**Linear Equations**

- system is inconsistent if it has no solution
- a system must have either none, one, or infinitely many solutions
- system is called Homogeneous if all the constant terms (right sides) are 0
- if coefficient matrix is invertible, there is one unique solution for the system

**Gaussian Elimination**

- first put system into matrix form
- Elementary Operations:
  - ∗ multiply a row by a nonzero scalar
  - ∗ add row multiplied by scalar to another row
  - ∗ swap rows
- Elementary Operations don't change any of the solutions
- try to make it into a diagonal matrix

**Row-Echelon form**

- all leading entries $\neq 0$, leading entries shift right as you go down
- from RE form you can use back substitution to get solutions
- leading entry of row: first non-zero element of row
- if there is any zero row, then the solution has a free variable

**Reduced Row-Echelon form**

- same as RE form, but all leading entries = 1, each column with a leading entry is zeros everywhere else
  - ∗ this isn't always the identity matrix; some columns could be missing leading entries entirely

**Gauss-Jordan reduction**

- use Gaussian Elimination to get matrix into Reduced Row-Echelon form
- if there are columns without leading entries, those are free variables
- take each row, transform back to equation, get solution

**Matrix**

- matrices $A, B \in M_{m,n}(R)$
- addition, scalar multiplication work like vectors
  - ∗ addition is only defined when matrix dimensions match
- **Dot Product:** $x * y = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=0}^{n} x_ky_k$
- **diagonal matrix:** only diagonal elements are non-zero
- identity matrix: $I$, diagonal of 1's
  - ∗ $AI = A$ for any A and properly sized $I$
- **Transpose:** just swap the rows and columns
  - ∗ represented as $A^T$
  - ∗ if $A = A^T$ then $A$ is symmetric

**Multiplication:** $A_{m\times n}, B_{n\times p}, C = AB$

- $c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$
- each element in $C$ is the dot product of that row in $A$ and column in $B$
- result has as many rows as $A$ and columns as $B$
- $AB$ is only the same size as $BA$ if they're both square; since the size depends on the matching dimensions of $A$ and $B$
- if $AB = BA$ then $A$ and $B$ commute
- $AB(C) = A(BC)$
  $(A + B)C = AC + BC$
  $C(A + B) = CA + CB$
  $(rA)B = A(rB) = r(AB)$

**Inverse Matrix**

- $AA^{-1} = A^{-1}A = I$
- no inverse: singular or non-invertible
- inverse exists: non-singular or invertible
- zero matrix is singular
- inverse of a matrix is unique
- inverse distributes over matrix multiplication
- inverse of diagonal matrix: reciprocal of each element
- inverse of $2 \times 2$ matrix $[a, b; c, d]$ is $\frac{1}{ac-bd}[d, -c; -b, a]$
- find inverse:
  - ∗ convert $A$ to Reduced Row-Echelon form
  - ∗ apply those same ordered Elementary Operations to $I$ to get $A^{-1}$
  - ∗ (you can do these two steps at the same time)
- **Elementary Matrix:** any matrix reachable by applying Elementary Operations to $I$

**These are Equivalent**

- $A$ is invertible
- $\det(A) \neq 0$
- $x = 0$ is the only solution to the equation $Ax = 0$
- $Ax = b$ has a unique solution for any column vector $b$
- Row-Echelon form of $A$ has no zero rows
- Reduced Row-Echelon form of $A$ is $I$
- the rows/columns of $A$ are linearly independent
- the columns of $A$ form a Basis of $R^n$

**Determinants**

- determinant of singleton matrix is single value

- determinant of $2 \times 2$ matrix $[a, b; c, d]$ is $ac - bd$
- determinant of diagonal matrix is product of diagonal entries
  * same for upper, lower triangular
- determinant of larger matrix can be broken down by a row or column:
  * for each element $a_{i,j}$, take $a_{i,j}$ times the determinant of the (smaller) matrix formed by leaving out row $i$, column $j$
  * and use the proper sign by the alternating method
- Elementary Operation Axioms:
  * D1: multiply row by $r \to$ multiply det by $r$
  * D2: add scalar multiple of one row to another $\to$ same det
  * D3: swapping rows of matrix $\to$ det changes sign
  * D4: $\det(I) = 1$
  * C1: if $A, B$ are square and $A$ is obtained by applying Elementary Operations to $B$, then $\det(A) = 0$ iff $\det(B) = 0$
  * C2: $\det(B) = 0$ whenever $B$ has a zero row
  * C3: $\det(A) = 0$ iff $A$ is not invertible
- Cramer's rule: explicit formula for solution to system of linear equations, using determinants. Not very useful because determinants.

**Wronskian**

- to show linear independence of functions in $C^\infty(R)$ (continuously differentiable functions)
- $W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$
  * functions in rows, derivatives down columns
- if $W(x)$ is not identically 0, then the functions are linearly independent
- alternatively (1): take derivatives of the top row until you get something that you can work with to solve
- alternatively (2): start with $af_1(x) + bf_2(x) + cf_3(x) = 0$ and show that the only solution is $a = b = c = 0$

**Basis**

- every vector space has a Basis
- it's like a coordinate system
- Basis = minimum spanning set = maximum set of linearly independent vectors
- can get one by adding linearly independent vectors to a too-small set or removing linearly dependent ones from a too-large one
- **Dimension:** $(\dim(V))$ number of basis vectors for a vector space
  * if $\dim(V) < \infty$ then every basis of $V$ is the same size

**Matrix Spaces**

- matrix $M_{m,n}$:
- **Row Space:** subspace of $R^n$ spanned by rows of $M$
  * **Rank:** = dimension of row space (number of linearly independent rows)
  * in Row-Echelon form, all non-zero rows are linearly independent
- **Column Space:** subspace of $R^m$ spanned by columns of $M$
- **Null Space:** $N(A)$: all $x$ such that $Ax = 0$
  * aka kernel
  * solution set of homogeneous equations with coefficients $A$
  * $N(A)$ is subspace of $R^n$
  * Nullity = $\dim(N(A))$ = number of free variables
- for any matrix, rank + nullity = number of columns

**Change of Basis (Coordinates):**

- basis $B_1 = \{u, v\}$ of $R^2$
- change basis of $(x, y)$: find $r_1, r_2$ such that $(x, y) = r_1 v + r_2 u$
- **Transition Matrix** $T = (u^T, v^T)$ has columns of $u$ and $v$, and maps $B_1 \to R^2$
  * that is, $T \cdot \binom{x'}{y'} = \binom{x}{y}$ when $(x, y) = x'u + y'v$
  * inverse works: $T^{-1} \cdot \binom{x}{y} = \binom{x'}{y'}$
- transition between general bases:
  * basis $B_1, B_2$ with transition matrix $T_1, T_2$
  * $f(x) : B_1 \to B_2 = T_2^{-1} \cdot T_1 \cdot x$
  * $f(x) : B_2 \to B_1 = T_1^{-1} \cdot T_2 \cdot x$
  * more generally, find one basis's coordinates in terms of another's, but then you'll need to solve $n^2$ equations

**Linear Relations**

- additivity: $L(ax + by) = aL(x) + bL(y)$
- homogeneity: $L(ax) = aL(x)$
- kernel: all $v$ such that $L(v) = 0$
- range: all possible output values

**Least Squares**

- express as overdetermined relation $Ax = b$ (where there are more rows than variables)
- Then left-multiply both sides by $A^{-1}$, getting $A^{-1}Ax = A^{-1}b$
- the resulting equation will be fully determined, so solve like normal
- at the end, you get values for x, which are the needful least squares coefficients

**Orthogonality**

- vectors are orthogonal if their dot product is 0
  - the zero vector (and only the zero vector) is orthogonal to itself
- **Orthogonal Compliment** sets (or subspaces) of vectors are orthogonal if every combination from the two is orthogonal
  - $R^3$: a line is orthogonal to a plane, and vice versa
- Orthonormal: vectors that are orthogonal and unit length
- any vector $x$ can be broken into $x = p + o$ where $p$ and $o$ are orthogonal, and $p$ is parallel to a known $y$
  - $p = \frac{x \cdot y}{y \cdot y} y$
  - $o = x - p$