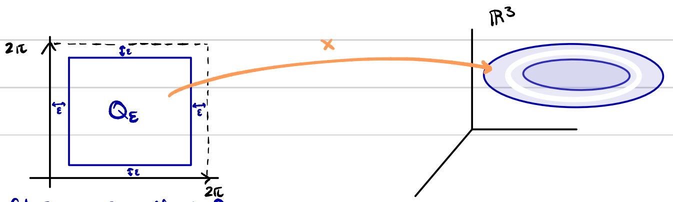
coefficients of the First Fundamental form

$$\|x_u \times x_v\| = \sqrt{\|x_u\|^2 \cdot \|x_v\|^2 - \langle x_u, x_v\rangle^2} = \sqrt{E \cdot G - F^2}$$

EXAMPLE: We intend to calculate the area of a torus obtained as a surface of revolution of a curve $\alpha(u) = (a + r \cdot \cos u, \, r \cdot \sin u)$ with respect to axe O_Z.

So we get the following parametrization:

$$x(u,v) = \left((a + r\cos u) \cdot \cos v, \, (a + r\cos u) \cdot \sin v, \, r \cdot \sin u \right) \qquad x, v \in (0, 2\pi)$$



Choose small $\varepsilon > 0$

Coefficients of the first fundamental form

$\frac{dx}{du} = x_u(u,v) = (-r\,\text{sen}(u)\cdot\cos(v), -r\,\text{sen}(u)\cdot\text{sen}(v), r\cos(u))$

$\frac{dx}{dv} = x_v(u,v) = (-(a+r\cos u)\,\text{sen}\, v, (a+r\cos u)\cos v, 0)$

$E = \langle x_u, x_u\rangle = r^2\text{sen}^2(u)\cos^2(v) + r^2\text{sen}^2(u)\cdot\text{sen}^2(v) + r^2\cos^2(u) =$
$= r^2\text{sen}^2(u) + r^2\cos^2(u) = r^2$

$F = \langle x_u, x_v\rangle = r\,\text{sen}(u)\cos(v)(a+r\cos u)\,\text{sen}(v) - (\dots)$
$(\dots) - r\,\text{sen}(u)\,\text{sen}(v)(a+r\cos u)\cos v + 0 = 0$

$G = \langle x_v, x_v\rangle = (a+r\cos u)^2\,\text{sen}^2 v + (a+r\cos u)^2\cos^2 v = (a+r\cos u)^2$

Find the area of the torus

$\text{Area}(Q_\varepsilon) = \iint_{Q_\varepsilon} \|x_u \times x_v\|\, du\,dv = \iint_{Q_\varepsilon} \sqrt{E(u,v)\cdot G(u,v) - F^2(u,v)}\, du\,dv =$

$= * \quad E\cdot G - F^2 = r^2(a+\cos u)^2 - 0^2 = \iint_{Q_\varepsilon} \sqrt{r^2(a+\cos u)^2}\, du\,dv =$

Fibonacci

$= \iint_{Q_\varepsilon} r(a+\cos u)\,du\,dv = \int_\varepsilon^{2\pi-\varepsilon}\left(\int_\varepsilon^{2\pi-\varepsilon} ra + r\cos u\, du\right) dv =$

$= (2\pi-\varepsilon-\varepsilon)\cdot\int_\varepsilon^{2\pi-\varepsilon} ra + r\cos u\, du = 2(\pi-\varepsilon)\left[rau + r\,\text{sen}\, u\right]_\varepsilon^{2\pi-\varepsilon} =$

$= 2(\pi-\varepsilon)\cdot\left[2(\pi-\varepsilon)ra + r\,\text{sen}(2\pi-\varepsilon) - r\,\text{sen}(\varepsilon)\right]$

So, $\text{Area}(T^2) = \lim_{\varepsilon\to 0} A(Q_\varepsilon) = 2\pi[2\pi ra + 0 - 0] = 4\pi^2 ra$

+

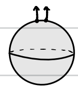
→ Consider a surface S with parametrization $x(u,v)$

$x_u, x_v \in T_pS \qquad x_u \times x_v \perp T_pS$

Let $\boxed{N(p) \overset{def}{:=} \dfrac{x_u \times x_v}{\|x_u \times x_v\|}(p)}$

DEF: By a differential field of unit normal vectors on an open set $\mathcal{U} \subset S$, we mean a differentiable map $N: \mathcal{U} \to \mathbb{R}^3$ which associates to each point $p \in \mathcal{U}$ a unit normal vector $\dfrac{x_u \times x_v}{\|x_u \times x_v\|}$

DEF: A regular surface $S \subset \mathbb{R}^3$ is orientable if, and only if, there exists a differentiable field of unit normal vectors $N: S \to \mathbb{R}^3$

EXAMPLE: 1) Sphere is an orientable surface 

2) Cylinder is an orientable surface 

EXAMPLE: 1) The Möbius trip is not orientable  la curva de la hormiguita de Escher

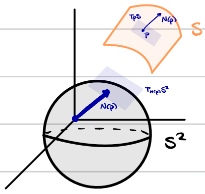
$x(u,v) = ((2 - v \cdot \sin u/2) \cdot \sin u, (2 - v \cdot \sin u/2) \cdot \cos u, \cos u/2)$ is a parametrization of a Möbius trip. We can see it is regular.

2) Klein bottle is not orientable

PROP: Assume S_1 and S_2 are regular surfaces and $\varphi: S_1 \to S_2$ is a differentiable map.
If S_1 is orientable, then S_2 is orientable aswell.

Gauss Map

Consider a regular surface S with parametrization $x(u,v)$. Assume that S is orientable.



$N(p) = \frac{x_u \times x_v}{\|x_u \times x_v\|}(p)$

The Gauss map is defined by $N: S \to S^2$.

To a point $p \in S$, first we associate vector $N(p)$. Next, by translation, we attach $N(p)$ to the origin of coordinate system. Then, since $N: S \to S^2$ is differentiable, we get $dN_p : T_pS \to T_{N(p)}S^2 \approx T_pS$

So $T_pS \parallel T_{N(p)}S^2$. Thus, we can identify $T_{N(p)}S^2$ with T_pS. Therefore, we can write $dN_p : T_pS \to T_pS$

DEF: Self-adjoint map.

Let V be a 2-dim vector space with scalar product $\langle , \rangle$. We say that a linear map $A: V \to V$ is self-adjoint if $\forall \vec{v}, \vec{w} \in V$, $\langle A(\vec{v}), \vec{w} \rangle = \langle \vec{v}, A(\vec{w}) \rangle$.

Remark: Notice that for orthonormal basis $\{\vec{e_1}, \vec{e_2}\}$ of V, we can identify linear transformation $A: V \to V$ with its matrix

$$M = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

Then, we get $\langle A(\vec{e_i}), \vec{e_j} \rangle = \alpha_{ij} = \langle \vec{e_i}, A(\vec{e_j}) \rangle = \alpha_{ji}$

So M is a symmetrical matrix

Remark: To each self-adjoint map $A: V \to V$ we can associate $B: V \times V \to \mathbb{R}$ given by

$$\forall \vec{v}, \vec{w} \in V, \; B(\vec{v}, \vec{w}) = \langle A(\vec{v}), \vec{w} \rangle$$

Then B is a 2-linear symetric form: $B(\vec{w}, \vec{v}) = B(\vec{v}, \vec{w})$

Remark: To each symetric 2-linear form $B: V \times V \to \mathbb{R}$ corresponds a quadratic form $Q: V \to \mathbb{R}$ given by: $\forall \vec{v} \in V \quad Q(\vec{v}) = B(\vec{v}, \vec{v})$
Notice that Q determines B completely, as:

$$B(\vec{u}, \vec{v}) = \frac{1}{2}\left[Q(\vec{u}+\vec{v}) - Q(\vec{u}) - Q(\vec{v})\right]$$

PROP: For differentiable Gauss map $N: S \to S^2$, the differential $dN_p: T_pS \to T_pS$ is a self-adjoint map.

Proof: Choose an orthonormal basis $\vec{w_1}, \vec{w_2}$ in T_pS. Since the differential dN_p is a linear map, it is sufficient to prove that
$$\langle dN_p(\vec{w_1}), \vec{w_2} \rangle = \langle \vec{w_1}, dN_p(\vec{w_2}) \rangle .$$

Let $x(u,v)$ be a regular parametrization of S, and x_u, x_v a basis of T_pS.
For a curve $\alpha(t) = x(u(t), v(t))$, where $t \in (-\varepsilon, \varepsilon)$, such that $\alpha(0) = p$ and $\alpha'(0) = x_u \cdot u' + x_v \cdot v'$, we can write:

$$dN_p(\alpha'(0)) = dN_p(x_u \cdot u' + x_v \cdot v') \overset{dN_p \text{ is linear}}{=} \underbrace{dN_p(x_u)}_{N_u} \cdot u' + \underbrace{dN_p(x_v)}_{N_v} \cdot v' = N_u \cdot u' + N_v \cdot v'$$

Now, to prove that dN_p is self-adjoint, we will prove that $\langle N_u, x_v \rangle = \langle x_u, N_v \rangle$. Notice that $\langle \underset{\text{from Gauss map}}{N}(p), x_u(p) \rangle$, so:

$$0 = \frac{\partial}{\partial v} \langle N, x_u \rangle = \langle N_v, x_u \rangle + \langle N, x_{uv} \rangle$$

On the other hand, $\langle N, x_v \rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial u} \langle N, x_v \rangle = \langle N_u, x_v \rangle + \langle N, x_{vu} \rangle$

$*\ x_{uv} = x_{vu}$

In this way, $\langle N_v, x_u \rangle + \langle N, x_{uv} \rangle = \langle N_u, x_v \rangle + \langle N, x_{vu} \rangle \Leftrightarrow$

$$\Leftrightarrow \langle N_v, x_u \rangle = \langle N_u, x_v \rangle \qquad +$$

DEF: The quadratic form $II_p : T_pS \to \mathbb{R}$ given by:

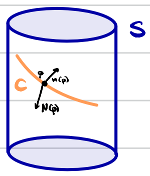
$$\forall \vec{v} \in T_pS \quad II_p(\vec{v}) = -\langle dN_p(\vec{v}), \vec{v} \rangle$$

is called the second fundamental form of S at point p.

DEF: Let C be a regular curve in S passing through a point $p \in S$. Denote by $\kappa(p)$ the curvature of C at point p

$n(p)$ the normal vector to C at p

$N(p)$ the normal vector to T_pS



- Let $\cos(\theta) = \langle n, N \rangle$
- The number $\kappa_n = \kappa(p) \cdot \langle n(p), N(p) \rangle$ is called the normal curvature of the curve $C \subset S$ at point $p \in S$

Interpretation of II_p

Let C be a curve with parametrization $\alpha(s)$ such that $\alpha(0) = p$. Denote by $N(s)$ the restriction of unit normal field N to $\alpha(s)$.

Since $N(s) \perp \alpha'(s)$, we get

$$\langle N(s), \alpha'(s) \rangle = 0 \Rightarrow \langle N(s), \alpha''(s) \rangle + \langle N'(s), \alpha'(s) \rangle = 0 \Rightarrow$$

$$\Rightarrow \langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle \; *$$

$$II_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle \overset{*}{=} \langle N(p), \overbrace{\alpha''(0)}^{\kappa(p) \cdot n(p)} \rangle =$$

$$= \kappa(p) \langle N(p), n(p) \rangle = \kappa_n = \text{normal curvature}$$

THM: Any two curves α, β such that tangent line to α and to β at point p is the same, have the same normal curve.

S - surface with parametrization $x(u,v) \rightarrow \{x_u, x_v\}$ is a basis of T_pS

$$N(p) = \text{normal vector field} = \frac{x_v \times x_u}{\|x_v \times x_u\|}(p)$$

Consider a curve on S with param. $\alpha(t) = x(u(t), v(t))$ s.t. $\alpha(0) = p$

$\alpha'(0) = x_u u'(0) + x_v \cdot v'(0)$

$$dN_p(\alpha'(0)) = dN_p(x_u \cdot u'(0) + x_v \cdot v'(0)) = N_u \cdot u'(0) + N_v \cdot v'(0) \quad *$$

LEMMA: $N_u(p), N_v(p) \in T_pS$

Proof: Since $\|N(p)\|=1$, we set $\langle N,N\rangle=1$, so, taking derivative:

$$0=\frac{\partial}{\partial u}\langle N,N\rangle=\langle N_u,N\rangle+\langle N,N_u\rangle=2\langle N_u,N\rangle=0 \Rightarrow N_u\perp N \Rightarrow N_u\in T_pS$$

We have $N_v\in T_pS$ taking derivative with respect to v instead to u.

$+$

COROLLARY: $\begin{cases} N_u = a_{11}x_u + a_{21}x_v \\ N_v = a_{12}x_u + a_{22}x_v \end{cases}$

$$dN_p(\alpha'(0)) = N_u\cdot u'(0)+N_v\cdot v'(0) =$$

$$= (a_{11}x_u+a_{21}x_v)\cdot u'(0) + (a_{12}x_u+a_{22}x_v)\cdot v'(0) =$$

$$= (a_{11}\cdot u'(0)+a_{12}\cdot v'(0))\cdot x_u + (a_{21}\cdot u'(0)+a_{22}x_v\cdot v'(0))\cdot x_v$$

Therefore, the matrix of linear transformation dN_p in basis $\{x_u,x_v\}$ can be written in this way:

$$dN_p\begin{pmatrix}u'(0)\\ v'(0)\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}u'(0)\\ v'(0)\end{pmatrix}$$

5 Dic 2023

THM: Assume that $A:V\to V$ is a self-adjoint linear transformation defined on 2-dimensional vectorial space V.
Then, there exists an orthonormal basis $\{v_1,v_2\}$ of V and $k_1,k_2\in\mathbb{R}$ such that:

$$A(v_1)=-k_1v_1$$
$$A(v_2)=-k_2v_2$$

Remark: The matrix of the linear transformation $A:V\to V$ with fix basis $\{v_1,v_2\}$ is:

$$\begin{bmatrix}-k_1 & 0\\ 0 & -k_2\end{bmatrix}$$

of

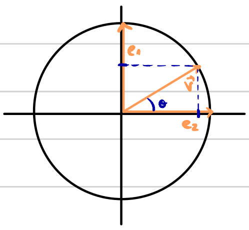
COROLLARY: For self-adjoint map $dN_p: T_pS \to T_pS$, there exists an orthonormal basis $\{e_1, e_2\}$ and $k_1, k_2 \in \mathbb{R}$ such that

$$dN_p(e_1) = -k_1 e_1$$
$$dN_p(e_2) = -k_2 e_2$$

DEF: The numbers k_1, k_2 are called the principal curvatures of the surface S at point $p \in S$, and corresponding directions given by eigen-vectors e_1 and e_2 are called principal directions.

Euler formula

Consider a regular parametrization $x(u,v)$ of a surface S such that $x_u = e_1$ and $x_v = e_2$ are orthonormal vectors of tangent space T_pS. Fix a vector $\vec{v} \in T_pS$ with $\|\vec{v}\| = 1$.



$\vec{v} = [v_1, v_2]$

$$\left.\begin{aligned} \frac{v_1}{\|\vec{v}\|} &= \cos\theta \\ \frac{v_2}{\|\vec{v}\|} &= \operatorname{sen}\theta \end{aligned}\right\} \Rightarrow \left| \begin{aligned} v_1 &= \cos\theta \\ v_2 &= \operatorname{sen}\theta \end{aligned}\right. \Rightarrow$$

$$\Rightarrow \vec{v} = [\cos\theta, \operatorname{sen}\theta] \text{ in basis } \{e_1, e_2\}$$
$$(\vec{v} = \cos\theta \cdot e_1 + \operatorname{sen}\theta \cdot e_2)$$

Now we are going to build the formula:

normal curvature along a vector $\vec{v}$

$$k_n = II_p(\vec{v}) = -\langle dN_p(\vec{v}), \vec{v}\rangle = -\langle dN_p(\cos\theta \cdot e_1 + \operatorname{sen}\theta \cdot e_2), \cos\theta\, e_1 + \operatorname{sen}\theta\, e_2\rangle =$$

$$= -\cos\theta \langle dN_p(e_1), \cos\theta \cdot e_1\rangle - \cos\theta \langle dN_p(e_1), \operatorname{sen}\theta \cdot e_2\rangle - (\dots)$$
$$(\dots) - \operatorname{sen}\theta \langle dN_p(e_2), \cos\theta \cdot e_1\rangle - \operatorname{sen}\theta \langle dN_p(e_2), \operatorname{sen}\theta \cdot e_2\rangle =$$

$$= -\cos^2\theta \overbrace{\langle dN_p(e_1), e_1\rangle}^{-k_1} - \cos\theta \operatorname{sen}\theta \underbrace{\langle dN_p(e_1), e_2\rangle}_{0 \text{ (perpendicular)}} - (\dots)$$
$$(\dots) - \cos\theta \operatorname{sen}\theta \underbrace{\langle dN_p(e_2), e_1\rangle}_{0 \text{ (perpendicular)}} - \operatorname{sen}^2 \underbrace{\langle dN_p(e_1), e_1\rangle}_{-k_2} =$$

$$= \boxed{k_1 \cdot \cos^2\theta + k_2 \cdot \operatorname{sen}^2\theta = k_n}$$

Remark: Let $V=\mathbb{R}^2$ be a 2D vector space with linear map $A:V\to V$. We fix a basis $\{v_1,v_2\}$ of V.

In that basis, we can identify the linear map with its matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Therefore, to the linear map A, we can associate its determinant

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and its trace

$$tr(A) = a_{11}+a_{22}$$

PROP: The determinant $\det(A)$ and trace (A) of a linear map $A:V\to V$ does not depend on the basis of V.

COROLLARY: $\det(dN) = \det \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} = k_1 \cdot k_2$

$$tr(dN) = tr \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} = (-k_1)+(-k_2)$$

DEF: Let $p\in S$ and $dN_p: T_pS\to T_pS$ be a differential of the Gauss map.

Then, $\det(dN_p)$ is called the Gaussian curvature (K) of S at $p\in S$.

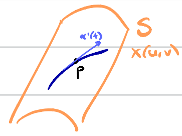
And $-\frac{1}{2}tr(dN_p)$ is called the mean curvature (H) of S at $p\in S$.

DEF: A point $p\in S$ is called:

a) an elliptic point if $\det(dN_p)>0$
b) an hyperbolic point if $\det(dN_p)<0$
c) a parabolic point if $\det(dN_p)=0$ and $dN_p\neq 0$
d) a planar point if $dN_p=0$

The Gauss curvature and mean curvature in local coordinates

Let $x(u,v)$ be a regular parametrization of S and a curve $\begin{cases} \alpha(t) = x(u(t), v(t)) \\ \alpha'(t) = x_u \cdot u' + x_v \cdot v' \end{cases}$ be a regular curve on S.



We will note: $dN_p(x_u) \overset{\text{def}}{:=} N_u \qquad dN_p(x_v) \overset{\text{def}}{:=} N_v$

LEMMA: $N_u, N_v \in T_pS$

Proof: $\|N\| = 1 = \langle N, N \rangle$

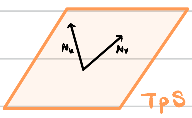
$$0 = \frac{\partial}{\partial u} \langle N, N \rangle = \langle N_u, N \rangle + \langle N, N_u \rangle = 2\langle N_u, N \rangle \Rightarrow N \perp N_u$$

in the same way...

$$0 = \frac{\partial}{\partial v} \langle N, N \rangle = \langle N_v, N \rangle + \langle N, N_v \rangle = 2\langle N_v, N \rangle \Rightarrow N \perp N_v$$

$\dagger$

$=$



$\{x_u(p), x_v(p)\}$ is a basis of T_pS.

Therefore, there exists $a_{ij} \in \mathbb{R}$ s.t. $\begin{cases} N_u = a_{11} \cdot x_u + a_{21} \cdot x_v \\ N_v = a_{12} \cdot x_u + a_{22} \cdot x_v \end{cases}$

$$dN_p(\alpha') = dN_p(x_u \cdot u' + x_v \cdot v') = \overbrace{dN_p(x_u)}^{N_u} \cdot u' + \overbrace{dN_p(x_v)}^{N_v} \cdot v' =$$

$$= N_u \cdot u' + N_v \cdot v' = (a_{11} x_u + a_{21} x_v) u' + (a_{12} x_u + a_{22} x_v) v' =$$

$$= (a_{11} u' + a_{12} v') x_u + (a_{21} u' + a_{22} v') x_v$$

Therefore, in the basis $\{x_u, x_v\}$ of T_pS, we get:

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

* The second fondamental form in basis $\{x_u, x_v\}$

$$II_p(\alpha') = -\langle dN_p(\alpha'), \alpha' \rangle = -\langle N_u \cdot u' + N_v \cdot v', x_u \cdot u' + x_v \cdot v' \rangle =$$

$$= -\langle N_u, x_u \rangle (u')^2 - \underbrace{\langle N_u, x_v \rangle}_{*} u' \cdot v' - \underbrace{\langle N_v, x_u \rangle}_{*} u' \cdot v' - \langle N_v, x_v \rangle (v')^2 =$$

$^{*}\langle N_u, x_v\rangle = \langle N_v, x_u\rangle$ because dN_p is self-adjoint

$$= \underbrace{-\langle N_u, x_u\rangle}_{e}(u')^2 + 2\,(\underbrace{-\langle N_u, x_v\rangle}_{f}\, u' \cdot v') \underbrace{-\langle N_v, x_v\rangle}_{g}(v')^2 =$$

$$= e\,(u')^2 + 2f\,(u'v') + g\,(v')^2$$

The numbers e, f, g are called the coefficients of II_p at point p.

Remark:

$N \perp N_u \Rightarrow \langle N, N_u\rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial v}\langle N, N_u\rangle = \langle N_v, N_u\rangle + \langle N, N_{uv}\rangle = 0$

$N \perp N_v \Rightarrow \langle N, N_v\rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial u}\langle N, N_v\rangle = \langle N_u, N_v\rangle + \langle N, \underbrace{N_{vu}}_{= N_{uv}}\rangle = 0$

DEF:

$e = -\langle N_u, x_u\rangle$

$f = -\langle N_u, x_v\rangle$

$g = -\langle N_v, x_v\rangle$

LEMMA: $e = \langle N, x_{uu}\rangle$; $f = \langle N, x_{uv}\rangle$; $g = \langle N, x_{vv}\rangle$

Proof:

•) $N \perp x_u \Rightarrow \langle N, x_u\rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial u}\langle N, x_u\rangle =$
$= \langle N_u, x_u\rangle + \langle N, x_{uu}\rangle = 0 \Rightarrow e = -\langle N_u, x_u\rangle = \langle N, x_{uu}\rangle$

•) $N \perp x_v \Rightarrow \langle N, x_v\rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial u}\langle N, x_v\rangle =$
$= \langle N_u, x_v\rangle + \langle N, x_{vu}\rangle = 0 \Rightarrow f = -\langle N_u, x_v\rangle = \langle N, x_{uv}\rangle$

•) $N \perp x_v \Rightarrow \langle N, x_v\rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial v}\langle N, x_v\rangle =$
$= \langle N_v, x_v\rangle + \langle N, x_{vv}\rangle = 0 \Rightarrow g = -\langle N_v, x_v\rangle = \langle N, x_{vv}\rangle$

$+$

LEMMA:

$-f = \langle N_u, x_v\rangle = a_{11}F + a_{21}G$

$-f = \langle N_v, x_u\rangle = a_{12}E + a_{22}F$

$-e = \langle N_u, x_u\rangle = a_{11}E + a_{21}F$

$-g = \langle N_v, x_v\rangle = a_{12}F + a_{22}G$

*we are using coefficients of the 1st Fund. Form

Proof:

·) $\langle N_u, x_v\rangle = \langle a_{11}\cdot x_u + a_{21}\cdot x_v, x_v\rangle = \langle a_{11} x_u, x_v\rangle + \langle a_{21} x_v, x_v\rangle =$
$= a_{11}\langle x_u, x_v\rangle + a_{21}\langle x_v, x_v\rangle = a_{11}F + a_{21}G$

·) $\langle N_v, x_u\rangle = \langle a_{12} x_u + a_{22} x_v, x_u\rangle = a_{12}\langle x_u, x_u\rangle + a_{22}\langle x_v, x_u\rangle =$
$= a_{12}E + a_{22}F$

·) $\langle N_u, x_u\rangle = \langle a_{11} x_u + a_{21} x_v, x_u\rangle = a_{11}\langle x_u, x_u\rangle + a_{21}\langle x_v, x_u\rangle =$
$= a_{11}E + a_{21}F$

·) $\langle N_v, x_v\rangle = \langle a_{12} x_u + a_{22} x_v, x_v\rangle = a_{12}\langle x_u, x_v\rangle + a_{22}\langle x_v, x_v\rangle =$
$= a_{12}F + a_{22}G$

+

LEMMA: $-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\cdot\begin{bmatrix} E & F \\ F & G \end{bmatrix}$

Proof:

·) $-e = a_{11}E + a_{21}F$ ✓
·) $-f = a_{11}F + a_{21}G$ ✓
·) $-f = a_{12}E + a_{22}F$ ✓
·) $-g = a_{12}F + a_{22}G$ ✓

COROLLARY: $-\begin{bmatrix} e & f \\ f & g \end{bmatrix}\cdot\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\cdot\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}\cdot\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\right) = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

Thus:

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = -\begin{bmatrix} e & f \\ f & g \end{bmatrix}\cdot\frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

Weingarten formulas

$N_u = a_{11} x_u + a_{21} x_v$
$N_v = a_{12} x_u + a_{22} x_v$

$a_{11} = -\frac{e\cdot G - f\cdot F}{EG-F^2}$ $a_{12} = -\frac{f\cdot G - g\cdot F}{EG-F^2}$

$a_{21} = -\frac{-e\cdot F + f\cdot E}{EG-F^2}$ $a_{22} = -\frac{-f\cdot F + g\cdot E}{EG-F^2}$

THM: The Gauss curvature in local coordinates is given by:

$$K = \det(dN) = \frac{e\cdot g - f^2}{EG - F^2}$$

Proof: $K = \det(dN) = \det\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = a_{11}\cdot a_{22} - a_{21}\cdot a_{12} \overset{\text{W.F.}}{=}$

$$= \frac{1}{(EG-F^2)^2}\Big(-\cancel{efGF} + egGE + f^2F^2 - \cancel{fgFE} + (\dots)$$

$$(\dots) + \cancel{feGF} - f^2GE - geF^2 + \cancel{gfFE}\Big) =$$

$$= \frac{1}{(EG-F^2)^2}\Big[eg\,(GE-F^2) + f^2(F^2-GE)\Big] =$$

$$= \frac{(eg-f^2)(\cancel{GE-F^2})}{(EG-F^2)^{\cancel{2}}} = \frac{eg-f^2}{EG-F^2}$$

THM: The mean curvature H in local coordinates is given by:

$$H = \tfrac{1}{2}(k_1 + k_2) = -\tfrac{1}{2}(a_{11} + a_{22}) = \tfrac{1}{2}\,\frac{eG - 2fF + gE}{EG - F^2}$$

Proof: $-\dfrac{1}{2(EG-F^2)}\Big[-eG + fF + fF - gE\Big] = \dfrac{1}{2}\,\dfrac{eG-2fF+gE}{EG-F^2}$

12 DEC 2023

Principal curvatures in local coordinates

Recall that dN is self-adjoint linear map. Therefore, there exists eingenvectors $\vec{v}_1, \vec{v}_2 \in T_pS$ and eingenvalues $-k_1, -k_2$ such that we have:

$$dN(\vec{v}_1) = -k_1 \cdot \vec{v}_1$$
$$dN(\vec{v}_2) = -k_2 \cdot \vec{v}_2$$

where k_1 and k_2 are principal curvatures.

$$dN(\vec{v}_i) = -k_i \cdot \vec{v}_i = -k_i \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \vec{v}_i$$

So: $\left(dN + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{v} = \vec{0}$ * null vector for some $\vec{v} \neq \vec{0}$

and the above formula has solutions.

The matrix of $dN + \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, in basis $\{\vec{v}_1, \vec{v}_2\}$, is the following:

$$\text{matrix } dN = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \Rightarrow dN + \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}+\kappa & a_{21} \\ a_{12} & a_{22}+\kappa \end{bmatrix}$$

Since $*$, we obtained that $\det \begin{bmatrix} a_{11}+\kappa & a_{21} \\ a_{12} & a_{22}+\kappa \end{bmatrix} = 0$. Thus, the matrix $\begin{bmatrix} a_{11}+\kappa & a_{21} \\ a_{12} & a_{22}+\kappa \end{bmatrix}$ is not invertible for $\kappa = \kappa_1$ or $\kappa = \kappa_2$

Computing the determinant, we have:

$$(a_{11}+\kappa)(a_{22}+\kappa) - a_{21} \cdot a_{12} = 0;$$
$$; a_{11} \cdot a_{22} + \kappa a_{11} + \kappa a_{22} + \kappa^2 - a_{21} \cdot a_{12} = 0;$$
$$; \kappa^2 + (a_{11}+a_{22})\kappa + (a_{11} \cdot a_{22} - a_{21} \cdot a_{12}) = 0.$$

Remark: Let us consider $x^2 + bx + c = 0$. Assume that $\Delta = \sqrt{b^2 - 4c} > 0$.

$$x_1 = \frac{-b + \sqrt{b^2-4c}}{2} \qquad x_2 = \frac{-b - \sqrt{b^2-4c}}{2}$$

If we take $x_1 + x_2 = \dfrac{-b + \Delta - b - \Delta}{2} = -b$

Remark: $H = \frac{1}{2}(\kappa_1 + \kappa_2) = -\frac{1}{2}(a_{11} + a_{22})$

Gauss Curvature

$$\kappa^2 - 2H\kappa + K$$

Therefore, $\kappa_i = H \pm \sqrt{H^2 - K} \Rightarrow \left| \begin{array}{l} \kappa_1 = H + \sqrt{H^2 - K} \\ \kappa_2 = H - \sqrt{H^2 - K} \end{array} \right] **$

Remark: The principal curvatures can be expressed by Gauss and mean curvature by the formula $**$

EXAMPLE: We will calculate the Gauss and mean curvature and principal curvatures of a torus

param given by revolution

$$x(u,v) = [(a + r\cos u) \cdot \cos v, (a + r\cos u) \cdot \sin v, r \cdot \sin u]$$

Partial derivates

$$x_u = [-r \operatorname{sen} u \cdot \cos v, -r \operatorname{sen} u \operatorname{sen} v, r\cos u]$$
$$x_v = [-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0]$$

Coefs of 1st FF

$$E = \langle x_u, x_u \rangle = r^2 \operatorname{sen}^2 u \cos^2 v + r^2 \operatorname{sen}^2 u \operatorname{sen}^2 v + r^2 \cos^2 v = \underline{\underline{r^2}}$$

$$F = \langle x_u, x_v \rangle = r(a + r\cos u)\operatorname{sen} v \operatorname{sen} u \cos v - r(a + r\cos u)\sin v \sin u \cos v = \underline{\underline{0}}$$

$$G = \langle x_v, x_v \rangle = (a + r\cos u)^2 \operatorname{sen}^2 v + (a + r\cos u)^2 \cos^2 v = \underline{\underline{(a + r\cos(u))^2}}$$

Normal vector

$$\cdot)\ x_u \times x_v = \left[\det \begin{vmatrix} -r\operatorname{sen} u \operatorname{sen} v & r\cos u \\ (a + r\cos u)\cos v & 0 \end{vmatrix}, -\det \begin{vmatrix} -r\operatorname{sen} u \cos v & r\cos u \\ -(a + r\cos u)\operatorname{sen} v & 0 \end{vmatrix}, \right.$$

$$\left. \det \begin{vmatrix} -r\operatorname{sen} u \cos v & -r\operatorname{sen} u \operatorname{sen} v \\ -(a + r\cos u)\operatorname{sen} v & (a + r\cos u)\cos v \end{vmatrix} \right] =$$

$$= [-r \underbrace{(a + r\cos u)}_{\alpha > 0} \cos u \cos v, -r(a + r\cos u)\cos u \operatorname{sen} v, -r(a + r\cos u)\operatorname{sen} u]$$

$$\cdot)\ \|x_u \times x_v\| = \sqrt{r^2\alpha^2 \cos^2 u \cos^2 v + r^2\alpha^2 \cos^2 u \operatorname{sen}^2 v + r^2\alpha^2 \operatorname{sen}^2 u} = r\alpha$$

$$N = \frac{x_u \times x_v}{\|x_u \times x_v\|} = [-\cos u \cos v, -\cos u \operatorname{sen} v, -\operatorname{sen} u]$$

Second partial derivates

$$x_{uu} = [-r\cos u \cos v, -r\cos u \operatorname{sen} v, -r\operatorname{sen} u]$$

$$x_{uv} = [r\operatorname{sen} u \operatorname{sen} v, -r\operatorname{sen} u \cos v, 0]$$

$$x_{vv} = [-\alpha \cos v, -\alpha \operatorname{sen} v, 0]$$

Coefs of 2nd FF

$$e = \langle N, x_{uu} \rangle = r\cos^2 u \cos^2 v + r\cos^2 u \operatorname{sen}^2 v + r\operatorname{sen}^2 u = \underline{\underline{r}}$$

$$f = \langle N, x_{uv} \rangle = -r\cos u \cos v \sin u \sin v + r\cos u \cos v \sin u \sin v + 0 = \underline{\underline{0}}$$

$$g = \langle N, x_{vv} \rangle = \alpha \cos^2 v \cos u + \alpha \operatorname{sen}^2 v \cos u = \underline{\underline{\alpha \cos u}}$$

Gauss curvature

$$K = \frac{eg - f^2}{EG - F^2} = \frac{r\alpha \cos u - 0^2}{r^2\alpha^2 - 0} = \frac{\cos u}{r(a + r\cos u)}$$

Mean curvature

$$H = \frac{1}{2}\,\frac{eG - 2fF - Eg}{EG - F^2} = \frac{1}{2}\,\frac{\cancel{r}\,a^2 - 2\cdot 0 + \cancel{a}\cdot \cos u\, r^2}{r^2 a^2} = \frac{1}{2}\,\frac{a + 2r\cos u}{r(a + r\cos u)}$$

Principal curvatures

- $K_1 = H + \sqrt{H^2 - K} = \frac{1}{2}\,\frac{a + 2r\cos u}{r(a + r\cos u)} + \sqrt{\frac{1}{4}\,\frac{(a + 2r\cos u)^2}{r^2 (a + r\cos u)^2} - \frac{\cos u}{r(a + r\cos u)}}$
- $K_2 = H - \sqrt{H^2 - K} = \frac{1}{2}\,\frac{a + 2r\cos u}{r(a + r\cos u)} - \sqrt{\frac{1}{4}\,\frac{(a + 2r\cos u)^2}{r^2 (a + r\cos u)^2} - \frac{\cos u}{r(a + r\cos u)}}$

+

DEF: We say that a regular parametrized surface S is minimal if its mean curvature is equal to zero everywhere

EXAMPLE: Enneper's curvature

Given by: $x(u,v) = \left(u - \frac{u^3}{3} + u\cdot v^2,\ v - \frac{v^3}{3} + v\cdot u^2,\ u^2 - v^2\right)$ is minimal surface

Partial derivates

$x_u = [1 - u^2 + v^2,\ 2uv,\ 2u]$
$x_v = [2uv,\ 1 - v^2 + u^2,\ -2v]$

Coefs 1st FF

$E = \langle x_u, x_u \rangle = (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2$
$F = \langle x_u, x_v \rangle = 2uv(1 - u^2 + v^2) + 2uv(1 - v^2 + u^2) - 4uv = 2uv(2-2) = 0.$
$G = \langle x_v, x_v \rangle = 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2$

Normal vector

- $x_u \times x_v = \left[\det\begin{vmatrix} 2uv & 2u \\ 1-v^2+u^2 & -2v \end{vmatrix},\ -\det\begin{vmatrix} 1-u^2+v^2 & 2u \\ 2uv & -2v \end{vmatrix},\ \det\begin{vmatrix} 1-u^2+v^2 & 2uv \\ 2uv & 1-v^2+u^2 \end{vmatrix}\right] =$

$= [-4uv^2 - 2u(1 - v^2 + u^2),\ -(-2v(1 - u^2 + v^2) - 4u^2v),\ (1 - u^2 + v^2)(1 - v^2 + u^2) - 4u^2v^2]$

$= [-4uv^2 - 2u + 2uv^2 - 2u^3,\ 2v - 2vu^2 + 2v^3 + 4u^2v,\ 1 - v^2 + u^2 - u^2 + u^2v^2 - u^4 + v^2 - v^4 + v^2u^2 - 4u^2v^2] =$

$= [-2u^3 - 2u - 2uv^2,\ 2v^3 + 2v + 2u^2v,\ 1 - 2u^2v^2 - u^4 - v^4]$

-) $\|x_u \times x_v\| = \sqrt{(-2u^3-2u-2uv^2)^2+(2v^3+2v+2u^2v)^2+(1-2u^2v^2-u^4-v^4)^2}$

No continuamos esto porque es un coñazo. Vamos a buscar otro ejemplo de superficie minimal. Consideramos el plano. Ver página 14 de los ejercicios.

EXAMPLE: A plane is a minimal surface

19 Dic 2023

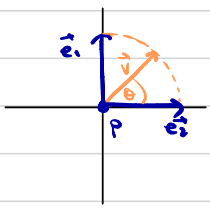
Some remarks on Gauss curvature

Recall that $dN_p : T_pS \to T_pS$ is self-adjoint map. Therefore:

$\forall p \in S$, there exists an orthonormal basis $\{\vec{e_1}, \vec{e_2}\}$ of T_pS such that $dN_p(\vec{e_1}) = -\kappa_1 \cdot \vec{e_1}$ and $dN_p(\vec{e_2}) = -\kappa_2 \cdot \vec{e_2}$ for $\kappa_1, \kappa_2 \in \mathbb{R}$

So, we can identify the linear map dN_p in basis $\{\vec{e_1}, \vec{e_2}\}$ with the matrix $\begin{bmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{bmatrix}$

Recall that the maximal number κ_1 and minimal number κ_2 are called principal curvatures.



Tangent space at point $p \in S$

So any unit vector $\vec{v} \in T_pS$ we can write as

$$\vec{v} = \cos\theta \cdot \vec{e_1} + \sin\theta \cdot \vec{e_2}$$

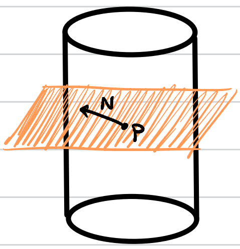
By Euler formula, we know that normal curvature with respect to vector $\vec{v}$ is given by

$$\kappa_n = \mathrm{II}_p(\vec{v}) = \kappa_1 \cdot \cos^2\theta + \kappa_2 \cdot \sin^2\theta$$

Fact: We know that $K = \kappa_1 \cdot \kappa_2$

Normal section on a cylinder / sphere / cone

CYLINDER

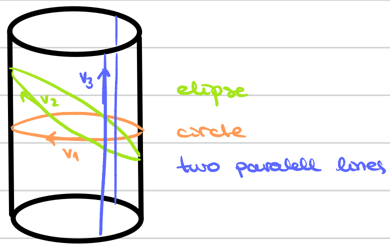


$N \perp T_pS$

$v \in T_pS$

Consider plane π_v determined by N and v

With different $\vec{v}$, we get different π_v. Therefore, different curves (sections)



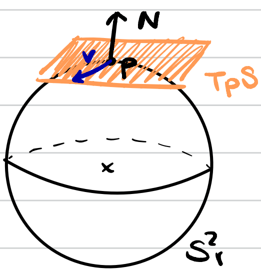
circle (max curvature)

$$\kappa_1 = 1/r \qquad \Rightarrow \qquad K = \kappa_1 \cdot \kappa_2 = 0 \quad \text{Gaussian curvature}$$

$$\kappa_2 = 0 \qquad\qquad H = \frac{\kappa_1 + \kappa_2}{2} = 1/2r \quad \text{Mean curvature}$$

segment (min curvature)

SPHERE of radius r



$v \in T_pS$

Consider plane π_v determined by N and v. The section of π_v with S^2_r is a great circle of our sphere (radius r).

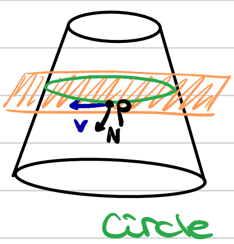
// This is because π_v contains the centre of S^2_r

$$\kappa_1 = \kappa_2 = 1/r \quad \Rightarrow \quad K = \kappa_1 \cdot \kappa_2 = 1/r^2$$

$$H = \frac{1/r + 1/r}{2} = \frac{2}{2r} = 1/r$$

the min and max curv. is the same

CONE



$v \in T_pS$

curvature of the circle. It depends on the radius of the circle. It depends on where p is.

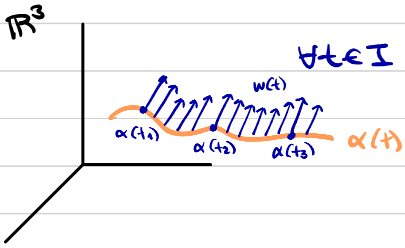
$$\kappa_1 > 0 \quad \Rightarrow \quad K = \kappa_1 \cdot \kappa_2 = 0$$

$$\kappa_2 = 0 \qquad H = \frac{\kappa_1 + \kappa_2}{2} = \kappa_1/2 > 0$$

(curvature of a segment)

Ruled surfaces

(intro:)



$\forall t \in I \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ regular curve

Let $w(t)$ be a regular vector field along curve $\alpha(t)$

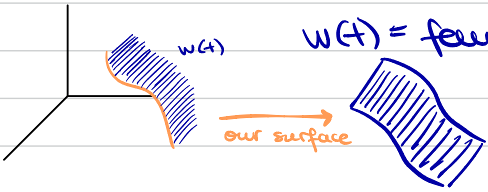
DEF: Given a differentiable one-parameter family $\{\alpha(t), w(t)\}_{t\in\mathbb{R}}$, where

$\alpha: \mathbb{R} \to \mathbb{R}^3$ is a differentiable, regular curve
$w: \mathbb{R} \to \mathbb{R}^3$ is a diff. vector field along the curve α

The surface S given by parametrization

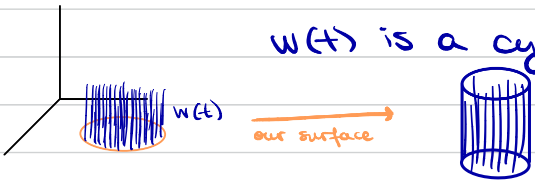
$x(t,v) = \alpha(t) + \underset{}{\overset{\in\mathbb{R}}{v}} \cdot w(t)$ is called a ruled surface.

EXAMPLE:



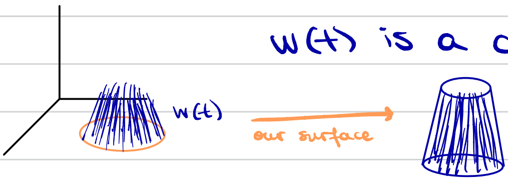
$w(t)$ = family of paralell vectors along the curve

EXAMPLE: Cylinder



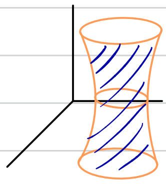
$w(t)$ is a cylinder

EXAMPLE: Cone



$w(t)$ is a cylinder

EXAMPLE: Hyperboloid of revolution



$$x^2+y^2-z^2=1$$

We choose the parametrization following the pattern of the definition of ruled surface:

$$x(s,v)=(\cos(s)-v\cdot\sin(s),\ \sin(s)+v\cdot\cos(s),\ v)=$$

$$=\underbrace{(\cos(s),\sin(s),0)}_{\text{param. of a circle}}+\underbrace{[-v\sin(s),\ v\cos(s),\ 0]}_{\text{param. of a line}}$$

$v[-\sin(s),\cos(s),0]$

We check it is, indeed, a good parametrization:

$$(\cos(s)-v\sin(s))^2+(\sin(s)+v\cos(s))^2-v^2=$$

$$=\cos^2(s)-2v\cos(s)\sin(s)+v^2\sin^2(s)+\sin^2(s)+2v\sin(s)\cos(s)+v^2\cos^2(s)-v^2=$$

$$=1+v^2-v^2=1 \quad\Rightarrow \text{it is OK}$$

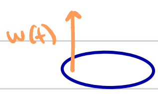
DEF: A ruled surface S given by $x(t,v)=\alpha(t)+v\cdot w(t)$ is called non cylindrical if $w'(t)\neq 0$.

* The Gaussian curvature of a non-cylindrical surface S satisfies $K\leq 0$.

DEF: A ruled surface S given by $x(t,v)=\alpha(t)+v\cdot w(t)$ generated by the family $\{\alpha(t),w(t)\}_{t\in\mathbb{R}}$ with $\|w(t)\|=1$ is said to be developable if $\langle w, w'\times\alpha'\rangle=0$.

THM: The Gaussian curvature of developable surfaces is $K=0$ everywhere.

EXAMPLE: Cylinder



$\alpha(t)$ is a param. of a circle

$w'(t)=0 \Rightarrow w'\times\alpha'=0 \Rightarrow \langle w,w'\times\alpha'\rangle=0 \Rightarrow$ cylinder is a developable surface

because $w(t)$ is a constant vector for each point we fix in the circle $\alpha(t)$

The Gauss Theorem and Equations of compatibility

Let S be a regular surface with parametrization $x(u,v)$.
We know that $x_u, x_v, N = \frac{x_u \times x_v}{\|x_u \times x_v\|}$ are a basis of $\mathbb{R}^3$.

In particular:

just a coef. to be determined

$$\left.\begin{aligned} x_{uu} &= \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + L_1 \cdot N \\ x_{uv} &= \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + L_2 \cdot N \\ x_{vv} &= \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + L_3 \cdot N \\ x_{vu} &= \Gamma_{21}^1 x_u + \Gamma_{21}^2 x_v + \bar{L}_2 \cdot N \end{aligned}\right\}$$

We know that: $N_u = a_{11} \cdot x_u + a_{21} x_v$
$N_v = a_{12} \cdot x_u + a_{22} x_v$

We obtain a system (1)

Γ_{ik}^j $\left\{ \begin{array}{l} [\, i,j,k = 1,2 \\ [\, i \to \text{derivate w.r.t. } u \\ \;\; k \to \text{derivate w.r.t. } v \\ [\, j \to \text{which vector of the basis we consider, but } j \neq 3 \end{array} \right.$

We call this Christofel Symbols

COROLARY: $\Gamma_{12}^1 = \Gamma_{21}^1$ and $\Gamma_{12}^2 = \Gamma_{21}^2$

Proof: $x_{uv} = x_{vu} \Rightarrow \Gamma_{12}^1 = \Gamma_{21}^1$ and $\Gamma_{12}^2 = \Gamma_{21}^2$ +

COROLARY: $L_1 = e$, $L_2 = \bar{L}_2 = f$, $L_3 = g$

Proof: 1) $e = \langle N, x_{uu} \rangle = \langle N, \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + L_1 N \rangle =$
$= \cancel{\langle N, \Gamma_{11}^1 x_u \rangle}^{0} + \cancel{\langle N, \Gamma_{11}^2 x_v \rangle}^{0} + \langle N, L_1 N \rangle = L_1 \underbrace{\langle N,N \rangle}_{=1} = L_1$

2) $f = \langle N, x_{uv} \rangle = \langle N, \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + L_2 N \rangle =$
$= \cancel{\langle N, \Gamma_{12}^1 x_u \rangle} + \cancel{\langle N, \Gamma_{12}^2 x_v \rangle} + \langle N, L_2 N \rangle = L_2 \langle N,N \rangle = L_2$

$f = \langle N, x_{vu} \rangle = \langle N, \Gamma_{21}^1 x_u + \Gamma_{21}^2 x_v + \bar{L}_2 N \rangle =$
$= \cancel{\langle N, \Gamma_{21}^1 x_u \rangle} + \cancel{\langle N, \Gamma_{21}^2 x_v \rangle} + \langle N, \bar{L}_2 N \rangle = \bar{L}_2 \langle N,N \rangle = \bar{L}_2$

3) $g = \langle N, x_{vv} \rangle = \langle N, \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + L_3 N \rangle =$
$= \cancel{\langle N, \Gamma_{22}^1 x_u \rangle} + \cancel{\langle N, \Gamma_{22}^2 x_v \rangle} + \langle N, L_3 N \rangle = L_3 \langle N,N \rangle = L_3$

+

Problem: Can we express Γ_{ik}^j in terms of E, F, G and their derivates?

Solution

COROLLARY:

a) $$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle x_{uu}, x_u \rangle = \tfrac{1}{2} E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle x_{uu}, x_v \rangle = F_u - \tfrac{1}{2} E_v \end{cases}$$

b) $$\begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \langle x_{uv}, x_u \rangle = \tfrac{1}{2} E_v \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G = \langle x_{uv}, x_v \rangle = \tfrac{1}{2} G_u \end{cases}$$

c) $$\begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = \langle x_{vv}, x_u \rangle = F_v - \tfrac{1}{2} E_u \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \langle x_{vv}, x_v \rangle = \tfrac{1}{2} G_v \end{cases}$$

Proof:

a.1) $$\langle x_{uu}, x_u \rangle = \langle \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + L_1 N, x_u \rangle =$$
$$= \Gamma_{11}^1 \underbrace{\langle x_u, x_u \rangle}_{E} + \Gamma_{11}^2 \underbrace{\langle x_v, x_u \rangle}_{F} + \cancel{L_1 \langle N, x_u \rangle} = \Gamma_{11}^1 E + \Gamma_{11}^2 F \,//$$

Now:

$$E_u = \frac{\partial E}{\partial u} = \frac{\partial}{\partial u} \langle x_u, x_u \rangle = \langle x_{uu}, x_u \rangle + \langle x_u, x_{uu} \rangle = 2 \langle x_u, x_{uu} \rangle \Rightarrow$$

$$\Rightarrow \tfrac{1}{2} E_u = \langle x_u, x_{uu} \rangle = \Gamma_{11}^1 E + \Gamma_{11}^2 F \,//$$

a.2) $$\langle x_{uu}, x_v \rangle = \langle \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + L_1 N, x_v \rangle =$$
$$= \Gamma_{11}^1 \underbrace{\langle x_u, x_v \rangle}_{F} + \Gamma_{11}^2 \underbrace{\langle x_v, x_v \rangle}_{G} + \cancel{L_1 \langle N, x_v \rangle} = \Gamma_{11}^1 F + \Gamma_{11}^2 G \,//$$

Now:

$$F_u - \tfrac{1}{2} E_v = \frac{\partial}{\partial u} \langle x_u, x_v \rangle - \tfrac{1}{2} \frac{\partial}{\partial v} \langle x_u, x_u \rangle =$$
$$= \langle x_{uu}, x_v \rangle + \langle x_u, x_{vu} \rangle - \tfrac{1}{2} \langle x_{uv}, x_u \rangle - \tfrac{1}{2} \langle x_u, x_{uv} \rangle =$$
$$= \langle x_{uu}, x_v \rangle + (\langle x_u, x_{vu} \rangle - \langle x_{uv}, x_u \rangle) = \langle x_{uu}, x_v \rangle \Rightarrow$$

$$\Rightarrow F_u - \tfrac{1}{2} E_v = \langle x_{uu}, x_v \rangle = \Gamma_{11}^1 F + \Gamma_{11}^2 G \,//$$

The rest of the proof follows similarly

$\dagger$

THM: **Gauss Formula**

$$-E\cdot K = (\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\cdot\Gamma^2_{11} + \Gamma^2_{12}\cdot\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}$$

THM: **Mainardi - Codazzi equations**

$$\left.\begin{aligned} e_v - f_u &= e\cdot\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} \\ f_v - g_u &= e\cdot\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} \end{aligned}\right\} \text{equations of compatibility}$$

THM: **Bonnet**

Let E, F, G, e, f, g be differentiable functions defined on an open subset $\mathcal{U} \subset \mathbb{R}^2$ with $E, G > 0$.

Assume that:

1) The functions E, F, G, e, f, g satisfy the eq. of compatibility
2) $EG - F^2 > 0$

Then, for any point $q \in \mathcal{U}$, there exists:

- an open neighborhood V_q (this is, V_q - open s.t. $q \in V_q \subset \mathcal{U}$)
- and a differentiable function $x: V_q \to \mathbb{R}^3$

such that the surface $x(V_q)$ has:

- coefficients of the I_p equal, resp., to E, F, G and
- coefficients of the II_p equal, resp., to e, f, g.

Geodesics and geodesic curvatures

Let S be a surface given by a regular parametrization $x(u,v)$ and let $S \ni p \mapsto w(p)$ be a vector field differentiable at p. It means

$$w(p) = a(p) \cdot x_u + b(p) \cdot x_v$$

$a, b : S \longrightarrow \mathbb{R}$ are differentiable

Now we fix a vector $v \in T_pS$, $v = w(p)$.

Consider a regular curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$.

Let $\tilde{w}(t) = w(\alpha(t))$ be the restriction of the vector field to α.

$$\frac{d\tilde{w}(t)}{dt}$$

$$P^{\perp}_{T_pS} \frac{d\tilde{w}(t)}{dt} \subset T_pS$$

Def. The projection of the derivative $\frac{d\tilde{w}}{dt}(0)$ onto the tangent space T_pS, i.e. $P^{\perp}_{T_pS} \frac{d\tilde{w}(0)}{dt}$ we call the covariant derivative of the vector field w relative to the vector v and we denote it by $D_v w(p)$

Remark. Assume that there are diff. curves $\alpha, \beta : (-\varepsilon, \varepsilon) \longrightarrow S$ such that $\alpha(0) = \beta(0)$ and $\alpha'(0) = r = \beta'(0)$. w a tangent diff. vector field. Then the covariant derivative of w w.r.t. β.

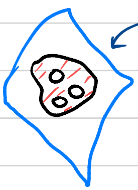
Def. Let w be a unit diff. vector field along $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S$. Then $\frac{dw(t)}{dt} \perp w(t)$, so $\frac{Dw(t)}{dt}$ is $\parallel$ to $N(t) \times w(t)$ so

$$\frac{Dw}{dt}(t) = K_g(t) \cdot N(t) \times w(t)$$

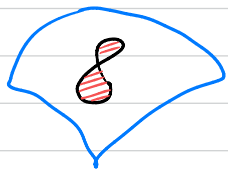
geodesic curvature

Triangulations

Assume that S is a regular surface without boundary. Consider a region $R \subset S$.
A region is a compact subset of a surface such that its boundary is a finite union of closed simple curves.



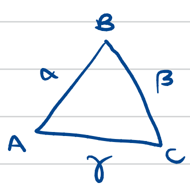
Example of a region



It is not a region

Triangle on a surface

Points A, B, C on a surface S
Regular curves α, β, γ on S



The geometric object is a triangle on S

Def. A triangulation T of a region $R \subset S$ is a finite family of triangules $T_1, T_2, \dots, T_k$ such that:

1) $\bigcup_{T_i \in T} T_i = R$ 2) $T_i \cap T_j = \begin{cases} \emptyset \\ \text{common vertex} \\ \text{common edge} \end{cases}$

For a triangulation T of R we define

F = the number of triangles in the triangulation T
E = the number of edges in " "
V = the number of vertices in " ".

The number

$\chi_T(R) \overset{\text{def}}{=} F - E + V$ we call the Euler characteristic of the region R

Thm. For any 2 triangulations T_1 and T_2 of a region R we get $\chi_{T_1}(R) = \chi_{T_2}(R)$

Cor. The Euler characteristic of R does not depend on a triangulation.

Thm. Given two regular surfaces S_1 and S_2.

S_1 is homeomorphic to $S_2 \iff \chi(S_1) = \chi(S_2)$

Prop. Any regular connected surface without boundary is homeomorphic to a sphere with k handles.

Prop. For any regular connected surface S without boundary $\chi(S) = \{2, 0, -2, -4, -6, \dots\}$

Example $\chi($  $) = 2$, $\chi(\text{torus}) = 0$, $\chi($  $) = -2$

Thm (Gauss-Bonet) Let S be a regular oriented surface and R a region of S such that boundary of R consists of regular simple curves $C_1, C_2, C_3, \dots, C_k$.
Let $\theta_1, \theta_2, \dots, \theta_m$ be external angles of the curves $C_1, \dots, C_k$

$$\sum_{i=1}^{k} \int_{C_i} k_g(s)\, ds + \sum_{i=1}^{m} \theta_i + \iint_R K\, d\sigma = 2\pi \cdot \chi(R)$$

geodesic curvature of curve C_i

external angle

Gauss curvature

Cor. If S is a regular connected surface with $K>0$ then S is homeomorphic to the sphere S^2.

<u>Proof</u>: We know $\chi(S) \in \{2,0,-2,-4,\dots\}$. Since $K>0$ we get $\iint_S K\,ds > 0$. On the other hand $\iint_S K\,ds = 2\pi \cdot \underset{2}{\underset{\parallel}{\chi(S)}}$

$2 = \chi(S^2)$

$\chi(S) = \chi(S^2)$ so S is homeomorphic to a sphere S^2.

1. Param of a regular surface or not
2. Gauss + Mean curvature
3. Area of a surface
4. Principal curvatures
5. Christoffel symbols in simple surfaces. $\left\{ \begin{array}{l} \text{plane} \\ \text{torus} \\ \text{sphere} \\ \text{cylinder} \\ \text{cone} \end{array} \right.$