

Complex Numbers

- $z = x + iy = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$
- $[r(\cos(\theta) + i\sin(\theta))]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$
- $z^n = (re^{i\theta})^n = r^n e^{in\theta}$
- $\sqrt[n]{z} = \sqrt[n]{r} e^{i\frac{\theta}{n} + \frac{2k\pi}{n}}$ for $n \in \mathbb{N}^*$ (ints ≥ 0)
- $e^{j\theta} = \cos(\theta) + j\sin(\theta)$
- $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$
- $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
- $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
-
- $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

Signals• **Even/Odd**

even: $x(-t) = x(t)$ for all t

odd: $x(-t) = -x(t)$ for all t

- **Auto Correlation:** compare signal with a time-delayed version of itself

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t) * x(t + \tau) dt$$

* peaks will be at multiples of the period

- **Cross Correlation:** like autocorrelation, but for two different signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x_1(t) * x_2(t + \tau) dt$$

* to easily tell if one signal is a shifted version of another

- Shifting and scaling: just always remember you're replacing **just** t with an expression involving t

• **Unit Step Signal**

$$* u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

• **Discrete Unit Impulse Signal**

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

* any discrete signal can be represented as a sum of shifted unit impulse signals

$$* \delta[n] = u[n] - u[n - 1]$$

• **Continuous Unit Impulse Signal**

$$x(t) = \delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

* discontinuous at $t = 0$

$$* \int_{-\infty}^{\infty} \delta(t) dt = 1$$

* pick out values from discrete function: (shifting property)

$$\int_{-\infty}^{\infty} \delta(t) * f(t) dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t - a) * f(t) dt = f(a)$$

• **Shifting Property:** $\int_{-\infty}^{\infty} x(t) \sigma(t - t_0) dt = x(t_0)$ • **Bounded:** $x(t) \leq M$ for all t , some M

* unbounded signals typically are infinite at some time instant

• **Causal** iff $x(t) = 0$ for all $t < 0$ • **Energy:** $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

* signal is an energy signal if $0 < E_x < \infty$

• **Power:** $P_x = \frac{1}{T} \int_T |x(t)|^2 dt$

* (for periodic signals)

* signal is a power signal if $0 < P_x < \infty$

Convolution

- $\sum_{k=-\infty}^{\infty} x(k)h(n - k)$ or $\int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$
- graphically:
 - * choose one function to be h
 - * flip around origin with $t \rightarrow -t$
 - * shift back and forth on form $h(t - \tau)$
 - * shift is reversed because the negative
 - * multiply by $x(t)$ and then sum
- if the system is LTI invariant, then the convolution of $x(t)$ with the impulse response $h(t)$ is the same as if $x(t)$ were the input of the system
- convolution with shifted unit impulse is the same as shifting the original system:

$$h(t) * \sigma(t - a) = h(t - a)$$
- Step response is just convolution with impulse response. worked out: $u(t) * h(t) = \int_{-\infty}^t h(\tau) d\tau$
 - * only works for LTI systems!

Geometric Series

$$* \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

* a is first term of the series

r is ratio between terms: $r = \frac{a_1}{a_0} = \frac{a_2}{a_1} \dots$

Systems

- A system is an operation that transforms an input signal into an output signal
 - * you can add/subtract signals
 - * composing signals (one input to another) is convolution (easier to just shift if input is shifted unit step (because LTI))
- **BIBO stability:** output is stable iff input signal is stable
 - * also if impulse response $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ (for LTI systems)
 - * bounded: $h(t) < M$ for all t and some M
- **Memory:** iff the system depends on past or future values of the input
- **Causality:** iff the output depends only on the current or past values of the input
 - * (cannot depend on future values of input)
- **Invertibility:** iff the system's input can be recovered from the output
- **Time Invariance:** iff shifting the input signal shifts the output
 - * integral is time invariant
- **Superposition:** additive commutativity
 - * $H\{x_1(t) + x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\}$
- **Homogeneity:**
 - * $H\{ax(t)\} = aH\{x(t)\}$
- **Linearity:** iff satisfies Superposition and Homogeneity
 - * $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$
 - * averaging filter is linear
- **LTI:** both Linear and Time Invariant

- * simplest systems
- system from block diagram:
 - * add/subtract signals just like you would
 - * for signals $h_1(t) \rightarrow h_2(t)$ (in series), you get $y(t) = h_1(t) * h_2(t)$ (convolution of the two signals)
- * basic method is to keep combining adjacent signal blocks using convolution, scaling, and addition until you get a single block
- system from differential equation:
 - * solve equation for $y(t)$
 - * stuff in terms of input goes on the left; output on the right
 - * add constants scaling to each output, and sum it all together

Linearity

- system is linear if it satisfies superposition (additive) and homogeneity (scalable)
 - * superposition: $h(a) + h(b) = h(a + b)$
 - * homogeneity: $ah(b) = h(ab)$

Noise

- unwanted signals generated externally or internally
- thermal noise is a thing

Impulse Response

- output of a system when the input is $\sigma(t)$
 - memoryless if $h(t) = c\sigma(t)$
 - causal if $h(t) = 0$ for $t < 0$
- BIBO stable if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
- invertible if $h(t) * h^{inv}(t) = \sigma(t)$
- * same for discrete time

even/odd signals

- $f(t) = f_e(t) + f_o(t)$
- $f_e(t) = \frac{1}{2}(f(t) + f(-t))$
- $f_o(t) = \frac{1}{2}(f(t) - f(-t))$

Fourier Series

- Harmonic: $e^{jk2\pi F_0 t}$
- Synthesis: $f(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk2\pi F_0 t}$
- Analysis: $X[k] = \frac{1}{T_p} \int_0^{T_p} x(t) e^{-jk2\pi F_0 t} dt$
 - * note the different sign!
- $X[k] = C_k$
- Parseval's theorem: (energy of a signal)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$
- FT of fraction of two polynomials: use partial fraction decomposition

Fourier Properties

- linearity:

$$z(t) = ax(t) + by(t) \leftrightarrow Z(k) = aX(k) + bY(k)$$
- time shift: $x(t - t_0) \leftrightarrow X(k) e^{-jk\omega_0 t_0}$
- frequency shift: $x(t) e^{jk_0 \omega_0 t} \leftrightarrow X(k - k_0)$
- time scaling: same coefficients, $x(at) \rightarrow \omega = a\omega_0$ (for $a > 0$)
- time reversal: $x(-t) \leftrightarrow X(-k)$
- convolution: $x(t) * z(t) \leftrightarrow TX(k)Z(k)$
- multiplication: $x(t)z(t) \leftrightarrow \sum_{l=-\infty}^{\infty} X(k)Z(k - l)$

- * similar to convolution

- derivative: $\frac{d}{dt}(x(t)) \leftrightarrow jk\omega_0 X(k)$
- integral: $\int_{-\infty}^t x(t) dt \leftrightarrow \frac{1}{jk\omega_0} X(k)$
- Symmetry: if $x(t) = x_r(t) + jx_i(t)$ then $x^*(t) = x_r(t) - jx_i(t)$
- if $x(t)$ is real and even, $X(k)$ is real and even
- if $x(t)$ is real and odd, $X(k)$ is imaginary and odd

Frequency Response

- how the system will respond to a particular frequency
- the Fourier Transform of the impulse response (we don't have to convolve it here, just multiply since it's frequency domain)
- find using $H(\omega) = \frac{Y(\omega)}{X(\omega)}$
 - * if the starting equation is expressed as a differential equation, you can (usually) derive this from that.
- usually represented as $H(\omega) = |H(\omega)| e^{j\theta_H(\omega)}$ (magnitude and phase)
 - * magnitude: $|H(\omega)|$, phase: $\theta_H(\omega)$
- magnitude and phase can be linearly combined

Filtering

- multiply signal by a filter to filter it
- pass: allow through (not filtered out)
- stop: filter out, remove
- *something* pass filter passes *something*; same for stop
- low pass filter: passes ω lower than b , drops higher

$$H(\omega) = \begin{cases} 1 & |\omega| < b \\ 0 & |\omega| > b \end{cases}$$
- high pass filter: opposite of low pass filter
- band pass filter: pass a specific band of frequency.
 - * That frequency is specified by magnitude, so it can be on the positive or negative side of the graph
- Notch filter: band stop filter with a narrow stop band
- An ideal filter has exact edges, but real filters don't
 - * This is impossible in practice. Typically there is vertical variation inside the pass band and stop band, and also a trans band (ω_s), as transition between pass and stop band.