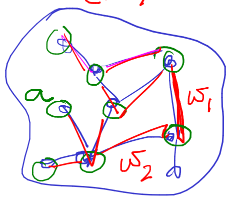
Alice graph $G=(V,E)$ Bob

1) Pick $p_v: V \to [0,1]$ probability distribution



2) Pick $w: E \to \mathbb{R}$

Bob wins iff $\exists t$ $\exists a$ s.t.

$$\boxed{\underbrace{|\psi\rangle} = \underbrace{e^{-iAt}|a\rangle}}$$

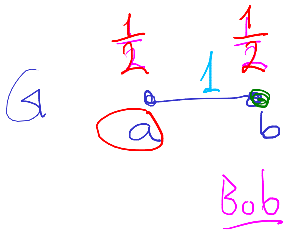
$$\underbrace{|\psi\rangle}_{\text{quantum state}} = \sum_{u\in V} \sqrt{p_u}\,|u\rangle$$

$|\psi\rangle \to \boxed{M} \Rightarrow p$

A = adjacency matrix of $\underbrace{(G,w)}_{\text{weighted graph}}$

$$A = u\begin{bmatrix} & \overset{v}{|} & \\ - & \boxed{w_{uv}} & - \\ & | & \end{bmatrix}$$ Symmetric matrix

Example 1



$$A = \begin{array}{c|c} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 0 & w \\ w & 0 \end{bmatrix} \end{array} = (+1)\,|+1\rangle\langle+1| + (-1)\,|-1\rangle\langle-1|$$

eigenvalues $= \pm 1$ $= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ✓

eigenvectors $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = |\pm 1\rangle$

$|a\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{matrix} a \\ b \end{matrix}$

$|b\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{matrix} a \\ b \end{matrix}$

$|+1\rangle\langle+1| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$|-1\rangle\langle-1| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$e^{-iAt} = e^{-i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}t}$$

$$= e^{-i(+1)t}\,\underbrace{|+1\rangle\langle+1|} + e^{-i(-1)t}\,|-1\rangle\langle-1|$$

$$= \frac{e^{-it}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{e^{it}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{c} \\ a \\ b \end{array}\begin{bmatrix} \boxed{\begin{matrix} \cos t \\ -i\sin t \end{matrix}} & \begin{matrix} -i\sin t \\ \cos t \end{matrix} \end{bmatrix}$$

$$\frac{e^{-it} + e^{it}}{2} = \cos t$$

$$\frac{e^{-it} - e^{it}}{2} = -i\sin t$$

$$e^{-iAt}|a\rangle = \begin{bmatrix} \cos t \\ -i\sin t \end{bmatrix} = [\cos(t)]\,|a\rangle + [-i\sin(t)]\,|b\rangle$$

↓ square $\frac{1}{2}$ ↓ square $\frac{1}{2}$

$$|\psi\rangle = \sum_u \sqrt{p_u}\,|u\rangle$$

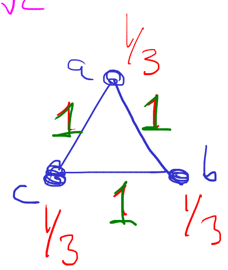
$\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$

$t = \frac{\pi}{4}$

Bob wins!

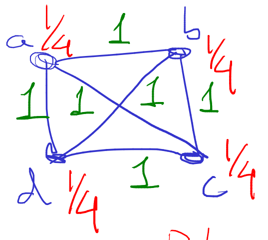
Example 2

CLIQUE (Complete Graph)



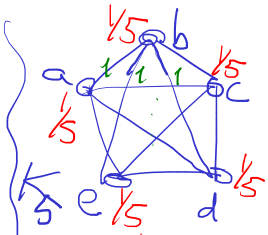
$$\begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

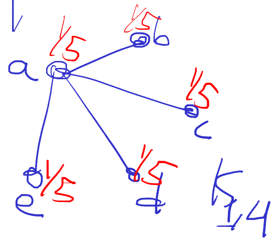
Bob also wins



Bob wins

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & & 0 \\ & & & 0 \end{bmatrix}$$





claw/star

Fact (2002) Bob loses if the weights are all one.

Fact (2007) Bob wins if he can use zero weights

$\boxed{K_{1,N} \text{ for any } N}$

What about paths?

P_5

$\boxed{P_N ?}$

Alastair Kay:

$|\psi\rangle = \sum_u \sqrt{p_u}\,|u\rangle$ Alice's challenge state

Suppose $|\lambda\rangle$ satisfies: $|\psi\rangle = \underbrace{(2|\lambda\rangle\langle\lambda| - I)}_{e^{iH\pi}}|a\rangle$

Suppose $\boxed{H}$ is a matrix whose eigenvalues are integers:

$$\underset{\substack{\shortparallel \\ 0}}{\lambda_1} \quad \underbrace{\lambda_2 \quad \lambda_3 \quad \dots \quad \lambda_N}_{2\mathbb{Z}+1}$$

0 odd integers

$$e^{iH\pi} = \underset{\lambda_1=0}{e^{i0\pi}}|\lambda_1\rangle\langle\lambda_1| + \overbrace{e^{i(2\ell+1)\pi}}^{-1}|\lambda_2\rangle\langle\lambda_2| + \dots$$

$$H = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k| \Rightarrow e^{iHt} = \sum_k e^{i\lambda_k t}|\lambda_k\rangle\langle\lambda_k|$$

$= |\lambda_1\rangle\langle\lambda_1| - (I - |\lambda_1\rangle\langle\lambda_1|)$

$= 2|\lambda_1\rangle\langle\lambda_1| - I$

$\sum_k |\lambda_k\rangle\langle\lambda_k| = I$

$H|\lambda_k\rangle = \lambda_k|\lambda_k\rangle$

eigenvalue eigenvector

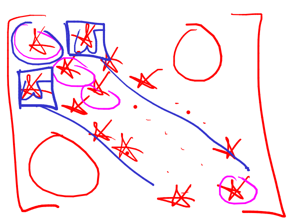
Goal: Create a Hermitian matrix H with integer eigenvalues $\boxed{0, 2\mathbb{Z}+1}$ where the eigenvector corresponding to eigenvalue 0 is given by $|\lambda\rangle$ s.t.

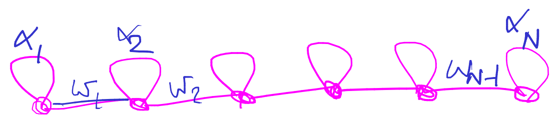
simple

$$\boxed{|\psi\rangle = (2|\lambda\rangle\langle\lambda| - I)|a\rangle}$$

given fixed?! fixed

Claim(?)

$H =$ 



$\| |\tilde{\psi}\rangle - |\psi\rangle \| < \varepsilon$

Want a matrix H:

eigenvalues: $0, \lambda_2 = \lambda_3, \dots = \lambda_N$ $\qquad \lambda_k \in 2\mathbb{Z}+1$

eigenvectors: $|\lambda\rangle$ ____?____

Spectral Theorem:

$$H = X D X^\dagger \iff H X = X D$$

$$D = \begin{bmatrix} \cancel{\lambda_1}\,0 & & & \\ & \lambda_2 & & 2\mathbb{Z}+1 \\ 0 & & \ddots & \\ & & & \lambda_N \end{bmatrix} \qquad X = \begin{bmatrix} |\cancel{\lambda_1}\rangle & |\lambda_2\rangle & \cdots & |\lambda_N\rangle \end{bmatrix}$$

$$H = \begin{bmatrix} |\lambda\rangle & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} 0 & & & 0 \\ & z_2 & & \\ & & z_3 & \\ 0 & & & \ddots \ z_N \end{bmatrix} \begin{bmatrix} \langle\lambda| \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}$$

Circulant Matrices:

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}. \qquad \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\ \vdots & a_{N-1} & a_0 & a_1 & \vdots \\ & & \ddots & \ddots & a_1 \\ a_1 & \cdots & & a_{N-1} & a_0 \end{bmatrix}$$

3×3 $\qquad\qquad$ $N\times N$

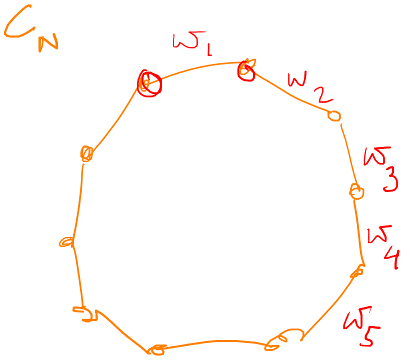
What are the eigenvectors of a circulant matrix

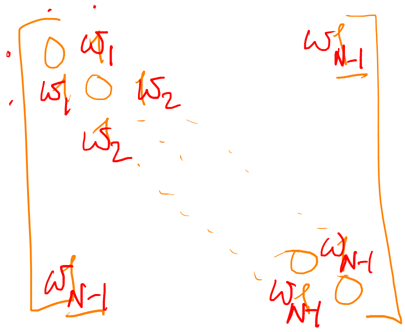
$$X^\dagger C X = \begin{bmatrix} \lambda_0 & & 0 \\ & \lambda_1 & \\ 0 & & \ddots \ \lambda_{N-1} \end{bmatrix} \qquad \omega = e^{2\pi i/N}$$

$$X = \text{Discrete Fourier Transform } F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \vdots \\ \vdots & & & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \end{bmatrix}$$

unitary, flat

$$\langle j|F_N|k\rangle = \frac{1}{\sqrt{N}}\,\omega^{jk} \qquad F_N^\dagger = F_N(\omega^{-1})$$

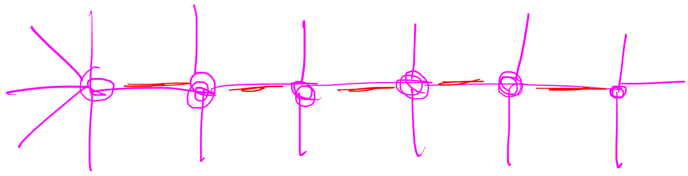


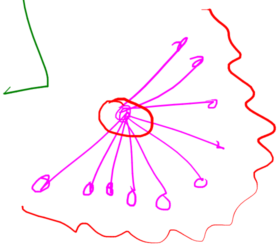


Design H

Sparse matrix

$$\begin{bmatrix} d & & 0 \\ & d & \\ 0 & & \ddots \; d \end{bmatrix}$$





$$e^{i(A+\underbrace{\cancel{dI}}_{\text{Commute}})t} = e^{iAt}\,e^{idIt} = \underbrace{\cancel{e^{idt}}}_{\text{global phase}}\,e^{iAt}$$

$$e^{idt} I$$

$$e^{A+B} \overset{\uparrow}{=} e^A e^B$$

A, B commute