

First Order Linear: Interval of Validity:

- find the x values for which the differential equation is undefined or discontinuous
- split up the number line into intervals among those points
- pick the interval that contains your initial condition

First Order Linear: Separable:

- coerce into form $f(y)dy = g(x)dx$
(use $y' = \frac{dy}{dx}$)
- integrate both sides
- solve for y

First Order Linear: Integrating Factor:

- form $y' + p(x)y = g(x)$
- $\mu(x) = e^{\int_0^x p(s)ds}$
- multiply both sides by $\mu(x)$
- now it's $\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$
 $\mu(x)y = \int \mu(x)g(x)dx$

First Order Linear: Exact Equations:

- form: $M(x, y) + N(x, y)\frac{dy}{dx} = 0$
- $\Psi_x = M, \Psi_y = N$, find $\Psi(x, y) = \dots$
- Exact if $M_y = N_x$
 $\Psi(x, y) = \int Mdx + h(y)$
 $\Psi(x, y) = \int Ndy + g(x)$

Inexact \rightarrow Exact:

- $\frac{d\mu(x)}{dx} = \frac{M_y - N_x}{N}\mu(x)$ only applies if y drops

out

(similar with $\mu(y)$)

Homogeneous:

- $y'' + p(t)y' + q(t)y = 0$
- Characteristic Equation: $ar^2 + br + c = 0$
- General Solution:
 $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$
- if $r_1 = r_2$ then $y_2 = te^{r_1t}$
- Complex roots:
 - $r = \lambda \pm \mu i$
 - $y_{1,2}(t) = e^{(\lambda \pm \mu i)t}$
 - $u(t) = e^{\lambda t} \cos(\mu t)$
 - $v(t) = e^{\lambda t} \sin(\mu t)$
 - $y(t) = c_1e^{\lambda t} \cos(\mu t) + c_2e^{\lambda t} \sin(\mu t)$
- λ ends up as negative because of $\frac{-b}{2a}$ in the quadratic formula
- μ is positive (the imaginary part of the solution)

Wronskian:

- $y'' + p(t)y' + q(t)y = 0$
- $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = Ce^{-\int_0^t p(s)ds}$

Non-Homogeneous: undetermined coefficients:

- $y'' + p(t)y' + q(t)y = g(t)$
- first replace $g(t)$ with 0 and find $y_1(t), y_2(t)$

(the complimentary solution $y_c(t)$)

- then find y_p : $y(t) = y_1(t) + y_2(t) + y_p(t)$
- guess a y_p (usually a combination from Ce^t , $C_1 \sin(C_2t)$, $C_1 \cos(C_2t)$, or polynomial)
- substitute $y(t) \rightarrow y_p$ in the original equation and
- solve for unknown coefficients
- final solution is $y(t) = y_c(t) + y_p(t)$

Variation of Parameters:

- $y'' + p(t)y' + q(t)y = g(t)$
First solve the complimentary homogeneous equation
- $y = u_1(t)y_1(t) + u_2(t)y_2(t)$

Damped Mass on a Spring:

- $mx'' + \gamma x' + kx = f(t)$
- equilibrium position: $x = 0$
- x = position; initial $x(0) = h$
- velocity: $v = x'$; initial $x'(0) = v_0$
- acceleration: $a = x''$
- k = spring constant
- m = mass
- ω = period = $\sqrt{\frac{k}{m}}$ (when undamped)
- friction determined by γ
 - (situation dependent)
 - has units force per velocity
- external force per time: $f(t)$

- if $f(t) = 0$ and $\gamma = 0$ (no friction)
 - $x(t) = h \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$
 - $x(t) = A \cos(\omega t + \sigma)$
 $A = \sqrt{h^2 + (\frac{v_0}{\omega})^2}$
 $\sigma = -\tan^{-1}(\frac{h v_0}{\omega})$
- damping type depends on how γ^2 relates to $4km$
- if $f(t) = 0$ and overdamped: $\gamma^2 > 4km$
 - happens when $\gamma^2 > 4km$
 - * Therefore $r_1, r_2 > 0$
 - $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
 - no oscillation, $x(t) \rightarrow 0$ as $t \rightarrow \infty$
- if $f(t) = 0$ and underdamped: $\gamma^2 < 4km$
 - happens when $\gamma^2 < 4km$
 - * r_1, r_2 are complex
 - oscillates forever, amplitude approaching 0
 - $x(t) = e^{-\frac{\gamma}{2m}t} (C_1 \cos(\omega t) + C_2 \sin(\omega t))$
 - $x(t) = A e^{-\frac{\gamma}{2m}t} \cos(\omega t + \sigma)$
 - * different A, σ than no friction scenario
- if $f(t) = 0$ and critically damped: $\gamma^2 = 4km$
 - happens when $\gamma^2 = 4km$
 - returns to $x = 0$ as quickly as possible without oscillating (exponentially decays toward $x = 0$ as $t \rightarrow \infty$)
 - $x(t) = e^{-\frac{\gamma}{2m}t} (C_1 + C_2 t)$
 - * only one r solution
 - * $x(0) = h = C_1$

- if C_1, C_2 have the same sign:
 - $x(t)$ is always on the same side of the x -axis
- if C_1, C_2 have opposite sign:
 - $x(t)$ must cross the x -axis exactly once
- if $f(t) = F_0$
 - constant external force
 - $x_p = A$
 - $x_c =$ complimentary solution for $f(t) = 0$
 - $x(t) = x_c + x_p$
 - $x(t) \rightarrow A$ as $t \rightarrow \infty$
- if $f(t) = F_0 \cos(\omega_2 t)$
 - external periodic force
 - $x_p = A \cos(\omega_2 t) + B \sin(\omega_2 t)$
 - oscillates with constant period $T = \frac{2\pi}{\omega_2}$ as $t \rightarrow \infty$ (because original oscillation dies out)
- if x_p solves x_c :
 - instead use $x_p = t(A \cos(\omega_2 t) + B \sin(\omega_2 t))$
 - means you have **resonance**; means that the force perfectly adds to the existing kinetic energy; if γ is small, amplitude can grow large

Electrical Vibrations: Series Circuit:

- equation: $LQ'' + RQ' + \frac{1}{C}Q = E(t)$
- $E(t)$: impressed (input) voltage
- derivative therefore in terms of current:

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

Series Solutions:

- For equation with order n , you need initial conditions for $y', y'', \dots, y^{(n-1)}$
- you can then find initial $y^{(n)}$ by solving the equation for it with the initial conditions
- And you can then repeatedly differentiate the equation with respect to x to find lots of initial condition derivatives
- then find the formula for the n^{th} derivative and plug it into the Taylor series formula
- **Taylor Series Formula:** $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$
- Maclaurin series is just the Taylor series with $a = 0$ (centered at 0)

$$\begin{aligned} f(x) &= \sum a_n x^n \\ f'(x) &= \sum_{n=1} (n) a_n x^{n-1} \\ f'(x) &= \sum_{n=0} (n+1) a_{n+1} x^n \\ f''(x) &= \sum_{n=2} n(n-1) a_n x^{n-2} \\ f''(x) &= \sum_{n=0} (n+2)(n+1) a_{n+2} x^n \\ f(x) + g(x) &= \sum (a_n + b_n) x^n \end{aligned}$$
- solve using: $y'' + Py' + Qy = 0 \rightarrow \sum [(n+2)(n+1)a_{n+2} + P(n+1)a_{n+1} + Qa_n]x^n = 0$

Matrix Solutions:

- works for any n^{th} order equation $y^{(n)} + P_{n-1}y^{(n-1)} + \dots + P_1y' + P_0y = 0$

Laplace Transform:

- $\mathcal{L}[f(t)](s) = F(s) = \int_0^{\infty} e^{-st} f(t) dt$
- convention is that an uppercase function in s

is the Laplace transform of a lowercase function in t

- generally use a lookup table
- n th derivative: $\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1}y(0) - s^{n-2}y'(0) \dots - y^{(n-1)}(0)$

- common Laplace transforms:

$f(t)$:	$F(s)$:	validity:
$f(t) + g(t)$	$F(s) + G(s)$	all s
$C * f(t)$	$C * F(s)$	all s
$t * f(t)$	$-F'(s)$	$s > 0?$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$f(t)e^{ct}$	$F(s-c)$	$s > c?$
$u(t-a)g(t-a)$	$G(s)e^{-as}$	$s > 0?$
$\delta(t-a)$	e^{-as}	
$\mathcal{L}[f(t)](s)$	$= Y(s)$	
$\mathcal{L}[f'(t)](s)$	$= sY(s) - y(0)$	
$\mathcal{L}[f''(t)](s)$	$= s^2Y(s) - sy(0) - y'(0)$	

- generally you don't do Laplace transforms by hand, you use a lookup table (same for inverse)

Laplace - Unit Step Function:

- $u(t) = \begin{cases} 1: t > 0 \\ 0: t < 0 \end{cases}$
- $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$
- $\mathcal{L}[u(t-a)g(t-a)] = G(s)e^{-as}$

Laplace - Convolution:

- $(f * g)(t) = \int_0^t f(t-T)g(T)dt$
- $(f * g)(t) = (g * f)(t)$
- $\mathcal{L}[(f * g)(t)] = F(s)G(s)$

Laplace - Derivative:

- $\mathcal{L}[tf(t)] = -F'(s)$
- $\mathcal{L}[t^2f(t)] = F''(s)$
- $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

Final Exam Stuff:

Linear Algebra:

- matrix manipulations:
 - for when you've got a square matrix (right hand side) and column matrix (left hand side)
 - multiply a row by a constant
 - add rows together and store in existing row
 - subtract rows, store in existing row
- equation as matrix:
 - a row is all 0's: solution is not unique
 - all 0's row except equal to non-zero: equations are inconsistent, no (real) solution exists
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