# Linear Programming

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# Example Problem

- Company produces three products:
  - $x_1, x_2, x_3$
  - Each has profit equal to 3,1,2 respectivley
- To produce it, the company needs raw material units, labour, and machine time
- Product $x_1$ requires:
  - 1 hour labour
  - 2 hours machine time
  - 4 units of raw material
- Product $x_2$ requires
  - 1 hour labour
  - 2 hours machine time
  - 1 units of raw material
- Product $x_3$ requires
  - 3 hour labour
  - 5 hours machine time
  - 2 units of raw material
- Company has a total of 30 hours of labour, 24 hours of machine time, and 36 units of raw material avaiable
- How many units of each product **maximizes profit**

- Let $z$ be the total profit, we need to maximize:

$$z = 3x_1 + x_2 + 2x_3$$

- Subject to:

$$\begin{aligned} x_1 + x_2 + 3x_3 &\leq 30 \\ 2x_1 + 2x_2 + 5x_3 &\leq 24 \\ 4x_1 + x_2 + 2x_3 &\leq 36 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

# Methods of Solving LP

- Simplex Method
  - Moves along the edges of the feasible region
- Interior Point Method
  - Moves through the interior of the feasible region
- **We only cover simplex**

# Standard Form

**Standard form of a LP Problem:**

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \text{ for } j = 1, 2, \ldots, n$$

**Matrix Notation:**

$$\text{maximize } z = \vec{c}\vec{x}$$

$$\text{subject to } \mathbf{A}\vec{x} \leq \vec{b}$$

$$\vec{x} \geq \vec{0}$$

where:

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

# Conversion to Standard Form

3. Standard Form
   - Conversion to Standard Form

- Replace $\min f$ by $\max -f$
  - Remember standard form only deals with maximisation
- For any constraint, $\sum_{j=1}^{n} a_{ij}x_j = b$ is replaced with $\sum_{j=1}^{n} a_{ij}x_j \leq b$ and $\sum_{j=1}^{n} a_{ij}x_j \geq b$
  - Ensures the equality
- For any constraint $\sum_{j=1}^{n} a_{ij}x_j \geq b$ is replaced by $\sum_{j=1}^{n} -a_{ij}x_j \leq -b$
- If a variable $x_j$ has no sign restriction, it is replaced by $x_j = x_j^+ - x_j^-$ where $x_j^+, x_j^- \geq 0$

# Slack Form

- **Slack form** converts the inequalities to equalities by introducing **slack** variables
- For each constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, we introduce a **slack variable** $s_i$ such that:

$$s_i = b_i - \sum_{j=1}^{n} a_{ij}x_j \text{ with } s_i \geq 0$$

- This transforms our inequality constraints into equality constraints:

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$$

- The slack variables represent the unused resources or "slack" in each constraint
- As we will see later, the slack variables will be mixed with the original variables during the iterations of the simplex method
- Therefore, we rename the slack variables $s_i$ as $x_{n+i}$ for a consistent variable naming scheme:

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \text{ for } i = 1, 2, \ldots, m$$

- The resulting system has:
  - $n$ original variables: $x_1, x_2, \ldots, x_n$
  - $m$ slack variables: $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$
  - All variables are non-negative: $x_j \geq 0$ for $j = 1, 2, \ldots, n+m$
- This transformed system with equality constraints is called the **slack form** of the linear program

# Basic and Non-Basic Variables

- In the slack form of a linear program, variables are classified as:
  - **Basic variables**: Variables that appear in the left-hand side of exactly one equation
  - **Non-basic variables**: Variables that are not basic
- Initially, after converting to slack form:
  - Slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are the basic variables
  - Original variables $x_1, x_2, \ldots, x_n$ are the non-basic variables
- A **basic solution** is obtained by setting all non-basic variables to zero and solving for the basic variables
- A **basic feasible solution (BFS)** is a basic solution where all variables are non-negative
- Key properties:
  - Every basic feasible solution corresponds to a vertex of the feasible region
  - The optimal solution to a linear program occurs at a vertex (if it exists)
  - The simplex method moves from one BFS to another by changing which variables are basic/non-basic
- During execution of the simplex method:
  - Variables can switch between basic and non-basic status
  - This corresponds to moving from one vertex to an adjacent one in the feasible region

# The Simplex Method

- The simplex method is an algorithm for finding the optimal solution to a linear programming problem
- Main idea: Start at a vertex of the feasible region and move to adjacent vertices that improve the objective function
- Algorithm overview:
  1. Convert the LP problem to slack form
  2. Identify initial basic variables (typically slack variables) and non-basic variables
  3. Check if the current solution is optimal
  4. If not optimal, select:
     - An entering variable: a non-basic variable to increase from zero
     - A leaving variable: a basic variable that will become zero
  5. Pivot: update the equations to exchange the roles of these variables
  6. Repeat until optimal solution is found
- Optimality criterion: If all coefficients in the objective function are $\leq 0$ , the current solution is optimal
- Selection rules:
  - Entering variable: non-basic variable with the largest positive coefficient in objective function
  - Leaving variable: determined by the ”minimum ratio test” to maintain feasibility
- Geometric interpretation: We’re moving along edges of the feasible region toward the ”highest point”

# Tableau Method

Take for example the following operations:

$$\begin{aligned} z - 3x_1 - x_2 - 2x_3 &= 0 \\ x_4 + x_1 + x_2 + 3x_3 &= 30 \\ x_5 + 2x_1 + 2x_2 + 5x_3 &= 24 \\ x_6 + 4x_1 + x_2 + 2x_3 &= 36 \end{aligned}$$

This can be represented in tableau form as:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $z$ | -3 | -1 | -2 | 0 | 0 | 0 | 0 |
| $x_4$ | 1 | 1 | 3 | 1 | 0 | 0 | 30 |
| $x_5$ | 2 | 2 | 5 | 0 | 1 | 0 | 24 |
| $x_6$ | 4 | 1 | 2 | 0 | 0 | 1 | 36 |

In this tableau:

- The first row represents the objective function
- The remaining rows represent constraints with slack variables
- Basic variables ($x_4$, $x_5$, $x_6$) are listed in the leftmost column
- All variables ($x_1$ through $x_6$) are included in the column headers
- The identity matrix formed by the basic variables is visible (the 1's along the diagonal)
- The right-hand side (RHS) column contains the constraint values

# Pivots in Tableu

7. Tableu Method
   - Pivots in Tableu
   - Tableu Operation

- To perform a pivot operation in the simplex tableau:
  1. **Select entering variable**: Choose a non-basic variable with a negative coefficient in the objective function row (maximizes improvement)
  2. **Select leaving variable**: Find the row with the smallest ratio of RHS/coefficient for the entering variable (maintains feasibility)
  3. **Perform pivot**: Exchange the roles of these variables by row operations
- Let's use our previous tableau to demonstrate:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $z$ | -3 | -1 | -2 | 0 | 0 | 0 | 0 |
| $x_4$ | 1 | 1 | 3 | 1 | 0 | 0 | 30 |
| $x_5$ | 2 | 2 | 5 | 0 | 1 | 0 | 24 |
| $x_6$ | 4 | 1 | 2 | 0 | 0 | 1 | 36 |

- **Selecting entering variable**: The non-basic variables that would increase the objective function are those with negative coefficients in the objective row ($x_1$, $x_2$, $x_3$)
- $x_1$ has the most negative coefficient (-3), so we select it as the entering variable
- **Selecting leaving variable**: Calculate ratios RHS/coefficient of $x_1$:

$$\text{Row } x_4 : 30/1 = 30$$
$$\text{Row } x_5 : 24/2 = 12$$
$$\text{Row } x_6 : 36/4 = 9$$

- The smallest ratio is 9 in the $x_6$ row, so $x_6$ is our leaving variable
- **The pivot element** is therefore at the intersection of the $x_1$ column and $x_6$ row: value 4
- This pivot represents moving from one basic feasible solution to an adjacent one that increases the objective value

# Tableu Operation

7. Tableu Method
   - Pivots in Tableu
   - Tableu Operation

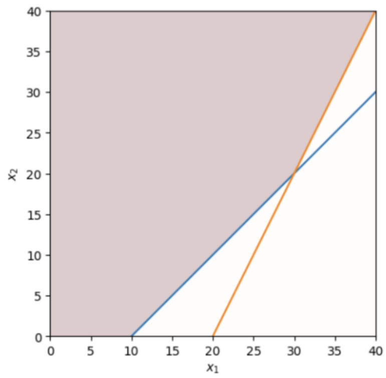
- **Executing the pivot operation**:
  1. **Pivot row operation**: Divide the entire pivot row by the pivot element to get a 1 in the pivot position
  2. **Other rows operation**: For each other row, subtract a multiple of the pivot row to get 0 in the pivot column
- After our pivot selection ($x_6$ leaving, $x_1$ entering), the operations would be:
  1. Divide row $x_6$ by 4 to get 1 in the $x_1$ column
  2. Subtract appropriate multiples from other rows to get 0s in the $x_1$ column
- The new tableau would have $x_1$ as a basic variable replacing $x_6$
- **New current solution**: Set all non-basic variables to 0, and read the values of basic variables from the RHS
- **Termination criteria**:
  - **Optimality**: Stop when all coefficients in the objective row (z-row) are non-negative
  - **Unboundedness**: If a column has a positive coefficient in the objective row but non-positive coefficients in all constraint rows, the problem is unbounded
  - **Infeasibility**: If we cannot maintain non-negative values for all basic variables
- **Reading the final solution**:
  - The optimal objective value is the value in the z-row, RHS column
  - For each basic variable (listed in leftmost column), its value is given in the RHS
  - Non-basic variables all equal zero
- **Cycling prevention**: Various techniques (like Bland's rule - select the lowest-indexed eligible variable) prevent the algorithm from cycling indefinitely
- Each iteration of the tableau method moves to an improved solution until optimality is reached or unboundedness/infeasibility is detected

# Unbounded Feasible Region

### What does it mean to have an Unbounded Feasible Region

- A feasible region of a simplex problem can e **unbounded**
- It means that the objective we are trying to maximize can increase indefinitley
  - (i.e.) there is no solution for the maximum

The objective function has **no maximum**. The simplex steps go from $(0,0)$ to $(10,0)$ to $(30,20)$. After that the objective function increases without bound.




### How to tell whether the feasible region is unbounded

- Look for the **most negative** value in the objective row - Call it $x_a$
- If the ratio $\frac{b_i}{a_{ia}} \leq 0$ for all $i$ - the problem is unbounded
- For example

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | 0 | $-4$ | 3 | 80 |
| $x_1$ | 1 | 0 | $-1$ | 1 | 30 |
| $x_2$ | 0 | 1 | $-2$ | 1 | 20 |

- $x_3$ has the most negative value in the objective row
- All of the ratios from the RHS to the values in the $x_3$ column are negative
- Therefore, the feasible region is unbounded

# Initial Basic Feasible Solution

**Initial Basic Solutions**

- The initial basic solution is **feasible** $\iff b_i \geq 0$ for all $i$
  - The initial solution would result in $0 \leq \mathbb{Z}^-$ in some of the formulae which is obviously not true
- $b_i$ reference the numbers on the RHS of the $\leq$ signs in the original LP formulation

**How to fix an infeasible initial basic solution**

- Use the **Two-Stage Simplex Method**
  - Phase 1 - Find feasible solution by minimizing an auxiliary function
  - Phase 2 - Use this new feasible solution to solve the original problem

# Two-Stage Simplex

### Phase 1 - Find the Auxillary Problem

- Introduce new variable $x_0$ and formulate the new auxillary problem:

$$\text{Maximize} - x_0$$
$$\text{Subject to} \sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i$$
$$x_j \geq 0$$

- Run simplex on this new problem until you reach an optimal solution and have maximized $-x_0$ (same as minimizing an artificial variable!)
- Once optimal solution is reached, remove the $x_0$ column completely

### Phase 2 - Getting the New Objective Row

- Once we have a feasible solution, drop the artificial variable $x_0$
- Replace the auxiliary objective function with the original objective function
- Substitute the basic variables from Phase 1 into the original objective function
- Continue with standard simplex method from this point
- The final solution will be optimal for the original problem

# Degeneracy

**What is degeneracy**

- An LP problem is degenerate if values on the RHS is zero
- A pivot is said to be degenerate when the leaving value has zero ratio
- Sometimes **BUT NOT ALWAYS** this causes the simplex method to cycle
- This could mean that the simplex will never terminate

**How to avoid Degeneracy**

- Use the **Bland Pivot Rule**
- Always choose the negative pivot that has the lowest index column