

Metric Prefixes

peta	P	10^{15}	1 000 000 000 000 000
tera	T	10^{12}	1 000 000 000 000
giga	G	10^9	1 000 000 000
mega	M	10^6	1 000 000
kilo	k	10^3	1 000
hecto	h	10^2	100
deca	da	10^1	10
one		10^0	1
deci	d	10^{-1}	0.1
centi	c	10^{-2}	0.01
milli	m	10^{-3}	0.001
micro	μ	10^{-6}	0.000 001
nano	n	10^{-9}	0.000 000 001
pico	p	10^{-12}	0.000 000 000 001
femto	f	10^{-15}	0.000 000 000 000 001

Complex Numbers

- $z = x + iy = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$
- $[r(\cos(\theta) + i\sin(\theta))]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$
- $z^n = (re^{i\theta}) = r^n e^{in\theta}$
- $\frac{1}{i} = -i$
- $\sqrt[n]{z} = \sqrt[n]{r} e^{\frac{\theta}{n} + \frac{2k\pi}{n}}$ for $n \in \mathbb{N}^*$ (ints ≥ 0)
- $e^{j\theta} = \cos(\theta) + j\sin(\theta)$
- $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$
- $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
- $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
- normalized: $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

Trig

- $\cos^2(a) + \sin^2(a) = 1$
- $\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
- $\sin(2a) = 2\sin(a)\cos(a)$
- $\cos^2(a) = \frac{1}{2}(1 + \cos(2a))$
- $\sin^2(a) = \frac{1}{2}(1 - \cos(2a))$

Signals**• Even/Odd**

even: $x(-t) = x(t)$ for all t
 odd: $x(-t) = -x(t)$ for all t

- **Auto Correlation:** compare signal with a time-delayed version of itself

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t) * x(t + \tau) dt$$

* peaks will be at multiples of the period

- **Cross Correlation:** like autocorrelation, but for two different signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x_1(t) * x_2(t + \tau) dt$$

* to easily tell if one signal is a shifted version of another

- Shifting and scaling: just always remember you're replacing **just** t with an expression involving t
- **Unit Step Signal**
 $* u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$
- **Discrete Unit Impulse Signal**

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

* any discrete signal can be represented as a sum of shifted unit impulse signals

$$* \delta[n] = u[n] - u[n - 1]$$

- **Continuous Unit Impulse Signal**

$$x(t) = \delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

* discontinuous at $t = 0$

$$* \int_{-\infty}^{\infty} \delta(t) dt = 1$$

* pick out values from discrete function: (shifting property)

$$\int_{-\infty}^{\infty} \delta(t) * f(t) dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t - a) * f(t) dt = f(a)$$

- **Shifting Property:** $\int_{-\infty}^{\infty} x(t)\sigma(t - t_0) dt = x(t_0)$

- **Bounded:** $x(t) \leq M$ for all t , some M

* unbounded signals typically are infinite at some time instant

- **Causal** iff $x(t) = 0$ for all $t < 0$

- **Energy:** $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

* signal is an energy signal if $0 < E_x < \infty$

- **Power:** $P_x = \frac{1}{T} \int_T |x(t)|^2 dt$

* (for periodic signals)

* signal is a power signal if $0 < P_x < \infty$

Convolution

- $\sum_{k=-\infty}^{\infty} x(k)h(n - k)$ or $\int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$

- graphically:

* choose one function to be h

* flip around origin with $t \rightarrow -t$

* shift back and forth on form $h(t - \tau)$

* shift is reversed because the negative

* multiply by $x(t)$ and then sum

- if the system is LTI invariant, then the convolution of $x(t)$ with the impulse response $h(t)$ is the same as if $x(t)$ were the input of the system

- convolution with shifted unit impulse is the same as shifting the original system: $h(t) * \sigma(t - a) = h(t - a)$

- Step response is just convolution with impulse response. worked out: $u(t) * h(t) = \int_{-\infty}^t h(\tau) d\tau$
 * only works for LTI systems!

Geometric Series

- $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

- a is first term of the series

r is ratio between terms: $r = \frac{a_1}{a_0} = \frac{a_2}{a_1} \dots$

Systems

- A system is an operation that transforms an input signal into an output signal

* you can add/subtract signals

* composing signals (one input to another) is convolution

(easier to just shift if input is shifted unit step (because LTI))

- **BIBO stability:** output is stable iff input signal is stable

* also if impulse response $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ (for LTI systems)

- * bounded: $h(t) < M$ for all t and some M
- **Memory:** iff the system depends on past or future values of the input
- **Causality:** iff the output depends only on the current or past values of the input
 - * (cannot depend on future values of input)
- **Invertibility:** iff the system's input can be recovered from the output
- **Time Invariance:** iff shifting the input signal shifts the output
 - * integral is time invariant
- **Superposition:** additive commutativity
 - * $H\{x_1(t) + x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\}$
- **Homogeneity:**
 - * $H\{ax(t)\} = aH\{x(t)\}$
- **Linearity:** iff satisfies Superposition and Homogeneity
 - * $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$
 - * averaging filter is linear
- **LTI:** both Linear and Time Invariant
 - * simplest systems
- system from block diagram:
 - * add/subtract signals just like you would
 - * for signals $h_1(t) \rightarrow h_2(t)$ (in series), you get $y(t) = h_1(t) * h_2(t)$ (convolution of the two signals)
 - * basic method is to keep combining adjacent signal blocks using convolution, scaling, and addition until you get a single block
- system from differential equation:
 - * solve equation for $y(t)$
 - * stuff in terms of input goes on the left; output on the right
 - * add constants scaling to each output, and sum it all together

Linearity

- system is linear if it satisfies superposition (additive) and homogeneity (scalable)
 - * superposition: $h(a) + h(b) = h(a + b)$
 - * homogeneity: $ah(b) = h(ab)$

Noise

- unwanted signals generated externally or internally
- thermal noise is a thing

Impulse Response

- output of a system when the input is $\sigma(t)$
 - memoryless if $h(t) = c\sigma(t)$
 - causal if $h(t) = 0$ for $t < 0$
- BIBO stable if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
- invertible if $h(t) * h^{inv}(t) = \sigma(t)$
 - * same for discrete time

even/odd signals

- $f(t) = f_e(t) + f_o(t)$
- $f_e(t) = \frac{1}{2}(f(t) + f(-t))$
- $f_o(t) = \frac{1}{2}(f(t) - f(-t))$

Fourier Series

- Harmonic: $e^{jk2\pi F_0 t}$
- Synthesis: $f(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk2\pi F_0 t}$
- Analysis: $X[k] = \frac{1}{T_p} \int_0^{T_p} x(t)e^{-jk2\pi F_0 t} dt$
 - * note the different sign!
- $X[k] = C_k$
- Parseval's theorem: (energy of a signal)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$
- FT of fraction of two polynomials: use partial fraction decomposition

Fourier Transform

- $x(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$
- $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$

Fourier Properties

- linearity:

$$z(t) = ax(t) + by(t) \leftrightarrow Z(k) = aX(k) + bY(k)$$
- time shift: $x(t - t_0) \leftrightarrow X(k)e^{-jk\omega_0 t_0}$
- frequency shift: $x(t)e^{jk_0\omega_0 t} \leftrightarrow X(k - k_0)$
- time scaling: same coefficients, $x(at) \rightarrow \omega = a\omega_0$ (for $a > 0$)
- time reversal: $x(-t) \leftrightarrow X(-k)$
- convolution: $x(t) * z(t) \leftrightarrow TX(k)Z(k)$
- multiplication: $x(t)z(t) \leftrightarrow \sum_{l=-\infty}^{\infty} X(k)Z(k - l)$
 - * similar to convolution
- derivative: $\frac{d}{dt}(x(t)) \leftrightarrow jk\omega_0 X(k)$
- integral: $\int_{-\infty}^t x(t) dt \leftrightarrow \frac{1}{jk\omega_0} X(k)$
- Symmetry: if $x(t) = x_r(t) + jx_i(t)$ then

$$x^*(t) = x_r(t) - jx_i(t)$$
- if $x(t)$ is real and even, $X(k)$ is real and even
- if $x(t)$ is real and odd, $X(k)$ is imaginary and odd
- unnormalized sinc: $\text{sinc}(t) = \frac{\sin(t)}{t}$
- normalized sinc: $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

Frequency Response

- how the system will respond to a particular frequency
- the Fourier Transform of the impulse response (we don't have to convolve it here, just multiply since it's frequency domain)
- find using $H(\omega) = \frac{Y(\omega)}{X(\omega)}$
 - * if the starting equation is expressed as a differential equation, you can (usually) derive this from that.
- usually represented as $H(\omega) = |H(\omega)|e^{j\theta_H(\omega)}$ (magnitude and phase)
 - * magnitude: $|H(\omega)|$, phase: $\theta_H(\omega)$
- magnitude and phase can be linearly combined

Sampling

- sampling: independent variable (input); continuous \rightarrow discrete
- quantization: dependent variable (output); continuous \rightarrow discrete
- aliasing: different signals being indistinguishable after sampling due to sampling rate.
- F_{CT} : continuous time frequency; f_{DT} : discrete time frequency, F_s : sampling frequency
- $F_{CT} \rightarrow f_{DT}$ is a many to one mapping
 - * Folding Frequency = $F_s/2$, where frequency

wraps-around

* restrict to one-to-one: satisfy $|F| < F_s/2$

- **Nyquist Rate:** signal with maximum frequency f_{max} can be recovered exactly if it is sampled at least $f_s > 2f_{max}$
- continuous to discrete:
 - * $x[n] = x_a(n/F_s)$
 - * $X(f) = F_s \sum_{k=-\infty}^{\infty} \infty X_a[(f - k)F_s]$
- sampling is equivalent to convolving with a delta chain (dirac comb)
- under-sampling: $f_s < 2f_{max}$. Generally bad

DFT

- discrete in both time domain and frequency domain
- zero padding (on the right) increases frequency domain resolution
- frequency domain is on range $[0, 2\pi)$
- TODO

DTFT

- Discrete Time Fourier Transform
- Discrete in time, continuous in frequency
- $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
- $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega$
- frequency domain is **periodic**, range $[-\pi, \pi]$ is repeated
 - * if aliasing happens ($F_s < 2f_{max}$) then parts will overlap, and won't work right
- maximum frequency $\rightarrow \pi$; sampling frequency $\rightarrow 2\pi$

Laplace Transform

- $X(s) = \int_0^{\infty} x(t)e^{st} dt$
- $s = \sigma - j\omega$
- relation to Fourier Transform: $X(j\omega) = X(s)|_{s=j\omega, \sigma=0}$
- zeros: numerator is 0, so value is 0
- poles: denominator is 0, so value is $+$ or $-\infty$
 - * effect on impulse response on $t > 0$:
 - poles left of $s = 0$: decaying exponential
 - poles right of $s = 0$: increasing exponential (reversed on $t < 0$)
- Specified on a Region of Convergence (ROC)
- ROC must contain no poles
- Unilateral: integral starts from 0
 - * assumes signals is 0 before $t = 0$
 - * i.e. multiplied by $u(t)$
- Bilateral: starts from $-\infty$
 - * same as unilateral iff $x(t) = 0$ for $t < 0$
- Shifting:
 - * $x(t - T) \leftrightarrow e^{-sT} X(s)$
 - * T such that $x(t - T)u(t) = x(t - T)u(t - T)$
 - i.e. doesn't shift any non-zero part of the signal left of $t = 0$
 - (Doesn't apply for Bilateral transform)
- Initial Value Theorem:
 - * $\lim_{s \rightarrow \infty} sX(s) = \lim_{t \rightarrow 0^+} x(t) = x(0^+)$
 - * order of numerator $<$ denominator

* $x(t) = 0$ for $t < 0$

Final Value Theorem:

- * $\lim_{s \rightarrow 0} sX(s) = \lim_{t \rightarrow \infty} x(t) = x(\infty)$
- * $x(t) = 0$ for $t < 0$, $x(t)$ finite as $t \rightarrow \infty$
- * all poles on left of plane with at most one at $s = 0$

System Response:

- * take Laplace Transform to get equation in form $a(s)Y(s) - c(s) = b(s)X(s)$
- * solve $Y(s) = \frac{b(s)X(s)}{a(s)} + \frac{c(s)}{a(s)} = Y^{(f)}(s) + Y^{(n)}(s)$
- * $Y^{(f)}(s)$: forced response; $Y^{(n)}(s)$: natural response
- * Frequency Response (transfer function):
$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

- Stability: system is stable \leftrightarrow all poles left of $s = 0$

Filters

- multiply signal by a filter to filter it
- pass: allow through (not filtered out)
- stop: filter out, remove
- *something* pass filter passes *something*; same for stop
- low pass filter: passes ω lower than b , drops higher
$$H(\omega) = \begin{cases} 0: |\omega| > b \\ 1: |\omega| < b \end{cases}$$
- high pass filter: opposite of low pass filter
- band pass filter: pass a specific band of frequency.
 - * That frequency is specified by magnitude, so it can be on the positive or negative side of the graph
- Notch filter: band stop filter with a narrow stop band
- An ideal filter has exact edges, but real filters don't
 - * This is impossible in practice. Typically there is vertical variation inside the pass band and stop band, and also a trans band (ω_s), as transition between pass and stop band.
- series RC circuit with $x(t)$ as input and $y(t)$ on capacitor: $RCy'(t) + y(t) = x(t)$ or $y'(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$
 - * capacitor: $I(t) = C * \frac{d}{dt}(V(t))$
 - * inductor: $V(t) = L * \frac{d}{dt}(I(t))$
- highpass and band-stop filters do not have bandwidth
- Absolute Bandwidth: width of passband
- 3-dB Bandwidth: (for low pass): width of passband where $H(j\omega) \geq |H(0)|/\sqrt{2}$
 - * because low pass has max at $\omega = 0$. Different for other filter types
 - * AKA half power bandwidth
- Bode Plot: frequency response ($H(s = j\omega, \sigma = 0)$) in dB as a function of logarithm of frequency
- ideal filter has linear phase response
 - * so that the response to different frequencies is the same
 - * constant delay is linear phase response (it's just $H(j\omega) = e^{-j\omega a}$)
- ideal filter magnitude response is 1 in passband and 0 in stopband
- real filter has E, δ and transition band (ω_p, ω_s) such that:

- * passband: $1 - E \leq H(j\omega) < 1$ for $0 \leq |\omega| \leq \omega_p$
- * stopband: $|H(j\omega)| < \delta$ for $\omega > \omega_s$
- FIR filter: Finite Impulse Response
 - * finite memory (lower startup transient time)
 - * always BIBO stable
 - * desired magnitude response with exactly linear phase response
- IIR filter: Infinite Impulse Response
 - * output governed by recursive linear constant coefficient differential equations
 - * uses z transform
 - * allows shorter (lower order?) filters
 - * has phase distortion (nonlinear phase response)
 - * non-finite transient startup
- Butterworth Filter
 - * looks like a nice simple slope, relatively large transition band though
 - * ω_c : cutoff frequency
 - * $|H(j\omega)|^2 = 1/(1 + (\frac{\omega}{\omega_c})^{1/(2K)})$, $K = 1, 2, 3, \dots$
 - $H(s)H(-s)|_{s=j\omega} = |H(j\omega)|^2$
 - * passband: $\omega_p = \omega_c(\frac{\epsilon}{1-\epsilon})^{1/(2K)}$
 - * stopband: $\omega_s = \omega_c(\frac{1-\delta}{\delta})^{1/(2K)}$
 - * poles: $\omega_c e^{i\pi(2k+1)/(2K)}$, $k = 0, 1, \dots, (2K-1)$
 - never pure imaginary
 - left of $s = 0$ belong to $H(s)$, right of $s = 0$ belong to $H(-s)$
 - * to get transfer function, find poles and make fraction from there
- Chebyshev Filter
 - * poles on ellipse in s -plane
 - * ripples in either passband or stopband, not both
 - * smaller transition band
- Elliptic Filter
 - * smallest transition band
 - * ripples in both passband and stopband

Communication Systems

- frequency range: what frequencies are best transmitted, and what are lost
- modulation: embedding a signal inside a carrier signal that propagates better
- simultaneous transmission (multiplexing): use modulation to transmit more than one signal simultaneously

Amplitude Modulation (AM)

- mix (convolve) payload and carrier in frequency domain (multiply in time domain)
- carrier is typically a high frequency cosine, which is two impulses in frequency domain, so the convolution makes two shifted copies of the input signals, each centered around \pm carrier frequency
- demodulation: recover original signal: shift it back to the origin and apply low-pass filter
- Frequency Modulation is another (different) kind of modulation