ECEN314 refsheet © Josh Wright September 23, 2017

Metric Prefixes

peta	P	10^{15}	1 000 000 000 000 000
tera	T	10^{12}	1 000 000 000 000
giga	G	10^{9}	1 000 000 000
mega	M	10^{6}	1 000 000
kilo	k	10^{3}	1 000
hecto	h	10^{2}	100
deca	da	10^{1}	10
one		10^{0}	1
deci	d	10^{-1}	0.1
centi	c	10^{-2}	0.01
milli	m	10^{-3}	0.001
micro	μ	10^{-6}	0.000 001
nano	n	10^{-9}	0.000 000 001
pico	p	10^{-12}	0.000 000 000 001
femto	f	10^{-15}	0.000 000 000 000 001

Complex Numbers

- $z = x + iy = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$
- $[r(\cos(\theta) + i\sin(\theta))]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$
- $z^n = (re^{i\theta}) = r^n e^{in\theta}$
- $\frac{1}{i} = -i$
- $\sqrt[n]{z} = \sqrt[n]{r}e^{\frac{\theta}{n}+\frac{2k\pi}{n}}$ for $n \in N^*$ (ints ≥ 0)
- $e^{j\theta} = \cos(\theta) + j\sin(\theta)$
- $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$
- $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
- $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
- normalized: $sinc(t) = \frac{\sin(\pi t)}{\pi t}$

Trig

- $\cos^2(a) + \sin^2(a) = 1$
- $$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$$
- $\sin(2a) = 2\sin(a)\cos(a)$
- $\cos^2(a) = \frac{1}{2}(1 + \cos(2a))$
- $\sin^2(a) = \frac{1}{2}(1 - \cos(2a))$

Signals

- **Even/Odd**
  even: $x(-t) = x(t)$ for all t
  odd: $x(-t) = -x(t)$ for all t
- **Auto Correlation:** compare signal with a time-delayed version of itself
  $\phi(\tau) = \int_{-\infty}^{\infty} x(t) * x(t+\tau)dt$
  - peaks will be at multiples of the period
- **Cross Correlation:** like autocorrelation, but for two different signals
  $\phi(\tau) = \int_{-\infty}^{\infty} x_1(t) * x_2(t+\tau)dt$
  - to easily tell if one signal is a shifted version of another
- Shifting and scaling: just always remember you're replacing **just** t with an expression involving t
- **Unit Step** Signal
  - $u(t) = \{^{0,n<0}_{1,n>0}$
- **Discrete Unit Impulse** Signal
  $\delta[n] = \{^{0,n\neq 0}_{1,n=0}$
  - any discrete signal can be represented as a sum of shifted unit impulse signals
  - $\delta[n] = u[n] - u[n-1]$
- **Continuous Unit Impulse** Signal
  $x(t) = \delta(t) = \{^{0,t\neq 0}_{\infty,t=0}$
  - discontinuous at $t = 0$
  - $\int_{-\infty}^{\infty} \delta(t)dt = 1$
  - pick out values from discrete function: (shifting property)
    $\int_{-\infty}^{\infty} \delta(t) * f(t)dt = f(0)$
    $\int_{-\infty}^{\infty} \delta(t-a) * f(t)dt = f(a)$
- **Shifting Property:** $\int_{-\infty}^{\infty} x(t)\sigma(t-t_0)dt = x(t_0)$
- **Bounded:** $x(t) \leq M$ for all t, some M
  - unbounded signals typically are infinite at some time instant
- **Causal** iff $x(t) = 0$ for all $t < 0$
- **Energy:** $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$
  - signal is an energy signal if $0 < E_x < \infty$
- **Power:** $P_x = \frac{1}{T}\int_T |x(t)|^2 dt$
  - (for periodic signals)
  - signal is an power signal if $0 < P_x < \infty$

Convolution

- $\sum_{k=-\infty}^{\infty} x(k)h(n-k)$ or $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
- graphically:
  - choose one function to be h
  - flip around origin with $t \to -t$
  - shift back and forth on form $h(t-\tau)$
  - shift is reversed because the negative
  - multiply by $x(t)$ and then sum
- if the system is LTI invariant, then the convolution of $x(t)$ with the impulse response $h(t)$ is the same as if $x(t)$ were the input of the system
- convolution with shifted unit impulse is the same as shifting the original system: $h(t) * \sigma(t-a) = h(t-a)$
- Step response is just convolution with impulse response. worked out: $u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$
  - only works for LTI systems!

Geometric Series

- $\sum_{k=0}^{\infty} ar^n = \frac{a}{1-r}$
- a is first term of the series
  r is ratio between terms: $r = \frac{a_1}{a_0} = \frac{a_2}{a_1} \ldots$

Systems

- A system is an operation that transforms an input signal into an output signal
  - you can add/subtract signals
  - composing signals (one input to another) is convolution
    (easier to just shift if input is shifted unit step (because LTI))
- **BIBO stability:** output is stable iff input signal is stable
  - also if impulse response $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ (for LTI systems)

- * bounded: $h(t) < M$ for all t and some M
- **Memory:** iff the system depends on past or future values of the input
- **Causality:** iff the output depends only on the current or past values of the input
  - * (cannot depend on future values of input)
- **Invertibility:** iff the system's input can be recovered from the output
- **Time Invariance:** iff shifting the input signal shifts the output
  - * integral is time invariant
- **Superposition:** additive commutativity
  - * $H\{x_1(t) + x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\}$
- **Homogeneity:**
  - * $H\{ax(t)\} = aH\{x(t)\}$
- **Linearity:** iff satisfies Superposition and Homogeneity
  - * $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$
  - * averaging filter is linear
- **LTI:** both Linear and Time Invariant
  - * simplest systems
- system from block diagram:
  - * add/subtract signals just like you would
  - * for signals $h_1(t) \to h_2(t)$ (in series), you get $y(t) = h_1(t) * h_2(t)$ (convolution of the two signals)
  - * basic method is to keep combining adjacent signal blocks using convolution, scaling, and addition until you get a single block
- system from differential equation:
  - * solve equation for $y(t)$
  - * stuff in terms of input goes on the left; output on the right
  - * add constants scaling to each output, and sum it all together

Linearity

- system is linear if it satisfies superposition (additive) and homogeneity (scalable)
  - * superposition: $h(a) + h(b) = h(a + b)$
  - * homogeneity: $ah(b) = h(ab)$

Noise

- unwanted signals generated externally or internally
- thermal noise is a thing

Impulse Response

- output of a system when the input is $\sigma(t)$
memoryless if	$h(t) = c\sigma(t)$
causal if	$h(t) = 0$ for $t < 0$
BIBO stable if	$\int_{-\infty}^{\infty} \lvert h(t)\rvert dt < \infty$
invertible if	$h(t) * h^{inv}(t) = \sigma(t)$

  - * same for discrete time

even/odd signals

- $f(t) = f_e(t) + f_o(t)$
- $f_e(t) = \frac{1}{2}(f(t) + f(-t))$
- $f_o(t) = \frac{1}{2}(f(t) - f(-t))$

Fourier Series

- Harmonic: $e^{jk2\pi F_0 t}$
- Synthesis: $f(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk2\pi F_0 t}$
- Analysis: $X[k] = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-jk2\pi F_0 t}dt$
  - * note the different sign!
- $X[k] = C_k$
- Parseval's theorem: (energy of a signal) $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$
- FT of fraction of two polynomials: use partial fraction decomposition

Fourier Transform

- $x(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
- $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Fourier Properties

- linearity: $z(t) = ax(t) + by(t) \leftrightarrow Z(k) = aX(k) + bY(k)$
- time shift: $x(t - t_0) \leftrightarrow X(k)e^{-jk\omega_0 t_0}$
- frequency shift: $x(t)e^{jk_0\omega_0 t} \leftrightarrow X(k - k_0)$
- time scaling: same coefficients, $x(at) \to \omega = a\omega_0$ (for $a > 0$)
- time reversal: $x(-t) \leftrightarrow X(-k)$
- convolution: $x(t) * z(t) \leftrightarrow TX(k)Z(k)$
- multiplication: $x(t)z(t) \leftrightarrow \sum_{l=-\infty}^{\infty} X(k)Z(k - l)$
  - * similar to convolution
- derivative: $\frac{d}{dt}(x(t)) \leftrightarrow jk\omega_0 X(k)$
- integral: $\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{jk\omega_0}X(k)$
- Symmetry: if $x(t) = x_r(t) + jx_i(t)$ then $x^*(t) = x_r(t) - jx_i(t)$
- if $x(t)$ is real and even, $X(k)$ is real and even
- if $x(t)$ is real and odd, $X(k)$ is imaginary and odd
- unnormalized sinc: $sinc(t) = \frac{sin(t)}{t}$
- normalized sinc: $sinc(t) = \frac{sin(\pi t)}{\pi t}$

Frequency Response

- how the system will respond to a particular frequency
- the Fourier Transform of the impulse response (we don't have to convolve it here, just multiply since it's frequency domain)
- find using $H(\omega) = \frac{Y(\omega)}{X(\omega)}$
  - * if the starting equation is expressed as a differential equation, you can (usually) derive this from that.
- usually represented as $H(\omega) = |H(\omega)|e^{j\theta_H(\omega)}$ (magnitude and phase)
  - * magnitude: $|H(\omega)|$, phase: $\theta_H(\omega)$
- magnitude and phase can be linearly combined

Sampling

- sampling: independent variable (input); continuous $\to$ discrete
- quantization: dependent variable (output); continuous $\to$ discrete
- aliasing: different signals being indistinguishable after sampling due to sampling rate.
- F_{CT}: continuous time frequency; f_{DT}: discrete time frequency, F_s: sampling frequency
- $F_{CT} \to f_{DT}$ is a many to one mapping
  - * Folding Frequency $= F_s/2$, where frequency

wraps-around
  - restrict to one-to-one: satisfy $|F| < F_s/2$
- **Nyquist Rate:** signal with maximum frequency f_{max} can be recovered exactly if it is sampled at least $f_s > 2f_{max}$
- continuous to discrete:
  - $x[n] = x_a(n/F_s)$
  - $X(f) = F_s \sum_{k=-\infty} \infty X_a[(f-k)F_s]$
- sampling is equivalent to convolving with a delta chain (dirac comb)
- under-sampling: $f_s < 2f_{max}$. Generally bad

DFT

- discrete in both time domain and frequency domain
- zero padding (on the right) increases frequency domain resolution
- frequency domain is on range $[0, 2\pi)$
- TODO

DTFT

- Discrete Time Fourier Transform
- Discrete in time, continuous in frequency
- $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$
- frequency domain is **periodic**, range $[-\pi, \pi]$ is repeated
  - if aliasing happens ($F_s < 2f_{max}$) then parts will overlap, and won't work right
- maximum frequency $\to \pi$; sampling frequency $\to 2\pi$

Laplace Transform

- $X(s) = \int_0^{\infty} x(t)e^{st}dt$
- $s = \sigma - j\omega$
- relation to Fourier Transform: $X(j\omega) = X(s)|_{s=j\omega,\sigma=0}$

- zeros: numerator is 0, so value is 0
- poles: denominator is 0, so value is + or $-\infty$
  - effect on impulse response on $t > 0$:
    poles left of $s = 0$: decaying exponential
    poles right of $s = 0$: increasing exponential
    (reversed on $t < 0$)
- Specified on a Region of Convergence (ROC)
- ROC must contain no poles
- Unilateral: integral starts from 0
  - assumes signals is 0 before $t = 0$
  - i.e. multiplied by $u(t)$
- Bilateral: starts from $-\infty$
  - same as unilateral iff $x(t) = 0$ for $t < 0$
- Shifting:
  - $x(t-T) \leftrightarrow e^{-sT}X(s)$
  - T such that $x(t-T)u(t) = x(t-T)u(t-T)$
    i.e. doesn't shift any non-zero part of the signal left of $t = 0$
    (Doesn't apply for Bilateral transform)
- Initial Value Theorem:
  - $\lim_{s\to\infty} sX(s) = \lim_{t\to 0^+} x(t) = x(0^+)$
  - order of numerator < denominator
  - $x(t) = 0$ for $t < 0$
- Final Value Theorem:
  - $\lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t) = x(\infty)$
  - $x(t) = 0$ for $t < 0$, $x(t)$ finite as $t \to \infty$
  - all poles on left of plane with at most one at $s = 0$
- System Response:
  - take Laplace Transform to get equation in form $a(s)Y(s) - c(s) = b(s)X(s)$
  - solve $Y(s) = \frac{b(s)X(s)}{a(s)} + \frac{c(s)}{a(s)} = Y^{(f)}(s) + Y^{(n)}(s)$
  - $Y^{(f)}(s)$: forced response; $Y^{(n)}(s)$: natural response
  - Frequency Response (transfer function): $H(\omega) = \frac{Y(\omega)}{X(\omega)}$
- Stability: system is stable $\leftrightarrow$ all poles left of $s = 0$

Filters

- multiply signal by a filter to filter it
- pass: allow through (not filtered out)
- stop: filter out, remove
- *something* pass filter passes *something*; same for stop

- low pass filter: passes ω lower than b, drops higher $H(\omega) = \{^{0:|\omega|>b}_{1:|\omega|<b}$
- high pass filter: opposite of low pass filter
- band pass filter: pass a specific band of frequency.
  - That frequency is specified by magnitude, so it can be on the positive or negative side of the graph
- Notch filter: band stop filter with a narrow stop band
- An ideal filter has exact edges, but real filters don't
  - This is impossible in practice. Typically there is vertical variation inside the pass band and stop band, and also a trans band (ω_s), as transition between pass and stop band.
- series RC circuit with $x(t)$ as input and $y(t)$ on capacitor: $RCy'(t) + y(t) = x(t)$ or $y'(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$
  - capacitor: $I(t) = C * \frac{d}{dt}(V(t))$
  - inductor: $V(t) = L * \frac{d}{dt}(I(t))$
- highpass and band-stop filters do not have bandwidth
- Absolute Bandwidth: width of passband
- 3-dB Bandwidth: (for low pass): width of passband where $H(j\omega) \geq |H(0)|/\sqrt{2}$
  - because low pass has max at $\omega = 0$. Different for other filter types
  - AKA half power bandwidth
- Bode Plot: frequency response ($H(s = jw, \sigma = 0)$) in dB as a function of logarithm of frequency
- ideal filter has linear phase response
  - so that the response to different frequencies is the same
  - constant delay is linear phase response (it's just $H(j\omega) = e^{-j\omega a}$)
- ideal filter magnitude response is 1 in passband and 0 in stopband
- real filter has E, δ and transition band (ω_p, ω_s) such that:

- ∗ passband: $1 - E \leq H(j\omega) < 1$ for $0 \leq |\omega| \leq \omega_p$
- ∗ stopband: $|H(j\omega)| < \delta$ for $\omega > \omega_s$
- FIR filter: Finite Impulse Response
  - ∗ finite memory (lower startup transient time)
  - ∗ always BIBO stable
  - ∗ desired magnitude response with exactly linear phase response
- IIR filter: Infinite Impulse Response
  - ∗ output governed by recursive linear constant coefficient differential equations
  - ∗ uses z transform
  - ∗ allows shorter (lower order?) filters
  - ∗ has phase distortion (nonlinear phase response)
  - ∗ non-finite transient startup
- Butterworth Filter
  - ∗ looks like a nice simple slope, relatively large transition band though
  - ∗ ω_c: cutoff frequency
  - ∗ $|H(j\omega)|^2 = 1/(1 + (\frac{\omega}{\omega_c})^{1/(2K)}), K = 1, 2, 3, \ldots$
    $H(s)H(-s)|_{s=j\omega} = |H(jw)|^2$
  - ∗ passband: $\omega_p = \omega_c(\frac{\epsilon}{1-\epsilon})^{1/(2K)}$
  - ∗ stopband: $\omega_s = \omega_c(\frac{1-\delta}{\delta})^{1/(2K)}$
  - ∗ poles: $\omega_c e^{i\pi(2k+1)/(2K)}, k = 0, 1, \ldots, (2K - 1)$
    - never pure imaginary
    - left of $s = 0$ belong to $H(s)$, right of $s = 0$ belong to $H(-s)$
  - ∗ to get transfer function, find poles and make fraction from there
- Chebyshev Filter
  - ∗ poles on ellipse in s-plane
  - ∗ ripples in either passband or stopband, not both
  - ∗ smaller transition band
- Elliptic Filter
  - ∗ smallest transition band
  - ∗ ripples in both passband and stopband

Communication Systems

- frequency range: what frequencies are best transmitted, and what are lost
- modulation: embedding a signal inside a carrier signal that propagates better
- simultaneous transmission (multiplexing): use modulation to transmit more than one signal simultaneously

Amplitude Modulation (AM)

- mix (convolve) payload and carrier in frequency domain (multiply in time domain)
- carrier is typically a high frequency cosine, which is two impulses in frequency domain, so the convolution makes two shifted copies of the input signals, each centered around ± carrier frequency
- demodulation: recover original signal: shift it back to the origin and apply low-pass filter
- Frequency Modulation is another (different) kind of modulation