SURFACES - EXERCISES

EX 1: Prove that the plane is a regular surface

$\pi \subset \mathbb{R}^3 = \{(p_1 + \alpha v_1 + \beta w_1,\ p_2 + \alpha v_2 + \beta w_2,\ p_3 + \alpha v_3 + \beta w_3 : \alpha, \beta \in \mathbb{R}\}$

$\tilde{x} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\tilde{x}(\alpha, \beta) = (p_1 + \alpha v_1 + \beta w_1,\ p_2 + \alpha v_2 + \beta w_2,\ p_3 + \alpha v_3 + \beta w_3)$

By definition, $\tilde{x}$ is a differentiable (1st) function, so $\exists$ is continuous. As $(\tilde{x}^{-1})$ exists and $\exists$ is continuous, we can affirm that $\tilde{x}$ is an homeomorphism. (2nd) because $\exists$ is a $\perp$ projection from π on the plane determined by axes X and Y

$\tilde{x}(\alpha, \beta) = (x(\alpha, \beta),\ y(\alpha, \beta),\ z(\alpha, \beta))$

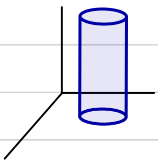
$$d\tilde{x}_q = \begin{bmatrix} \partial x/\partial \alpha & \partial x/\partial \beta \\ \partial y/\partial \alpha & \partial y/\partial \beta \\ \partial z/\partial \alpha & \partial z/\partial \beta \end{bmatrix} \Rightarrow$$

$\tilde{x}$ is 1-to-1 (3rd) because vectors $\vec{v}$ and $\vec{w}$ are linearly independent ($\exists$ is because of the definition of π)

Therefore, a plane is an example of a regular surface.

Reposo.

Ex 2: Prove that a cylinder is a regular surface



First of all, we need a parametrization.

$$S=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=r^2,\ z\in\mathbb{R}\}$$

Let's consider $\tilde{x}:]0,2\pi[\times\mathbb{R}\to\mathbb{R}^3$

$$\tilde{x}(u,v)=(r\cos(u), r\sin(u), v) \qquad u\in]0,2\pi[,\ v\in\mathbb{R}$$

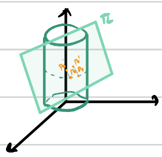
We know that the maps

$$\left.\begin{array}{l} x(u,v) = r\cos(u) \\ y(u,v) = r\,\mathrm{sen}(u) \\ z(u,v) = v \end{array}\right\} \text{ are differentiable}$$

$\Downarrow$ 1st

$\tilde{x}$ is diff $\Rightarrow$ $\tilde{x}$ is cont.

Let's consider $p_0=(r\cos(t_0), r\sin(t_0), z_0)$ and $p_1=(r\cos(t_1), r\sin(t_1), z_1)$



and suppose they are very close. That will mean that the projection of the points on the plane π given by $p_0'=(t_0,z_0)$, $p_1'=(t_1,z_1)$ which will also be very close.

So $(\tilde{x})^{-1}$ exists and it is cont.

$$\left.\begin{array}{l} \tilde{x} \text{ cont} \\ + \\ (\tilde{x})^{-1} \text{ cont} \end{array}\right\} \Rightarrow \tilde{x} \text{ homeomorphism}$$ (2nd)

$$d\tilde{x}_q=\begin{bmatrix} -r\sin u & 0 \\ r\cos u & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightarrow$ We have two possibilities:

- $-r\sin(u)=0 \Leftrightarrow u=\pi$
- $r\cos(u)=0 \Leftrightarrow u=\pi/2,\ 3\pi/4$

So, if $u\neq\pi/2, \pi, 3\pi/4$, $\tilde{x}$ is 1-to-1 (3rd)

$\Rightarrow$ A cylinder is a regular surface.

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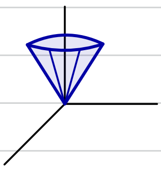
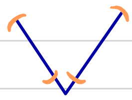
Ex 3 Write a parametrization of a cone and check if its param. gives a regular surface



La base NO!

If we consider v, we are not going to have a regular surface, but if we consider  , then we obtain a regular surface.

Curve: $(t, at) = \gamma(t) \qquad a > 0$

Surface of revolution: $(t\,\mathrm{sen}\varphi, t\cos\varphi, at) = \tilde{x}(t,\varphi)$

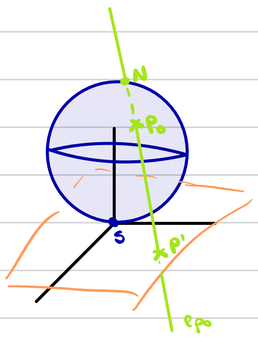
$$d\tilde{x}_q = \begin{bmatrix} \mathrm{sen}\varphi & t\cos\varphi \\ \cos\varphi & -t\,\mathrm{sen}\varphi \\ a & 0 \end{bmatrix}$$

$\Longrightarrow$ When $t = 0$, $d\tilde{x}_q$ is not a one-to-one function, so the cone is not a regular surface

EX 4: Given a unit sphere ($r=1$) and the plane tangent to the sphere at the south pole, fix a point P_0 of the sphere and consider a half-line ℓ_P starting at the north and going through point P_0.

The line ℓ_P cuts the tangent plane π at point P'. The transformation which associates P to P' is a stereographic projection

Write the analytic formula of the stereographic projection.



$N=(0,0,2)$
$P_0=(x,y,z)$
$C=(0,0,1)$
$S=(0,0,0)$

$S^2=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+(z-1)^2=1\}$

$\ell_{p_0}=\{p_0+\alpha\,\overrightarrow{p_0N} : \alpha\in\mathbb{R}\}$

Calculate ℓ

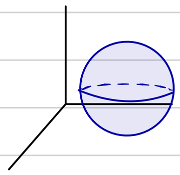
$$\left.\begin{array}{l}\overrightarrow{NP}=[x,y,z-2] \\ \text{As } N=(0,0,2)\in\ell\end{array}\right\} \Rightarrow \begin{array}{l}\ell=\{(0,0,2)+t(x,y,z-2) : t\in\mathbb{R}\} = \\ =\{(tx,ty,t(z-2)+2) : t\in\mathbb{R}\}\end{array}$$

Let's find the point $\in\ell$ where $z=0$

$$2+t(z-2)=0 \; ; \; t=\frac{-2}{z-2} \; ; \; t=\frac{2}{2-z}$$

So $X(P)=X(x,y,z)=\left(\frac{2x}{2-z}, \frac{2y}{2-z}, 0\right)$

EX 5: Calculate the area of the sphere of radius r and center $(0,0,0)$.



→ Parametrization

It is a surface of revolution of a semicircle $\alpha(u) = (r\cos(u), r\,\text{sen}(u))$ $u \in [-\pi/2, \pi/2]$

So $x(u,v) = (r\cos(u)\cdot\sin(v), r\cos(u)\cdot\cos(v), r\sin(u))$

→ Partial derivates

$$x_u = (-r\,\text{sen}(u)\cdot\text{sen}(v), -r\,\text{sen}(u)\cos(v), r\cos(u))$$

$$x_v = (r\cos(u)\cos(v), -r\cos(u)\,\text{sen}(v), 0)$$

→ Coefficients FFF

$$E = \langle x_u, x_u\rangle = r^2\text{sen}^2(u)\,\text{sen}^2(v) + r^2\text{sen}^2(u)\cos^2(v) + r^2\cos^2(u) = r^2\text{sen}^2(u) + r^2\cos^2(u) = r^2$$

$$F = \langle x_u, x_v\rangle = -r^2\text{sen}(u)\,\text{sen}(v)\cos(u)\cos(v) + (\dots)$$
$$(\dots) + r^2\text{sen}(u)\cos(v)\cos(u)\,\text{sen}(v) + 0 = 0$$

$$G = \langle x_v, x_v\rangle = r^2\cos^2(u)\cos^2(v) + r^2\cos^2(u)\,\text{sen}^2(v) + 0 = r^2\cos^2(u)$$

→ Area of the sphere

$$A(\mathbb{S}^2) = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \|x_u \times x_v\|\,du\,dv = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\sqrt{E\cdot G - F^2}\,du\,dv =$$

$$= * \; E\cdot G - F^2 = r^2 r^2\cos^2(u) - 0^2 = \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\pi} r^2|\cos(u)|\,du\right)dv =$$

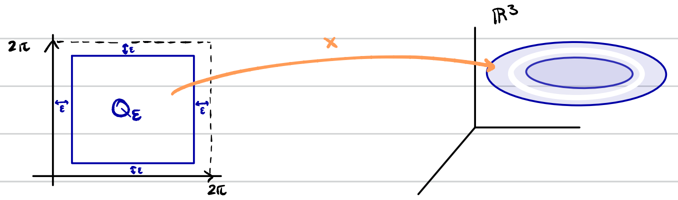
$$= (\pi/2 + \pi/2)\int_0^{2\pi} r^2|\cos(u)|\,du = r^2\pi\cdot\left(\int_{-\pi/2}^{\pi/2}\cos(u)\,du - \int_{\pi/2}^{3\pi/2}\cos(u)\,du\right) =$$

$$= 2\pi r^2\int_{-\pi/2}^{\pi/2}\cos(u)\,du = 2\pi r^2\left[\text{sen}(u)\right]_{-\pi/2}^{\pi/2} = 2\pi r^2(1-(-1)) = 4\pi r^2$$

EXAMPLE: We intend to calculate the area of a torus obtained as a surface of revolution of a curve $\alpha(u) = (a + r\cdot\cos u,\ r\cdot\sin u)$ with respect to axe O_z.

So we get the following parametrization:

$$x(u,v) = ((a + r\cos u)\cdot\cos v,\ (a + r\cos u)\cdot\sin v,\ r\cdot\sin u) \qquad x, v \in (0, 2\pi)$$



Choose small $\varepsilon > 0$

Coefficients of the first fundamental form

$$\frac{dx}{du} = x_u(u,v) = (-r\,\mathrm{sen}(u)\cdot\cos(v),\ -r\,\mathrm{sen}(u)\cdot\mathrm{sen}(v),\ r\cos(u))$$

$$\frac{dx}{dv} = x_v(u,v) = (-(a + r\cos u)\,\mathrm{sen}\,v,\ (a + r\cos u)\cos v,\ 0)$$

$$E = \langle x_u, x_u\rangle = r^2\,\mathrm{sen}^2(u)\cos^2(v) + r^2\,\mathrm{sen}^2(u)\cdot\mathrm{sen}^2(v) + r^2\cos^2(u) =$$
$$= r^2\,\mathrm{sen}^2(u) + r^2\cos^2(u) = r^2$$

$$F = \langle x_u, x_v\rangle = r\,\mathrm{sen}(u)\cos(v)(a + r\cos u)\,\mathrm{sen}(v) - (\dots)$$
$$(\dots) - r\,\mathrm{sen}(u)\,\mathrm{sen}(v)(a + r\cos u)\cos v + 0 = 0$$

$$G = \langle x_v, x_v\rangle = (a + r\cos u)^2\,\mathrm{sen}^2 v + (a + r\cos u)^2\cos^2 v = (a + r\cos u)^2$$

Find the area of the torus

$$\text{Area}(Q_\varepsilon) = \iint_{Q_\varepsilon} \|x_u \times x_v\|\,du\,dv = \iint_{Q_\varepsilon} \sqrt{E(u,v)\cdot G(u,v) - F^2(u,v)}\,du\,dv =$$

$$= *\ E\cdot G - F^2 = r^2(a + \cos u)^2 - 0^2 = \iint_{Q_\varepsilon} \sqrt{r^2(a + \cos u)^2}\,du\,dv =$$

$$= \iint_{Q_\varepsilon} r(a + \cos u)\,du\,dv \overset{\text{Fibonacci}}{=} \int_\varepsilon^{2\pi-\varepsilon}\left(\int_\varepsilon^{2\pi-\varepsilon} ra + r\cos u\,du\right)dv =$$

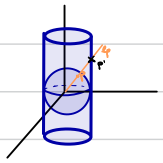
$$= (2\pi - \varepsilon - \varepsilon)\cdot\int_\varepsilon^{2\pi-\varepsilon} ra + r\cos u\,du = 2(\pi - \varepsilon)\left[rau + r\,\mathrm{sen}\,u\right]_\varepsilon^{2\pi-\varepsilon} =$$

$$= 2(\pi - \varepsilon)\cdot\left[2(\pi - \varepsilon)ra + r\,\mathrm{sen}(2\pi - \varepsilon) - r\,\mathrm{sen}(\varepsilon)\right]$$

So, $\text{Area}(\mathbb{T}^2) = \lim_{\varepsilon\to 0} A(Q_\varepsilon) = 2\pi[2\pi ra + 0 - 0] = 4\pi^2 ra$

EXERCISE: We have a unit sphere S^2 with centre at $(0,0,0)$ and a cylinder given by:

$$C = \{(\sin(t), \cos(t), s)\} \quad s \in \mathbb{R} \quad t \in (0, 2\pi)$$

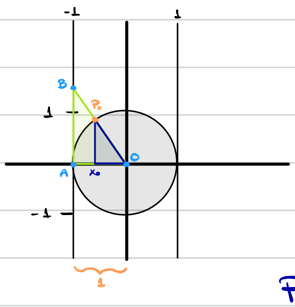


For fixed $p \in S^2$, we draw a half line ℓ_p such that $p, (0,0,0) \in \ell_p$. The intersection point $\ell_p \cap C = p'$

Write the analytic formula for the transformation

$$f: S^2 \setminus \{N, S\} \to C \quad \text{s.t. } f(p) = p'$$

We search a parametrization of the sphere, which we are going to get by revolution.



We assume that a sphere is given as a surface of revolution, so its param. is:

$$x(u,v) = (\cos(u) \cdot \cos(v), \cos(u) \cdot \sin(v), \sin(u))$$

$$P_0 = (\cos(t_0) \cdot \cos(s_0), \cos(t_0) \cdot \operatorname{sen}(s_0), \operatorname{sen}(t_0))$$

$$\frac{\|\vec{OB}\|}{\|\vec{OP_0}\|} = \frac{1}{\|\vec{Ox_0}\|} \Rightarrow \vec{OB} = \frac{\vec{OP_0}}{\|\vec{Ox_0}\|}$$

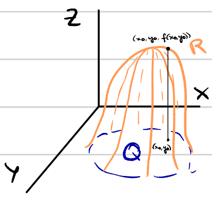
$$\operatorname{dist}(O, x_0) = \sqrt{\cos^2(t_0) \cdot \cos^2(s_0) + \cos^2(t_0) \cdot \sin^2(s_0)} = \cos(t_0)$$

$$B = (x, y, z) = \frac{[\cos(t_0) \cdot \cos(s_0), \cos(t_0) \cdot \operatorname{sen}(s_0), \operatorname{sen}(t_0)]}{\cos(t_0)}$$

$$(t_0, s_0) \to [\cos(s_0), \sin(s_0), \tan(t_0)]$$

EXERCISE: Show that the area of a bounded region R of the surface given by parametrization $\tilde{x}(x,y) = (x, y, f(x,y))$, where $f: Q \subset \mathbb{R}^2 \to \mathbb{R}$ is a differenciable map and Q is the normal projection of R onto XY plane.

Area is given by $\iint_Q \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$



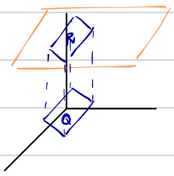
$\tilde{x} = (x, y, f(x,y))$

$\tilde{x}_x = (1, 0, f_x(x,y))$
$\tilde{x}_y = (0, 1, f_y(x,y))$

$$\left.\begin{aligned} E &= \langle \tilde{x}_x, \tilde{x}_x \rangle = 1 + 0 + f_x^2 \\ F &= \langle \tilde{x}_x, \tilde{x}_y \rangle = 0 + 0 + f_x f_y \\ G &= \langle \tilde{x}_y, \tilde{x}_y \rangle = 0 + 1 + f_y^2 \end{aligned}\right\} \quad EG - F^2 = 1 + f_y^2 + f_x^2 + f_x^2 f_y^2 - f_x^2 f_y^2 =$$

$$= \underline{\underline{1 + f_y^2 + f_x^2}} \quad \checkmark$$

EXERCISE: Calculate the area of the plane given by $x(u,v) = (u, v, a\cdot u + b\cdot v)$, where $a,b \in \mathbb{R}$ over $Q = [-1,1]\times[-1,1]$



Orthogonal projection of R equals to Q

$f(u,v) = a\cdot u + b\cdot v$ $\qquad f_u = a$

$\qquad\qquad\qquad\qquad\qquad f_v = b$

$$\text{Area}(R) = \iint_Q \sqrt{1 + f_u^2 + f_v}\ du\,dv =$$

$$= \iint_Q \sqrt{1+a^2+b^2}\ du\,dv = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1+a^2+b^2} = 4\sqrt{1+a^2+b^2}$$

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EXERCISE: Calculate Gauss and mean curvature of

a) a cylinder: $x(u,v) = (r\cdot \sin(u), r\cdot \cos(u), v)$

b) a sphere: $x(u,v) = (r\cdot\cos(u)\cdot\sin(v), r\cdot\cos(u)\cdot\cos(v), r\cdot\sin(u))$

c) a plane: $x(u,v) = (p_1 + uv_1 + vw_1, p_2 + uv_2 + vw_2, p_3 + uv_3 + vw_3)$

a) 1st: partial derivates → coef. 1stFF

$x_u = (r\cos u, -r\,\text{sen}\,u, 0)$
$x_v = (0, 0, 1)$

$E = \langle x_u, x_u\rangle = r^2\cos^2 u + r^2\text{sen}^2 u = r^2$

$F = \langle x_u, x_v\rangle = 0$

$G = \langle x_v, x_v\rangle = 1$

2nd: normal vector and more derivates → coef 2ndFF

$$x_u \times x_v = \left[\det\begin{vmatrix} -r\,\text{sen}\,u & 0 \\ 0 & 1\end{vmatrix}, -\det\begin{vmatrix} r\cos u & 0 \\ 0 & 1\end{vmatrix}, \det\begin{vmatrix} r\cos u & -r\,\text{sen}\,u \\ 0 & 0\end{vmatrix}\right] =$$

$$= [-r\,\text{sen}\,u, -r\cos u, 0]$$

$$\|x_u \times x_v\| = \sqrt{r^2\text{sen}^2 u + r^2\cos^2 u} = r$$

$$N = \frac{x_u \times x_v}{\|x_u \times x_v\|} = [-\text{sen}\,u, -\cos u, 0]$$

$x_{uu} = (-r\,\text{sen}\,u, -r\cos u, 0)$
$x_{vv} = (0, 0, 0)$
$x_{uv} = (0, 0, 0)$

$e = \langle N, x_{uu}\rangle = r\,\text{sen}^2 u + r\cos^2 u = r$
$f = \langle N, x_{uv}\rangle = 0$
$g = \langle N, x_{vv}\rangle = 0$

3rd: applying formulas

Gauss curvature:

$$K = \frac{e \cdot g - f^2}{EG - F^2} = \frac{r \cdot 0 - 0^2}{r^2 \cdot 1 - 0} = \boxed{0}$$

Mean curvature

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} = \frac{1}{2} \frac{r \cdot 1 - 2 \cdot 0 \cdot 0 + 0 \cdot r^2}{r^2 \cdot 1 - 0} = \boxed{\frac{1}{2r}}$$

b) $x(u,v) = (r\cdot\cos(u)\cdot\sin(v),\ r\cdot\cos(u)\cdot\cos(v),\ r\cdot\sin(u))$

1st: partial derivates → coef I_p

$$x_u = (-r\cdot\sin(u)\cdot\sin(v),\ -r\cdot\sin(u)\cdot\cos(v),\ r\cdot\cos(u))$$
$$x_v = (r\cdot\cos(u)\cdot\cos(v),\ -r\cdot\cos(u)\cdot\sin(v),\ 0)$$

$$E = \langle x_u, x_u\rangle = r^2\sin^2(u)\cdot\sin^2(v) + r^2\sin^2(u)\cos^2(v) + r^2\cos^2(u) = r^2$$

$$F = \langle x_u, x_v\rangle = -r^2\sin(u)\sin(v)\cos(u)\cos(v) + (\dots)$$
$$(\dots) + r^2\sin(u)\cos(v)\cos(u)\sin(v) + 0 = 0$$

$$G = \langle x_v, x_v\rangle = r^2\cos^2(u)\cos^2(v) + r^2\cos^2(u)\sin^2(v) = r^2\cos^2(u)$$

2nd: normal vector and more derivates → coef II_p

•) $$x_u \times x_v = \left[\det\begin{vmatrix} -r\sin(u)\cos(v) & r\cos(u) \\ -r\cos(u)\sin(v) & 0 \end{vmatrix},\ -\det\begin{vmatrix} -r\sin(u)\sin(v) & r\cos(u) \\ r\cos(u)\cos(v) & 0 \end{vmatrix},\ \det\begin{vmatrix} -r\sin(u)\sin(v) & -r\sin(u)\cos(v) \\ r\cos(u)\cos(v) & -r\cos(u)\sin(v) \end{vmatrix}\right] =$$

$$= [r^2\cos^2(u)\sin(v),\ r^2\cos^2(u)\cos(v),\ r^2\sin^2(v)\sin(u)\cos(u) + r^2\cos^2(v)\sin(u)\cos(u)] =$$

$$= [r^2\cos^2(u)\sin(v),\ r^2\cos^2(u)\cos(v),\ r^2\sin(u)\cos(u)]$$

•) $$\|x_u \times x_v\| = \sqrt{r^4\cos^4(u)\sin^2(v) + r^4\cos^4(u)\cos^2(v) + r^4\sin^2(u)\cos^2(u)} =$$

$$= r^2\sqrt{\cos^4(u) + \sin^2(u)\cos^2(u)} = r^2\sqrt{\cos^2(u)(\cos^2(u) + \sin^2(u))} =$$

$$= r^2|\cos(u)|$$

•) $$N = \frac{x_u \times x_v}{\|x_u \times x_v\|} = [\cos(u)\sin(v),\ \cos(u)\cos(v),\ \sin(u)]$$

$$x_{uu} = (-r \cdot \cos(u) \cdot \sin(v),\ -r \cdot \cos(u) \cdot \cos(v),\ -r \cdot \sin(u))$$

$$x_{uv} = (-r \cdot \sin(u) \cdot \cos(v),\ r \cdot \sin(u) \cdot \sin(v),\ 0)$$

$$x_{vv} = (-r \cdot \cos(u) \cdot \sin(v),\ -r \cdot \cos(u) \cdot \cos(v),\ 0)$$

$$e = \langle N, x_{uu} \rangle = -r\cos^2(u)\sin^2(v) - r\cos^2(u)\cos^2(v) - r\sin^2(u) = \underline{\underline{-r}}$$

$$f = \langle N, x_{uv} \rangle = -r\cos(u)\sin(v)\sin(u)\cos(v) + r\cos(u)\cos(v)\sin(u)\sin(v) = \underline{\underline{0}}$$

$$g = \langle N, x_{vv} \rangle = -r\cos^2(u)\sin^2(v) - r\cos^2(u)\cos^2(v) + 0 = \underline{\underline{-r\cos^2(u)}}$$

3rd : applying formulas

<u>Gauss curvature</u>

$$K = \frac{e \cdot g - f^2}{EG - F^2} = \frac{r^2 \cancel{\cos^2(u)} - 0}{r^2 r^2 \cancel{\cos^2(u)} - 0} = \boxed{\frac{1}{r^2}}$$

<u>Mean curvature</u>

$$H = \frac{1}{2}\,\frac{eG - 2fF + Eg}{EG - F^2} = \frac{1}{2}\,\frac{r^3\cos^2(u) - 0 - r^3\cos^2(u)}{r^4\cos^2(u)} = \boxed{0}$$

$\Downarrow$

minimal surface

c) $x(u,v) = (p_1 + uv_1 + vw_1,\ p_2 + uv_2 + vw_2,\ p_3 + uv_3 + vw_3)$

We assume that $[v_1, v_2, v_3] \perp [w_1, w_2, w_3]$ and $\|[v_1, v_2, v_3]\| =$
$= \|[w_1, w_2, w_3]\| = 1$

Partial derivates

$x_u = [v_1, v_2, v_3]$

$x_v = [w_1, w_2, w_3]$

Coefs I_p

$E = \langle x_u, x_u \rangle = \|x_u\|^2 = 1$

$F = \langle x_u, x_v \rangle = 0$ (because they are $\perp$)

$G = \langle x_v, x_v \rangle = \|x_v\|^2 = 1$

Second partial derivates

$x_{uu} = [0, 0, 0]$

$x_{uv} = [0, 0, 0]$

$x_{vv} = [0, 0, 0]$

Normal vector is not needed in this case

Coefs II_p

$e = \langle N, x_{uu} \rangle = 0$

$f = \langle N, x_{uv} \rangle = 0$

$g = \langle N, x_{vv} \rangle = 0$

Gauss curvature

$$K = \frac{eg - f^2}{EG - F^2} = \boxed{0}$$

Mean curvature

$$H = \frac{1}{2} \frac{eG - 2fF + Eg}{EG - F^2} = \boxed{0}$$

We can say that a plane is a minimal surface.

EXERCISE: Compute Christofel symbols for surface of revolution given by:

$$x(u,v) = (f(v)\cdot\cos(u),\ f(v)\cdot\sin(u),\ g(v))$$

First, we calculate coef. of Ip of the surface of rev.

$$x_u = (-f(v)\sin(u),\ f(v)\cos(u),\ 0)$$

$$x_v = (f'(v)\cos(u),\ f'(v)\sin(u),\ g'(v))$$

$$E = \langle x_u, x_u\rangle = f^2(v)\sin^2(u) + f^2(v)\cos^2(u) + 0 = f^2(v)$$

$$F = \langle x_u, x_v\rangle = -f(v)f'(v)\sin(u)\cos(u) + f(v)f'(v)\cos(u)\sin(u) + 0 = 0$$

$$G = \langle x_v, x_v\rangle = (f'(v))^2\cos^2(u) + (f'(v))^2\operatorname{sen}^2(u) + (g'(v))^2 = (f'(v))^2 + (g'(v))^2$$

Now, the derivates of those coefs:

$$E_u = 0$$
$$E_v = 2f(v)\cdot f'(v)$$

$$F_u = 0$$
$$F_v = 0$$

$$G_u = 0$$
$$G_v = 2f'(v)f''(v) + 2g'(v)g''(v)$$

And now, we apply the corollary from page 30 in the notes (the systems of equations)

$$\rightarrow \text{a)} \begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2}E_u = 0 \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2}E_v = -f(v)f'(v) \end{cases}$$

We will solve the system of equations above with unknown Γ_{11}^1 and Γ_{11}^2

$$\Gamma_{11}^1 = \frac{\begin{vmatrix} 0 & F \\ -f(v)f'(v) & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\overset{0}{F}\cdot f(v)f'(v)}{EG - F^2} = 0$$

$$T^2_{11} = \frac{\begin{vmatrix} E & 0 \\ F & -f(v)f'(v) \end{vmatrix}}{EG - F^2} = \frac{-E f(v) f'(v)}{EG - F^2} = \frac{-f^{\cancel{3}}(v)\cdot f'(v)}{\cancel{f^2(v)}\left[(f'(v))^2 + (g'(v))^2\right]} = \boxed{\frac{-f(v) f'(v)}{(f'(v))^2 + (g'(v))^2}}$$

We repeat the process:

$$\to \text{b)} \begin{cases} T^1_{12} E + \cancel{T^2_{12} F} = \frac{1}{2} E_v = f(v) f'(v) = T^1_{12} \cdot f^2(v) \Rightarrow \boxed{T^1_{12}} = \frac{f(v) f'(v)}{f^2(v)} = \boxed{\frac{f'(v)}{f(v)}} \\ \cancel{T^1_{12} F} + T^2_{12} G = \frac{1}{2} G_u = 0 = T^2_{12} \cdot \left[(f'(v))^2 + (g'(v))^2\right] \Rightarrow \boxed{T^2_{12} = 0} \end{cases}$$

$$\to \text{c)} \begin{cases} T^1_{22} E + \cancel{T^2_{22} F} = F_v - \frac{1}{2} G_u = 0 - 0 = T^1_{22}\, f^2(v) \Rightarrow \boxed{T^1_{22} = 0} \\ \cancel{T^1_{22} F} + T^2_{22} G = \frac{1}{2} G_v = f'(v) f''(v) + g'(v) g''(v) = T^2_{22} \cdot \left[(f'(v))^2 + (g'(v))^2\right] \Rightarrow \end{cases}$$

$$\Rightarrow \boxed{T^2_{22} = \frac{(f'(v))^2 + (g'(v))^2}{f'(v) f''(v) + g'(v) g''(v)}}$$

Practical lesson
(simplest)

The <u>pseudosphere</u>, is a surface of revolution of a tractix, it has parametrization

$x(w,v) = (w - \tanh(w), \operatorname{sech}(w)\cdot\cos(v), \operatorname{sech}(w)\cdot\cos(v))$

We intend to prove that Gauss curvature of pseudosphere is equal to -1.

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$, $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e + e^{-x}}$

$\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$

$(\tanh)' = \ldots = \dfrac{4}{(e^x + e^{-x})^2}$, $\operatorname{sech}(x) = \dfrac{2}{e^x + e^{-x}}$, $\operatorname{sech}'(x) = \dfrac{-(e^x - e^{-x})\cdot 2}{(e^x + e^{-x})^2}$

- Gaussian curvature $K = \dfrac{eg - f^2}{EG - F^2}$

$$x_w = \left[1 - \frac{4}{(e^x + e^{-x})^2}, \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}\cos(v), \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}\cdot\cos(v)\right]$$

$$x_v = [0, -\operatorname{sen}(v)\operatorname{sech}(w), -\operatorname{sen}(v)\cdot\operatorname{sech}(w)]$$

$$E = \langle x_w, x_w\rangle = \left(1 - \frac{4}{(e^x + e^{-x})^2}\right)^2 + \frac{8(e^x - e^{-x})^2}{(e^x + e^{-x})^4}\cos^2(v)$$

$$F = \langle x_w, x_v\rangle = 4\operatorname{sen}(v)\cos(v)\cdot\frac{(e^x - e^{-x})}{(e^x + e^x)^3}$$

$$G = 2\operatorname{sen}^2(v)\operatorname{sech}^2(w)$$

$$N = \frac{x_w \times x_v}{\| x_w \times x_v \|}$$

$$x_w \times x_v = \left[0, -\operatorname{sen}(v)\frac{2}{e^x + e^{-x}} + \operatorname{sen}(v)\cdot\frac{8}{(e^x + e^{-x})^3}, -\operatorname{sen}(v)\frac{2}{e^x + e^{-x}} + \operatorname{sen}(v)\cdot\frac{8}{(e^x + e^{-x})^3}\right]$$

$$\| x_w \times x_v \| = \sqrt{0^2 + 2\left(\operatorname{sen}^2(v)\left(\frac{2(e^x + e^{-x})^2 + 8}{(e^x + e^{-x})^3}\right)^2\right)}$$

$$N = \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$x_{ww} = \left[\frac{42(e^x - e^{-x})}{(e^x + e^{-x})^2}, \frac{-2(e^x + e^{-x})(e^x + e^{-x})^2 + 2(e^x - e^{-x})^2)\cos(v)}{(e^x + e^{-x})^4}, \text{''}\right]$$

$$x_{wv} = \left[0, \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}\operatorname{sen}(v), \frac{2(e^x - e^{-x})\operatorname{sen}(v)}{(e^x + e^{-x})^2}\right]$$

$$x_{vv} = [0, -\cos(v)\operatorname{sech}(w), -\cos(v)\operatorname{sech}(w)]$$

$$e = \langle N, x_{ww}\rangle = 0 + \sqrt{2}\cos(v)\left(\frac{-2(e^x + e^{-x})^3 + 2(e^x - e^{-x})^2}{(e^x + e^{-x})^4}\right)$$

$$f = \langle N, X_{wv} \rangle = 0 + 2\sqrt{2}\, \text{sen}(v) \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}$$

$$g = \langle N, X_{vv} \rangle = 0 + \sqrt{2} \cos(v) \,\text{sech}(w)$$

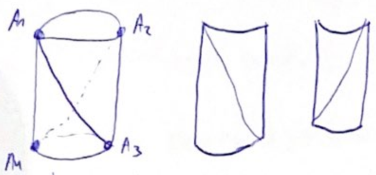
$$K = \frac{2\cos^2(v) \dfrac{(-2(e^x+e^{-x})^3 + 2(e^x - e^{-x})^2}{(e^x+e^{-x})^3} - \dfrac{8\,\text{sen}^2(v)(e^x-e^{-x})^2}{(e^x+e^{-x})^2}}{\left(1 - \dfrac{4}{(e^x+e^{-x})^2}\right)^2 2\,\text{sen}^2(v) \cdot 4 + 16\,\text{sen}^2(v) \cdot 4\cos^2(v)(e^x - e^{-x})^2 - 16\,\text{sen}^2(v)\cos^2(v)\dfrac{(e^x-e^{-x})^2}{(e^x+e^{-x})^4}} =$$

= after simplification $= -1$

Ex Calculate the Euler characteristic of S

a) S = cylinder b) S = sphere c) S = torus

a) S = cylinder

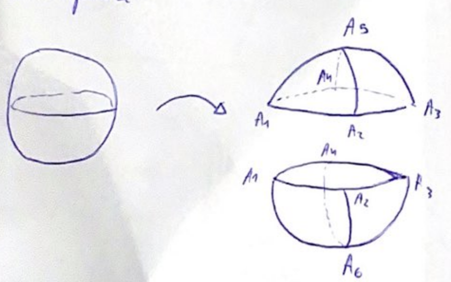


↳ lo atravesamos con un plano y lo dividimos en 2

$$\begin{cases} F = 4 \\ E = 8 \\ V = 4 \end{cases} \qquad F - E - V$$

$\chi(\text{cylinder}) = 4 - 8 + 4 = 0$

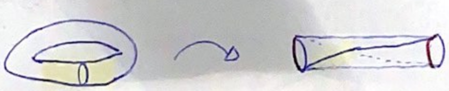
b) S = sphere



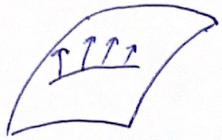
$$\begin{cases} V = 6 \\ F = 8 \\ E = 8 + 4 = 12 \end{cases}$$

$\chi(\text{sphere}) = 8 - 12 + 6 = 2$

c) S = torus



$$\begin{cases} F: 4 \\ E = 6 \\ V = 2 \end{cases} \qquad \chi(\text{torus}) = 4 - 6 + 2 = 0$$



We say that a curve $\gamma : (-\varepsilon, \varepsilon) \to S$ is a geodesic if

$$P^{\top}_{T_pS} \frac{d\gamma'(t)}{dt} = 0$$

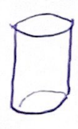
$\gamma(t) = \text{point} + t \cdot \vec{v}$

$\gamma(t) = (p_1, p_2) + t \cdot [v_1, v_2] = (p_1 + t \cdot v_1,\ p_2 + t \cdot v_2)$

$\gamma'(t) = (v_1, v_2)$, $\gamma''(t) = (0,0)$

Cor 1) Any line on a plane is a geodesic

Cor 2)  Great circles are geodesics on a sphere

Cor 3)  $\gamma(t) = (r \cdot \sin(t),\ r \cdot \cos(t),\ c \cdot t)$