Assignment 1

Joshua Hwang (44302650)

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1 Westeros

Let B be the set of all in the Brotherhood and let D be the set of people the tickler detects as a Brotherhood member. From the question we get

$$\begin{aligned} P(B) &= 0.01 \\ P(D|B) &= 0.95 \\ P(\ D|\ B) &= 0.4 \end{aligned}$$

and we have to find $P(B|D)$.

We first find $P(D)$ using Bayes theorem and the union of two events, $P(A \cup B) =$

$P(A) + P(B) - P(A \cap B)$.

$$
\begin{aligned}
P(\ D|\ B) &= \frac{P(\ D \cap\ B)}{P(\ B)} \\
0.4 &= \frac{P(\ (D \cup B))}{1 - P(B)} \\
&= \frac{1 - P(D \cup B)}{1 - P(B)} \\
&= \frac{1 - (P(D) + P(B) - P(D \cap B))}{1 - P(B)} \\
&= \frac{1 - (P(D) + P(B) - P(D|B)P(B))}{1 - P(B)} \\
&= \frac{1 - (P(D) + 0.01 - 0.95 \times 0.01)}{1 - 0.01} \\
&= \frac{1 - P(D) - 0.01 + 0.0095}{0.99} \\
0.396 &= 0.9995 - P(D) \\
P(D) &= 0.6035
\end{aligned}
$$

Now we can find $P(B|D)$.

$$
\begin{aligned}
P(B|D) &= \frac{P(D|B)P(B)}{P(D)} \\
&= \frac{0.95 \times 0.01}{0.6035} \\
&= 0.01574
\end{aligned}
$$

Therefore there is only a 0.01574 chance of this person being in the brotherhood.

2 Coin flips

2.1 List outcomes

All in A_3

$$\begin{gathered} 0100 \\ 0101 \\ 0110 \\ 0111 \\ 1100 \\ 1101 \\ 1110 \\ 1111 \end{gathered}$$

2.2 Mutually independent

Let us choose sets A_i and A_k where $i \neq k$. They are independent if $P(A_i \cap A_k) = P(A_i)P(A_k)$.

Each set A has 2^3 elements since you may choose to flip any of the three remaining bits (one of the bits must stay on as determined by the set definition) where order is important.

To be in two different sets we must set two of the bits as on. Now we have two unset bits to play with giving 2^2 options.

The size of Ω is all possible combinations of four ordered bits, 2^4.

$$\begin{gathered} P(A_i \cap A_k) = P(A_i)P(A_k) \\ \frac{2^2}{\Omega} = \frac{2^3}{\Omega} \times \frac{2^3}{\Omega} \\ \frac{2^2}{2^4} = \frac{2^3}{2^4} \times \frac{2^3}{2^4} \\ \frac{2^2}{2^4} = \frac{2^6}{2^8} \\ \frac{2^2}{2^4} = \frac{2^2}{2^4} \\ \text{LHS} = \text{RHS} \end{gathered}$$

Since $P(A_i \cap A_k) = P(A_i)P(A_k)$ we know any two of these sets are mutually independent.

2.3 Solve an expression

$$\begin{aligned}
P(A_1 \cap A_2 | A_1) &= \frac{P(A_1 \cap A_2 \cap A_1)}{P(A_1)} \\
&= \frac{P(A_1 \cap A_2)}{P(A_1)} \\
&= \frac{2^{-2}}{2^{-1}} \\
&= 2^{-1} \\
&= 0.5
\end{aligned}$$

$$\begin{aligned}
P(A_1 \cup A_3 | A_2) &= \frac{P((A_1 \cup A_3) \cap A_2)}{P(A_2)} \\
&= \frac{(P(A_1) + P(A_3) - P(A_1 \cap A_2)) \times P(A_2)}{P(A_2)} \\
&= \frac{(2^{-1} + 2^{-1} - 2^{-2}) \times 2^{-1})}{2^{-1}} \\
&= 0.75
\end{aligned}$$

3 Plane

The plane needs an engine on each side to fly. The plane will crash only when the complete left wing(L) or complete right wing(R) fail. A complete wing will fails on when both engines fail independently, $0.1 \times 0.1 = 0.01$.

$$\begin{aligned}
P(L \cup R) &= P(L) + P(R) - P(L \cap R) \\
&= P(L) + P(R) - P(L \cap R) \\
&= 0.01 + 0.01 - 0.01 \times 0.01 \\
&= 0.0199
\end{aligned}$$

Thus the probability the plane remains in the air is $1 - 0.0199 = 0.9801$.

4 Annulus

Since R can't exist outside the domain $[1, 4]$ the cdf will be 0 at $R \leq 1$ and 1 at $R \geq 4$. Every point has an equal chance of being chosen but as we increase in distance there are more points to choose from (the circumference of a circle at that radius is larger than one with a smaller radius).

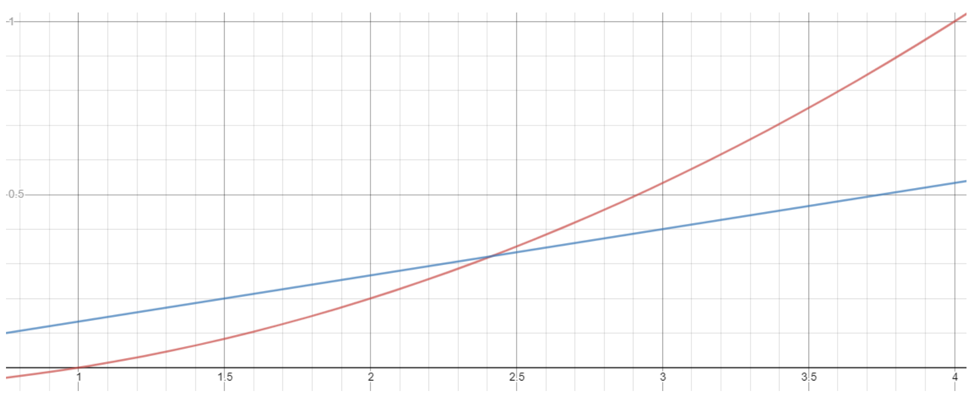
With these facts we produce a cdf from the area of the circle. Ω will be the area of the entire domain, $16\pi - \pi = 15\pi$. The cdf at R will be

$$= \frac{R^2\pi - \pi}{15\pi}$$
$$= \frac{R^2 - 1}{15} \qquad [R \in [1, 4]]$$

The pdf is the derivative of the cdf thus we differentiate our result.

$$= \frac{2R}{15} \qquad [R \in [1, 4]]$$

The blue graph is the pdf while the red is the cdf.



The expectation for a continuous random variable is given as,

$$\begin{aligned}
\int_1^4 R \times \text{pdf} dR &= \int_1^4 R \times \frac{2R}{15} dR \\
&= \int_1^4 \frac{2R^2}{15} dR \\
&= \frac{2}{45}[64 - 1] \\
&= \frac{126}{45} \\
&= 2.8
\end{aligned}$$

5 PGF

The properties of a probability generating function are as follows,

$$\begin{aligned}
\text{Expectation} = G'(1) &&= \frac{2}{3} \\
\text{Variance} = G''(1) + G'(1) - (G'(1))^2 &&= \frac{5}{9} \\
G''(1) + \frac{2}{3} - \frac{4}{9} &&= \frac{5}{9} \\
G''(1) &&= \frac{1}{3}
\end{aligned}$$

We know try find the generating function.

$$\begin{aligned}
G(z) &= E(z^x) \\
&= \sum_{x=0}^{\infty} z^x P(X = x) \\
&= z^0 P(X = 0) + z^1 P(X = 1) + z^2 P(X = 2) \\
G'(z) &= P(X = 1) + 2zP(X = 2) \\
G''(z) &= 2P(X = 2)
\end{aligned}$$

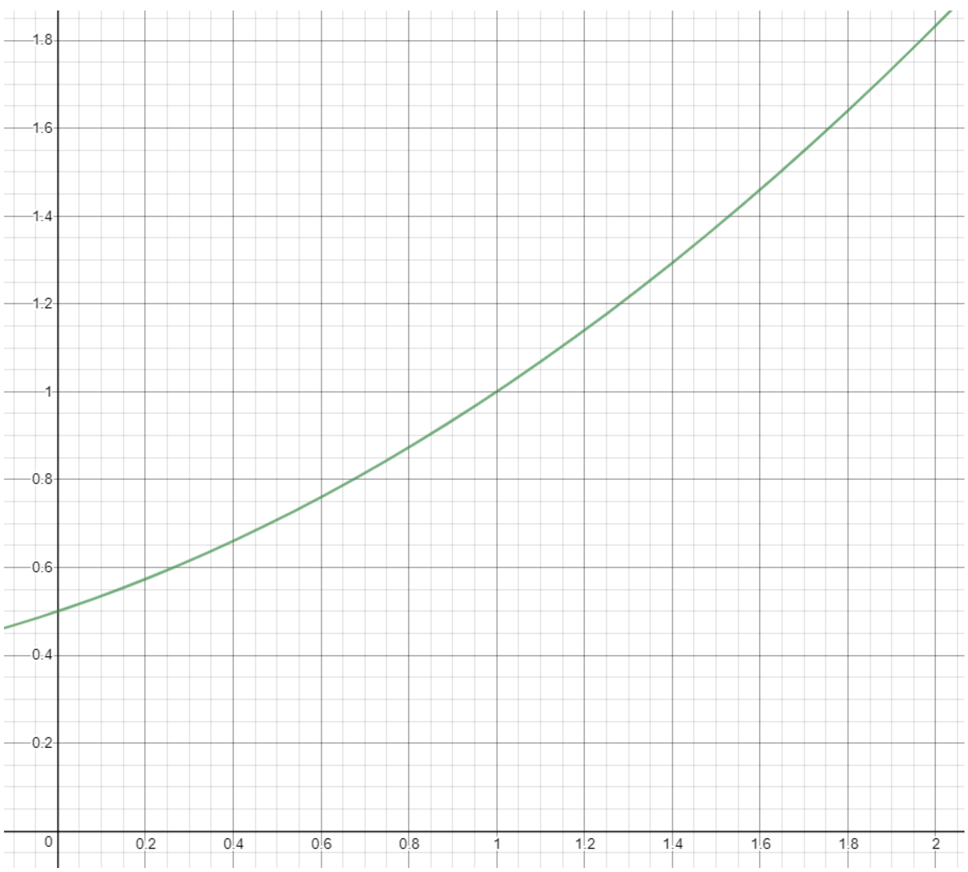
We now use our derivatives to solve the pgf.

$$\begin{aligned}
G''(1) = 2P(X = 2) &= \frac{1}{3} \\
P(X = 2) &= \frac{1}{6}
\end{aligned}$$

$$
\begin{aligned}
G'(1) = P(X=1) + 2P(X=2) &= \frac{2}{3} \\
P(X=1) + \frac{1}{3} &= \frac{2}{3} \\
P(X=1) &= \frac{1}{3}
\end{aligned}
$$

The remaining probability must be $P(X=0) = \frac{1}{2}$. Thus the pgf is

$$
\begin{aligned}
G(z) &= z^0 P(X=0) + z^1 P(X=1) + z^2 P(X=2) \\
&= \frac{1}{2} + z^1 \frac{1}{3} + z^2 \frac{1}{6}
\end{aligned}
$$



6 PDF

6.1 Find c

A pdf has an integral the adds to one. Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= 1 \\
\int_{2}^{\infty} \frac{c}{x^4}dx &= 1 \\
\frac{-c}{3}\left[\frac{1}{x^3}\right]_{x=2}^{x\to\infty} &= 1 \\
\frac{-c}{3}\left[0 - \frac{1}{8}\right] &= 1 \\
\frac{c}{24} &= 1 \\
c &= 24
\end{aligned}$$

6.2 Expectation

We use the standard expectation and variance formulas

$$\begin{aligned}
\text{Expectation} &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{2}^{\infty} x\frac{24}{x^4}dx \\
&= 24\int_{2}^{\infty} \frac{1}{x^3}dx \\
&= -12\left[\frac{1}{x^2}\right]_{x=2}^{x\to\infty} \\
&= -12\left[0 - \frac{1}{4}\right] \\
&= 3
\end{aligned}$$

$$
\begin{aligned}
\text{Second moment} &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_{2}^{\infty} x^2 \frac{24}{x^4} dx \\
&= 24 \int_{2}^{\infty} \frac{1}{x^2} dx \\
&= -24 \left[\frac{1}{x}\right]_{x=2}^{x\to\infty} \\
&= -24 \left[0 - \frac{1}{2}\right] \\
&= 12
\end{aligned}
$$

$$
\begin{aligned}
\text{Variance} &= E(X^2) - (EX)^2 \\
&= 12 - 3 \\
&= 9
\end{aligned}
$$

6.3 Evaluate an expression

$$
\begin{aligned}
P(X > 4 | X > 3) &= \frac{P(X > 3 | X > 4) P(X > 4)}{P(X > 3)} \\
&= \frac{P(X > 4)}{P(X > 3)} \\
&= \frac{\text{cdf}(X > 4)}{\text{cdf}(X > 3)} \\
&= \frac{\int_4^{\infty} f(x) dx}{\int_3^{\infty} f(x) dx} \\
&= \frac{\left[0 - \frac{1}{64}\right]}{\left[0 - \frac{1}{27}\right]} \\
&= \frac{27}{64} \\
&= 0.421875
\end{aligned}
$$

7 Birthday

I used the following code to simulate birthdays.

```
#!/bin/env python3
# Note: python2 works just as well

import random
SEED = 1

NUM_TESTS = 100000
NUM_DAYS = 365

def simulate_bday():
    # Create a generating list of birthdays
    birthdays = (random.randint(0,NUM_DAYS-1) for x in range(NUM_DAYS))
    birthday_counter = [0 for x in range(NUM_DAYS)]
    for birthday in birthdays:
        if birthday_counter[birthday] > 0:
            # This day has been counted before
            return sum(birthday_counter)
        birthday_counter[birthday] += 1

random.seed(SEED)
tests = [simulate_bday() for x in range(NUM_TESTS)]
mean = sum(tests)/float(len(tests))

print("Expectation:", mean)
```

It gave the following output in Python3

```
Expectation: 23.6113
```