Ecen 321 test 2 Joshua Benfell

1 RV X over space a to b with pdf $f_X(x)$

by definition

a $\mu_X = E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx$

~~let $f_X(x)$~~

$= \cancel{\int_a^b x f_X(x)\,dx} = \cancel{\int_a^b \int_a^b f_X(x)}$

as it is uniforming distributed, the pdf $f_X(x) = \frac{1}{b-a}$ over a-b

$\therefore$

$$\mu_X = E\{X\} = \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{1}{b-a}\left(\frac{(b+a)(ab-a)}{2}\right)$$

$$= \frac{b+a}{2}$$

b Variance σ_X^2

by definition

$$\sigma_X^2 = E\{(X-\mu_X)^2\}$$

$$= E\{(X - E\{X\})^2\}$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \mu_x^2$$

$$\rightarrow \sigma_X^2 = \int_a^b x^2 \frac{1}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2$$

$$= \left.\frac{x^3}{3}\right|_a^b \frac{1}{b-a} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{b^3-a^3}{3}\,\frac{1}{b-a} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{a^2+ab+b^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3b^2-6ab-3a^2}{2}$$

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1b cont $\sigma_x^2 = \dfrac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}$

$= \dfrac{a^2 - 2ab + b^2}{12}$

$= \dfrac{(b-a)^2}{12}$ in this form as b is typically referred as the higher limit.

2 $y = \dfrac{(x_2 - x_1)}{x_1}$

a $x_1 = 1.11 \pm 0.01$, $x_2 = 0.99 \pm 0.01$

Value of $y = \dfrac{0.99 - 1.11}{1.11}$

$= \cancel{-0.10909}$

$= -0.108$

b uncertainty of y, $\sigma_y = \sqrt{\left(\dfrac{\partial Y}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\dfrac{\partial Y}{\partial x_2}\right)^2 \sigma_{x_2}^2}$

$y = \dfrac{x_2}{x_1} - 1$

$\dfrac{\partial y}{\partial x_2} = \dfrac{1}{x_1} = \dfrac{1}{1.11}$

$\dfrac{\partial y}{\partial x_1} = \dfrac{-x_2}{x_1^2}$

$= \dfrac{-0.99}{1.11^2}$

$\sigma_y = \sqrt{\left(\dfrac{-0.99}{1.11^2}\right)^2 \times 0.01^2 + \left(\dfrac{1}{1.11}\right)^2 \times 0.01^2}$

$= \cancel{0.0145}$

$= 0.01207$

$= 0.0121$ 4dp.

$y = -0.108 \pm 0.0121$

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3a ~~#~~ exponentiall dist

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\int y v' = yv - \int y' v$$

$$\mu_X = E\{X\} = \int_0^\infty x\lambda e^{-\lambda x}\, dx$$

let $u = x\lambda$

$\frac{du}{dx} = \lambda$

$dx = \frac{du}{\lambda}$

$$= \frac{1}{\lambda}\int_0^\infty u e^{-u}\, du$$

let $y = u$

$\frac{dy}{du} = 1$

let $v' = e^{-u}$

$v = -e^{-u}$

$$= \frac{1}{\lambda}\left(-ue^{-u}\Big|_0^\infty + \int_0^\infty e^{-u}\, du\right)$$

$$= \frac{1}{\lambda}\left(-ue^{-u}\Big|_0^\infty - e^{-u}\Big|_0^\infty\right)$$

$$\lim_{u\to\infty} ue^{-u} = 0$$

$$\mu_X = \frac{1}{\lambda}\left(\lim_{u\to\infty} -ue^{-u} + 0 \times e^0 - \left(\lim_{u\to\infty} e^{-u} - e^0\right)\right)$$

$$= \frac{1}{\lambda}(0 + 0 - 0 + 1)$$

$$= \frac{1}{\lambda}$$

b $\mu = 10m$ $\therefore$ $\lambda = 1/10$

find $P(X > 15)$

$CDF = 1 - e^{-\lambda x}$

$$P(X > 15) = \cancel{1 - P(X < 15)}$$

$$= 1 - P(X < 15)$$

$$= 1 - (1 - e^{-15 \times 0.1})$$

$$= e^{-0.1 \times 15}$$

$$= 0.223$$

$$= 22.3\ \%$$

4 Estimation

a Accurate - low bias
Precise - low standard deviation.

b bias = sample mean of estimator - true mean.

$b = E\{\hat{\theta}\} - \theta$

$\sigma_{\hat{\theta}} = E\{\hat{\theta} - E(\hat{\theta})\}$ for the variance of the estimator

c an accurate low bias estimator has its mean close to the mean of the value it's estimating having this low allows us to more accurately determine the true mean or a value that's a close approximation of it.

high precision is desirable. as we can then be more confident that the values of the mean are the true values of this estimator.

5 $X \sim Bin$

$$P(X=x;p) = \begin{cases} \frac{N!}{x!(N-x)!} p^x (1-p)^{n-x} & , x = 0 \dots N \\ 0 , \text{ otherwise} \end{cases}$$

$\frac{N!}{x!(N-x)!}$ is constant so we shall opt to ignore it

for N trials, k successes, find $\hat{p}$

$$P(X=k;\hat{p}) = p^k (1-p)^{(N-k)}$$

$$\log(P) = k \log \hat{p} + (N-k) \log(1-\hat{p})$$

$$\frac{\partial \log P}{\partial \hat{p}} = \frac{k}{\hat{p}} - \frac{N-k}{1-\hat{p}} = 0$$

$$= \cancel{\frac{k - k\hat{p} - N\hat{p} + k\hat{p}}{\hat{p}(1-\hat{p})}}$$

$$\frac{k}{\hat{p}} = \frac{N-k}{1-\hat{p}}$$

$$= \frac{k-N\hat{p}}{\hat{p}(1-\hat{p})}$$

$$\frac{1-\hat{p}}{\hat{p}} = \frac{N-k}{k}$$

$$\cancel{\frac{k-N\hat{p}}{\hat{p}(1-\hat{p})} = 0}$$

$$\frac{1}{\hat{p}} - 1 = \frac{N}{k} - 1$$

$$\cancel{k - N\hat{p} = \hat{p}(1-\hat{p})}$$

$$\frac{1}{\hat{p}} = \frac{N}{k}$$

$$\hat{p} = \frac{k}{N}$$

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6 Small S

$n_s = 15$

$k_s = 11$

$P_s = 11/15$

Large L

$n_L = 1\cancel{4}2$

$k_L = \cancel{4}4$

$P_L = \frac{4}{12}$

$Z_{\alpha/2} = 2.055$
From table.

Using CLT

$S \sim N(\underbrace{\cancel{44}\ 11}_{np}, \underbrace{\tfrac{44}{15}}_{np(1-p)})$ $\quad L \sim N(4, \frac{8}{3})$

then inflate p and n for L and S

$\tilde{n}_s = 15+2 = 17$

$\tilde{p}_s = \frac{11+1}{17}$

$= 12/17$

$\tilde{n}_L = \cancel{8}\ 12+2 = 14$

$\tilde{p}_L = \frac{4+1}{14}$

$= 5/14$

CI

$$\tilde{p}_s - \tilde{p}_L \pm Z_{\alpha/2}\sqrt{\frac{\tilde{p}_s(1-\tilde{p}_s)}{\tilde{n}_s} + \frac{\tilde{p}_L(1-\tilde{p}_L)}{\tilde{n}_L}}$$

$$\frac{12}{17} - \frac{5}{14} \pm 2.055\sqrt{\frac{\frac{12}{17}(1-\frac{12}{17})}{17} + \frac{\frac{5}{14}(1-\frac{5}{14})}{14}}$$

$= \frac{83}{238} \pm 2.055 \times 0.16915$

$= 0.3487 \pm 0.3476$

interval

$(0.001135, 0.69634)$

7 $N = 49$ $\bar{x} = 20\mu V$ $s = 36\mu V$

a $H_0 : \mu = 0$ we want to know if it's uncalibrated so we attempt to prove it is calibrated assuming it's H_0, attempting to disprove it being calibrated.

$H_a : \mu \neq 0$

b $$Z = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}$$ normal dist as $N > 30$

$$= \frac{20 - 0}{\frac{36}{\sqrt{49}}}$$

$$= \frac{35}{9}$$

$$= 3.889$$

$P(0 < Z < 3.889) = 0.4999$

$P(Z > 3.89) = 0.0001$

as its two tailed

$p = 2 \times 0.0001$

$= 0.0002.$

c this p value is much less than a 0.1% confidence interval so it is reasonably implausible that the instrument is calibrated thus we can reject the null hypothesis and the instrument needs to be recalibrated

8 rate, λ = 3 calls per minute

Erlang B as the leave immediately on congestion

μ duration, h = 7 minutes

= 7/60 hours

~~E = 3x7~~

~~= 21 calls Erlangs Erlangs~~

a ~~E = 3x7/60~~

~~= 21 call minutes per minute~~

~~= 7/20 call minutes per hour.~~

E = 3x7

= 21 call minutes per 1 minute

$E \times \frac{60}{60}$ = 21 call hours per hour.

= 21 Erlangs.

b Using the erlang B table with Pr[B] = 0.05 we go down this column and find values 20.94 or 21.9 we use 21.9 Erlangs as it's higher than our required number of erlangs

this corresponds to an N of 27 callers.