Algorithms

1) Bell state circuit ✓ ✓ ✓
2) Teleportation ✓ ✓ ✓
3) Deutsch-Josza ? ? ✓
4) Bernstein-Vazirani ? ? ✓
5) Simon ✓ ? ✓
6) Quantum Fourier Transform ✓ ✓ ✓ ✓
7) Phase Estimation ①
8) Shor almost
9) Grover / Amplitude Amplification ② ✓
10) HHL / QLSA

Jupyter notebook
Visual studio code
atom

[Pyquil
Qiskit] Anaconda
Ubuntu 14.04

[Liquid]

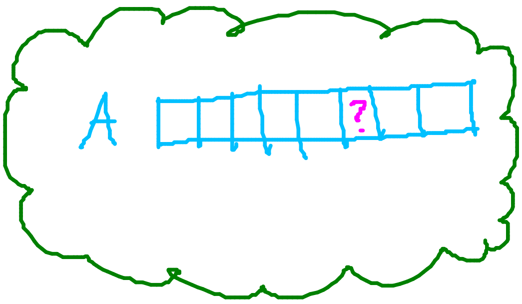
[Quipper]

Grover search (Grover 1996)

input: An array A of size N and an item x

output: Yes if x appears in A or No otherwise

index k s.t. $A[k] = x$



item x

oracle

j index

Yes if $A[j] = x$

No otherwise

Algorithm B

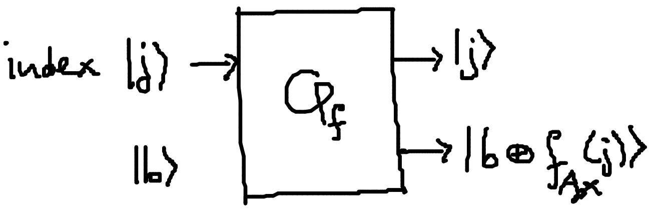
How many queries must B make to solve the problem?

$\sqrt{N}$ queries needed

Quantum queries

$\sqrt{N}$ vs N

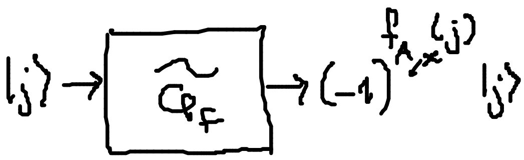
quantum oracle



reversible

$$f_{A,x}(j) = \begin{cases} 1 & \text{if } A[j] = x \\ 0 & \text{else} \end{cases}$$

phase oracle

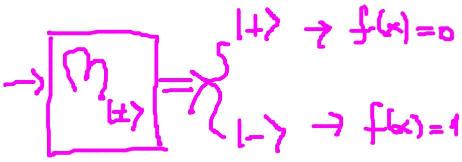


if $A[k] = x$: $|k\rangle \xrightarrow{\tilde{O}_f} -|k\rangle$

$|j\rangle \longrightarrow +|j\rangle$ for $j \neq k$

$$|0\rangle \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

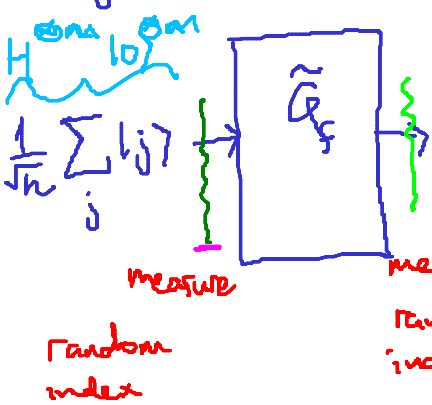
$$\xrightarrow{O_f} \frac{|0\rangle + (-1)^{f(x)}|1\rangle}{\sqrt{2}} = \begin{cases} |+\rangle & \text{if } f(x) = 0 \\ |-\rangle & \text{if } f(x) = 1 \end{cases}$$



m-bit $j = j_1 j_2 \cdots j_m$ $\quad \boxed{m = \log_2 n}$

$|j\rangle = |j_1\rangle|j_2\rangle \cdots |j_m\rangle$

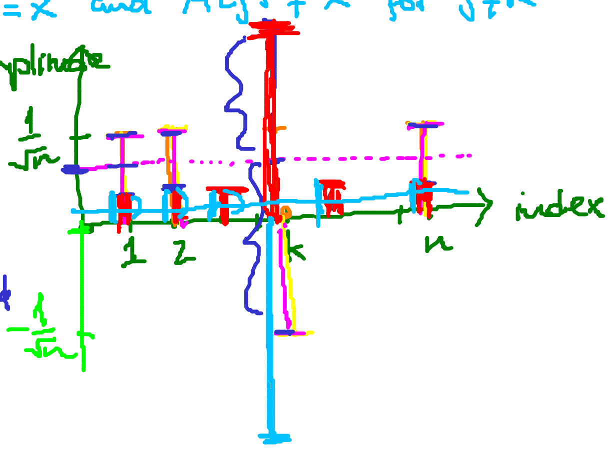
assumption: $A[k] = x$ and $A[j] \neq x$ for $j \neq k$

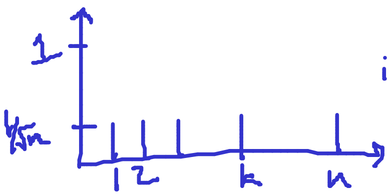


$$\frac{1}{\sqrt{n}} \sum_{j \neq k} |j\rangle - \frac{1}{\sqrt{n}} |k\rangle$$

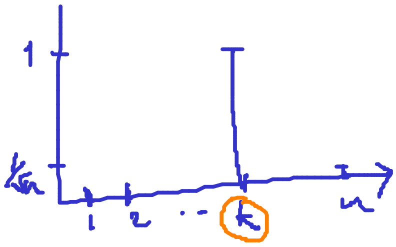
- ask oracle
- take average
- reflect around average

repeat





Grover iteration $\Rightarrow \dots \Rightarrow$



measure

1) ask oracle
2) invert around average

Analysis

$$|0_m\rangle \xrightarrow{H^{\otimes m}} \frac{1}{\sqrt{n}} \sum_j |j\rangle$$

$m = \log n$

$|u\rangle \doteq \frac{1}{\sqrt{n}} \sum_j |j\rangle$

projectors: $|u\rangle\langle u|$ span$\{|u\rangle\}$

$|k\rangle\langle k|$ span$\{|k\rangle\}$

reflectors: $R_u = 2|u\rangle\langle u| - I$ inversion around average

$R_k = I - 2|k\rangle\langle k|$ oracle

Grover iterator:

$$|\psi\rangle = G^{\ell} |u\rangle$$

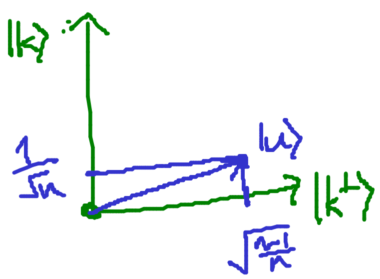
Grover step: $G = R_u R_k$

Claim: If $\ell \sim \sqrt{n}$ then

$$|\langle k|\psi\rangle| \geq 1 - \varepsilon \text{ for some small } \varepsilon$$

"proof": Initial state: $|u\rangle = \frac{1}{\sqrt{n}} \sum_j |j\rangle = \frac{1}{\sqrt{n}} |k\rangle + \frac{1}{\sqrt{n}} \underbrace{\sum_{j \neq k} |j\rangle}_{n-1 \text{ items}}$

2-dim picture



$$= \frac{1}{\sqrt{n}} \underbrace{|k\rangle}_{\checkmark} + \frac{\sqrt{n-1}}{\sqrt{n}} \underbrace{\left(\frac{1}{\sqrt{n-1}} \sum_{j \neq k} |j\rangle\right)}_{|k^{\perp}\rangle \checkmark}$$

$|k\rangle \perp |k^{\perp}\rangle$

$$|u\rangle \stackrel{\downarrow}{=} \frac{1}{\sqrt{n}} |k\rangle + \sqrt{\frac{n-1}{n}} |k^{\perp}\rangle$$

$$R_k = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{array}{c} |k\rangle \\ |k^{\perp}\rangle \end{array} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \xrightarrow{R_k} \begin{bmatrix} -\alpha \\ \beta \end{bmatrix}$$

product of 2 reflections is a rotation

$$R_u = 2|u\rangle\langle u| - I = 2 \begin{bmatrix} 1/\sqrt{n} \\ \sqrt{\frac{n-1}{n}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} & \sqrt{\frac{n-1}{n}} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} \frac{1}{n} & \frac{\sqrt{n-1}}{n} \\ \frac{\sqrt{n-1}}{n} & \frac{n-1}{n} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{n} - 1 & \frac{2}{n}\sqrt{n-1} \\ \frac{2}{n}\sqrt{n-1} & 1 - \frac{2}{n} \end{bmatrix}$$

$$G = R_u R_k = \begin{bmatrix} 1 - \frac{2}{n} & \frac{2}{n}\sqrt{n-1} \\ -\frac{2}{n}\sqrt{n-1} & 1 - \frac{2}{n} \end{bmatrix} \stackrel{\downarrow}{=} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

rotation

$\cos\theta \doteq 1 - \frac{2}{n}$

$\sin\theta = \sqrt{1 - \left(1 - \frac{2}{n}\right)^2} = \frac{2}{n}\sqrt{n-1}$

$1 - \left(1 - \frac{2}{n}\right)^2 = 1 - \left(1 - \frac{4}{n} + \frac{4}{n^2}\right) = \frac{4}{n} - \frac{4}{n^2} = \frac{4(n-1)}{n^2}$ $\sqrt{\ }$

$$G^{\ell}|u\rangle = \begin{bmatrix} \cos(\ell\theta) & \sin(\ell\theta) \\ -\sin(\ell\theta) & \cos(\ell\theta) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \sqrt{\frac{n-1}{n}} \end{bmatrix}$$

$\sin\theta \sim \theta$

$\theta \sim \frac{2}{\sqrt{n}}$

$$= \begin{bmatrix} \frac{1}{\sqrt{n}} \underbrace{\cos(\ell\theta)}_{=0} + \underbrace{\sqrt{\frac{n-1}{n}}}_{} \underbrace{\sin(\ell\theta)}_{=1} \\ \dots \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{array}{c} |k\rangle \\ |k^{\perp}\rangle \end{array}$$

$\sin(\ell\theta) = 1$

$\ell\theta = \frac{\pi}{2} \Leftrightarrow \frac{2\ell}{\sqrt{n}} = \frac{\pi}{2}$

$\Leftrightarrow \ell = \frac{\pi}{4}\sqrt{n}$ $\square$

Notes:

1) if $n=4$ then Grover is exact (recovers answers w/ prob 1)

2) what if $x \notin A$? repeat several times & check.

3) what if x appears in A more than once

(Grover becomes "worse"?)

4) generalization of Grover: Amplitude Amplification

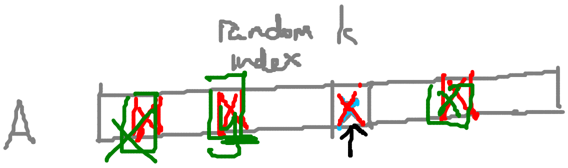
(Brassard, Høyer, et al)

$\boxed{\text{Dürr \& Høyer}}$ MINIMUM FINDING via Grover

input: array A of size N (of integers)

output: The minimum integer in A

Sketch:

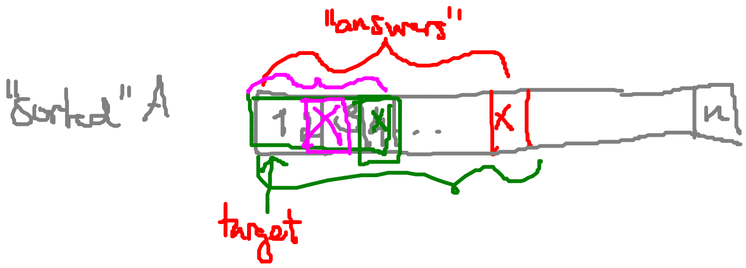


$$f_y(j) = \begin{cases} 1 & \text{if } A[j] \leq \cancel{x}\, y \\ 0 & \text{else} \end{cases}$$

$y \leq x$

oracle $\quad |j\rangle \to \boxed{O} \to (-1)^{f(j)}|j\rangle$

Use Grover to find a "random" marked element $\quad O(\sqrt{N})$



$$\sqrt{N} + \sqrt{\frac{N}{2}} + \sqrt{\frac{N}{4}} + \cdots \approx \sqrt{N}$$

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{8}} + \cdots \leq ?$$

$$\sum_{m=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^m = \frac{1}{1-\frac{1}{\sqrt{2}}} = \text{constant}$$

5) How to construct the Grover circuit?

R_u, R_k (what if k is not unique)

6) There is a strong connection between Grover search and discrete-time Quantum Walk on graphs (Szegedy) 2004

classical Markov Chain $\Rightarrow$ quantum Markov Chain