EXERCICES DIF. GEOM

Ex 1 Given points $A=(1,0,2)$, $B=(1,2,3)$, $C=(-1,1,5)$.
Write the equation of plane π such that $A, B, C \in \pi$

$$\vec{AB} = (0,2,1) = \vec{u}$$
$$\vec{AC} = (-2,1,3) = \vec{v}$$

$$\vec{u} \times \vec{v} = \left[\det\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, -\det\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \det\begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} \right] = [5,-2,4] = \vec{n}$$

$\vec{n} \perp \pi. \Rightarrow \pi := 5x - 2y + 4z + D = 0$

Since $A \in \pi \Rightarrow 5 \cdot 1 - 2 \cdot 0 + 4 \cdot 2 + D = 13 + D = 0;\ D = -13$ //

EX 2: Given a plane $\pi: 2x+4y+7z=0$, find:

a) a plane π_1 such that $\pi_1 \parallel \pi$ and $(2,1,5) \in \pi_1$

$\pi_D: 2x+4y+7z+D=0$ is parallell to π, $D \in \mathbb{R}$

If we want $(2,1,5) \in \pi_1$, then we should adjust D.

$2\cdot 2+4\cdot 1+7\cdot 5 = 4+4+35 = 43 = -D$

Then: $\pi_1: 2x+4y+7z-43=0$

b) a line ℓ such that $\ell \perp \pi$ and $(3,1,7) \in \ell$

We can take $\vec{n} \perp \pi$, which is $\vec{n}=[2,4,7]$.
That's the vector of ℓ.

We want $(3,1,7) \in \ell$. Finally:

$\ell = \{(3,1,7)+\alpha\cdot[2,4,7] : \alpha \in \mathbb{R}\} = \{(3+2\alpha, 1+4\alpha, 7+7\alpha) : \alpha \in \mathbb{R}\}$

c) coordinates of the point $E = \ell \cap \pi$

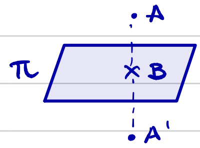
Any point of ℓ is given by $(3+2\alpha, 1+4\alpha, 7+7\alpha)$, $\alpha \in \mathbb{R}$.
Then, we need to force it to belong to π_1:

$2\cdot(3+2\alpha)+4\cdot(1+4\alpha)+7\cdot(7+7\alpha)=0$;

; $6+4\alpha+4+16\alpha+49+49\alpha=0$; $59+69\alpha=0$; $\alpha = -59/69$

Finally, $E = (3+2(-59/69), 1+4(-59/69), 7+7(-59/69))$

EX 3: Given a plane $\pi: 2x+3y+5z+7=0$ and a point $A=(1,1,2)$, we know that A' is a reflection of A with respect to plane π. Find A'.



$\vec{n}=[2,3,5] \perp \pi$.

1) Find ℓ such as $\ell \perp \pi$ and $A \in \ell$.

$$\ell = \{(1+2\alpha, 1+3\alpha, 2+5\alpha) : \alpha \in \mathbb{R}\}$$

2) Find $B = \ell \cap \pi \iff B \in \ell \wedge B \in \pi$

$$2(1+2\alpha)+3(1+3\alpha)+5(2+5\alpha)+7=0;$$

$$; 2+4\alpha+3+9\alpha+10+25\alpha+7=0; \; 22+38\alpha=0; \; \alpha=-{}^{11}/_{19}$$

Then, $B=(1+2(-{}^{11}/_{19}), 1+3(-{}^{11}/_{19}), 2+5(-{}^{11}/_{19}))$

3a) $\vec{AB}=\vec{BA'}$, as B is in the middle of A and A'.

$$\vec{AB}=[1+2(-{}^{11}/_{19})-1, 1+3(-{}^{11}/_{19})-1, 2+5(-{}^{11}/_{19})-1]$$

$$=[-{}^{22}/_{19}, -{}^{33}/_{19}, -{}^{55}/_{19}]$$

Then,

$$\vec{BA'}=[x-1+{}^{22}/_{19}, y-1+{}^{33}/_{19}, z-2+{}^{55}/_{19}]$$

We have:

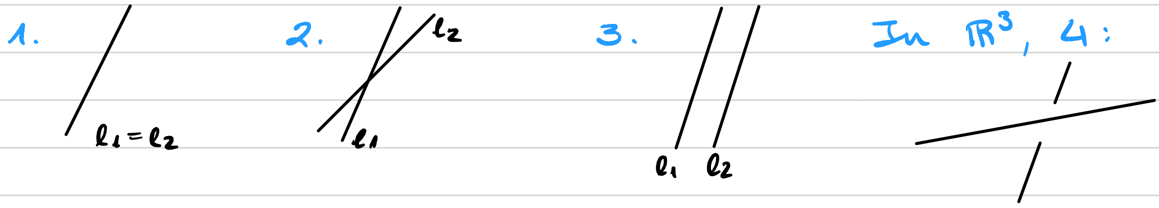
$$\left.\begin{array}{l} x-1+{}^{22}/_{19}=-{}^{22}/_{19}; \quad x=1-{}^{44}/_{19} \\ y-1+{}^{33}/_{19}=-{}^{33}/_{19}; \quad y=1-{}^{66}/_{19} \\ z-2+{}^{55}/_{19}=-{}^{55}/_{19}; \quad z=2-{}^{110}/_{19} \end{array}\right\} A'=(-{}^{25}/_{19}, -{}^{47}/_{19}, -{}^{72}/_{19})$$

3b) Another way of doing it.

$$\left.\begin{array}{l} A=(a_1,a_2,a_3) \\ B=(b_1,b_2,b_3) \\ A'=(c_1,c_2,c_3) \end{array}\right\} \Rightarrow \frac{a_i+c_i}{2}=b_i \quad \forall i=1,2,3$$

Then we can calculate c_i $\forall i=1,2,3$, as we know both A and B.

Possible positions of lines in $\mathbb{R}^2$



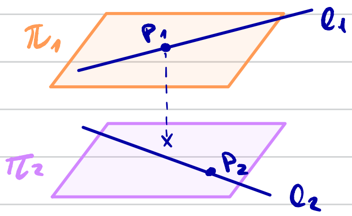
Coincidentes Secantes Paralelas Cruzadas

EX 4: Given l_1, l_2, two skew-lines,

$$l_1 = \{P_1 + \alpha \vec{v_1} : \alpha \in \mathbb{R}\} \quad \text{and} \quad l_2 = \{P_2 + \beta \vec{v_2} : \beta \in \mathbb{R}\}$$

Find the distance between them.

Let us consider $\pi_1 = \{P_1 + \alpha \vec{v_1} + \beta \vec{v_2} : \alpha, \beta \in \mathbb{R}\}$
$\pi_2 = \{P_2 + \alpha \vec{v_1} + \beta \vec{v_2} : \alpha, \beta \in \mathbb{R}\}$



$$\begin{cases} l_1 \subset \pi_1 \\ l_2 \subset \pi_2 \\ \pi_1 \parallel \pi_2 \end{cases}$$

Now, we have that $d(l_1, l_2) = d(\pi_1, \pi_2) = d(P_1, \pi_2)$
To be able to use the formula, we just need to translate π_2 parametric equations into $Ax \dots$

Ex 5: Given two lines

$l_1 = \{(1,2,3) + t[1,7,2] : t \in \mathbb{R}\}$
$l_2 = \{(2,0,7) + s[2,0,9] : s \in \mathbb{R}\}$

Find $d(l_1, l_2)$

Let us consider $\pi_1 = \{(1,2,3) + t[1,7,2] + s[2,0,9] : t, s \in \mathbb{R}\}$
$\pi_2 = \{(2,0,7) + t[1,7,2] + s[2,0,9] : t, s \in \mathbb{R}\}$

Then, let us express π_2 in an implicit way.
Let's find $\vec{n_2}$:

$$\vec{n_2} = \left[\det\begin{bmatrix} 7 & 2 \\ 0 & 9 \end{bmatrix}, -\det\begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}, \det\begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix}\right] =$$

$$= [63, -5, -14] \Rightarrow \pi_2 : 63x - 5y - 14z + D_2 = 0$$

As $P_2 \in \pi_2$, let's find D_2:

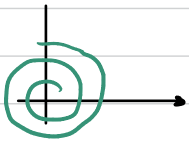
$$63 \cdot 2 - 5 \cdot 0 - 14 \cdot 7 = 28 \Rightarrow \pi_2 : 63x - 5y - 14z - 28 = 0$$

Finally, we can use the formula:

$$d(P_1, \pi_2) = \frac{|Ap_1 + Bp_2 + Cp_3 + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|63 \cdot 1 - 5 \cdot 2 - 14 \cdot 3 - 28|}{\sqrt{63^2 + 5^2 + 14^2}} = \frac{17}{\sqrt{4190}} \simeq 0'26$$

CURVES

Ex: Logarithmspiral $\gamma(t) = (e^{k\cdot t}\cdot \cos(t),\ e^{kt}\cdot \sin(t), 0)$



Calculate the length of the spiral.

$$\gamma'(t) = (k e^{kt}\cdot \cos t - e^{kt} \operatorname{sen} t,\ k e^{kt} \operatorname{sen} t + e^{kt} \cos t,\ 0)$$

Now, we calculate the norm.

$$\|\gamma'(t)\| = \sqrt{e^{2kt}(k\cos t - \operatorname{sen} t)^2 + e^{2kt}(k \operatorname{sen} t + \cos t)^2} =$$

$$= e^{kt}\sqrt{k^2\cos^2 t + \operatorname{sen}^2 t \cancel{- 2k\cos t \operatorname{sen} t} + k^2 \operatorname{sen}^2 t + \cos^2 t \cancel{+ 2k \operatorname{sen} t \cos t}} =$$

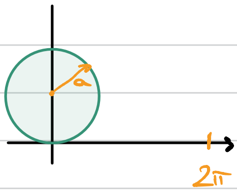
$$= e^{kt}\sqrt{k^2(\cos^2 t + \operatorname{sen}^2 t) + (\operatorname{sen}^2 t + \cos^2 t)} = e^{kt}\sqrt{k^2+1} \neq 0$$ → regular curve

The lenght:

$$L(\gamma) = \int_0^{t_0} \|\gamma'(t)\|\, dt = \int_0^{t_0} e^{kt}\sqrt{k^2+1}\, dt = \left[\tfrac{1}{k}\sqrt{k^2+1}\, e^{kt}\right]_0^{t_0} =$$

$$= \tfrac{1}{k}\sqrt{k^2+1}\,(e^{kt_0} - e^0) = \tfrac{1}{k}\sqrt{k^2+1}\,(e^{kt_0} - 1)$$

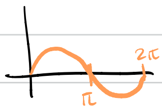
ex: Calculate the legth of cycloid given by $\gamma(t) = (a(t - \sin(t)), a(1-\cos(t)))$ where $a>0$ $t \in [0, 2\pi]$



$$\gamma'(t) = [a(1-\cos t), a \, \text{sen}\, t]$$

$$\|\gamma'(t)\| = \sqrt{a^2(1-\cos t)^2 + a^2 \text{sen}^2 t} = a\sqrt{1 - 2\cos t + \cos^2 t + \text{sen}^2 t} =$$

$$= \sqrt{2}\, a \sqrt{1-\cos t} = \sqrt{2}\, a \sqrt{2\,\text{sen}^2(t/2)} = 2a\, |\text{sen}\,(t/2)| =$$



$$= 2a\, \text{sen}\,(t/2)$$ $t/2 \in [0, \pi] \Rightarrow \text{sen}\, t/2 > 0 \quad \forall t \in [0, 2\pi]$

$$\ell(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\, dt = 2a \int_0^{2\pi} \text{sen}\, t/2 \, dt = 2a \left[-2\cos t/2\right]_0^{2\pi} =$$

$$= -4a\,(\cos \pi - \cos 0) = -4a(-1-1) = \underline{8a}$$

Ex. Characterize all regular curves $c:(a,b) \longrightarrow \mathbb{R}^3$ such that $\forall t \in (a,b)$ $c''(t) = [0,0,0]$

$$c(t) = (c_1(t), c_2(t), c_3(t))$$

$$c_1 : (a,b) \longrightarrow \mathbb{R} \Rightarrow c_1''(t) = 0$$
$$c_2 : (a,b) \longrightarrow \mathbb{R} \Rightarrow c_2''(t) = 0 \quad \text{at any } t \in (a,b).$$
$$c_3 : (a,b) \longrightarrow \mathbb{R} \Rightarrow c_3''(t) = 0$$

Let $f:(a,b) \longrightarrow \mathbb{R}$ such that $\forall t \in (a,b)$ $f'(t) = 0$, so $f(t) = \text{constant}$

$$c_1''(t) \equiv 0 \Rightarrow c_1'(t) = p_1 \ (cte) \Rightarrow c_1(t) = tp_1 + q_1$$
$$c_2''(t) \equiv 0 \Rightarrow c_2'(t) = p_2 \ (cte) \Rightarrow c_2(t) = tp_2 + q_2$$
$$c_3''(t) \equiv 0 \Rightarrow c_3'(t) = p_3 \ (cte) \Rightarrow c_3(t) = tp_3 + q_3$$

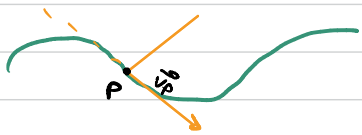
$$c(t) = (c_1(t), c_2(t), c_3(t)) = (tp_1 + q_1, tp_2 + q_2, tp_3 + q_3) = (q_1, q_2, q_3) + t[p_1, p_2, p_3]$$

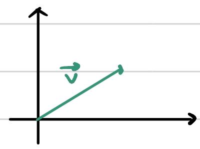
If a regular curve c satisfies condition $c''(t)$ for any $t \in (a,b)$, then c is a parametrization of a segment in $\mathbb{R}^3$.

Ex: Find the tangent and normal lines to the plane curve

$\gamma(t) = (2\cos(t) - \cos(2t), 2\sin(t) - \sin(2t))$

where the normal line to a curve at a point p is the stripth line passin throught p and $\perp$ to tangent line at p.





$\vec{v} = [v_1, v_2]$

$\vec{w} = [-v_2, v_1]$ $\Rightarrow \vec{w} \perp \vec{v}$

$[\vec{v}, \vec{w}] = -v_1v_2 + v_1v_2 = \vec{0}$

$\gamma'(t) = (-2\operatorname{sen} t + 2\operatorname{sen} 2t, 2\cos t - 2\cos 2t)$

$L_t = \{\gamma(t_0) + \gamma'(t_0) \cdot r \; : r \in \mathbb{R}\}$ // tangent line

$\gamma''(t) = (-2\cos t + 4\operatorname{sen} 2t, -2\operatorname{sen} t + 4\operatorname{sen} 2t)$

$L_n = \{\gamma(t_0) + \gamma''(t_0) \cdot r \; : r \in \mathbb{R}\}$ // normal line

$\gamma'(t_0) = (-2\operatorname{sen}(t_0) + 2\operatorname{sen}(2t_0), 2\cos(t_0) - 2\cos(2t_0))$

$P = \gamma(t_0)$

Tangent line l_1 determined by vector $\gamma'(t_0)$ and passing through the point $\gamma(t_0)$ has the following description

$L_1 = \{\gamma(t_0) + t\gamma'(t_0), t \in \mathbb{R}\}$

The normal line $L_2 = \{\gamma(t_0) + t[\cancel{2}\cos(2t_0) - 2\cos(t_0), -2\operatorname{sen}(t_0) + 2\operatorname{sen}(2t_0), t \in \mathbb{R}\}$

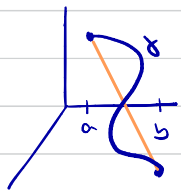
EX: Prove that the segment joining points A and B is the shortest curve joining these two points.

Consider γ a curve.

regular

$\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ γ diff. $[a,b] \subset I$.

$\underbrace{\|\gamma(a) - \gamma(b)\|}_{\text{lenght of the segment}} = \| \int_a^b \gamma'(t)\, dt \| \leq \int_a^b \|\gamma'(t)\|\, dt = l(\gamma) =$ lenght of $\gamma : [a,b] \to \mathbb{R}^3$

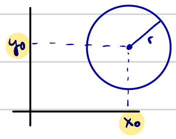


Ex 1: [1] Find a parametrization with unit speed of a circle

$$S = \{(x,y)\in\mathbb{R}^2 : (x-x_0)^2+(y-y_0)^2 = r^2\}$$

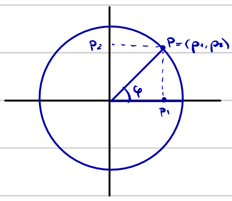
with radius r and center (x_0, y_0).

[2] Calculate tangent, normal, binormal vectors and curvature with torsion.



1

We are going to simplify the case. We assume $(x_0,y_0)=(0,0)$



$P_1/r = \cos\varphi$

$P_2/r = \operatorname{sen}\varphi$

$\varphi \to (r\cos\varphi,\ r\operatorname{sen}\varphi)$

$$\forall\varphi\in[0,2\pi],\quad \alpha(\varphi)=(r\cos\varphi,\ r\operatorname{sen}\varphi)$$

Con esto, encontramos una parametrización del círculo centrado en el origen. Para el círculo original:

$$\forall\varphi\in[0,2\pi],\quad \beta(\varphi) = (x_0+r\cos\varphi,\ y_0+r\operatorname{sen}\varphi)$$

Ahora vemos si es un vector unit speed.

$$\beta'(\varphi) = [-r\operatorname{sen}\varphi,\ r\cos\varphi] \Rightarrow \|\beta'(\varphi)\| = \sqrt{r^2\operatorname{sen}^2\varphi + r^2\cos^2\varphi} = r$$

So, in general $(r\neq 1)$, it is not a parametrization with unit speed.

Consider the parametrization:

$$\forall\varphi\in[0,2\pi],\quad \gamma(\varphi) = (x_0+r\cos\tfrac{\varphi}{r},\ y_0+r\operatorname{sen}\tfrac{\varphi}{r})$$

$$\gamma'(\varphi) = [-\operatorname{sen}\varphi/r,\ \cos\varphi/r] \Rightarrow \|\gamma'(\varphi)\| = \sqrt{\operatorname{sen}^2\varphi/r + \cos^2\varphi/r} = 1$$

This parametrization has unit speed! :)

2 Tenemos que considerar una coordenada más pq estamos en $\mathbb{R}^3$

$\forall \varphi \in [0,2\pi]$, $\beta(\varphi) = (x_0 + r\cos\varphi,\ y_0 + r\,\text{sen}\,\varphi,\ 0$

$\forall \varphi \in [0,2\pi]$, $\gamma(\varphi) = (x_0 + r\cos\frac{\varphi}{r},\ y_0 + r\,\text{sen}\,\frac{\varphi}{r},\ 0)$

Tangent vector: $t(\varphi) = \gamma'(\varphi) = [-\sin\varphi/r,\ \cos\varphi/r,\ 0]$

Normal vector: $\gamma''(\varphi) = [-1/r\cos\varphi/r,\ -1/r\,\text{sen}\,\varphi/r,\ 0]$

Now we calculate its lenght:

$$\|\gamma''(\varphi)\| = \sqrt{1/r^2\cos^2\varphi/r + 1/r^2\,\text{sen}^2\varphi/r + 0^2} = 1/r$$

The normal vector, $n(\varphi) = \dfrac{\gamma''(\varphi)}{\|\gamma''(\varphi)\|} = [-\cos\varphi/r,\ -\text{sen}\,\varphi/r,\ 0]$

¿? The curvature $\kappa(\varphi) = \|\gamma''(\varphi)\| = 1/r$ of a circle at any point.

To calculate torsion, we need the binormal vector.

If $r \to \infty$, $\kappa(\varphi) \to 0$. Un circulo de radio ∞ tiene curvatura 0, localmente parece el segmento de una recta.

Binormal vector:

$$b(\varphi) = t(\varphi) \times n(\varphi) = \left[\det\begin{vmatrix}\cos\varphi/r & 0\\ -\sin\varphi/r & 0\end{vmatrix},\ -\det\begin{vmatrix}-\sin\varphi/r & 0\\ -\cos\varphi/r & 0\end{vmatrix},\ \det\begin{vmatrix}-\sin\varphi/r & \cos\varphi/r\\ -\cos\varphi/r & -\sin\varphi/r\end{vmatrix}\right] =$$

$$= [0,\ 0,\ \sin^2\varphi/r + \cos^2\varphi/r] = [0,0,1]$$

$\forall \varphi \in [0,2\pi]$, $b(\varphi) = [0,0,1] \Rightarrow b'(\varphi) = [0,0,0]$

Therefore, $\forall \varphi \in [0,2\pi]$, we get that $\tau(\varphi) = 0$

Because $\underset{0}{\underset{\parallel}{b'(\varphi)}} = \tau(\varphi)\cdot \underset{0}{\underset{\neq}{n(\varphi)}}$

Ex 2: Find a parametrization of a helix with unit speed.

We know that a possible param. is:

$$\alpha(t) = (r\cos t,\ r\,\text{sen}\,t,\ at) \qquad a \in \mathbb{R}.$$

Does it have unit speed?

$$\alpha'(t) = (-r\,\text{sen}\,t,\ r\cos t,\ a) \Rightarrow \|\alpha'(t)\| = \sqrt{r^2\text{sen}^2 t + r^2\cos^2 t + a^2} = \sqrt{r^2+a^2} \neq 1$$

In general, $\|\alpha'(t)\| \neq 1 \Rightarrow \alpha$ is a param. of the helix but not with unit speed.

Another option:

$$\beta(t) = \left(r\cos\frac{t}{\sqrt{r^2+a^2}},\ r\,\text{sen}\,\frac{t}{\sqrt{r^2+a^2}},\ \frac{at}{\sqrt{r^2+a^2}}\right)$$

$$\beta'(t) = \left(\frac{-r}{\sqrt{r^2+a^2}}\,\text{sen}\,\frac{t}{\sqrt{r^2+a^2}},\ \frac{r}{\sqrt{r^2+a^2}}\cos\frac{t}{\sqrt{r^2+a^2}},\ \frac{a}{\sqrt{r^2+a^2}}\right)$$

$$\|\beta'(t)\| = \sqrt{\frac{r^2}{r^2+a^2} + \frac{a^2}{r^2+a^2}} = 1 \ \checkmark$$

Tangent vector

$$\beta'(t) = \left(\frac{-r}{\sqrt{r^2+a^2}}\,\text{sen}\,\frac{t}{\sqrt{r^2+a^2}},\ \frac{r}{\sqrt{r^2+a^2}}\cos\frac{t}{\sqrt{r^2+a^2}},\ \frac{a}{\sqrt{r^2+a^2}}\right)$$

Normal vector

$$\beta''(t) = \left(\frac{-r}{r^2+a^2}\cos\frac{t}{\sqrt{r^2+a^2}},\ \frac{-r}{r^2+a^2}\,\text{sen}\,\frac{t}{\sqrt{r^2+a^2}},\ 0\right)$$

$$\|\beta''(t)\| = \sqrt{\frac{r^2}{(r^2+a^2)^2}} = \frac{r}{r^2+a^2} = \kappa(t)$$ curvature of an helix in parametrization $\beta(t)$

$$n(t) = \frac{\beta''(t)}{\|\beta''(t)\|} = \left(-\cos\frac{t}{\sqrt{r^2+a^2}},\ -\text{sen}\,\frac{t}{\sqrt{r^2+a^2}},\ 0\right)$$

Torsion

$$\langle b'(s), n(s)\rangle = \langle \tau(s) n(s), n(s)\rangle = \tau(s)\overbrace{\langle n(s), n(s)\rangle}^{=1} = \tau(s)$$

Let's find $b(s)$:

$$b(s) = t(s) \times n(s) = \left[\frac{a}{\sqrt{r^2+a^2}} \operatorname{sen} \frac{s}{\sqrt{r^2+a^2}}, -\frac{a}{\sqrt{r^2+a^2}} \cos \frac{s}{\sqrt{r^2+a^2}}, \frac{r}{\sqrt{r^2+a^2}}\right]$$

$$\beta'(t) = \left(\frac{-r}{\sqrt{r^2+a^2}} \operatorname{sen} \frac{t}{\sqrt{r^2+a^2}}, \frac{r}{\sqrt{r^2+a^2}} \cos \frac{t}{\sqrt{r^2+a^2}}, \frac{a}{\sqrt{r^2+a^2}}\right)$$

$$n(t) = \frac{\beta''(t)}{\|\beta''(t)\|} = \left(-\cos \frac{t}{\sqrt{r^2+a^2}}, -\operatorname{sen} \frac{t}{\sqrt{r^2+a^2}}, 0\right)$$

Then, we have:

$$b'(s) = \left[\frac{a}{r^2+a^2} \cos \underbrace{\frac{s}{\sqrt{r^2+a^2}}}_{\mu}, \frac{a}{r^2+a^2} \operatorname{sen} \frac{s}{\sqrt{r^2+a^2}}, 0\right]$$

Finally:

$$\tau(s) = \langle b'(s), n(s)\rangle = -\frac{a}{r^2+a^2} \cos^2 \mu - \frac{a}{r^2+a^2} \operatorname{sen}^2 \mu + 0 = -\frac{a}{r^2+a^2}$$

EX 3: Given a parametrization $\alpha(s) = \left(\frac{1}{\sqrt{2}}\cos(s), \operatorname{sen}(s), \frac{1}{\sqrt{2}}\cos(s)\right)$.

Calculate curvature and torsion and identify the curve.

$$\alpha'(s) = \left(-\tfrac{1}{\sqrt{2}}\operatorname{sen}(s), \cos(s), -\tfrac{1}{\sqrt{2}}\operatorname{sen}(s)\right) = t(s)$$

$$\|\alpha'(s)\| = \sqrt{\tfrac{1}{2}\operatorname{sen}^2(s) + \cos^2(s) + \tfrac{1}{2}\operatorname{sen}^2(s)} = 1.$$

$$\alpha''(s) = \left(-\tfrac{1}{\sqrt{2}}\cos(s), -\operatorname{sen}(s), -\tfrac{1}{\sqrt{2}}\cos(s)\right) = n(s)$$

$$\|\alpha''(s)\| = \sqrt{\tfrac{1}{2}\cos^2(s) + \operatorname{sen}^2(s) + \tfrac{1}{2}\cos^2(s)} = 1 = \kappa(s) \text{ curvature.}$$

$$b(s) = t(s) \times n(s) =$$

$$= \left[\det\begin{vmatrix} \cos(s) & -\frac{1}{\sqrt{2}}\operatorname{sen}(s) \\ -\operatorname{sen}(s) & -\frac{1}{\sqrt{2}}\cos(s) \end{vmatrix}, -\det\begin{vmatrix} -\frac{1}{\sqrt{2}}\operatorname{sen}(s) & -\frac{1}{\sqrt{2}}\operatorname{sen}(s) \\ -\frac{1}{\sqrt{2}}\cos(s) & -\frac{1}{\sqrt{2}}\cos(s) \end{vmatrix}, \det\begin{vmatrix} -\frac{1}{\sqrt{2}}\operatorname{sen}(s) & \cos(s) \\ -\frac{1}{\sqrt{2}}\cos(s) & -\operatorname{sen}(s) \end{vmatrix}\right] =$$

$$= \left[-\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right]$$

We know that $b'(s) = \tau(s) \cdot n(s)$. But $b'(s) = [0,0,0]$ and, since $n(s) \neq [0,0,0]$, necessarily $\tau(s) = 0$.

What curve is it

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Ex 1: Find a parametrization of a hyperbolic curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hint: Consider $\alpha(t) = (a\cosh(t), b\operatorname{senh}(t))$

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \operatorname{senh}(t) = \frac{e^t - e^{-t}}{2}$$

$$\left(a\,\frac{e^t+e^{-t}}{2},\; b\,\frac{e^t-e^{-t}}{2}\right) \rightarrow$$

$$\rightarrow \frac{\cancel{a^2} \cdot \frac{(e^t+e^{-t})^2}{2^2}}{\cancel{a^2}} - \frac{\cancel{b^2} \cdot \frac{(e^t-e^{-t})^2}{2^2}}{\cancel{b^2}} = \cosh^2(t) - \operatorname{senh}^2(t) = 1 \checkmark$$

Ex 2: Calculate the lenght of the catenary $\gamma(t) = (t, \cosh(t))$

$\gamma'(t) = (1, \text{senh}(t))$

$\|\gamma'(t)\| = \sqrt{1^2 + \text{senh}^2(t)} = \sqrt{\cosh^2(t)} = \cosh(t).$

$$\ell_\gamma(t) = \int_{t_1}^{t_2} \|\gamma'(t)\| \, dt = \int_{t_1}^{t_2} \cosh(t) \, dt = \text{senh}(t)\Big]_{t_1}^{t_2} = \text{senh}(t_2) - \text{senh}(t_1)$$

Ex 3: Find tangent and normal lines to the curve

$\gamma(t) = (2\cos t - \cos(2t),\ 2\operatorname{sen} t - \operatorname{sen}(2t),\ 0)$

// tangent vector

$\gamma'(t) = (-2\operatorname{sen} t + 2\operatorname{sen}(2t),\ 2\cos t - 2\cos(2t),\ 0)$

// equation of a tangent line to curve γ at time t_0

$L_1 = \{\gamma(t_0) + s\,\gamma'(t_0) : s \in \mathbb{R}\}$

// equation of a normal line to curve γ at time t_0

$\gamma''(t) = (-2\cos t + 4\cos(2t),\ -2\operatorname{sen} t - 4\cos(2t),\ 0)$

$L_2 = \{\gamma(t_0) + s \cdot \gamma''(t_0) : s \in \mathbb{R}\}$

EX 4: Calculate curvature and torsion of a curve

a) $\alpha(t) = \left(1/3\,(1+t)^{3/2},\ 1/3\,(1-t)^{3/2},\ t/\sqrt{2}\right)$

b) $\beta(t) = \left(4/3 \cos t,\ 1-\operatorname{sen} t,\ -3/5 \cos t\right)$

a) $\alpha'(t) = \left(1/2\,(1+t)^{1/2},\ -1/2\,(1-t)^{1/2},\ 1/\sqrt{2}\right)$

$\|\alpha'(t)\| = \sqrt{1/4\,(1+t) + 1/4\,(1-t) + 1/2} = 1 \Rightarrow$ curve with unit speed.

$\alpha''(t) = \left(1/4\,(1+t)^{-1/2},\ 1/4\,(1-t)^{-1/2},\ 0\right)$ curvature

$$\|\alpha''(t)\| = \sqrt{\frac{1}{16(1+t)} + \frac{1}{16(1-t)}} = 1/4\sqrt{\frac{1-t}{1-t^2} + \frac{1+t}{1-t^2}} = 1/4\sqrt{\frac{2}{1-t^2}} = \kappa(t)$$

$\tau(t) = b'(t)\cdot n(t)$ de las fórmulas de Frenet.

$$\bullet)\ n(t) = \frac{\alpha''(t)}{\|\alpha''(t)\|} = \left(\sqrt{\frac{1-t}{2}},\ \sqrt{\frac{1+t}{2}},\ 0\right)$$

$\bullet)\ b(t) = \alpha'(t) \times n(t) =$

$$= \left[\det\begin{vmatrix} -1/2\sqrt{1-t} & 1/\sqrt{2} \\ \sqrt{1+t}/\sqrt{2} & 0 \end{vmatrix},\ -\det\begin{vmatrix} 1/2\sqrt{1+t} & 1/\sqrt{2} \\ \sqrt{1-t}/\sqrt{2} & 0 \end{vmatrix},\ \det\begin{vmatrix} 1/2\sqrt{1+t} & -1/2\sqrt{1-t} \\ \sqrt{1-t}/\sqrt{2} & \sqrt{1+t}/\sqrt{2} \end{vmatrix}\right] =$$

$$= \left[-\frac{\sqrt{1+t}}{2},\ \frac{\sqrt{1-t}}{2},\ \frac{1}{2\sqrt{2}}(1+t) + \frac{1}{2\sqrt{2}}(1-t)\right] = \left[-\frac{\sqrt{1+t}}{2},\ \frac{\sqrt{1-t}}{2},\ \frac{1}{\sqrt{2}}\right]$$

$$\hookrightarrow b'(t) = \left[-\frac{1}{4\sqrt{1+t}},\ -\frac{1}{4\sqrt{1-t}},\ 0\right]$$

$$\bullet)\ \tau(t) = b'(t)\cdot n(s) = -\frac{\sqrt{1-t}}{8\sqrt{1+t}} - \frac{\sqrt{1+t}}{8\sqrt{1-t}} = -\frac{1}{8}\left(\frac{(1-t)+(1+t)}{\sqrt{1-t^2}}\right) =$$

$$= -1/8 \cdot \frac{2}{\sqrt{1-t^2}} = -\frac{1}{4\sqrt{1-t^2}} = \tau(t)$$

+

$\beta(t) = \left(\frac{4}{3}\cos t,\ 1 - \operatorname{sen} t,\ -\frac{3}{5}\cos t\right)$

b) $\beta'(t) = \left(-\frac{4}{3}\operatorname{sen} t,\ -\cos t,\ \frac{3}{5}\operatorname{sen} t\right)$

$\|\beta'(t)\| = \sqrt{\frac{16}{9}\operatorname{sen}^2 t + \cos^2 t + \frac{9}{25}\operatorname{sen}^2 t}$

$\beta''(t) = \left(-\frac{4}{3}\cos t,\ \operatorname{sen} t,\ \frac{3}{5}\cos t\right)$

$\|\beta''(t)\| = \sqrt{\frac{16}{9}\cos^2 t + \operatorname{sen}^2 t + \frac{9}{25}\cos^2 t} = \sqrt{\frac{481}{225}\cos^2 t + \operatorname{sen}^2 t} =$

$= \boxed{\sqrt{1 + \frac{256}{225}\cos^2 t} = \kappa(t)}$

$\tau(t) = b'(s) \cdot n(s)$

$n(s) = \frac{\beta''(t)}{\|\beta''(t)\|} =$