MATH 308 Reference Sheet Last Updated: November 3, 2016

Paul's Online Math Notes: http://tutorial.math.lamar.edu/Classes/DE/DE.aspx

Latex Symbols: http://oeis.org/wiki/List_of_LaTeX_mathematical_symbols

First Order Linear: Interval of Validity:

- find the x values for which the differential equation is undefined or discontinuous
- split up the number line into intervals among those points
- pick the interval that contains your initial condition

First Order Linear: Separable:

- coerce into form $f(y)dy = g(x)dx$ (use $y' = \frac{dy}{dx}$)
- integrate both sides
- solve for y

First Order Linear: Integrating Factor:

- form $y' + p(x)y = g(x)$
- $\mu(x) = e^{\int_0^x p(s)ds}$
- multiply both sides by $\mu(x)$
- now it's $\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$
  $\mu(x)y = \int \mu(x)g(x)dx$

First Order Linear: Exact Equations:

- form: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$
- $\Psi_x = M, \Psi_y = N$, find $\Psi(x,y) = \ldots$
- Exact if $M_y == N_x$
  $\Psi(x,y) = \int M dx + h(y)$
  $\Psi(x,y) = \int N dx + g(x)$

Inexact → Exact:

- $\frac{d\mu(x)}{dx} = \frac{M_y - N_x}{N}\mu(x)$ only applies if y drops out
  (similar with $\mu(y)$)

Homogeneous:

- $y'' + p(t)y' + q(t)y = 0$
- Characteristic Equation: $ar^2 + br + c = 0$
- General Solution:
  $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
- if $r_1 = r_2$ then $y_2 = te^{rt}$
- Complex roots:
  - $r = \lambda \pm \mu i$
  - $y_{1,2}(t) = e^{(\lambda \pm \mu i)t}$
  - $u(t) = e^{\lambda t}\cos(\mu t)$
  - $v(t) = e^{\lambda t}\sin(\mu t)$
  - $y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$
- λ ends up as negative because of $\frac{-b}{2a}$ in the quadratic formula
- μ is positive (the imaginary part of the solution)

Wronskian:

- $y'' + p(t)y' + q(t)y = 0$
- $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = Ce^{-\int_0^t p(s)ds}$

Non-Homogeneous: undetermined coefficients:

- $y'' + p(t)y' + q(t)y = g(t)$
- first replace $g(t)$ with 0 and find $y_1(t), y_2(t)$ (the complimentary solution $y_c(t)$)
- then find y_p: $y(t) = y_1(t) + y_2(t) + y_p(t)$
- guess a y_p (usually a combination from Ce^t, $C_1\sin(C_2 t)$, $C_1\cos(C_2 t)$, or polynomial)
- substitute $y(t) \to y_p$ in the original equation and
- solve for unknown coefficients
- final solution is $y(t) = y_c(t) + y_p(t)$

Variation of Parameters:

- $y'' + p(t)y' + q(t)y = g(t)$
  First solve the complimentary homogeneous equation
- $y = u_1(t)y_1(t) + u_2(t)y_2(t)$

Damped Mass on a Spring:

- $mx'' + \gamma x' + kx = f(t)$
- equilibrium position: $x = 0$
- x = position; initial $x(0) = h$
- velocity: $v = x'$; initial $x'(0) = v_0$
- acceleration: $a = x''$
- k = spring constant
- m = mass
- ω = period = $\sqrt{\frac{k}{m}}$ (when undamped)
- friction determined by γ
  - (situation dependent)
  - has units force per velocity
- external force per time: $f(t)$

- if $f(t) = 0$ and $\gamma = 0$ (no friction)
  - $x(t) = h\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$
  - $x(t) = A\cos(\omega t + \sigma)$
    $A = \sqrt{h^2 + (\frac{v_0}{\omega})^2}$
    $\sigma = -\tan^{-1}\left(\frac{hv_0}{\omega}\right)$
- damping type depends on how γ^2 relates to $4km$
- if $f(t) = 0$ and overdamped: $\gamma^2 > 4km$
  - happens when $\gamma^2 > 4km$
    - Therefore $r_1, r_2 > 0$
  - $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
  - no oscillation, $x(t) \to 0$ as $t \to \infty$
- if $f(t) = 0$ and underdamped: $\gamma^2 < 4km$
  - happens when $\gamma^2 < 4km$
    - r_1, r_2 are complex
  - oscillates forever, amplitude approaching 0
  - $x(t) = e^{-\frac{\gamma}{2m}t}\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right)$
  - $x(t) = Ae^{-\frac{\gamma}{2m}t}\cos(\omega t + \sigma)$
    - different A, σ than no friction scenario
- if $f(t) = 0$ and critically damped: $\gamma^2 = 4km$
  - happens when $\gamma^2 = 4km$
  - returns to $x = 0$ as quickly as possible without oscillating (exponentially decays toward $x = 0$ as $t \to \infty$)
  - $x(t) = e^{-\frac{\gamma}{2m}t}(C_1 + C_2 t)$
    - only one r solution
    - $x(0) = h = C_1$
  - if C_1, C_2 have the same sign:
    $x(t)$ is always on the same side of the x-axis
  - if C_1, C_2 have opposite sign:
    $x(t)$ must cross the the x-axis exactly once
- if $f(t) = F_0$
  - constant external force
  - $x_p = A$
  - x_c = complimentary solution for $f(t) = 0$
  - $x(t) = x_c + x_p$
  - $x(t) \to A$ as $t \to \infty$
- if $f(t) = F_0\cos(\omega_2 t)$
  - external periodic force
  - $x_p = A\cos(\omega_2 t) + B\sin(\omega_2 t)$
  - oscillates with constant period $T = \frac{2k}{\omega_2}$ as $t \to \infty$ (because original oscillation dies out)
- if x_p solves x_c:
  - instead use $x_p = t(A\cos(\omega_2 t) + B\sin(\omega_2 t))$
  - means you have **resonance**; means that the force perfectly adds to the existing kinetic energy; if γ is small, amplitude can grow large

Electrical Vibrations: Series Circuit:

- equation: $LQ'' + RQ' + \frac{1}{C}Q = E(t)$
- $E(t)$: impressed (input) voltage
- derivative therefore in terms of current:
  $LI'' + RI' + \frac{1}{C}I = E'(t)$

Series Solutions:

- For equation with order n, you need initial conditions for $y', y'', \ldots, y^{(n-1)}$
- you can then find initial $y^{(n)}$ by solving the equation for it with the initial conditions
- And you can then repeatedly differentiate the equation with respect to x to find lots of initial condition derivatives
- then find the formula for the n^{th} derivative and plug it into the taylor series formula
- **Taylor Series Formula:** $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
- Maclaurin series is just the taylor series with $a = 0$ (centered at 0)
- $$\begin{aligned} f(x) &= \sum a_n x^n \\ f'(x) &= \sum_{n=1} (n) a_n x^{n-1} \\ f'(x) &= \sum_{n=0} (n+1) a_{n+1} x^n \\ f''(x) &= \sum_{n=2} n(n-1) a_n x^{n-2} \\ f''(x) &= \sum_{n=0} (n+2)(n+1) a_{n+2} x^n \\ f(x) + g(x) &= \sum (a_n + b_n) x^n \end{aligned}$$
- solve using: $y'' + Py' + Qy = 0 \to \sum[(n+2)(n+1)a_{n+2} + P(n+1)a_{n+1} + Qa_n]x^n = 0$

Matrix Solutions:

- works for any n^{th} order equation $y^{(n)} + P_{n-1}y^{(n-1)} \ldots P_1 y' + P_0$

Laplace Transform:

- $\mathscr{L}[f(t)](s) = F(s) = \int_0^{\infty} e^{-st} f(t)dt$
- convention is that an uppercase function in s

is the Laplace transform of a lowercase function in t
- generally use a lookup table
- nth derivative: $\mathscr{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}y(0) - s^{n-2}y'(0) \ldots - y^{(n-1)}(0)$
- common Laplace transforms:

$f(t)$:	$F(s)$:	validity:
$f(t)+g(t)$	$F(s)+G(s)$	all s
$C*f(t)$	$C*F(s)$	all s
$t*f(t)$	$-F'(s)$	$s>0$?
e^{at}	$\frac{1}{s-a}$	$s>a$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s>0$
1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	$s>0$
$f(t)e^{ct}$	$F(s-c)$	$s>c$?
$u(t-a)g(t-a)$	$G(s)e^{-as}$	$s>0$?
$\delta(t-a)$	e^{-as}	

$$\begin{aligned}\mathscr{L}[f(t)](s) &= Y(s)\\ \mathscr{L}[f'(t)](s) &= sY(s)-y(0)\\ \mathscr{L}[f''(t)](s) &= s^2Y(s)-sy(0)-y'(0)\end{aligned}$$

- generally you don't do Laplace transforms by hand, you use a lookup table (same for inverse)

Laplace - Unit Step Function:
- $u(t) = \{ {}^{1:t>0}_{0:t<0}$
- $\mathscr{L}[u(t-a)] = \frac{1}{s}e^{-as}$
- $\mathscr{L}[u(t-a)g(t-a)] = G(s)e^{-as}$

Laplace - Convolution:
- $(f*g)(t) = \int_0^t f(t-T)g(T)dt$
- $(f*g)(t) = (g*f)(t)$
- $\mathscr{L}[(f*g)(t)] = F(s)G(s)$

Laplace - Derivative:
- $\mathscr{L}[tf(t)] = -F'(s)$
- $\mathscr{L}[t^2f(t)] = F''(s)$
- $\mathscr{L}[t^nf(t)] = (-1)^nF^{(n)}(s)$

Final Exam Stuff:

Linear Algebra:
- matrix manipulations:
  - for when you've got a square matrix (right hand side) and column matrix (left hand side)
  - multiply a row by a constant
  - add rows together and store in existing row
  - subtract rows, store in existing row
- equation as matrix:
  - a row is all 0's: solution is not unique
  - all 0's row except equal to non-zero: equations are inconsistent, no (real) solution exists
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