ECEN315 - Open Loop Response of a Motorised, Propellor Driven Pendulum

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1 Introduction

This report covers the derivation of the theoretical open loop response for a motorised, propellor driven pendulum arm, for the purpose of later designing controllers. To do this the resultant angular displacement from an applied voltage needs to be quantifiable.

2 Background

Fig. 1. Block diagram for full system

An open loop system is a system where the output is not fed back into the system. The alternative is a closed loop system where the input is fed back into the system, typically subtracted from the input to get the error. The full system (fig. 1) that this paper will be modelling can be done so as an open loop system. While the overarching system is an open loop system, some of the systems that make up the blocks are themselves, closed loop systems, while others are open loop systems.

One of the evaluated characteristics of this system is it's stability. Stability of a system is defined as whether or not the output grows without bounds, if it does then the system is unstable, whereas it is stable if it settles on a value. This is regardless of any oscillation that would occur in the system. Stability can also be pre-determined by looking at the poles of the system, i.e. the values of s that make the denominator of the transfer function 0. If the real component of the poles is negative then the system is stable and vise versa if it is positive.

The response of the system will be classified against a step input. This input is a suitable approximation for flicking a switch in a practical system.

3 Method

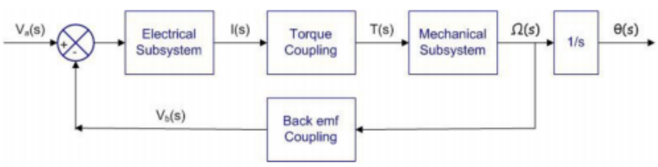


Fig. 2. Block diagram that describes the motor response to an applied voltage. [1]

The block diagram in fig. 1 illustrates an approximation of the full powered pendulum system. This system takes an input voltage into the motor causing the shaft to spin with angular velocity $\omega(t)$. Due to the propellor on the motor, this speed at which the shaft spins will generate a force which is represented by the constant K_p. The torque is then obtained by mutliplying by the distance from the pivot to the center of the motor. Finally this torque is applied to the pendulum arm, resulting in an angular displacement of the arm.

The first block to look at is the one that describes how the motor responds to an applied

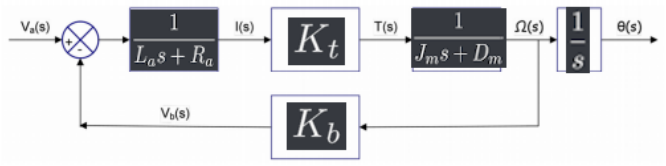


Fig. 3. Block diagram for motor and prop with equations in corresponding blocks [1]

voltage. This behaviour can be broken down into smaller sub-systems as shown in fig. 2.Each of those systems can be filled with the appropriate transfer function consisting of measurable properties of the motor with propellor as show in fig. 3 where:

- L_a is the inductance of the armature
- R_a is the resistance of the armature
- K_t is the torque constant that relates the current to the torque produced
- J_m is the inertia of the load (propellor)
- D_m is the damping coefficient of the load (propellor)
- K_b is the back emf constant, relating the angular velocity to an induced voltage.

The block diagram can be simplified into a generic transfer function (eq. (1)) the represents the response of the motor to an applied voltage (See Appendix C for full derivation). This transfer function was then plugged into MATLAB and evaluated with step inputs of amplitude 2, 3, 4, 5 and 6V.

$$\frac{\Omega(s)}{V(s)} = \frac{\frac{K_t}{J_m L_a}}{s^2 + \frac{J_m R_a + D_m L_a}{J_m L_a} s + \frac{R_a D_m + K_t K_b}{J_m L_a}} \tag{1}$$

The next block to expand upon is the one that represents the pendulums response, as the two in the middle are just constants. For this block, a force is being applied (from the motor) which results in the angular displacement of the pendulum arm. An expression for net torque ($\tau_{net} = \alpha J_p$) must then be derived. Using the diagram in fig. 4, it can be seen that there are three main torques acting on the arm; The torque from the propellored motor ($\tau_m = Fr$), the torque induced by friction around the pivot when the arm is moving ($\tau_f = c\frac{\mathrm{d}\theta}{\mathrm{d}t}$) and the component of the gravity vector that is perpendicular to the arm ($\tau_g = dmg\sin\theta$).

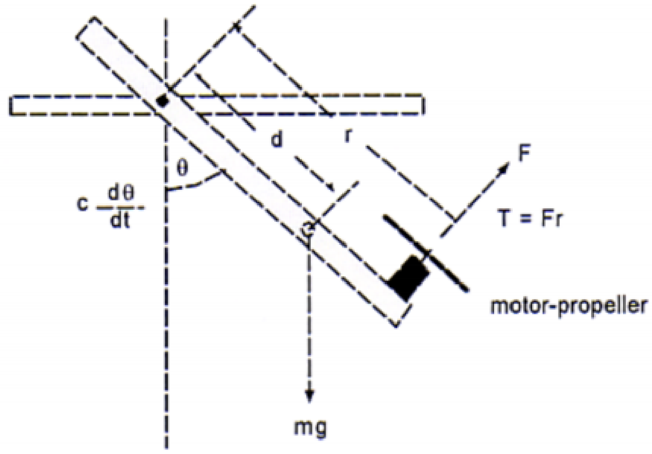


Fig. 4. Diagram of the powered pendulum

Combining the different torques of the system into one equation allows us to find an expression for the torque produced by the motor (eq. (2)). After making the assumption that $\sin\theta = \theta$ and that α is the second derivative of angular displacement, eq. (2) takes the form of a second order homogeneous ordinary differential equation. Applying the Laplace transform to this equation allows us to rearrange it so that it resembles a transfer function with angular displacement as the output to an applied torque (eq. (3)).

$$\begin{aligned} \tau_{net} &= \tau_m - \tau_f - \tau_g \\ \tau_m &= \tau_{net} + \tau_f + \tau_g \\ \tau_m &= \alpha J_p + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + dmg\sin\theta \\ \tau_m &= J_p\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + dmg\sin\theta \\ \tau_m &= J_p\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + dmg\theta \end{aligned} \tag{2}$$

$$\frac{\Theta(s)}{\mathrm{T(s)}} = \frac{\frac{1}{J_p}}{s^2 + \frac{c}{J_p}s + \frac{dmg}{J_p}} \tag{3}$$

Before plotting this in MATLAB for evaluation, the value for each variable need to be found. For d, m and g this is relatively easy as it is just measuring the distance to the center of mass, weighing it and using the constant value for gravity on earth of $9.81\frac{m}{s^2}$. To find J_p and c the pendulum was let to fall and the oscillation recorded. An exponential was then fit through the peaks of the plot to find the values of A and B in eq. (4) using the non-linear model (nlm) function in MATLAB (Can also see Appendix D for a manual method). eq. (4) is the equation that models the motion of the pendulum where ω is given by eq. (5). When finding A and B, only the non-oscillitary component of eq. (4) is important and used. Once found, eq. (5) can be rearranged for J_P and then $c = 2BJ_p$.

$$\begin{aligned} y &= Ae^{-\frac{c}{2J_p}t}\cos(\omega t + \phi) \\ &= Ae^{-Bt}\cos(\omega t + \phi) \end{aligned} \tag{4}$$

$$\begin{aligned} \omega &= \sqrt{\frac{mgd}{J_p} - \left(\frac{c}{2J_p}\right)^2} \\ &= \sqrt{\frac{mgd}{J_p} - B^2} \end{aligned} \tag{5}$$

4 Results

TABLE 1
Table of Motor Parameters

Motor Parameters	Value
$R_a(\Omega)$	6.3
$La(H)$	0.797
$Kb(V.s/rad) = Kt(N.m/A)$	0.0043
$Dm(N.m.s/rad)$	0.00000553
$Jm(kgm2)$	0.00000241

5 Discussion

6 Conclusion

References

[1] G. Gouws, *Understanding small, permanent magnet brushed DC motors*. Victoria University of Wellington, 2008. [Online]. Available: https://ecs.wgtn.ac.nz/foswiki/pub/Courses/ECEN315_2020T1/Assignments/DCmotor_transfer_function.pdf

Appendix A
Matlab Code

```matlab
%Script for ECEN315 Lab 2
clear
clf
R_a = 6.3; %Ohms
L_a = 0.797; %H
K_b = 0.0043; %Vs/rad
K_t = K_b; %Nm/A
D_m = 0.00000553; %Nms/rad
J_m = 0.00000241; %Kgm^2

numerator = K_t/(L_a*J_m);
sSqCoef = 1;
sCoef = (L_a * D_m + R_a * J_m)/(L_a * J_m);
coef3 = (R_a * D_m + K_t * K_b)/(L_a * J_m);

sys = tf(numerator , [sSqCoef sCoef coef3])
stepCoefs = [1 2 3 4 5 6];
steps = 6;

Legend = cell(steps, 1);

for i = 1:steps
    step(i * sys, 10)
    hold on
    Legend{i} = strcat(num2str(i), 'V Step');
end
hold off
ylabel("\omega_m (rad/s)");
xlabel("Time (s)");
title("Step Response of the Motor");
legend(Legend)

stepResponseInfo = stepinfo(stepCoefs.*sys)%, 'SettlingTimeThreshold', 0.000005); %Include this setting to know approx when oscillation ends

%units of steady state gains rad/Vs
Gain = numerator/coef3
steadyStateValue = Gain * stepCoefs

SettlingTime = [stepResponseInfo(1:6).SettlingTime] %The inductance of the motor is likely too high which is causing the critical damping/overdamping and thus quick settling time.
damp(sys)
```

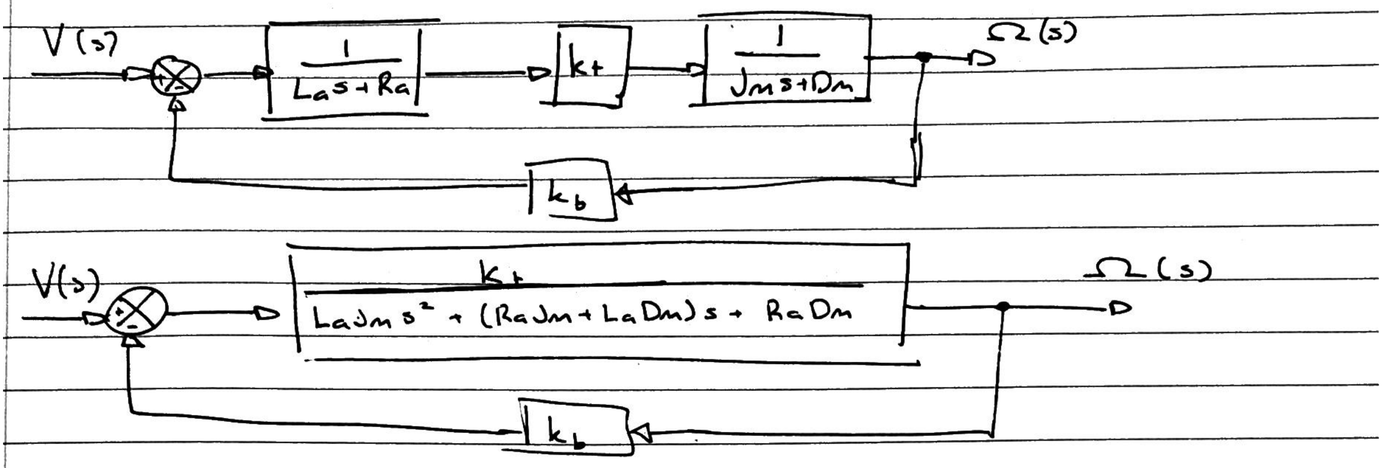
```matlab
% Finding Coefficients from un-driven Damped pendulum
```

```
clear
clf

m = 0.168; %kg
g = 9.81;  %m/s^2
d = 0.14; %m distance from pivot to cog
r = 0.165; %m length of pendulum arm
T = 0.97; %Period in seconds
f = 1/T; %freq
w = 2*pi*f; %angular frequency

data = readtable("Data.csv"); % load csv data

figure(1)
plot(data.Var1, data.Var2); % plot csv data
xlabel("Time (s)")
ylabel("y(t)")

positiveValues = abs(data.Var2); % make all peaks positive

[peaks, locs] = findpeaks(positiveValues); % find all peaks
peaksTable = table(data.Var1(locs), peaks); % convert to table

hold on
plot(peaksTable.Var1, peaksTable.peaks); % plot line connecting all
    peaks
hold off

%Fitting coefficients to the function
modelfun = @(b,x) b(1)*exp(-b(2).*x(:, 1)); % function to model
beta0 = [200, 1]; % initial Guesses
model = fitnlm(peaksTable, modelfun, beta0); % the non-linear model we
    get out
coefs = model.Coefficients{:, 'Estimate'}; % Coefficients we are
    looking for


A = coefs(1); % Our A coefficient
B = coefs(2); % Our B coefficient

j_p = (m*d*g) / (power(w,2)+power(B,2)) % moment of inertia we are
    looking for
c = 2 * B * j_p % damping coeff we are looking for

% Transfer Function

numerator = 1/j_p;
coefB = c/j_p;
coefC = (d*m*g)/j_p;
```

```

sys = tf(numerator, [1, coefB, coefC])
figure(2)
step(sys)


% Combined Transfer function
% Lab 2 Transfer Function
R_a = 6.3; %Ohms
L_a = 0.797; %H
K_b = 0.0043; %Vs/rad
K_t = K_b; %Nm/A
D_m = 0.00000553; %Nms/rad
J_m = 0.00000241; %Kgm^2

numeratorLab2 = K_t/(L_a*J_m);
coefALab2 = 1;
coefBLab2 = (L_a * D_m + R_a * J_m)/(L_a * J_m);
coefCLab2 = (R_a * D_m + K_t * K_b)/(L_a * J_m);

sysLab2 = tf(numeratorLab2 , [coefALab2 coefBLab2 coefCLab2]);
ylabel("\theta (rads)")

k_p = 0.0053;

sys3 = sysLab2 * sys * k_p * r
Leg = cell(3, 1);
figure(3)
hold on
for i = 3:5
    step(i * sys3);
    Leg{i-2} = strcat(num2str(i), 'V Step');
end
hold off
ylabel("\theta (rads)")
legend(Leg)
```

Appendix B
Risk Assessment

Appendix C
Derivation of the Motor Transfer Function



$$\frac{\Omega(s)}{V(s)} = \frac{\dfrac{K_t}{L_aJ_m s^2 + (R_aJ_m + L_aD_m)s + R_aD_m}}{1 + \dfrac{k_t k_b}{L_aJ_m s^2 + (R_aJ_m + L_aD_m)s + R_aD_m}}$$

$$= \frac{k_t}{L_aJ_m s^2 + (R_aJ_m + L_aD_m)s + R_aD_m + k_t K_b}$$

$$= \frac{\dfrac{K_t}{L_aJ_m}}{s^2 + \dfrac{(R_aJ_m + L_aD_m)}{L_aJ_m}s + \dfrac{R_aD_m + k_t k_b}{L_aJ_m}}$$

Fig. 5. Full derivation of fig. 3 from block diagram to transfer function

APPENDIX D
MANAUL DERIVATION OF A AND B FOR AN UNPOWERED DAMPED PENDULUM

Model an exponential by drawing a line through the peaks we can ignore the sinusoidal component of the function

$y(t) = Ae^{-Bt}$

take two points

$(0.47, 184.6632)$
and
$(3.85, 18.0729)$

$A_1 \quad 184.6632 = A e^{-B(0.47)}$

$A_2 \quad 18.0729 = A e^{-B(3.85)}$

$$\frac{A_1}{A_2} = \frac{184.6632}{18.0729} = \frac{e^{-B(0.47)}}{e^{-B(3.85)}}$$

$$= e^{-B(0.47)} e^{B(3.85)}$$

$$= e^{B(\cancel{0.47}\, 3.85 - 0.47)}$$

$$10.217685 = e^{3.38B}$$

$$\ln(10.217685) = 3.38B$$

$$B = \frac{\ln(10.217685)}{3.38}$$

$$= 0.6876$$

$= 0.688$ agrees with NLM.

~~184.66~~

$A = 255.113$

Fig. 6. Working for finding A and B using two peaks of a damped sinusoid