Las líneas de la geometría euclídea son curvas, mientras que los planos son superficies.

Points and vectors in R3

$\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$

$\mathbb{R}^3$ as vector space consists of vectors $[x,y,z]$, $x,y,z \in \mathbb{R}$

$P = (x,y,z)$
$\vec{OP} = [x,y,z]$ Notación

Sean 2 vectores de $\mathbb{R}^3$: $\vec{v} = [v_1, v_2, v_3]$ $\vec{w} = [w_1, w_2, w_3]$

¿Cómo se suman? Coordenada a coordenada. Gráficamente se hace lo del paralelogramo

Sea $\alpha \in \mathbb{R}$, $\alpha \cdot \vec{v} = [\alpha \cdot v_1, \alpha \cdot v_2, \alpha \cdot v_3]$

Def: Given two vectors $\vec{v} = [v_1, v_2, v_3]$, $\vec{w} = [w_1, w_2, w_3]$. We define the scalar product $\langle \vec{v}, \vec{w} \rangle$ of vectors $\vec{v}$ and $\vec{w}$ by $(\vec{v} \cdot \vec{w})$.

$$\langle \vec{v}, \vec{w} \rangle \overset{def}{=} v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3 \in \mathbb{R}$$

The norm (or length) of a vector $\vec{v}$ is defined by $\|\vec{v}\| \overset{def}{=} \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_1^2 + v_2^2 + v_3^3}$

The angle φ between vectors $\vec{v}$ and $\vec{w}$ is defined by $\cos(\varphi) \overset{def}{=} \dfrac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|}$

$\vec{v}$ and $\vec{w}$ are perpendicular if $\langle \vec{v}, \vec{w} \rangle = 0$. Notación: $\vec{v} \perp \vec{w}$

Thm (Properties of scalar product) Given vectors $\vec{v}, \vec{w}, \vec{u} \in \mathbb{R}^3$. Then:

1) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

2) $\langle \vec{v} + \vec{w}, \vec{u} \rangle = \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle$

3) $\forall \alpha \in \mathbb{R}$ $\langle \alpha \cdot \vec{v}, \vec{w} \rangle = \alpha \cdot \langle \vec{v}, \vec{w} \rangle$

4) $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \iff \vec{v} = [0,0,0]$

5) $\forall \alpha \in \mathbb{R} \quad \|\alpha \cdot \vec{v}\| = \alpha \cdot \|\vec{v}\|$

6) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

7) If $\vec{v} \perp \vec{w}$,

$\Longrightarrow \vec{v} + \vec{z} = \vec{w} \Longrightarrow \vec{z} = \vec{w} - \vec{v}$

Thm. of Pitagoras: $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|w - v\|^2$

Proof: $\|\vec{w} - \vec{v}\|^2 = \langle \vec{w} - \vec{v}, \vec{w} - \vec{v} \rangle = \langle \vec{w}, \vec{w} - \vec{v} \rangle + \langle -\vec{v}, \vec{w} - \vec{v} \rangle$

$= \langle \vec{w}, \vec{w} \rangle - \langle \vec{w}, \vec{v} \rangle + \langle -\vec{v}, \vec{w} \rangle + \langle -\vec{v}, -\vec{v} \rangle = \|\vec{w}\|^2 - 2 \cdot \cancelto{0}{\langle \vec{v}, \vec{w} \rangle} + \|\vec{v}\|^2$

Vector product of v and w

$\vec{v} = [v_1, v_2, v_3] \quad \vec{w} = [w_1, w_2, w_3]$

$$\vec{v} \times \vec{w} = \left[\det \begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix}, -\det \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix}, \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right] \longleftarrow \text{vector!}$$

Thm (Properties of vector product) given $\vec{v}, \vec{u}, \vec{w} \in \mathbb{R}^3$, then

1) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ Antisimétrico

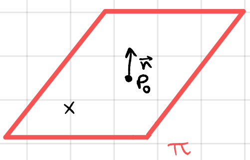
2) $\vec{v} \perp \vec{v} \times \vec{w} \perp \vec{w}$

3) $\forall \alpha \in \mathbb{R} \quad (\alpha \cdot \vec{v}) \times \vec{w} = \alpha \cdot (\vec{v} \times \vec{w}) = \vec{v} \times (\alpha \cdot \vec{w})$

4) $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

5) $\|\vec{v} \times \vec{w}\|$ = area of parallelogram built on $\vec{v}$ and $\vec{w}$

Equation of a plane in R3



$\vec{n} = [A, B, C]$

$P_0 = (p_1, p_2, p_3)$

$X = (x, y, z)$

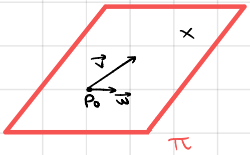
$\overrightarrow{P_0X} = [x - p_1, y - p_2, z - p_3]$

$X \in \pi \iff \overrightarrow{P_0X} \perp \vec{n} \iff \langle [x - p_1, y - p_2, z - p_3], [A, B, C] \rangle = 0$

$\iff A(x - p_1) + B(y - p_2) + C(z - p_3) = 0 \iff Ax + By + Cz \underbrace{- (Ap_1 + Bp_2 + Cp_3)}_{+D} = 0$

Equation of a plane passing by P_0 and perpendicular to $\vec{n}$

$$\pi = \{(x,y,z) \in \mathbb{R}^3 : Ax + By + Cz + D = 0\}$$



$$x \in \pi \iff \exists \alpha,\beta \in \mathbb{R} / \overrightarrow{P_0X} = \alpha \cdot \vec{v} + \beta \cdot \vec{w} \iff \exists_{\alpha,\beta\in\mathbb{R}} X = P_0 + \overrightarrow{P_0X} \iff$$

$$\iff \exists_{\alpha,\beta\in\mathbb{R}} X = P_0 + \alpha\vec{v} + \beta\vec{w}$$

$$\iff \begin{cases} x = p_1 + \alpha \cdot v_1 + \beta \cdot w_1 \\ y = p_2 + \alpha \cdot v_2 + \beta \cdot w_2 \\ z = p_3 + \alpha \cdot v_3 + \beta \cdot w_3 \end{cases}$$

Parametric description of the plane

Equation of a line in R3

1) $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$

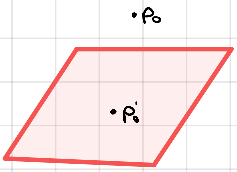
$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$

Assume that $\pi_1 \not\parallel \pi_2$, then π_1 and π_2 intersect in a line.

Line $$\ell = \pi_1 \cap \pi_2 = \left\{(x,y,z) \in \mathbb{R}^3 : \begin{matrix} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{matrix}\right\}$$

Parametric description of a line: $$\left\{(x,y,z) \in \mathbb{R}^3 : \begin{matrix} x = a_1 + \alpha \cdot v_1 \\ y = a_2 + \alpha \cdot v_2 \\ z = a_3 + \alpha \cdot v_3 \end{matrix}, \alpha \in \mathbb{R}\right\}$$

Distance between point $P_0 = (p_1, p_2, p_3)$ and plane $\pi : Ax + By + Cz + D = 0$



$$d(P_0, \pi) = d(P_0, P_0') = \frac{|A \cdot p_1 + B \cdot p_2 + C \cdot p_3 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Ejercicios

Ex 1. $A = (1,0,2)$, $B = (1,2,3)$, $C = (-1,1,5)$

Write the equation of plane π such that $A, B, C \in \pi$

$$\underbrace{\overrightarrow{AB} = [0,2,1]}_{= \vec{u}} \qquad \underbrace{\overrightarrow{AC} = [-2,1,3]}_{= \vec{v}}$$

$$\vec{u} \times \vec{v} = [5,-2,-4] = \vec{w} \perp \pi \Rightarrow \pi : 5x - 2y - 4z + D = 0$$

$$5 \cdot 1 - 2 \cdot 0 - 4 \cdot 2 + D = 0$$

$$D = 3$$

Ex 2. Given a plane $\pi: 2x+4y+7z=0$. Find

a) plane π' such that $\pi_1 \parallel \pi$ and $(2,1,5) \in \pi$

b) line ℓ such that $\ell \perp \pi$ and $(3,1,7) \in \ell$

a) $2x+4y+7z+D=0$

$2\cdot 2+4\cdot 1+7\cdot 5+D=0 \Rightarrow D=-43$

b) $\vec{v}=[2,4,7] \quad C(3,1,7) \in \ell$

$$\ell=\{C+\alpha\cdot\vec{v} : \alpha\in\mathbb{R}\}=\left\{(x,y,z)\in\mathbb{R}^3 : \begin{array}{l} 3+\alpha\cdot 2=x \\ 1+\alpha\cdot 4=y \\ 7+\alpha\cdot 7=z \end{array}\right\}$$

c) Find coordinates of the point $\ell \cap \pi$. Lo he hecho yo

Ex 3. $\pi: 2x+3y+5z+7=0$

$A(1,1,2)$

$$\ell\perp\pi \Rightarrow \ell=\left\{(x,y,z)\in\mathbb{R}^3 : \begin{array}{l} x=1+2\cdot\alpha \\ y=1+3\cdot\alpha \\ z=2+5\cdot\alpha \end{array}\right\}$$

$$2(1+2\alpha)+3(1+3\alpha)+5(2+5\alpha)+7=0$$

$$4\alpha+9\alpha+25\alpha+2+3+10+7=0$$

$$38\alpha=-22 \Rightarrow \alpha=\frac{-22}{38}$$

Y por último, el punto dado por α es el pto. medio

$$\left(-\frac{3}{19}, -\frac{14}{19}, -\frac{17}{19}\right)$$

$$\left(\frac{1+x'}{2}, \frac{1+y'}{2}, \frac{2+z'}{2}\right)=\left(-\frac{3}{19}, -\frac{14}{19}, -\frac{17}{19}\right)$$

$$x'=-\frac{25}{19} \qquad y'=-\frac{47}{19} \qquad z'=-\frac{72}{19}$$

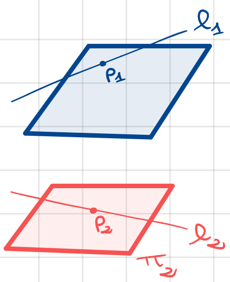
Ex 4. Given 2 skew lines, calculate the distance between them.

$\ell_1 = \{P_1 + \alpha \cdot \vec{v_1} : \alpha \in \mathbb{R}\}$ $\quad \ell_2 = \{P_2 + \beta \cdot \vec{v_2} : \beta \in \mathbb{R}\}$

distance $(\ell_1, \ell_2) = ?$

We consider plane $\pi_1 = \{P_1 + \alpha \cdot \vec{v_1} + \beta \cdot \vec{v_2} : \alpha, \beta \in \mathbb{R}\}$

$\pi_2 = \{P_2 + \alpha \cdot \vec{v_1} + \beta \cdot v_2 : \alpha, \beta \in \mathbb{R}\}$



distance $(\ell_1, \ell_2) = \text{dist}(\pi_1, \pi_2) = \text{dist}(P_1, \pi_2)$

$\ell_1 \subset \pi_1 \quad \ell_2 \subset \pi_2 \quad \pi_1 \parallel \pi_2$

Homework

Given 2 lines $\ell_1 = \{(1,2,3) + t \cdot [1,7,2] : t \in \mathbb{R}\}$

$\ell_2 = \{(2,0,7) + s \cdot [2,0,9] : s \in \mathbb{R}\}$

Find distance between ℓ_1 and ℓ_2.

If we consider what we have studied in class, the distance between these two skew lines will be the distance between

$P_1 = (1,2,3)$ and $\pi_2 = \{(2,0,7) + s \cdot [2,0,9] + t \cdot [1,7,2] : s,t \in \mathbb{R}\}$

Let's calculate the equation of π_2:

$$\left[\det\begin{pmatrix} 0 & 9 \\ 7 & 2 \end{pmatrix}, \det\begin{pmatrix} 2 & 9 \\ 1 & 2 \end{pmatrix}, \det\begin{pmatrix} 2 & 0 \\ 1 & 7 \end{pmatrix} \right] = [\underset{\substack{\shortparallel \\ A}}{-63}, \underset{\substack{\shortparallel \\ B}}{5}, \underset{\substack{\shortparallel \\ C}}{14}]$$

$\pi_2 : -63x + 5y + 14z + D = 0$

Given that $(2,0,7) \in \pi_2$, we calculate D:

$$-63 \cdot 2 + 5 \cdot 0 + 14 \cdot 7 = -28 \Rightarrow D = 28$$

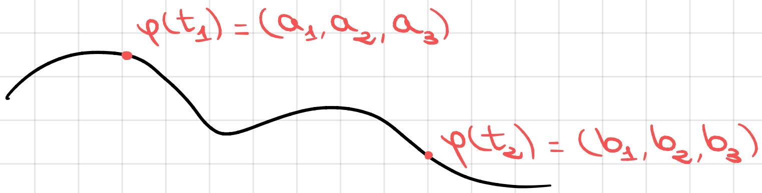
And now we can use the formula $d(P_0,\pi) = d(P_0,P_0') = \frac{|A\cdot P_1 + B\cdot P_2 + C\cdot P_3|}{\sqrt{A^2+B^2+C^2}}$:

$$d(\ell_1,\ell_2) = d(P_1,\pi_2) = \frac{|-63\cdot 1 + 5\cdot 2 + 14\cdot 3|}{\sqrt{(-63)^2+5^2+14^2}} = \frac{11}{64.73} \approx 0.17u$$

Curves

From a physical point of view:

Curve = trajectory of a moving point



Def.: A continuous application $c:(a,b) \longrightarrow \mathbb{R}^3$ is called a curve.

Remark: In algebraic geometry, a curve is a set of points. In differential geometry, a curve is a transformation.

Def.: Given two curves $c_1: I_1 \longrightarrow \mathbb{R}^3$, $c_2: I_2 \longrightarrow \mathbb{R}^3$. These two are equivalent if $c_1 = c_2 \circ f$, where $f: I_1 \longrightarrow I_2$ is a homeomorphism.

Remark: If two curves $c_i: I_i \longrightarrow \mathbb{R}^3$, $i=1,2$ are equivalent, then $c_1(I_1) = c_2(I_2)$

Def.: A curve $c:(a,b) \longrightarrow \mathbb{R}^3$, where $\overset{\forall t\in(a,b)}{c(t)} = (c_1(t), c_2(t), c_3(t))$ is called regular if apps $c_i:(a,b) \longrightarrow \mathbb{R}$ $i=1,2,3$ is differentiable and

$$\forall t\in(a,b) \quad c'(t) = (c_1'(t), c_2'(t), c_3'(t)) \neq [0,\dots,0]$$

$c'(t)$ is called tangent vector to curve c at point $c(t)$ (at time t).

We say that a regular curve $c:(a,b) \longrightarrow \mathbb{R}^3$ has a unit speed if $\forall t \in (a,b)$ $\|c'(t)\| = 1$

Examples of regular curves

Assume $[v_1, v_2, v_3] \neq [0,0,0]$

1) $\forall t \in \mathbb{R}$ $c(t) = (p_1 + t \cdot v_1, p_2 + t \cdot v_2, p_3 + t \cdot v_3) = (p_1, p_2, p_3) + t[v_1, v_2, v_3]$

$c'(t) = [v_1, v_2, v_3] \neq [0,0,0]$ So c is a line in $\mathbb{R}^3$

So c is a regular curve

2) $\tilde{c}: (0, 2\pi) \longrightarrow \mathbb{R}^3$

$\forall t \in (0, 2\pi)$ $\tilde{c}(t) = (x_0 + r\cos t, y_0 + r\,\text{sen}\,t, 0), r > 0$

$\tilde{c}'(t) = (-r\,\text{sen}\,t, r\cos t, 0)$

$\|\tilde{c}'(t)\| = \sqrt{r^2 \text{sen}^2 t + r^2 \cos^2 t + 0^2} = r \neq 0$

$\tilde{c}$ is a parametrization of a circle of radius r and center $(x_0, y_0, 0)$

3) $\hat{c}: (a,b) \longrightarrow \mathbb{R}^3$

$\forall t \in (a,b)$ $\hat{c}(t) = (r\cos t, r\,\text{sen}\,t, 2t)$

These are differentiable

$\hat{c}'(t) = (-r\,\text{sen}\,t, r\cos t, 2)$

$\|\hat{c}'(t)\| = \sqrt{r^2 \text{sen}^2 t + r^2 \cos^2 t + 4}$

Def.: A length of a regular curve $c: [a,b] \longrightarrow \mathbb{R}^3$ is a number, $\mathcal{L}(c) = \int_a^b \|c'(t)\| \, dt$

Thm.: Regular equivalent curves have the same length

Proof: $c_1 = c_2 \circ f$, $\left.\begin{array}{l} c_1: [a_1, b_1] \longrightarrow \mathbb{R}^3 \\ c_2: [a_2, b_2] \longrightarrow \mathbb{R}^3 \end{array}\right\}$ regular curves and $f:[a_1,b_1]\longrightarrow[a_2,b_2]$ is a diffeomorphism.

Then, since $c_1=c_2\circ f$ we get $\|c_1'(t)\|=\|c_2'(f(t))\cdot f'(t)\|$
$=\|c_2'(f(t))\|\cdot\|f'(t)\|$

$$\text{So } \mathcal{L}(c_1)=\int_{a_1}^{b_1}\|c_1'(t)\|\,dt=\int_{a_1}^{b_1}\|c_2'(f(t))\|\cdot\|f'(t)\|\,dt\overset{u=f(t)}{=}$$

$$=\int_{a_2}^{b_2}\|c_2'(u)\|\,du=\mathcal{L}(c_2)$$

Thm. For any regular curve c there exists a curve $\hat{c}$ with unit speed such that $\hat{c}$ is equivalent to c.

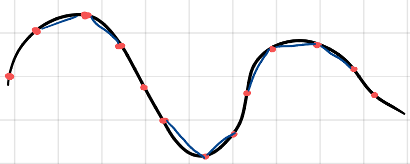
PROOF. Let $c:(a,b)\longrightarrow\mathbb{R}^3$ be a regular curve. Let $s(t)=\int_a^t\|c'(u)\|\,du$
so $s:(a,b)\to\mathbb{R}$ is differentiable and $s'(t)=\|c'(t)\|>0$,
so s is increasing function and 1-1 function.

Moreover $s:[a,b]\xrightarrow[\text{inyective}]{\text{surjective}}[0,\mathcal{L}(c)]$ and 1-1 function,
so there exists (biyective) inverse function $\varphi:[0,\mathcal{L}(c)]\to[a,b]$
such that

1) $c\circ\varphi$ is equivalent to c.
2) $\|(c\circ\varphi)'(u)\|=\|c'(\varphi(u))\|\cdot\|\varphi'(u)\|=\dfrac{1}{\|c'(\varphi(u))\|}\cdot\|c'(\varphi(u))\|=1$

Therefore, for a given regular curve c, there exists regular curve $\hat{c}=c\circ\varphi$ such that $\hat{c}$ has unit speed.

Another approach to length of curves



Given a continuous curve $c:[a,b]\longrightarrow\mathbb{R}^3$. A partition $\mathcal{P}$ of $[a,b]$ is sequence (t_n) $a=t_0<t_1<t_2<\cdots<t_n=b$

$$\sup\left\{\sum_{i=0}^{n-1}d(c(t_i),c(t_{i+1})):(t_0,\dots,t_n)\text{ is a partition of }[a,b]\right\}$$

Def.: We say that a continuous curve $c:[a,b]\longrightarrow\mathbb{R}^3$ is rectifyable (has a finite length) if $\tilde{\mathcal{L}}(c)<\infty$

Thm.: If $c:(a,b)\longrightarrow \mathbb{R}^3$ is C-diff. curve, then $\mathcal{L}(c) = \tilde{\mathcal{L}}(c)$

Ejercicios

1. $\gamma(t) = (e^{k\cdot t}\cos t, e^{k\cdot t}\operatorname{sen} t, 0)$ spiral

Calculate the length of the spiral

$$\gamma'(t) = (ke^{kt}\cos t - e^{kt}\operatorname{sen} t, ke^{kt}\operatorname{sen} t + e^{kt}\cos t, 0)$$

$$\|\gamma'(t)\| = \sqrt{(e^{kt})^2(k\cos t - \operatorname{sen} t)^2 + (e^{kt})^2(k\operatorname{sen} t + \cos t)^2}$$

$$= e^{kt}\sqrt{k^2\cos^2 t + \operatorname{sen}^2 t - 2k\cos t\operatorname{sen} t + k^2\operatorname{sen}^2 t + \cos^2 t + 2k\operatorname{sen} t\cos t}$$

$$= e^{kt}\sqrt{k^2+1} \neq 0$$

$$\mathcal{L}(\gamma) = \int_0^{t_0}\|\gamma'(t)\|\,dt = \int_0^{t_0} e^{kt}\sqrt{k^2+1}\,dt = \frac{1}{k}\sqrt{k^2+1}\cdot e^{kt}\Big]_0^{t_0} =$$

$$\frac{1}{k}\sqrt{k^2+1}\cdot e^{kt_0} - \frac{1}{k}\sqrt{k^2+1} = \frac{1}{k}\sqrt{k^2+1}\,(e^{kt_0}-1)$$

2. $\gamma(t) = (a(t-\operatorname{sen} t), a(1-\cos t))$ cycloid $a>0$, $t\in[0,2\pi]$

Calculate the length

$$\gamma'(t) = (a(1-\cos t), a\operatorname{sen} t)$$

$$\|\gamma'(t)\| = \sqrt{a^2(1-\cos t)^2 + a^2\operatorname{sen}^2 t} = a\sqrt{1+\cos^2 t - 2\cos t + \operatorname{sen}^2 t} =$$

$$= a\sqrt{2-2\cos t} = 2a\sqrt{\operatorname{sen}^2(t/2)} \underset{\uparrow}{=} 2a\operatorname{sen}(t/2)$$

$\frac{t}{2}\in[0,\pi]$, sen is positive

$$\mathcal{L}(\gamma) = \int_0^{2\pi} 2a\operatorname{sen}(t/2)\,dt = 2a\int_0^{2\pi}\operatorname{sen}\left(\frac{t}{2}\right)dt = -4a\cos(t/2)\Big]_0^{2\pi} =$$

$$= -4a(\cos(\pi) - \cos(0)) = (-4a)\cdot(-2) = 8a$$

3. Characterize all regular curves

$$c : (a,b) \longrightarrow \mathbb{R}^3$$

such that $\forall t \in (a,b) \quad c''(t) = [0,0,0]$

$c(t) = (c_1(t), c_2(t), c_3(t))$

$c_1 : (a,b) \longrightarrow \mathbb{R} \Rightarrow c_1''(t) = 0$

$c_2 : (a,b) \longrightarrow \mathbb{R} \Rightarrow c_2''(t) = 0$

$c_3 : (a,b) \longrightarrow \mathbb{R} \Rightarrow c_3''(t) = 0$

$c_1''(t) = 0 \Rightarrow c_1'(t) = \text{const.} = p_1 \Rightarrow c_1(t) = t \cdot p_1 + q_1$

Same with c_2 and c_3. As a result:

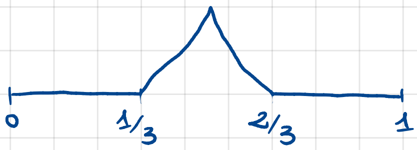
$c(t) = (q_1 + tp_1, q_2 + tp_2, q_3 + tp_3) = (q_1, q_2, q_3) + t[p_1, p_2, p_3]$

c is a parametrization of a segment in $\mathbb{R}^3$.

Homework

Proof that the segment that connects a and b is the shortest curve that unites them.

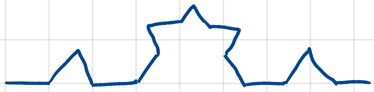
Lemma. The Koch curve has ∞ length

1) 

$\mathcal{L}_m$ = length of Koch's curve of level m

d_m = length of each segment of the broken line of level m

n_m = number of segments in broken line at level m

2) 

$d_1 = 1/3 \qquad n_1 = 4 \qquad \mathcal{L}_1 = n_1 \cdot d_1$

$d_2 = (1/3)^2 \qquad n_2 = 4 \cdot n_1$

Assume that we know $d_m, n_m, \mathcal{L}_m$:

$$d_{m+1} = \frac{d_m}{3} = \left(\frac{1}{3}\right)^{m+1}$$

$$n_{m+1} = 4 \cdot n_m = 4^{m+1}$$

$$\mathcal{L}_{m+1} = d_{m+1} \cdot n_{m+1} = \left(\frac{4}{3}\right)^{m+1}$$

$$\lim_{m \to \infty} \mathcal{L}_m = \lim_{m \to \infty} \left(\frac{4}{3}\right)^m = \infty$$

Def.: Warsawsin function is a curve definite

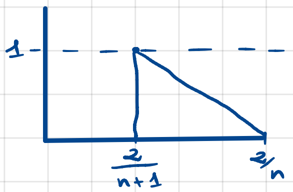
$$\gamma : (0,1) \longrightarrow \mathbb{R}^2 \qquad \forall_{t \in (0,1)} \ \gamma(t) = \left(t, \operatorname{sen} \frac{\pi}{t}\right)$$

$$\gamma_n : \left[\frac{2}{n+1}, \frac{2}{n}\right] \longrightarrow \mathbb{R}^2 \qquad \mathcal{L}(\gamma) = \sum_{n=1}^{\infty} \ell(\gamma_n)$$

Assume that n is even number
$n+1$ is odd number

$$\gamma_n\left(\frac{2}{n+1}\right) = \left(\frac{2}{n+1}, \operatorname{sen}\left(\frac{\pi}{2}\right)n\right) = \left(\frac{2}{n+1}, \pm 1\right)$$

$$\gamma_n\left(\frac{2}{n}\right) = \left(\frac{2}{n}, \operatorname{sen}\frac{\pi}{2}n\right) = \left(\frac{2}{n}, 0\right)$$



$\Rightarrow \ell(\gamma_n) \geq 1$

Local theory of regular curves with unit speed

Let $\alpha : (a,b) \longrightarrow \mathbb{R}^3$ be parath. curve with unit speed, i.e.

$$\forall_{t\in(a,b)} \ \|\alpha'(t)\| = 1$$

The vector $\alpha'(t)$ is a tangent vector to α at time t. the line $\{\alpha(t) + s\cdot\alpha'(t) : s \in \mathbb{R}\}$ is a tangent line to α at time t (or at point $\alpha(t)$)

$\alpha''(t)$ measures how rapidly the curve pulls away from the tangent line.

Def.: Let $\alpha : (a,b) \longrightarrow \mathbb{R}^3$ be a regular curve with unit speed. The number $\|\alpha''(t)\| \overset{def}{=} k(t)$ and it is called the curvature of α at time t.

Examples

1) $\alpha : (a,b) \longrightarrow \mathbb{R}^3$

$$\alpha(t) = (p_1, p_2, p_3) + t\cdot[v_1, v_2, v_3]$$

$\|\alpha''(t)\| = 0 \Rightarrow$ At any point, the curvature is equal to zero

$\beta : (a,b) \longrightarrow \mathbb{R}^3$ with $k(t) = 0$ $\forall t \in (a,b)$.

Then, $\beta''(t) = [0,0,0]$ $\forall t \in (a,b)$

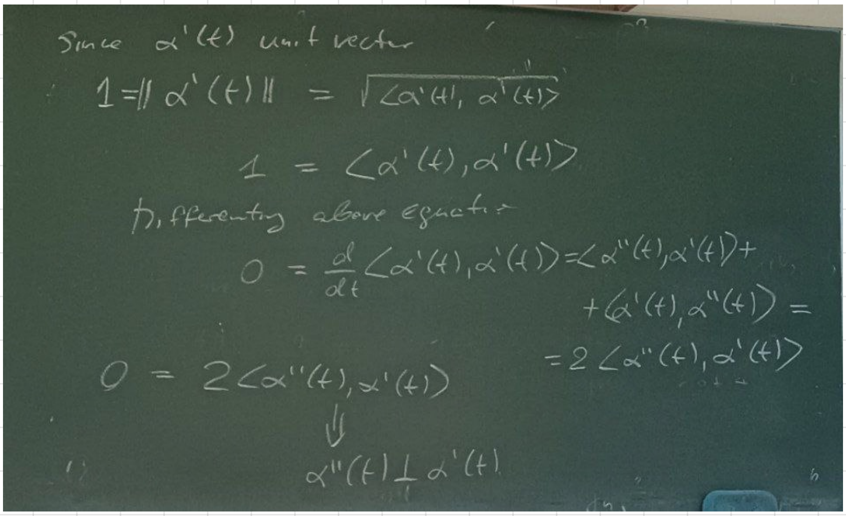
Remarks

1) We consider a curve $\alpha : (a,b) \longrightarrow \mathbb{R}^3$ regular with unit speed. At any point of this curve, where $k(t) \neq 0$ a unit vector $n(t)$ in direction of $\alpha''(t)$ is well defined by the equation:

$$\alpha''(t) = k(t) \cdot n(t)$$

2) We claim that for unit speed vector $\alpha'(t)$ we set that

$\alpha''(t) \perp \alpha'(t)$.



Def.: The vector $n(t)$ that is normal to $\alpha'(t)$ is called the normal vector at t to the curve α. Moreover, $n(t) = \dfrac{\alpha''(t)}{\|\alpha''(t)\|}$

Def.: The plane determined by $\alpha'(t)$ and vector $n(t)$ is called an oscilating plane. Notice that at any point $\alpha(s)$ such that $k(s) = 0$. The oscilating plane is not defined

Def.: The unit vector $t(s) = \alpha'(s)$ is called a tangent vector to α at time s. We defined binormal vector to α at time s by $b(s) \overset{def}{=} t(s) \times n(s)$. Therefore, $t(s) \perp b(s) \perp n(s)$

Remark. If $\|b(s)\| = 1 \Longrightarrow \langle b(s), b(s) \rangle = 1 \Longrightarrow b'(s) \perp b(s)$

Remark: $b'(s) = [b(s)\times n(s)]' = \cancel{t'(s)\times n(s)}^{0} + t(s)\times n'(s) =$

$= t(s)\times n'(s)$

Because $b'(s)\perp t(s)$

$b'(s) \parallel n(s)$

Def.: Given a parath. curve with unit speed and $\forall_{t\in(a,b)}\ \alpha''(t)\neq 0$

The number $\tau(s)$, given by formula $b'(s) = \tau(s)\cdot n(s)$, is called a torsion of curve α at time s.

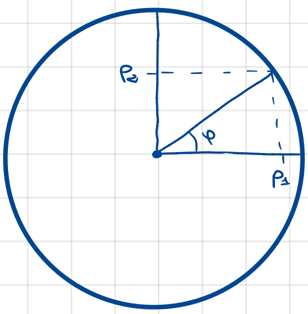
Remark. If $\alpha:(a,b)\longrightarrow\mathbb{R}^3$ is a curve like above, then the unit vector $t(s), b(s), n(s)$ are well defined and they are paircase perpendicular

Def.: $\alpha:(a,b)\longrightarrow\mathbb{R}^3$ is a plane curve if there exists a plane π such that $\alpha(a,b)\subset\pi$

Thm. A curve given by parametrization with unit speed and $\alpha''(t)\neq 0$ is a plane curve $\Longleftrightarrow$ the binormal vector $b(s)$ is constant

Ejercicios

1. Find a parametrization with unit speed of a circle $S = \{(x,y)\in\mathbb{R}^2 : (x-x_0)^2+(y-y_0)^2 = r^2\}$ of radius r and center $*$. Calculate tangent, normal, binomial vector and curvature with torsion.



$\frac{P_1}{r} = \cos(\varphi)$

$\frac{P_2}{r} = \operatorname{sen}(\varphi)$

$\varphi \to (r\cdot\cos(\varphi), r\cdot\operatorname{sen}(\varphi))$

$\forall_{\varphi\in[0,2\pi]}\ \alpha(\varphi) = [r\cos(\varphi), r\operatorname{sen}(\varphi)]$

$$\underset{\varphi\in[0,2\pi]}{\forall}\ \beta(\varphi) = (x_0 + r\cos(\varphi), y_0 + r\,\text{sen}(\varphi))$$

$$\beta'(\varphi) = (-r\,\text{sen}(\varphi), r\cos(\varphi))$$

$$\|\beta'(\varphi)\| = \sqrt{r^2\cdot\cos^2\varphi + r^2\text{sen}^2\varphi} = r$$

Consider parametrization

$$\gamma(\varphi) = \left(x_0 + r\cos\left(\frac{\varphi}{r}\right), y_0 + r\,\text{sen}\left(\frac{\varphi}{r}\right)\right)$$

$$\gamma'(\varphi) = \left[-\text{sen}\left(\frac{\varphi}{r}\right), \cos\left(\frac{\varphi}{r}\right)\right]$$

$\|\gamma'(\varphi)\| = 1$ $\quad\varphi$ - parametrization with unit speed

Para estar en $\mathbb{R}^3$, añadimos 0 como tercera coordenada.

$$\gamma''(\varphi) = \left(-\cos\left(\frac{\varphi}{r}\right)\cdot\frac{1}{r}, -\text{sen}\left(\frac{\varphi}{r}\right)\cdot\frac{1}{r}, 0\right)$$

$\|\gamma''(\varphi)\| = \frac{1}{r} = k(\varphi)$ curvature of a circle at any point

The normal vector $n(\varphi) = \dfrac{\gamma''(\varphi)}{\|\gamma''(\varphi)\|} = \left(-\cos\left(\frac{\varphi}{r}\right), -\text{sen}\left(\frac{\varphi}{r}\right), 0\right)$

$$b'(s) = \tau(s)\cdot n(s)$$

$$\langle b'(s), n(s)\rangle = \langle \tau(s)\cdot n(s), n(s)\rangle = \tau(s)\cdot\underbrace{\langle n(s), n(s)\rangle}_{=1} = \tau(s)$$

Tangent vector $t(\varphi) = \left[-\text{sen}\left(\frac{\varphi}{r}\right), \cos\left(\frac{\varphi}{r}\right), 0\right]$

Normal vector $n(\varphi) = \left[-\cos\left(\frac{\varphi}{r}\right), -\text{sen}\left(\frac{\varphi}{r}\right), 0\right]$

Binormal vector $b(\varphi) = t(\varphi)\times n(\varphi) = [0,0,1]$

$$b'(\varphi) = [0,0,0]$$

So $\tau(\varphi) = \langle b'(\varphi), n(\varphi)\rangle = [0,0,0]$

2. Find a parametrization with unit speed of a helix

$\alpha(t) = (r \cdot \cos t, r \operatorname{sen} t, a \cdot t)$ a - fixed number

$\alpha'(t) = (-r \operatorname{sen} t, r \cos t, a)$

$\|\alpha'(t)\| = \sqrt{r^2 + a^2} \neq 1 \Longrightarrow$ No unit speed

$$\beta(t) = \left(r \cdot \cos\left(\frac{t}{\sqrt{r^2+a^2}}\right), r \cdot \operatorname{sen}\left(\frac{t}{\sqrt{r^2+a^2}}\right), \frac{a \cdot t}{\sqrt{r^2+a^2}}\right)$$

$$\underset{\substack{\uparrow \\ \text{TANGENT}}}{\beta'(t)} = \left(-\frac{r}{\sqrt{r^2+a^2}} \operatorname{sen} \frac{t}{\sqrt{r^2+a^2}}, \frac{r}{\sqrt{r^2+a^2}} \cos \frac{t}{\sqrt{r^2+a^2}}, \frac{a}{\sqrt{r^2+a^2}}\right)$$

$\|\beta'(t)\| = 1$

$$\beta''(t) = \left(-\frac{r}{r^2+a^2} \cos \frac{t}{\sqrt{r^2+a^2}}, -\frac{r}{r^2+a^2} \operatorname{sen} \frac{t}{\sqrt{r^2+a^2}}, 0\right)$$

$\|\beta''(t)\| = \frac{r}{r^2+a^2} = k(t)$ curvature of a helix in param $\beta(t)$

$$n(t) = \frac{\beta''(t)}{\|\beta''(t)\|} = \left[-\cos \frac{t}{\sqrt{r^2+a^2}}, -\operatorname{sen} \frac{t}{\sqrt{r^2+a^2}}, 0\right]$$

$$b(t) = \beta(t) \times n(t) = \left[\frac{a}{\sqrt{r^2+a^2}} \operatorname{sen}\left(\frac{t}{\sqrt{r^2+a^2}}\right), -\frac{a}{\sqrt{r^2+a^2}} \cos\left(\frac{t}{\sqrt{r^2+a^2}}\right), \frac{r}{\sqrt{r^2+a^2}}\right]$$

$$b'(t) = \left[\frac{a}{r^2+a^2} \cos\left(\frac{t}{\sqrt{r^2+a^2}}\right), \frac{a}{r^2+a^2} \operatorname{sen}\left(\frac{t}{\sqrt{r^2+a^2}}\right), 0\right]$$

$$\tau(t) = \langle b'(t), n(t) \rangle = -\frac{a}{r^2+a^2}$$

3. Given a parametrization

$$\alpha(s) = \left(\frac{1}{\sqrt{2}}\cos(s), \operatorname{sen}(s), \frac{1}{\sqrt{2}}\cos(s)\right)$$

Calculate curvature and torsion and identify the curve

$$\alpha'(s) = \left(-\frac{1}{\sqrt{2}}\operatorname{sen}(s), \cos(s), -\frac{1}{\sqrt{2}}\operatorname{sen}(s)\right)$$

$$\|\alpha'(s)\| = 1$$

$$\alpha''(s) = \left(-\frac{1}{\sqrt{2}}\cos(s), -\operatorname{sen}(s), -\frac{1}{\sqrt{2}}\cos(s)\right)$$

$$\|\alpha''(s)\| = 1 = k(s) \quad \text{curvature}$$

$$b(s) = t(s) \times n(s) =$$

$$\beta(t) = \left(\frac{4}{5}\cos t, 1 - \operatorname{sen} t, -\frac{3}{5}\cos t\right)$$

$$\beta'(t) = \left(-\frac{4}{5}\operatorname{sen} t, -\cos t, \frac{3}{5}\operatorname{sen} t\right) = t(t)$$

$$\|\beta'(t)\| = \sqrt{\frac{16}{25}\operatorname{sen}^2 t + \cos^2 t + \frac{9}{25}\operatorname{sen}^2 t} = \sqrt{\operatorname{sen}^2 t + \cos^2 t} = 1$$

It's a curve with unit speed

$$\beta''(t) = \left(-\frac{4}{5}\cos t, \operatorname{sen} t, \frac{3}{5}\cos t\right)$$

$$\|\beta''(t)\| = \sqrt{\frac{16}{25}\cos^2 t + \operatorname{sen}^2 t + \frac{9}{25}\cos^2 t} = 1 = k(t)$$

Curvature at any point

$$n(t) = \frac{\beta''(t)}{\|\beta''(t)\|} = \left(-\frac{4}{5}\cos t, \operatorname{sen} t, \frac{3}{5}\cos t\right)$$

$$b(t) = t(t) \times n(t) = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4/5\operatorname{sen} t & -\cos t & 3/5\operatorname{sen} t \\ -4/5\cos t & \operatorname{sen} t & 3/5\cos t \end{pmatrix} =$$

$$= \left(\frac{3}{5}\cos^2 t + \frac{3}{5}\operatorname{sen}^2 t\right)\vec{i} + \left(-\frac{12}{25}\operatorname{sen} t\cos t + \frac{12}{25}\operatorname{sen} t\cos t\right)\vec{j} + \left(-\frac{4}{5}\operatorname{sen}^2 t - \frac{4}{5}\cos^2 t\right)\vec{k}$$

$$= \frac{3}{5}\vec{i} - \frac{4}{5}\vec{k} = \left[\frac{3}{5}, 0, -4/5\right]$$

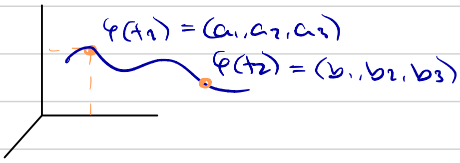
$$b'(t) = [0,0,0]$$

$$\tau(t) = b'(t)\cdot n(t) = 0 \Rightarrow \text{Torsion}$$

CURVES

From a physical point of view,

curve = trajectory of a moving point



DEF: A continuous map $c: (a,b) \to \mathbb{R}^3$ is called a curve.

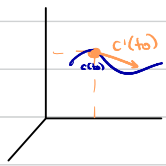
Remark: In algebraic geometry, a curve is a set of points. In differential geometry, a curve is a transformation

DEF: Assume that $c_1: I_1 \to \mathbb{R}^3$, $c_2: I_2 \to \mathbb{R}^3$, $c_3: I_3 \to \mathbb{R}^3$ are curves and $c_1 = c_2 \circ f$, where $f: I_1 \to I_2$ is an homeomorphism. Then, we say that curves c_1 and c_2 are equivalent

Remark: If the curves $c_i: I_i \to \mathbb{R}^3$, $i = 1, 2$ are equivalent, then $c_1(I_1) = c_2(I_2)$

DEF: A curve $c: (a,b) \to \mathbb{R}^3$, where $\forall t \in (a,b)$, $c(t) = (c_1(t), c_2(t), c_3(t))$ is called a regular curve if maps $c_i: (a,b) \to \mathbb{R}$, $i = 1, 2, 3$ are differentiable and $\forall t \in (a,b)$, $c'(t) = (c_1'(t), c_2'(t), c_3'(t)) \neq [0,0,0]$

The vector $c'(t)$ is called a tangent vector to curve c at point $c(t)$ (at time t)



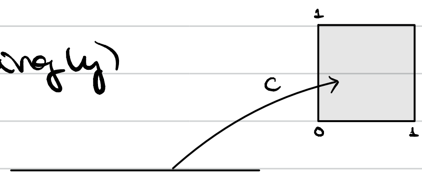
We say that a regular curve $c: (a,b) \to \mathbb{R}^3$ has a unit speed if $\forall t \in (a,b)$, $\|c'(t)\| = 1$.

EX: Pathological Curves

1) Peano's Curve (space fillingly)

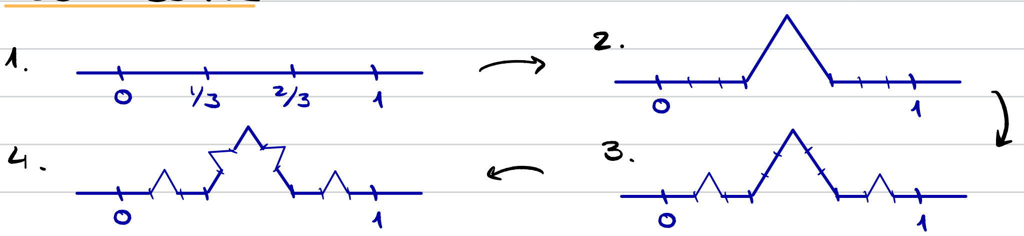
$c: (-\infty, \infty) \to \mathbb{R}^2$

$c(-\infty, \infty) = [0,1] \times [0,1]$



It is surjective but not injective.
It is not a regular curve.

2) Koch curve



... In a limit case  is called Koch curve.
It has infinite lenght.

EX: Regular curves

so c is a line in $\mathbb{R}^3$

1) • $\forall t \in (-\infty, \infty)$, $c(t) = (p_1 + t v_1, p_2 + t v_2, p_3 + t v_3) = (p_1, p_2, p_3) + t(v_1, v_2, v_3)$ * continuous ✓

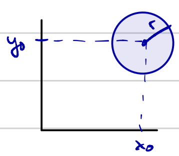
• $c'(t) = [v_1, v_2, v_3] \neq [0,0,0]$ ✓ $\Rightarrow$ c is regular.

2) • $\tilde{c} : (0, 2\pi) \to \mathbb{R}^3$, $\forall t \in (0, 2\pi)$, $c(t) = (x_0 + r\cos t, y_0 + r \operatorname{sen} t, 0)$ cont ✓

• $\tilde{c}'(t) = [-r \operatorname{sen} t, r\cos t, 0] \neq [0,0,0]$ ✓ $\Rightarrow \tilde{c}$ is regular

porque la norma no es 0.

$$\|\tilde{c}(t)\| = \sqrt{r^2 \operatorname{sen}^2 t + r^2 \cos^2 t + 0^2} = \sqrt{r^2} = r \neq 0$$



$\tilde{c}$ is a parametrization of a circle of radius r and centre $(x_0, y_0, 0)$

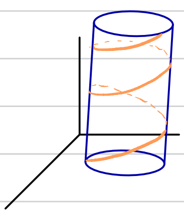
3) $\hat{c} : (a,b) \to \mathbb{R}^3$, $\forall t \in (a,b)$ $\hat{c}(t) = (r\cos t, r \operatorname{sen} t, 2t)$

$t \mapsto r\cos t$
$t \mapsto r \operatorname{sen} t$
$t \mapsto 2t$

these are differentiable $\Rightarrow \hat{c}$ is regular

$\hat{c}'(t) = (-r \operatorname{sen} t, r\cos t, 2) \neq [0,0,0]$ ✓

$$\|\hat{c}'(t)\| = \sqrt{r^2 \operatorname{sen}^2 t + r^2 \cos^2 t + 4} = \sqrt{r^2 + 4} \neq 0$$



This curve is called "helix".

DEF: The lenght of a regular curve $c:[a,b]\to\mathbb{R}^3$ is a number, $L(c)$, defined by:

$$L(c) \overset{def}{:=} \int_a^b \|c'(t)\|\, dt$$

THM: Regular equivalent curves have the same lenght.

Proof: Let $c_1=c_2\circ f$, $\left.\begin{matrix} c_1:[a_1,b_1]\to\mathbb{R}^3 \\ c_2:[a_2,b_2]\to\mathbb{R}^3 \end{matrix}\right\}$ regular curves

para que sean regulares.

and $f:[a_1,b_1]\to[a_2,b_2]$ a diffeomorphism.

Then, since $c_1=c_2\circ f$, we get $\|c_1'(t)\|=\|c_2'(f(t))\cdot f'(t)\| =$
$= \|c_2'(f(t))\|\cdot\|f'(t)\|$

So, $L(c_1) = \int_{a_1}^{b_1} \|c_1'(t)\|\, dt = \int_{a_1}^{b_1} \|c_2'(f(t))\|\cdot\|f'(t)\|\, dt$

Taking $u=f(t)$, $L(c_2)=\int_{a_2}^{b_2} \|c_2'(u)\|\, du$

$+$

Remark: Assume that $c:(a,b)\to\mathbb{R}^3$ is a regular curve with unit speed. Then,

$$\int_a^b \overbrace{\|c'(t)\|}^{=1}\, dt = \int_a^b 1\, dt = b-a$$

THM: For any regular curve c, there exists a curve $\hat{c}$ with unit speed such that $\hat{c}$ is equivalent to c.

Proof: Let $c:(a,b)\to\mathbb{R}^3$ be a regular curve.

Let $s(t)=\int_a^t \|c'(u)\|\, du$ so $s:(a,b)\to\mathbb{R}$ is differentiable.

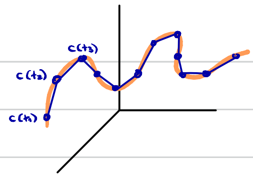
And $s'(t)=\|c'(t)\|>0$ so s is increasing function and 1-1 function.

Moreover, $s:[a,b]\xrightarrow[inj]{surj}[0,L(c)]$ is bijective, so there exists an inverse function $\varphi:[0,L(c)]\to[a,b]$, such that:

1. $c\circ\varphi$ is equivalent to c
2. $\|(c\circ\varphi)'(u)\| = \|c'(\varphi(u))\|\cdot\|\varphi'(u)\| = \dfrac{1}{\|c'(\varphi(u))\|}\cdot\|c'(\varphi(u))\|=1.$

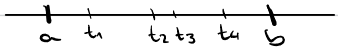
Therefore, for a given regular curve c, there exists a regular curve $\hat{c}=c\circ\varphi$ such that $\hat{c}$ has unit speed. $+$

Another approach to lenght of curves.



DEF: Given a continuous curve $c:[a,b]\to\mathbb{R}^3$, a partition $\mathcal{P}$ of $[a,b]$ is a sequence (t_n) s.t. $a=t_0<t_1<\dots<t_{n-1}<t_n=b$.

Graphically:



We can approximate the lenght of a curve by:

$$\tilde{L}(c) = \sup\left\{ \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) : (t_0,t_1,\dots,t_n) \text{ is a partition of } [a,b] \right\}$$

DEF: We say that a continuous curve $c:(a,b)\to\mathbb{R}^3$ is rectyficable if $\tilde{L}(c)<\infty$ (has a finite lenght).

THM: If $c:(a,b)\to\mathbb{R}^3$ is C^3-diff curve, then $L(c)=\tilde{L}(c)$

EX: We will calculate the lenght of a circle of radius r and centre $(0,0,0)$.

The curve $c:(0,2\pi)\to\mathbb{R}^3$, $\forall t\in(0,2\pi)$ $c(t)=(r\sin t, r\cdot\cos t, 0)$ is a parametrization of the circle.

$$L(c) = \int_0^{2\pi} \|c'(t)\|\,dt = \int_0^{2\pi} \|[r\cos t, -r\sin t, 0]\|\,dt = \int_0^{2\pi} \sqrt{r^2\cos^2 t + r^2\sin^2 t}\,dt$$

$$= \int_0^{2\pi} r\,dt = 2\pi r \,//$$

LEMMA: The Koch curve has infinite lenght.

Proof:

l_m = lenght of Koch's curve of level m.

d_m = lenght of each segment of the broken line of level m.

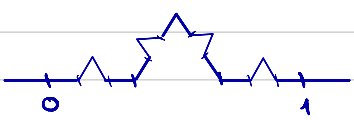
n_m = number of segments in broken line at level m.

$// \; l_m = d_m \cdot n_m$

1.

$d_1 = 1/3 \quad n_1 = 4$

2.



$d_2 = (1/3)^2 \quad n_2 = 4\cdot n_1$

Assume that we know d_m, n_m, l_m:

$$\left.\begin{aligned} d_{m+1} &= \frac{d_m}{3} = \left(1/3\right)^{m+1} \\ n_{m+1} &= 4\cdot n_m = 4^{m+1} \\ l_{m+1} &= d_{m+1}\cdot n_{m+1} = \left(4/3\right)^{m+1} \end{aligned}\right\} \lim_{m\to\infty} \left(4/3\right)^{m+1} = \infty.$$

+

DEF: A warsawsin function is a curve definite

$$\gamma : (0,1) \to \mathbb{R}^2 \qquad \forall t\in(0,1),\ \gamma(t) = (t, \operatorname{sen} \pi/t)$$

We have to prove that $\ell(\gamma) = \infty$:

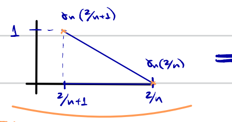
We consider $\gamma_n : \left[\frac{2}{n+1}, \frac{2}{n}\right] \longrightarrow \mathbb{R}^2$

We know that $\ell(\gamma) = \sum_{n=1}^{\infty} \ell(\gamma_n)$

Assume that n is an even number $\to$ $n+1$ is an odd number

$$\gamma_n\left(\frac{2}{n+1}\right) = \left(\frac{2}{n+1}, \operatorname{sen}\left(\pi/2\right)^{n+1}\right) = \left(\frac{2}{n+1}, \pm 1\right)$$

$$\gamma_n\left(\frac{2}{n}\right) = \left(\frac{2}{n}, \operatorname{sen}\left(\pi/2\right)^{n}\right) = \left(2/n, 0\right)$$



$$\Rightarrow \ell(\gamma_n) \geq 1. \Rightarrow L(\gamma) = \sum_{n\geq 1}^{\infty} \ell(\gamma_n) = \infty.$$

Esto es la curva muy muy ampliada, que se ve como un segmento cuya longitud es superior a la unidad.

The segment is the shortest possible lenght for a curve. If it's ≥ 1, any curve it ≥ 1.

Local theory of regular curves with unit speed.

Let $\alpha : (a,b) \to \mathbb{R}^3$ be a par. curve with unit speed, i.e. $\forall t\in(a,b)$, $\|\alpha'(t)\| = 1$.

DEF: The vector $\alpha'(t)$ is a tangent vector to α at time t. The line $\{\alpha(t) + s\cdot\alpha'(t) : s\in\mathbb{R}\}$ is a tangent line to α at time t (or at point $\alpha(t)$).

$\alpha''(t)$ measures how rapidly the curve pulls away from the tangent line.

DEF: Let $\alpha:(a,b)\to\mathbb{R}^3$ be a regular curve with unit speed. The number $\|\alpha''(t)\| \overset{\text{def}}{=} \kappa(t)$ is called curvature of α at time t.

EX: 1) $\alpha:(a,b)\to\mathbb{R}^3$
$\alpha(t)=(p_1,p_2,p_3)+t\cdot[v_1,v_2,v_3]$
$\|\alpha'(t)\|=0 \Rightarrow$ At any point, the curvature is 0.

2) $\beta:(a,b)\to\mathbb{R}^3$ with $\kappa(t)=0 \quad \forall t\in(a,b)$
Then, $\beta''(t)=[0,0,0] \quad \forall t\in(a,b)$.

Remark: We consider a curve $\alpha:(a,b)\to\mathbb{R}^3$ regular with unit speed. At any point of this curve, where $\kappa(t)\neq 0$, a unit vector $n(t)$ in direction of $\alpha''(t)$ is well defined by the equation:

$$\alpha''(t)=\kappa(t)\cdot n(t)$$

Remark: We claim that for unit speed vector $\alpha'(t)$, we set that $\alpha''(t)\perp\alpha'(t)$.

Proof: Since $\alpha'(t)$ is a unit vector,

$$1=\|\alpha'(t)\|=\sqrt{\langle\alpha'(t),\alpha'(t)\rangle} \Rightarrow \langle\alpha'(t),\alpha'(t)\rangle=1.$$

Diff. above equation:

$$0=\frac{d}{dt}\langle\alpha'(t),\alpha'(t)\rangle=\langle\alpha''(t),\alpha'(t)\rangle+\langle\alpha'(t),\alpha''(t)\rangle=2\langle\alpha''(t),\alpha'(t)\rangle$$

Then: $0=2\langle\alpha''(t),\alpha'(t)\rangle \Rightarrow \alpha''(t)\perp\alpha'(t)$. +

DEF: The vector $n(t)$ that is normal to $\alpha'(t)$ is called the normal vector at t to the curve $\alpha(t)$. Moreover,

$$n(t)=\frac{\alpha''(t)}{\|\alpha''(t)\|}.$$

DEF: The plane determined by $\alpha'(t)$ and vector $n(t)$ is called an oscilating plane.
Notice that at any point $\alpha(s)$ such that $\kappa(s)=0$, the oscilating plane is not defined.

DEF: The unit vector $t(s)=\alpha'(s)$ is called a tangent vector to α at time s.

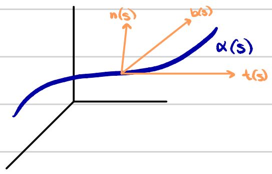
We define binormal vector to α at time s by $b(s) \overset{def}{=} t(s)\times n(s)$

Therefore, $t(s) \perp b(s) \perp n(s)$.

Remark: If $\|b(s)\|=1 \Rightarrow \langle b(s), b(s)\rangle = 1 \Rightarrow b'(s) \perp b(s)$.

Remark: $b'(s) = [t(s)\times n(s)]' = \cancel{t'(s)\times n(s)} + t(s)\times n'(s) = t(s)\times n'(s)$

Remark: If $\alpha:(a,b)\to\mathbb{R}^3$ is a curve like above, then the unit vector $t(s), n(s), b(s)$ are well defined and they are pairwise perpendicular.



DEF: We say that a curve $\alpha:(a,b)\to\mathbb{R}^3$ is a plane curve if there exists a plane π such that $\alpha(a,b)\subset\pi$.

THM: A curve $\alpha:(a,b)\to\mathbb{R}^3$ given by parametrization with unit speed and $\alpha''(t)\neq 0$ is a plane curve if and only if the binormal vector $b(s)$ is constant.

Frenet formulas

Assume that $\alpha: I\to\mathbb{R}^3$ is a regular curve with unit speed. Then, we have:

curvature of α $\quad \kappa(s)=\|\alpha''(s)\|$

normal vector $\quad n(s)=\dfrac{\alpha''(s)}{\|\alpha''(s)\|}$

$\alpha''(s)=\kappa(s)\cdot n(s)$

tangent vector $\quad t(s)=\alpha'(s)$

binormal vector $\quad b(s)=t(s)\times n(s)$

Remark: $t(s), n(s), b(s)$ are pairwise perpendicular and of length 1. So they create an orthonormal basis of $\mathbb{R}^3$.

→ Is it possible to write vectors $t'(s), n'(s), b'(s)$ in basis $(t(s), n(s), b(s))$? Yes, it is. Let's see how:

1. By definition, we have $t'(s) = \kappa(s)\cdot n(s)$ *

2. Computation of $b'(s)$. First of all, $\forall s\in I$, $\|b(s)\| = 1 = \sqrt{\langle b(s), b(s)\rangle}$. Taking the derivate of the length:

primero vemos la orientación

- $0 = \langle b'(s), b(s)\rangle + \langle b(s), b'(s)\rangle = 2\langle b(s), b'(s)\rangle \Rightarrow b(s) \perp b'(s)$.

y luego lo calculamos

- $b'(s) = (t(s)\times n(s))' = \underbrace{t'(s)\times n(s)}_{\text{por def de prod. vectorial}} + t(s)\times n'(s) =$

$* \; t'(s) = \alpha''(s) = \boxed{\kappa(s)\cdot n(s) = 0}$

$= t(s)\times n'(s) \Rightarrow \left\{\begin{array}{l} b'(s) \perp t(s) \\ b'(s) \perp n'(s) \end{array}\right\} \Rightarrow b'(s) \parallel n(s) \Rightarrow$

$\Rightarrow \exists \tau(s): \; b'(s) = \tau(s)\, n(s)$. TORSIÓN

Podemos entender la torsión por "cuanto sale una curva de un plano"

3. Computation of $n'(s)$:

regla de los tres dedos

$b(s) = t(s)\times n(s) \Rightarrow n(s) = b(s)\times t(s)$. Taking derivative:

$n'(s) = (b(s)\times t(s))' = \underbrace{b'(s)\times t(s)}_{*} + \underbrace{b(s)\times t'(s)}_{*} =$

→ We use $\vec{w}\times\vec{v} = -\vec{v}\times\vec{w}$

→ We use $b'(s) = \tau(s)\times n(s)$

$* \; [\underbrace{\tau(s)}_{\in\mathbb{R}}\cdot \underbrace{n(s)}_{\in\mathbb{R}^3}]\times \underbrace{t(s)}_{\in\mathbb{R}^3} = -\tau(s)\cdot t(s)\times n(s)$

$* \; b(s)\times[\kappa(s)\cdot n(s)] = \kappa(s)\cdot b(s)\times n(s) = -\kappa(s)\cdot n(s)\times b(s)$

$= -\tau(s)\cdot \overbrace{t(s)\times n(s)}^{b(s)} - \kappa(s)\cdot \overbrace{n(s)\times b(s)}^{t(s)} = -\tau(s)\cdot b(s) - \kappa(s)\cdot t(s) = n'(s)$

Summing up...

THM: Frenet formulas. For a regular curve $\alpha: I\to\mathbb{R}^3$ with unit speed, we have:

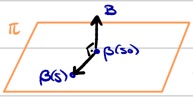
$$t'(s) = \kappa(s)\cdot n(s)$$
$$b'(s) = \tau(s)\cdot n(s)$$
$$n'(s) = -\tau(s)\cdot b(s) - \kappa(s)\cdot t(s)$$

, where $\kappa(s)$ is a curvature of α at s and $\tau(s)$ is a torsion of α at s.

DEF: We say that a curve $\beta: I \to \mathbb{R}^3$ is a plane curve if there exists a plane π such that $\underset{s\in I}{\forall}\ \beta(s) \in \pi$ ($\Leftrightarrow \beta(I) \subset \pi$)

THM: Consider a regular curve $\beta: I \to \mathbb{R}^3$ with unit speed. Then, $\forall s \in I$ $\tau(s) = 0 \Leftrightarrow \beta$ is a plane curve.

Proof: $\Rightarrow)$ Assume that $\forall s\in I$, $\tau(s)=0$. By one of the Frenet formulas, we have $b'(s) = \tau(s)\cdot n(s)$. Thus, $b'(s) = 0 \Rightarrow$ $\Rightarrow b(s)$ is a constant vector, B.



We claim that for any point $s\in I$, $\beta(s)$ belongs to the plane $\pi \perp B$ and such that $\beta(s_0) \in \pi$.

We want to show that $\forall s \in I$, $\beta(s) \in \pi$ or, equivalently, $\vec{B} \perp \overline{\beta(s_0)\beta(s)}$. We claim that $\forall s\in I$, $(\beta(s)-\beta(s_0))\cdot \vec{B} = 0$.

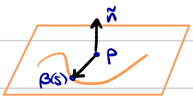
Consider the derivate:

$$((\beta(s)-\underset{cte}{\beta(s_0)})\cdot \underset{cte}{\vec{B}})' = \beta'(s)\cdot\vec{B} + \cancel{\beta(s)\cdot\vec{B}'}^{\,0} = \beta'(s)\cdot b(s) = t(s)\cdot b(s) = 0$$

So $((\beta(s)-\beta(s_0))\cdot\vec{B})' = 0 \Rightarrow (\beta(s)-\beta(s_0))\cdot\vec{B} = cte$
Taking $s=s_0$, we have $(\beta(s_0)-\beta(s_0))\cdot\vec{B} = 0$ $\Big\}$ $(\beta(s)-\beta(s_0))\cdot\vec{B} = 0 \Rightarrow$

$\Rightarrow (\beta(s)-\beta(s_0)) \perp \vec{B} \Rightarrow \forall s\in I$, $\beta(s)\in\pi$, so $\beta(s)$ is a plane curve

$\Leftarrow)$ Assume that $\beta: I \to \mathbb{R}^3$ is a plane curve, i.e. there exists a plane π such that $\beta(I)\subset\pi$.



Assume that point $p\in\pi$ and $\tilde{n}\perp\pi$. Therefore, for any $s\in I$, $(\beta(s)-p)\cdot\underset{cte}{\tilde{n}} = 0$. Taking derivative:

$$0 = \beta'(s)\cdot\tilde{n} + \cancel{\beta(s)\cdot(\tilde{n})'}^{\,0} \Rightarrow \forall s\in I,\ \beta'(s)\cdot\tilde{n} = 0 \xrightarrow{der} \beta''(s)\cdot\tilde{n} + \cancel{\beta'(s)\cdot\tilde{n}'} = 0$$

$$\overset{\beta'(s)}{t(s)}\cdot\tilde{n} = 0 \Rightarrow t(s)\perp\tilde{n} \qquad \Big| \ \underbrace{\beta''(s)}_{\substack{t'(s)=\kappa(s)\cdot n(s) \\ \text{Fórmulas de Frenet}}}\cdot\tilde{n} = 0 \Rightarrow (\kappa(s)\cdot n(s))\cdot\tilde{n} = 0$$

$$\Downarrow$$

$$n(s)\perp\tilde{n}$$

$\forall s\in I$, $n(s)\perp\tilde{n}$ and $t(s)\perp\tilde{n} \Rightarrow b(s) = t(s)\times n(s) \parallel \tilde{n}$. Therefore, $\exists c\in\mathbb{R}$ such that $b(s) = c\cdot\tilde{n}$. Taking derivatives of both sides, $b'(s) = 0$. By Frenet formula, $0 = b'(s) = \tau(s)\cdot\underline{n(s)}$, so $\forall s\in I$, $\tau(s) = 0$.

y pq no es ese?

+

THM: Fundamental thm of local theory of curves

Consider two differentiable functions $\kappa:(a,b)\to\mathbb{R}^+$, and $\tau:(a,b)\to\mathbb{R}$. Then, there exists a regular curve $\alpha:(a,b)\to\mathbb{R}^3$ with unit speed such that $\kappa(s)$ is its curvature at point s and $\tau(s)$ is its torsion at point s.

Moreover, if there exists two curves $\alpha_1:(a,b)\to\mathbb{R}^3$ and $\alpha_2:(a,b)\to\mathbb{R}^3$ satisfying above conditions, then they are the same up to rigid motion.

Global results of the theory of plane curves

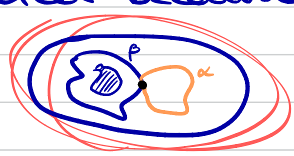
DEF: A closed plane curve $\alpha:[a,b]\to\mathbb{R}^2$ is a regular curve such that $\alpha(a)=\alpha(b)$, $\alpha'(a)=\alpha'(b)$, $\dots$, $\alpha^{(n)}(a)=\alpha^{(n)}(b)$.

We say that a closed plane curve is a simple curve if it has no self-intersection. It means that there are two points $t_1,t_2\in[a,b]$, with $t_1<t_2$, such that $\alpha(t_1)=\alpha(t_2)$.

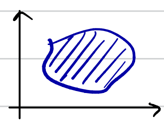
For example, the following curve is not a simple one: 

DEF: An open set $A\subset\mathbb{R}^2$ is called a region in $\mathbb{R}^3$ if it is non-empty and simple connected.

For example, this set is not simple connected because the loop β cannot be to a point



THM: Jordan. Any simple closed planar curve bounds a region



THM: Isopermetric inequality

Let c be a planar simple closed curve with lenght ℓ and let A_ℓ be the area of the region bounded by c.

Then, $\ell^2 - 4\pi A_\ell \geq 0$.

The equality holds if and only if the curve c is a circle.

DEF: Given a simple closed planar curve $\alpha : [a,b] \to \mathbb{R}^2$, we say that α has a vertex at point t if $\kappa'(t) = 0$.

Ex: Circle

$$\alpha(t) = (R \operatorname{sen}(t/R), R\cos(t/R))$$

$$\alpha'(t) = (\cos(t/R), -\operatorname{sen}(t/R))$$

$$\alpha''(t) = (-1/R \operatorname{sen}(t/R), -1/R \cos(t/R))$$

$$\kappa(t) = \|\alpha''(t)\| = \sqrt{1/R^2 \operatorname{sen}^2(t/R) + 1/R^2 \cos^2(t/R)} = 1/R \quad \text{cte}$$

$\kappa'(t) = 0 \Rightarrow$ Any point of the circle is a vertex.

Ex: Elipse $= \{(x,y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 = 1\}$

$\alpha(t) = (a \operatorname{sen} t, b \cos t)$ (It is a param. bc it verifies the equality)

THM: Any simple closed planar curve has at least 4 vertices.

Regular surfaces

Def.: A subset $S \subset \mathbb{R}^3$ is a regular surface if

$\forall_{p \in S}$ $\exists$ open $V_p \subset \mathbb{R}^3$ $\exists$ open $U_p \subset \mathbb{R}^2$ $\exists$ $\tilde{x}: U_p \to V_p \cap S$

such that 3 conditions hold:

1) $\tilde{x}$ is differentiable
2) $\tilde{x}$ is homeomorphism
3) different. $\forall_{q \in U_p} d\tilde{x}_q : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one map.

The map $\tilde{x}$ is a regular parametrization of the surface S

Remarks

1) $\tilde{x}(u,v) = (x(u,v), y(u,v), z(u,v))$

Differentiability of $\tilde{x}$ means that all maps

$x: U_p \to \mathbb{R}$
$y: U_p \to \mathbb{R}$
$z: U_p \to \mathbb{R}$

have partial derivatives of all orders

2) $\tilde{x}: U_p \to V_p \cap S$ is a homeomorphism $\Leftrightarrow$ $\tilde{x}$ is continuous, injective and surjective

3) $\tilde{x}$ is different. $\Rightarrow$ $\tilde{x}$ is cont.

4) We compute matrix of $d\tilde{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ in basis $e_1 = (1,0), e_2 = (0,1)$ in $\mathbb{R}^2$

$f_1 = (1,0,0)$ $f_2 = (0,1,0)$ $f_3 = (0,0,1)$ in $\mathbb{R}^3$

$$d\tilde{x}_q \approx \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

$d\tilde{x}_q$ is one-to-one $\iff$ $\begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix}$ and $\begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix}$ are linear independent $\iff$

$$\iff \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \neq 0 \text{ or } \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \neq 0 \text{ or } \det \begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \neq 0$$

Examples of regular surfaces

1) Plane
2) Sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\}$
   radius r center (a,b,c)
3) Cylinder $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, z \in \mathbb{R}\}$
4) Torus
5) Many more examples

Ex.: Prove that the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$ is a regular surface

$$z = \pm\sqrt{r^2 - x^2 - y^2}$$

$$f(x,y) = \sqrt{r^2 - x^2 - y^2}$$

The function $f(x,y)$ has partial derivatives of arbitrary order $\Rightarrow f$ is differentiable

$$D = \{(x,y,0) : x^2 + y^2 < r^2\}$$
open set in $\mathbb{R}^2$

f is differentiable $\Rightarrow f$ is continuous

$f^{-1} : S^2_+ \longrightarrow D$ cont.

$$f : D \longrightarrow S^2_+$$

f cont. $+ f^{-1}$ cont. $\Rightarrow f$ is homeomorphism

Calculate the matrix of df.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial\sqrt{r^2-x^2-y^2}}{\partial x} & \dfrac{\partial\sqrt{r^2-x^2-y^2}}{\partial y} \end{bmatrix}$$

$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

so df is one to one

Therefore, the upper hemisphere is a regular surface.

The map $f_1 : D \longrightarrow S^2_-$, $f_1(x,y) = -\sqrt{r^2-x^2-y^2}$

the same arguments, as in the case of f, gives that f_1 is a regular parametrization of the lower hemisphere.

Remark

It is easy to check that

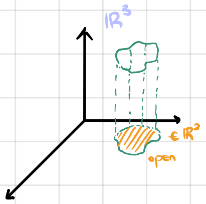
$\vec{x}(\varphi_1,\varphi_2) = (\operatorname{sen}\varphi_1 \cos\varphi_2, \operatorname{sen}\varphi_1 \operatorname{sen}\varphi_2, \cos\varphi_1)$ is a regular parametrization of the unit sphere

Thm. If $f : U \longrightarrow \mathbb{R}$, where U is an open set in $\mathbb{R}^2$, is a differentiable function, then the graph of f i.e.

$$\{(x,y,f(x,y)) : (x,y) \in U\} \text{ is a regular surface}$$

PROOF.

Let $\tilde{x}(u,v) = (u,v,f(u,v))$, then $\tilde{x}$ is differentiable $\overset{(1)}{\Longrightarrow}$ $\tilde{x}$ is continuous



$(\tilde{x})^{-1}$ exists and it is continuos

$\tilde{x}$ and $\tilde{x}^{-1}$ continuos $\Longrightarrow$ $\tilde{x}$ homeomorphism (2)

Notice that $\frac{\partial\tilde{x}}{\partial u} = [1,0,f_u]$, $\frac{\partial\tilde{x}}{\partial v} = [0,1,f_v]$, those two vectors are linearly independent, so $d\tilde{x}_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is one-to-one. (3)

Def.: Given a differentiable map $F: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, we say that a point $p \in U$ is a critical point of F if $dF_p: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is not surjective. Then $F(p)$ is called critical value of F. A point $q \in \mathbb{R}^m$ which is not a critical value, is called a regular value of F.

Thm. If $f: U \longrightarrow \mathbb{R}$ where U is an open subset of $\mathbb{R}^3$ is differentiable function and a point $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface.

Cor.: The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is a regular surface

1. Proyecciones en planos y cosas de rectas

2. Longitud de curvas

3. Cálculo de curvaturas

4. " " torsiones

5. Sorpresa!! :)

Exercises

1. Proof that the plane is a regular surface

$$\pi^{\subset \mathbb{R}^3} = \{(p_1+\alpha v_1+\beta w_1, p_2+\alpha v_2+\beta w_2, p_3+\alpha v_3+\beta w_3 : \alpha, \beta \in \mathbb{R}\}$$

$$\tilde{x} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\tilde{x}(\alpha, \beta) = (p_1+\alpha v_1+\beta w_1, p_2+\alpha v_2+\beta w_2, p_3+\alpha v_3+\beta w_3)$$

By definition, $\tilde{x}$ is a differentiable function, so it is continuous. As $(\tilde{x})^{-1}$ exists and $\underbrace{\text{it is continuous}}_{\text{because it is a } \perp \text{ projection from } \pi \text{ on the plane determines by axes } x \text{ and } y}$, we can affirm that $\tilde{x}$ is an homeomorphism (2nd condition)

$$\tilde{x}(\alpha, \beta) = (x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta))$$

$$d\tilde{x}_q = \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \\ \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} \end{bmatrix}$$

, $\tilde{x}$ is one-to-one because vectors $\vec{v}$ and $\vec{w}$ are linearly independent (it is because of the definition of π)

Therefore, a plane is an example of regular surface.

2. Prove that a cylinder is a regular surface

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = r^2\}$

$\tilde{x}(u,v) = (r \cdot \cos u, r \cdot \operatorname{sen} u, v)$ with $u \in [0, 2\pi]$, $v \in \mathbb{R}$

Since every element is diff, $\tilde{x}$ is too $\Longrightarrow \tilde{x}$ is continuous

$\tilde{x}^{-1}$ is continuous since it is a projection

$$\left.\begin{array}{l} \tilde{x} \text{ continuous} \\ \tilde{x}^{-1} \text{ continuous} \end{array}\right\} \Longrightarrow \tilde{x} \text{ homeomorphism}$$

$$d\tilde{x}_q = \begin{bmatrix} -r\operatorname{sen} u & 0 \\ r\cos u & 0 \\ 0 & 1 \end{bmatrix}$$

We have 2 possibilities:

•) $\cos u = 0 \iff u = \frac{\pi}{2}, \frac{3\pi}{2}$

•) $\operatorname{sen} u = 0 \iff u = \cancel{0}, \pi, \cancel{2\pi}$

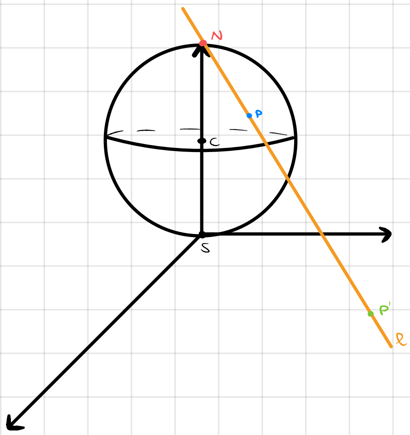
So if $u = \frac{\pi}{2}, \frac{3\pi}{2}$ we check $\det \begin{bmatrix} -r\operatorname{sen} u & 0 \\ 0 & 1 \end{bmatrix} \neq 0$ and

if $v = \pi$ we can check $\det \begin{bmatrix} r\cos u & 0 \\ 0 & 1 \end{bmatrix} \neq 0$

• When $u \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ it does not matter which one to check

3. Write a parametrization of a cone and check if this parametrization gives a regular surface.

4. Given a unit sphere ($r = 1$) and the plain tangent to the sphere at the south pole. Fix a point P of the sphere and consider a half-line ℓ_P starting at the north and going through point P. The line ℓ_P cuts the tangent plane π at point P'. The transformation which associates P to P' is a stereographic projection. Write the analitic formula of the stereographic projection.

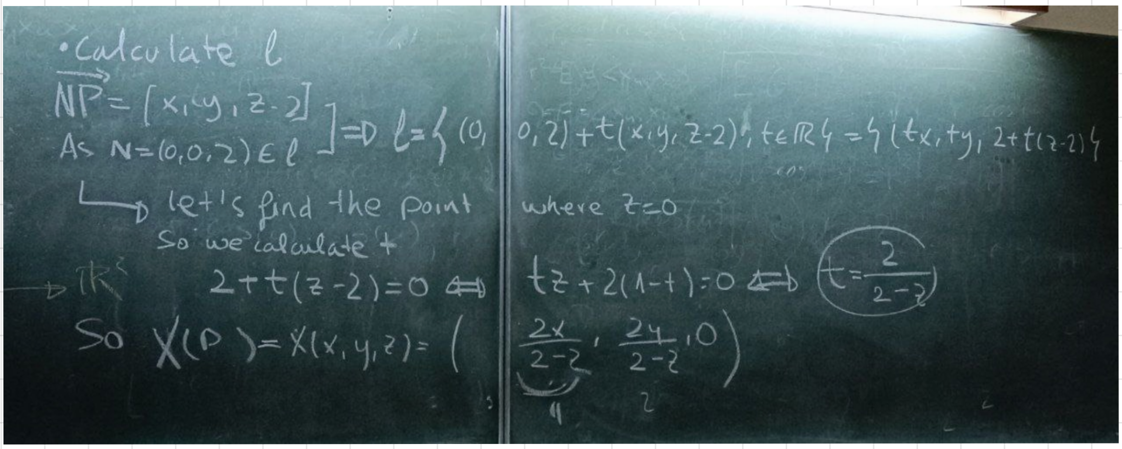


$N = (0,0,z)$

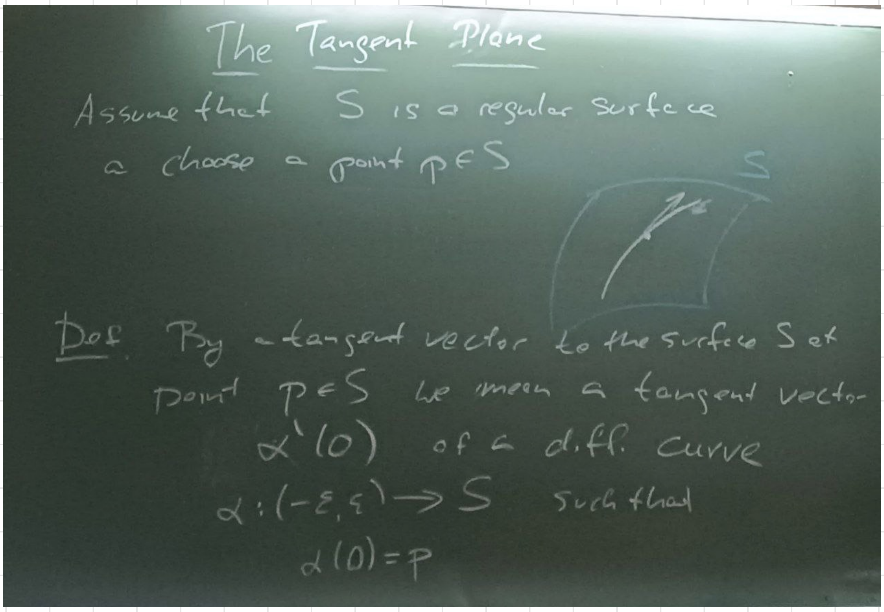
$P = (1,y,z)$

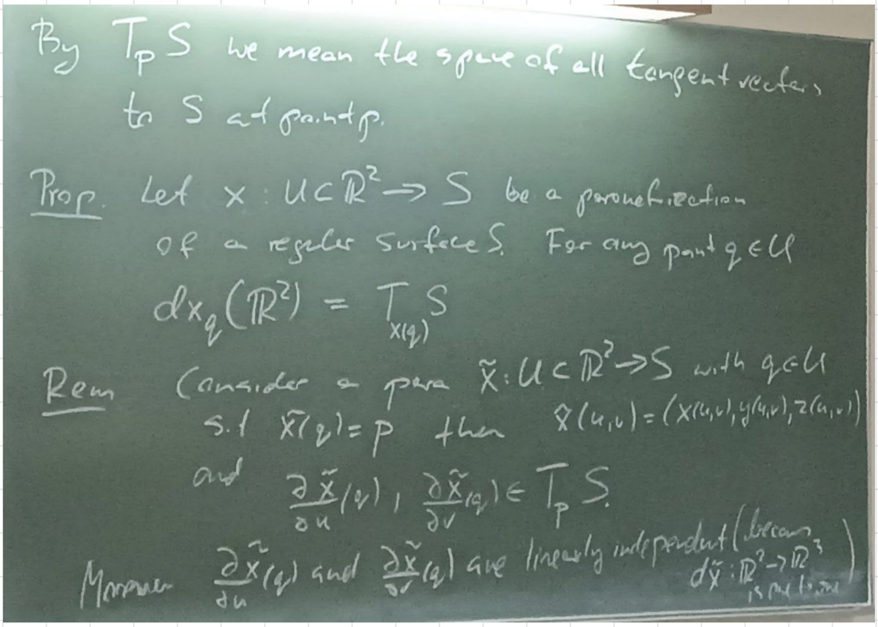
$C = (0,0,1)$

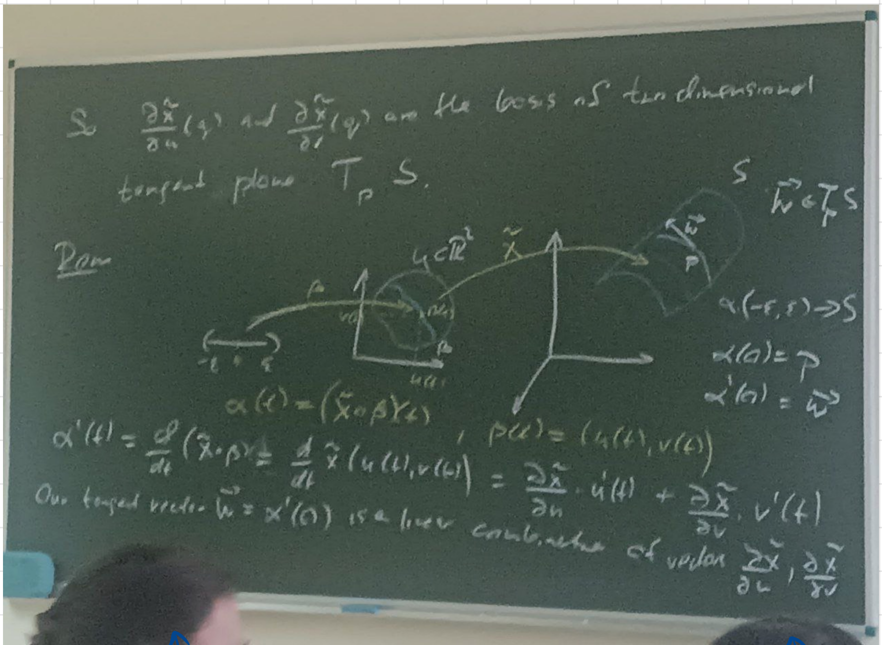
$S = (0,0,0)$



The tangent plane







Soledad

Clara

The first Fundamental Form

S – regular surface, $p \in S$, T_pS = tangent plane to S at p.

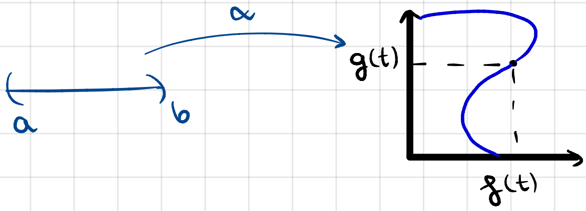
In $\mathbb{R}^3$ we have a scalar product $\langle \ , \ \rangle$
so we can use the scalar product
for any vectors $v, w \in T_pS$

Def The quadratic form
$$I_p : T_pS \to \mathbb{R}$$
defined by
$$I_p(w) \overset{def}{=} \langle w, w \rangle, \text{ for any } w \in T_pS,$$
is called the first fundamental form, of a surface S with parametriz. $\tilde{x}$ at point p

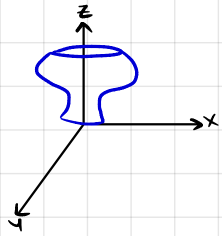
Since $w \in T_pS$ there exists
a curve $\alpha(t) = \tilde{x}(u(t), v(t))$ such that
$t \in (-\varepsilon, \varepsilon)$, $\alpha(0) = p$ $\alpha'(0) = w$

Therefore,
$$I_p(w) = I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle = (*)$$
since $\alpha'(0) = \frac{\partial}{\partial t}(\tilde{x}(u(t), v(t))) = \underbrace{\frac{\partial \tilde{x}}{\partial u}}_{\tilde{x}_u} \cdot u' + \underbrace{\frac{\partial \tilde{x}}{\partial v}}_{\tilde{x}_v} \cdot v'$
we get

$$(*) = \langle \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v', \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v' \rangle =$$
$$= u' \cdot \langle \tilde{x}_u, \tilde{x}_u \cdot u' + \tilde{x}_v \cdot v' \rangle + v' \cdot \langle \tilde{x}_v, \tilde{x}_u u' + \tilde{x}_v v' \rangle =$$
$$\langle \tilde{x}_u, \tilde{x}_u \rangle \cdot (u')^2 + \langle \tilde{x}_u, \tilde{x}_v \rangle \cdot u' \cdot v' + \langle \tilde{x}_v, \tilde{x}_u \rangle \cdot v' \cdot u' + \langle \tilde{x}_v, \tilde{x}_v \rangle \cdot (v')^2$$



$\xrightarrow[\text{rotation}]{\text{after}}$



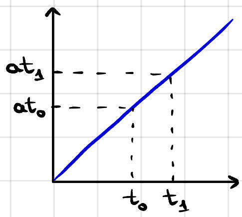
$\underset{t\in(a,b)}{\forall} \alpha(t) = (f(t), g(t))$ $\qquad x(t,\varphi) = (f(t)\cdot \text{sen}(\varphi),\ f(t)\cdot\cos(\varphi),\ g(t))$

Surface of revolution

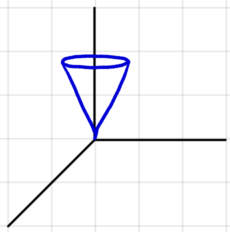
$\underset{t>0}{\forall} \alpha(t) = (t, a\cdot t)$ $\qquad x(t,\varphi) = (t\cdot \text{sen}(\varphi),\ t\cdot \cos(\varphi),\ at)$

Parametrization of a cone

$\varphi \in (0, 2\pi)\ ,\ t \geq 0$

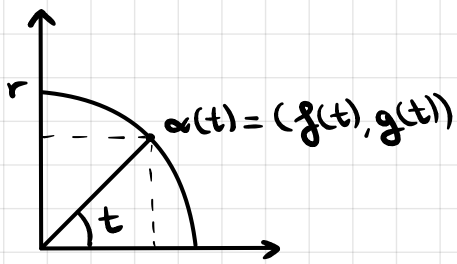


$\xrightarrow[\text{rotation}]{\text{after}}$



It is not a regular surface

Sphere as a surface of revolution



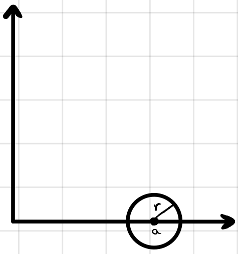
$\cos t = \dfrac{f(t)}{r}$

$\text{sen}\, t = \dfrac{g(t)}{r}$

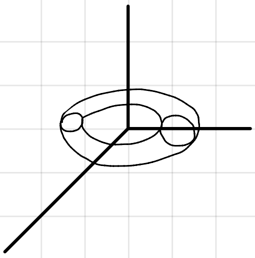
Parametrization of a hemi-circle: $\underset{t\in(-\frac{\pi}{2},\frac{\pi}{2})}{\forall} \alpha(t) = (r\cdot\cos t,\ r\cdot \text{sen}\, t)$

Parametrization of a sphere:

$x(t,\varphi) = (r\cdot \cos t\cdot \text{sen}(\varphi),\ r\cdot \text{sen}\, t\cdot \cos(\varphi),\ r\cdot \text{sen}\, t)$



$\forall\, \alpha(t) = (a + r\cos t,\ r\,\mathrm{sen}\,t)$
$t \in (0, 2\pi)$



Torus has the following parametrization:

$x(t,\varphi) = ((a + r\cos t)\cdot \mathrm{sen}(\varphi),\ (a + r\cos t)\cdot \cos(\varphi),\ r\,\mathrm{sen}\,t)$

$$E = \langle x_s, x_s\rangle = \langle v, v\rangle = \|v\|^2$$

$$F = \langle x_s, x_T\rangle = \langle v, w\rangle = 0$$

$$G = \langle x_T, x_T\rangle = \langle w, w\rangle = \|w\|^2$$

Def.: The numbers E, F, G are called the coefficients of the first fundamental form I_p of the surface S with parametrization $x(u,v)$

Examples 1) Given a plane Π with parametrization

$$\Pi = \{ P + s\cdot v + t\cdot w : s, t \in \mathbb{R} \}$$

$$= \{ (P_1 + s\cdot v_1 + t\cdot w_1,\ P_2 + s\cdot v_2 + t\cdot w_2,\ P_3 + s\cdot v_3 + t\cdot w_3) : t, s \in \mathbb{R} \}$$

$v \perp w$

$v = [v_1, v_2, v_3]$

$w = [w_1, w_2, w_3]$

Parametrization of the plane

$$X(s,t) = (P_1 + s\cdot v_1 + t\cdot w_1,\ P_2 + s\cdot v_2 + t\cdot w_2,\ P_3 + s\cdot v_3 + t\cdot w_3)$$

2) Consider cylinder $C = \{(r\cos t, r\,\mathrm{sen}\,t, s) : t \in (0, 2\pi), s \in \mathbb{R}\}$

$$\underset{x(t,s)}{\parallel}$$

So $x_T = (-r\,\mathrm{sen}\,t, r\cos t, 0)$ $\quad x_s = (0,0,1)$

$$E = \langle x_T, x_T \rangle = r^2\,\mathrm{sen}^2 t + r^2 \cos^2 t = r^2$$

$$F = \langle x_T, x_s \rangle = 0$$

$$G = \langle x_s, x_s \rangle = 0 + 0 + 1 = 1$$

Applications of the First Fundamental Form

1) The length of a parath. curve on S

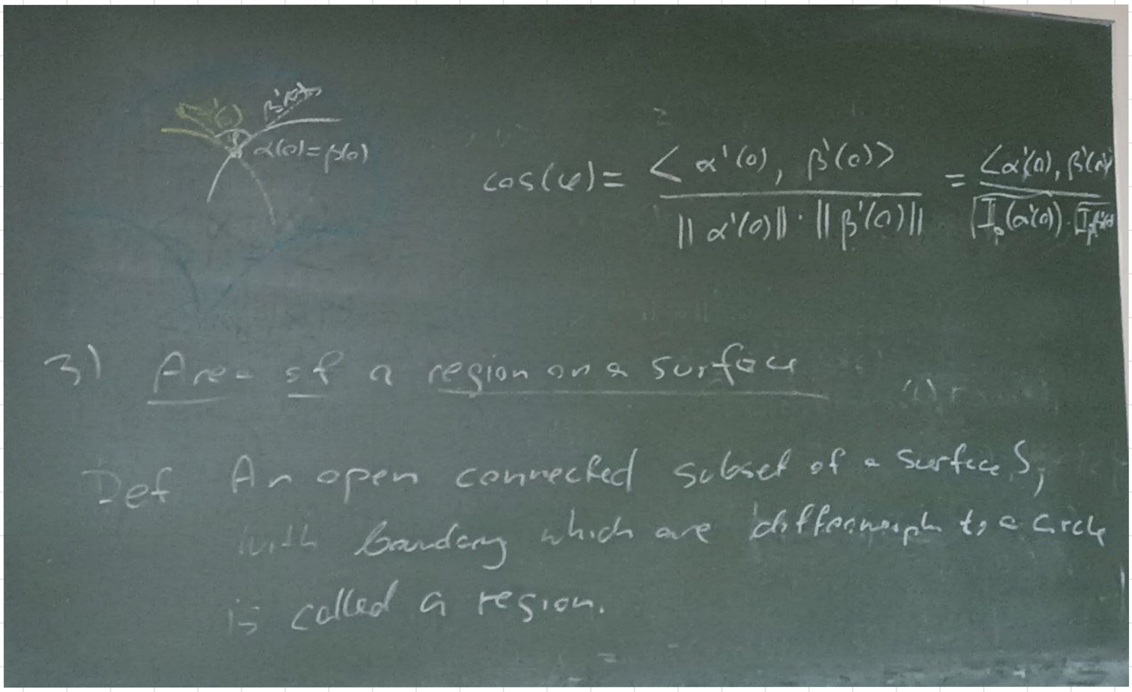
$$\alpha : (a,b) \longrightarrow S$$

$$\mathcal{L}(\alpha) = \int_a^b \|\alpha'(t)\|\, dt = \int_a^b I_p(\alpha'(t))\, dt$$

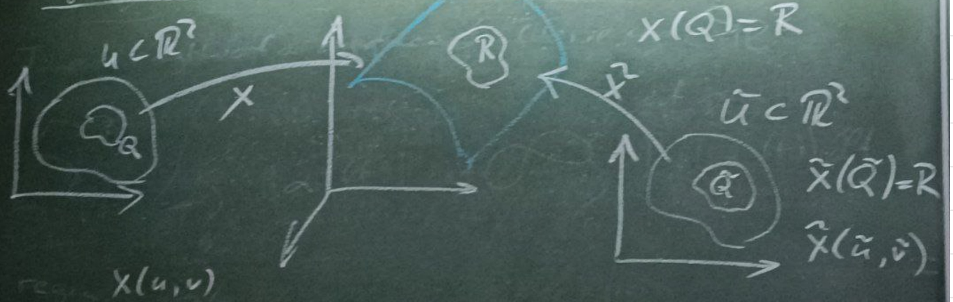
2) The angle of intersection of two curves

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow S \quad \text{and} \quad \beta : (-\varepsilon, \varepsilon) \longrightarrow S$$

$$\alpha(0) = \beta(0)$$



Applications of the First Fundamental Form



Our claim is that $\iint_Q \|X_u \times X_v\| \, du \, dv$ does not depend on parametrization.

$$\iint_Q \|X_u \times X_v\| \, du \, dv \overset{\text{calculus}}{=} \iint_Q \|X_u \times X_v\| \cdot \left| \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} \right| d\tilde{u} \, d\tilde{v} =$$

$$= \iint_{\tilde{Q}} \|\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}}\| \, d\tilde{u} \, d\tilde{v}$$

Def Let $R \subset S$ be a region in S of a regular surface S given by parametrization $X(u,v)$. Then the number

$$\text{Area}(R) = \iint_Q \|X_u \times X_v\| \, du \, dv$$

is called the area of region $R = X(Q)$.

Lemma For any two vectors $v, w \in \mathbb{R}^3$ we have

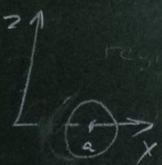
$$\|v \times w\|^2 + \langle v, w \rangle^2 = \|v\|^2 \cdot \|w\|^2$$

so $\|v \times w\| = \sqrt{\|v\|^2 \cdot \|w\|^2 - \langle v, w \rangle^2}$

Cor. So in particular taking vectors X_u, X_v partial derivatives of a parametrization $X(u,v)$ we get:

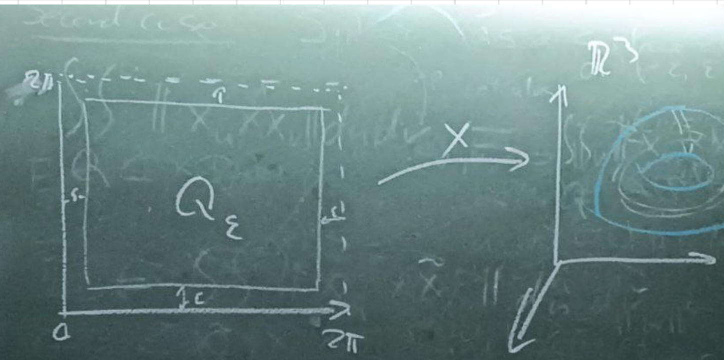
$$\|X_u \times X_v\| = \sqrt{\|X_u\|^2 \cdot \|X_v\|^2 - \langle X_u, X_v \rangle^2} = \sqrt{E \cdot G - F^2}$$

Example We intend to calculate the area of a torus obtain as a surface of revolution of a curve $\alpha(u) = (a + r\cdot\cos(u), r\sin(u))$ w.r.t. axe OZ. So we set the following parametrization



$$X(u,v) = \big((a + r\cos(u))\cdot\cos(v),\ (a + r\cos(u))\cdot\sin(v),\ r\sin(u)\big)$$

where $v, u \in (0, 2\pi)$



Choose small $\varepsilon > 0$

Coefficients of the first fundamental form.

$$\frac{dx}{du} = X_u(u,v) = (-r\,\mathrm{sen}(u)\cdot\cos(v),\ -r\sin(u)\cdot\sin(v),\ r\cos(u))$$

$$\frac{dx}{dv} = X_v(u,v) = (-(a + r\cos(u))\cdot\sin(v),\ (a + r\cos(u))\cdot\cos(v),\ 0)$$

$$E = \langle x_u, x_u\rangle = r^2\sin^2(u)\cos^2(v) + r^2\sin^2(u)\sin^2(v) + r^2\cos^2(u) = r^2\sin^2(u) + r^2\cos^2(u) = r^2 = E$$

$$F = \langle x_u, x_v\rangle = +r\,\mathrm{sen}(u)\cos(v)(a + r\cos(u))\cdot\sin(v) - (\ )$$
$$(\ ) - r\sin(v)\sin(u)(a + r\cos(u))\cdot\cos(v) + 0 = 0$$

$$G = \langle x_v, x_v\rangle = (a + r\cos(u))^2\sin^2(v) + (a + r\cos(u))^2\cos^2(v) = (a + r\cos(u))^2$$

$$\mathrm{Area}(Q_\varepsilon) = \iint_{Q_\varepsilon} \|x_u \times x_v\|\,du\,dv = \iint_{Q_\varepsilon} \sqrt{E(u,v)\cdot G(u,v) - F(u,v)^2} =$$

$$= \iint_{Q_\varepsilon} r\cdot(a + r\cos u)\,du\,dv = \int_\varepsilon^{2\pi-\varepsilon}\left(\int_\varepsilon^{2\pi-\varepsilon} ra + r^2\cos u\,du\right)dv =$$

$$= (2\pi - \varepsilon - \varepsilon) \int_{\varepsilon}^{2\pi-\varepsilon} (ra + r^2 \cos u)\, du = (2\pi - 2\varepsilon) \cdot \left[ra \cdot u + r^2 \operatorname{sen} u \right]_{\varepsilon}^{2\pi-\varepsilon} =$$

$$= 2(\pi - \varepsilon) \cdot \left[ra \cdot 2(\pi - \varepsilon) + r^2 \operatorname{sen}(2\pi - \varepsilon) - r^2 \operatorname{sen} \varepsilon \right]$$

$$\text{Area}(T^2) = \lim_{\varepsilon \to 0} \text{Area}(Q_\varepsilon) =$$

$$\lim_{\varepsilon \to 0} 2(\pi - \varepsilon)\left[ra \cdot 2(\pi - \varepsilon) + r^2 \operatorname{sen}(2\pi - \varepsilon) - r^2 \operatorname{sen} \varepsilon \right] =$$

$$= 2\pi \left[ra \cdot 2\pi + 0 - 0 \right] = 4\pi^2 ra$$

Ex. Calculate the area of the sphere of radius r and center $(0,0,0)$

$$x(t,\varphi) = (r \cdot \cos t \cdot \operatorname{sen}(\varphi), r \cdot \cos t \cdot \cos(\varphi), r \cdot \operatorname{sen} t)$$

$$x_T = (-r \operatorname{sen} t \operatorname{sen} \varphi, -r \operatorname{sen} t \cos\varphi, r \cos t)$$

$$x_\varphi = (r \cos t \cos\varphi, -r \operatorname{sen}\varphi \cos t, 0)$$

$$E = \langle x_T, x_T \rangle = r^2 \operatorname{sen}^2 t (\operatorname{sen}^2 \varphi + \cos^2 \varphi) + r^2 \cos^2 t = r^2$$

$$F = \langle x_T, x_\varphi \rangle = 0$$

$$G = r^2 \cos^2 t \cos^2 \varphi + r^2 \operatorname{sen}^2 \varphi \cos^2 t = r^2 \cos^2 t$$

$$A(S^2) = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \| x_T \times x_\varphi \| \, dt\, d\varphi = \iint \sqrt{E \cdot G - F^2} \, dt\, d\varphi =$$

$$= \int_0^{2\pi} \left(\int_{-\pi/2}^{\pi/2} r^2 |\cos t| \, dt \right) d\varphi = 2\pi r^2 \int_{-\pi/2}^{\pi/2} |\cos t| \, dt = 2\pi r^2 \int_{-\pi/2}^{\pi/2} \cos t \, dt =$$

$$= 2\pi r^2 \operatorname{sen} u \Big]_{-\pi/2}^{\pi/2} = 4\pi r^2$$

Given a surface S with parametrization $x(u,v)$

$x_u, x_v \in T_pS \qquad x_u \times x_v \perp T_pS$

Let $N(p) \overset{\text{def}}{=} \dfrac{x_u \times x_v}{\|x_u \times x_v\|}(p)$

Def. By a differentiable field of unit normal vectors on an open set $U \subset S$ we mean a differentiable map $N: U \longrightarrow \mathbb{R}^3$ which associates to each point $p \in U$ a unit normal vector

$$\frac{x_u \times x_v}{\|x_u \times x_v\|}(p)$$

Def. A regular surface $S \subset \mathbb{R}^3$ is orientable if and only if there exists a differentiable field of unit normal vectors $N: S \longrightarrow \mathbb{R}^3$.

Example: 1) Sphere 2) Cylinder

Example: The Möbius trip is not orientable

Klein bottle " "

$$x(u,v) = \left(\left(2 - v \cdot \operatorname{sen}\frac{u}{2}\right) \cdot \operatorname{sen} u, \left(2 - v \cdot \operatorname{sen}\frac{u}{2}\right) \cdot \cos u, \cos\frac{u}{2}\right)$$

is a parametrization of the Möbius strip.

Proposition. Assume that S_1 and S_2 are regular surfaces and $\varphi: S_1 \longrightarrow S_2$ is differentiable map. If S_1 is orientable then S_2 is orientable as well.

Gauss map

Consider a regular surface S with parametrization $x(u,v)$.
Assume that S is orientable. The Gauss map is defined by $N: S \longrightarrow S^2$ to a point $p \in S$ first we associate vector $N(p)$ next by translation we attach $N(p)$ to the origin of coordinate system. Since $N: S \longrightarrow S^2$ is differentiable we get $dN_p : T_pS \longrightarrow T_{N(p)}S^2 \simeq T_pS$

Self-adjoint map

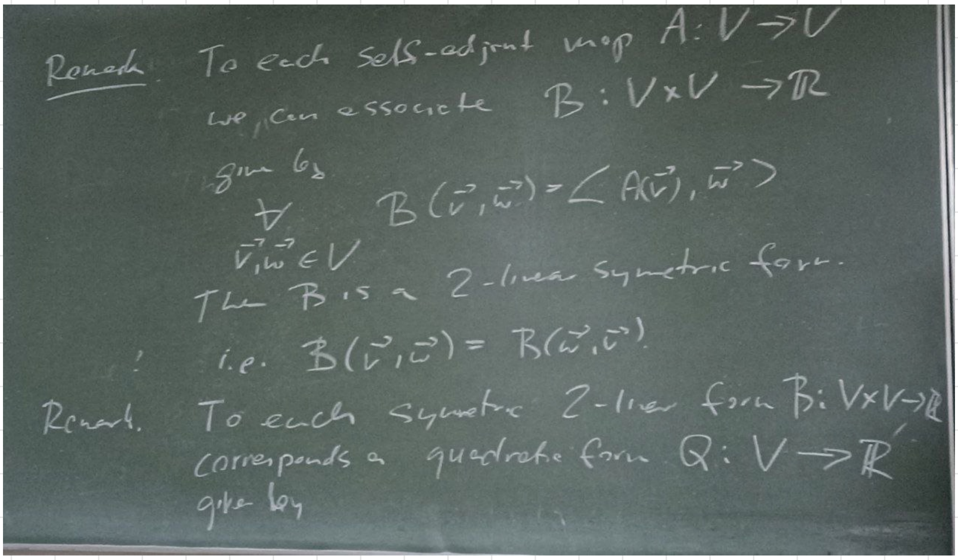
Let V be two dimensional vector space with scalar product $\langle , \rangle$.

Def. We say that a linear map $A: V \longrightarrow V$ is self-adjoint if $\forall_{\vec{v}, \vec{w} \in V} \langle A(\vec{v}), \vec{w} \rangle = \langle \vec{v}, A(\vec{w}) \rangle$

Notice that for orthonormal basis $\{\vec{e}_1, \vec{e}_2\}$ of V we can identify linear transformation $A: V \longrightarrow V$ with its matrix $M = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$

Then we get $\langle A(\vec{e}_i), \vec{e}_j \rangle = \alpha_{ij} = \langle \vec{e}_i, A(\vec{e}_j) \rangle = \alpha_{ji}$

So M is a symetric matrix



$\forall\, \vec{v} \in V \quad Q(\vec{v}) = B(\vec{v}, \vec{v}).$

Notice that Q determines B completely,

$$B(\vec{u}, \vec{v}) = \frac{1}{2}\left[Q(\vec{u}+\vec{v}) - Q(\vec{u}) - Q(\vec{v})\right].$$

Prop. For differentiable Gauss map $N: S \to S^2$ the differential $dN_p: T_pS \to T_pS$ is self-adjoint map.

Proof. Choose an orthonormal basis $\vec{w_1}, \vec{w_2}$ in T_pS. Since dN_p is a linear map, it is sufficient to prove

$$\langle dN_p(\vec{w_1}), \vec{w_2} \rangle = \langle \vec{w_1}, dN_p(\vec{w_2}) \rangle$$

Let $x(u,v)$ be a regular parametrization of S and x_u, x_v a basis of T_pS.

For a curve $\alpha(t) = x(u(t), v(t))$, where $t \in (-\varepsilon, \varepsilon)$, such that $\alpha(0) = p$ and $\alpha'(0) = x_u \cdot u' + x_v \cdot v'$,

$$dN_p(\alpha'(0)) = dN_p(x_u \cdot u' + x_v \cdot v') =$$

$$dN_p(x_u) \cdot u' + dN_p(x_v) \cdot v' = N_u \cdot u' + N_v \cdot v'$$

Now, to prove that dN_p is self-adjoint we will prove that $\langle N_u, x_v \rangle = \langle x_u, N_v \rangle$.

Notice that

$$\langle N, x_u \rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial v}\langle N, x_u \rangle = \langle N_v, x_u \rangle + \langle N, x_{uv} \rangle$$

$$\langle N, x_v \rangle = 0 \Rightarrow 0 = \frac{\partial}{\partial u}\langle N, x_v \rangle = \langle N_u, x_v \rangle + \langle N, x_{vu} \rangle$$

Since $\langle N, x_{uv} \rangle = \langle N, x_{vu} \rangle$ we get that $\langle N_v, x_u \rangle = \langle N_u, x_v \rangle$

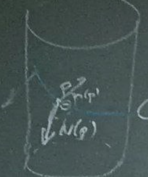
Def. The quadratic form $II_p : T_pS \to \mathbb{R}$ given by

$$\forall\, \vec{v} \in T_pS \qquad II_p(\vec{v}) = -\langle dN_p(\vec{v}), \vec{v} \rangle$$

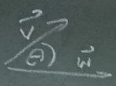
is called the second fundamental form of S at point p.

Def. Let C be a regular curve in S passing through a point $p \in S$.

Denote by $\kappa(p)$ the curvature of C at point p,
$n(p)$ the normal vector to C at point p,
$N(p)$ the normal vector to T_pS



N



$$\cos(\theta) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

Let $\cos(\theta) = \langle n, N \rangle$

The number $k_n = \kappa(p) \cdot \langle n(p), N(p) \rangle$ is called the normal curvature of the curve $C \subset S$ at point $p \in S$

Interpretation of II_p

Let C be a curve with unit speed parametrization $\alpha(s)$ such that $\alpha(0) = p$

Denote by $N(s)$ the restriction of unit normal field N to $\alpha(s)$

Since $N(s) \perp \alpha'(s)$ we get

$$\langle N(s), \alpha'(s) \rangle = 0 \Rightarrow \langle N(s), \alpha''(s) \rangle + \langle N'(s), \alpha'(s) \rangle = 0$$

$$\Rightarrow \langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle \quad (*)$$

$$II_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle \overset{(*)}{=}$$

$$= \langle N(0), \underbrace{\alpha''(0)}_{\kappa(p) \cdot n(p)} \rangle = \langle N(p), \kappa(p) \cdot n(p) \rangle =$$

$$= \kappa(p) \langle N(p), n(p) \rangle =$$

$$= k_n = \text{normal curvature}$$

α

$= k_n$

Theorem Any two curves α, β such that tangent line to α and to β at point p is the same, have the same normal curvature.

S – surface with parametrization $X(u,v)$

$N(p)$ = normal vector field $= \dfrac{X_v \times X_u}{\| X_v \times X_u \|}(p)$ $\quad \{X_u, X_v\}$ is a basis T_pS

Consider a curve on S, with parametrization

$$\alpha(t) = X(u(t), v(t)) \quad \text{s.t. } \alpha(0) = p$$

$$\alpha'(0) = X_u u'(0) + X_v \cdot v'(0)$$

$$dN_p(\alpha'(0)) = dN_p(X_u \cdot u'(0) + X_v \cdot v'(0)) = N_u \cdot u'(0) + N_v \cdot v'(0)$$

Lemma $N_u(p), N_v(p) \in T_pS$

Proof Since $\|N(p)\| = 1$, we get $\langle N, N\rangle = 1$, so

$$0 = \frac{\partial}{\partial u}\langle N, N\rangle = \langle N_u, N\rangle + \langle N, N_u\rangle = 2\langle N_u, N\rangle \Rightarrow N_u \perp N \Rightarrow N_u \in T_pS$$

$$0 = \frac{\partial}{\partial v}\langle N, N\rangle = 2\langle N_v, N\rangle \Rightarrow N_v \perp N \Rightarrow N_v \in T_pS.$$

Cor.
$$\begin{cases} N_u = a_{11} X_u + a_{21} X_v \\ N_v = a_{12} X_u + a_{22} \cdot X_v \end{cases}$$

$$dN_p(\alpha'(0)) = N_u \cdot u'(0) + N_v \cdot v'(0) = (a_{11} X_u + a_{21} X_v) \cdot u'(0) + (a_{12} X_u + a_{22} X_v) \cdot v'(0) =$$

$$= (a_{11} \cdot u'(0) + a_{12} \cdot v'(0)) X_u + (a_{21} \cdot u'(0) + a_{22} \cdot v'(0)) \cdot X_v$$

Therefore the matrix of linear transformation dN_p, in basis $\{X_u, X_v\}$ is

$$dN_p\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

Ex. We have a unit sphere S^2 with center at $(0,0,0)$ and a cylinder given by

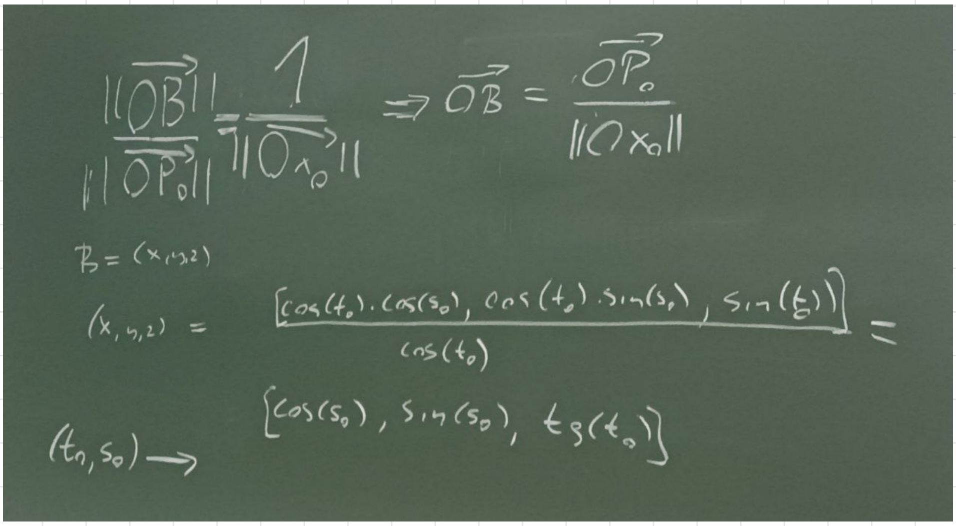
$$C = \{(\operatorname{sen} t, \cos t, s),\ s \in \mathbb{R},\ t \in (0, 2\pi)\}$$

For fixed $p \in S^2$ we draw a half line ℓ_p such that $p, (0,0,0) \in \ell_p$. The intersection point $\ell_p \cap C = p'$. Write the analitic formula for the transformation $f: S^2 \setminus \{N, S\} \longrightarrow C$ $f(p) = p'$

We assume that a sphere is given as a surface of revolution so its parametrization is:

$$x(u,v) = (\cos u \cdot \cos v,\ \cos u \cdot \operatorname{sen} v,\ \operatorname{sen} u)$$

$$\operatorname{dist}(0, x_0) = \sqrt{\cos^2 t_0 \cos^2 s_0 + \cos^2 t_0 \operatorname{sen}^2 s_0} = \cos t_0$$



Ex. Show that the area of a bounded region R of the surface given by parametrization

$$\tilde{x}(x,y) = (x, y, f(x,y))$$

where $f: Q \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable map and Q is the normal projection of R onto xy plane. Area is given by $\iint_Q \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$

$$\tilde{x}(x,y) = (x, y, f(x,y))$$
$$\tilde{x}_x = (1, 0, f_x(x,y))$$
$$\tilde{x}_y = (0, 1, f_y(x,y))$$

$$E = \langle \tilde{x}_x, \tilde{x}_x \rangle = 1 + f_x^2(x,y)$$
$$G = \langle \tilde{x}_y, \tilde{x}_y \rangle = 1 + f_y^2(x,y)$$

$$F = \langle \tilde{x}_x, \tilde{x}_y \rangle = f_x(x,y) \cdot f_y(x,y)$$

$$\text{Area}(R) = \iint_Q \sqrt{E \cdot G - F^2}\, dx\, dy$$

Ex. Calculate the area of the plane given $x(u,v) = (u, v, a \cdot u + b \cdot v)$, where $a, b \in \mathbb{R}$ over $Q = [-1,1] \times [-1,1]$

orthogonal projection of R is equal to Q

$$f(u,v) = a \cdot u + b \cdot v$$
$$f_u = a \qquad f_v = b$$

$$\text{Area}(R) = \iint_Q \sqrt{1 + f_u^2 + f_v^2}\, du\, dv = \iint_Q \sqrt{1 + a^2 + b^2}\, du\, dv =$$

$$= \int_{-1}^{1} \left(\int_{-1}^{1} \sqrt{1 + a^2 + b^2}\, du \right) dv = 4 \cdot \sqrt{1 + a^2 + b^2}$$