Math 251 Reference Sheet © Josh Wright 2015
Latex Symbols: http://oeis.org/wiki/List_of_LaTeX_mathematical_symbols Last Updated: February 7, 2017
Paul's Online Math Notes: http://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx

3D vectors:

- vector $v = \langle a, b, c \rangle$
- Magnitude (length): $|v| = \sqrt{a^2 + b^2 + c^2} =$
- **Dot Product:** $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = |a||b| \cos \theta$
  if $a \cdot b = 0$, a and b are perpendicular
- **Cross Product:** $a \times b = \langle a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x \rangle$
  if $a \times b = 0$, a and b are parallel
  $a \times b = n|a||b| \sin \theta$ where n is a vector perpendicular to both a and b in direction given by right hand rule
  $a \times (b + c) = a \times b + a \times c$
- **Angle** between (nonzero) vectors: $\theta = \cos^{-1}(\frac{a \cdot b}{|a||b|})$
- Unit Vector: $\hat{a} = \frac{a}{|a|}$
  $\hat{a}$ is a vector of length 1 parallel to vector a
- Scalar triple product: $a \cdot (b \times c)$
- Vector triple product: $a \times (b \times c)$
- areas and volumes:
  - area of parallelogram with sides $a, b = |a \times b|$
  - area of triangle with sides $a, b = \frac{1}{2}|a \times b|$
  - volume of box with sides $a, b, c = a \cdot (b \times c)$

Lines:

- **Vector Equation:** $L(t) = r_0 + vt$
  r_0 is a point on the line and v is a vector parallel to the line
  $L(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$
- **Parametric Equation:** $L(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
  point on line: (x_0, y_0, z_0)
  vector parallel to line: $\langle a, b, c \rangle$

Planes:

- **Standard (linear) form:** $ax + by + cz = d$
  $d = ax_0 + by_0 + cz_0$ where $P(x_0, y_0, z_0)$ is a point in the plane
  normal vector: $n = \langle a, b, c \rangle$
- **Scalar form:** $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
  normal vector: $n = \langle a, b, c \rangle$
  point in plane: $P(x_0, y_0, z_0)$
- **Distance** from point $P(x, y, z)$ to plane: $D = \frac{|ax+by+cz-d|}{\sqrt{a^2+b^2+c^2}}$
  (assuming plane is in linear form above)

Quadratic Surfaces:

- **Ellipsoid** .... $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
  - **All** traces are ellipses
- **Elliptic Paraboloid** .... $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
  - **Horizontal** traces are ellipses
  - **Vertical** traces are parabolas
- **Hyperboloid** of one sheet .... $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
  - **Horizontal** traces are ellipses
  - **Vertical** traces are hyperbolas
- **Hyperboloid** of two sheets .... $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
  - **Horizontal** traces are ellipses
  - **Vertical** traces are hyperbolas
  - some traces do not exist because graph has a gap centered around the origin
- **Cone** .... $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
  - **Horizontal** traces are ellipses.
  - **Vertical** traces are pair of lines if x or y is 0, otherwise hyperbolas
- **Hyperbolic Paraboloid** .... $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
  - **Horizontal** traces are hyperbolas.
  - **Vertical** traces are parabolas

Vector Functions:

- **Arc length** from $t = a$ to $t = b$: $\int_a^b |r'(t)| dt$
  - Conversion back to the similar form from 2D:
  - $|r'(t)| = \sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}$
- **Arc Length Function:** $s(t) = \int_a^t |r'(u)| du$
- **Unit Tangent Vector:** $T(t) = \frac{r'(t)}{|r'(t)|}$
  unit-length vector tangent to the curve $r(t)$
- **Unit normal vector:** $N(t) = \frac{T'(t)}{|T'(t)|}$
  unit-length vector perpendicular to $r(t)$

Derivatives:

- $z = f(x, y)$
- Notation: $\frac{\partial z}{\partial x} = f_x = \frac{\partial f}{\partial x}$ etc...
  - Same for second derivatives: $f_{xy} =$
- **Gradient Vector:** $\nabla f = \langle f_x, f_y, f_z \rangle$
- **Tangent plane** at $P(a, b, c)$: $f_x(a, b) + f_y(a, b) = z - c$
- Chain Rule:
  - $\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x} = \frac{F_z}{F_x}$ (∂F cancels out, fraction flips)
  - $\frac{\partial z}{\partial t} = -\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}$ (∂x cancels out)
- Directional Derivative, parallel to $\langle a, b, c \rangle$ at $P(x, y, z)$: $f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$

Double Integrals:

- volume under the function $f(x, y)$ on the rectangle $R = [a, b] \times [c, d]$
- $\iint_R f(x, y) dA$ on $R = [a, b] \times [c, d] = \int_c^d \int_a^b f(x, y) dx dy$
  solve the inner integral first, then the outer
  if $f(x, y)$ is continuous on R, then you can flip the order of the integrals
- if $f(x, y) = g(x) \cdot h(y)$ then $\int_c^d \int_a^b f(x, y) dx dy = \int_a^b g(x) dx \cdot \int_c^d h(y) dy$
- **General Regions:**
  Only difference is whether x or y has its limits defined in terms of the other
  For these, you must evaluate the inner integral first, you can't swap them
  - Type 1: bounds of x are constants, bounds of y are defined as functions of x
    $D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$
    $= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$
  - Type 2: bounds of y are constants, bounds of x are defined as functions of y
    $D = \{(x, y) | h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$
    $= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$

Polar Coordinates:

- Polar → Cartesian:
  $r = \pm\sqrt{x^2 + y^2}$
  $\theta = \tan^{-1}(\frac{y}{x})$
  - You need to be careful with the sign of r and multiples of θ because Polar Coordinates are **not** unique.
- Cartesian → Polar:
  $x = r \cos \theta$
  $y = r \sin \theta$
  - You don't have to worry about quadrants or anything because Cartesian Coordinates **are** unique.
- **Double Integrals in Polar Coordinates**
  works best when D is in a polar-coordinate-friendly shape
  - Do the following replacements:

    * dA or $dxdy$ or $dydx \to rdrd\theta$
    * $x \to r\cos\theta$
    * $y \to r\sin\theta$
    * $x^2 + y^2 \to r$
    * Translate limits
  - Should end up with something that looks like one of these general regions:
    $\int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)dxdy$
    $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)drd\theta$
  - integrate as normal with new function and new limits stuff will probably cancel out everywhere
- **Arc Length in Polar Coordinates:**
  $L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

Cylindrical Coordinates:
- $dV = rdrd\theta dz$
- usually:
  $r \geq 0$
  $0 \leq \theta \leq 2\pi$
- Cylindrical $\to$ Cartesian:
  $x = r\cos\theta$
  $y = r\sin\theta$
  $z = z$
- Cartesian $\to$ Cylindrical:
  (be careful about the quadrant of θ)
  $r = \sqrt{x^2 + y^2}$
  $r^2 = x^2 + y^2$
  $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
  $z = z$

Spherical Coordinates:
- $dV = \rho^2 \sin\phi$
- ϕ is the angle from the $+z$ axis down to ρ
  - usually:
    $-\pi/2 \leq \rho \leq \pi/2$
    $\rho \geq 0$
    $0 \leq \theta \leq 2\pi$
- Spherical $\to$ Cartesian:
  $x = \rho\sin\phi\cos\theta$
  $y = \rho\sin\phi\sin\theta$
  $z = \rho\cos\phi$
- Cartesian $\to$ Spherical:
  $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
  $\rho^2 = x^2 + y^2 + z^2$
  $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$
- Other:
  $\rho^2 = x^2 + y^2 + z^2$
  $r = \rho\sin\phi$

Minimum and Maximum:
Find critical points by solving $\nabla f = \langle 0, 0\rangle$
For each point, find $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
$D > 0$ and $f_{xx} > 0$: relative minimum at (a, b)
$D > 0$ and $f_{xx} < 0$: relative maximum at (a, b)
$D < 0$: saddle point at (a, b)
$D = 0$: can't tell (probably won't see)

Line Integrals:
- **2D:**
  $C = \{r(t) | a \leq r \leq b\}$
  $r(t) = \langle x, y, z\rangle$
  Scalar function $f(x, y)$:
  $\int_C f(x,y)ds = \int_a^b f(r(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt$
  $= \int_a^b f(r(t))|r'(t)|dt$
  $\int_C f(x,y)dx = \int_a^b f(r(t))x'(t)dt$
  Vector field $F(x,y) = \langle P, Q\rangle$:
  $\int_C F(x,y) \cdot ds = \int_b^a P(r(t))dx + Q(r(t))dy$
- **3D:**
  $C = \{r(t) | a \leq r \leq b\}$
  (scalars are the same)
  Vector field $F(x,y,z) = \langle P, Q, R\rangle$:
  $\int_C F(x,y,z) \cdot ds = \int_a^b F(r(t)) \cdot r'(t)dt$
  $= \int_C Pdx + Qdy + Rdy$
  $= \int_a^b P\frac{\partial r}{\partial x}dx + Q\frac{\partial r}{\partial y}dy + R\frac{\partial r}{\partial z}dz$
  $\int_C F(x,y,z) \cdot dr = \int_a^b F(r(t)) \cdot r'(t)dt$

Fundamental Theorem for Line Integrals:
$C: r(t), a \leq r \leq b$, C is simple. Domain is simply-connected
$F(x,y,z) = \langle P, Q, R\rangle$
If there exists f such that $\nabla f = F$, then:
$\int_C F \cdot dr = f(r(b)) - f(r(a))$

Green's Theorem: (2D only, doesn't work in 3D)
C: curve with $r(t) = \langle x(t), y(t), z(t)\rangle$ on $a \leq t \leq b$; C is closed and simple; D: region enclosed by C
$F(x,y) = \langle P(x,y), Q(x,y)\rangle$
$\int_C F \cdot dr = \int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

Vector Field: Curl and Divergence:
$curl F = \nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \langle P, Q, R\rangle$
$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$
$div F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = P_x + Q_y + R_z$
$div(curl(F)) = 0, curl(div(f)) = 0$
if $curl(F) = 0$, then F is irrotational (causes no rotation); and therefore $\nabla f = F$ exists

Surface Integrals:
Surface $S: r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for $(u,v) \in D$
scalar function $f(x,y,z)$, vector field $F(x,y,z)$
$\hat{n} = (r_u \times r_v)/|r_u \times r_v| = \frac{r_u \times r_v}{|r_u \times r_v|}$
Scalar Function f:
$\iint_S f dS = \iint_D f(r(u,v))|r_u \times r_v|dA$
When $f(x,y,z) : z = g(x,y)$:
$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}dA$
Area of $S : \iint_D |r_u \times r_v|dA$
Vector Function F:
$\iint_S F \cdot dS = \iint_S (F \cdot \hat{n})dS = \iint_D F(r(u,v)) \cdot (r_u \times r_v)dA$

Stokes' Theorem: (works in 3D)
Surface S bounded by curve $C : g(t), a \leq t \leq b$
vector field $F(x,y,z)$
$\int_C F \cdot dg = \iint_S curl(F) \cdot dS$

Divergence Theorem:
surface S is boundary of solid region E
F is vector field
$\iint_S F \cdot dS = \iiint_E div(F)dV$

test this stuff like editing in emacs vim spacemacs