Zadanie 1. The random vector $(X, Y, Z)^T$ has distribution $\mathcal{N}\left((1,0,3)^T, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right)$. Are there constants a, b such that the random variable $X + Y + aZ + 3$ is independent of the random vector $(X + 2Z + 4,\ X - bY + Z)^T$? If so, determine these constants.

Zadanie 2. The random vector (X, Y) has the normal disribution with the parameters $EX = 0 = EY = 0$, $D^2X = D^2Y = 1$ and $Cov(X, Y) = \rho \in [-1, 1]$. Calculate $D^2(XY)$ and $Cov(X^2, Y^2)$.

Zadanie 3. Let (X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n) be independent random vectors with the same normal distribution with the following parameters: unknown expectation $EX_i = EY_i = m$, the variance $Var\ X_i = \frac{1}{4}Var\ Y_i = 1$ and the correlation coefficient $corr(X_i, Y_i) = \frac{1}{2}$. Separately, based on random samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, two confidence intervals were built for the expected value m, each at the confidence level of 0.8. Calculate the probability that the intervals constructed in this way turn out to be disjoint.

Zadanie 4.

We have independent observations from three normal distributions with the same unknown variance. For each of the distributions we have a certain group of observations. For each of these groups, the mean and variance from the sample were determined separately. The obtained values, together with the sizes of individual groups, are given in the table:

n_l	$\overline{X}_l$	$S_l^2 = \frac{1}{n_l-1}\sum_{i=1}^{n_l}(X_i^l - \overline{X}_l)^2$
5	1,8	0,8
8	2,6	0,5
7	0,9	0,6

We perform an analysis of variance test at the significance level of $\alpha = 0.05$, where the null hypothesis is that the expected values in all three groups are the same. What is the result of this test?

Zadanie 5. We assume that $(X_1, X_2, \ldots, X_n)$ is a simple sample from the normal distribution $\mathcal{N}(\mu, \gamma^2\mu^2)$, where $\gamma^2 > 0$ is known and $\mu \in \mathbb{R}$ is unknown. Find an estimator of μ of the form $\hat{\mu} = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ that would have the smallest mean squared error $E(\hat{\mu} - \mu)^2$ among such estimators.

Zadanie 1. The random vector $(X,Y,Z)^T$ has distribution $\mathcal{N}\left((1,0,3)^T, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right)$. Are there constants a, b such that the random variable $X+Y+aZ+3$ is independent of the random vector $(X+2Z+4,\ X-bY+Z)^T$? If so, determine these constants.

$$(X,Y,Z)^T \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right)$$

$a, b \in \mathbb{R}$ s.t. $X+Y+aZ+3$ indep. of $\begin{pmatrix} X+2Z+4 \\ X-bY+Z \end{pmatrix}$

$$\begin{pmatrix} X+Y+aZ+3 \\ X \quad +2Z+4 \\ X-bY+Z \end{pmatrix}^{*} = \underbrace{\begin{pmatrix} 1 & 1 & a \\ 1 & 0 & 2 \\ 1 & -b & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}}$$

This is an affine transformation of $(X,Y,Z)^T$, which follows a 3-dim. normal dist. Therefore, the result * is also a 3-dim. normal dist.

This allows us to apply a proposition that says that, in this case,

$$X+Y+aZ+3 \text{ indep of. } \begin{pmatrix} X+2Z+4 \\ X-bY+Z \end{pmatrix}$$

$$\Updownarrow$$

$$\mathrm{Cov}(X+Y+aZ+3,\ X+2Z+4) = 0$$
$$\mathrm{Cov}(X+Y+aZ+3,\ X-bY+Z) = 0.$$

Ya sólo se trata de aplicar propiedades de la Cov y fijarnos en la matriz de cov. para despejar a y b.

Ejercicio hecho en la relación.

+

Zadanie 2. The random vector (X, Y) has the normal disribution with the parameters $EX = 0 = EY = 0$, $D^2X = D^2Y = 1$ and $Cov(X, Y) = \rho \in [-1, 1]$. Calculate $D^2(XY)$ and $Cov(X^2, Y^2)$.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

$var(XY)$

$Cov(X^2, Y^2)$

$\rho \in [-1, 1]$.

$var\, X = EX^2 - (EX)^2$

$Cov(X,Y) = E(X \cdot Y) - EXEY$

•) $var(XY) = E(XY)^2 - (EXY)^2 = E(X^2Y^2) - (E(XY))^2$

$\rightarrow E(XY) = Cov(X,Y) + EXEY = \rho + 0 \cdot 0 \Rightarrow (E(XY))^2 = \rho^2 \in [0,1]$

$\rightarrow E(X^2Y^2)$

We need to standarize it. Take $U, V \sim \mathcal{N}(0,1)$ indep.
We want $L \in M_2(\mathbb{R})$ and $b \in \mathbb{R}^2$ such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim L\begin{pmatrix} U \\ V \end{pmatrix} + b$$

We have $E\begin{pmatrix} X \\ Y \end{pmatrix} = b \Rightarrow b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Also, $var\begin{pmatrix} X \\ Y \end{pmatrix} = LL^T$. So $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ verifies:

$$\left.\begin{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \\ \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} \end{matrix}\right\} \Rightarrow \begin{matrix} a^2+b^2 = 1 \\ ac+bd = \rho \\ c^2+d^2 = 1 \end{matrix} \Rightarrow \left| \underbrace{\begin{matrix} a=1 \quad b=0 \\ c=\rho \\ d=\sqrt{1-\rho^2} \end{matrix}}_{\text{possible solution}} \right.$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U \\ \rho U + \sqrt{1-\rho^2}\, V \end{pmatrix}$$

Now, we have:

$$E(X^2Y^2) = E(U^2(\rho U + \sqrt{1-\rho^2}\,V)^2) =$$

$$= E(U^2(\rho^2 U^2 + \rho\sqrt{1-\rho^2}\,UV + (1-\rho^2)V^2)) =$$

$$= \rho^2 E(U^4) + \rho\sqrt{1-\rho^2}\,E(U^3V) + (1-\rho^2)E(U^2V^2) = \circledast$$

$$\to E(U^4) = \frac{4!}{\frac{4}{2}!\cdot 2^{4/2}} = \frac{\cancel{4}\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot \cancel{4}} = 3$$

$$\to E(U^3V) \overset{\text{indep}}{=} E(U^3)E(V) = E(U^3)\cdot 0 = 0$$

$$\to E(U^2V^2) = E(U^2)E(V^2) = \left(\frac{2!}{1!\cdot 2^1}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$$

$$\circledast = 3\rho + (1-\rho^2) = -\rho^2 + 3\rho + 1 = E(X^2Y^2)$$

Therefore:

$$\boxed{\mathrm{var}(XY)} = E(XY)^2 - (EXY)^2 = -\rho^2 + 3\rho + 1 - \rho^2 = \boxed{-2\rho^2 + 3\rho + 1}$$

=

Now, we intend to compute $\mathrm{cov}(X^2, Y^2)$.

$$\boxed{\mathrm{cov}(X^2,Y^2)} = \underbrace{E(X^2Y^2)}_{\downarrow} - \underbrace{EX^2EY^2}_{\downarrow\ *} =$$

$$= -\rho^2 + 3\rho + 1 - 1 = \boxed{-\rho^2 + 3\rho}$$

$*$ $EX^2 = E(X-0)^2 = E(X-\mu)^2 = 1\cdot\sigma^2 = 1\cdot 1$

Same for $EY^2 = 1$

+

Zadanie 3. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent random vectors with the same normal distribution with the following parameters: unknown expectation $EX_i = EY_i = m$, the variance $Var\ X_i = \frac{1}{4} Var\ Y_i = 1$ and the correlation coefficient $corr(X_i, Y_i) = \frac{1}{2}$. Separately, based on random samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, two confidence intervals were built for the expected value m, each at the confidence level of 0.8. Calculate the probability that the intervals constructed in this way turn out to be disjoint.

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} m \\ m \end{pmatrix}, \begin{pmatrix} 1 & c \\ c & 4 \end{pmatrix}\right) \qquad corr(X_i, Y_i) = 1/2$$

Two intervals from $<^X_Y$ for m. conf $= 0'8$. $\Rightarrow \alpha = 0'2$

We know that $corr(X_i, Y_i) = \frac{Cov(X_i, Y_i)}{\sqrt{Var X_i}\sqrt{Var Y_i}} \Rightarrow Cov(X_i, Y_i) = 1/2 \cdot \sqrt{4} \cdot \sqrt{1} = 1$

Take $\left(\bar{X} - \frac{\sqrt{Var X_i}}{\sqrt{n}} u_{1-\alpha/2}, \bar{X} + \frac{\sqrt{Var X_i}}{\sqrt{n}} u_{1-\alpha/2}\right) =$

$= \left(\bar{X} - \frac{1}{\sqrt{n}} u_{0'9}, \bar{X} + \frac{1}{\sqrt{n}} u_{0'9}\right)$ for X_i sample

Take $\left(\bar{Y} - \frac{\sqrt{Var Y_i}}{\sqrt{n}} u_{1-\alpha/2}, \bar{Y} + \frac{\sqrt{Var Y_i}}{\sqrt{n}} u_{1-\alpha/2}\right) =$

$= \left(\bar{Y} - \frac{2}{\sqrt{n}} u_{0'9}, \bar{Y} + \frac{2}{\sqrt{n}} u_{0'9}\right)$ for Y_i sample.

We want the prob of:

$\bar{X} - \frac{1}{\sqrt{n}} u_{0'9} \geq \bar{Y} + \frac{2}{\sqrt{n}} u_{0'9}$ or $\bar{Y} - \frac{2}{\sqrt{n}} u_{0'9} \geq \bar{X} + \frac{1}{\sqrt{n}} u_{0'9} \Leftrightarrow$

$\Leftrightarrow \bar{X} - \bar{Y} \geq \frac{3}{\sqrt{n}} u_{0'9}$ or $\bar{X} - \bar{Y} \leq -\frac{3}{\sqrt{n}} u_{0'9}$

$P(\bar{X} - \bar{Y} \geq 3/\sqrt{n}\, u_{0'9}) + P(\bar{X} - \bar{Y} \leq -3/\sqrt{n}\, u_{0'9})$

$\bar{Z} = \bar{X} - \bar{Y}$ s.t. $Z_i = X_i - Y_i \Rightarrow$ what distribution?

Normal, ofc (because it is a lin. transf. of two normal).

$E Z_i = E X_i - E Y_i = m - m = 0$,

$\text{Var}\, Z_i = \text{var}\, X_i + \text{Var}\, Y_i - 2\,\text{cov}(X_i, Y_i) = 1 + 4 - 2 \cdot 1 = 3$

$\Downarrow$

$\bar{Z} \sim \mathcal{N}(0, 3/n)$

$\circ)$ $P\left(\bar{Z} \geq \frac{3}{\sqrt{n}} u_{0'9}\right) = P\left(\frac{\bar{Z}}{\sqrt{3/n}} \geq \frac{3/\cancel{\sqrt{n}}}{\sqrt{3}/\cancel{\sqrt{n}}} u_{0'9}\right) = P\left(\gamma \geq \sqrt{3}\, u_{0'9}\right) =$

$= 1 - P(\gamma \leq \sqrt{3}\, u_{0'9}) = 1 - P(\gamma \leq \sqrt{3} \cdot 1'29) = 0'0132$

$\circ)$ $P\left(\bar{Z} \leq -\frac{3}{\sqrt{n}} u_{0'9}\right) = P(\gamma \leq -\sqrt{3}\, u_{0'9}) = 1 - P(\gamma \leq \sqrt{3}\, u_{0'9} = 0'0132$

So: $P(\text{intervals disjoint}) = \underline{\underline{0'0264}}$

+

Zadanie 5. We assume that $(X_1, X_2, \ldots, X_n)$ is a simple sample from the normal distribution $\mathcal{N}(\mu, \gamma^2\mu^2)$, where $\gamma^2 > 0$ is known and $\mu \in \mathbb{R}$ is unknown. Find an estimator of μ of the form $\hat{\mu} = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ that would have the smallest mean squared error $E(\hat{\mu} - \mu)^2$ among such estimators.

$X_1, \ldots, X_n \sim \mathcal{N}(\mu, \gamma^2\mu^2)$ $\quad$ $\gamma^2 > 0$ known

$\mu \in \mathbb{R}$ unknown.

Estimator $\hat{\mu} = a_1 X_1 + \ldots + a_n X_n$

such that $E(\hat{\mu} - \mu)^2$ is minimized.

We will take $\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ to try. We know it is biased point estimator.

$MSE = E((\hat{\mu}-\mu)^2) = Var(\hat{\mu}-\mu) + (E(\hat{\mu}-\mu))^2 =$

$= Var\,\hat{\mu} + (E(\hat{\mu}) - E(\mu))^2 = \sum a_i^2 Var(X_i) + (\sum a_i E(X_i) - \mu)^2 =$

$= \gamma^2\mu^2 \sum a_i^2 + (\mu(\sum a_i - 1))^2 = \gamma^2\mu^2\sum a_i^2 + \mu^2(\sum a_i)^2 - \mu^2\sum a_i + \mu^2$?? how to minimize ??

If $a_i = 1 \; \forall i \Rightarrow \gamma^2\mu^2 \cdot n + \mu^2 \cdot n^2 - \mu^2 \cdot n + \mu^2 =$

$= \mu^2(\gamma^2 n + n^2 - n + 1)$ $\quad$ X

If $a_i = 1/n \; \forall i \Rightarrow \gamma^2\mu^2 \cdot 1/n + \mu^2 \cdot 1/n^2 - \mu^2 \cdot 1/n + \mu^2 =$

$= \mu^2(\gamma^2/n + 1/n^2 - 1/n + 1)$ $\quad$ It's better than before.