DTU

Perception for Autonomous Systems 31392:

State Estimation - Kalman Filter

Lecturer: Evangelos Boukas—PhD

Sum-UP so far

- State Estimation
- Markov Localization
- Probability
- Bayes
- Total Probability

Coming UP:

- Kalman Filter

What is State Estimation

Goal:

- Given a State Vector of a system
- Estimate over time the state using input of external sensors

Useful for:

- Localization
- Tracking
- Prediction
- Sensor Fusion
- …

Catching up

- Last part, we did state estimation specifically Localization

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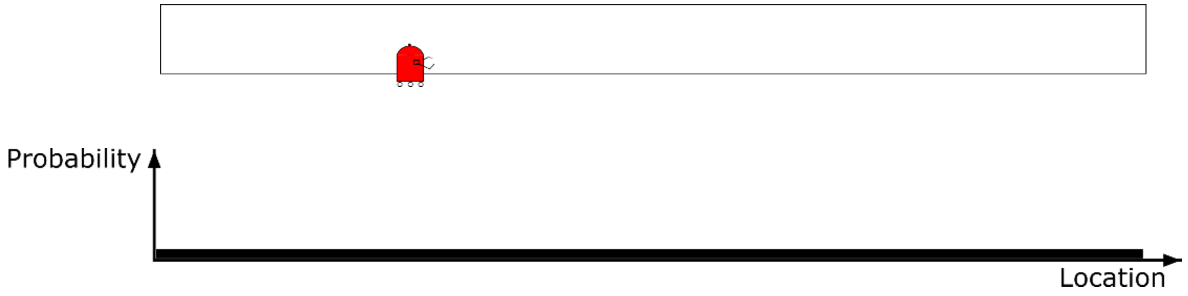

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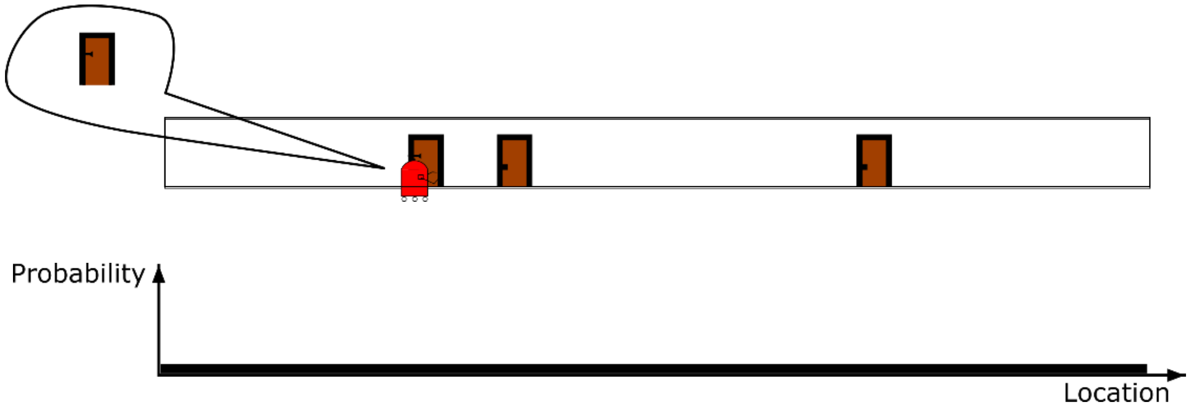

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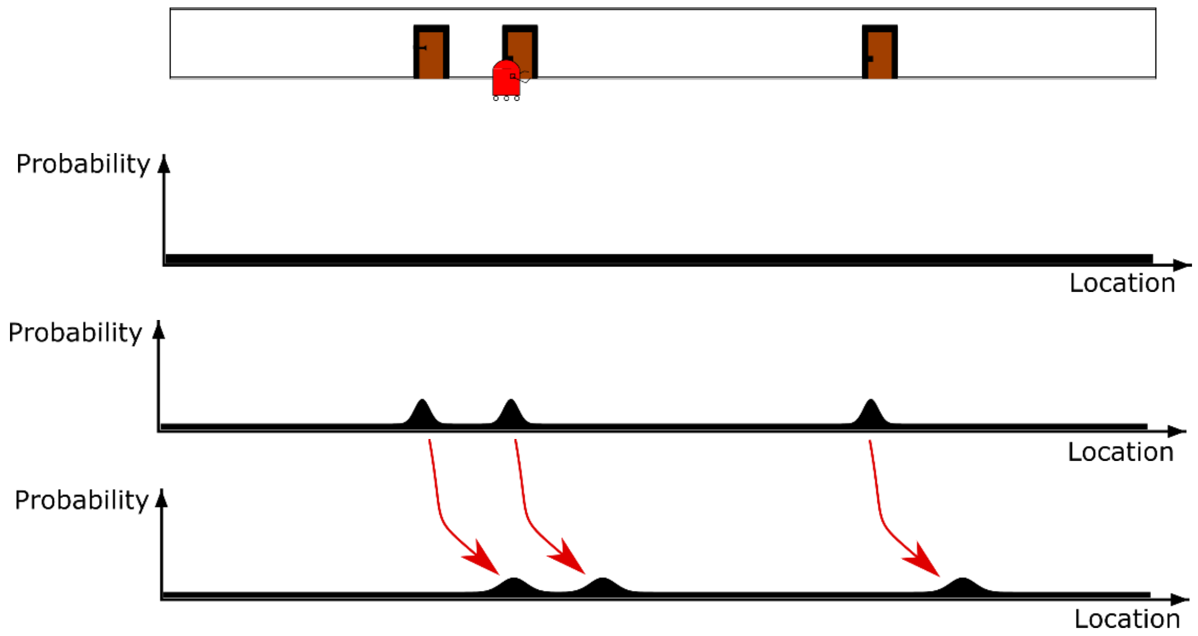

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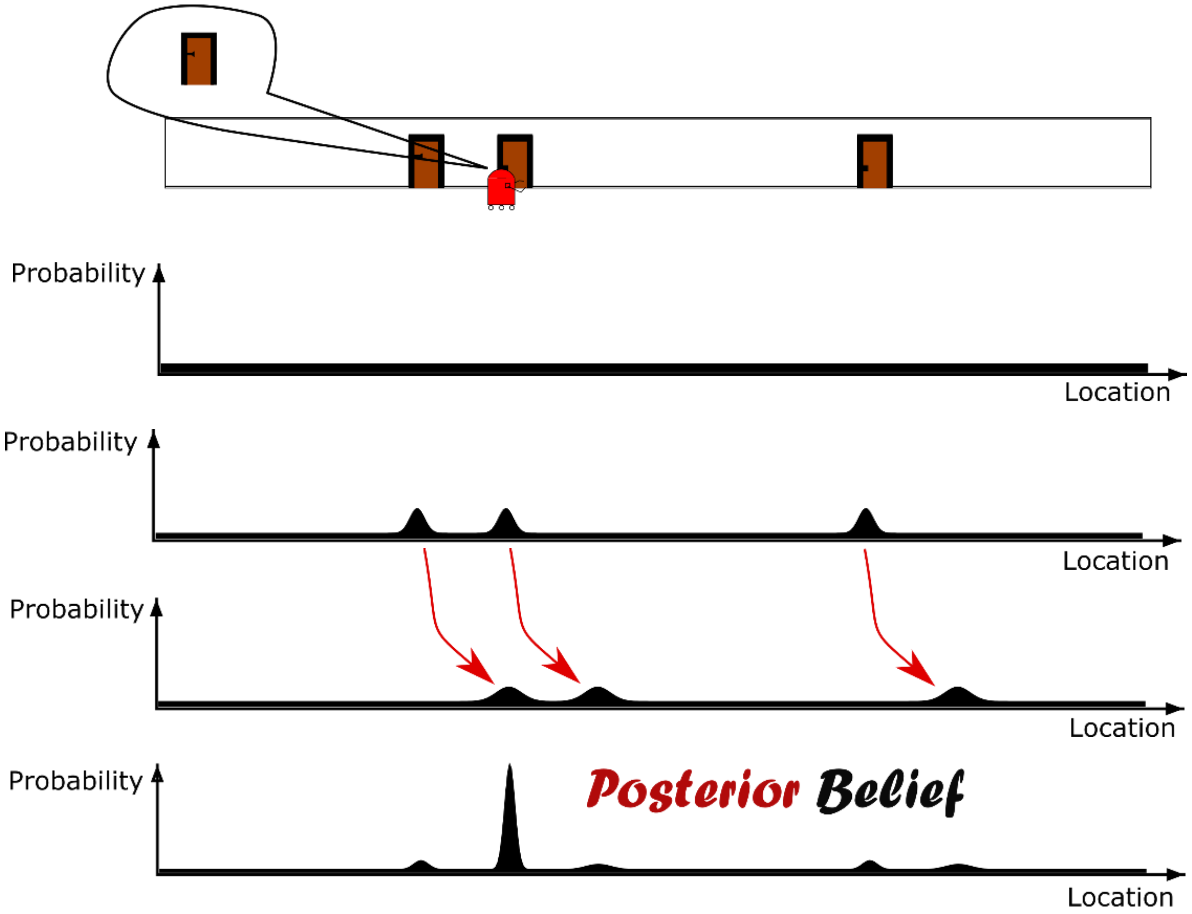

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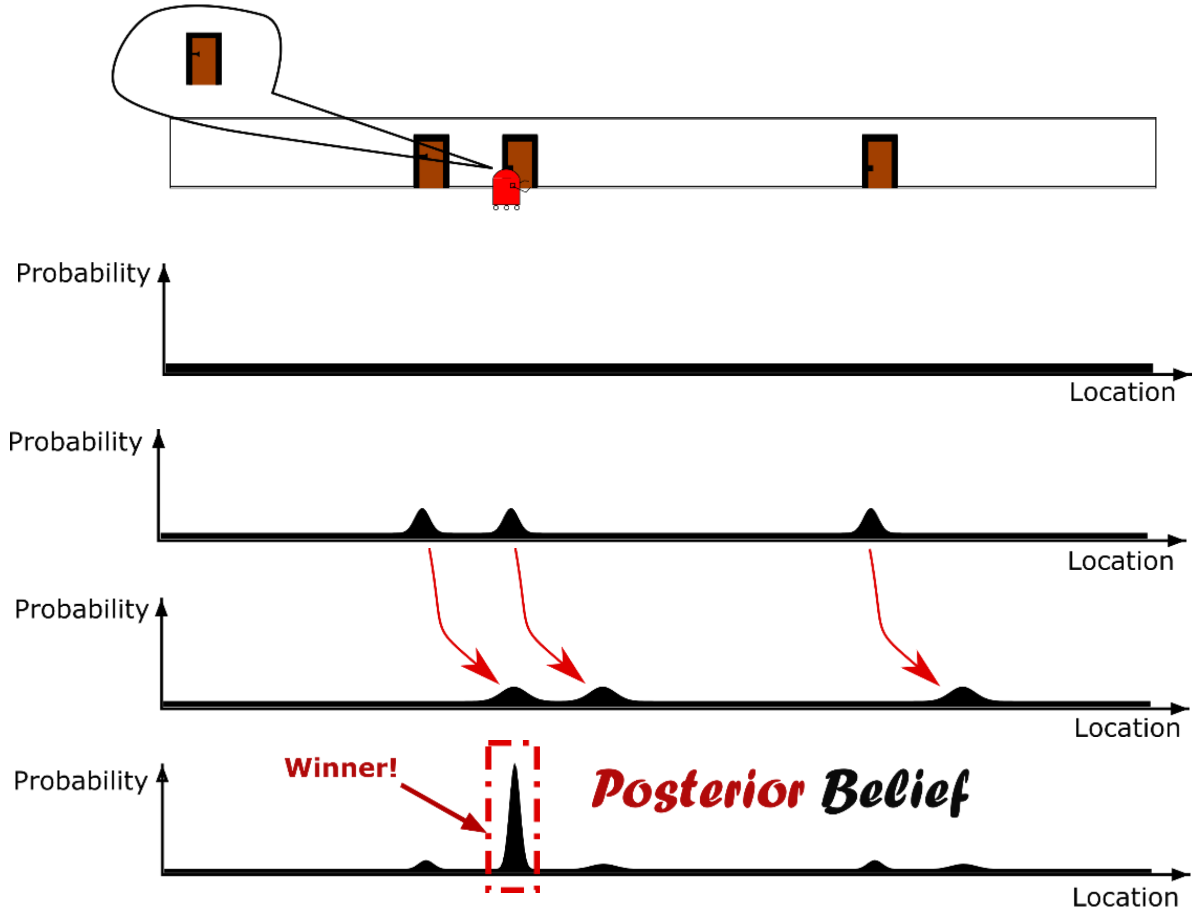

Catching up

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  specifically Localization
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  to Location estimation

Catching up

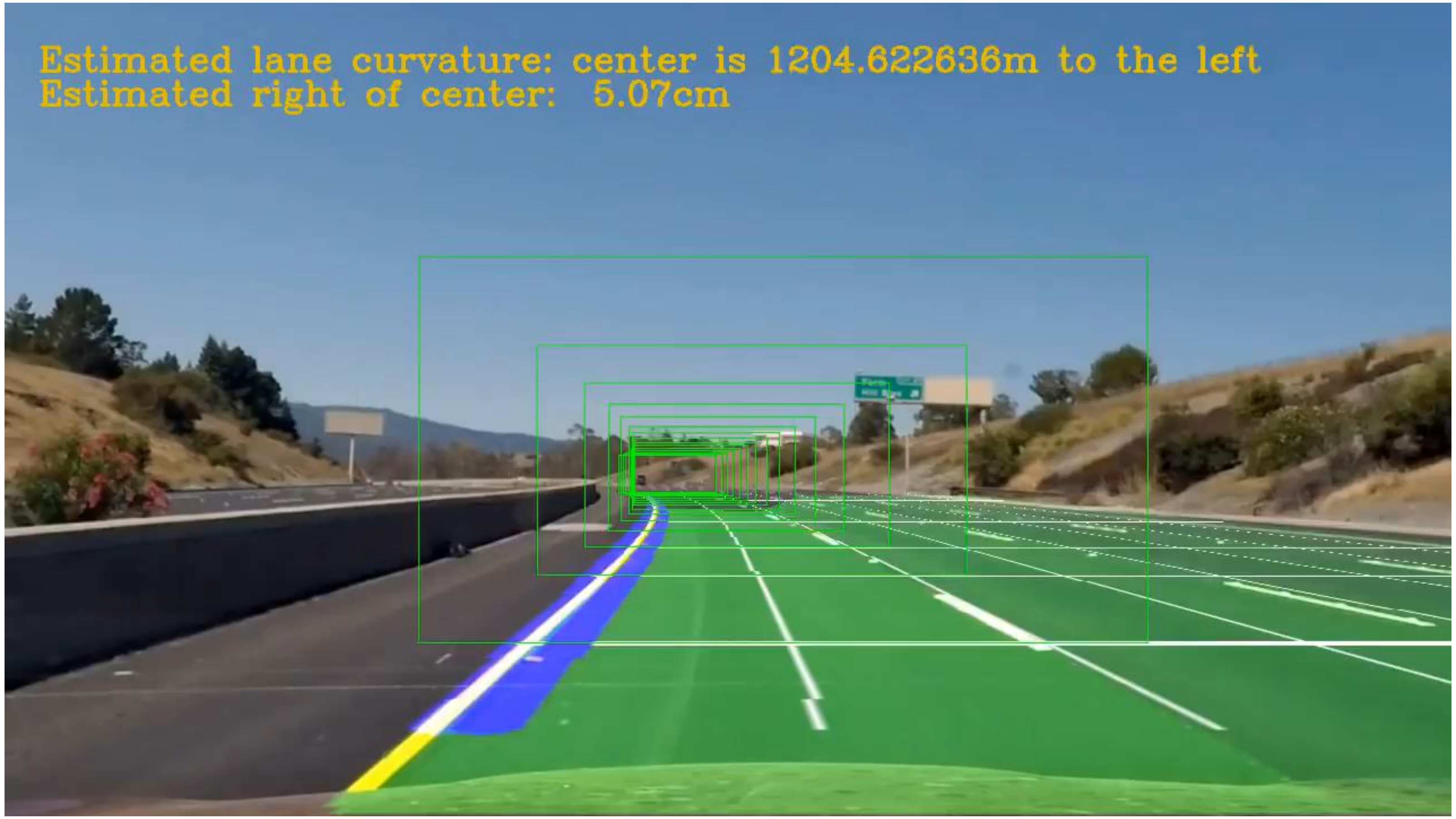
- Last part, we did state estimation
  specifically Localization
- Maximum confusion
  to Location estimation
- In other words:
  “Used sensor information to manipulate an original belief (a uniform distribution) into a high confidence probability density function centered on our correct location“

In this part..

- We will see how we can track objects.
- Not only their location as in the previous localization case,
- But also infer their speed.

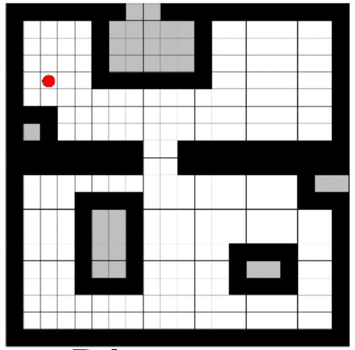
- In the case of autonomous driving it is quite important to track and predict the movement of objects.
  - Why?
  - Which other examples can we find?

Cases of Tracking

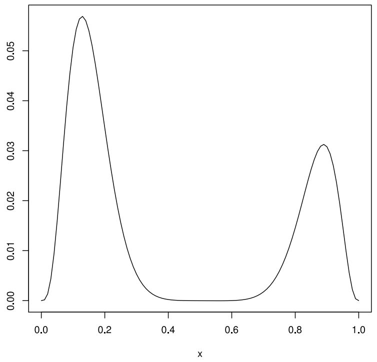


Differences between state estimation filters

Histogram filter

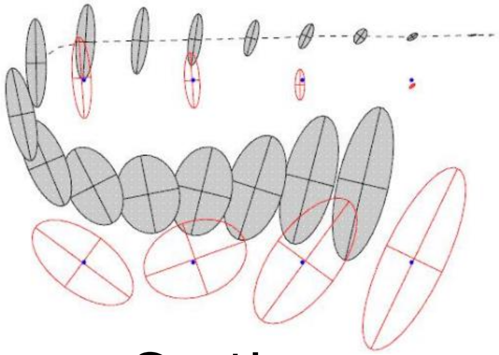


Discrete

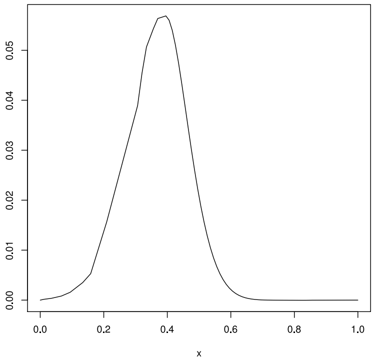


Multimodal

Kalman Filter



Continuous



Unimodal

Let’s see tracking as an example

- Assuming there is a point in space like this:
- The object moves like this:

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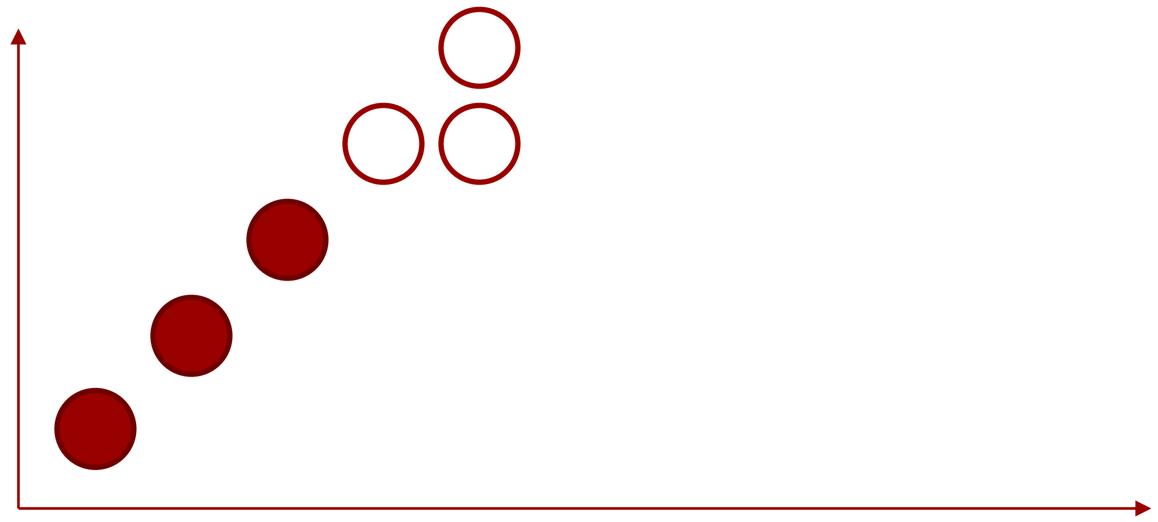

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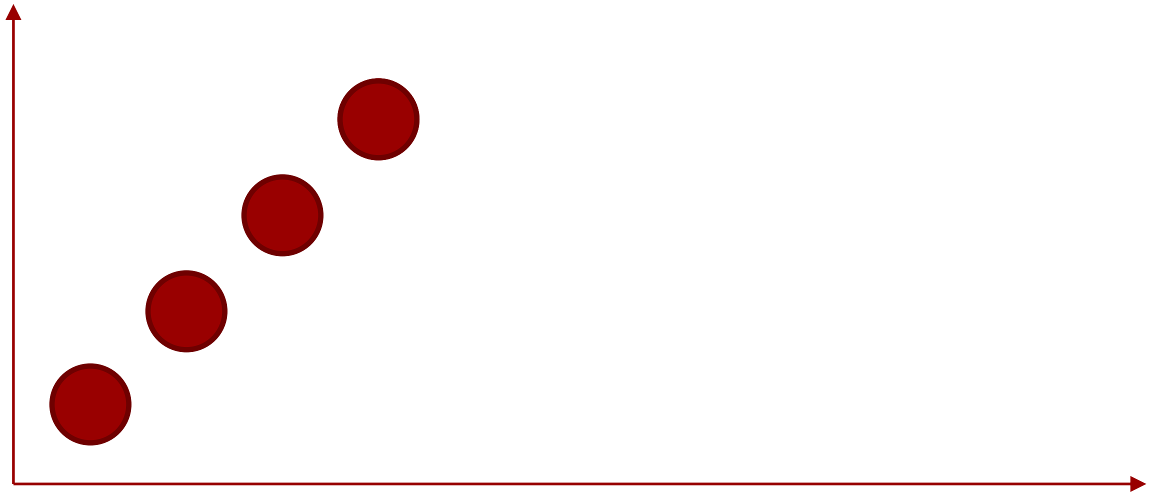
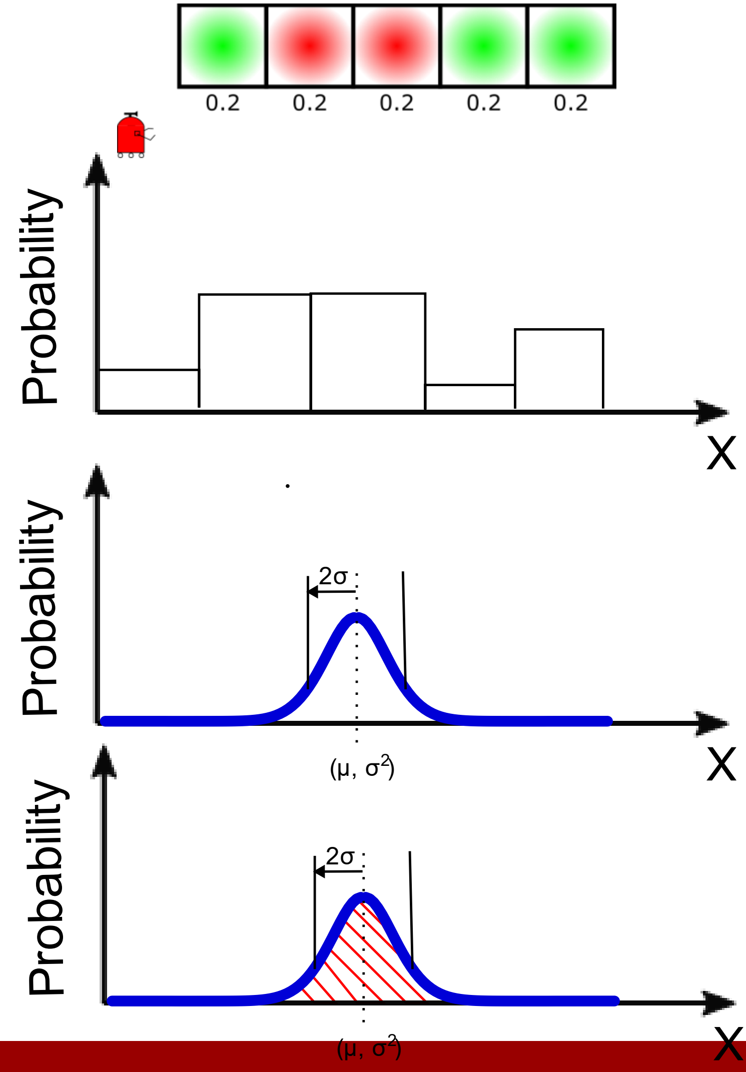


- For you it is easy! How about a machine?

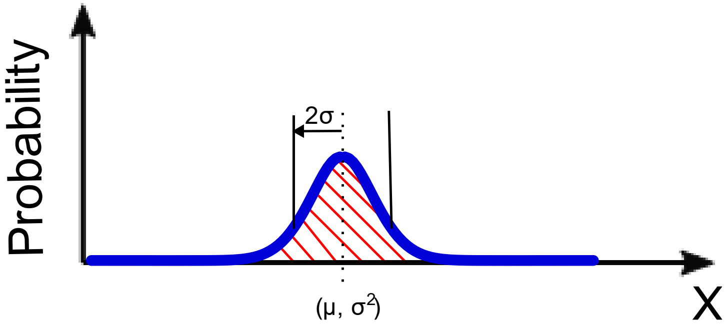
Kalman has a thing for Gaussians

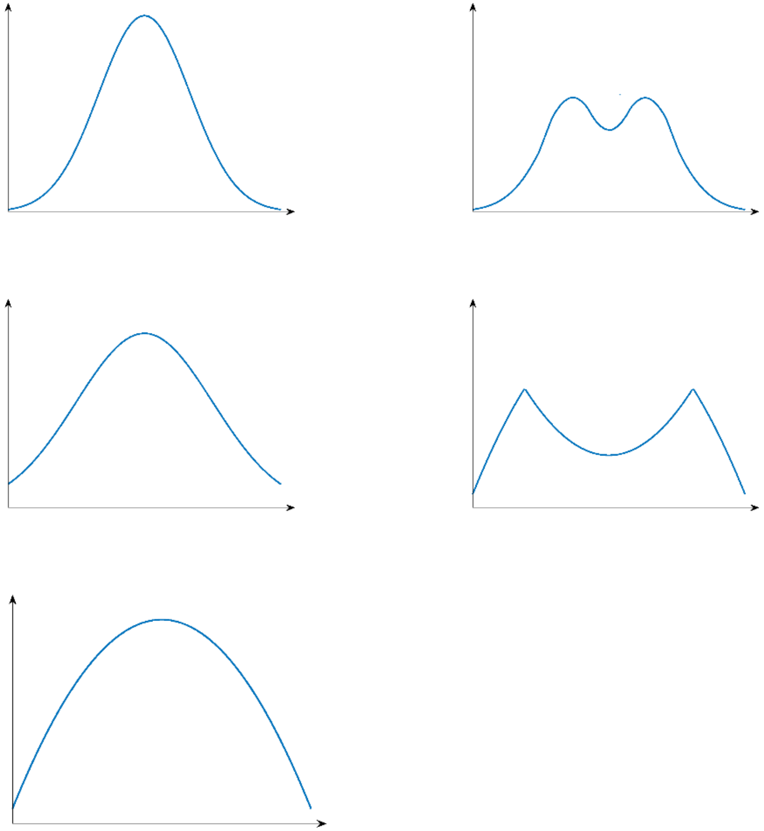
- In our Markov model the world was divided into discrete grids and each grid had a probability
- This is called a histogram:

- In Kalman Filter we describe the distribution as a Gaussian:
  - It is a continuous function and
  - The area under the curve is: 1



Gaussian





- 1D Gaussian is described by the pair: (μ, σ^2)

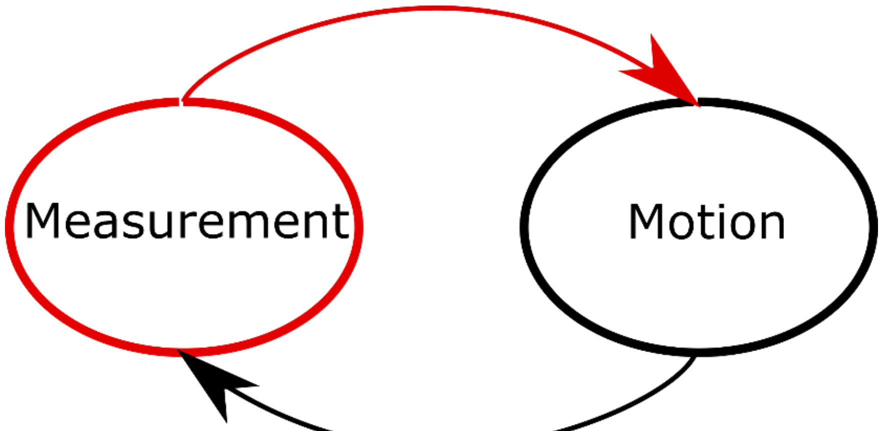
$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

- Which of the following are Gaussians?

- Which of the following have small, medium, larger (co)variance?
- When doing state estimation which one do we prefer?

Kalman

- Kalman, as with the histogram filter involves the measurement ⇔ motion cycle:



- Which one requires **convolution** and which one a **product**?

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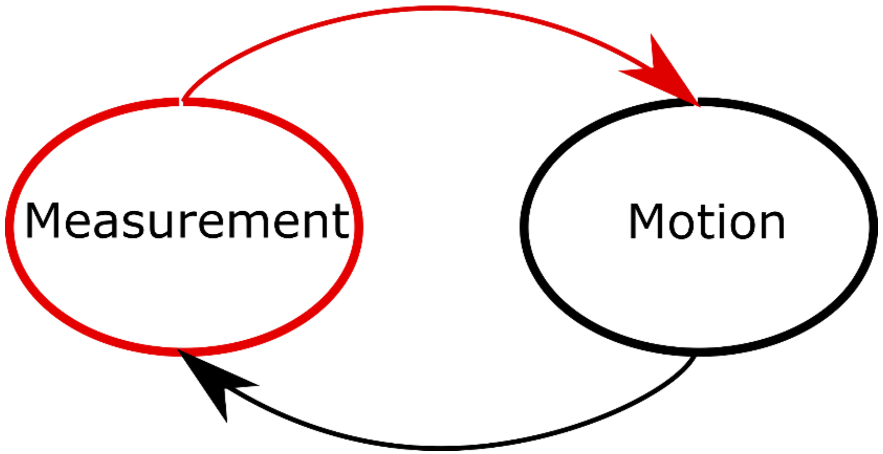

- Which one requires **convolution** and which one a **product**?
  – Measurement ⇔ Product
  – Motion ⇔ Convolution

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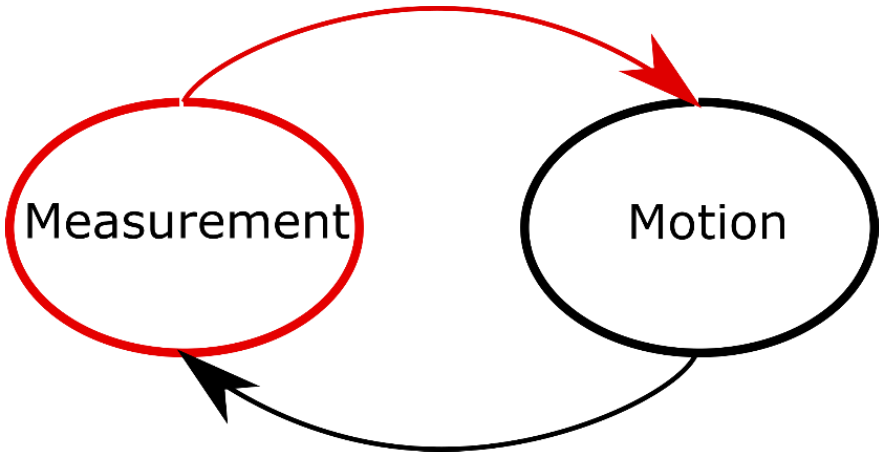

- Which one applies **Bayes Rule** and which one **Total Probability**?

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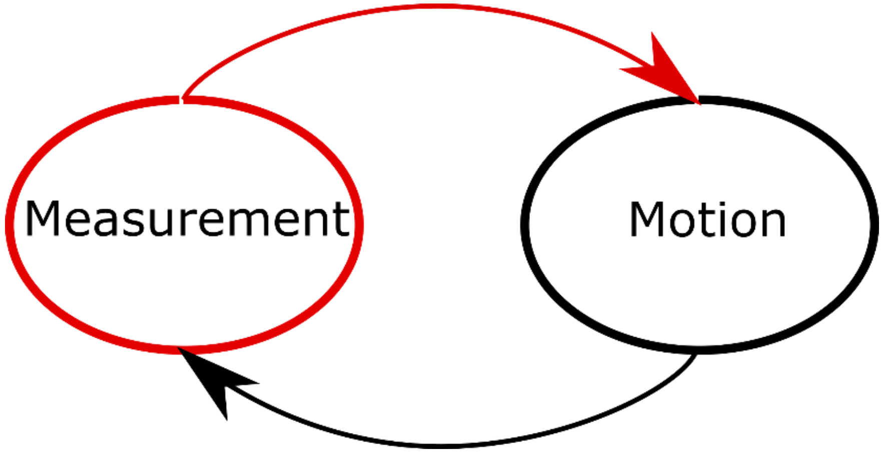
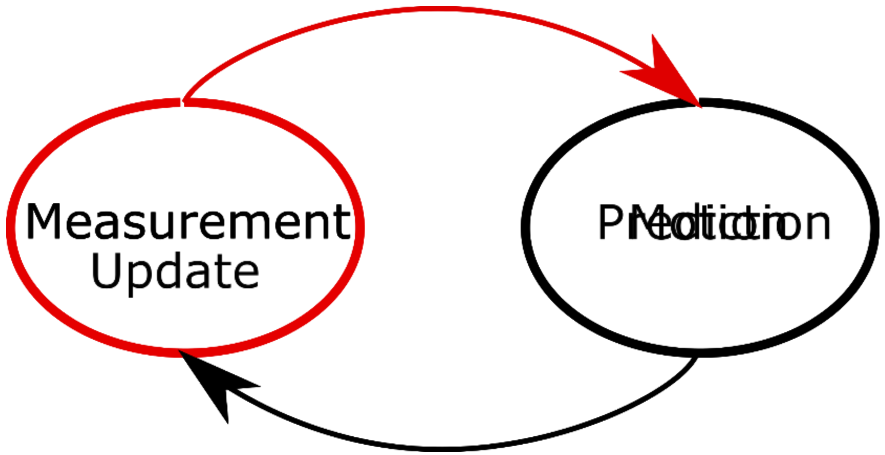


- Which one applies **Bayes Rule** and which one **Total Probability**?
  - Measurement ⇔ Bayes Rule
  - Motion ⇔ Total Probability

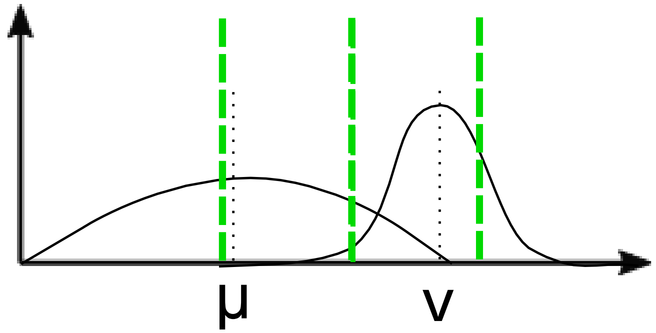
Kalman



- In Kalman we call them “Measurement Update” and “Prediction”
- Both of these involve the Gaussians

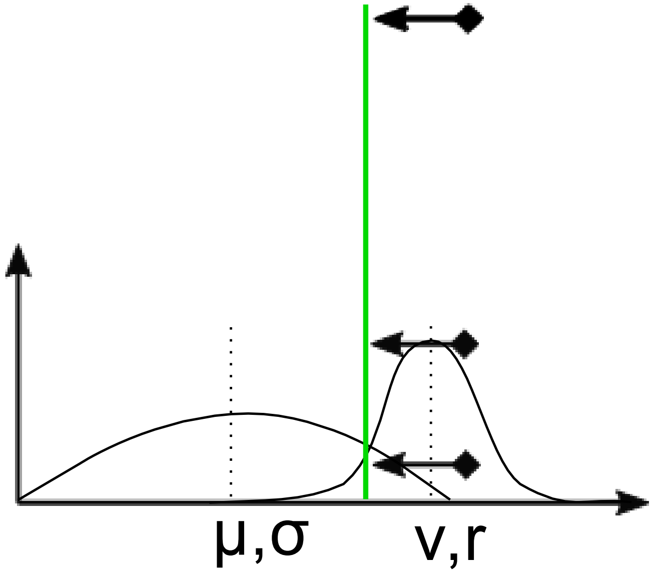
Kalman Measurement Update

- Assume we are localizing another robot with a prior as follows:



- Then we have a measurement which inform us that we have this location:
- Where will the new mean be?

Kalman Measurement Update



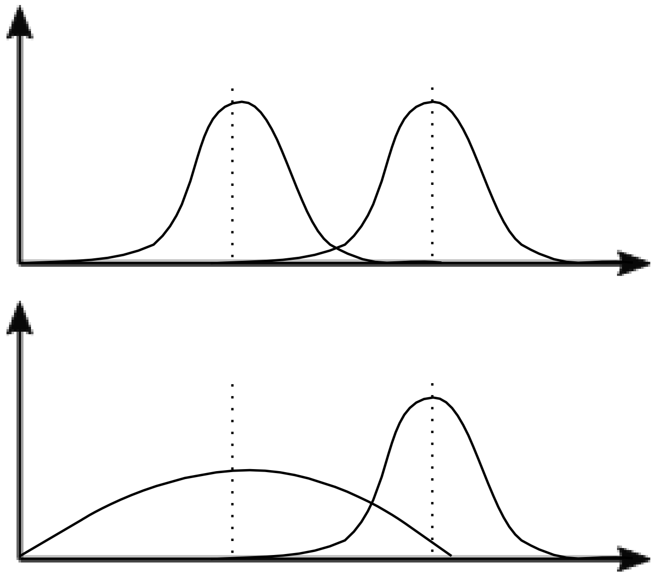
- Where will the new peak be?
- The higher one -> as we gain information
- Let's prove it

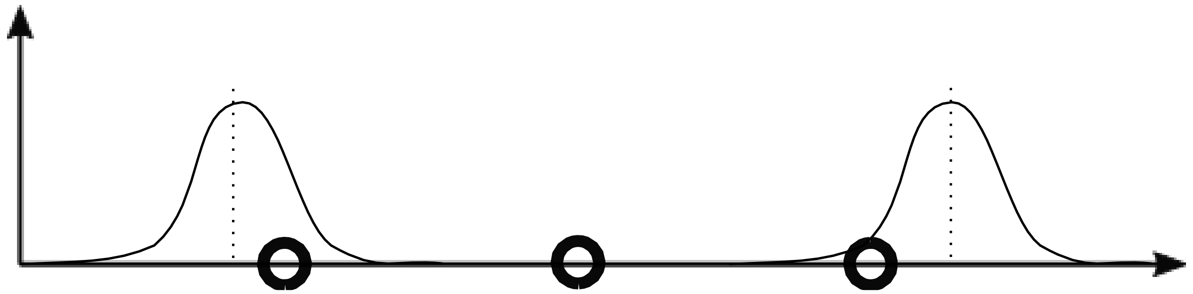
$$\mu' = \frac{r^2\mu + \sigma^2 v}{r^2 + \sigma^2}$$

$$\sigma'^2 = \frac{1}{\frac{1}{r^2} + \frac{1}{\sigma^2}}$$

Gaussians

- Assuming these gaussians:
  - $\mu = 10$, $\sigma^2 = 4$
  - $v = 12$, $r^2 = 4$
- Assuming these gaussians:
  - $\mu = 10$, $\sigma^2 = 8$
  - $v = 13$, $r^2 = 2$
- Assuming these gaussians:





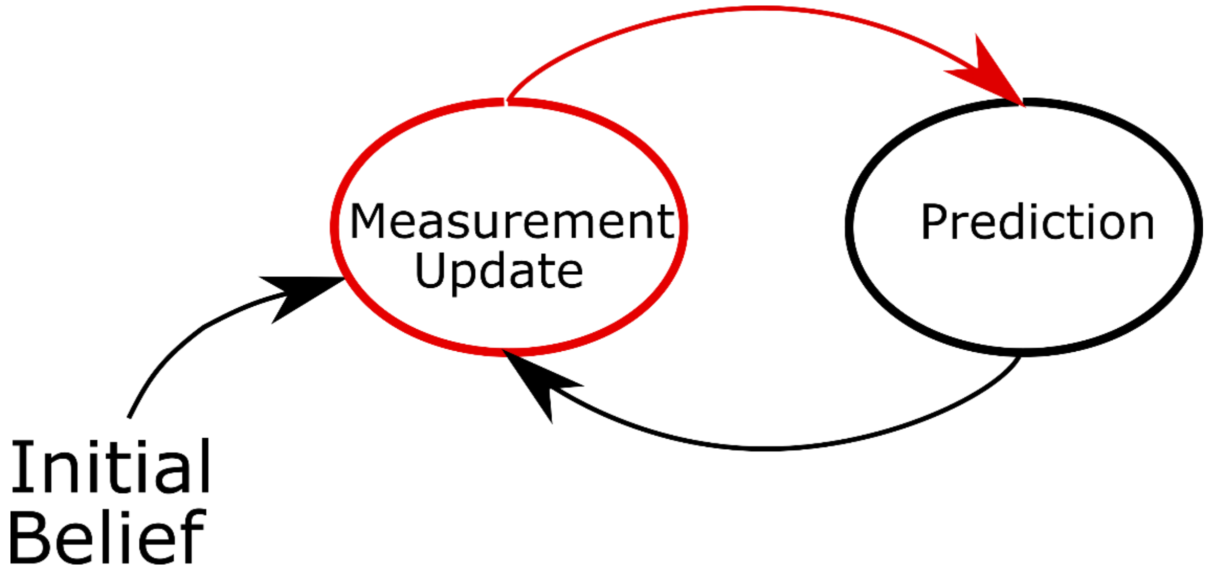
Motion Update

- Called also prediction:
- As we move we lose some information:

- Assuming a gaussian before the prediction:
  - $\mu = 8$, $\sigma^2 = 4$
- And a movement gaussian
  - $v = 10$, $r^2 = 6$
- What's the Gaussian after the update?

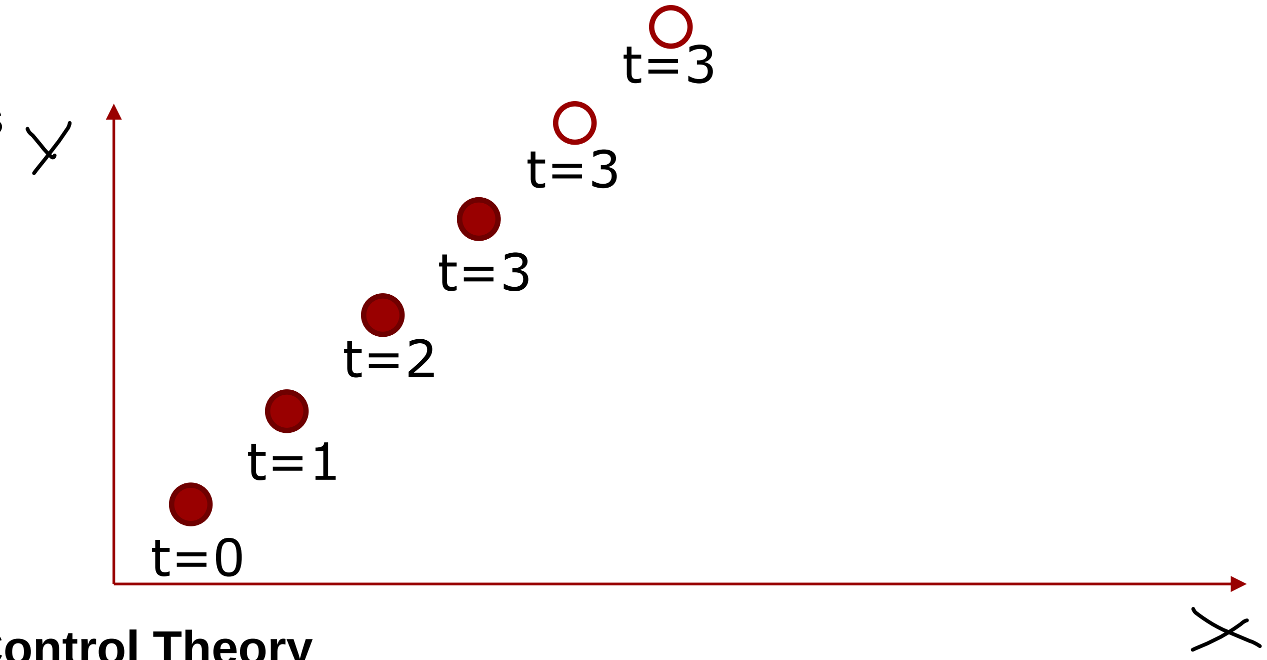
Let’s code the 1D Kalman

- Start with a initial belief of:
  - u=0,
  - σ^2=10000
- Motion:
  - [1, 1, 2, 1, 1]
  - Uncertainty: 2
- Measurement:
  - [5, 6, 7, 9, 10]
  - Uncertainty: 4



From 1D to Many D’s

- We just implemented a Full 1D Kalman filter.
- However the Kalman Filter shines in Many D’s
- Let’s see an example:
  - A camera
  - Or a pedestrian in front of a car
  - Where should it be at t=3?



- That is the power of Kalman!!! ➔ **AI** and **Control Theory**

Multivariate Gaussians

$$\frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma}|}}$$

As promised we have married the Gaussians today

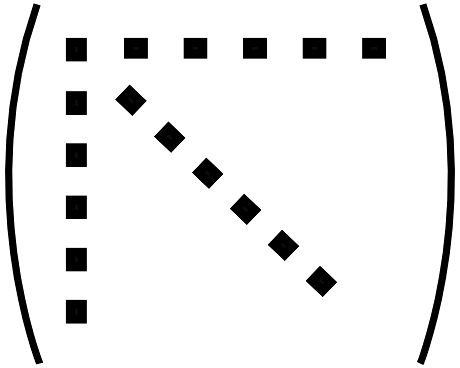
Multi-Dimensional Gaussians

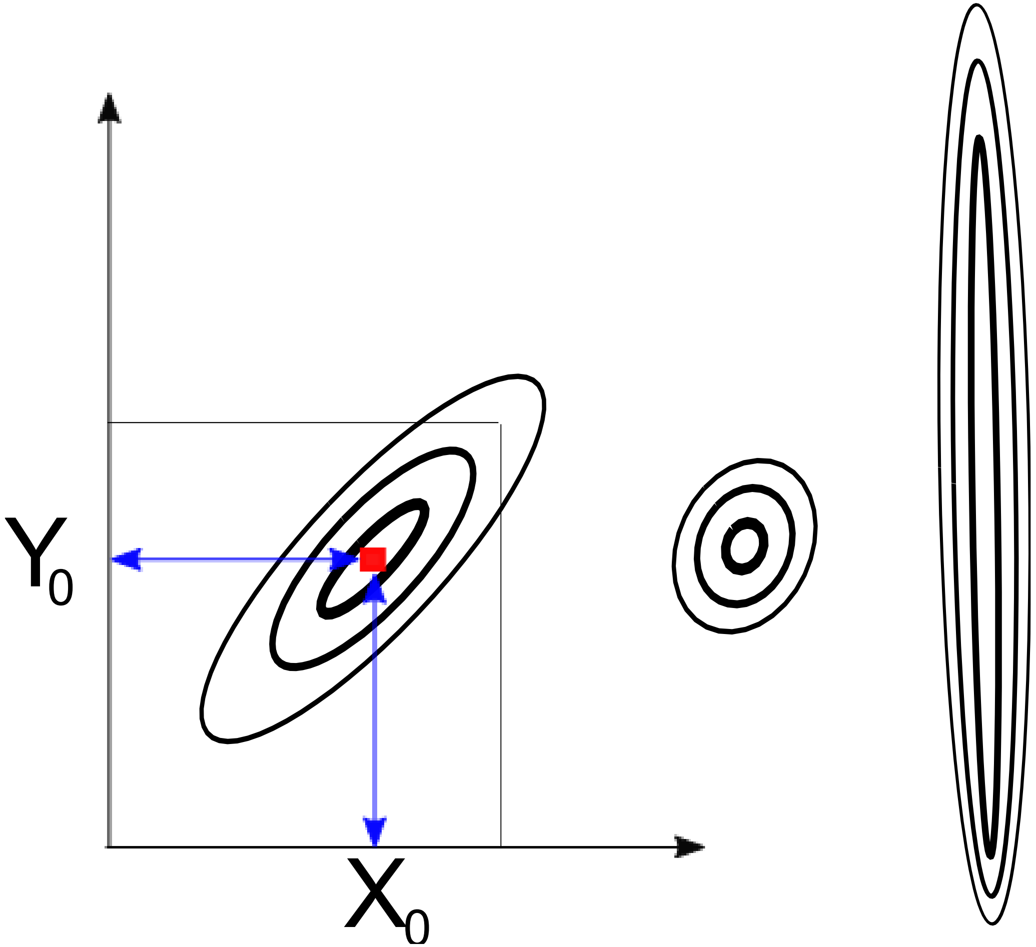
- The mean is a vector:

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_D \end{pmatrix}$$

- The variance now is called covariance and is a matrix:

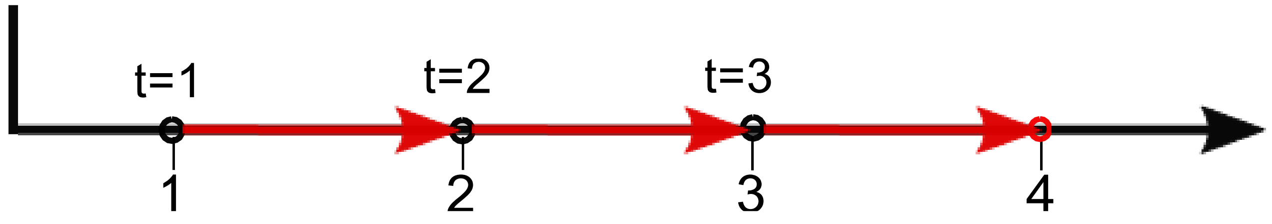
$\boldsymbol{\Sigma} =$



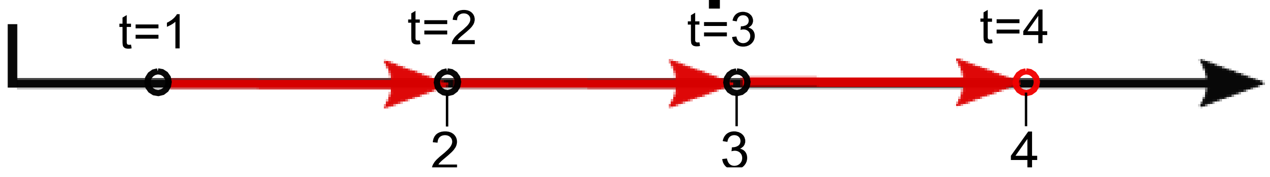


Multivariate Gaussians

- Let's start with an one dimensional motion example:



Multivariate Gaussians - Kalman State Space



- Let’s go at the Kalman state space:

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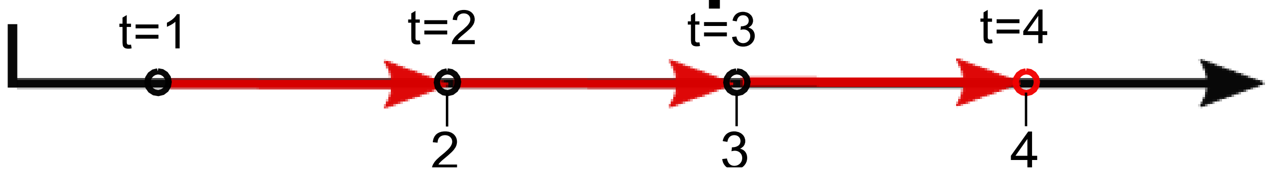

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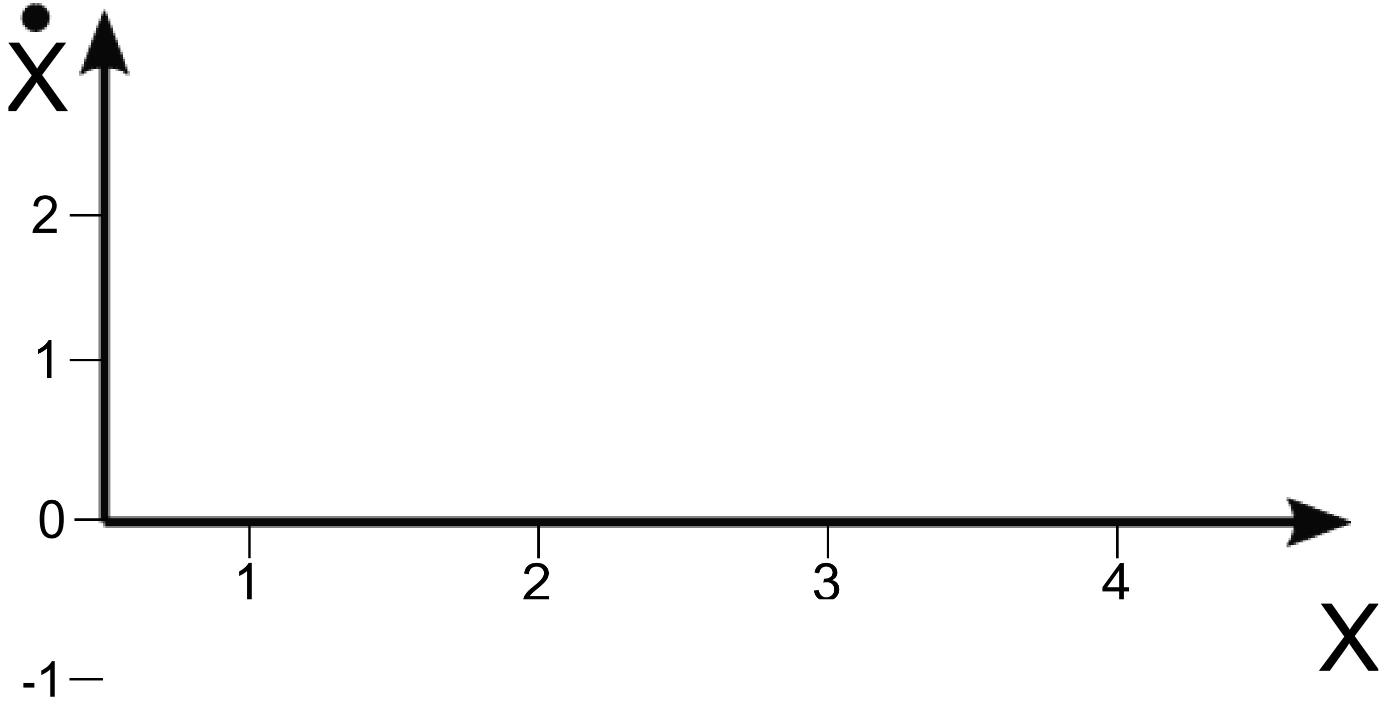

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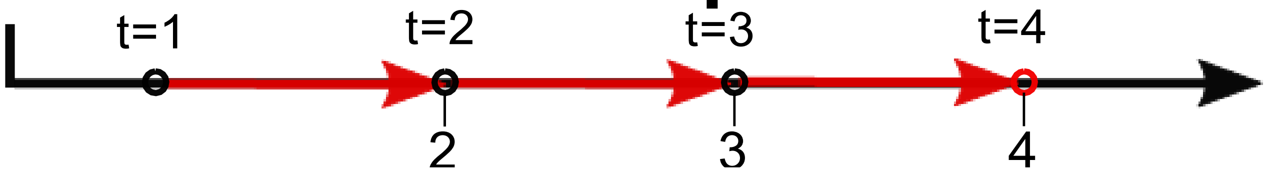
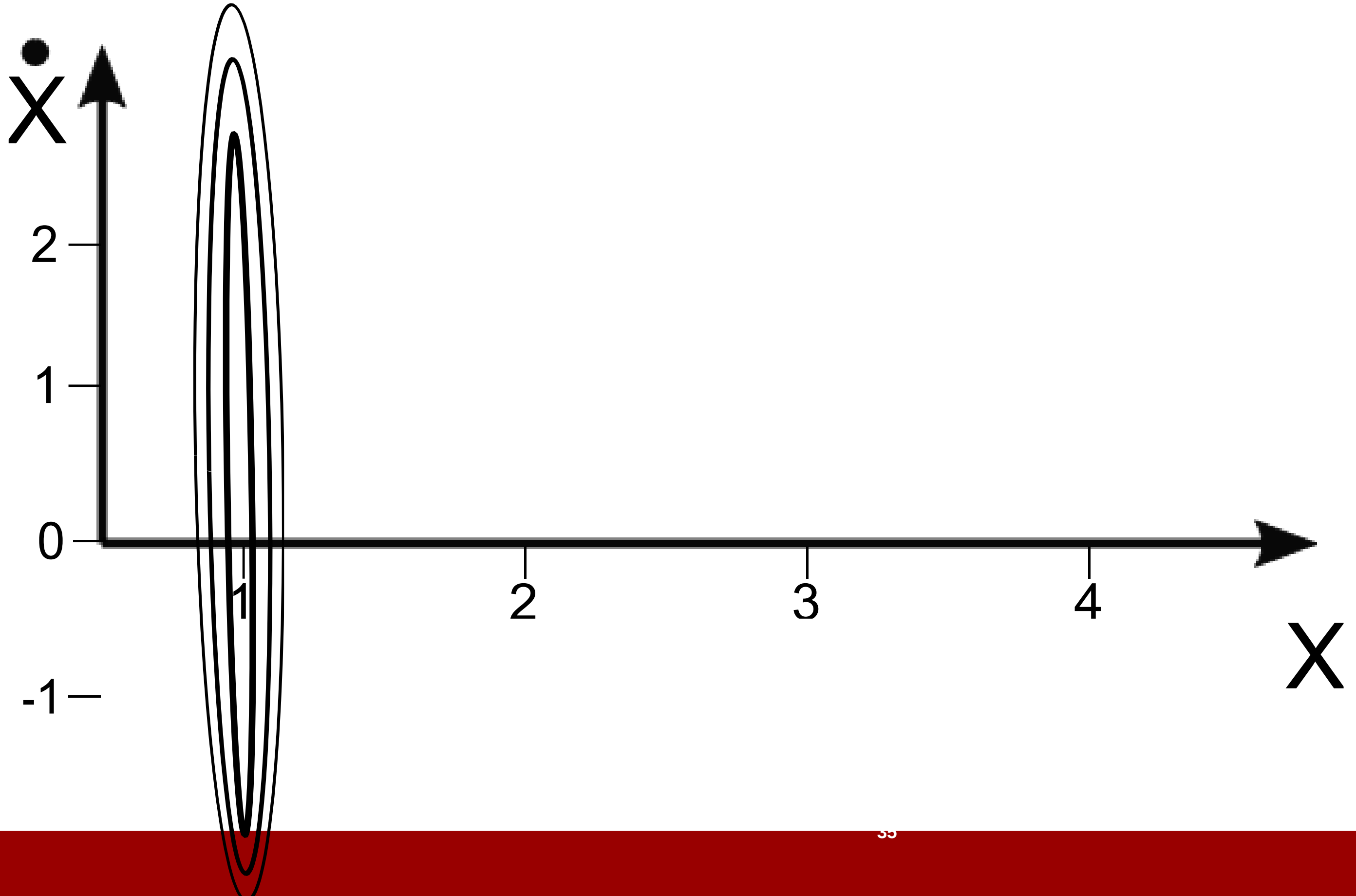


- Let's go at the Kalman state space:
- Prior:
  - Location: 1



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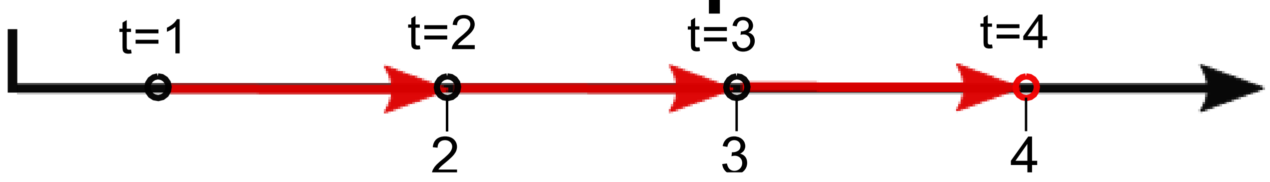
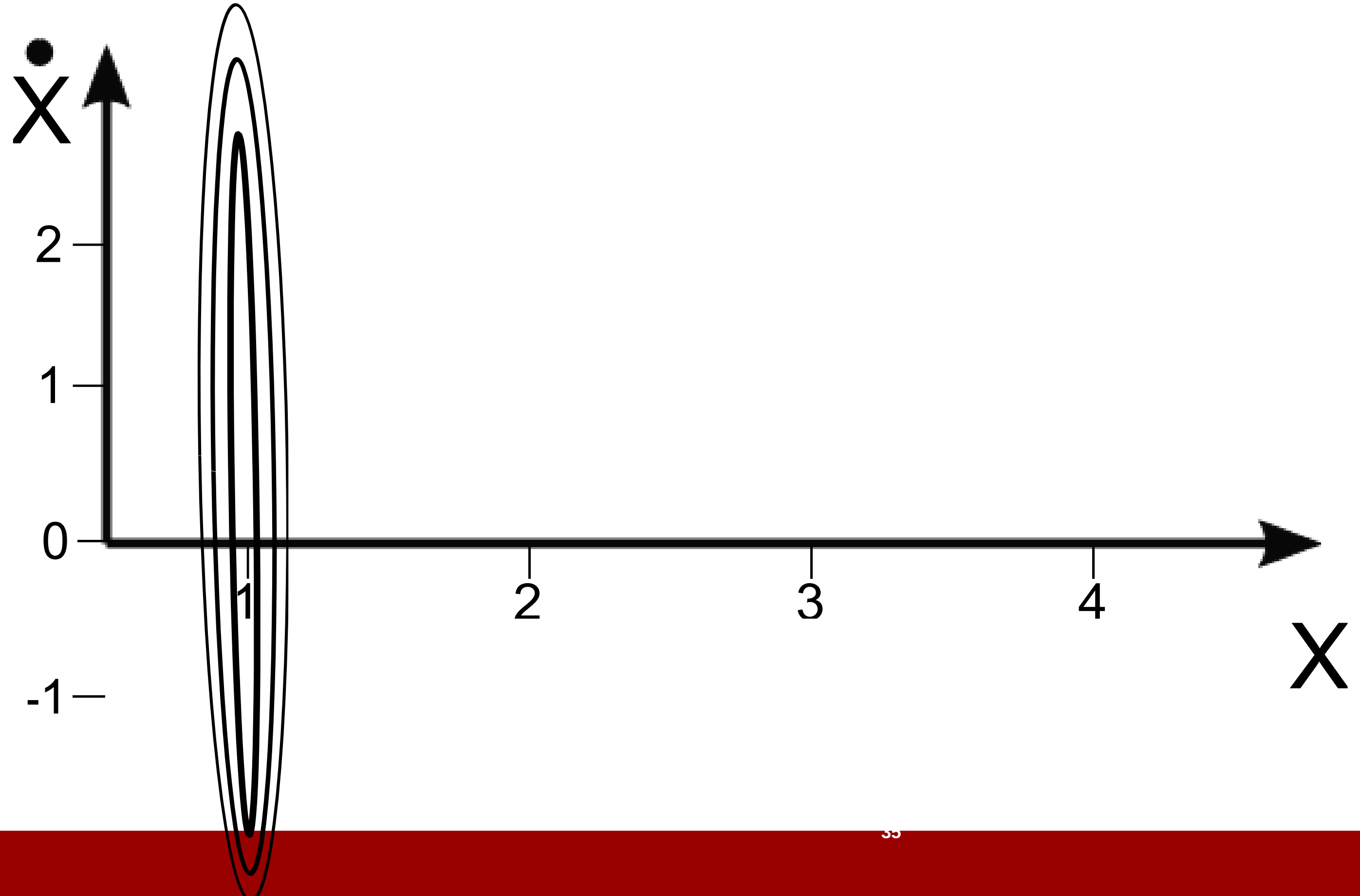


- Let's go at the Kalman state space:
- Prior:
  - Location: 1
- Prediction:
  - Velocity: 0



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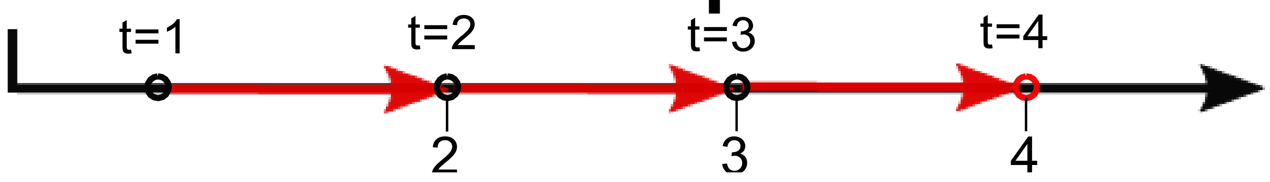

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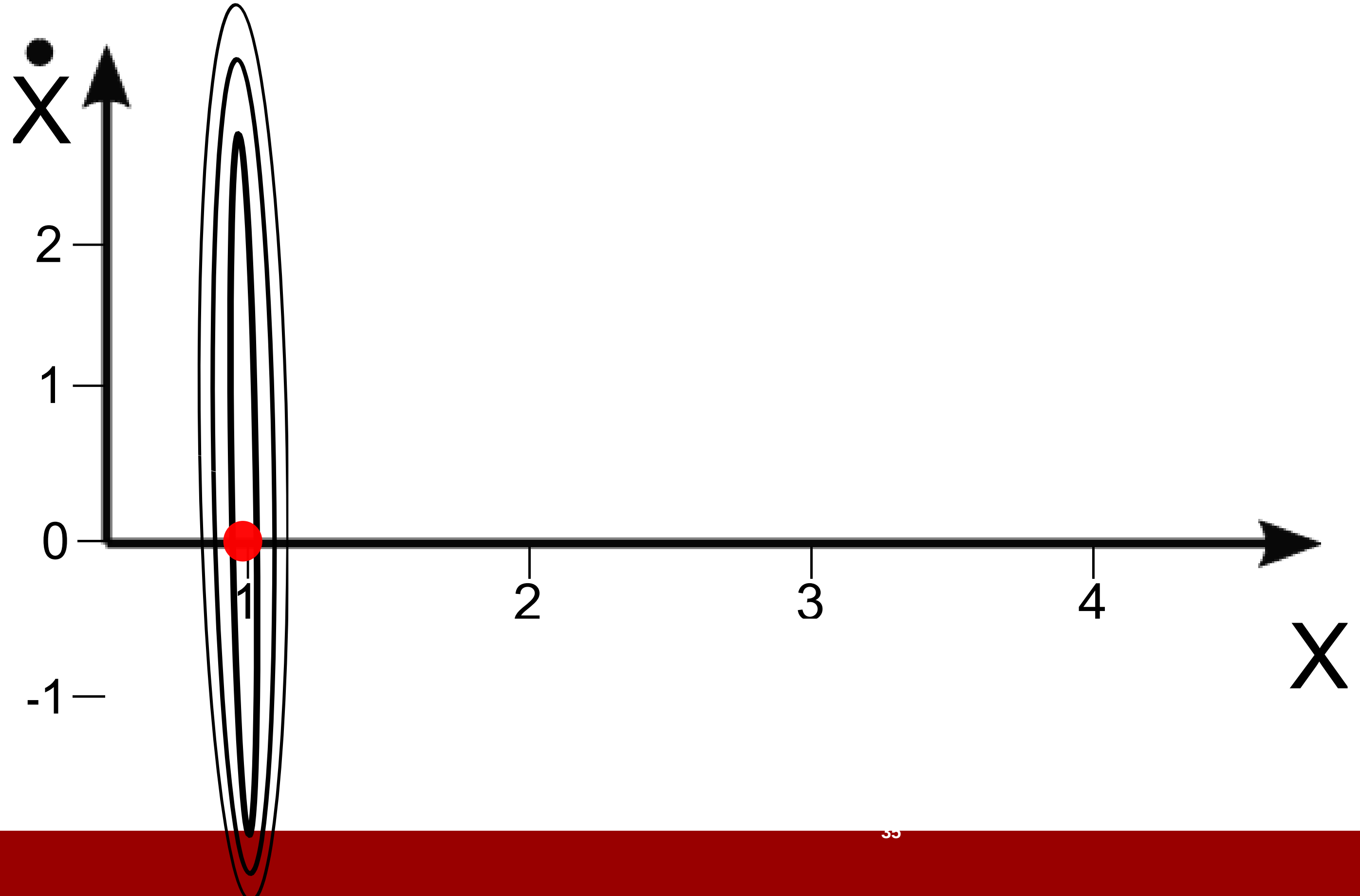

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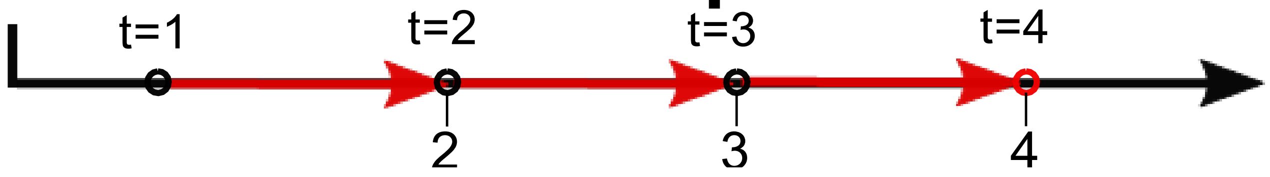

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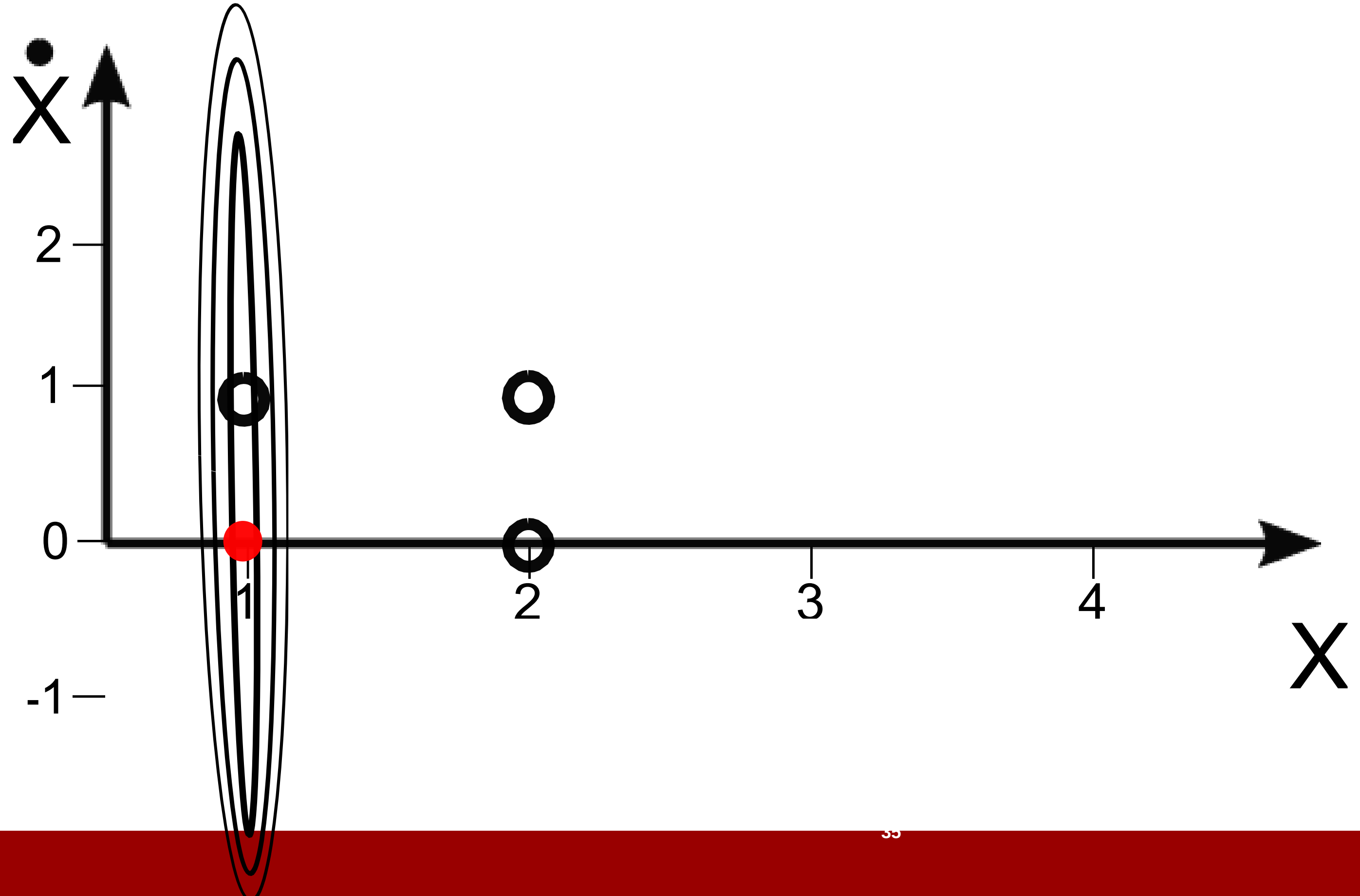

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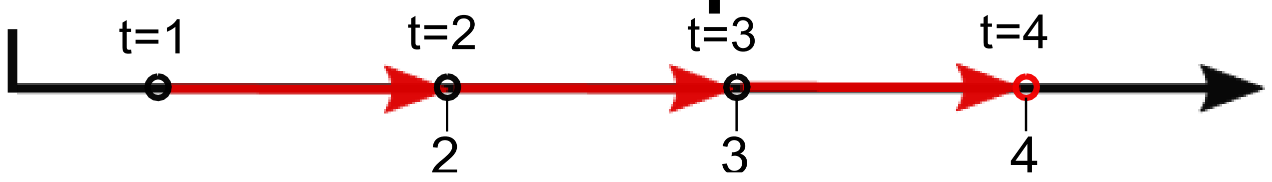

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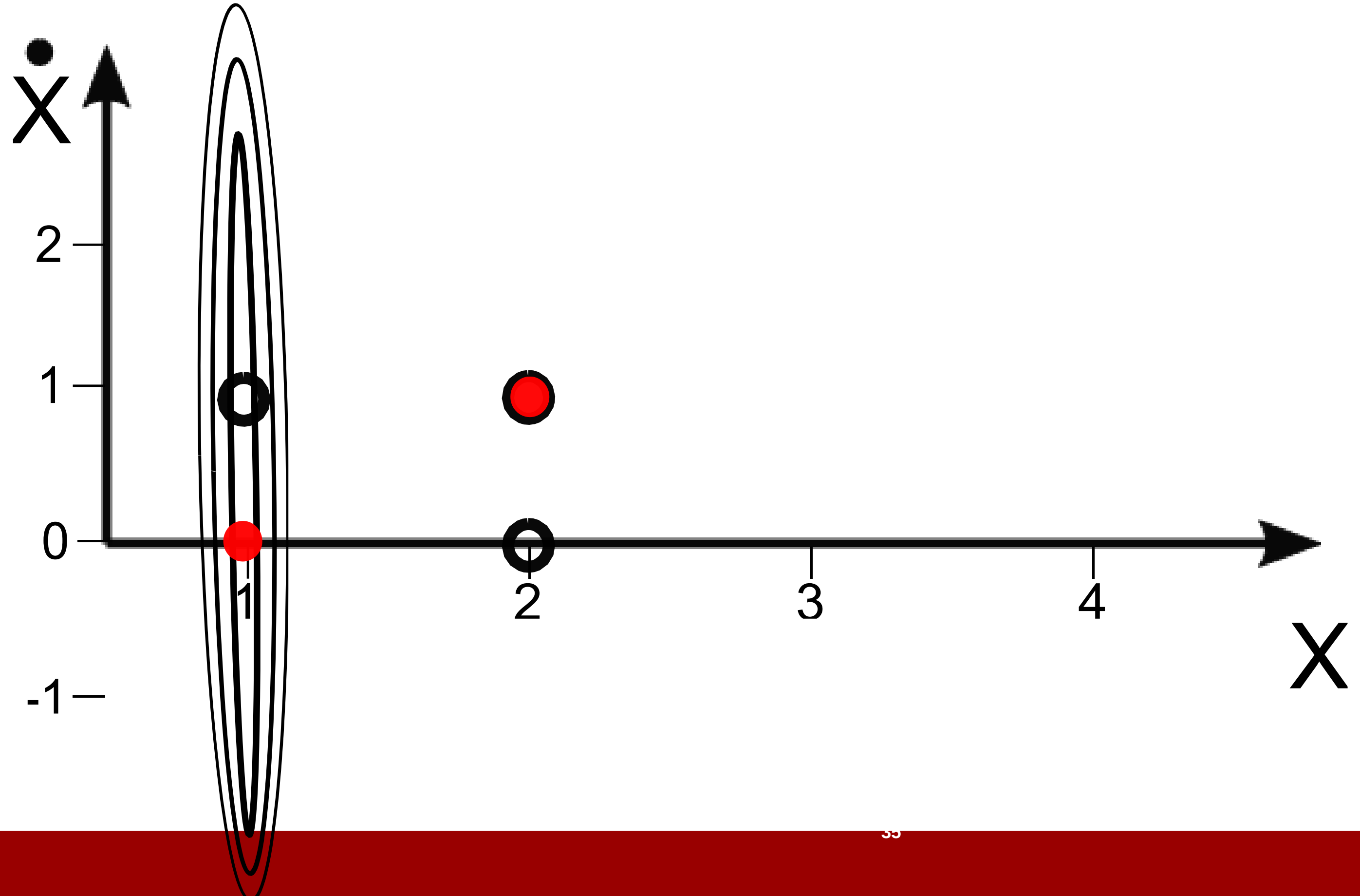

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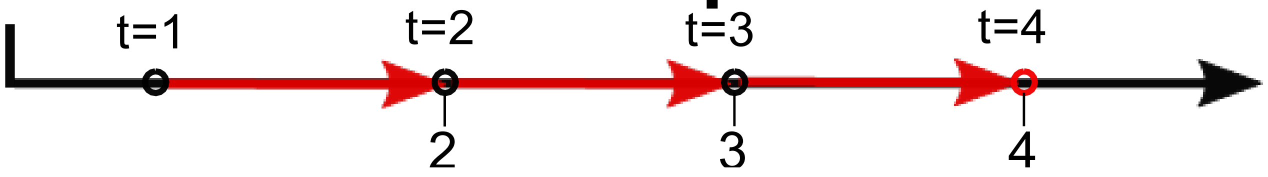

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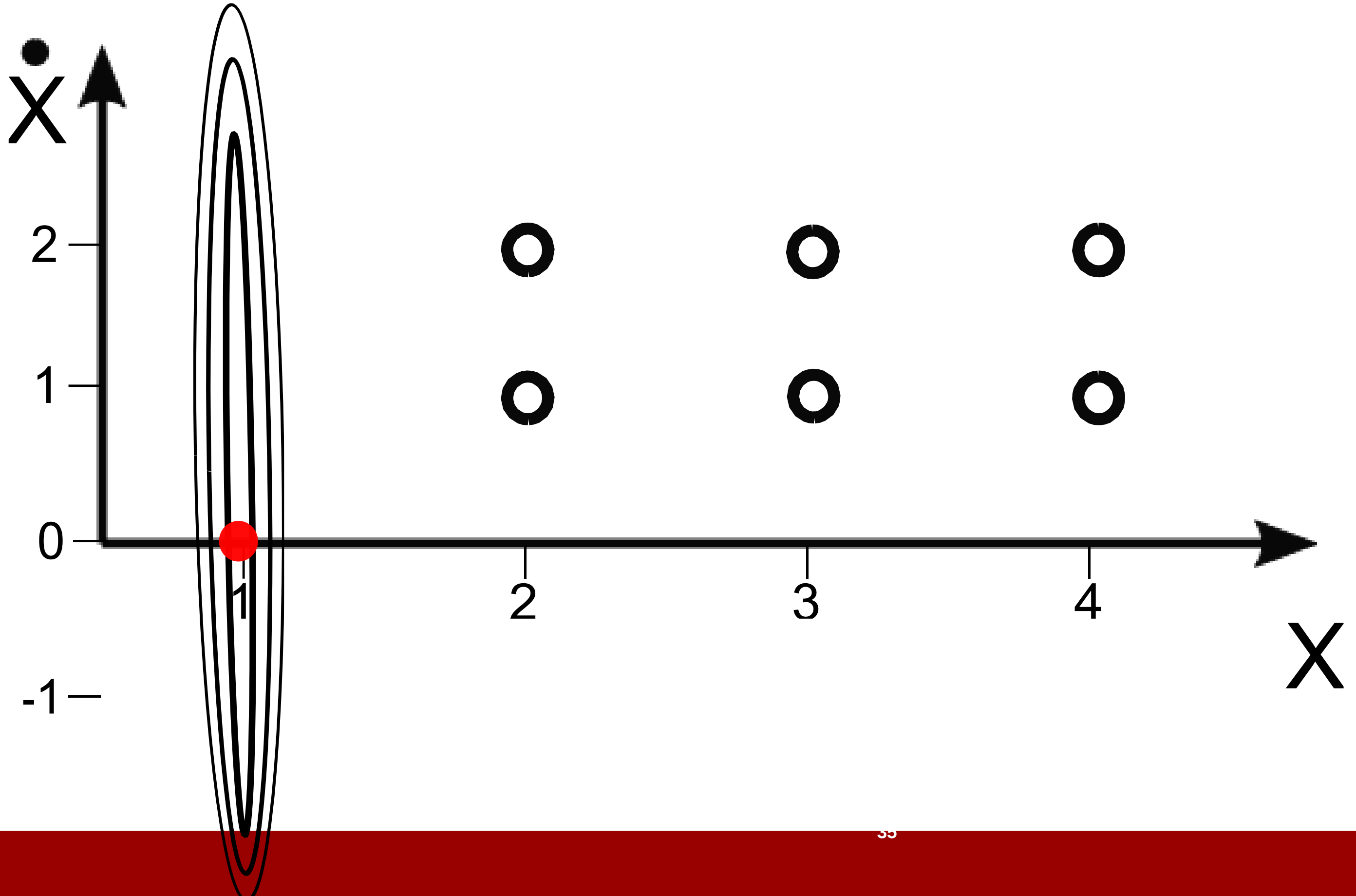
- Prior:
  - Location: 1
- Prediction:
  - Velocity: 0
  - Velocity: 1
  - Velocity: 2



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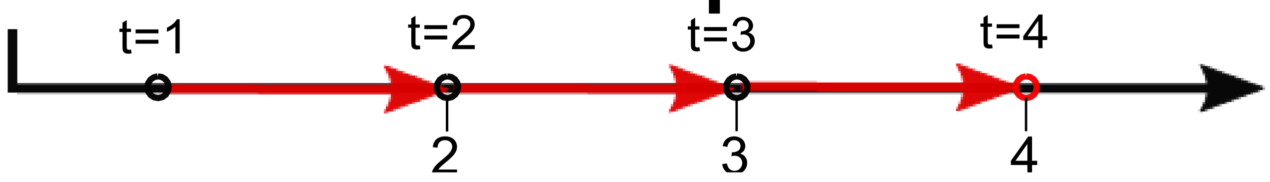

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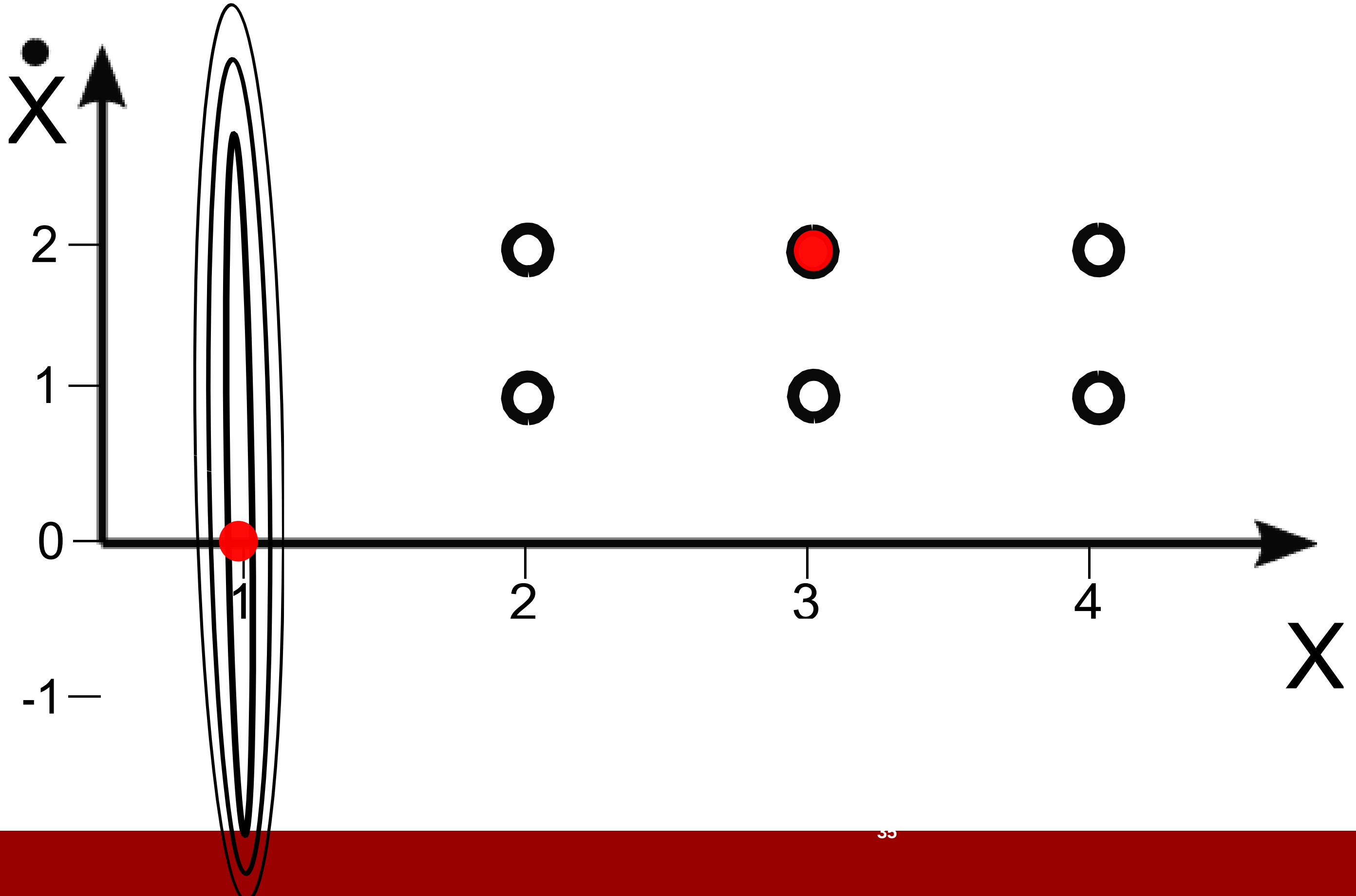

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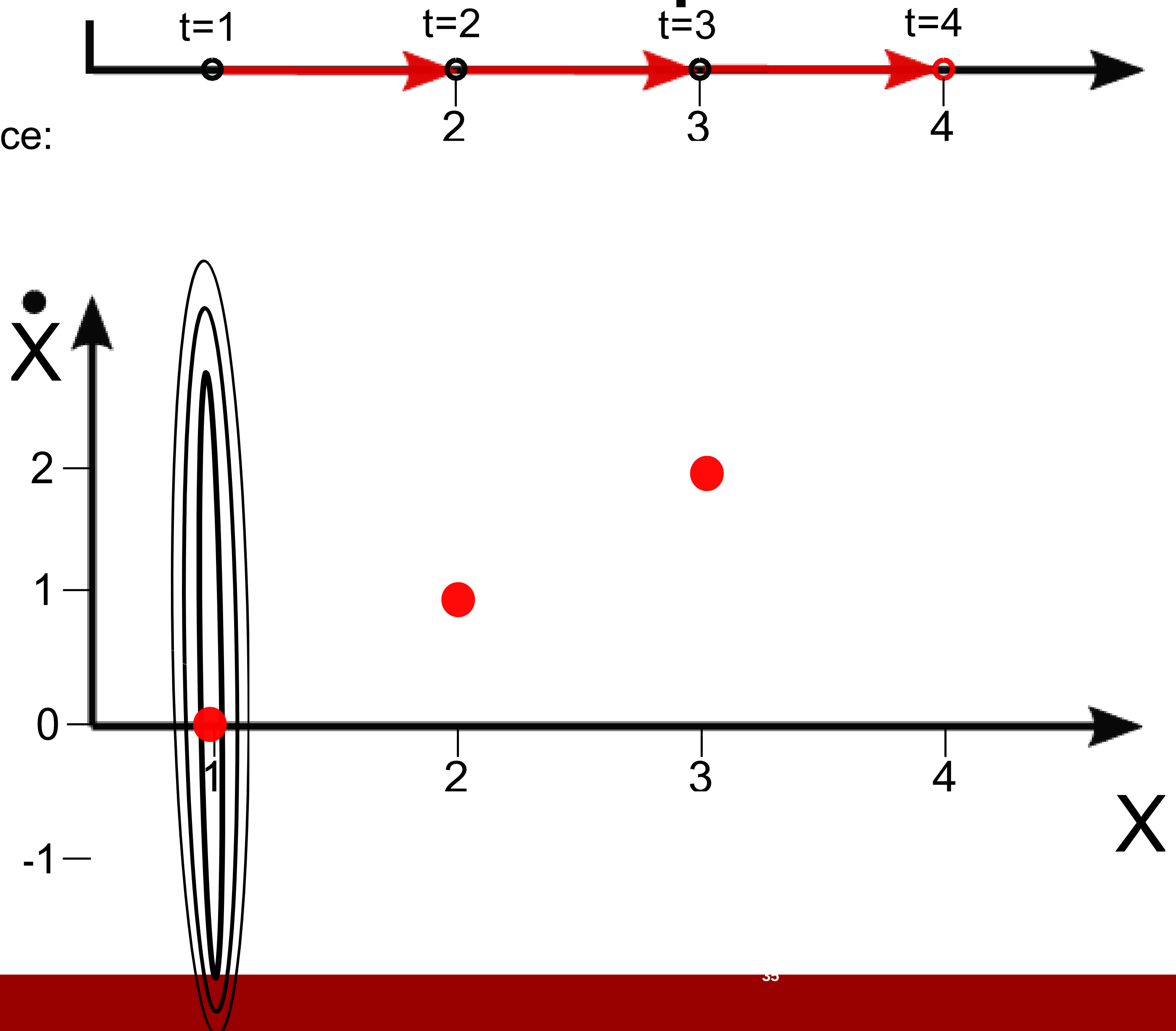

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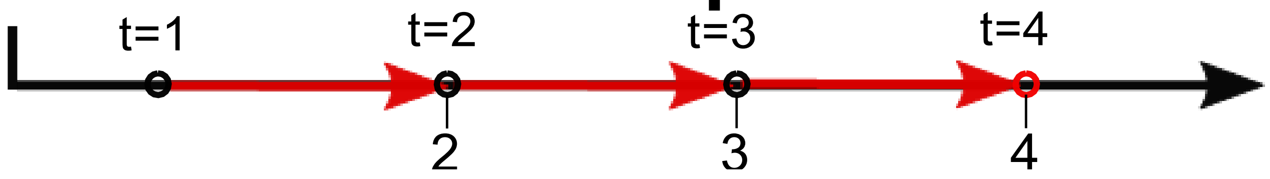

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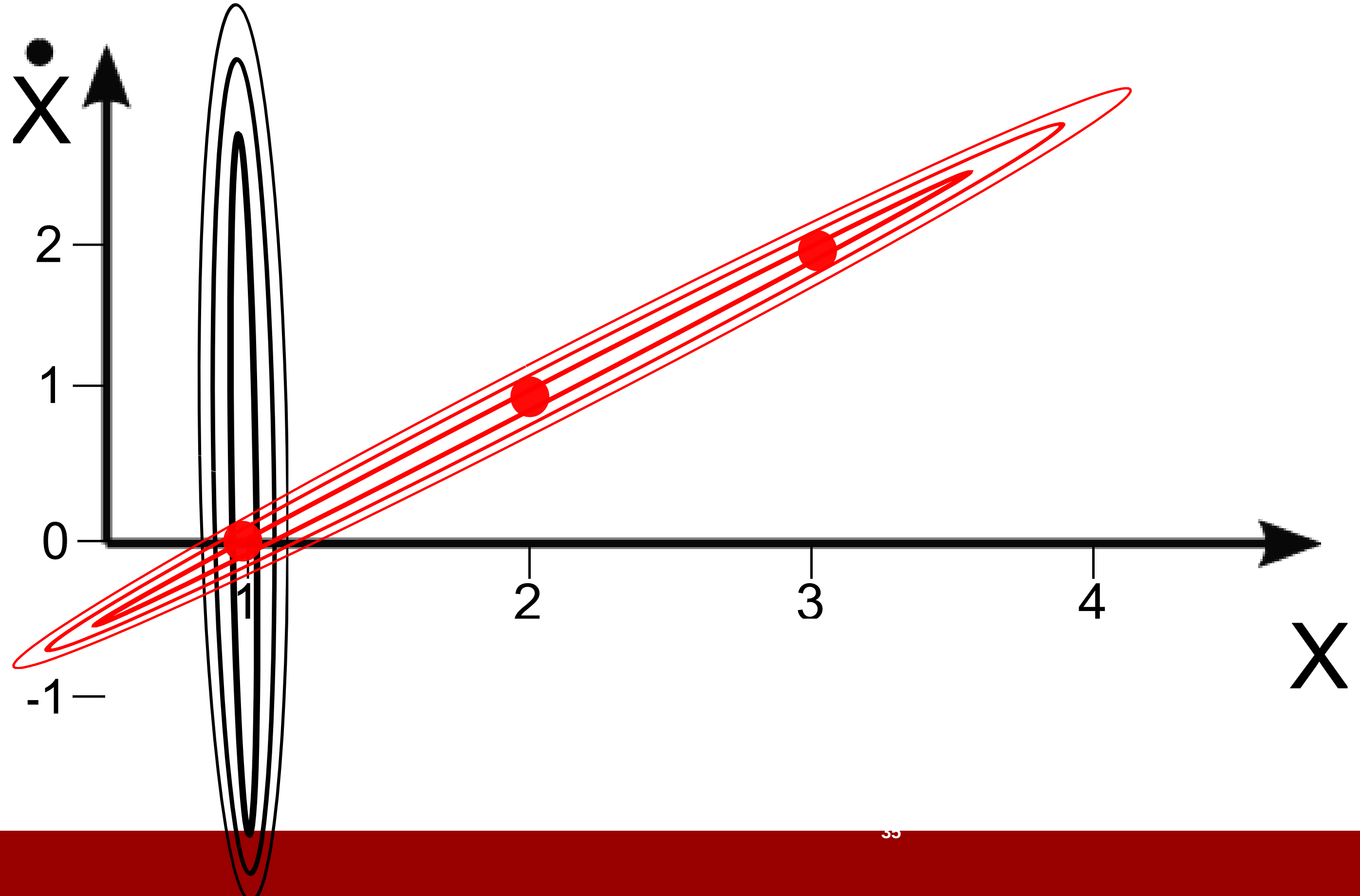
Consider this Gaussian



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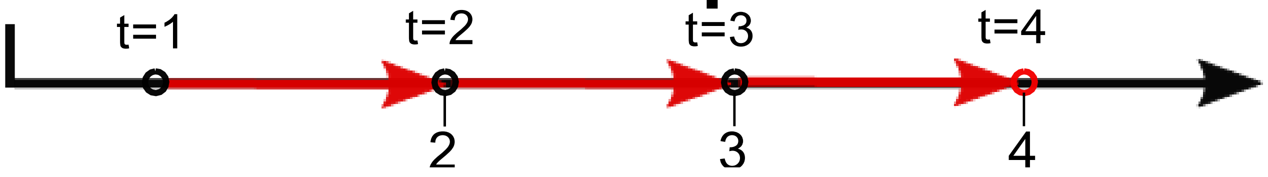

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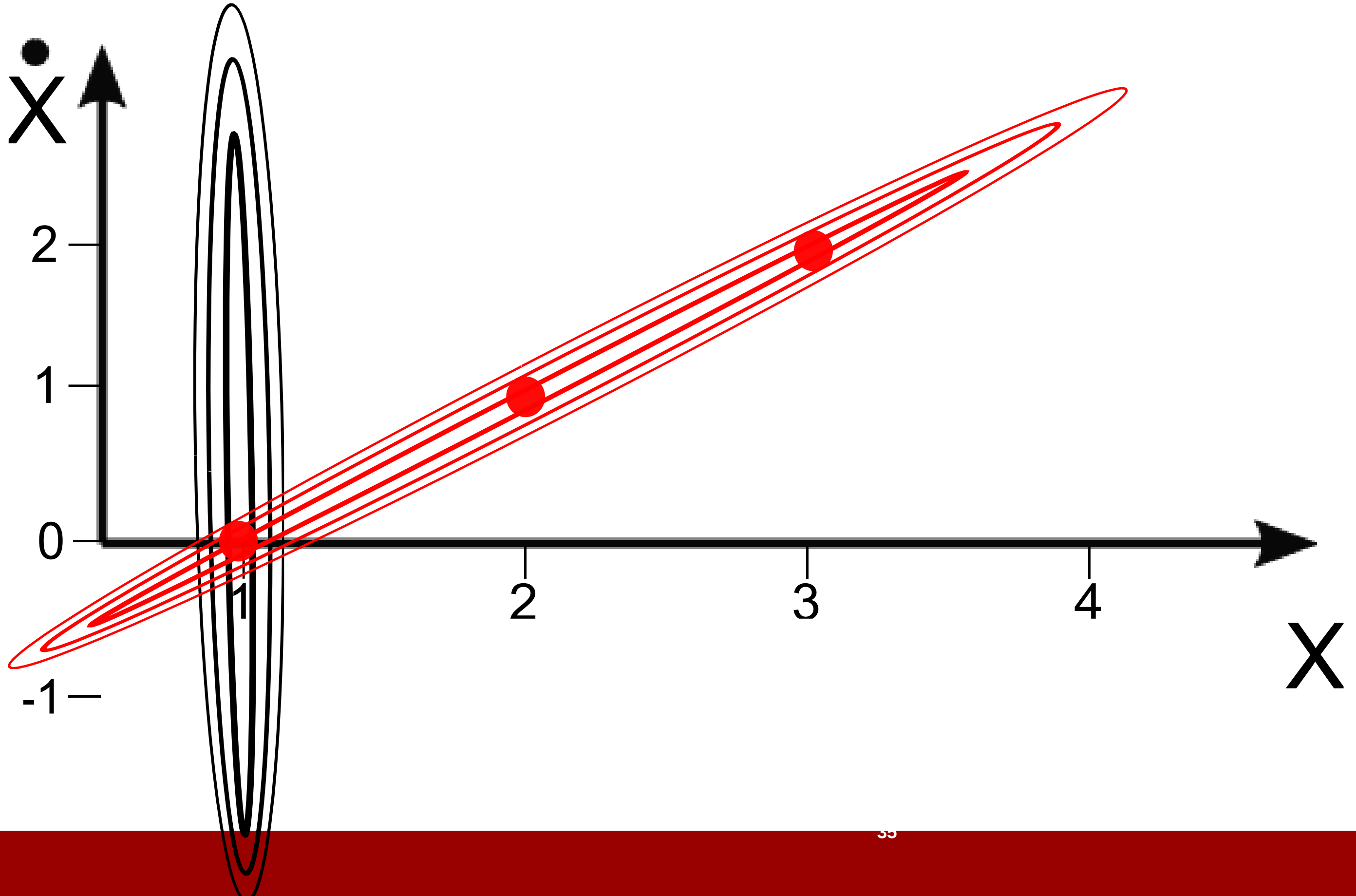
- Measurement:
  – Z = 2



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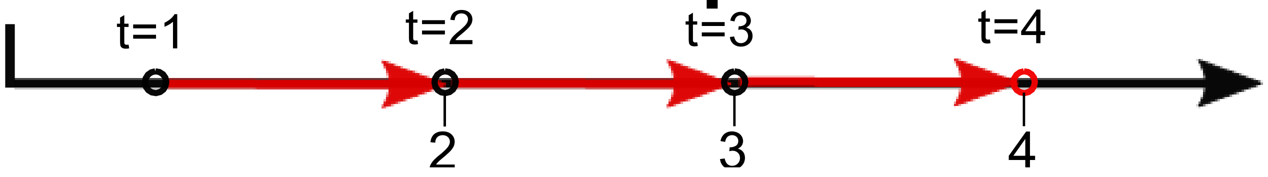

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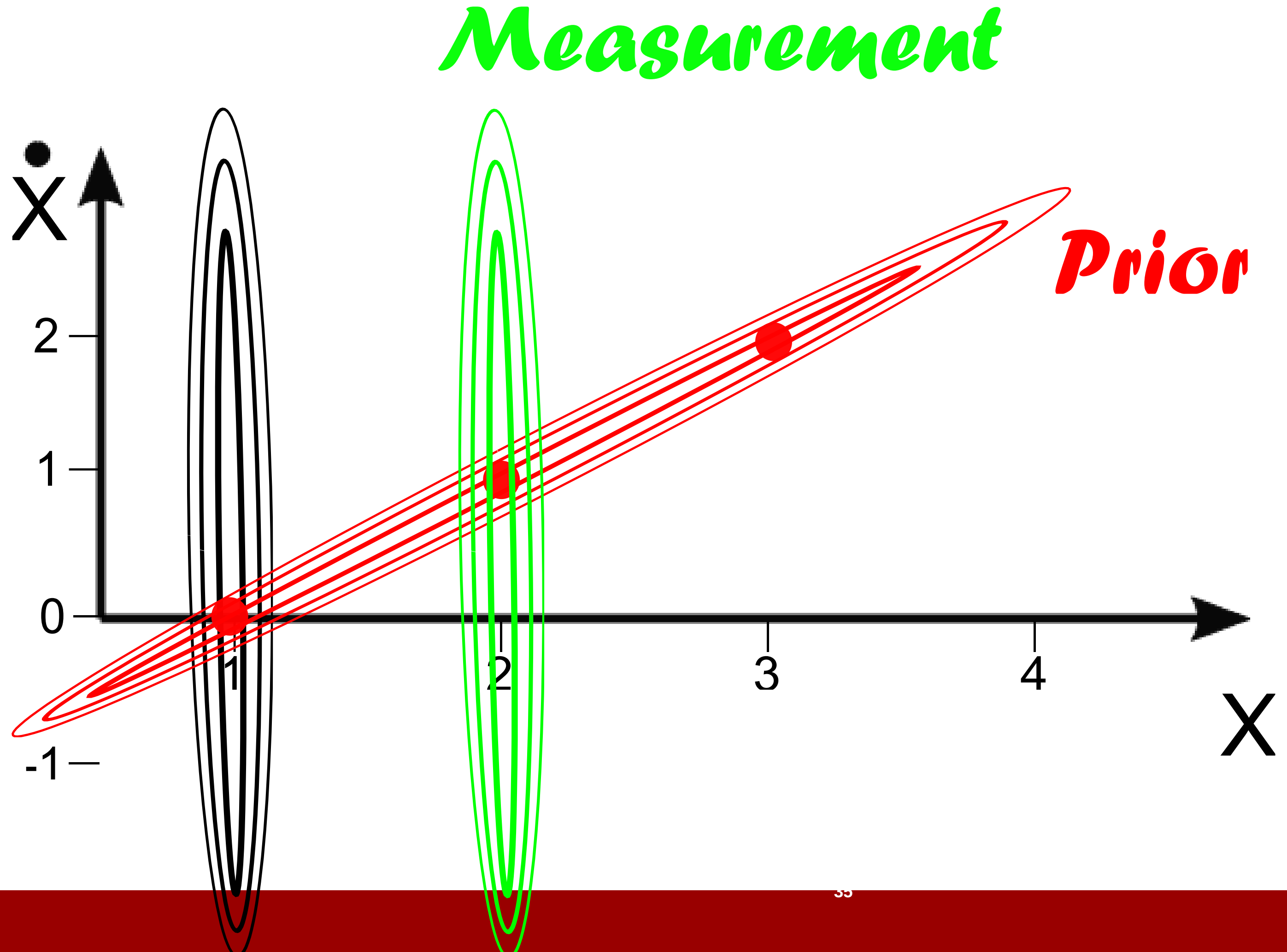

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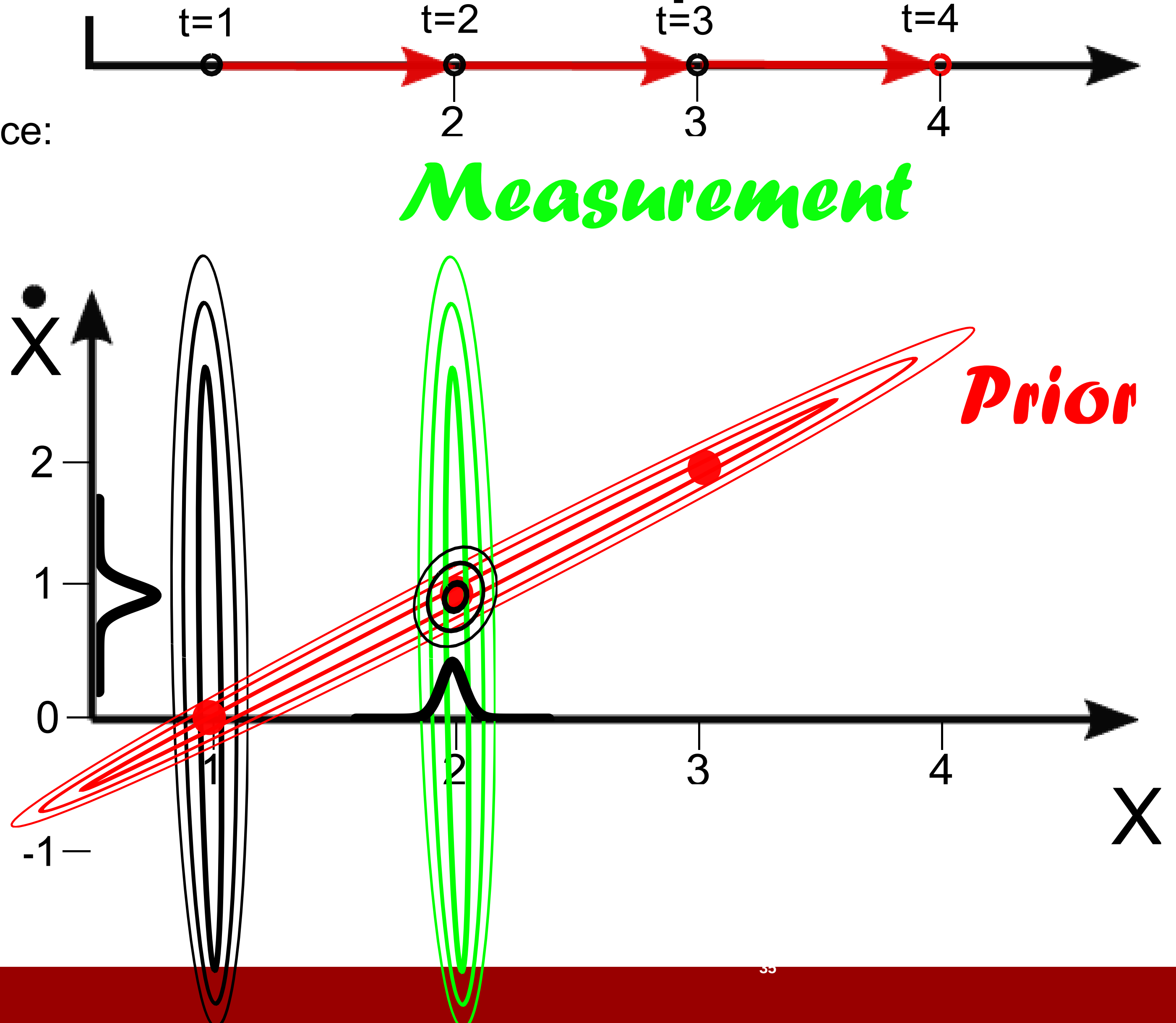
- Measurement:
  - Z = 2

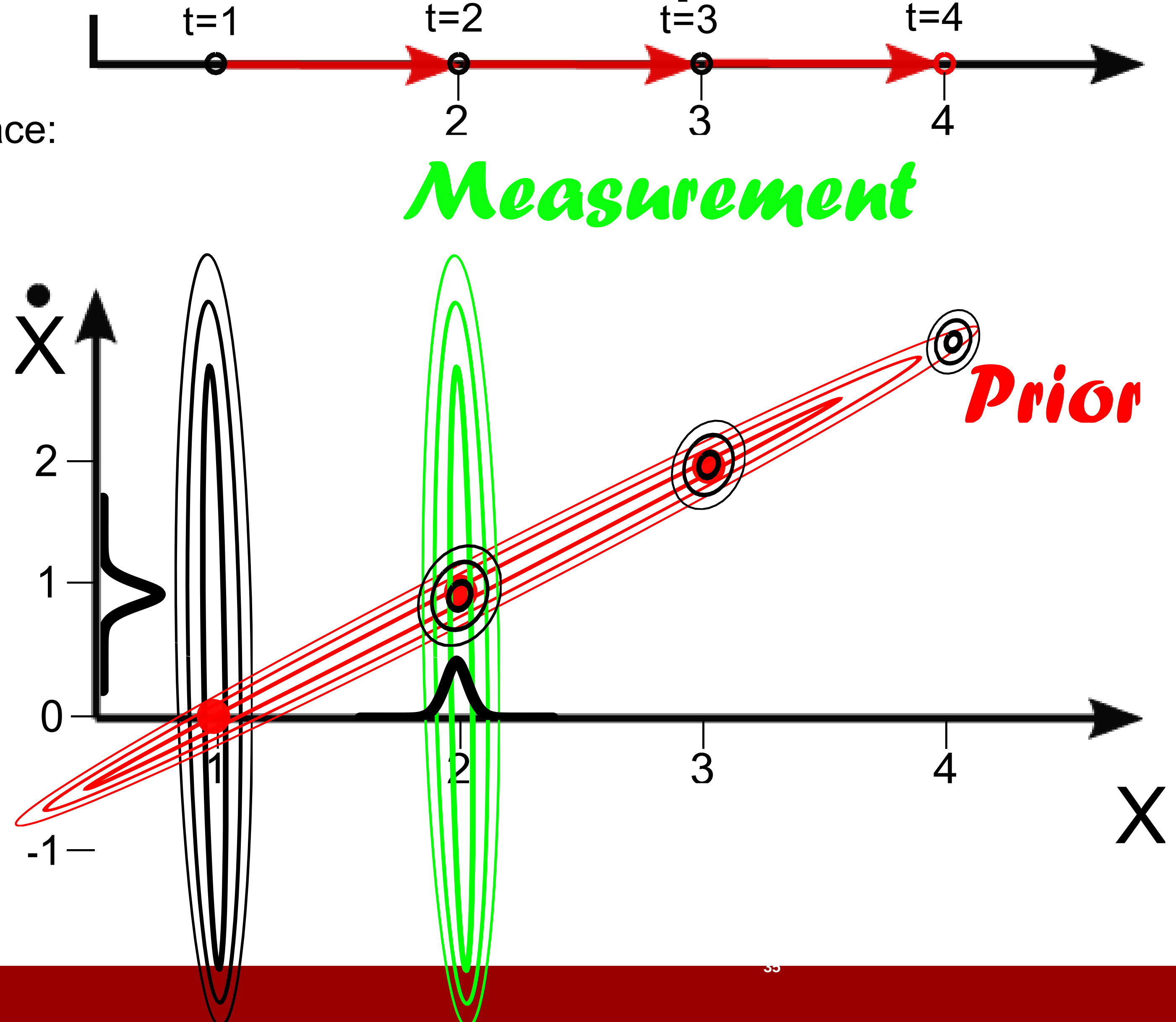


Multivariate Gaussians - Kalman State Space

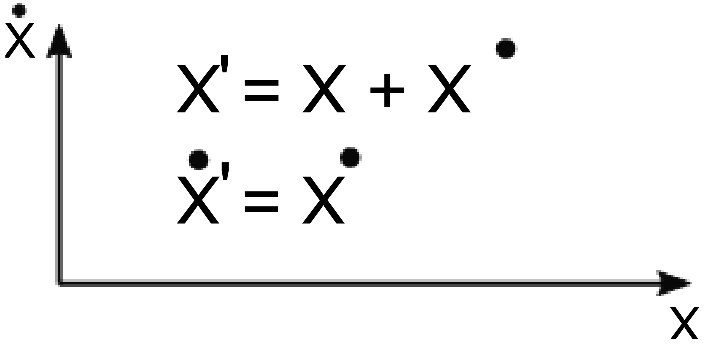
- Let's go at the Kalman state space:
- Prior:
  - Location: 1
- Prediction:
  - Velocity: 0
  - Velocity: 1
  - Velocity: 2

Consider this Gaussian

- Measurement:
  - Z = 2



Design of a Kalman Filter



$$x' = x + \dot{x}$$
$$\dot{x}' = \dot{x}$$

- Two types of States variables:
  - Observables
  - Hidden
- State Transition Function:
  - Matrix
- Measurement Function:
  - Vector (Usually be Matrix)

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

$$\begin{pmatrix} x' \\ \dot{x}' \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

$$\begin{pmatrix} z' \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

Linear Algebra Formulation For For Kalman Filter

(No need to memorize these:)

Prediction (Ingredients):

- X: State Vector (Including our Prior Info)
- P: Uncertainty Covariance (Incl. Prior Info)
- F: State Transition Matrix (we just discussed it)
- u: External Motion (E.g. Deceleration from car)

Measurement Update (Ingredients): **:**

- Z: Measurement
- H: Measurement Matrix
- R: Measurement Noise
- y: Error
- K: Gain
- I : Identity Matrix

Recipe

$$X' = F X + u$$

$$P' = F \cdot P \cdot F^T$$

$$y = Z - H \cdot X$$

$$S = H \cdot P \cdot H^T + R$$

$$K = P \cdot H^T \cdot S^{-1}$$

$$X' = X + K \cdot Y$$

$$P' = (I - K \cdot H) \cdot P$$

Conclusion

- We’ve acquired an amazing skill!!!
- We now understand what state estimation is
- We understand what a covariance represents
- We know how to track objects in space

  (We can handle even occlusions)
- We know how to estimate our position (attitude) over time

- We’ll come back on this (project) to do an end-to-end example!

Perception for Autonomous Systems 31392:

State Estimation - Kalman Filter

Lecturer: Evangelos Boukas—PhD