3. (More difficult version of Question 1) Consider the matrix $A \in \mathbb{R}^{4\times 4}$,

$$A = \begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 2 & 1 \\ -3 & -1 & 0 & 0 \end{bmatrix}.$$

It has an eigenvalue $\lambda = 1$ of algebraic multiplicity 2 and geometric multiplicity 1. Find an eigenvector $\mathbf{b}_1$ and a generalised eigenvector $\mathbf{b}_2$ for λ, such that $\mathbf{b}_1, \mathbf{b}_2$ form a Jordan chain for $\lambda = 1$ (so $\lambda\mathbf{b}_1 = A\mathbf{b}_1$ and $\lambda\mathbf{b}_2 + \mathbf{b}_1 = A\mathbf{b}_2$, see below for instructions). Let $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ be a Jordan normal form basis for A, where the first two elements equal your previously computed Jordan chain $\mathbf{b}_1, \mathbf{b}_2$ for $\lambda = 1$. The vector $\mathbf{v} = (-1, 1, 0, 0)^T$ can be written as a linear combination $\mathbf{v} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3 + \alpha_4\mathbf{b}_4$. Find the coefficients α_1 and α_2.

(Hints) The general procedure for finding Jordan chains simplifies to the following steps if we know that λ has algebraic multiplicity 2 and geometric multiplicity 1:

(1) compute REFs for $A - \lambda I$ and $(A - \lambda I)^2$, and call them R_1 and R_2. The matrix R_1 has one non-pivot column (call its number k), the matrix R_2 has one *additional* pivot column (call its number ℓ).

(2) One possible candidate for generalised eigenvector $\mathbf{b}_2$ is the solution $\mathbf{x}$ of $R_2\mathbf{x} = 0$ with $\mathbf{x}_k = 0$ and $\mathbf{x}_\ell = 1$ (or any non-zero number).

(3) The corresponding eigenvector $\mathbf{b}_1$ is then $\mathbf{b}_1 = (A - \lambda I)\mathbf{b}_2$.

Note that the choice how we pick $\mathbf{b}_2$ in step 2 is somewhat arbitrary. This suggestion works best for easy computability on paper. For finding the coefficients α_1, α_2, you may plan to repeat the above procedure for A^T, but for the left generalised eigenvector the choice suggested in step 2 is not particularly good.

Solution 3. We first observe that the approach from Question 1 cannot work for this problem (according to the result in Question 2), since any left eigenvector (let's call it $\mathbf{c}_1$ here) and any right eigenvector $\mathbf{b}_1$ for the same eigenvectors satisfy $\mathbf{c}_1^T\mathbf{b}_1 = 0$ as soon as there are generalised eigenvectors of higher order present.

First let us calculate right eigenvector $\mathbf{b}_1$ and generalised right eigenvector $\mathbf{b}_2$. We take $L_1 = A - \lambda I =$

```
[ -2, -2, -1, -1]
[  1, -1, -1,  0]
[  1,  2,  1,  1]
[ -3, -1,  0, -1]
```

and perform Gauss elimination to REF. The $\mathrm{RREF}_{nz}(L_1) = R_1 =$

```
[ 1, 0, 0,  0]
[ 0, 1, 0,  1]
[ 0, 0, 1, -1]
```

such that the non-pivot column is $k = 4$. for the second order we compute $L_2 = R_1 L_1 =$

```
[ -2, -2, -1, -1]
[ -2, -2, -1, -1]
[  4,  3,  1,  2]
```

which has the same nullspace as L_1^2. Its $\mathrm{RREF}_{nz}(L_2) = R_2 =$

```
[ 2, 0, -1, 1]
[ 0, 1,  1, 0]
```

which has the non-pivot column (in addition to k) $\ell = 3$. So, we fix $b_{2,k} = b_{2,4} = 0$ and set $b_{2,\ell} = b_{2,3} \neq 0$ to find the solution $\mathbf{b}_2 = (1, -2, 2, 0)^\mathsf{T}$. The corresponding eigenvector $\mathbf{b}_1$ is $L_1\mathbf{b}_2 = (0, 1, -1, -1)^\mathsf{T}$, such that we have overall the right Jordan chain

$$(\mathbf{b}_1, \mathbf{b}_2) = (L_1\mathbf{b}_2, \mathbf{b}_2) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \\ -1 & 2 \\ -1 & 0 \end{bmatrix}.$$

For the left Jordan chain, we know that no matter how we pick the left eigenvector $\mathbf{c}_1$, it will satisfy $\mathbf{b}_1^\mathsf{T}\mathbf{c}_1 = 0$. So, if we pick our generalised left eigenvector $\mathbf{c}_2$ such that $\mathbf{b}_1^\mathsf{T}\mathbf{c}_2 = 1$ and $\mathbf{b}_2^\mathsf{T}\mathbf{c}_2 = 0$, then $\mathbf{c}_2$ cannot be in the nullspace of A^T (spanned by $\mathbf{c}_1$). Hence, we compute $\hat{L}_2 = (L_1^\mathsf{T})^2 =$

```
[ 4, -4, -2, 8]
[ 5, -3, -3, 8]
[ 3, -1, -2, 4]
[ 2, -2, -1, 4]
```

and find its REF, which will have two zero rows, dropped in $\hat{R}_2 = \mathrm{RREF}_{nz}(\hat{L}_2) =$

```
[ 4, 0, -3,  4]
[ 0, 4, -1, -4]
```

We replace the zero rows with $(\mathbf{b}_1, \mathbf{b}_2)^\mathsf{T}$ to get $R_{ext} =$

```
[ 4,  0, -3,  4]
[ 0,  4, -1, -4]
[ 0,  1, -1, -1]
[ 1, -2,  2,  0]
```

and solve

$$R_{ext}\mathbf{c}_2 = \begin{bmatrix} \hat{R}_2 \\ \mathbf{b}_1^\mathsf{T} \\ \mathbf{b}_2^\mathsf{T} \end{bmatrix} \mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

to find $\mathbf{c}_2 = (0, -4/3, -4/3, -1)^\mathsf{T}$, and, hence, $\mathbf{c}_1 = L_1^\mathsf{T}\mathbf{c}_2 = (1/3, -1/3, 0, -1/3)^\mathsf{T}$ to get the complete left Jordan chain (sorted in reverse order)

$$(\mathbf{c}_2, \mathbf{c}_1) = (\mathbf{c}_2, L_1^\mathsf{T}\mathbf{c}_2) = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{4}{3} & -\frac{1}{3} \\ -\frac{4}{3} & 0 \\ -1 & -\frac{1}{3} \end{bmatrix}.$$

We may check that $(\mathbf{c}_2, \mathbf{c}_1)^T(\mathbf{b}_1, \mathbf{b}_2) = I_2$ and then apply $\mathbf{w}_1^T = \mathbf{c}_2^T$ and $\mathbf{w}_2^T = \mathbf{c}_1^T$ to any vector $\mathbf{v}$ to get the coefficients α_1 and α_2. In particular for $\mathbf{v} = (-1, 1, 0, 0)^T$ this is

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}\mathbf{w}_1^T\\ \mathbf{w}_2^T\end{bmatrix}\begin{bmatrix}-1\\ 1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}0 & -4/3 & -4/3 & -1\\ 1/3 & -1/3 & 0 & -1/3\end{bmatrix}\begin{bmatrix}-1\\ 1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}-4/3\\ -2/3\end{bmatrix}$$

An alternative way to find $(\mathbf{w}_1, \mathbf{w}_2)$ is to construct a Jordan chain $(\mathbf{c}_1, \mathbf{c}_2)$ for $L_1^T = A^T - \lambda I$ in the same way as we constructed $(\mathbf{b}_1, \mathbf{b}_2)$ for L_1 and then notice that the 2×2 matrix

$$P_2 = \begin{bmatrix}\mathbf{c}_1^T\\ \mathbf{c}_2^T\end{bmatrix}\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}$$

is invertible but possibly different from the identity. Then setting

$$\begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{bmatrix} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix}\left(P_2^{-1}\right)^T.$$

Then $(\mathbf{w}_1, \mathbf{w}_2)$ still span the two-dimensional nullspace of $(A^T - \lambda I)^2$ and satisfy

$$\begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{bmatrix}^T\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

such that $\mathbf{w}_1^T\mathbf{v} = \alpha_1$ and $\mathbf{w}_2\mathbf{v} = \alpha - 2$ for all $\mathbf{v}$.

Powers and functions of matrices

4. Consider the matrix

$$A = \begin{pmatrix}\frac{2}{5} & \frac{3}{5}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}.$$

(i) Show that the row-sums (i.e., the sum of the elements in each row) of A^n are 1 for all $n \in \mathbb{N}$.

(ii) Compute A^{10},

(iii) Show that $A^n \to \begin{pmatrix}\frac{10}{19} & \frac{9}{19}\\ \frac{10}{19} & \frac{9}{19}\end{pmatrix}$ as $n \to \infty$.

(iv) Inspired by this example, show that, if a matrix $A \in \mathbb{C}^{n\times n}$ has one eigenvalue $\lambda_1 = 1$ of algebraic multiplicity 1 with right eigenvector $\mathbf{v}_1$ and left eigenvector $\mathbf{w}_1^T$ (scaled such that $\mathbf{w}_1^T\mathbf{v}_1 = 1$), and all other eigenvalues of A have modulus smaller than 1, then $\lim_{k\to\infty} A^k = \mathbf{v}_1\mathbf{w}_1^T$. (This question is easier when one assumes that A is diaogonalisable, so this should be attempted first.)

Solution 4.

(i) Proof by induction. Clearly A itself has row-sums equal to 1. Assume

$$A^k = (a_{ij}^{(k)})_{i,j=1}^2$$