

Study notes

Joshua Hwang (44302650)

June 5, 2019

1 Important theorems

- Tail of a sequence
- Finding delta when things are complicated
- When you can simplify limits

2 Important proofs

2.1 Every convergent sequence is bounded

pg 46

Summary of proof: To fully complete the proof it must be done for below and above. A convergent sequence, by definition, will have an N^{th} term that will be extremely close to whatever the convergence is. Prior this point there are a finite number of elements less than N . We take the maximum of this set of finite elements (with the addition of the convergent value) and consider this the upper bound.

2.2 A sequence has at most one limit

pg 45

Summary of proof: Consider the sequence has two limits L and R . We attempt to prove that L and R approach each other.

To do this we take $|L - R| = |L - x_n + x_n - R| = |(L - x_n) - (R - x_n)|$. The rest should be obvious.

2.3 Montone Convergence Theorem

pg 47

“A monotone sequence is bounded if and only if it is convergent”

Summary of proof: This theorem is an if and only if. Thus it is important to consider both ways. The proof is done for the greater than case but the less than is exactly the same.

Consider a monotone bounded sequence. We take the lowest upper bound L . By definition of the lowest upper bound we know there exists $x > L - \epsilon$. Not only that but every element after x also satisfies this statement because of the monotone part.

Now we consider a monotone sequence that converges. Luckily we have already proven that convergent sequences are bounded.

2.4 Continuous function on closed interval is bounded

pg 114

Summary of proof: To prove this we must use the contrapositive and the Bolzano-Weierstrass theorem.

We first suppose the function is unbounded, then conclude the function must be discontinuous. We construct an infinite sequence x_n where $f(x_n) > n$. But since

our x s are bounded we can use Bolzano-Weierstrass theorem to find a convergent subsequence, essentially x_∞ . But this means our function isn't continuous since the definition of continuity states our limit must equal the value of the function at that point, $f(x) = \lim_{x \rightarrow x_\infty} f(x)$. (Forgot to mention our function maps to \mathbb{R} which certainly doesn't have infinity in it)

2.5 Extreme Value Theorem

pg 115

"If f is a bounded continuous function then f achieves both an absolute maximum and absolute minimum on the interval"

Summary of proof: We use the previous theorem to prove this one. This proof does not give us a way to find such maximum or minimum but rather proves the existence of an $f(x)$ and $f(y)$ that give global maximum and minimum.

We first note that f is bounded meaning we can create a convergent sequence of $f(x)$ s that converge to both the infimum and supremum. The issue is that the sequence of x s that let our f converge to the infimum or supremum may not converge to anything. Luckily the Bolzano-Weierstrass theorem states that our bounded sequence of x s will have a convergent subsequence.

Thus we have proven the existence of an x that will satisfy the supremum/infimum.

2.6 Differentiable implies continuous

pg 138

Summary of proof: This particular proof is very easy barring a single trick. All that is used are basic definitions.

We first write the definition of the derivative and work towards proving the numerator, $\lim_{x \rightarrow c} f(x) - f(c)$ exists and is 0. From there we would be able to write $\lim_{x \rightarrow c} f(x) = f(c)$ which is the definition of continuity. We do this by

the following,

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$$

Since the right hand side limit exists so does $\lim_{x \rightarrow c} f(x) - f(c)$.

Note: it is important to state that $\lim_{x \rightarrow c} f(x) - f(c)$ exists which is why we are able to perform our operations.

2.7 Product rule for derivatives

pg 140

Summary of proof: This is done through manipulating the definition of the derivative. The key point is the use the following fact, $f(x)g(x) - f(c)g(c) = f(x)(g(x) - g(c)) + g(x)(f(x) - f(c))$.

2.8 Rolle's Theorem

pg 144

“Consider a continuous differentiable function on (a, b) such that $f(a) = f(b)$. There there exists $f'(c) = 0$ between a and b ”

Summary of proof: We consider three cases of what happens in the interval (a, b) ; there's a point greater than $f(a) = f(b)$, there's a point less than $f(a) = f(b)$ or all points are equal to $f(a) = f(b)$. All three cases gives us the respective conclusion, a local maxima, a local minima or flat. In all three cases we conclude there exists a point, c , where the derivative is 0.

2.9 Continuous functions are integrable

pg 169

Note: I only have a proof for a closed bounded interval.

Summary of proof: We make use of the fact a continuous function is uniformly continuous on a closed bound allowing us to create a small difference between partitions that.

We find a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. We now consider a partition with each partition's width being at least $\Delta x < \delta$. From here we consider each partition as a closed interval. We know the function reaches a absolute maximum and absolute minimum in this region but we also know that as x and y get closer $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ will still hold. Thus our maximum M and our minimum m in this region gives us $M - m < \frac{\epsilon}{b-a}$.

From here we find the difference between the upper sum and the lower sum and attempt to place $M - m$ in there somewhere. By doing this we show that as the partitions get smaller our upper and lower sums get closer too.

2.10 Monotone function are integrable

Summary of proof: We can prove easily through the definition of the upper sum and lower sum. Computing these values will be simple for a monotonic function. Once we have both we find what happens to the difference as we increase the number of partitions.

2.11 Fundamental Theorem of Calculus

There are two forms of FTC. We first examine $\int_a^b f = F(b) - F(a)$.

2.11.1 First form $\int_a^b f = F(b) - F(a)$

We're given F is a continuous, differentiable function and $f = F'$.

Summary of proof: We use the Mean Value Theorem on each partition to get $F'(c_i)\Delta x_i = F(x_i) - F(x_{i+1})$. From here we produce an inequality involving

the absolute minimum and maximum in that partition.

$$m_i \Delta x_i \leq F(x_i) - F(x_{i+1}) \leq M_i \Delta x_i$$

From here we sum across all partitions which results in the middle part cancelling to give $F(b) - F(a)$.

$$\sum m_i \Delta x_i \leq F(b) - F(a) \leq \sum M_i \Delta x_i$$

Which you can clearly see are the lower and upper sums. Since we know our function is Riemann integrable the left and right inequalities converge. Thus we apply sandwiching and conclude $\int_a^b f = F(b) - F(a)$.

2.11.2 Second form $F'(c) = f(c)$

We define $F(x) = \int_a^x f$. We're given f is continuous and differentiable (obviously). From here we prove that F is continuous and if f is continuous then F is differentiable and is $F' = f$ at that point.

Summary of proof: There are a lot of tricks that need to be memorised here. The end of the proof intends to show that the difference between $F'(c)$ and $f(c)$ becomes arbitrarily small.

We first consider a small region around $f(c)$. We create an inequality with some $f(x)$ floating within this region, $f(c) - \epsilon \leq f(x) \leq f(c) + \epsilon$. Since f is continuous we also know $|x - c| < \delta$.

We then multiply our inequality by $x - c$. You may be confused by the swap from $f(x)(x - c) \rightarrow \int_c^x f$. They are not equal but rather the inequality will still hold for $\int_c^x f$. A helpful visual is to note the right hand side is the upper approximation rectangle while the left hand side is the opposite. The middle

part is the true value of the area.

$$\begin{aligned}
f(c) - \epsilon &\leq f(x) && \leq f(c) + \epsilon \\
(f(c) - \epsilon)(x - c) &\leq \int_c^x f && \leq (f(c) + \epsilon)(x - c) \\
f(c) - \epsilon &\leq \frac{\int_c^x f}{x - c} && \leq f(c) + \epsilon \\
f(c) - \epsilon &\leq \frac{F(x) - F(c)}{x - c} && \leq f(c) + \epsilon
\end{aligned}$$

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \epsilon$$

2.12 Weierstrauss M-test

The Weierstrauss M-test states that if we take the absolute value of the upper bound of each function in the series, i.e. take $M_i > |f_i(x)|$ for all x . If we then sum all our M s and get a non-infinite result our series converges absolutely and uniformly.

Summary of proof: The trickiest part is proving uniform convergence. This is achieved by taking the tail of our sum (Cauchy convergence).

It's obvious that if $\sum |f_i(x)| \leq \sum M_i$ and $\sum M_i$ converges **absolutely** then $\sum f_i(x)$.

$$\begin{aligned}
|S_n(x) - S_m(x)| &= \left| \sum_{i=m+1}^n f_i(x) \right| \\
&\leq \sum_{i=m+1}^n |f_i(x)| \\
&\leq \sum_{i=m+1}^n M_i && \leq \epsilon
\end{aligned}$$

The final step makes use of the Cauchy criterion (which is available in the reals). A sequence is convergent in the reals if and only if it is Cauchy.

Note: for some strange reason we're allowed to have \leq despite never allowing it previously.

2.13 Uniform limits of continuous functions are continuous

Summary of proof: we attempt to prove that the limit function f reaches is continuous by use of the definition of continuity, $|f(x) - f(x_0)| < \epsilon$. From here we attempt to find a $\delta > 0$ for our corresponding ϵ . We now add the terms $f_n(x)$ and $f_n(x_0)$ to the $|f(x) - f(x_0)|$. Then we can split them up with the triangle inequality. We should eventually find every term is a known limit with δ s that exist.

3 Multivariate and vector functions

Please note: (x, y) is referring to the inner product.

3.1 Useful inequalities

Cauchy inequality

$$|(x, y)| \leq \|x\| \|y\|$$

Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

We attempt to prove the above.

First prove Cauchy inequality. First consider the case where $x = 0$ or $y = 0$, then in inequality is obvious. Now consider $x \neq 0$ and $y \neq 0$. Consider two numbers

$a, b \in \mathbb{R}$.

$$\begin{aligned} 0 &\leq (|a| - |b|)^2 \\ &= a^2 - 2|a||b| + b^2 \\ |ab| &\leq \frac{1}{2}(a^2 + b^2) \end{aligned}$$

Now we take $a = \frac{|x_i|}{||x||}$ And $b = \frac{|y_i|}{||y||}$.

$$\left| \frac{x_i y_i}{||x|| ||y||} \right| \leq \frac{1}{2} \left(\frac{|x_i|^2}{||x||^2} + \frac{|y_i|^2}{||y||^2} \right)$$

We sum these inequalities over all i to get,

$$\begin{aligned} \sum \left| \frac{x_i y_i}{||x|| ||y||} \right| &\leq \frac{1}{2} \left(\sum \frac{|x_i|^2}{||x||^2} + \sum \frac{|y_i|^2}{||y||^2} \right) \\ &= \frac{1}{2||x||^2} \sum |x_i|^2 + \frac{1}{2||y||^2} \sum |y_i|^2 \\ &= \frac{1}{2||x||^2} ||x||^2 + \frac{1}{2||y||^2} ||y||^2 \\ \sum \left| \frac{x_i y_i}{||x|| ||y||} \right| &\leq 1 \\ \sum |x_i y_i| &\leq ||x|| ||y|| \end{aligned}$$

Using the triangle inequality (of reals which is already proven).

$$\begin{aligned} \left| \sum x_i y_i \right| &\leq \sum |x_i y_i| \leq ||x|| ||y|| \\ \left| \sum x_i y_i \right| &\leq ||x|| ||y|| \\ |(x, y)| &\leq ||x|| ||y|| \end{aligned}$$

Thus we have proven Cauchy inequality. Note: for the dot product we actually have an equality with a cosine term, this inequality however is more general.

Now for the proof of the triangle inequality,

$$\begin{aligned}
||x + y||^2 &= (x + y, x + y) \\
&= (x, x) + (y, y) + (x, y) + (y, x) \quad \text{Through elementwise expansion} \\
&= ||x||^2 + ||y||^2 + 2(x, y) \\
&\leq ||x||^2 + ||y||^2 + 2|(x, y)| \\
&\leq ||x||^2 + ||y||^2 + 2||x|| ||y|| \quad \text{From our previous proof} \\
&= (||x|| + ||y||)^2 \\
||x + y||^2 &\leq (||x|| + ||y||)^2 \\
||x + y|| &\leq ||x|| + ||y||
\end{aligned}$$

Please note: in general the null factor law does not hold, $(x, y) = 0$ implies $x = 0$ or $y = 0$.

3.2 Definition of an open domain

A domain, Ω , is considered open if for all $x \in \Omega$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $y \in \Omega$.

Visually, this is like a blurred edge or border. No matter where you pick along the border, there will always be a radius of points surrounding x that will also be part of the region. This definition of open will help with derivatives since the derivative requires a neighbourhood of points, this definition always guarantees a neighbourhood.

3.3 Jacobian

We make use of the fact that a multivariate function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be changed into a set of $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$, a function for every dimension of the output.

The Jacobian has a different function for every row and a different variable for every column. We create a rectangular (not necessarily square) matrix. If our input and output dimensions are equal then we can take a determinant.

3.4 Derivative of multivariate

We make the startling discovery that the Jacobian is the derivative! We alter the definition of the derivative to make more sense in the context of matrices. J is the Jacobian (and derivative).

$$\lim_{h \rightarrow 0} \frac{|f(x_0 + h) - f(x_0) - Jh|}{|h|} = 0$$

The definition had to be changed since a division of matrices does not always make sense (especially nonsquare). But we shall see that this generalisation was derived quite naturally. Consider the single variable case where $J \in \mathbb{R}$.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{|f(x_0 + h) - f(x_0) - Jh|}{|h|} &= 0 \\ \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) - Jh}{h} &= 0 \\ \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} - J &= 0 \\ \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} &= J \end{aligned}$$

3.4.1 Remark

Even if a hypothetical Jacobian can be created (all partials are available) the function may not be differentiable (the definition above fails).

4 Inverse function theorem

4.1 Single variable

If we have a function $f: I \rightarrow \mathbb{R}$ that is strictly monotone, is differentiable for all $x \in I$ and $f'(x) \neq 0$. Then the inverse function, $f^{-1}(x)$ is also differentiable at $y = f(x)$ and is in fact $\frac{1}{f'(x)}$. Note: we can construct such a strictly monotone function by **restricting** our initial function.

The proof is on pg 153 but we will outline it here. We take two points in J and

note that since f is strictly monotone our inverse function will be continuous (pg 133). From there we consider the limit as these two points approach each other and we end up proving our theorem.

The most obvious use of this fact is that we may now swap between the derivative of an inverse and the inverse of a derivative interchangeably when talking about continuous functions.

4.2 Multivariable

Consider a function with a square Jacobian ($f: \mathbb{R}^n \rightarrow \mathbb{R}^n$)

It's easy to see that $f' = J$ and f' is only invertible when $\det(J) \neq 0$.

If $\det(J) \neq 0$ at location x_0 then there exists an open ball (domain) around x_0 where f is a bijection. Also the same conclusion is made as the previous section.