

3D vectors:

- vector $v = \langle a, b, c \rangle$
- Magnitude (length): $|v| = \sqrt{a^2 + b^2 + c^2}$
- **Dot Product:** $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 = |a||b| \cos \theta$
if $a \cdot b = 0$, a and b are perpendicular
- **Cross Product:** $a \times b = \langle a_yb_z - a_zb_y, a_zb_x - a_xb_z, a_xb_y - a_yb_x \rangle$
if $a \times b = 0$, a and b are parallel
 $a \times b = n|a||b| \sin \theta$ where n is a vector perpendicular to both a and b in direction given by right hand rule
 $a \times (b + c) = a \times b + a \times c$
- **Angle** between (nonzero) vectors: $\theta = \cos^{-1}(\frac{a \cdot b}{|a||b|})$
- Unit Vector: $\hat{a} = \frac{a}{|a|}$
 \hat{a} is a vector of length 1 parallel to vector a
- Scalar triple product: $a \cdot (b \times c)$
- Vector triple product: $a \times (b \times c)$
- areas and volumes:
 - area of parallelogram with sides $a, b = |a \times b|$
 - area of triangle with sides $a, b = \frac{1}{2}|a \times b|$
 - volume of box with sides $a, b, c = a \cdot (b \times c)$

Lines:

- **Vector Equation:** $L(t) = r_0 + vt$
 r_0 is a point on the line and v is a vector parallel to the line
 $L(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$
- **Parametric Equation:** $L(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
point on line: (x_0, y_0, z_0)
vector parallel to line: $\langle a, b, c \rangle$

Planes:

- **Standard (linear) form:** $ax + by + cz = d$
 $d = ax_0 + by_0 + cz_0$ where $P(x_0, y_0, z_0)$ is a point in the plane
normal vector: $n = \langle a, b, c \rangle$
- **Scalar form:** $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
normal vector: $n = \langle a, b, c \rangle$
point in plane: $P(x_0, y_0, z_0)$
- **Distance** from point $P(x, y, z)$ to plane: $D = \frac{|ax + by + cz - d|}{\sqrt{a^2 + b^2 + c^2}}$
(assuming plane is in linear form above)

Quadratic Surfaces:

- **Ellipsoid** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
 - All traces are ellipses
- **Elliptic Paraboloid** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
 - **Horizontal** traces are ellipses
 - **Vertical** traces are parabolas
- **Hyperboloid of one sheet** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
 - **Horizontal** traces are ellipses
 - **Vertical** traces are hyperbolas
- **Hyperboloid of two sheets** $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
 - **Horizontal** traces are ellipses
 - **Vertical** traces are hyperbolas
 - some traces do not exist because graph has a gap centered around the origin
- **Cone** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
 - **Horizontal** traces are ellipses.
 - **Vertical** traces are pair of lines if x or y is 0, otherwise hyperbolas
- **Hyperbolic Paraboloid** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
 - **Horizontal** traces are hyperbolas.
 - **Vertical** traces are parabolas

Vector Functions:

- **Arc length** from $t = a$ to $t = b$: $\int_a^b |r'(t)| dt$
 - Conversion back to the similar form from 2D: $|r'(t)| = \sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}$
- **Arc Length Function:** $s(t) = \int_a^t |r'(u)| du$
- **Unit Tangent Vector:** $T(t) = \frac{r'(t)}{|r'(t)|}$
unit-length vector tangent to the curve $r(t)$
- **Unit normal vector:** $N(t) = \frac{T'(t)}{|T'(t)|}$
unit-length vector perpendicular to $r(t)$

Derivatives:

- $z = f(x, y)$
- Notation: $\frac{\partial z}{\partial x} = f_x = \frac{\partial f}{\partial x}$ etc. . .
 - Same for second derivatives: $f_{xy} =$
- **Gradient Vector:** $\nabla f = \langle f_x, f_y, f_z \rangle$
- **Tangent plane** at $P(a, b, c)$: $f_x(a, b) + f_y(a, b) = z - c$
- Chain Rule:
 - $\frac{\partial x}{\partial z} = -\frac{\partial F / \partial z}{\partial F / \partial x} = \frac{F_z}{F_x}$ (∂F cancels out, fraction flips)
 - $\frac{\partial z}{\partial t} = -\frac{\frac{\partial z}{\partial x} \frac{\partial x}{\partial t}}{\frac{\partial z}{\partial y} \frac{\partial y}{\partial t}}$ (∂x cancels out)
- Directional Derivative, parallel to $\langle a, b, c \rangle$ at $P(x, y, z)$:
 $f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$

Double Integrals:

- volume under the function $f(x, y)$ on the rectangle $R = [a, b] \times [c, d]$
- $\iint_R f(x, y) dA$ on $R = [a, b] \times [c, d] = \int_c^d \int_a^b f(x, y) dx dy$
solve the inner integral first, then the outer
if $f(x, y)$ is continuous on R , then you can flip the order of the integrals
- if $f(x, y) = g(x) \cdot h(y)$ then $\int_c^d \int_a^b f(x, y) dx dy = \int_a^b g(x) dx \cdot \int_c^d h(y) dy$
- **General Regions:**
Only difference is whether x or y has its limits defined in terms of the other
For these, you must evaluate the inner integral first, you can't swap them

- Type 1: bounds of x are constants, bounds of y are defined as functions of x

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

- Type 2: bounds of y are constants, bounds of x are defined as functions of y

$$D = \{(x, y) | h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Polar Coordinates:

- Polar \rightarrow Cartesian:
 $r = \pm \sqrt{x^2 + y^2}$
 $\theta = \tan^{-1}(\frac{y}{x})$
 - You need to be careful with the sign of r and multiples of θ because Polar Coordinates are **not** unique.
- Cartesian \rightarrow Polar:
 $x = r \cos \theta$
 $y = r \sin \theta$
 - You don't have to worry about quadrants or anything because Cartesian Coordinates are **unique**.
- **Double Integrals in Polar Coordinates**
works best when D is in a polar-coordinate-friendly shape
 - Do the following replacements:

- * dA or $dx dy$ or $dy dx \rightarrow r dr d\theta$
- * $x \rightarrow r \cos \theta$
- * $y \rightarrow r \sin \theta$
- * $x^2 + y^2 \rightarrow r$
- * Translate limits

- Should end up with something that looks like one of these general regions:

$$\int_a^b \int_{g_1(r)}^{g_2(r)} f(r \cos \theta, r \sin \theta) dx dy$$

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) dr d\theta$$

- integrate as normal with new function and new limits
stuff will probably cancel out everywhere

• Arc Length in Polar Coordinates:

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Cylindrical Coordinates:

- $dV = r dr d\theta dz$
- usually:
 - $r \geq 0$
 - $0 \leq \theta \leq 2\pi$
- Cylindrical \rightarrow Cartesian:
 - $x = r \cos \theta$
 - $y = r \sin \theta$
 - $z = z$
- Cartesian \rightarrow Cylindrical:
 - (be careful about the quadrant of θ)
 - $r = \sqrt{x^2 + y^2}$
 - $r^2 = x^2 + y^2$
 - $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
 - $z = z$

Spherical Coordinates:

- $dV = \rho^2 \sin \phi$
- ϕ is the angle from the $+z$ axis down to ρ
 - usually:
 - $-\pi/2 \leq \phi \leq \pi/2$
 - $\rho \geq 0$
 - $0 \leq \theta \leq 2\pi$
- Spherical \rightarrow Cartesian:
 - $x = \rho \sin \phi \cos \theta$
 - $y = \rho \sin \phi \sin \theta$
 - $z = \rho \cos \phi$
- Cartesian \rightarrow Spherical:
 - $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
 - $\rho^2 = x^2 + y^2 + z^2$
 - $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$
- Other:
 - $\rho^2 = x^2 + y^2 + z^2$
 - $r = \rho \sin \phi$

Minimum and Maximum:

Find critical points by solving $\nabla f = \langle 0, 0 \rangle$
 For each point, find $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
 $D > 0$ and $f_{xx} > 0$: relative minimum at (a, b)
 $D > 0$ and $f_{xx} < 0$: relative maximum at (a, b)
 $D < 0$: saddle point at (a, b)
 $D = 0$: can't tell (probably won't see)

Line Integrals:

- **2D:**
 - $C = \{r(t) | a \leq t \leq b\}$
 - $r(t) = \langle x, y, z \rangle$

Scalar function $f(x, y)$:

$$\int_C f(x, y) ds = \int_a^b f(r(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$= \int_a^b f(r(t)) |r'(t)| dt$$

$$\int_C f(x, y) dx = \int_a^b f(r(t)) x'(t) dt$$

Vector field $F(x, y) = \langle P, Q \rangle$:

$$\int_C F(x, y) \cdot ds = \int_a^b P(r(t)) dx + Q(r(t)) dy$$

• **3D:**

$$C = \{r(t) | a \leq t \leq b\}$$

(scalars are the same)

Vector field $F(x, y, z) = \langle P, Q, R \rangle$:

$$\int_C F(x, y, z) \cdot ds = \int_a^b F(r(t)) \cdot r'(t) dt$$

$$= \int_C P dx + Q dy + R dz$$

$$= \int_a^b P \frac{\partial r}{\partial x} dx + Q \frac{\partial r}{\partial y} dy + R \frac{\partial r}{\partial z} dz$$

$$\int_C F(x, y, z) \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$$

Fundamental Theorem for Line Integrals:

$C : r(t), a \leq t \leq b$, C is simple. Domain is simply-connected

$$F(x, y, z) = \langle P, Q, R \rangle$$

If there exists f such that $\nabla f = F$, then:

$$\int_C F \cdot dr = f(r(b)) - f(r(a))$$

Green's Theorem: (2D only, doesn't work in 3D)

C : curve with $r(t) = \langle x(t), y(t), z(t) \rangle$ on $a \leq t \leq b$; C is closed and simple; D : region enclosed by C

$$F(x, y) = \langle P(x, y), Q(x, y) \rangle$$

$$\int_C F \cdot dr = \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Vector Field: Curl and Divergence:

$$\text{curl} F = \nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = P_x + Q_y + R_z$$

$$\text{div}(\text{curl}(F)) = 0, \text{curl}(\text{div}(f)) = 0$$

if $\text{curl}(F) = 0$, then F is irrotational (causes no rotation); and therefore $\nabla f = F$ exists

Surface Integrals:

Surface $S : r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for $(u, v) \in D$

scalar function $f(x, y, z)$, vector field $F(x, y, z)$

$$\hat{n} = (r_u \times r_v) / |r_u \times r_v| = \frac{r_u \times r_v}{|r_u \times r_v|}$$

Scalar Function f :

$$\iint_S f dS = \iint_D f(r(u, v)) |r_u \times r_v| dA$$

When $f(x, y, z) : z = g(x, y)$:

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{g_x^2 + g_y^2 + 1} dA$$

$$\text{Area of } S : \iint_D |r_u \times r_v| dA$$

Vector Function F :

$$\iint_S F \cdot dS = \iint_S (F \cdot \hat{n}) dS = \iint_D F(r(u, v)) \cdot (r_u \times r_v) dA$$

Stokes' Theorem: (works in 3D)

Surface S bounded by curve $C : g(t), a \leq t \leq b$

vector field $F(x, y, z)$

$$\int_C F \cdot dg = \iint_S \text{curl}(F) \cdot dS$$

Divergence Theorem:

surface S is boundary of solid region E

F is vector field

$$\iint_S F \cdot dS = \iiint_E \text{div}(F) dV$$

test this stuff like editing in emacs vim spacemacs