Linear Equations
- system is inconsistent if it has no solution
- a system must have either none, one, or infinitely many solutions
- system is called Homogeneous if all the constant terms (right sides) are 0
- if coefficient matrix is invertible, there is one unique solution for the system

Gaussian Elimination
- first put system into matrix form
- Elementary Operations:
  * multiply a row by a nonzero scalar
  * add row multiplied by scalar to another row
  * swap rows
- Elementary Operations don't change any of the solutions
- try to make it into a diagonal matrix

Row-Echelon form
- all leading entries $\neq 0$, leading entries shift right as you go down
- from RE form you can use back substitution to get solutions
- leading entry of row: first non-zero element of row
- if there is any zero row, then the solution has a free variable

Reduced Row-Echelon form
- same as RE form, but all leading entries = 1, each column with a leading entry is zeros everywhere else
  * this isn't always the identity matrix; some columns could be missing leading entries entirely

Gauss-Jordan reduction
- use Gaussian Elimination to get matrix into Reduced Row-Echelon form
- if there are columns without leading entries, those are free variables
- take each row, transform back to equation, get solution

Matrix
- matrices $A, B \in M_{m,n}(R)$
- addition, scalar multiplication work like vectors
  * addition is only defined when matrix dimensions match
- **Dot Product:** $x * y = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{k=0}^{n} x_ky_k$
- **diagonal matrix:** only diagonal elements are non-zero
- identity matrix: I, diagonal of 1's
  * $AI = A$ for any A and properly sized I
- **Transpose:** just swap the rows and columns
  * represented as A^T
  * if $A = A^T$ then A is symmetric

Multiplication: $A_{m\times n}, B_{n\times p}, C = AB$
- $c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$
- each element in C is the dot product of that row in A and column in B
- result has as many rows as A and columns as B
- AB is only the same size as BA if they're both square; since the size depends on the matching dimensions of A and B
- if $AB = BA$ then A and B commute
- $AB(C) = A(BC)$
  $(A + B)C = AC + BC$
  $C(A + B) = CA + CB$
  $(rA)B = A(rB) = r(AB)$

Inverse Matrix
- $AA^{-1} = A^{-1}A = I$
- no inverse: singular or non-invertible
- inverse exists: non-singular or invertible
- zero matrix is singular
- inverse of a matrix is unique
- inverse distributes over matrix multiplication
- inverse of diagonal matrix: reciprocal of each element
- inverse of 2×2 matrix $[a, b; c, d]$ is $\frac{1}{ac-bd}[d, -c; -b, a]$
- find inverse:
  * convert A to Reduced Row-Echelon form
  * apply those same ordered Elementary Operations to I to get A^{-1}
  * (you can do these two steps at the same time)
- **Elementary Matrix:** any matrix reachable by applying Elementary Operations to I

These are Equivalent
- A is invertible
- $det(A) \neq 0$
- $x = 0$ is the only solution to the equation $Ax = 0$
- $Ax = b$ has a unique solution for any column vector b
- Row-Echelon form of A has no zero rows
- Reduced Row-Echelon form of A is I
- the rows/columns of A are linearly independent
- the columns of A form a Basis of R^n

Determinants
- determinant of singleton matrix is single value

- determinant of 2×2 matrix $[a, b; c, d]$ is $ac - bd$
- determinant of diagonal matrix is product of diagonal entries
  - * same for upper, lower triangular
- determinant of larger matrix can be broken down by a row or column:
  - * for each element $a_{i,j}$, take $a_{i,j}$ times the determinant of the (smaller) matrix formed by leaving out row i, column j
  - * and use the proper sign by the alternating method
- Elementary Operation Axioms:
  - * D1: multiply row by $r \rightarrow$ multiply det by r
  - * D2: add scalar multiple of one row to another $\rightarrow$ same det
  - * D3: swapping rows of matrix $\rightarrow$ det changes sign
  - * D4: $det(I) = 1$
  - * C1: if A, B are square and A is obtained by applying Elementary Operations to B, then $det(A) = 0$ iff $det(B) = 0$
  - * C2: $det(B) = 0$ whenever B has a zero row
  - * C3: $det(A) = 0$ iff A is not invertible
- Cramer's rule: explicit formula for solution to system of linear equations, using determinants. Not very useful because determinants.

Wronskian

- to show linear independence of functions in $C^\infty(R)$ (continuously differentiable functions)
- $W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$
  - * functions in rows, derivatives down columns
- if $W(x)$ is not identically 0, then the functions are linearly independent
- alternatively (1): take derivatives of the top row until you get something that you can work with to solve
- alternatively (2): start with $af_1(x) + bf_2(x) + cf_3(x) = 0$ and show that the only solution is $a = b = c = 0$

Basis

- every vector space has a Basis
- it's like a coordinate system
- Basis = minimum spanning set = maximum set of linearly independent vectors
- can get one by adding linearly independent vectors to a too-small set or removing linearly dependent ones from a too-large one
- **Dimension:** $(dim(V))$ number of basis vectors for a vector space
  - * if $dim(V) < \infty$ then every basis of V is the same size

Matrix Spaces

- matrix $M_{m,n}$:
- **Row Space:** subspace of R^n spanned by rows of M
  - * **Rank:** = dimension of row space (number of linearly independent rows)
  - * in Row-Echelon form, all non-zero rows are linearly independent
- **Column Space:** subspace of R^m spanned by columns of M
- **Null Space:** $N(A)$: all x such that $Ax = 0$
  - * solution set of homogeneous equations with coefficients A
  - * $N(A)$ is subspace of R^n
  - * Nullity $= dim(N(A))$ = number of free variables
- for any matrix, rank + nullity = number of columns

Change of Basis (Coordinates):

- basis $B_1 = \{u, v\}$ of R^2
- change basis of (x, y): find r_1, r_2 such that $(x, y) = r_1 v + r_2 u$
- **Transition Matrix** $T = (u^T, v^T)$ has columns of u and v, and maps $B_1 \rightarrow R^2$
  - * that is, $T \cdot \binom{x'}{y'} = \binom{x}{y}$ when $(x, y) = x'u + y'v$
  - * inverse works: $T^{-1} \cdot \binom{x}{y} = \binom{x'}{y'}$
- transition between general bases:
  - * basis B_1, B_2 with transition matrix T_1, T_2
  - * $f(x) : B_1 \rightarrow B_2 = T_2^{-1} \cdot T_1 \cdot x$
  - * $f(x) : B_2 \rightarrow B_1 = T_1^{-1} \cdot T_2 \cdot x$
  - * more generally, find one basis's coordinates in terms of another's, but then you'll need to solve n^2 equations

Linear Relations

- additivity: $L(ax + by) = aL(x) + bL(y)$
- homogeneity: $L(ax) = aL(x)$
- kernel: all v such that $L(v) = 0$
- range: all possible output values

Least Squares

- express as overdetermined relation $Ax = b$ (where there are more rows than variables)
- Then left-multiply both sides by A^{-1}, getting $A^{-1}Ax = A^{-1}b$
- the resulting equation will be fully determined, so solve like normal
- at the end, you get values for x, which are the needful least squares coefficients