## Overview

My research interests lie in the study of the wave dynamics of lattices with random material components with emphasis on long wave approximations. I use the knowledge and methods of partial differential equations, homogenization theory, and probability theory.

## Research Background

Let's start with a famous yet simple lattice, the linearized Fermi-Pasta-Ulam-Tsingou (FPUT) [9] lattice. Imagine a sequence of masses in a line and connected to their nearest neighbors via a spring as in Figure 1.

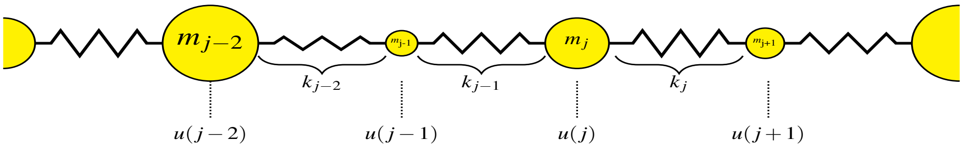


Figure 1: One-dimensional infinite spring-mass system with spatially varying masses and springs

The equations of motion of such a lattice can be easily derived from Newton's second law along with Hooke's law. We find

$$m(j)\ddot{u}(j,t) = k(j)(u(j+1,t)-u(j,t)-l(j)) - k(j-1)(u(j,t)-u(j-1,t)-l(j-1)), \tag{1}$$

where the force of $j^{\text{th}}$ spring upon the $j^{\text{th}}$ mass, $m(j)$, is proportional to the amount the spring is stretched, $u(j+1)-u(j)$, minus the relaxation length of the spring, $l(j)$. We call the $k(j)$ the spring constant. Lattices like this one have a long history of being studied [3] since they are simple yet exhibit features that arise in more complicated systems [15]. For example, similar equation can be derived for an LC circuit [2].

I have been interested in studying the behavior of the lattice when the material coefficients, i.e. the spring constants or the masses are chosen randomly. In particular I am interested in rederiving and justifying well known approximation theorems after some form of randomness has been introduced. In general, the randomness can affect how good the approximation is, as well as the details in deriving and proving the approximation. As we will overview below, dimension of the system can become an important factor when there are random coefficients. One of the main difficulties in dealing with random coefficients in this discrete setting, is that it is not clear how to apply Fourier methods, so we proceed through the framework of homogenization theory [5].

As a starting point, consider a simpler version of (1),

$$m(j)\ddot{u}(j,t) = u(j+1,t) - 2u(j,t) + u(j-1,t), \tag{2}$$

where only the masses vary and the relaxation length is taken to be 0. It is already known that an effective wave equation yields a good approximation to the macroscopic dynamics if the masses are chosen to vary periodically [16]. More specifically we have the following informal theorem.

**Theorem 1** *Let the initial conditions of u be "macroscopic" relative to the lattice as in*

$$u(j,0) = \varepsilon^{-1}\phi(\varepsilon j) \text{ and } \dot{u}(j,0) = \psi(\varepsilon j). \tag{3}$$

*Let $X = \varepsilon j$ and $T = \varepsilon t$ be the macroscopic variables. Then*

$$\tilde{u}_\varepsilon(j,t) = U(\varepsilon j, \varepsilon t) \tag{4}$$

*is a good approximation for $|t| \leq O(1/\varepsilon)$ to u if*

$$\bar{m}U_{TT} = U_{XX} \tag{5}$$

*and $U(X,0) = \varepsilon^{-1}\phi(X)$ and $U_T(X,0) = \psi(X)$.*

The goodness of this approximation is dependent upon how the masses are chosen and also what we mean by approximation. Theorems are usually easiest to prove in $\ell^2$ since one has access to energy arguments, such as the one in [12], and one usually considers the relative error, that is one measures

$$\text{Relative Error} := \frac{||\tilde{u}_\varepsilon - u||_{\ell^2}}{||u||_{\ell^2}}, \tag{6}$$

because the solution becomes larger as $\varepsilon$ becomes smaller. The effective wave speed in this case is given by $\sqrt{\frac{1}{\bar{m}}}$ where $\bar{m}$ is the average of $m$ over the period for periodic masses or the expected value of $m$ for random masses (i.e. average of $m$ over the entire lattice). One finds that the approximation is actually worse when the masses are random than when they are periodic, at least in one dimension. For example, in our setup, one can show that the relative $\ell^2$ error for periodic masses is $O(\varepsilon)$ but is $O(\sqrt{\varepsilon \log\log\varepsilon})$ for random i.i.d. masses when one wants the approximation to be true almost surely. The $\log\log$ comes from the law of the iterated logarithm for sums of i.i.d. random variables (see [8]), which plays an important role in making the approximation work. See [18] for more details. When one wants the approximation to be true in mean, one finds the error is $O(\sqrt{\varepsilon})$ as well as that the error converges to the error for the constant mass lattice as the standard deviation of the masses approaches 0. These upper bounds can be verified to be nearly sharp empirically through numerical simulation.

My work has also looked at an analogous lattice for two dimensions [19]. The original method of proof for one dimension does not easily extend to two dimensions. This is because one generally needs to solve and bound $\chi$ where

$$\Delta\chi = m - \bar{m} \tag{7}$$

and $\Delta$ is the discrete Laplacian for $\mathbb{Z}^d$. The equation (7) is easy to solve in 1D for all of $\mathbb{Z}$ but there seems to be no analytic solution for higher dimensions. However, one can get away with just solving for $\chi$ on the largest finite domain that is accessible to the wave for the times scales of interest. This domain then depends on $\varepsilon$ since the time scale of interest depends on $\varepsilon$, and this in turn makes the analysis more interesting yet also more difficult. One "solves" for $\chi$ by using the fundamental solution of $\Delta$. The form of $\chi$ is then given by a convolution of $m(x) - \bar{m}$ and the fundamental solution,

$$\chi_\varepsilon(x) = \sum_{y \in D_\varepsilon} \phi(y-x)(m(y) - \bar{m}). \tag{8}$$

There is no exact formula for the fundamental solution, $\phi$, of the discrete Laplacian (in 2D or higher) but one has bounds on it. This in turn allows one to bound $\chi$ but not without first appealing to some important probabilistic theory. To establish an almost sure bound when the masses are independent and bounded away from 0 and $\infty$, one can use Hoeffding's Theorem [17] to find a Chernoff type bound for (8) in $\varepsilon$. After a little bit of massaging, the bound says that the probability that (8) is larger than $O(\varepsilon^{-1}L(\varepsilon^{-1}))$ ($L$ is a slowly varying function like log) for all $x \in D_\varepsilon$ is summable allowing one to apply the Borel-Cantelli Lemma. One then comes to the amazing result that the relative error is $O(\varepsilon L(\varepsilon^{-1}))$, which is a half power better in $\varepsilon$ than it is in 1D. The approximations for both the constant mass lattice and the periodic mass lattice do not exhibit this dimensionality dependence. Note that if $m$ is periodic, then from (7), $\chi$ is also periodic and thus simply bounded. The approximation for the random mass lattice does depend on the dimension, because the size of $\chi$ is dependent on the fundamental solution which is different in each dimension, just like for the continuous Laplacian. As a heuristic, one can compute, using the independence of the masses, that

$$(\mathrm{Var}\,[\chi_\varepsilon(x)])^{1/2} = \left( E\left[ \left( \sum_{y \in D_\varepsilon} \phi(y-x)(m(y) - \bar{m}) \right)^2 \right] \right)^{1/2} \leq b \left( \sum_{y \in D_\varepsilon} \phi(y-x)^2 \right)^{1/2} \tag{9}$$

where it is assumed $E((m(x) - \bar{m})^2) \leq b^2$. In one dimension, $\phi(x) = \frac{1}{2}|x|$. In two dimensions, $\phi(x) \approx \log(|x|)$ [7], and in three dimensions $\phi(x) \approx \frac{1}{|x|}$. (Note $\phi(0) = 0$ for the discrete Laplacian in $\mathbb{Z}^d$.) On then finds that $(\mathrm{Var}\,[\chi_\varepsilon(x)])^{1/2}$ is $O(\varepsilon^{d/2-2}L(\varepsilon^{-1}))$ which becomes smaller i.e. better as the dimension increases.

It should be noted that much has already been said about the propagation of acoustic waves through a continuous random medium. For an extensive resource, see [11]. However, sizes of errors are rarely addressed. Furthermore, almost sure convergence results in the discrete setting require different techniques. In the continuous setting, convergence theorems can be achieved more directly through the law of large numbers. In our setting, one cannot use the law of large numbers outside of the formal calculation and must define what one means by convergence. In order to achieve a rigorous rate of convergence, one must prove the types of bounds on the stochastic error terms discussed above and then look at the limit through the lens of "coarse-graining", which is also used in [16] to prove convergence in the periodic problem.

## Ongoing and future research

## Longer time scales and higher dimensions

In [23] and [12], it is proven that the Korteweg-de Vries (KdV) equation governs the dynamics of the FPUT lattice (with a non-linearity force between the masses and in one dimension) when the masses and spring constants are periodic. One uses a different scaling regime i.e. $\tilde{u}_\varepsilon = \varepsilon U(\varepsilon j - c\varepsilon t, \varepsilon^3 t)$ and then finds that the travelling wave with a speed of sound $c$, $U$, satisfies the KdV equation in $w := \varepsilon j - c\varepsilon t$ and $\tau := \varepsilon^3 t$. The goal of such approximation is to prove the existence of traveling waves or sometimes more specifically, solitons, for timescales that are $O(\varepsilon^{-3})$ or longer. In [12], many of the elements of the derivation and the proof, including solving for $\chi$ and using an energy argument, are similar to what is seen for the wave equation albeit a bit more involved. One would then hope that maybe, with the help of some of the probabilistic tools used in proving the wave equation approximation, one could prove that the KdV approximation also holds for random masses and springs. This, however, does not seem to be possible, probably because of the phenomenon known as Anderson localization [24] where localized eigenfunctions [1] of the random operator $\frac{1}{m}\Delta$ inhibit the propagation of waves. Although dynamical Anderson localization hasn't been proven specifically for FPUT with random coefficients, if one simulates the system for such large time scales, one can see considerable attenuation of the wave for these timescales. There has been some work in trying to account for such attenuation in an analogous setting of shallow water waves moving over a random bottom [4] where formal calculations lead to a modified KdV equation that predicts the attenuation, but it is unclear how to translate such results into the FPUT setting. A starting point could be looking at numerical simulations.

One of the main issues one runs into in trying to derive the KdV approximation is that $\chi$ turns out to be too large for the longer timescales causing the error to be too large. Recall that the size of $\chi$ depends upon the size of the domain accessible to the wave. If one goes back to (9) and uses the fundamental solution for the 1D Laplacian, one finds that $\chi$ is $O(\varepsilon^{-9/2})$ for $O(\varepsilon^{-3})$ timescales. As an ad hoc work around, one may take the masses to be random in a special way. First, generate a sequence of independent, mean zero random variables $\gamma \in [a,b]$. Then generate the masses from $m = \Delta\gamma + \bar{m}$ where $\bar{m}$ so that $m$ is always bounded away from 0. Then $\chi$ equals $\gamma$ solves (7) and is bounded in $\varepsilon$. There are more technical details that still need to be resolved, but since $\chi$ is bounded, one should be able to prove the KdV approximation for this special case, and simulations agree with this prediction.

Thus, the size of $\chi$ seems to be the key factor in determining how good an approximation is in general. This leads to the interesting question of whether, in a sufficiently higher dimensional analog of the FPUT system, one can prove some sort of result regarding longer time existence of travelling waves that would not be possible to prove in 1D. For example, it can be shown that the dynamics of a scalar FPUT system may be approximated by the KP-II equation [21]. I am especially interested in trying to prove a similar result where the masses are random. There is also a kind of precedent for this regarding Anderson localization since it is known to be weaker in 3D and thus one may expect traveling waves to exist for longer times without attenuating.

## Other Projects

In addition to KdV, it is known that FPUT can be approximated by a nonlinear Schrödinger equation for the correct kind of initial data [22]. I am interested if this could be extended to the case of random coefficients. I am also interested in seeing how random coefficients may play a role in other types of lattice differential equations such as discrete reaction diffusion equations. Without randomness, these systems are known to exhibit travelling fronts [13] and are important in modeling many systems such as Cellular Neural Networks [6] and species invasion in patchy landscapes [14].

Finally, I am interested in parameter estimation for discrete systems which exhibit wave-like behavior. That is, given some observed solution or parts of a solution to such a system, can one estimate the coefficients of the system? For example, if approximate waves are observed in (2), can one determine what the value of the masses are through which the wave passes? In the formal derivation of an approximate traveling wave for (2), one finds that

$$u(j,t) - \varepsilon^{-1}\phi(\varepsilon j - \frac{1}{\sqrt{\bar{m}}}\varepsilon t) \approx \chi(j)\phi'\left(\varepsilon j - \frac{1}{\sqrt{\bar{m}}}\varepsilon t\right). \tag{10}$$

[18]. That is, formally, the next order of $\varepsilon$ is greatly influenced by $\chi$. Thus I am interested in using variation between the true solution and effective wave to estimate the masses or features of the masses through which the wave passes. Such parameter estimation using the observations of waves has seemingly been limited to seismic waves [20], and it would be nice to extend it to this and other discrete settings.

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