Complex Numbers

- $z = x + iy = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$
- $[r(\cos(\theta) + i\sin(\theta))]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$
- $z^n = (re^{i\theta}) = r^n e^{in\theta}$
- $\sqrt[n]{z} = \sqrt[n]{r}e^{\frac{\theta}{n}+\frac{2k\pi}{n}}$ for $n \in N^*$ (ints ≥ 0)
- $e^{j\theta} = \cos(\theta) + j\sin(\theta)$
- $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$
- $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
- $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
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- $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

Signals

- **Even/Odd**
  even: $x(-t) = x(t)$ for all t
  odd: $x(-t) = -x(t)$ for all t
- **Auto Correlation:** compare signal with a time-delayed version of itself
  $\phi(\tau) = \int_{-\infty}^{\infty} x(t) * x(t+\tau)dt$
  - peaks will be at multiples of the period
- **Cross Correlation:** like autocorrelation, but for two different signals
  $\phi(\tau) = \int_{-\infty}^{\infty} x_1(t) * x_2(t+\tau)dt$
  - to easily tell if one signal is a shifted version of another
- Shifting and scaling: just always remember you're replacing **just** t with an expression involving t
- **Unit Step** Signal
  - $u(t) = \{^{0, n<0}_{1, n>0}$
- **Discrete Unit Impulse** Signal
  $\delta[n] = \{^{0, n\neq 0}_{1, n=0}$
  - any discrete signal can be represented as a sum of shifted unit impulse signals
  - $\delta[n] = u[n] - u[n-1]$
- **Continuous Unit Impulse** Signal
  $x(t) = \delta(t) = \{^{0, t\neq 0}_{\infty, t=0}$
  - discontinuous at $t = 0$
  - $\int_{-\infty}^{\infty} \delta(t)dt = 1$
  - pick out values from discrete function: (shifting property)
    $\int_{-\infty}^{\infty} \delta(t) * f(t)dt = f(0)$
    $\int_{-\infty}^{\infty} \delta(t-a) * f(t)dt = f(a)$
- **Shifting Property:** $\int_{-\infty}^{\infty} x(t)\sigma(t-t_0)dt = x(t_0)$
- **Bounded:** $x(t) \leq M$ for all t, some M
  - unbounded signals typically are infinite at some time instant
- **Causal** iff $x(t) = 0$ for all $t < 0$
- **Energy:** $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$
  - signal is an energy signal if $0 < E_x < \infty$
- **Power:** $P_x = \frac{1}{T}\int_T |x(t)|^2 dt$
  - (for periodic signals)
  - signal is an power signal if $0 < P_x < \infty$

Convolution

- $\sum_{k=-\infty}^{\infty} x(k)h(n-k)$ or $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
- graphically:
  - choose one function to be h
  - flip around origin with $t \to -t$
  - shift back and forth on form $h(t-\tau)$
  - shift is reversed because the negative
  - multiply by $x(t)$ and then sum
- if the system is LTI invariant, then the convolution of $x(t)$ with the impulse response $h(t)$ is the same as if $x(t)$ were the input of the system
- convolution with shifted unit impulse is the same as shifting the original system:
  $h(t) * \sigma(t-a) = h(t-a)$
- Step response is just convolution with impulse response. worked out: $u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$
  - only works for LTI systems!

Geometric Series

- $\sum_{k=0}^{\infty} ar^n = \frac{a}{1-r}$
- a is first term of the series
  r is ratio between terms: $r = \frac{a_1}{a_0} = \frac{a_2}{a_1} \dots$

Systems

- A system is an operation that transforms an input signal into an output signal
  - you can add/subtract signals
  - composing signals (one input to another) is convolution
    (easier to just shift if input is shifted unit step (because LTI))
- **BIBO stability:** output is stable iff input signal is stable
  - also if impulse response $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ (for LTI systems)
  - bounded: $h(t) < M$ for all t and some M
- **Memory:** iff the system depends on past or future values of the input
- **Causality:** iff the output depends only on the current or past values of the input
  - (cannot depend on future values of input)
- **Invertibility:** iff the system's input can be recovered from the output
- **Time Invariance:** iff shifting the input signal shifts the output
  - integral is time invariant
- **Superposition:** additive commutativity
  - $H\{x_1(t) + x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\}$
- **Homogeneity:**
  - $H\{ax(t)\} = aH\{x(t)\}$
- **Linearity:** iff satisfies Superposition and Homogeneity
  - $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$
  - averaging filter is linear
- **LTI:** both Linear and Time Invariant

- simplest systems
- system from block diagram:
  - add/subtract signals just like you would
  - for signals $h_1(t) \to h_2(t)$ (in series), you get $y(t) = h_1(t) * h_2(t)$ (convolution of the two signals)

  - basic method is to keep combining adjacent signal blocks using convolution, scaling, and addition until you get a single block
- system from differential equation:
  - solve equation for $y(t)$
  - stuff in terms of input goes on the left; output on the right
  - add constants scaling to each output, and sum it all together

Linearity
- system is linear if it satisfies superposition (additive) and homogeneity (scalable)
  - superposition: $h(a) + h(b) = h(a + b)$
  - homogeneity: $ah(b) = h(ab)$

Noise
- unwanted signals generated externally or internally
- thermal noise is a thing

Impulse Response
- output of a system when the input is $\sigma(t)$
memoryless if	$h(t) = c\sigma(t)$
causal if	$h(t) = 0$ for $t < 0$
BIBO stable if	$\int_{-\infty}^{\infty} \lvert h(t)\rvert dt < \infty$
invertible if	$h(t) * h^{inv}(t) = \sigma(t)$
  - same for discrete time

even/odd signals
- $f(t) = f_e(t) + f_o(t)$
- $f_e(t) = \frac{1}{2}(f(t) + f(-t))$
- $f_o(t) = \frac{1}{2}(f(t) - f(-t))$

Fourier Series
- Harmonic: $e^{jk2\pi F_0 t}$
- Synthesis: $f(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk2\pi F_0 t}$
- Analysis: $X[k] = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-jk2\pi F_0 t}dt$
  - note the different sign!
- $X[k] = C_k$
- Parseval's theorem: (energy of a signal) $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$
- FT of fraction of two polynomials: use partial fraction decomposition

Fourier Properties
- linearity: $z(t) = ax(t) + by(t) \leftrightarrow Z(k) = aX(k) + bY(k)$
- time shift: $x(t - t_0) \leftrightarrow X(k)e^{-jk\omega_0 t_0}$
- frequency shift: $x(t)e^{jk_0\omega_0 t} \leftrightarrow X(k - k_0)$
- time scaling: same coefficients, $x(at) \to \omega = a\omega_0$ (for $a > 0$)
- time reversal: $x(-t) \leftrightarrow X(-k)$
- convolution: $x(t) * z(t) \leftrightarrow TX(k)Z(k)$
- multiplication: $x(t)z(t) \leftrightarrow \sum_{l=-\infty}^{\infty} X(k)Z(k - l)$
  - similar to convolution
- derivative: $\frac{d}{dt}(x(t)) \leftrightarrow jk\omega_0 X(k)$
- integral: $\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{jk\omega_0}X(k)$
- Symmetry: if $x(t) = x_r(t) + jx_i(t)$ then $x^*(t) = x_r(t) - jx_i(t)$
- if $x(t)$ is real and even, $X(k)$ is real and even
- if $x(t)$ is real and odd, $X(k)$ is imaginary and odd

Frequency Response
- how the system will respond to a particular frequency
- the Fourier Transform of the impulse response (we don't have to convolve it here, just multiply since it's frequency domain)
- find using $H(\omega) = \frac{Y(\omega)}{X(\omega)}$
  - if the starting equation is expressed as a differential equation, you can (usually) derive this from that.
- usually represented as $H(\omega) = |H(\omega)|e^{j\theta_H(\omega)}$ (magnitude and phase)
  - magnitude: $|H(\omega)|$, phase: $\theta_H(\omega)$
- magnitude and phase can be linearly combined

Filtering
- multiply signal by a filter to filter it
- pass: allow through (not filtered out)
- stop: filter out, remove
- *something* pass filter passes *something*; same for stop
- low pass filter: passes ω lower than b, drops higher $H(\omega) = \{^{0:|\omega|>b}_{1:|\omega|<b}$
- high pass filter: opposite of low pass filter
- band pass filter: pass a specific band of frequency.
  - That frequency is specified by magnitude, so it can be on the positive or negative side of the graph
- Notch filter: band stop filter with a narrow stop band
- An ideal filter has exact edges, but real filters don't
  - This is impossible in practice. Typically there is vertical variation inside the pass band and stop band, and also a trans band (ω_s), as transition between pass and stop band.

DTFT
- Discrete Time Fourier Transform
- $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
- $x[n] = \frac{1}{2\pi}\int -\pi\pi X(e^{j\Omega})e^{j\Omega n}d\Omega$