PROBLEMS

EX 1: Let X_1 and X_2 be two independent r.v. such that $EX_1 = EX_2 = m$. The value of m is unknown. Moreover, we know $\text{Var}\, X_1 = \sigma_1^2 > 0$ and $\text{Var}\, X_2 = \sigma_2^2 > 0$. Find numbers $a_1, a_2, b \in \mathbb{R}$ s.t. the r.v. $\tilde{X} = a_1 X_1 + a_2 X_2 + b$ satisfies:

1. $E\tilde{X} = m$
2. $\text{Var}\, \tilde{X}$ is as small as possible.

$$m = E\tilde{X} = E(a_1X_1 + a_2X_2 + b) = a_1 EX_1 + a_2 EX_2 + Eb = a_1 m + a_2 m + b;$$

$$; m = (a_1 + a_2)m + b; \quad (a_1 + a_2 - 1)m = -b. \quad \forall m \in \mathbb{R}$$

it does not depend on m.

Linear function of m = constant $\Rightarrow$ coeficient must be 0 $\Rightarrow$

$$\Rightarrow \begin{cases} a_1 + a_2 - 1 = 0; \quad \underline{a_1 + a_2 = 1} \\ \underline{b = 0} \end{cases}$$

We have $\tilde{X} = a_1X_1 + a_2X_2$ with $a_1 + a_2 = 1$.

Now we want $\text{Var}\, \tilde{X} \to \min$.

$$\text{Var}\, \tilde{X} = \text{Var}(a_1X_1 + a_2X_2 + b) \overset{\text{indep}}{=} a_1^2 \text{Var}\, X_1 + a_2^2 \text{Var}\, X_2 + \cancel{\text{Var}\, b}^{\,0};$$

$$; \text{Var}\, \tilde{X} = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 \quad *$$

Knowing that $a_1 + a_2 = 1$, we want to minimize $*$

$$a_1 + a_2 = 1; \quad a_2 = 1 - a_1 \to \text{Var}\, \tilde{X} = a_1^2\sigma_1^2 + (1-a_1)^2\sigma_2^2 =$$

$$= (\sigma_1^2 + \sigma_2^2)a_1^2 - 2\sigma_2^2 a_1 + \sigma_2^2$$ // $ax^2 + bx + c \to \min$ for $x = \frac{-b}{2a}$

The minimum is obtained for $a_1 = \dfrac{\cancel{2}\sigma_2^2}{\cancel{2}(\sigma_1^2 + \sigma_2^2)} = \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = a_1$

Thus, $a_2 = 1 - \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \dfrac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = a_2$

And, $b = 0$, as we knew.

Finally:

$$\text{Var}\, \tilde{X} = \frac{\sigma_1^2\sigma_2^4}{(\sigma_1^2 + \sigma_2^2)^2} + \frac{\sigma_1^4\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{\sigma_1^2\sigma_2^2\cancel{(\sigma_1^2 + \sigma_2^2)}}{(\sigma_1^2 + \sigma_2^2)^{\cancel{2}}} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Let's generalize the obtained result:

We have n independent r.v. $X_1, \dots, X_n$, satisfying $EX_1 = \cdots = EX_n = m$ (unknown). We also know $\operatorname{Var} X_i = \sigma_i^2 > 0$ for $i = 1, \dots, n$. Consider $\tilde{X} = a_1 X_1 + \dots + a_n X_n + b = \sum_{i=1}^{n} a_i X_i + b$.

Find $a_1, \dots, a_n, b \in \mathbb{R}$, s.t. $E\tilde{X} = m$ and $\operatorname{Var}\tilde{X} \to \min$.

This can be solved in a similar way. But first, let's consider:

$$a_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{1/\sigma_1^2}{1/\sigma_1^2 + 1/\sigma_2^2}$$

$$a_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \frac{1/\sigma_2^2}{1/\sigma_1^2 + 1/\sigma_2^2}$$

$$\operatorname{Var}\tilde{X} = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{1}{1/\sigma_1^2 + 1/\sigma_2^2}$$

These are much easier to generalize.

Computating, we would obtain:

→ $b = 0$

→ $$a_i = \frac{1/\sigma_i^2}{\sum_{j=1}^{n} 1/\sigma_j^2} = \frac{1/\operatorname{Var} X_i}{\sum_{j=1}^{n} 1/\operatorname{Var} X_j}$$

$\tilde{X}$ is the weighted mean of $X_1, \dots, X_n$ with the weights proportional to reciprocals of variances.

Noting $\alpha = \sum_{j=1}^{n} 1/\operatorname{Var} X_j$, we have that $a_i = 1/\alpha \cdot 1/\operatorname{Var} X_i$

Moreover, $\operatorname{Var}\tilde{X} = \frac{1}{\sum_{j=1}^{n} \operatorname{Var} X_j}$ and $\frac{1}{\operatorname{Var}\tilde{X}} = \sum_{j=1}^{n} 1/\operatorname{Var} X_j$.

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Recall: Let $X_1, \dots, X_n$ be independent r.v. Assume $X_1, \dots, X_n$ have the same distribution with $EX_i = m$ $\forall i = 1, \dots, n$ and $\operatorname{Var} X_i = \sigma_i^2$ $\forall i = 1, \dots, n$.

We want to know m and σ^2.

The unbiased estimation of m is $\underbrace{\hat{m}}_{\text{this is a r.v}} = \bar{X} = 1/n \cdot \sum_{i=1}^{n} X_i$

$E\bar{X} = m$

$ES^2 = \sigma^2$

$E\tilde{S}^2 = \sigma^2$

The unbiased estimations of σ^2 are:

If we don't know m: $\quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad n>0$

If we know m: $\quad \tilde{S}^2 = 1/n \sum_{i=1}^{n} (X_i - m)^2$

$* \operatorname{Var} X = E((X-EX)^2) = E((X-m)^2)$

EX 2: Let $X_1, \dots, X_n$ be independent r.v. s.t. $EX_i = m$ $\forall i = 1, \dots, n$ (m is unknown), and $\operatorname{Var} X_i = \frac{\sigma^2}{w_i}$ where $\sigma^2 > 0$ is unknown and $w_1, \dots, w_n > 0$ are known.

Give examples of "good", unbiased estimations of m and σ^2.

We still start with m. We can try to use the answer to the previous problem:

$$\hat{m} = \tilde{X} = a_1 X_1 + \dots + a_n X_n + b$$

$$a_i = \frac{\frac{1}{\operatorname{Var} X_i}}{\sum_{j=1}^{n} 1/\operatorname{Var} X_j}$$

Unfortunately, we don't know $\operatorname{Var} X_i \overset{!!}{=} \frac{\sigma^2}{w_i}$

$$\Vert$$

$$a_i = \frac{w_i/\sigma^2}{\sum_{j=1}^{n} w_j/\sigma^2} = \frac{w_i}{\sum_{j=1}^{n} w_j}$$

Luckily, it does not depend on σ^2

It follows that $\hat{m} = \tilde{X} = \frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i}$ is the best (variance is minimal), linear ($\tilde{X} = \sum a_i X_i + b$) and unbiased ($E\hat{m} = m$) estimation of m.

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$$\operatorname{Var}\tilde{X} = \frac{1}{1/\sigma_i^2 + \dots}$$

Remark: If $\operatorname{Var}X_1 = \dots = \operatorname{Var}X_n = \sigma^2$ ($\Leftrightarrow w_1 = \dots = w_n = 1$), then we have:

$$\hat{m} = \tilde{X} = \frac{\sum_{i=1}^n 1 \cdot X_i}{\sum_{i=1}^n 1} = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}$$

Let's try to estimate σ^2. We want "to mimic" the formula

when the variances are the same

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2 \approx \sigma^2 = \operatorname{Var} X_i$$

Let's try to find the constant c such that:

when the variances are different.

$$\hat{\sigma}^2 = c \cdot \sum_{i=1}^n w_i (X_i - \tilde{X})^2 \approx \frac{\sigma^2}{w_i} = \operatorname{Var} X_i$$

is an unbiased estimator of σ^2.

$$\text{We want } E\hat{\sigma}^2 = \sigma^2 \ (\hat{\sigma}^2 \text{ unbiased}) \Leftrightarrow c \cdot \overbrace{E\left(\sum_{i=1}^n w_i (X_i - \tilde{X})^2\right)}^{\text{what's this?}} = \sigma^2$$

$$E\left(\sum_{i=1}^n w_i (X_i - \tilde{X})^2\right) = \sum_{i=1}^n w_i\, E(X_i - \tilde{X})^2$$

$*\ \operatorname{Var} X = E(X^2) - (EX)^2 \Rightarrow E(X^2) = \operatorname{Var} X + (EX)^2$

$$E(X_i - \tilde{X})^2 = \operatorname{Var}(X_i - \tilde{X}) + (E(X_i - \tilde{X}))^2 = (**)$$

$$E(X_i - \tilde{X}) = EX_i - E\tilde{X} = m - m = 0$$

Unfortunately, X_i and $\tilde{X}$ are not independent, so we shall use:

$$\operatorname{Var}(X \pm Y) = \operatorname{Var} X + \operatorname{Var} Y \pm 2\operatorname{Cov}(X, Y)$$

$$\operatorname{Var}(X_i - \tilde{X}) = \operatorname{Var} X_i + \operatorname{Var}\tilde{X} - 2\operatorname{Cov}(X_i, \tilde{X}) =$$

$$= \frac{\sigma^2}{w_i} + \frac{\sigma^2}{\sum_j w_j} - 2\frac{\sigma^2}{\sum_j w_j} = \frac{\sigma^2}{w_i} - \frac{\sigma^2}{\sum_j w_j}$$

$$\operatorname{Cov}(X_i, \tilde{X}) = \operatorname{Cov}\left(X_i, \frac{\sum w_j X_j}{\sum w_j}\right) = \frac{\sum w_j \boxed{\operatorname{Cov}(X_i, X_j)}}{\sum w_j} =$$

$j = i \to \operatorname{Var} X_i$

$j \neq i \to X_i, X_j$ are independent $\to \operatorname{Cov} = 0$.

$$= \frac{w_i \operatorname{Var} X_i}{\sum w_j} = \frac{w_i \cdot \sigma^2 / w_i}{\sum w_j} = \frac{\sigma^2}{\sum w_j}$$

x_j independent

$$\Rightarrow \operatorname{Var} \tilde{x} = \operatorname{Var} \frac{\sum w_j x_j}{\sum w_j} = \left(\frac{1}{\sum w_j}\right)^2 \operatorname{Var}\left(\sum w_j x_j\right) =$$

$$= 1/(\sum w_j)^2 \cdot \sum \operatorname{Var}(w_j x_j) = \frac{\sum w_j^2 \operatorname{Var} x_j}{(\sum w_j)^2} = \frac{\sum w_j^{\cancel{2}} \frac{\sigma^2}{\cancel{w_j}}}{(\sum w_j)^2} =$$

$$= \frac{\sigma^2 \cancel{\sum w_j}}{(\sum w_j)^{\cancel{2}}} = \frac{\sigma^2}{\sum w_j}$$

Finally:

$$E\left(\sum_{i=1}^{n} w_i (x_i - \tilde{x})^2\right) = \sum w_i E\left((x_i - \tilde{x})^2\right) = \sum w_i \left(\frac{\sigma^2}{w_i} - \frac{\sigma^2}{\sum w_j}\right) =$$

$$= \sum \sigma^2 - \sum \frac{w_i \sigma^2}{\sum w_j} = n\sigma^2 - \frac{\sigma^2 \cancel{\sum w_i}}{\cancel{\sum w_j}} = n\sigma^2 - \sigma^2$$

So: $c \cdot E\left(\sum_{i=1}^{n} w_i (x_i - \tilde{x})^2\right) = c(n-1)\sigma^2 = \sigma^2 \Rightarrow c = \frac{1}{n-1}$

Therefore, the unbiased estimation of σ^2 is

$$\hat{\sigma^2} = \frac{1}{n-1} \sum_{i=1}^{n} w_i (x_i - \tilde{x})^2$$

Remark: If $\operatorname{Var} x_1 = \dots = \operatorname{Var} x_n = \sigma^2$ ($\equiv w_1 = \dots = w_n = 1$), then we obtain:

$$\hat{\sigma}_2 = \frac{1}{n-1} \sum 1 \cdot (x_i - \bar{x})^2 = s^2$$

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Ex: Assume that $X, Y \sim \mathcal{N}(0,1)$ are independent. Find $a, b \in \mathbb{R}$ such that $X+2Y+3$ and $X+aY+b$ are indep. r.v.

Let's try to see that $(X+2Y+3, X+aY+b)^T$ has 2-dim normal distribution.

We know that X, Y have normal dist. and are independent.

$\Downarrow$

$(X,Y)^T$ has 2-dim normal dist.

$\Downarrow$

$$\begin{pmatrix} X+2Y+3 \\ X+aY+b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & a \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 3 \\ b \end{pmatrix}}_{\text{affine transformation of } (X,Y)^T}$$ also has 2-dim normal dist.

It follows that:

$X+2Y+3$, $X+aY+b$ are independent $\Leftrightarrow$ $\mathrm{Cov}(X+2Y+3, X+aY+b) = 0$.

Which is something easier to prove. Let's do it:

$$\mathrm{Cov}(X+2Y+\underbrace{3}_{\text{const}}, X+aY+\underbrace{b}_{\text{const}}) = \mathrm{Cov}(X+2Y, X+aY) = \mathrm{Cov}(X, X+aY) + \mathrm{Cov}(2Y, X+aY) =$$

$$= \mathrm{Cov}(X,X) + \mathrm{Cov}(X, aY) + \mathrm{Cov}(2Y, X) + \mathrm{Cov}(2Y, aY) =$$

$$= \mathrm{Var}(X) + a\,\mathrm{Cov}(X,Y) + 2\,\mathrm{Cov}(Y,X) + 2a\,\mathrm{Cov}(Y,Y) =$$

$$= \mathrm{Var}\,X + (2+a)\overbrace{\mathrm{Cov}(X,Y)}^{=0} + 2a\,\mathrm{Var}\,Y = 1 + 2a$$

Therefore:

$\mathrm{Cov}(X+2Y+3, X+aY+b) = 0 \Leftrightarrow 1+2a = 0 \Leftrightarrow a = -1/2$

Finally, $X+2Y+3$, $X+aY+b$ are independent if and only if $a = -1/2$ and $b \in \mathbb{R}$.

EX:

Consider $(X,Y,Z)^T \sim \mathcal{N}\left(\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}2 & -1 & 0\\ -1 & 4 & 0\\ 0 & 0 & 3\end{pmatrix}\right)$

Are there constants $a,b \in \mathbb{R}$ s.t. the r. variable $X+Y+aZ+3$ is independent of the r. vector $(X+2Z+4, X-bY+Z)$? If so, determine these constants.

First, we will try to show that $\begin{pmatrix} X+Y+aZ+3 \\ X+2Z+4 \\ X-bY+Z \end{pmatrix}$ has normal dist.

$$\begin{pmatrix} X+Y+aZ+3 \\ X+2Z+4 \\ X-bY+Z \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 & a \\ 1 & 0 & 2 \\ 1 & -b & 1 \end{pmatrix}\begin{pmatrix} X\\Y\\Z\end{pmatrix} + \begin{pmatrix}3\\4\\0\end{pmatrix}}^{\text{affine transformation of } (X,Y,Z)}$$

has a normal dist, since $(X,Y,Z)^T$ has it too.

It follows that $X+Y+aZ+3$ is independent of $\begin{pmatrix} X+2Z+4 \\ X-bY+Z\end{pmatrix}$ if, and only if,

$$\mathrm{Cov}(X+Y+aZ+3,\ X+2Z+4) = 0$$
$$\mathrm{Cov}(X+Y+aZ+3,\ X-bY+Z) = 0$$

Let's compute it:

•) $\mathrm{Cov}(X+Y+aZ+3,\ X+2Z+4) = \mathrm{Cov}(X+Y+aZ,\ X+2Z) =$

$$= \overset{=2}{\mathrm{Var}\,X} + 2\overset{=0}{\mathrm{Cov}(X,Z)} + \overset{=-1}{\mathrm{Cov}(Y,X)} + 2\overset{=0}{\mathrm{Cov}(Y,Z)} + a\overset{=0}{\mathrm{Cov}(Z,X)} + 2a\overset{=3}{\mathrm{Var}\,Z} =$$

$$= 2 - 1 + 6a = 1 + 6a = 0 \iff \underline{a = -1/6}$$

$\mathrm{Var}\begin{pmatrix}X\\Y\\Z\end{pmatrix} = \begin{pmatrix}2 & -1 & 0\\ -1 & 4 & 0\\ 0 & 0 & 3\end{pmatrix}$

•) $\mathrm{Cov}(X+Y+aZ+3,\ X-bY+Z) = \overset{=2}{\mathrm{Var}\,X} - b\overset{=-1}{\mathrm{Cov}(X,Y)} + \overset{=0}{\mathrm{Cov}(X,Z)} + \ldots$

$$\ldots + \overset{=-1}{\mathrm{Cov}(Y,X)} - b\overset{=4}{\mathrm{Var}\,Y} + \overset{=0}{\mathrm{Cov}(Y,Z)} + a\overset{=0}{\mathrm{Cov}(Z,X)} - ab\overset{=0}{\mathrm{Cov}(Z,Y)} + a\overset{=3}{\mathrm{Var}(Z)} =$$

$$= 2 + b - 1 - 4b + 3a = 1 - 3b - 1/2 = 1/2 - 3b = 0 \iff \underline{b = 1/6}$$

Therefore, the random variable $X+Y-aZ+3$ is independent of $(X+2Z+4,\ X-bY+Z)$ if, and only if, $a = -1/6$, $b = 1/6$

EX: Assume that $\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\right)$

Compute $\mathrm{cov}(X,Y)$, $E(XY)$, $\mathrm{cov}(X^2,Y^2)$

•) $\mathrm{cov}(X,Y) = -3$

•) $\mathrm{cov}(X,Y) = E(XY) - EX \cdot EY \rightarrow E(XY) = \underbrace{\mathrm{cov}(X,Y)}_{=-3} + \underbrace{EX}_{=0}\underbrace{EY}_{=0} = -3$

•) $\mathrm{cov}(X^2,Y^2) = E(X^2Y^2) - EX^2 EY^2 \overset{*}{=} E(X^2Y^2) - 10$ (**)

*) $\begin{cases} EX^2 = \mathrm{Var}\,X + (EX)^2 = 2 + 0 \\ EY^2 = \mathrm{Var}\,Y + (EY)^2 = 5 + 0 \end{cases}$

**) $\mathrm{Var}(XY) = E((XY)^2) - (\underbrace{E(XY)}_{=-3})^2 = E(X^2Y^2) - 9$:(

$E(X^2Y^2) = ¿?$

Let's try to standarize this random vector. Let's find a 2x2 real matrix, L, and vector b, such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim L\begin{pmatrix} U \\ V \end{pmatrix} + b, \text{ where } U,V \sim \mathcal{N}(0,1) \text{ independent.}$$

por teoria

↳ We have $b = E\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} \sim L\begin{pmatrix} U \\ V \end{pmatrix}$. Also $\mathrm{Var}\begin{pmatrix} X \\ Y \end{pmatrix} = LL^T$

We don't need to find all such matrices L. It is enough to find one of them.

Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We want to have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

$\begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$

We have: $\begin{cases} a^2+b^2 = 2 \\ ac+bd = -3 \\ c^2+d^2 = 5 \end{cases} \xrightarrow{\text{possible solution}} \begin{cases} a = b = -1 \\ c = 1 \\ d = 2 \end{cases} \Rightarrow L = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$

Then,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} -U-V \\ U+2V \end{pmatrix},$$

where $U,V \sim \mathcal{N}(0,1)$ are independent.

$E(X^2Y^2) = E\left((-U-V)^2 (U+2V)^2\right) = E\left((U^2+2UV+V^2)(U^2+4UV+4V^2)\right) =$

$= E(U^4 + 4V^4 + 13U^2V^2 + 12UV^3 + 6U^3V) =$ * $E(U)=0$, $E(V)=0$ bc $U,V \sim \mathcal{N}(0,1)$

U,V indep.

$= EU^4 + 4EV^4 + 13\underbrace{E(U^2)}_{E(U^2) = Var\, U + (EU)^2 = 1+0=1} E(V^2) + 12\overbrace{E(U)}^{=0}E(V^3) + 6E(U^3)\overbrace{E(V)}^{=0} =$

$= EU^4 + 4EV^4 + 13.$

Assume that $U \sim \mathcal{N}(0,1)$. Then:

$$E(U^k) = \begin{cases} 0 & \text{when } k \text{ is odd} \\ \dfrac{k!}{(k/2)! \cdot 2^{k/2}} & \text{when } k \text{ is even} \quad (1\cdot 3\cdot 5 \cdot \cdots \cdot k-1) \end{cases}$$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$E((X-\mu)^k) = \begin{cases} 0 & k \text{ is odd} \\ 1\cdot 3 \cdot \cdots \cdot (k-1)\cdot \sigma^k & k \text{ is even} \end{cases}$$

Then, $E(U^4) = 1\cdot 3 = 3 = E(V^4) \Rightarrow E(X^2Y^2) = 3 + 12 + 13 = 28$

Therefore: (*) $Cov(X^2, Y^2) = 28 - 10 = 18$ //

$Var(XY) = 28 - 9 = 19$ //

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EX: Let $X_1, \dots, X_{20}$ be independent random variables such that $\text{Var}(X_i) = \sigma^2$ for $i = 1, \dots, 20$ and $E(X_i) = \mu_1$ for $i = 1, \dots, 10$

$E(X_i) = \mu_2$ for $i = 11, \dots, 20$

Let $\bar{X}_1 = 1/10 \sum_{i=1}^{10} X_i$, $\bar{X}_2 = 1/10 \sum_{i=11}^{20} X_i$, $\bar{X} = 1/20 \sum_{i=1}^{20} X_i$

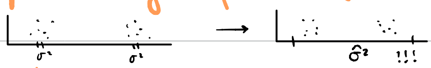
Find numbers α, β * such that

$$\hat{\sigma}^2 = \frac{1}{20-1} \sum_{i=1}^{20} (X_i - \bar{X})^2$$ is a classical unbiased estimation of σ^2

(classical estimation) ← but it has a problem.

problem: σ^2 very small but μ_1, μ_2 very apart from !!

the estimator will grow !!



solution: substract distance between means

$$\hat{\sigma}_2 = \frac{1}{20-1} \sum_{i=1}^{20} (X_i - \bar{X})^2 - (\mu_1 - \mu_2)^2$$

problem: μ_1, μ_2 are unknown

solution: we use estimators

$$\hat{\sigma}_2 = \frac{1}{20-1} \sum_{i=1}^{20} (X_i - \bar{X})^2 - (\bar{X}_1 - \bar{X}_2)^2$$

problem: how do we know we have to substract that exact amount?

$$\hat{\sigma}_2 = \alpha \sum_{i=1}^{20} (X_i - \bar{X})^2 + \beta (\bar{X}_1 - \bar{X}_2)^2 \; *$$ ← solution: we calibrate.

is an unbiased estimation of σ^2

We need to find α, β such that

$$\sigma^2 = E\hat{\sigma}_2 = \alpha \cdot E \sum_{i=1}^{20} (X_i - \bar{X})^2 + \beta E((\bar{X}_1 - \bar{X}_2)^2) \qquad \forall \mu_1, \mu_2, \sigma^2$$

$$E((\bar{X}_1 - \bar{X}_2)^2) = \text{Var}(\bar{X}_1 - \bar{X}_2) + (E(\bar{X}_1 - \bar{X}_2))^2 =$$

$$= \text{Var}(\bar{X}_1 - \bar{X}_2) + (E\bar{X}_1 - E\bar{X}_2)^2 =$$

independent because X_i are

$$= \text{Var}(\bar{X}_1 - \bar{X}_2) + (\mu_1 - \mu_2)^2 =$$

$\downarrow$ $\text{Var}(\bar{X}_1) - \underbrace{2\text{Cov}(\bar{X}_1, \bar{X}_2)}_{0} + \text{Var}(\bar{X}_2)$

$\text{Var}(1/10 \sum_{i=1}^{10} X_i) \rightarrow$

$$= \text{Var}\, \bar{X}_1 + \text{Var}\, \bar{X}_2 + (\mu_1 - \mu_2)^2 =$$

independent / independent.

$$= 1/100\, \text{Var}\left(\sum_{i=1}^{10} X_i\right) + 1/100\, \text{Var}\left(\sum_{i=11}^{20} X_i\right) + (\mu_1 - \mu_2)^2 =$$

$$= 1/100 \sum_{i=1}^{20} \underbrace{\text{Var}(X_i)}_{=\sigma^2} + (\mu_1 - \mu_2) = \sigma^2/5 + (\mu_1 - \mu_2)^2$$

$$E\left(\left(\sum_{i=1}^{20}(x_i-\bar{x})^2\right)\right) = \sum_{i=1}^{20} E((x_i-\bar{x})^2) = 10E((x_1-\bar{x})^2) + 10E((x_{11}-\bar{x})^2) =$$

$$= 10\left[Var(x_1-\bar{x}) + Var(x_{11}-\bar{x})\right] + 10\left[\underbrace{E(x_1-\bar{x})}_{=})^2 + (\overbrace{E(x_{11}-\bar{x})}^{=\frac{\mu_2-\mu_1}{2}})^2\right] =$$

$$E X_1 - E\bar{X} = \mu_1 - E\,\frac{\bar{X}_1+\bar{X}_2}{2} = \mu_1 - \frac{\mu_1+\mu_2}{2} = \frac{\mu_1-\mu_2}{2}$$

$$= 10\left[\overbrace{\underbrace{Var(X_1-\bar{X})}_{= \text{ see } 1^\circ}}^{= \text{ see } 2^\circ:\ \sigma^2 - 2\frac{\sigma^2}{20} + \frac{\sigma^2}{20} = \frac{19}{20}\sigma^2} + Var(x_{11}-\bar{x}) + 2\left(\frac{\mu_1-\mu_2}{2}\right)^2\right] = 10\left[2\,\frac{19}{20}\sigma^2 + \frac{(\mu_1-\mu_2)^2}{2}\right] =$$

$$Var\,X_1 - 2\,Cov(X_1,\bar{X}) + Var\,\bar{X}$$

$$Var\left(\tfrac{1}{20}\sum_{i=1}^{20} x_i\right) = \frac{1}{400}\sum_{i=1}^{20} Var(x_i) = \frac{\sigma^2}{20}$$

$$Cov\left(X_1, \tfrac{1}{20}\sum_{i=1}^{20} x_i\right) = \frac{1}{20}\sum_{i=1}^{20} Cov(X_1,X_i) = \tfrac{1}{20}\,Var\,X_1 = \frac{\sigma^2}{20}$$

$$= 19\sigma^2 + 5(\mu_1-\mu_2)^2$$

We are looking for $\alpha, \beta \in \mathbb{R}$ such that:

$$\sigma^2 = \alpha\left(19\sigma^2 + 5(\mu_1-\mu_2)^2\right) + \beta\left(\sigma^2/5 + (\mu_1-\mu_2)^2\right) \quad \forall \mu_1,\mu_2,\sigma^2;$$

$$;\ \sigma^2 = \sigma^2\,(19\alpha + \beta/5) + (\mu_1-\mu_2)^2\,(5\alpha+\beta) \Rightarrow$$

$$\Rightarrow \begin{cases} 19\alpha + \beta/5 = 1 & ;\ 19\alpha - \alpha = 1;\ \alpha = 1/18 \\ 5\alpha + \beta = 0 & ;\ \beta = -5\alpha;\ \beta = -5/18 \end{cases}$$

$\beta < 0$, as expected

Finally:
$$\hat{\sigma}^2 = \tfrac{1}{18}\sum_{i=1}^{20}(x_i-\bar{x})^2 - \tfrac{5}{18}(\bar{x}_1-\bar{x}_2)^2 =$$

$$= \tfrac{1}{2}\left[\tfrac{1}{9}\sum_{i=1}^{10}(x_i-\bar{x}_1)^2 + \tfrac{1}{9}\sum_{i=11}^{20}(x_i-\bar{x}_2)^2\right]$$

EX: Let $X_1, X_2 \sim \mathcal{U}nif(0,1)$ independent.

Compute $\mu = E|X_1 - X_2|$ and $\sigma^2 = Var|X_1 - X_2|$

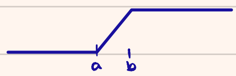
A) $\mu = 1/3$ $\quad \sigma^2 = 1/36$
B) $\mu = 1/2$ $\quad \sigma^2 = 1/12$
C) $\mu = 1/2$ $\quad \sigma^2 = 1/24$
D) $\mu = 1/3$ $\quad \sigma^2 = 1/18$
E) $\mu = 1/3$ $\quad \sigma^2 = 1/6$

Reminder

$\mathcal{U}nif(a,b)$ dist with $a<b$

density: $f(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & x \notin (a,b) \end{cases}$

c.d.f: $F(x) = P(X \leq x)$



$X \sim \mathcal{U}nif(a,b)$ $\quad EX = \frac{a+b}{2}$
$\quad Var\, X = \frac{(b-a)^2}{12}$

If we didn't have the abs. value, it would be so easy:

$E(X_1 - X_2) = EX_1 - EX_2 = 1/2 - 1/2 = 0.$

But we do. So...

$\sigma^2 = Var|X_1 - X_2| = E(|X_1-X_2|^2) - (\underbrace{E|X_1 - X_2|}_{\mu})^2 = E((X_1-X_2)^2) - \mu^2 =$

$= \underbrace{Var(X_1 - X_2)}_{indep} + (\underbrace{E(X_1-X_2)}_{=0})^2 - \mu^2 = Var\, X_1 + Var\, X_2 - \mu^2 = 1/12 + 1/12 - \mu^2 =$

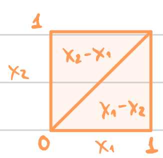
$= 1/6 - \mu^2$

opt 1: $\mu = 1/2 \Rightarrow \sigma^2 = 1/6 - 1/4 = 2/12 - 3/12 = -1/12$!! $\Rightarrow \mu \neq 1/2 \Rightarrow \mu = 1/3$

opt 2: $\mu = 1/3 \Rightarrow \sigma^2 = 1/6 - 1/9 = 3/18 - 2/18 = 1/18 \Rightarrow$ correct answer: D

En el examen haríamos esto para no perder tiempo calculando μ.

$|X_1 - X_2| = \begin{cases} X_1 - X_2 & X_1 \geq X_2 \\ X_2 - X_1 & X_2 > X_1 \end{cases}$



The density of the random vector (X_1, X_2) is

$f(x_1, x_2) = \underbrace{f(x_1)}_{dens\ X_1} \cdot \underbrace{f(x_2)}_{dens\ X_2} = \begin{cases} 1 & x_1, x_2 \in (0,1) \\ 0 & \text{in other cases} \end{cases}$

$$\mu = E\,|X_1 - X_2| = \overset{\triangledown}{\iint} (x_2 - x_1)\cdot 1 \, dx_1\, dx_2 + \overset{\triangle}{\iint} (x_1 - x_2)\cdot 1 \, dx_1\, dx_2 =$$

$$= \int_0^1 \int_0^{x_2} (x_2 - x_1)\, dx_1\, dx_2 + \int_0^1 \int_0^{x_1} (x_1 - x_2)\, dx_1\, dx_2 =$$

$$= 2 \int_0^1 \int_0^{x_2} (x_2 - x_1)\, dx_1\, dx_2 = 2 \int_0^1 \left(x_1 x_2 - \frac{x_1^2}{2}\right)\Big]_0^{x_2} dx_2 =$$

$$= 2 \int_0^1 \frac{x_2^2}{2}\, dx_2 = \frac{x_2^3}{3}\Big]_0^1 = 1/3$$

Ex: Let $X_1, \dots, X_{n+m}$ be a simple sample of $\mathcal{N}(\mu, \sigma^2)$ (i.e. $X_i \sim \mathcal{N}(\mu,\sigma^2)$ indepently $\forall i \in \{1, \dots, n+m\}$).

We know $X_1, \dots, X_n$ and $\bar{X} = \frac{1}{n+m} \sum_{i=1}^{n+m} X_i$

Find $\alpha \in \mathbb{R}$ such that $\hat{\sigma}^2 = \alpha \sum_{i=1}^{n} (X_i - \bar{X})^2$ is an unbiased estimator of σ^2.

$\left.\begin{matrix} X_1, \dots, X_n, \cancel{X_{n+1}, \dots, X_{n+m}} \end{matrix}\right\} \to \bar{X}$ ¿?

We are looking for $\alpha \in \mathbb{R}$ such that

$$\sigma^2 = E\hat{\sigma}^2 = \alpha E \sum_{i=1}^{n} (X_i - \bar{X})^2 = \alpha \sum_{i=1}^{n} E((X_i - \bar{X})^2) =$$

$$= \alpha \sum_{i=1}^{n} \left(Var(X_i - \bar{X}) + (\underbrace{E(X_i - \bar{X})}_{EX_i - E\bar{X} = 0})^2\right) =$$

$$= \alpha n \, Var(X_1 - \bar{X}) = \alpha n \, (\overset{\sigma^2}{Var X_1} - 2 \underbrace{Cov(X_1, \bar{X})}_{} + \overset{\sigma^2}{Var \bar{X}}) =$$

$$Cov\left(X_1, \frac{1}{n+m}\sum_{i=1}^{n+m} X_i\right) = \frac{1}{n+m} \sum Cov(X_1, X_i) = \frac{\sigma^2}{n+m}$$

$$= \alpha n \left(\sigma^2 - \frac{2}{n+m}\sigma^2 + \sigma^2\right) = \alpha \frac{n(n+m-1)}{n+m} \sigma^2 \Rightarrow \boxed{\alpha = \frac{n+m}{n(n+m-1)}}$$

EX: Let $X_1, \dots, X_{100}$ independent $EX_i = \mu$, $Var\, X_i = \sigma^2$ $\forall i = 1, \dots, 100$.

We do not know $X_1, \dots, X_{100}$. We know:

$$Y_1 = X_1 + \dots + X_{10}$$
$$Y_2 = X_{11} + \dots + X_{20}$$
$$\dots$$
$$Y_{10} = X_{91} + \dots + X_{100}$$

Find $a \in \mathbb{R}$ s.t. $\hat{\sigma}^2 = a \sum_{i=1}^{10} (Y_i - \bar{Y})^2$ is an unbiased est. of σ^2.

$$\bar{Y} = \frac{1}{10} \sum_{i=1}^{10} Y_i$$

$EY_i = 10\mu$ for $i = 1, \dots, 10$

$Var\, Y_i = 10\sigma^2 = Var\left(\sum_{i=1}^{10} X_i\right) \overset{indep}{=} \sum_{i=1}^{10} Var\, X_i = 10\, Var\, X_1 = 10\,\sigma^2$

$E\left(1/9 \sum_{i=1}^{10} (Y_i - \bar{Y})^2\right) = Var\, Y_i = 10\sigma^2$

$E\left(\underbrace{1/90}_{= \alpha} \sum_{i=1}^{10} (Y_i - \bar{Y})^2\right) = \sigma^2$ ✓

$$E\left[\sum_{i=1}^{10} (Y_i - \bar{Y})^2\right] = \sum E\left((Y_i - \bar{Y})^2\right) =$$
$$= \sum Var(Y_i - \bar{Y}) + \cancel{E(Y_i - \bar{Y})^2}^{\,0}$$
$$= \sum Var\, Y_i + Var\, \bar{Y} - 2\, Cov(Y_i, \bar{Y}) =$$
$$\sum 10\sigma^2 + \sigma^2 - \frac{2}{10} \underbrace{Var\, Y_i}_{10\sigma^2} = \sum_{i=1}^{10} 9\sigma^2 = 90\,\sigma^2$$

15-NOV-2023

EX: $Y \sim Unif(0,1)$.

Conditional distribution of $X/Y=y$.

Compute $P(X<1/2)$ $\sim Unif(0,y)$

a) 0'5 b) 0'622 c) 0'75 d) 0'847 e) 0'911

This problem has nothing to do with normal dist, but with conditional probability.

a) cannot be, for sure.

$y<1/2 \Rightarrow X<1/2$ because $X/Y=y \sim Unif(0,y)$ implies $0 \le X \le Y$.

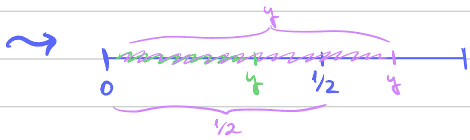
$$P(X<1/2) = \underbrace{P(Y<1/2)}_{=0'5} + \underbrace{P(Y \geq 1/2, X<1/2)}_{>0} > 0'5 \Rightarrow \text{a) not possible.}$$

Let's compute the probability.

$$P(X<1/2) = E(P(X<1/2 \overset{\text{condition}}{|} Y)) = \int_0^1 \underbrace{P(X<1/2 \overset{\text{condition.}}{|} Y=y)}_{X|Y=y \sim Unif(0,1)} \cdot \overset{\text{density of } Y \text{ on } (0,1)}{1} \, dy = *$$

* We have to compute

$$P(X<1/2 \mid Y=y) = \begin{cases} 1 & y<1/2 \\ \frac{1/2}{y} = \frac{1}{2y} & 1/2 \le y < 1 \end{cases} \mapsto \text{Why is that?} \rightsquigarrow$$

$\rightsquigarrow$ 

OPTION 1: $y<1/2$

OPTION 2: $y \geq 1/2$ (we divide the length of the longer interval, y, by the length of the shorter interval, $1/2$)

$\ln 1 - \ln 1/2 = \ln \frac{1}{1/2} = \ln 2$

$$* = \int_0^{1/2} 1 \, dy + \int_{1/2}^1 \frac{1}{2y} \, dy = 1/2 + 1/2 \ln y \Big]_{1/2}^1 = 1/2 + 1/2 \ln 2 =$$

$$= \boxed{1/2 \, (1 + \ln 2) = P(X<1/2) \simeq 0'847} \Rightarrow d)$$

EX: $(X_1,Y_1),\dots,(X_n,Y_n)$ have 2-dim normal distribution and are independent (the pairs, not the variables inside the pairs)

More precisely, $EX_i = EY_i = \mu$
$\text{Var}\,X_i = \text{Var}\,Y_i = \sigma^2 \qquad \forall i = 1,\dots,n$
$\text{Cov}(X_i,Y_i) = \rho\cdot\sigma^2$

Unfortunately, each of the pairs (X_i,Y_i) have been separated and mixed.

We still know $X_1,\dots,X_n$ but we don't know $Y_1,\dots,Y_n$. Instead, we know $Z_1,\dots,Z_n$, which is a random permutation of $Y_1,\dots,Y_n$

Compute $\text{Cov}(X_i,Z_i)$

a) 0 Hay más opciones pero no las ha puesto.

Comment: $\text{Cov}(X_i,Y_i)=\rho\sigma^2$, but, unfortunately $\text{Cov}(X_i,Y_j)=0$ when $i\neq j$, because both variables are independent.

No sé qué, dice que seguramente el valor correcto esté entre 0 y $\rho\sigma^2$.

$$\text{Cov}(X_i,Z_i) = \underbrace{E(X_i\cdot Z_i)}_{\shortparallel} - \overbrace{EX_i}^{\mu}\,\overbrace{EZ_i}^{=EY_j=\mu\quad j\in\{1,\dots,n\}} = *$$

$$P(Y_i=Z_i)=1/n \qquad j\neq i$$

$$\frac{1}{n}E(X_iY_i) + \frac{n-1}{n}E(X_iY_j) =$$

$$= 1/n\,(\underbrace{\text{Cov}(X_i,Y_i)}_{=\rho\sigma^2} + \underbrace{EX_i}_{=\mu}\,\underbrace{EY_i}_{=\mu}) + \frac{n-1}{n}(\underbrace{\text{Cov}(X_i,Y_j)}_{=0} + \underbrace{EX_i}_{=\mu}\,\underbrace{EY_j}_{=\mu})$$

$$* = \left(\frac{\rho\sigma^2}{n} + \frac{\mu^2}{n}\right) + \left(\frac{n-1}{n}\mu^2\right) - (\mu^2) = \frac{\rho\sigma^2}{n} + \mu^2 - \mu^2 = \boxed{\frac{\rho\sigma^2}{n}}$$

EX: $EX_1 = \dots = EX_n = \mu$

$Var\, X_1 = \dots = Var\, X_n = \sigma^2$

$Cov(X_i, X_j) = \rho\sigma^2$ for $i \neq j$ $i,j = 1, \dots, n$

A) Compute $Var(\sum_{i=1}^{n} X_i)$ easy part

B) Let $\varepsilon_1, \dots, \varepsilon_m$ be independent and independent of $X_1, \dots, X_n$

We know that $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$. $\Rightarrow \varepsilon_i = \pm 1$

compute $Var(\sum_{i=1}^{n} \varepsilon_i X_i)$

A) We can use formulas we already know:

$Var(X+Y) = Var\, X + 2Cov(X,Y) + Var\, Y$

In our case,

we don't have the coefficient 2 because in Σ we already repeat the Cov

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var\, X_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} Cov(X_i, X_j) = \sum_{i=1}^{n} Var\, X_i + 2\sum_{\substack{i,j=1 \\ i<j}}^{n} Cov(X_i, X_j) =$$

$$= \sum_{i=1}^{n} \sigma^2 + 2\sum_{\substack{i,j=1 \\ i<j}}^{n} \rho\sigma^2 = n\sigma^2 + n(n-1)\rho\sigma^2 = n\sigma^2(1 + (n-1)\rho)$$

as before

$$B) \quad Var(\sum_{i=1}^{n} \varepsilon_i X_i) = \sum_{i=1}^{n} Var(\varepsilon_i X_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} Cov(\varepsilon_i X_i, \varepsilon_j X_j) =$$

the result will be the same

$$= n\, Var(\varepsilon_1 X_1) + n(n-1)\, Cov(\varepsilon_1 X_1, \varepsilon_2 X_2)$$

Now, we compute:

1 because $\varepsilon_i = \pm 1$ — independent

- $$Var(\varepsilon_1 X_1) = E(\varepsilon_1^2 X_1^2) - (E(\varepsilon_1 X_1))^2 = E(X_1^2) - (E\varepsilon_1 \cdot EX_1)^2 =$$

$P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2 \Rightarrow E(\varepsilon_1) = 1 \cdot 1/2 + (-1) \cdot 1/2 = 0$.

$$= E(X_1^2) - (0 \cdot \mu)^2 = E(X_1^2) = Var(X_1) + (E(X_1))^2 = \sigma^2 + \mu^2$$

Note: Generally, X_1, X_2 are not independent, so we have to compute the following Cov

- $$Cov(\varepsilon_1 X_1, \varepsilon_2 X_2) = E(\varepsilon_1 \cdot \varepsilon_2 \cdot X_1 \cdot X_2) - E(\varepsilon_1 X_1) \cdot E(\varepsilon_2 X_2) =$$ using independence

$$= \underbrace{\overset{=0}{E(\varepsilon_1)} \cdot \overset{=0}{E(\varepsilon_2)} \cdot E(X_1 \cdot X_2)}_{0} - \underbrace{\overset{=0}{E(\varepsilon_1)} \cdot E(X_1) \cdot \overset{=0}{E(\varepsilon_2)} \cdot E(X_2)}_{0} = 0$$

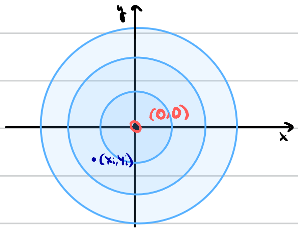
Remember: Cov=0 does not imply independence.

Finally,

$$\operatorname{var}\left(\sum_{i=1}^{n} \varepsilon_i X_i\right) = n \operatorname{var}(\varepsilon_1 X_1) + n(n-1)\operatorname{cov}(\varepsilon_1 X_1, \varepsilon_2 X_2) =$$

$$= n(\sigma^2 + \mu^2) + n(n-1)\cdot 0 = n(\sigma^2+\mu^2)$$

EX: A shooter is aiming at the target and shooting.



He repeats this procedure n times, indepently. Coordinates of i-th shoot is (x_i, y_i).

We assume that $(X_1, Y_1)^T, \dots, (X_n, Y_n)^T \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\right)$

Compute the expectation of the distance of the best shoot from (0,0).

because $x \mapsto \sqrt{x}$ is an increasing map.

$$E\Big(\min\big(\underbrace{\sqrt{X_1^2+Y_1^2}, \dots, \sqrt{X_n^2+Y_n^2}}_{\text{distances of the shoots from } (0,0)}\big)\Big) = E\left(\sqrt{\min\left[(X_1^2+Y_1^2), \dots, (X_n^2+Y_n^2)\right]}\right)$$

Let $Z = X^2+Y^2$, where $(X,Y)^T \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\right)$

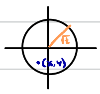
What is the distribution of Z? We'll find F_Z (c.d.f of Z).

The density of $(X,Y)^T$ is:

$$f(x,y) = \frac{1}{(\sqrt{2\pi})^2\sqrt{\sigma^4}} \cdot e^{-\frac{(x,y)\cdot\begin{pmatrix}1/\sigma^2 & 0\\ 0 & 1/\sigma^2\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}}{2}} = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Since $Z = X^2+Y^2$, we obtain that $Z \geq 0$ and, for every $t<0$, $F_Z(t) = P(Z \leq t) = 0$.

Let $t \geq 0 \Rightarrow F_Z(t) = P(Z \leq t) = P(X^2+Y^2 \leq t) = P(\sqrt{X^2+Y^2} \leq \sqrt{t}) =$



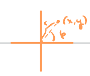
$$= P\big((X,Y) \in \text{Disk}((0,0), \sqrt{t})\big) = \iint_{\text{Disk}} f(x,y)\, dx\, dy =$$

$$= \iint_{\text{Disk}} \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2+y^2}{2\sigma^2}}\, dx\, dy =$$

idea: switch to polar coordinates

$x, y \rightsquigarrow r, \varphi$

$\begin{cases} y = r\cdot\sin\varphi \\ x = r\cdot\cos\varphi \\ x^2+y^2 = r^2 \end{cases}$



$$= \underbrace{\int_0^{2\pi}\int_0^{\sqrt{t}} \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} \cdot r\, dr\, d\varphi}_{\text{it does not depend on } \varphi \Rightarrow \text{ we integrate directly.}} = \cancel{2\pi} \int_0^{\sqrt{t}} \frac{1}{\cancel{2\pi}\sigma^2} \cdot e^{\frac{-r^2}{2\sigma^2}} \cdot r\, dr =$$

$$= \left\{\begin{array}{l} \text{C.V.} \\ s = \frac{r^2}{2\sigma^2} \\ ds = \frac{r}{\sigma^2}\, dr \end{array}\right\} = \int_0^{t/2\sigma^2} \frac{1}{\cancel{\sigma^2}} \cdot e^{-s} \cdot \cancel{r} \cdot \frac{\cancel{\sigma^2}}{\cancel{r}}\, ds = \int_0^{t/2\sigma^2} e^{-s}\, ds = -e^{-s}\Big]_0^{t/2\sigma^2} =$$

$$= -e^{-t/2\sigma^2} + 1 = \underline{\underline{1 - e^{-t/2\sigma^2}}}$$

Finally:
$$F_Z(t) = \begin{cases} 0 & t<0 \\ 1-e^{-t/2\sigma^2} & t\geq 0 \end{cases}$$

We can recognise Z distribution, $Z \sim \mathcal{E}(1/2\sigma^2) = Exp(1/2\sigma^2)$

exponential distribution with parameter $1/2\sigma^2$

$\Rightarrow$ expectation $2\sigma^2$] no confondir

Recall: EXPONENTIAL DISTRIBUTION $X \sim \mathcal{E}(\beta)$ $\beta > 0$

$EX = 1/\beta$

c.d.f. $F_X(t) = \begin{cases} 0 & t<0 \\ 1-e^{-\beta t} & t\geq 0 \end{cases}$

density $f(x) = \begin{cases} 0 & t<0 \\ \beta e^{-\beta t} & t \geq 0 \end{cases}$

Applying a theorem, we see that $\min\big(\underbrace{(X_1^2+Y_1^2)}_{\sim \mathcal{E}(1/2\sigma^2)}, \dots, \underbrace{(X_n^2+Y_n^2)}_{\sim \mathcal{E}(1/2\sigma^2)}\big) = V$

they are independent!

$V \sim \mathcal{E}\left(\sum_{i=1}^{n} 1/2\sigma^2\right) = \mathcal{E}(n/2\sigma^2)$

We want to compute $E(\sqrt{V})$, where $V \sim \mathcal{E}(n/2\sigma^2)$

$$E(\sqrt{V}) = \int_0^\infty \sqrt{v} \cdot \underbrace{\frac{n}{2\sigma^2} e^{-\frac{n}{2\sigma^2} v}}_{\text{density}} dv = \left\{\begin{array}{l} \text{C.V.} \ u = \sqrt{v} \\ v = u^2,\ u \geq 0 \\ dv = 2u\, du \end{array}\right\} = \int_0^\infty u \frac{n}{2\sigma^2} \cdot e^{-\frac{nu^2}{2\sigma^2}} \cdot 2u\, du =$$

$$= \int_0^\infty \underbrace{\frac{n}{\sigma^2} u^2 e^{-\frac{nu^2}{2\sigma^2}}}_{\substack{\text{even function} \\ \text{(simetrical resp. 0)}}} du = \frac{1}{2} \int_{-\infty}^{\infty} \frac{n}{\sigma^2} u^2 e^{-\frac{nu^2}{2\sigma^2}} du =$$

similar to density of normal dist.

Let $\mathcal{U} \sim \mathcal{N}(0, \sigma^2/n)$

$$= \sqrt{\frac{\pi}{2\sigma^2/n}} \int_{-\infty}^{\infty} u^2 \underbrace{\frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}} e^{-\frac{u^2}{2\cdot \sigma^2/n}}}_{\text{density of } \mathcal{N}(0,\sigma^2/n)} du = \sqrt{\frac{\pi}{2\sigma^2/n}} E(\mathcal{U}^2) =$$

$$= \sqrt{\frac{\pi}{2\sigma^2/n}} \left[\operatorname{Var} \mathcal{U} + (E\mathcal{U})^2\right] = \sqrt{\frac{\pi}{2\sigma^2/n}} \left[\sigma^2/n + 0^2\right] = \boxed{\sqrt{\frac{\pi}{2n}}\, \sigma = E(\sqrt{V})}$$

■

EX: Let $(X_1,Y_1), \ldots, (X_n,Y_n)$ be independent random vectors with the same normal distribution and the following parameters:
$EX_i = EY_i = m$, $\operatorname{var} X_i = 1/4$ $\operatorname{var} Y_i = 1$, $\operatorname{corr}(X_i,Y_i) = 1/2$.

Separately, based on the random samples $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, two confidence intervals were built for the parameter m, each at the confidence level $1-\alpha = 0'8$.

Compute the probability that the intervals are disjoint.

From $X_1, \ldots, X_n$ we obtain the confidence interval

$$\left(\bar{X} - \frac{\sqrt{\operatorname{var} X_i}}{\sqrt{n}} u_{1-\alpha/2}, \bar{X} + \frac{\sqrt{\operatorname{var} X_i}}{\sqrt{n}} u_{1-\alpha/2}\right) =$$

$1-\alpha = 0'8$; $\alpha = 0'2$; $\alpha/2 = 0'1 \Rightarrow 1-\alpha/2 = 0'9$
$\operatorname{var} X_i = 1$.

$$= \left(\bar{X} - \frac{1}{\sqrt{n}} u_{0'9}, \bar{X} + \frac{1}{\sqrt{n}} u_{0'9}\right)$$

$u_{0'9}$ is the quantile of $\mathcal{N}(0,1)$ of order $0'9$ (the 9th percentile)

From $(Y_1, \ldots, Y_n)$ we obtain the confidence interval

$$\left(\bar{Y} - \frac{2}{\sqrt{n}} u_{0'9}, \bar{Y} + \frac{2}{\sqrt{n}} u_{0'9}\right) \qquad (\operatorname{var} Y_i = 4 \Rightarrow \sqrt{\operatorname{var} Y_i} = 2)$$

The intervals are disjoint if:

$$\bar{X} + \frac{1}{\sqrt{n}} u_{0'9} \le \bar{Y} - \frac{2}{\sqrt{n}} u_{0'9} \quad \text{or} \quad \bar{Y} + \frac{2}{\sqrt{n}} u_{0'9} \le \bar{X} - \frac{1}{\sqrt{n}} u_{0'9} \iff$$

$$\iff \frac{3}{\sqrt{n}} u_{0'9} \le \bar{Y} - \bar{X} \quad \text{or} \quad \frac{3}{\sqrt{n}} u_{0'9} \le \bar{X} - \bar{Y} \iff$$

$$\iff \bar{X} - \bar{Y} \le -\frac{3}{\sqrt{n}} u_{0'9} \quad \text{or} \quad \bar{X} - \bar{Y} \geq \frac{3}{\sqrt{n}} u_{0'9} \iff$$

We need to find:

$$P\left(\bar{X} - \bar{Y} \le -\frac{3}{\sqrt{n}} u_{0'9}\right) \quad \text{or} \quad P\left(\bar{X} - \bar{Y} \geq \frac{3}{\sqrt{n}} u_{0'9}\right)$$

What is the distribution of $\bar{X} - \bar{Y}$?

$\bar{X} - \bar{Y} = \bar{Z}$, where $Z_i = X_i - Y_i$ for $i = 1, 2, \ldots, n$ (clearly, $Z_1, \ldots, Z_n$ are independent)

Clearly, $Z_i \sim \mathcal{N}(0, ?)$ 3

$E Z_i = E X_i - E Y_i = m - m = 0$

$\text{Var } Z_i = \text{Var}(X_i - Y_i) = \overset{=1}{\text{Var } X_i} + \overset{=4}{\text{Var } Y_i} - 2 \overset{=1}{\text{Cov}(X_i, Y_i)} = 5 - 2 = 3 = ?$ *

$\bar{Z} = \bar{X} - \bar{Y} \sim \mathcal{N}(0, 3/n)$

So, now:

$$P\left(\bar{X} - \bar{Y} \leq -\frac{3}{\sqrt{n}} u_{0'9}\right) + P\left(\bar{X} - \bar{Y} \geq \frac{3}{\sqrt{n}} u_{0'9}\right) =$$

$$= P\left(\underbrace{\frac{\bar{X} - \bar{Y}}{\sqrt{3/n}}}_{\mathcal{N}(0,1)} \leq -\sqrt{3}\, u_{0'9}\right) + P\left(\frac{\bar{X} - \bar{Y}}{\sqrt{3/n}} \geq \sqrt{3}\, u_{0'9}\right) =$$

By the symetry of $\mathcal{N}(0,1)$, both prob. are the same.

$$= 2 \cdot \phi(-\sqrt{3}\, u_{0'9}) = 2 \cdot (1 - \phi(\sqrt{3}\, u_{0'9})) = 2 \cdot 0'0132 = 0'0264 \simeq \underline{0'026}$$

using the table, $u_{0'9} = 1'2816$

$\phi(\sqrt{3}\, u_{0'9}) \simeq 0'9868$

* We know that $\text{corr}(X_i, Y_i) = \dfrac{\text{Cov}(X_i, Y_i)}{\text{Var } X_i \text{ Var } Y_i} \Rightarrow \text{Cov}(X_i, Y_i) = 1/2 \cdot \sqrt{4} \cdot \sqrt{1} = 1$

EX: Let $X_1, \ldots, X_n$ be a sample from $Exp(\theta)$, $\theta > 0$. ($(X_1, \ldots, X_n) \sim \mathcal{E}(\theta)$ are independent).

Let $\hat\theta = \dfrac{5}{X_1 + \cdots + X_5}$. Find numbers $a, b \in \mathbb{R}$ such that

$$\left(\frac{a}{X_1 + \cdots + X_5}, \frac{b}{X_1 + \cdots + X_5} \right)$$

is a confidence interval on the level $0'95$, satisfying:

$$\begin{cases} P\left(\theta < \dfrac{a}{X_1 + \cdots + X_5}\right) = 0'025 \\ P\left(\theta > \dfrac{b}{X_1 + \cdots + X_5}\right) = 0'025 \end{cases} \rightarrow \quad \begin{aligned} \alpha &= 0'05 \\ \beta &= \alpha/2 \\ \gamma &= \alpha/2 \end{aligned}$$

$$0'025 = P\left(\theta \le \frac{a}{X_1 + \cdots + X_5}\right) = P(\theta X_1 + \cdots + \theta X_5 \le a)$$

If $X \sim Exp(\beta)$ and $a > 0$, then $a \cdot X \sim Exp(\beta/a)$

Proof: $t > 0$ $\quad P(aX \le t) = P(X \le t/a) = 1 - e^{-\beta(t/a)} = \underbrace{1 - e^{-\beta/a\, t}}_{\text{c.d.f. of } Exp(\beta/a)}$

So, we have:

$$\left.\begin{array}{l} X_i \sim Exp(\theta) \\ \theta \cdot X_i \sim Exp(\theta/\theta) = Exp(1) \sim \Gamma(1,1) \end{array}\right\} \Rightarrow \theta X_1 + \cdots + \theta X_5 \sim \Gamma(5,1)$$

a is the quantile of $\Gamma(5,1)$ of order $0'025$

Most of the tables don't have these exact values.

Also, we have: $P(\theta X_1 + \cdots + \theta X_5 \ge b) = 0'025$

$$\Downarrow$$

$$P(\underbrace{\theta X_1 + \cdots + \theta X_5}_{\sim \Gamma(5,1)} \le b) = 0'975$$

b is the quantile of $\Gamma(5,1)$ of order $0'975$

What should we do if we don't know the above quantile?

We know $\theta X_i \sim Exp(1) \Rightarrow 2\theta X_i \sim Exp(1/2)$, so:

$$P(\underbrace{2\theta X_1 + \cdots + 2\theta X_5}_{\sim \Gamma(5, 1/2) = \Gamma(10/2, 1/2) = \chi^2_{10}} \le 2a) = 0'025 \Rightarrow \begin{array}{l} 2a = \chi^2_{10}(0'025) \\ 2b = \chi^2_{10}(0'975) \end{array}$$

$\chi^2_n = \Gamma(n/2, 1/2)$

These values are in the table

We look in the tables and:

$$\left| \begin{array}{l} 2a \simeq 3'247 \\ 2b \simeq 20'483 \end{array} \right\} \Rightarrow \begin{array}{l} a \simeq 1'6235 \simeq 1'62 \\ b \simeq 10'2415 \simeq 10'24 \end{array}$$

+

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Ex: Let $X_1, \dots, X_{10}$ be independent and $X_i \sim \mathcal{N}(\mu, 1/i)$ $i = 1, \dots, 10$. μ is unknown. Construct a $\boxed{95\%}$ confident interval for μ of the following form:

$\to 1-\alpha = 0'95$

$(\hat{\mu} - \alpha, \hat{\mu} + \alpha)$, where $\alpha \in \mathbb{R}$ and $\hat{\mu}$ is the MLE (maximum likelihood estimator) for μ

*Note: our sample is not a simple sample because there random variables have not the same distribution.

We start from writing down the formula for the density function of the sample:

$$f_n(x_1, \dots, x_{10}) = \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi \, 1/i}} \, e^{-\frac{(x_i-\mu)^2}{2 \cdot 1/i}} = \frac{\sqrt{10!}}{(2\pi)^{5}} \cdot e^{-\frac{1}{2}\sum_{i=1}^{10} i(x_i-\mu)^2}$$

$5 = \frac{10}{2}$

The likelihood function MLE is the value of μ which maximizes

$$\mathcal{L}_{x_1, \dots, x_{10}}(\mu) = \frac{\sqrt{10!}}{(2\pi)^5} \cdot e^{-\frac{1}{2}\sum_{i=1}^{10} i(x_i-\mu)^2}$$

$$\overbrace{\frac{\sqrt{10!}}{(2\pi)^5}}^{\text{coef.}} \cdot \underbrace{e^{-\frac{1}{2}\sum_{i=1}^{10} i(x_i-\mu)^2}}$$

As the coefficient is >0, it is enough with maximizing

$$e^{-\frac{1}{2}\boxed{\sum_{i=1}^{10} i(x_i-\mu)^2}}$$

As the function is $\nearrow$, it is enough to minimize $\boxed{\cdot}$

$$\sum_{i=1}^{10} i(x_i-\mu)^2 = \sum_{i=1}^{10} i(x_i^2 - 2x_i\mu + \mu^2) = \underbrace{\left(\sum_{i=1}^{10} i\right)}\mu^2 - 2\left(\sum_{i=1}^{10} i x_i\right)\cdot\mu + \left(\sum_{i=1}^{10} i x_i^2\right)$$

this coef is >0. The function is $\cup$ with min in $-b/2a$

So the minimum is at:

$$\mu = -\frac{-\cancel{2}\sum_{i=1}^{10} i x_i}{\cancel{2}\sum_{i=1}^{10} i} \Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{10} i x_i}{\sum_{i=1}^{10} i} = \frac{\sum_{i=1}^{10} i x_i}{55} = \hat{\mu}$$

Now, we want to find $\alpha \in \mathbb{R}$ s.t. $(\hat{\mu} - \alpha, \hat{\mu} + \alpha)$ is a 95% confidence interval for μ.

$\downarrow$

$$-\hat{\mu}+\alpha > -\mu > -\hat{\mu}-\alpha \quad {\scriptstyle +\hat{\mu}+\mu}$$

$$\cdot(-1) \quad \mu+\alpha > \hat{\mu} > \mu-\alpha$$

$$0'95 = P(\mu \in (\hat{\mu}-\alpha, \hat{\mu}+\alpha)) = P(\hat{\mu}-\alpha < \mu < \hat{\mu}+\alpha) = P(\mu-\alpha < \hat{\mu} < \mu+\alpha)$$

pasamos de calcular una prob de μ a calcular una de $\hat{\mu}$.

What is the distribution of $\hat{\mu}$? We know it is normal because $X_1, \dots, X_{10}$ has normal dist. and independence and $\frac{\sum_{i=1}^{10} i x_i}{55}$ is a linear transformation. We have to find its parameters:

·) $$E\hat{\mu} = \frac{\sum_{i=1}^{10} i \overbrace{EX_i}^{=\mu}}{55} = \mu \frac{\cancel{\sum_{i=1}^{10} i}}{\cancel{55}} = \mu.$$

·) $$\text{var}\,\hat{\mu} = \frac{\text{var} \sum_{i=1}^{10}(i x_i)}{55^2} \underset{\substack{\uparrow \\ x_1, \dots, x_{10} \\ \text{independent}}}{=} \frac{\sum_{i=1}^{10} i^2 \overbrace{\text{var}(x_i)}^{=1/i}}{55^2} = \frac{\sum_{i=1}^{10} i^{\cancel{2}} \frac{1}{\cancel{i}}}{55^2} = \frac{1}{55}$$

Then, we have:

$$0'95 = P(\mu-\alpha < \hat{\mu} < \mu+\alpha) \underset{\text{standarization of } \hat{\mu}}{=} P\left(-\sqrt{55}\,\alpha < \frac{\hat{\mu}-\mu}{\sqrt{1/55}} < \sqrt{55}\,\alpha\right) =$$

$$= \phi(\sqrt{55}\,\alpha) - \phi(-\sqrt{55}\,\alpha) = \phi(\sqrt{55}\,\alpha) - (1-\phi(\sqrt{55}\,\alpha)) = 2\phi(\sqrt{55}\,\alpha) - 1;$$

$$;\ \phi(\sqrt{55}\,\alpha) = \frac{1+0'95}{2} = 0'975 \overset{\text{table}}{\Rightarrow} \sqrt{55}\,\alpha \simeq 1'96 \Rightarrow \boxed{\alpha \simeq 0'264}$$

Ex: $X_1, \ldots, X_n$ is a simple sample from $\mathcal{E}(\beta)$ (density:

$f(x) = \beta e^{-\beta x}$, $x > 0$)

Let $\hat{\beta}$ be the MLE for the parameter β.

Find the minimal size n of the sample such that

$P\left(\frac{|\hat{\beta}-\beta|}{\beta} \le 0'01\right) \simeq 0'95$ $\quad$ $\beta \in (\hat{\beta} - 0'01\beta, \hat{\beta} + 0'01\beta)$

Hint: use the normal approximation.

$$f_\beta(x_1, \ldots, x_n) = \begin{cases} \prod_{i=1}^{n} \beta e^{-\beta x_i} & x_1, \ldots, x_n > 0 \\ 0 & x_1 \le 0 \text{ or } \ldots \text{ or } x_n \le 0 \end{cases}$$

In our case: $\quad$ we can assume this because they are taken from our sample.

$$\underbrace{L_{x_1, \ldots, x_n}(\beta)}_{L(\beta)} = \beta^n e^{-\beta \sum_{i=1}^{n} x_i} \quad (x_1, \ldots, x_n > 0)$$

→ we want to maximize this function.
aplicamos logaritmos.

$$\ln(L(\beta)) = n \cdot \ln(\beta) - \beta \sum_{i=1}^{n} x_i$$ → to maximize, we derivate

igualamos a 0.

$$[\ln(L(\beta))]' = \frac{n}{\beta} - \sum_{i=1}^{n} x_i = 0$$

β		$n/\sum_{i=1}^{n} x_i$	
$\ln(L(\beta))'$	+	0	-
$\ln(L)$	↗	$\boxed{\max}$	↘

// derivada
// función
¡HAY MÁX! :)

$$\beta = \frac{n}{\sum_{i=1}^{n} x_i} \Rightarrow \boxed{\hat{\beta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}}$$

Now, we write:

$Y_i := \beta \cdot X_i \sim \mathcal{E}(\beta/\beta) = \mathcal{E}(1) \Rightarrow$ $\quad$ ($X_i \sim \mathcal{E}(\beta)$)

$$\frac{|\hat{\beta}-\beta|}{\beta} = \left|\frac{\hat{\beta}}{\beta} - 1\right| = \left|\frac{n}{\beta \sum X_i} - 1\right| = \left|\frac{n}{\sum Y_i} - 1\right|$$

$$\Rightarrow P\left(\frac{|\hat{\beta}-\beta|}{\beta} \le 0'01\right) = P\left(-0'01 \le \frac{n}{\sum Y_i} - 1 \le 0'01\right) =$$

$$= P\left(0'99 \le \frac{n}{\sum Y_i} \le 1'01\right) = P\left(\frac{n}{0'99} \le \sum_{i=1}^{n} Y_i \le \frac{n}{1'01}\right)$$

We will use CLT (central limit theorem)

We note $\mu = EY_i = 1$ → because $Y_i \sim \mathcal{E}(1)$

$$\frac{\sum Y_i - n\mu}{\sqrt{n\cdot\sigma^2}} \simeq \mathcal{N}(0,1) \quad \text{for large } n$$

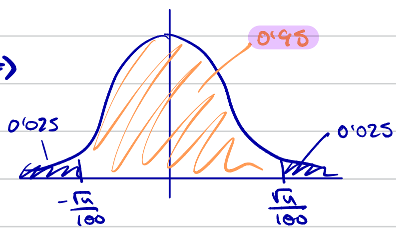
$\| \;$ — var $Y_i = \sigma^2 = 1$

$$\frac{\sum Y_i - n}{\sqrt{n}}$$

. . . so . . .

$$-\frac{\sqrt{n}}{101} = \frac{\left(\frac{1}{1'01} - 1\right)n}{\sqrt{n}} \le \frac{\sum Y_i - n}{\sqrt{n}} \le \frac{\left(\frac{1}{0'99} - 1\right)}{\sqrt{n}} = \frac{\sqrt{n}}{99} \Rightarrow$$

$$-\frac{\sqrt{n}}{101} \ge -\frac{\sqrt{n}}{100} \qquad \frac{\sqrt{n}}{99} \ge \frac{\sqrt{n}}{100}$$

$\Rightarrow$ 

$$\Rightarrow \frac{\sqrt{n}}{100} = u_{0'975} \simeq 1'96 \Rightarrow n = (100 \cdot 1'96)^2 =$$

quantile of N(0,1) — table

$$= \underline{\underline{38100 = n}}$$

Problem. We have 2 independent observations X_1 and X_2 from the normal distribution. One of them has the distribution $N(\mu, \sigma^2)$ and the other one $N(2\mu, 2\sigma^2)$. Unfortunately, we lost information about which observation comes from which distribution. All the parameters are unknown. In this situation we consider the following estimator $\hat{\sigma^2}$ of the parameter σ^2:

$$\hat{\sigma^2} = a(X_1 - X_2)^2 + b(X_1 + X_2)^2 \text{ where } a, b \in \mathbb{R}$$

Find a and b if the estimator $\hat{\sigma^2}$ is unbiased.

First possibility: $X_1 \sim N(\mu, \sigma^2)$ $X_2 \sim N(2\mu, 2\sigma^2)$

Second possibility: $X_1 \sim N(2\mu, 2\sigma^2)$ $X_2 \sim N(\mu, \sigma^2)$

$$E(\hat{\sigma^2}) = aE(X_1 - X_2)^2 + bE(X_1 + X_2)^2$$

$$E(X_1 + X_2)^2 = Var(X_1 + X_2) + (E(X_1 + X_2))^2 = Var X_1 + Var X_2$$

$$+ (EX_1 + EX_2)^2 = \sigma^2 + 2\sigma^2 + (\mu + 2\mu)^2 = 3\sigma^2 + 9\mu^2$$

$$E(X_1 - X_2)^2 = Var(X_1 - X_2) + (E(X_1 - X_2))^2 =$$
$$Var X_1 + Var X_2 + (EX_1 - EX_2)^2 = \sigma^2 + 2\sigma^2 + (\mu - 2\mu)^2 =$$
$$= 3\sigma^2 + \mu^2$$

We want this

$$E(\hat{\sigma^2}) = a(3\sigma^2 + \mu^2) + b(3\sigma^2 + 9\mu^2) = \sigma^2$$

$$\underset{\overset{\shortparallel}{1}}{(3a+3b)}\sigma^2 + \cancelto{0}{(a+9b)}\mu^2$$

$$\begin{cases} 3a + 3b = 1 \\ a + 9b = 0 \end{cases} \Longrightarrow a = 3/8 \quad b = -1/24$$

Y luego se haría el segundo caso de forma análoga.