# Assignment 2

Joshua Hwang (44302650)

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## 1 Show the limit

Given $f(x) = 2x^2 + 5$ show,

$$\lim_{x \to 1} f(x) = 7$$

If the limit is 7 then for all $\epsilon > |f(x) - 7|$ there exists $\delta > 0$ such that $\delta > |x - 1|$.

$$\begin{aligned}
\epsilon &> |f(x) - 7| \\
&= |2x^2 + 5 - 7| \\
&= |2x^2 - 2| \\
&= 2|x - 1||x + 1|
\end{aligned}$$

Try $\delta_1 = 1$ so,

$$\begin{aligned}
\delta &> |x - 1| & & \\
1 &> |x - 1| & > |x| - 1 & \qquad \text{reverse triangle inequality} \\
2 &> |x| & &
\end{aligned}$$

So

$$
\begin{aligned}
2|x-1||x+1| &\leq 2|x-1|(|x|+|1|) && \text{triangle inequality} \\
&< 2|x-1|(2+|1|) \\
&< 6|x-1|
\end{aligned}
$$

Now we choose $\delta_2 = \frac{\epsilon}{6}$. Thus our final choice of delta is $\delta = min\{1, \frac{\epsilon}{6}\}$. We now verify our results.

$$
\begin{aligned}
|2x^2+5-7| &= 2|x-1||x+1| \\
&< 2\delta|x+1| && \delta > |x-1| \\
&= 2\delta|(x-1)+2| \\
&< 2\delta|1+2| && 1 \geq \delta > |x-1| \\
&< 6\delta \\
&< \epsilon && \delta \leq \frac{\epsilon}{6}
\end{aligned}
$$

We have been able to contruct a delta for any given $\epsilon$. Therefore the limit holds.

# 2 Show absolute of function is continuous

Knowing $f(x)$ is continuous we have for any $\epsilon > |f(x) - f(c)|$ there exists a $\delta > |x-c|$. Using the reverse triange inequality we get

$$
\begin{aligned}
&||f(x)| - |f(c)|| \leq |f(x) - f(c)| < \epsilon \\
&||f(x)| - |f(c)|| < \epsilon
\end{aligned}
$$

Thus the $\delta$ that exists for $f(x)$ is equally valid for $F(x)$ thus it is continuous.

## 3 There exists an interval that is positive

Since $f(x)$ is continuous at $a$ then by definition we have for all $|f(x) - f(a)| < \epsilon$ there will exist a $\delta < |x - a|$. We choose $\epsilon = f(a)$ so the range of $f(x)$ is now $(f(a) - f(a), f(a) + f(a)) = (0, 2f(a))$. Which is a positive range. For such an epsilon we know there exists a $\delta$.

## 4 Continuity of functions

We know all polynomials are continuous and we know $\log x$ is continuous. When we add two continuous functions we produce another continuous function. Now we convert our problem to use IVT. Let $g(x) = f(x) - 3$ so if there exists and $x$ where $g(x) = 0$ we will find $f(x) = 3$.

$$\begin{aligned} g(x) &= 0 \\ f(x) - 3 &= 0 \\ f(x) &= 3 \end{aligned}$$

Using a calculator we try $g(0.5) = \frac{1}{64} - \log(2) - 3$ which is less than 0. Now take $g(1.5) = \frac{729}{64} + \log\left(\frac{3}{2}\right) - 3$ which is greater than 0. Since $g(x)$ is a sum of continuous functions it is continuous and the IVT can be applied. There is a point which is greater than 0 and a point less than 0 thus there exists a point where it is 0. Therefore, there is a point, $x$, where $x^6 + \log(x) = 3$.

## 5 Fixed point

We use the same technique as before. Construct $g(x) = f(x) - x$ so when $g(x) = 0 \to f(x) = x$. Since $f(x)$ is continuous, $g(x)$ is also continuous.

Try $g(0)$. Since the domain and range of $f(x)$ is $[0, 1]$, $f(x) \geq 0$. If $f(0) = 0$

we're done thus our only interesting option is $f(0) > 0$. Thus $g(0) \geq 0$. Now try $g(1)$ Since the domain and range of $f(x)$ is $[0,1]$, $f(x) \leq 1$. If $f(1) = 1$ we're done thus our only interesting option is $f(1) < 1$. Thus $g(1) \leq 0$.

We now apply IVT to show there exists $g(x) = 0$ which means $f(x) = x$.

# 6 Restricted function

Counter example, $f(x) = \left|\frac{1}{0.5+x}\right|$. We show it is bounded on the integers but unbounded on the reals.

Since the function is absolute it is bounded below by 0. It is also monotone increasing until it hits 0 then is monotone decreasing. We first try to show the monotone increasing to 0. For clarity let $a = -x$

$$\begin{aligned}
f(x) &\leq f(x+1) && x < 0 \\
f(-a) &\leq f(-a+1) && a > 0 \\
\left|\frac{1}{0.5-a}\right| &\leq \left|\frac{1}{0.5-a+1}\right| \\
\left|\frac{1}{0.5-a}\right| &\leq \left|\frac{1}{1.5-a}\right| \\
\frac{1}{|0.5-a|} &\leq \frac{1}{|1.5-a|} \\
|0.5-a| &\geq |1.5-a| \\
|a-0.5| &\geq |a-1.5|
\end{aligned}$$

Since $a > 0$, the inequality holds. For $x < 0$ the function is monotone increasing. A finite monotone increasing function will have it's upper bound as its final answer. In this case it is $x = -1$ so $f(-1) = 2$.

The other side is monotone decreasing as we will demonstrate.

$$\begin{aligned}
f(x) &\geq f(x+1) && x \geq 0 \\
\left|\frac{1}{0.5+x}\right| &\geq \left|\frac{1}{0.5+x+1}\right| \\
\frac{1}{|0.5+x|} &\geq \frac{1}{|0.5+x+1|} \\
|0.5+x| &\leq |0.5+x+1| \\
|0.5+x| &\leq |1.5+x|
\end{aligned}$$

Since $x \geq 0$, the inequality holds. For $x \geq 0$ the function is monotone decreasing. Recall the function is absolute so it is bounded below by 0 and it's maximum value is the first possible answer. In this case it is $x = 0$ so $f(0) = 2$.

Thus we have a lower bound and upper bound that exist for this function restricted to the integers. We now show such limits do not exist in the reals.

Assume there exists an upper bound $L$. Choose $x = \frac{1}{L+1} - 0.5$ so $f\left(\frac{1}{L+1} - 0.5\right) = L+1$ which is higher than the upper bound. Therefore there is no upper bound and it's unbounded over $\mathbb{R}$.