4.2 Let R be a commutative ring with unit, such that for every $r \in R$ there is an integer $n > 1$ (possibly depending upon r) such that $r^n = r$. Show that every prime ideal in R is maximal.

Let I be a prime ideal of R. Using the proposition 4.5.1, if we demonstrate that R/I is a field, then we will have that I is maximal.

Let $\bar{r} = r + I$ be a nonzero element of R/I and let's prove that it has a multiplicative inverse so that we can conclude that R/I is a field:

$$\bar{r}^n = (r+I)^n = r^n + I = r + I = \bar{r}$$

So we have that $\bar{r}^n - \bar{r} = 0$

$$\bar{r}(\bar{r}^{n-1} - 1) = 0$$

Now, using the proposition 4.6.1, since I is a prime ideal, R/I is an integral domain. This, together with $\bar{r}(\bar{r}^{n-1} - 1) = 0$ implies that $\bar{r}^{n-1} - 1 = 0$, so $\bar{r}^{n-1} = 1$. This also means that $\bar{r} \cdot \bar{r}^{n-2} = 1$, so we have found the multiplicative inverse of $\bar{r}$, a random nonzero element from R/I. We can conclude that R/I is a field, so I is a maximal ideal.

4.8 Let R be a commutative ring with unit. Suppose R contains an *idempotent* element r other than 0 or 1. (That is, $r^2 = r$.) Show that every prime ideal in R contains an idempotent other than 0 or 1.

Let's consider a prime ideal I in R. Since I is an ideal, $0 \in I$ and $r - r^2 \underset{r^2 = r}{=} r - r = 0 \in I$.

$r - r^2 = r(1-r) = 0 \Rightarrow I$ is a prime ideal, so $r \in I$ or $1 - r \in I$. We have to check that $1-r$ is idempotent:

$(1-r)^2 = 1 + r^2 - 2r = 1 + r - 2r = 1 - r \Rightarrow 1-r$ is idempotent

Is $1-r$ different from 0 and 1?

•) If $1 - r = 0 \Rightarrow r = 1$ which is not true

•) If $1 - r = 1 \Rightarrow r = 0$ which is not true

In conclusion, I contains r or $1-r$, which are both idempotent and different from 0 and 1.

Ex. If the map of the previous exercise is a superjective homomorphism, proof the following: If I is an ideal of B then $A/f^{-1}(I)$ and B/I are isomorphics.

From the 3rd exercise of Andrea, we know that if I is an ideal of B, then $f^{-1}(I)$ is an ideal of A. Now, we are going to build a superjective homomorphism from A to B/I:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{p} & B/I \\ a & \longmapsto & f(a) & \longmapsto & f(a)+I \end{array}$$

Since f and p are both superjective homomorphisms, the composition of these two $(p \circ f)$ is also a superjective homomorphism, so $\mathrm{Im}(p \circ f) = B/I$

Let's also compute the $\mathrm{Ker}(p \circ f)$:

$$\mathrm{Ker}(p \circ f) = \{a \in A \,/\, (p \circ f)(a) = 0\} = \{a \in A \,/\, f(a) + I = 0\} =$$

$$= \{a \in A \,/\, f(a) \in I\} = \{a \in A \,/\, f(a) \in I\} =$$

$$= \{a \in A \,/\, a \in f^{-1}(I)\} = f^{-1}(I)$$

Finally, we use the following theorem:

First Isomorphism Theorem

THEOREM

Let G and H be two groups and let $\phi\colon G \to H$ be a group homomorphism. Then the kernel $\ker(\phi)$ is a normal subgroup of G, and

$$G/\ker(\phi) \simeq \mathrm{Im}(\phi).$$

So we can conclude that:

$$A/_{\mathrm{Ker}(p\circ f)} \simeq \mathrm{Im}(p\circ f) \Longrightarrow A/_{f^{-1}(I)} \simeq B/_{I}$$