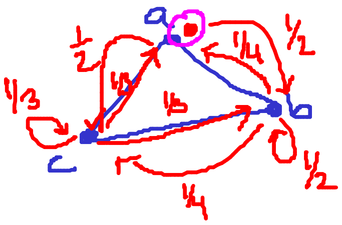
Szegedy's Quantum Walk: generalization of Grover

Markov chain $\underline{\underline{P}} = (p_{jk})$ stochastic matrix



$$\begin{array}{c|ccc} & a & b & c \\ \hline a & & & \\ b & & & \\ c & & & \end{array}\left[\begin{array}{ccc} 0 & 1/4 & 1/3 \\ 1/2 & 1/2 & 1/3 \\ 1/2 & 1/4 & 1/3 \end{array}\right] = \underset{\sim}{P}$$

initial state

$$\underset{p(0)}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \rightarrow \underset{p(1)=P\,p(0)}{\begin{bmatrix}0\\1/2\\1/2\end{bmatrix}} \rightarrow \underset{p(2)}{\begin{bmatrix}1/8+1/6\\1/4+1/6\\1/8+1/6\end{bmatrix}}$$

k-step random walk:

$p(k) = P^k p(0)$

$p(2) = P\,p(1) = P^2 p(0)$

$= \begin{bmatrix}14/48\\10/24\\14/48\end{bmatrix}$ prob. vector

quantum walk based on P ? $\quad P = (p_{jk})$ ← row j, column k

P_{jk} = probability $k \to j$

$\forall j: \quad |\eta_j\rangle = \sum_k \sqrt{P_{kj}}\,|k\rangle$ superposition of all neighbors of j

Grover: $P = (\dots)$

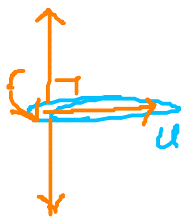
$|\psi_j\rangle = |j\rangle|\eta_j\rangle = \sum_k \sqrt{P_{kj}}\,|j\rangle|k\rangle$ start end

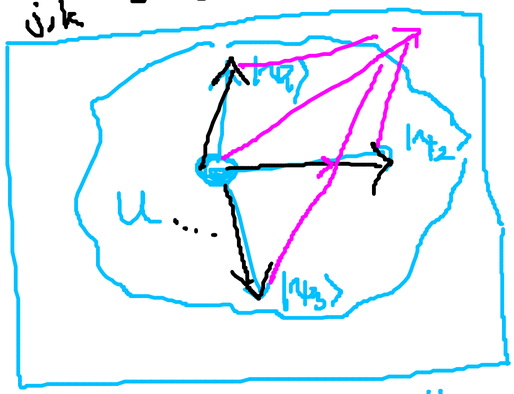
swap operator: $S :: |j\rangle|k\rangle \mapsto |k\rangle|j\rangle$ $\quad S = \sum_{j,k} |k\,j\rangle\langle j\,k|$

$\mathcal{U} = \mathrm{span}\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$ N states

Projector onto $\mathcal{U}$: $P_{\mathcal{U}} = \sum_j |\psi_j\rangle\langle\psi_j|$

reflector around $\mathcal{U}$: $R_{\mathcal{U}} = 2P_{\mathcal{U}} - I$





walk operator: $W = S R_{\mathcal{U}}$

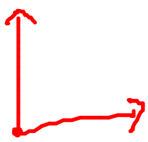
2-step walk operator: $\Phi = W^2 = (S R_{\mathcal{U}} S) R_{\mathcal{U}} = R_{\mathcal{V}} R_{\mathcal{U}}$ Szegedy walk

$\mathcal{V} = \mathrm{span}\{S|\psi_1\rangle, \dots, S|\psi_N\rangle\}$

Amplitude Amplification:

$-A S_0 A^{-1} S_\chi$

2-dimensional problem



qubit

$A = H^{\otimes n}$ quipper

(3) Szegedy, (1) Grover & (2) AA are quantum walk based on some Markov Chain

Childs [2008]: discrete-time / continuous-time quantum walk — random walk on graphs

Hamiltonian simulation: Chuang & Low [2016] unitary operation

(4) qubitization:

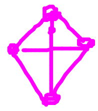
Hermitian matrix

$H \longrightarrow e^{iHt} \sim$ discrete-time quantum walk (Szegedy)

graph / Markov chain $\quad$ quantum walk continuous-time

Szegedy technique: discrete-time Markov Chain $\rightsquigarrow$ discrete-time quantum Markov Chain

Grover walk: complete graph

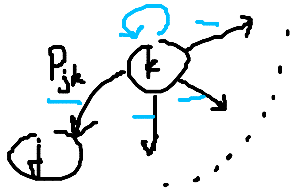


state space (vertices of a graph)

connected graph

Markov Chain $\underset{N\times N}{P} = (P_{jk}) = \begin{array}{c|c} & 1 \quad 2 \quad \cdots \quad k \quad \cdots \quad N \\ \begin{matrix}1\\2\\ \vdots \\ j \\ N\end{matrix} & \begin{bmatrix} & & \vdots & & \\ & & P_{jk} & & \\ & & \vdots & & \end{bmatrix} \end{array}$ $\quad P_{jk} = \text{Prob}(k \to j)$

N states



$$\underset{N\times N}{P} \underset{N\times 1}{\overset{p(0)}{\begin{bmatrix} - \\ - \\ - \\ \vdots \\ - \end{bmatrix}}} = \underset{N\times 1}{\overset{p(1)}{\begin{bmatrix} - \\ - \\ - \\ \vdots \\ - \end{bmatrix}}}$$

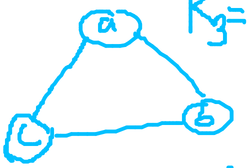
Column-stochastic

column $\to$ row

k-step random walk

$$\forall k: \quad P^k p(0) = p(k) \qquad \lim_{k\to\infty} p(k)$$

ex: $K_3 = C_3$



$$P = \begin{array}{c|c} & a \quad\; b \quad\; c \\ \begin{matrix} a\\ b\\ c\end{matrix} & \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}\end{array}$$

$$\overset{p(0)}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \to \overset{p(1)}{\begin{bmatrix}0\\1/2\\1/2\end{bmatrix}} \to \overset{p(2)}{\begin{bmatrix}1/2\\1/4\\1/4\end{bmatrix}} \to \overset{p(3)}{\begin{bmatrix}1/4\\3/8\\3/8\end{bmatrix}} \cdots \to \overset{p(\infty)}{\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}}$$

Simple random walk: "goto a random neighbor" (uniform)

$\boxed{\text{circulant}}$

$$P = \frac{1}{2}\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \frac{1}{2}\left[2 \underbrace{\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}[1\ 1\ 1]}_{\frac{1}{3}J} + (-1)\,\underbrace{\frac{1}{3}(I-J)}\right]$$

$$A^k = \sum_r \lambda_r^k E_r$$

$$\downarrow \qquad |z_r\rangle\langle z_r|$$

Spectrum: $2^{(1)}, (-1)^{(2)}$

$$\sum_r E_r = I = \underbrace{E}_{J} + \underbrace{(I-E)}_{I-J}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \checkmark$$

$1+\omega^2 = -\omega$ ✓ $\qquad \omega+\omega^2 \overset{?}{=} -1$ ✓

$1+\omega = -\omega^2$ ✓ $\qquad 1+\omega+\omega^2 \overset{?}{=} 0$ ✓

$$\boxed{1+\omega+\omega^2 = \frac{\omega^3-1}{\omega-1} = 0}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix}1\\ \omega\\ \omega^2\end{bmatrix}}_{\frac{1}{\sqrt{3}}} \overset{?}{=} -\frac{1}{\sqrt{3}}\begin{bmatrix}1\\ \omega\\ \omega^2\end{bmatrix} \checkmark$$

$$1 \neq \omega = e^{2\pi i/3} \Rightarrow \omega^3 = 1$$

Discrete Fourier Transform

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} = F_3$$

$\omega^2+\omega = -1$ ✓

$1+\omega = -\omega^2$ ✓

$1+\omega^2 = -\omega$ ✓

$$\omega^4 = \omega^{3+1} = \omega^1 \qquad \omega^3 = 1$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix}1\\ \omega^2\\ \omega\end{bmatrix} = -\begin{bmatrix}1\\ \omega^2\\ \omega\end{bmatrix} \checkmark$$

$$P^k = \left(\tfrac{1}{2}A\right)^k = \frac{1}{2^k}A^k = \frac{1}{2^k}\left[2^k \frac{J}{3} + (-1)^k \left(I - J/3\right)\right]$$

$$P = \tfrac{1}{2}A, \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad = \frac{1}{3}J + \underbrace{\left(-\frac{1}{2}\right)^k \left(I - J/3\right)}_{\to 0} \xrightarrow[k\to\infty]{} \frac{1}{3}J$$

$$\begin{array}{ccc} a & b & c \\ \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} & \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} & \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \end{array}$$

Caveat: will not work if P is not symmetric

$$P = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1/3 & 1/2 & 1/4 \\ 1/3 & 0 & 1/4 \\ 1/3 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

"nice" stochastic matrix

[Perron-Frobenius Theorem]

eigenvalues: $\lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_N|$

$\lambda_1 = 1$

$|z_1\rangle \quad |z_2\rangle \quad \cdots \quad |z_N\rangle$

basis

$$P^s p(0) = \sum_k a_k P^s |z_k\rangle$$

$$= \sum_k a_k \lambda_k^s |z_k\rangle = a_1 |z_1\rangle + a_2 \lambda_2^s |z_2\rangle + \cdots + a_N \lambda_N^s |z_N\rangle$$

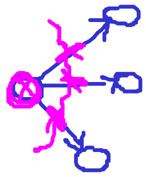
$$s \to \infty \quad \to a_1 |z_1\rangle$$

stationary distribution

$\boxed{P = (P_{jk})}$ classical walk

current next   current prev

$|\eta_j\rangle = \sum_k \sqrt{P_{kj}}\,|k\rangle$, $|\psi_j\rangle = |j\rangle|\eta_j\rangle = \sum_j \sqrt{P_{kj}}\,|j\rangle|k\rangle$, $S|\psi_j\rangle = \sum_j \sqrt{P_{kj}}\,|k\rangle|j\rangle$



Swap operator $S|j\rangle|k\rangle = |k\rangle|j\rangle$

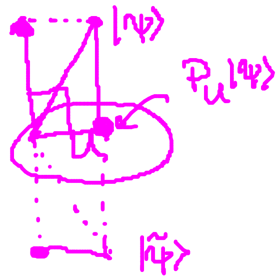
outer products

$\mathcal{U} = \text{span}\{|\psi_j\rangle : j \in [N]\}$ projector $P_{\mathcal{U}} = \sum_j |\psi_j\rangle\langle\psi_j|$

reflector $R_{\mathcal{U}} = 2P_{\mathcal{U}} - I$

1-step walk $W = SR_{\mathcal{U}}$

2-step walk $\mathcal{D} = WW = \underbrace{(SR_{\mathcal{U}}S)}_{R_V}R_{\mathcal{U}}$



$S^\dagger = S = S^T$
$S^2 = I$

$V = \text{span}\{S|\psi_j\rangle : j \in [N]\}$

$SR_{\mathcal{U}}S = S(2P_{\mathcal{U}} - 1)S = S(2\sum_j |\psi_j\rangle\langle\psi_j| - 1)S$

$= \underbrace{2\sum_j S|\psi_j\rangle\langle\psi_j|S}_{P_V} - 1 = R_V$

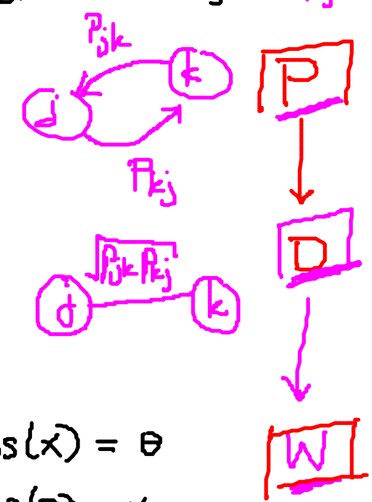
Operator $T : |j\rangle \xrightarrow{\text{lift}} |\psi_j\rangle$ output input selector $T = \sum_j |\psi_j\rangle\langle j|$ $T|\ell\rangle = |\psi_\ell\rangle$

vertex   edges

Properties: (a) $T^\dagger T = I$ (c) $T^\dagger S T = D = (d_{jk})$, $d_{jk} = \sqrt{P_{jk}P_{kj}} = d_{kj}$

(b) $TT^\dagger = P_{\mathcal{U}}$

discriminant $N\times N$ matrix
symmetric



Theorem (Szegedy)

Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of D.

$D|\lambda_j\rangle = \lambda_j|\lambda_j\rangle$

Then the eigenvalues of W are ± 1 and $e^{\pm i \arccos(\lambda_j)}$.

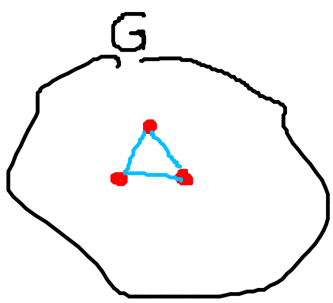
spectral information about W (discrete-time quantum walk)

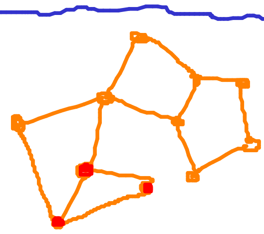
$\arccos(x) = \theta$
$\cos(\theta) = x$

Quipper: Quantum walk connections

- Triangle Finding: TF

$\binom{n}{3} \sim n^{3} \to 3/2$



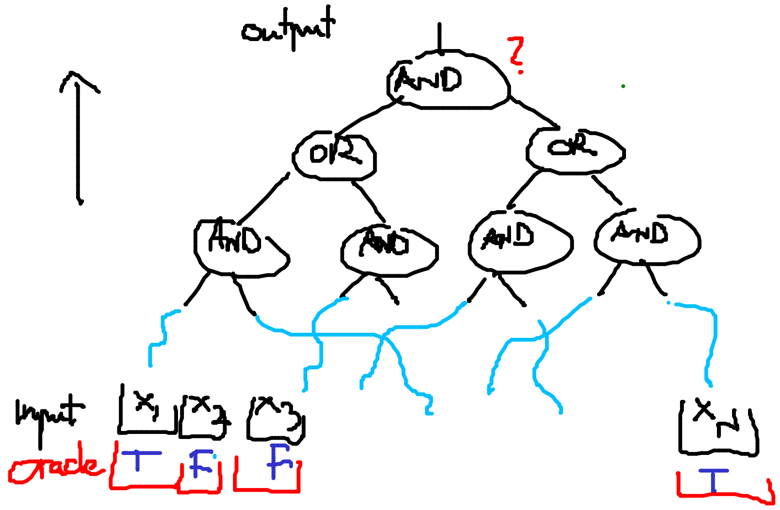


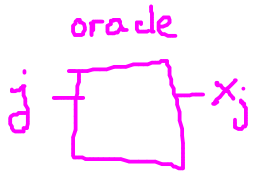
Quipper. Algorithms. TF

edge oracle

$a, b \to \boxed{O} \to \begin{cases} 1 & \text{if } (a,b) \in E \\ 0 & \text{otherwise} \end{cases}$

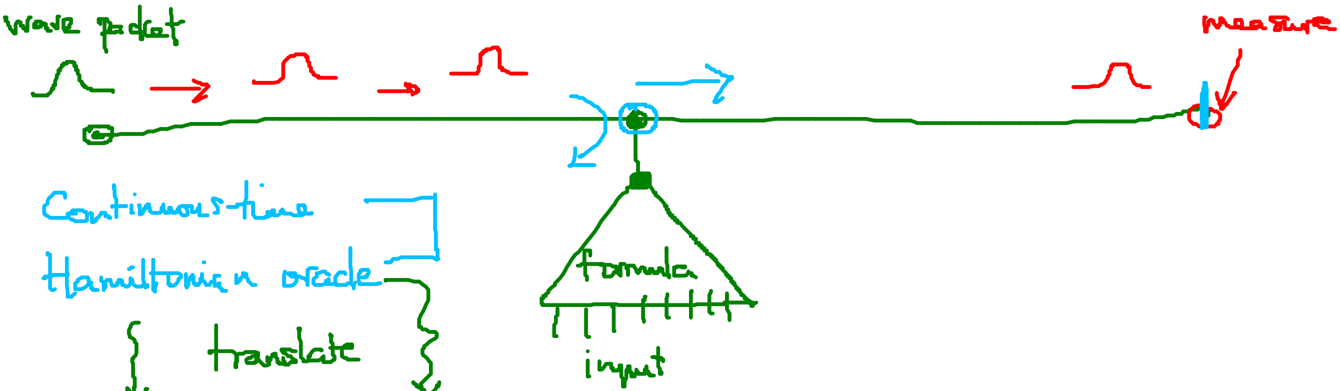
– Boolean Formula Evaluation: BF





How many input values does the algorithm need to know to evaluate the formula?

Farhi, Goldstone & Guttmann (2008):



Continuous-time Hamiltonian oracle

translate

circuit-model discrete-time

Standard oracle