Derivation of the $Q\bar{Q}$ Potential in $AdS_5 \times S^5$: A Study of Confinement in $\mathcal{N} = 4$ Super Yang-Mills

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Abstract

The quark-antiquark potential is derived using the Wilson Loop approach in the $AdS_5 \times S^5$ background of $\mathcal{N} = 4$ Super Yang-Mills theory. The resulting energy scales linearly for large quark separation L, indicating confinement. For small L the potential is found to be non-confining, agreeable with conformal field theory.

Keywords: Confinement, inter-quark potential, string theory, Wilson Loops, Holography

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1 Introduction

In the mid-1960s, a group of physicists began working on a theory of hadrons — the particles that make up atomic nuclei — in an effort to replicate the success of Quantum Electrodynamics (QED), the quantum field theory that accurately describes interactions between photons and charged fermions. Their goal was to construct a framework that could explain the internal structure and dynamics of hadrons, effectively formulating a theory of the strong nuclear force [1].

Around the same time, the idea of quarks was proposed — not initially as physical particles, but rather as abstract mathematical entities introduced to account for the observed symmetries and quantum numbers of hadrons. In the early 1970s, deep inelastic scattering experiments at SLAC [1] demonstrated that protons have point-like constituents which were interpreted as quarks. Despite the fact that the existence of quarks was implied by the SLAC experiments, no isolated (free) quarks have ever been observed. This is the problem of confinement, and dates back to the 1960s, when Gell-Mann and Zweig attempted to explain the classification of hadrons with the *Eightfold Way*[1].) Earlier models such as Yukawa's meson exchange [2] could describe nuclear forces, but were unable to formulate a consistent theory describing quark confinement.

Later on in the 1970s, a non-Abelian SU(3) gauge theory with quarks and gluons was proposed called Quantum Chronodynamics. This theory was successful for two important reasons: (i) it explains why quarks behave like free-particles at high energies, and (ii) it explains why quarks are never seen as isolated particles at low energies [3]. Despite the success of QCD describing quark behavior at high and low energies, no complete proof of confinement has been found. At low energies, quark-gluon interactions become extremely strong (coupled) making analytical methods like perturbation theory extremely difficult. This was solved by formulating Lattice QCD, which discretizes spacetime (into a "lattice"), and performs the complex calculations needed for QCD on powerful computers.

During the next couple of years, many new hadrons were discovered in particle accelerators around the globe (including the PS at CERN), and it was noticed that when plotting the mass-squared versus angular momentum of these particles, the relationship was approximately linear.

This observation - known as a Regge Trajectory - indicated that these particles can be modeled as rotating, string-like objects, where higher spin states correspond to higher masses.

In 1968, Gabriele Veneziano postulated an analytic formula to describe the scattering amplitudes associated with Regge Trajectories:

$$\mathcal{A}(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{1.0.1}$$

where $\alpha(s)$ and $\alpha(t)$ are the Regge Trajectories in the s- and t-channels respectively, for which Veneziano postulated the linear form

[1] The Eightfold Way is way of classifying hadrons based on their fundamental properties. Particles are ordered into multiplets based on their charge, strangeness, and isospin. The classifications utilises the flavour symmetry of SU(3) group theory.

$$\alpha(s) = \alpha(0) + \alpha' s \tag{1.0.2}$$

Here, α' is the Regge slope and $\alpha(0)$ is the intercept.

Veneziano's celebrated formula was a ground-breaking discovery that led to early models of duality - the idea that the same scattering amplitude could be described by different particle exchange channels - and, eventually, to string theory. String theory was initially developed as a model of the strong force, but when QCD was introduced, strings fell out of favor [1]. There were many reasons for the sudden loss of interest in string theory but the principle ones were that it demanded an unrealistic number of spacetime dimensions, and the existence of a spin-2 massless boson. It was only in 1974, when Scherk and Schwarz [4] decided to take string theory at face value and consider the spin-2 particle to represent the graviton, thus opening entirely new avenues of research into quantum gravity and string theory.

The goals of this project are to derive the quark-antiquark potential, for small and large quark separation L, using an object called a Wilson Loop Operator, and to prove analytically that the potential goes as

$$E \sim \begin{cases} -1/L, & \text{for small } L \\ \sigma L, & \text{for large } L \end{cases} \tag{1.0.3}$$

where, σ is a constant to be found, and $E \sim \sigma L$ indicates confinement.

2 Background

2.1 Discussion

The evolution of physics is dotted with unifications. In the 1600s, Newton formulated a theory of gravity unifying planetary motion with gravitational effects on Earth. Around 2 centuries later, Maxwell unified light, electricity and magnetism into Electromagnetism. In 1905, Einstein formulated his groundbreaking theory of special relativity, providing a framework for classical dynamics and electrodynamics. (He later won the 1921 Nobel Prize in physics, but oddly enough not for his work in special relativity, but for his discovery of the photoelectric effect, a crucial part of establishing quantum theory in the early 20th century.) In 1915, Einstein presented his theory of General Relativity to the Prussian Academy of Sciences [5], demonstrating that gravity arises from the curvature of spacetime due to the effects of mass and energy.

In the 1960s, Weinberg and others proved that electromagnetism and the weak nuclear force are really two aspects of a single force, dubbed the electroweak theory, which was experimentally verified with the discovery of the W and Z boson at the LHC, CERN in 1983. Finally, in the 1970s, the strong nuclear force was married to the weak and electromagnetic forces in what became the Standard Model of Particle physics, with the latest discovery being the Higgs boson in 2012.

String theory is widely regarded as one of the most promising frameworks which unify quantum gravity with gauge interactions and matter [6]. The origins of string theory can be traced back to Veneziano's proposal of a simple expression for the scattering amplitude of a $2 \to 2$ interaction (1.0.1).

String theory postulates that fundamental particles are not treated as point-like but as one-dimensional objects called strings, where higher vibrational modes correspond to higher mass. An incredible feature of string theory is that the number of spacetime dimensions is not arbitrary - it in fact emerges from a calculation (which will not be explored here). For superstring theory, which describes both fermions and bosons, this number is 10; for bosonic string theory, which deals only with bosons, it is 26. These extra dimensions are compactified (rolled up).

This section covers all the background physics relevant to the project. In Section 2.2, we present a proof of why string theory is able to explain why certain groups of particles sit on a linear slope (called a Regge trajectory) when plotting the experimentally derived mass-squared and spin values. In Section 2.3, we discuss scattering amplitudes and present the intuition behind the Veneziano amplitude. Section 2.4 is a discussion on non-relativistic strings. Section 2.5 is a brief overview of symmetries and classical Lagrangian mechanics where the concept of the action is introduced. In Section 2.6, the Wilson Loop Operator is defined, and its connection to the string world-sheet is explored. Section 2.7 isa discussion on relativistic strings where the Nambu-Goto action is derived.

2.2 Regge Trajectories

Here we provide a proof for the $M^2 \propto J$ relationship that was discovered in the 1960s when plotting the mass-squared and spin values of a family of mesons, and that string theory is, so far, the only theory that can explain this relationship.

We start with a string of length L, mass $M = T_0 \cdot L$ (where T_0 is the string tension), rotating with angular velocity v. The spin is given as

$$J = Mv\frac{L}{2} = Mc\frac{L}{2} \tag{2.2.1}$$

where we took $v = c$, so the string is rotating at the speed of light (relativistically). With $L = M/T_0$, the spin becomes

$$\begin{aligned} J &= \frac{Mc}{2} \cdot \frac{M}{T_0} \\ &= \frac{1}{2}\frac{M^2 c}{T_0} \end{aligned} \tag{2.2.2}$$

$$\therefore \quad M^2 = \frac{2T_0}{c} J \tag{2.2.3}$$

and since T_0 is a constant,

$$M^2 \propto J \tag{2.2.4}$$

It is therefore clear that linear Regge trajectories emerge directly from a calculation involving a rotating string, indicating that string theory is the only model (that we know of) that can explain this relationship.

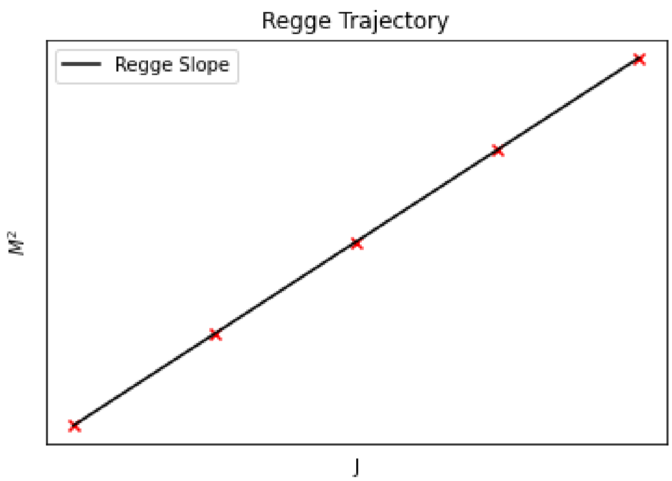

Figure 1: Regge Trajectory

2.3 Scattering Amplitudes

Consider an elastic scattering process involving 4 fermions. In Quantum Field Theory, a one-loop tree diagram can be expressed as a infinite sum of tree diagrams, each representing a different 'channel' of the interaction.

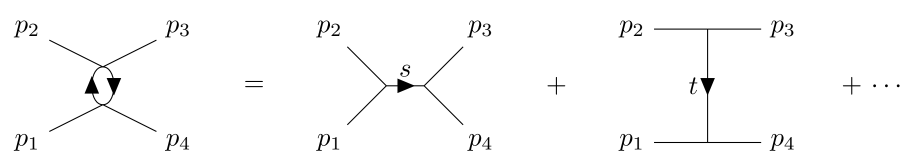

Figure 2: An elastic scattering process written in terms of the s- and t-channel diagrams. (This figure was made using the TikZ-Feynman LaTeX package)

The leading contributions to the scattering amplitude are the s, t, and u channels (see Figure 2), defined as

$$
\begin{aligned}
s &= -(p_1 + p_2)^2 \\
t &= -(p_2 + p_3)^2 \\
u &= -(p_1 + p_3)^2
\end{aligned} \tag{2.3.1}
$$

which are the Mandelstam variables. These obey the identity

$$s + t + u = \sum m_i^2 \tag{2.3.2}$$

The scattering amplitude $\mathcal{A}$ is then written as a sum of the three channels:

$$\mathcal{A}(s, t, u) = \mathcal{A}_s + \mathcal{A}_t + \mathcal{A}_u \tag{2.3.3}$$

Consider the t-channel diagram in Figure 2. Let us denote the external particles as ϕ, and the virtual, exchanged particle as σ. If σ is of spin 0, we can generally right the scattering amplitude as

$$\mathcal{A}(s, t) = -\frac{g^2}{t - M^2} \tag{2.3.4}$$

Suppose, however, that the particle has spin J, and taking the external particles to be scalars, the amplitude becomes [7]

$$\mathcal{A}_J(s, t) = -\frac{g^2(-s)^J}{t - M^2} \tag{2.3.5}$$

for high energies. This new amplitude now contains s-channel dependence due to the crossing symmetry of the $s-$ and $t-$channels (i.e. one amplitude can describe both the annihilation process in the s-channel, and the exchange process in the t-channel). We must also consider that there may be other strongly interacting particles of varying mass and spin that are exchanged in the $t-$channel, so in fact we must sum up all the possible particles exchanged in the $t-$channel:

$$\mathcal{A}(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - M_J^2} \tag{2.3.6}$$

We can also construct an amplitude analogous to (2.3.6) but containing $s-$channel poles rather than $t-$channel poles:

$$\mathcal{A}'(s,t) = -\sum_J \frac{g_J^2(-t)^J}{s - M_J^2} \tag{2.3.7}$$

Now, if the couplings g_J and masses M_J are chosen such that $\mathcal{A}(s,t)$ and $\mathcal{A}'(s,t)$ are equal, we could write the entire amplitude as a sum over only $s-$channel poles or only $t-$channel poles. This equality was first postulated in 1968 by Dolen, Horn, and Schmid [8, 7], who argued that the equality $\mathcal{A}(s,t) = \mathcal{A}'(s,t)$ was approximately true for *small* values of s and t. This was known as the duality hypothesis, and there was a lot of discourse at this time around whether this is an approximation, or a principle.

It seems almost impossible to choose values of g_J and M_J that obey this equality, but this is where Veneziano enters. In the Introduction, we mentioned his infamous formula for the scattering amplitude (1.0.1), but for the benefit of the reader, we will write it again here:

$$\mathcal{A}(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \tag{2.3.8}$$

where Γ is the Euler gamma function

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt \tag{2.3.9}$$

and of course $\alpha(s)$ is the Regge slope (1.0.2). The Veneziano amplitude is widely considered the birth of modern string theory, since it gave rise to decades of research on models of duality, and enabled the strong force to be well described by the mathematics of string theory.

2.4 Non-relativistic strings

In this section we take a brief moment to discuss non-relativistic strings, and basic string dynamics.

We need to define two new parameters; string tension, T_0, and mass per unit length, μ_0. Tension has units of force, thus it has units of $\frac{\text{Energy}}{\text{Length}}$:

$$
\begin{aligned}
[T_0] &= \frac{[Energy]}{L} \\
&= \frac{M}{L}[v]^2 \\
&= [\mu_0][v]^2
\end{aligned}
\tag{2.4.1}
$$

where T_0 and μ_0 are adjustable parameters.

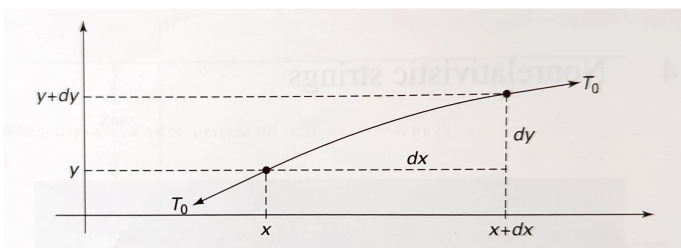

Figure 3: A short piece of classical non-relativistic string vibrating transversely. [9]

We wish to derive an equation of motion. First, we must assume that oscillations along the string are small compared to the length of the string, l_s, i.e.

$$\frac{\partial y}{\partial x} \ll 1 \tag{2.4.2}$$

and that this holds for all points along the string. Next, consider the string stretching an infinitesimal amount, i.e.

$$
\begin{aligned}
y &\to y + dy \\
x &\to x + dx
\end{aligned} \tag{2.4.3}
$$

At any given time t, the transverse displacement is $y(t,x)$ at x and $y(t,x{+}dx)$ at $x+dx$. In order to obtain an equation of motion, we need to compute the force on the string. Since we're dealing with transverse oscillations we need only consider the *net vertical force*, and since $\frac{\partial y}{\partial x}$ is dimensionless, we characterise the force as

$$
T_0 \frac{\partial y}{\partial x} \tag{2.4.4}
$$

and, at $x + dx$ this force is pointing upwards, and at x it is pointing downwards (see Figure 3). Thus, the net vertical force is

$$
dF_v = T_0 \frac{\partial y}{\partial x}\bigg|_{x+dx} - T_0 \frac{\partial y}{\partial x}\bigg|_{x} \tag{2.4.5}
$$

Given (2.4.2), we can Taylor expand around x:

$$
dF_v = T_0 \left(\frac{\partial y}{\partial x}\bigg|_{x} + \frac{\partial^2 y}{\partial x^2} dx + \mathcal{O}(dx^2) \right) - T_0\, \frac{\partial y}{\partial x}\bigg|_{x} \tag{2.4.6}
$$

So, by neglecting the terms of order dx^2 and higher, we have

$$
dF_v = T_0 \frac{\partial^2 y}{\partial x^2} dx \tag{2.4.7}
$$

Newton's second law tells us that $dF = mda$, and here $m = \mu_0 dx$, so

$$
T_0 \frac{\partial^2 y}{\partial x^2} dx = (\mu_0 dx) \frac{\partial^2 y}{\partial t^2} \tag{2.4.8}
$$

and, canceling dx on both sides, we have arrived at our equation of motion:

$$T_0 \frac{\partial^2 y}{\partial x^2} = \mu_0 \frac{\partial^2 y}{\partial t^2} \tag{2.4.9}$$

This is analogous to the wave equation in 2D,

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2} \frac{\partial^2 y}{\partial t^2} = 0 \tag{2.4.10}$$

where v_0 is the velocity of transverse waves, so for our string

$$\boxed{v_0 = \sqrt{\frac{T_0}{\mu_0}}} \tag{2.4.11}$$

(2.4.11) suggests that, for a relativistic string, T_0 and μ_0 are related by

$$T_0 = \mu_0 c^2 \tag{2.4.12}$$

2.5 Lagrangian Mechanics

In Quantum Mechanics, symmetries are linked to conservation laws, and for each conservation law there is a conserved quantity. Spacial translation symmetry results in conservation of linear momentum. Rotational translation symmetry results in conservation of angular momentum. Time translation symmetry results in conservation of energy.

In this section, a brief overview of classical Lagrangian mechanics is provided, where it is shown that an equation of motion for a particle can be derived by an infinitesimal variation of the path of least action.

Consider a particle of mass m moving along the x-axis under a potential $V(x)$. The Lagrangian is given by

$$L = T - V \tag{2.5.1}$$

where T is kinetic energy, and V is potential energy, and

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}(t))^2 \\
V &= V(x(t))
\end{aligned}
\tag{2.5.2}
$$

where $\dot{x}(t) \equiv \frac{dx(t)}{dt}$.

So,

$$
L(t) = \frac{1}{2}m(\dot{x}(t))^2 - V(x(t)) \tag{2.5.3}
$$

Now, the action is defined as

$$
S = \int L(t)dt \tag{2.5.4}
$$

Note that the action is a *functional*, meaning it takes a function as an argument and returns a number. Substituting (2.5.3) into (2.5.4), we have

$$
S[x] = \int_{t_i}^{t_f} \left\{ \frac{1}{2}m\dot{x}^2 - V(x) \right\} dt \tag{2.5.5}
$$

The action S can be calculated for any path $x(t)$, and as such, is a very powerful tool in finding all the paths that can be physically realized, not just those that are *mathematically* possible.

Keeping the endpoints unchanged, i.e.

$$
\begin{aligned}
x_i &= x(t_i) \\
x_f &= x(t_f)
\end{aligned}
\tag{2.5.6}
$$

infinitesimal changes along the path leave the action unchanged:

$$\delta x(t_i) = \delta x(t_f) = 0 \tag{2.5.7}$$

Now, we compute the action $S[x + \delta x]$ for the perturbation $x(t) + \delta x(t)$:

$$\begin{aligned} S[x+\delta x] &= \int_{t_i}^{t_f} \left\{ \frac{m}{2} \left(\frac{d}{dt}(x(t) + \delta x(t)) \right)^2 - V(x(t) + \delta x(t)) \right\} \\ &= S[x] + \int_{t_i}^{t_f} \left\{ m\dot{x}(t) \frac{d}{dt} \delta x(t) - V'(x(t))\delta x(t) \right\} dt + \mathcal{O}((\delta x)^2) \end{aligned} \tag{2.5.8}$$

In the last line of (2.5.8) we expanded V in a Taylor series about $x(t)$. Now, we have $S[x + \delta x]$ written as $S + \delta S$, where δS is

$$\delta S = \int_{t_i}^{t_f} \left\{ m\dot{x}(t) \frac{d}{dt}(\delta x(t)) - V' \delta x(t) \right\} dt \tag{2.5.9}$$

In order to find an equation of motion, we have to get δS in the form $\delta S = \int \delta x(t) dt \{...\}$:

$$\begin{aligned} \delta S &= \int_{t_i}^{t_f} m\dot{x}(t) \frac{d}{dt}(\delta x(t)) dt - \int_{t_i}^{t_f} V' \delta x(t) dt \\ &= \int_{t_i}^{t_f} \frac{d}{dt}(m\dot{x}(t)\delta x(t)) dt - \int_{t_i}^{t_f} m\ddot{x}(t)\delta x(t) dt - \int_{t_i}^{t_f} V' \delta x(t) dt \\ &= m\dot{x}(t)\delta x(t)|_{t_i}^{t_f} + \int_{t_i}^{t_f} \delta x(t)(-m\ddot{x}(t) - V'(x(t))) dt \end{aligned} \tag{2.5.10}$$

Using (2.5.7), this reduces to

$$\delta S = \int_{t_i}^{t_f} \delta x(t)(-m\ddot{x}(t) - V'(x(t))) dt \tag{2.5.11}$$

Recall that earlier we mentioned that, given fixed endpoints, infinitesimal changes leave the action unchanged, implying that $\delta S = 0$, therefore the terms multiplying into $\delta x(t)$ in (2.5.11) must vanish:

$$-m\ddot{x}(t) - V'(x(t)) = 0 \tag{2.5.12}$$

Rearranging:

$$m\ddot{x}(t) = -\frac{d}{dt}V(x(t)) \tag{2.5.13}$$

which is simply Newton's second law for a particle under a potential $V(x)$. So, by varying the action for infinitesimal perturbations, we have recovered the equation of motion.

2.6 Wilson Loop Operators

In gauge theories, physical observables must be gauge-invariant. Since we are trying to calculate the potential between the $Q\bar{Q}$ pair, how do we do so in a gauge-invariant way? A natural way to do this is with Wilson Loops.

In Quantum Field Theory, a Wilson Loop is defined as the trace of a path-ordered exponential of the gauge field A_μ, along a closed contour $\mathcal{C}$:

$$W(\mathcal{C}) = Tr\mathcal{P}\text{exp}\left(i\oint_{\mathcal{C}} A_\mu dx^\mu\right) \tag{2.6.1}$$

where $\mathcal{P}$ is a path-ordering operator. Wilson Loops are global, gauge-invariant operators that, in Yang-Mills theory, represent an infinitely massive quark-antiquark pair in the vacuum.

For a rectangular loop with dimensions T (temporal) and L (spacial), the the expectation value [10, 11] is

$$\langle W(\mathcal{C})\rangle \sim e^{-TE(L)} \tag{2.6.2}$$

where $E(L)$ is the quark-antiquark potential, and T is the total time of the Wilson Loop.

The area law tells us that the expectation value of the Wilson loop can also be written as

$$\langle W(C) \rangle \propto e^{-\sigma A} \tag{2.6.3}$$

In previous work by Maldacena [12], the expectation value of the Wilson loop is related to the world-sheet area S by

$$\langle W(C) \rangle \sim e^{-S} \tag{2.6.4}$$

leading to the relation

$$e^{-TE(L)} = e^{-S} \tag{2.6.5}$$

This will be an important consideration in the coming sections.

2.7 The Nambu Goto Action

In this section, we will derive the form of the Nambu-Goto action that was first proposed in 1970, from the framework of relativistic strings. Let us consider for a moment an open string.

The action associated with the string is proportional to the area of its world-sheet (see Section 2.6). To calculate this area, we first determine an infinitesimal area element on the world-sheet. This requires re-parameterising the surface into a spatial domain rather than a spacetime representation. (Since the action is reparametrisation invariant, we can save ourselves a lot of work by using parameters that simplify the problem)

The parameterized surface is described by

$$\vec{x}(\xi^1, \xi^2) = (x^1(\xi^1, \xi^2), x^2(\xi^1, \xi^2), x^3(\xi^1, \xi^2)) \tag{2.7.1}$$

We need to find an expression for the area element of the target space. To do this, we look at an infinitesimal area on the parameter space, where the sides of the shape are denoted by $d\xi^1$ and $d\xi^2$ (Figure 4).

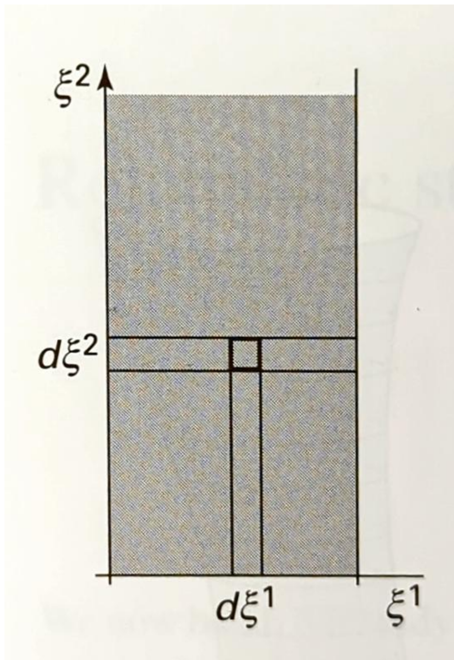

Figure 4: An area of parameter space with dimensions (ξ^1, ξ^2). The infinitesimal area of target space is shown, with side lengths $d\xi^1$ and $d\xi^2$ [9].

Since the world-sheet is not assumed to be perfectly flat, the infinitesimal area of target space is, in general, a parallelogram. This area of target space will have sides $d\vec{v}_1$ and $d\vec{v}_2$, expressed as

$$d\vec{v}_1 = \frac{\partial \vec{x}}{\partial \xi^1} d\xi^1, \qquad d\vec{v}_2 = \frac{\partial \vec{x}}{\partial \xi^2} d\xi^2 \tag{2.7.2}$$

We can then write the area element,

$$dA = |d\vec{v}_1||d\vec{v}_2||sin\theta| = |d\vec{v}_1||d\vec{v}_2|\sqrt{1 - cos^2\theta} \tag{2.7.3}$$

Using spatial dot-products, we have

$$dA = \sqrt{(d\vec{v}_1 \cdot d\vec{v}_1)(d\vec{v}_2 \cdot d\vec{v}_2) - (d\vec{v}_1 \cdot d\vec{v}_2)^2} \tag{2.7.4}$$

Making use of (2.7.2),

$$dA = d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial\vec{x}}{\partial\xi^1} \cdot \frac{\partial\vec{x}}{\partial\xi^1}\right)\left(\frac{\partial\vec{x}}{\partial\xi^2} \cdot \frac{\partial\vec{x}}{\partial\xi^2}\right) - \left(\frac{\partial\vec{x}}{\partial\xi^1} \cdot \frac{\partial\vec{x}}{\partial\xi^2}\right)^2} \tag{2.7.5}$$

Therefore,

$$A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial\vec{x}}{\partial\xi^1} \cdot \frac{\partial\vec{x}}{\partial\xi^1}\right)\left(\frac{\partial\vec{x}}{\partial\xi^2} \cdot \frac{\partial\vec{x}}{\partial\xi^2}\right) - \left(\frac{\partial\vec{x}}{\partial\xi^1} \cdot \frac{\partial\vec{x}}{\partial\xi^2}\right)^2} \tag{2.7.6}$$

We have now arrived at the general expression for the area functional of a parameterised spatial surface. We want to end up with the area functional for a spacetime surface, in this case the surface of a string world-sheet. The world-sheet is not that different from the spacial surface - it is 2 dimensional and requires 2 parameters, only now we will call these τ and σ rather than ξ^1 and ξ^2.

Similar to earlier, the spacetime here is described by the mapping functions

$$X^\mu(\tau, \sigma) = (X^0(\tau, \sigma), X^1(\tau, \sigma), ..., X^d(\tau, \sigma)) \tag{2.7.7}$$

We can call X^μ the *string coordinates.*

To compute the area element, we proceed as we did before, but this time using relativistic notation. Explicitly,

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau \quad , \quad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma \tag{2.7.8}$$

Therefore, our area functional for spacetime becomes

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}\right)\left(\frac{\partial X^\nu}{\partial \sigma}\frac{\partial X_\nu}{\partial \sigma}\right)} \tag{2.7.9}$$

Notice how the terms inside the square root have been switched. This is because, for a surface where at every point P there is a τ and σ direction, the above quantity inside the square root is always greater than zero.

Now, by letting

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \quad , \quad X^{\mu'} = \frac{\partial X^\mu}{\partial \sigma} \tag{2.7.10}$$

we have

$$A = \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{2.7.11}$$

This is now the proper area of the world-sheet. We are now in a position to express the action in terms of this area, but first, some dimensional analysis is required. The action, S, has units

$$[S] = [Energy] \cdot [Time] = \frac{ML^2}{T} \tag{2.7.12}$$

Therefore, the proper area (with units L^2) needs to be multiplied by a quantity with units $\frac{M}{T}$. If we multiply this quantity by $\frac{L}{L} \cdot \frac{T}{T}$,

$$\frac{ML}{T^2} \cdot \frac{T}{L} = \frac{[Force]}{[Velocity]} \tag{2.7.13}$$

where, for a relativistic string, the velocity is the speed of light c. It would not be unreasonable to suggest the force acting on the string here is the string tension T_0, so the quantity to be multiplied into the area is

$$\frac{T_0}{c} \tag{2.7.14}$$

Thus, we have arrived at the Nambu-Goto action,

$$S = -\frac{T_0}{c}\int d\tau d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} \tag{2.7.15}$$

in the form originally proposed by Nambu and Goto [9].

2.7.1 Reparameterization invariance of the action

We need to write this action in a reparameterization-invariant way. Consider a string of mass m moving through D-dimensional Minkowski spacetime. The string will trace a world-line $x^\mu(\tau)$ parametrized by the proper time τ with an invariant length of $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric. The dynamics are characterized by the action [6]

$$S = -m\int ds = -m\int d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{2.7.16}$$

We now extend this to a world-sheet having parameters of proper time τ and proper length σ. The action is then rewritten as

$$S = -T_0\int d\tau d\sigma\sqrt{-det(\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu)} \tag{2.7.17}$$

where $T_0 = \frac{1}{2\pi\alpha'}$ is the string tension. This is the reparameterization-invariant form of the Nambu-Goto action, for flat Minkowski spacetime.

3 Methodology

3.1 Discussion

In the 1990s, methods of constructing the quark-antiquark potential in terms of Wilson Loops $\langle e^{-S} \rangle$ (see Section 2.6) became very popular, lead by the work of Juan Maldacena [12, 13]. It was shown in Section 2.7 that the action is directly related to the area of the string world-sheet via Wilson loop operators. In this section, we will be deriving a non-confining potential for an infinitely heavy quark-antiquark pair.

In this section, we will be deriving the quark-antiquark potential by using a rectangular Wilson Loop whose boundary encloses the world-sheet.

We need to introduce the system we're working with. The approach we will take to compute the quark-antiquark potential is valid in the gravity/gauge duality framework. The string between the two quarks can be imagined as having its endpoints attached to the boundary of the S^5 geometry, allowing it to settle with its minimum point somewhere inside the five-sphere. We are going to model the quarks as infinitely heavy W-bosons, as described in [12]. We consider $T \to \infty$ for the rectangular Wilson loop, so effectively it becomes two infinitely long antiparallel lines.

The spacetime metric [12, 13] associated with AdS_5 is

$$\begin{aligned} ds^2 &= dx^\mu dx^\nu G_{\mu\nu} \\ &= -G_{00}(U(x))dt^2 + G_{ii}(U(x))dx_i^2 + G_{UU}(U(x))dU^2 \end{aligned} \tag{3.1.1}$$

where the induced metric $g_{\mu\nu}$ is given as

$$g_{\mu\nu} = dx^\mu dx^\nu G_{\mu\nu} \tag{3.1.2}$$

which explicitly is

$$
\begin{bmatrix}
\frac{\partial X^0}{\partial \tau}\frac{\partial X^0}{\partial \tau}G_{00} & 0 & 0 \\
0 & \frac{\partial X^i}{\partial \sigma}\frac{\partial X^i}{\partial \sigma}G_{ii} & 0 \\
0 & 0 & \frac{\partial U}{\partial \sigma}\frac{\partial U}{\partial \sigma}G_{UU}
\end{bmatrix}
\tag{3.1.3}
$$

where X^i are the ordinary (x, y, z) coordinates, and U is a radial coordinate with dimensions of energy, as depicted in figure 5.

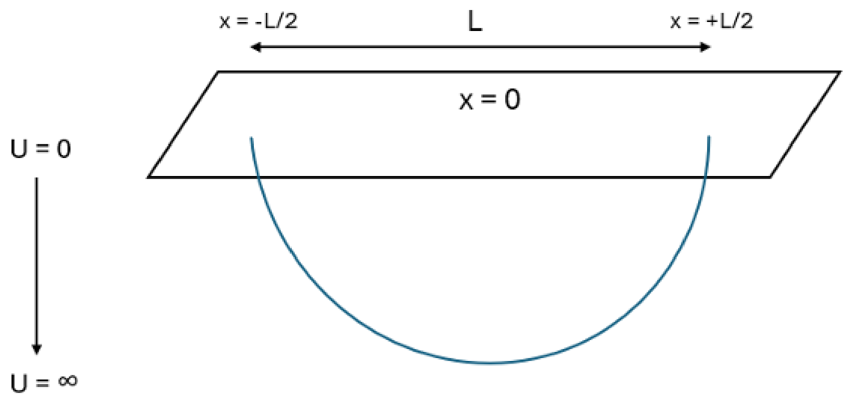

Figure 5: The configuration of the quark-antiquark pair.

The Nambu-Goto action now takes the form:

$$
S = \int d\tau d\sigma \sqrt{\det(\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu})}
\tag{3.1.4}
$$

We now define:

$$
\begin{aligned}
f^2(U) &= G_{00}(U)G_{ii}(U) \\
g^2(U) &= G_{00}(U)G_{UU}(U)
\end{aligned}
\tag{3.1.5}
$$

Setting $\tau = t$ and $\sigma = x$, the action now reads

$$S = T\int dx\sqrt{f^2(U)+g^2(U)(\partial_x U)^2} \tag{3.1.6}$$

3.2 Non-confining potential

In the $\mathcal{N}=4$ SYM case, see [12, 13], the functions $f(U)$ and $g(U)$ are given as:

$$\begin{aligned} f(U) &= (2\pi)^{-1}(U^2/R^2) \\ g(U) &= (2\pi)^{-1} \end{aligned} \tag{3.2.1}$$

so the metric becomes

$$ds^2 = \alpha'\left[\frac{U^2}{R^2}(dt^2+dx_i dx_i)+R^2\frac{dU^2}{U^2}\right] \tag{3.2.2}$$

By comparing (3.1.1) with (3.2.2), one finds that

$$\begin{aligned} G_{00} = G_{ii} &= \frac{U^2}{R^2} \\ G_{UU} &= \frac{R^2}{U^2} \end{aligned} \tag{3.2.3}$$

Therefore, the Nambu-Goto action will take the form,

$$S = \frac{T}{2\pi}\int dx\sqrt{(\partial_x U)^2+\frac{U^4}{R^4}} \tag{3.2.4}$$

Recall that T is the *total time* of the Wilson Loop, not the tension! Here,

$$\mathcal{L} = \sqrt{(\partial_x U)^2+\frac{U^4}{R^4}} \tag{3.2.5}$$

Since $\mathcal{L}$ does not depend explicitly on x, we have a conserved quantity in the form of the Hamiltonian:

$$\mathcal{H} = P_U \cdot U' - \mathcal{L} = \text{constant} \tag{3.2.6}$$

where

$$P_U \equiv \frac{\partial \mathcal{L}}{\partial(\partial_x U)} = \frac{\partial_x U}{\sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}}} \tag{3.2.7}$$

So,

$$\begin{aligned}\mathcal{H} &= \frac{(\partial_x U)^2}{\sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}}} - \sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}} = \text{constant} \\ &= \frac{-\frac{U^4}{R^4}}{\sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}}} = \text{constant}\end{aligned} \tag{3.2.8}$$

We fix the constant by setting $U = U_0$, so $\partial_x U = 0$:

$$\begin{aligned}\text{constant} &= \frac{-\frac{U_0^4}{R^4}}{\sqrt{0 + \frac{U_0^4}{R^4}}} \\ &= -\frac{U_0^2}{R^2}\end{aligned} \tag{3.2.9}$$

With the constant fixed, we can then rearrange (3.2.8) for $\partial_x U$,

$$
\begin{aligned}
\frac{\frac{U^4}{R^4}}{\sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}}} &= -\frac{U_0^2}{R^2} \\
\frac{\frac{U^8}{R^8}}{(\partial_x U)^2 + \frac{U^4}{R^4}} &= \frac{U_0^4}{R^4} \\
(\partial_x U)^2 + \frac{U^4}{R^4} &= \frac{U^8}{R^4 U_0^4} \\
(\partial_x U)^2 &= \frac{U^4}{R^4}\left[\left(\frac{U}{U_0}\right)^4 - 1\right] \\
\frac{dU}{dx} &= \frac{U^2}{R^2}\sqrt{\left(\frac{U}{U_0}\right)^4 - 1}
\end{aligned}
\tag{3.2.10}
$$

Thus,

$$
dx = \frac{R^2 \cdot dU}{U^2\sqrt{\left(\frac{U}{U_0}\right)^4 - 1}} \tag{3.2.11}
$$

We can simplify the problem with a substitution: $U \to y = \frac{U}{U_0}$, so $dy = \frac{dU}{U_0}$:

$$
\begin{aligned}
dx &= \frac{R^2 \cdot U_0 dy}{U_0^2 y^2 \sqrt{y^4 - 1}} \\
&= \frac{R^2}{U_0}\frac{dy}{y^2\sqrt{y^4 - 1}}
\end{aligned}
\tag{3.2.12}
$$

Integrating from 1 to $y = \frac{U}{U_0}$, we obtain an equation for x:

$$
x = \frac{R^2}{U_0}\int_1^{U/U_0} \frac{dy}{y^2\sqrt{y^4 - 1}} \tag{3.2.13}
$$

We can take the special case, evaluating x at one of the endpoints (quarks) where $U \to \infty$:

$$\frac{L}{2} = \frac{R^2}{U_0} \int_0^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \frac{R^2}{U_0} \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} \tag{3.2.14}$$

where we solved the integral using Beta and Gamma functions (see Appendix A). We are now at a point where we can reconstruct the action and find the potential energy. Plugging (3.2.10) and (3.2.13) into the action (3.2.4):

$$\begin{aligned}
S &= \frac{T}{2\pi} \int \frac{R^2}{U_0} \frac{dy}{y^2\sqrt{y^4-1}} \sqrt{\left[\frac{U^4}{R^4}\left(\left(\frac{U}{U_0}\right)^4 - 1\right)\right] + \frac{U^4}{R^4}} \\
&= \frac{T}{2\pi} \int \frac{R^2}{U_0} \frac{dy}{y^2\sqrt{y^4-1}} \cdot \frac{U^2}{R^2} \left(\frac{U}{U_0}\right)^2 \\
&= \frac{T}{2\pi} \int \frac{dy}{U_0 y^2\sqrt{y^4-1}} \cdot (U_0 y)^2 y^2 \\
&= \frac{U_0 T}{2\pi} \int dy \frac{y^2}{\sqrt{y^4-1}}
\end{aligned} \tag{3.2.15}$$

Thus, by making use of the relation $e^{-TE} = e^{-S}$,

$$E = \frac{U_0}{2\pi} \int dy \frac{y^2}{\sqrt{y^4-1}} \tag{3.2.16}$$

However, this result will be infinity. We need to subtract the mass of the quarks (in a sense, implementing a cut-off for U, i.e. instead of the string stretching all the way to $U = \infty$, we will instead set a limit $U_{\max}$). Doing this, we find

$$E = \frac{U_0}{\pi} \left[\int_1^\infty dy \left(\frac{y^2}{\sqrt{y^4-1}} - 1 \right) - 1 \right] \tag{3.2.17}$$

In [12], this result comes out to be

$$E = -\frac{4\pi^2 (2g_{YM}^2 N)^{1/2}}{\Gamma(1/4)^4 L} \tag{3.2.18}$$

3.3 Confining potential

From now on, we will take $R = \alpha' = 1$. The AdS_5 metric now assumes the form:

$$ds^2 = U^2(dt^2 + dx_i dx_i) + \frac{dU^2}{U^2} \tag{3.3.1}$$

Now, we're going to generalize the problem, by introducing a perturbation onto U:

$$U^2 \to U^2 + \tilde{U}^2 \tag{3.3.2}$$

where $\tilde{U}^2$ is a constant and $U >> \tilde{U}$. Now, we have

$$\begin{aligned} f(U) &= (2\pi)^{-1}(U^2 + \tilde{U}^2) \\ g(U) &= (2\pi)^{-1} \end{aligned} \tag{3.3.3}$$

We now implement this change in the Nambu-Goto action and recompute the energy. The new action is then

$$S = \frac{T}{2\pi} \int dx \sqrt{(\partial_x U)^2 + (U^2 + \tilde{U}^2)^2} \tag{3.3.4}$$

Since the Lagrangian does not depend *explicitly* on x, we have a constant Hamiltonian $H = p_U \partial_x U - L =$ constant, where

$$P_U \equiv \frac{\partial L}{\partial(\partial_x U)} = \frac{\partial_x U}{\sqrt{(\partial_x U)^2 + (U^2 + \tilde{U}^2)^2}} \tag{3.3.5}$$

So the full Hamiltonian is

$$H = \frac{(\partial_x U)^2}{\sqrt{(\partial_x U)^2 + (U^2 + \tilde{U}^2)^2}} - \sqrt{(\partial_x U)^2 + (U^2 + \tilde{U}^2)^2} = C \tag{3.3.6}$$

where $C =$ constant. Simplifying down, we have

$$C = -\frac{(U^2 + \tilde{U}^2)^2}{\sqrt{(\partial_x U)^2 + (U^2 + \tilde{U}^2)^2}} \tag{3.3.7}$$

The constant is fixed by setting $U = U_0$, so $\partial_x U = 0$:

$$C = -(U_0^2 + \tilde{U}^2) \tag{3.3.8}$$

Substituting back into (3.3.6), we find

$$\frac{dU}{dx} = \sqrt{\frac{(U^2 + \tilde{U}^2)^4}{(U_0^2 + \tilde{U}^2)^2} - (U^2 + \tilde{U}^2)^2} \tag{3.3.9}$$

leading to

$$dx = \frac{dU}{\sqrt{\frac{(U^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2} - (U^2 + \tilde{U}^2)^2}} \tag{3.3.10}$$

We can verify this by setting $\tilde{U}^2 = 0$, the equation should return to (3.2.12), but of course with $R = 1$,

$$dx = \frac{dU}{\sqrt{\frac{U^8}{U_0^4} - U^4}} \tag{3.3.11}$$

letting $y = \frac{U}{U_0}$,

$$
\begin{aligned}
dx &= \frac{U_0 dy}{\sqrt{U^4 y^4 - U_0^4 y^4}} \\
&= \frac{U_0 dy}{y^2 \sqrt{U^4 - U_0^4}} \\
&= \frac{U_0 dy}{U_0^2 y^2 \sqrt{(\frac{U}{U_0})^4 - 1}} \\
&= \frac{dy}{U_0 y^2 \sqrt{y^4 - 1}}
\end{aligned} \tag{3.3.12}
$$

With (3.3.10) verified, we find

$$
\frac{L}{2} = \int_{U_0}^{\infty} \frac{dU}{\sqrt{\frac{(U^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2} - (U^2 + \tilde{U}^2)^2}} \tag{3.3.13}
$$

Now, we will use the same substitution used in Section 3.2, setting $y = \frac{U}{U_0}$:

$$
\frac{L}{2} = U_0 \int_{1}^{\infty} \frac{dy}{\sqrt{\frac{(U_0^2 y^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2} - (U_0^2 y^2 + \tilde{U}^2)^2}} \tag{3.3.14}
$$

Then, using (3.3.4), along with (3.3.9) and (3.3.14), we can construct the action:

$$
\begin{aligned}
S &= \frac{U_0 T}{2\pi} \int_{1}^{\infty} dy \frac{\sqrt{\frac{(U_0^2 y^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2} - (U_0^2 y^2 + \tilde{U}^2)^2 + (U_0^2 y^2 + \tilde{U}^2)^2}}{\sqrt{\frac{(U_0^2 y^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2} - (U_0^2 y^2 + \tilde{U}^2)^2}} \\
&= \frac{U_0 T}{2\pi} \int_{1}^{\infty} dy \frac{\frac{(U_0^2 y^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2}}{\sqrt{\frac{(U_0^2 y^2+\tilde{U}^2)^4}{(U_0^2+\tilde{U}^2)^2} - (U_0^2 y^2 + \tilde{U}^2)^2}}
\end{aligned} \tag{3.3.15}
$$

Now, again with our knowledge of Wilson Loops and the relation between the action and the potential $e^{-TE} = e^{-S}$, we have our energy:

$$E = \frac{U_0}{2\pi} \int_1^\infty dy \frac{\frac{(U_0^2 y^2 + \tilde{U}^2)^4}{(U_0^2 + \tilde{U}^2)^2}}{\sqrt{\frac{(U_0^2 y^2 + \tilde{U}^2)^4}{(U_0^2 + \tilde{U}^2)^2} - (U_0^2 y^2 + \tilde{U}^2)^2}} \tag{3.3.16}$$

As before, we must subtract the masses of the quarks:

$$E = \frac{U_0}{2\pi} \left[\int_1^\infty dy \left(\frac{\frac{(U_0^2 y^2 + \tilde{U}^2)^4}{(U_0^2 + \tilde{U}^2)^2}}{\sqrt{\frac{(U_0^2 y^2 + \tilde{U}^2)^4}{(U_0^2 + \tilde{U}^2)^2} - (U_0^2 y^2 + \tilde{U}^2)^2}} - 1 \right) - 1 \right] \tag{3.3.17}$$

3.3.1 Confinement Condition

In [13], it was found that the general condition for confinement is that confinement occurs if and only if $f(0) > 0$.

For small L, L is inversely proportional to U_0 ($L \sim \frac{1}{U_0}$), and since U_0 has dimensions of energy, we have

$$E \sim -\frac{1}{L} \tag{3.3.18}$$

When L is large, we do not see the string stretch down as before, but rather it rests along a "floor" at $U = U_0$, as shown in Figure 6:

Figure 6: The quark-antiquark string for large L. This configuration minimizes the action (proper area of the world-sheet).

For large L, we can take the generalized result from [13], stating that

$$E = f(0)L \tag{3.3.19}$$

with $f(0)$ being the string tension. Since $f(U) = (2\pi)^{-1}(U^2 + \tilde{U}^2)$, this leads to the result:

$$E = \frac{\tilde{U}^2}{2\pi}L \tag{3.3.20}$$

4 Results

$L(U_0, \tilde{U})$ and $E(U_0, \tilde{U})$ were plotted against each other for a range of U_0 values. The results are shown below:

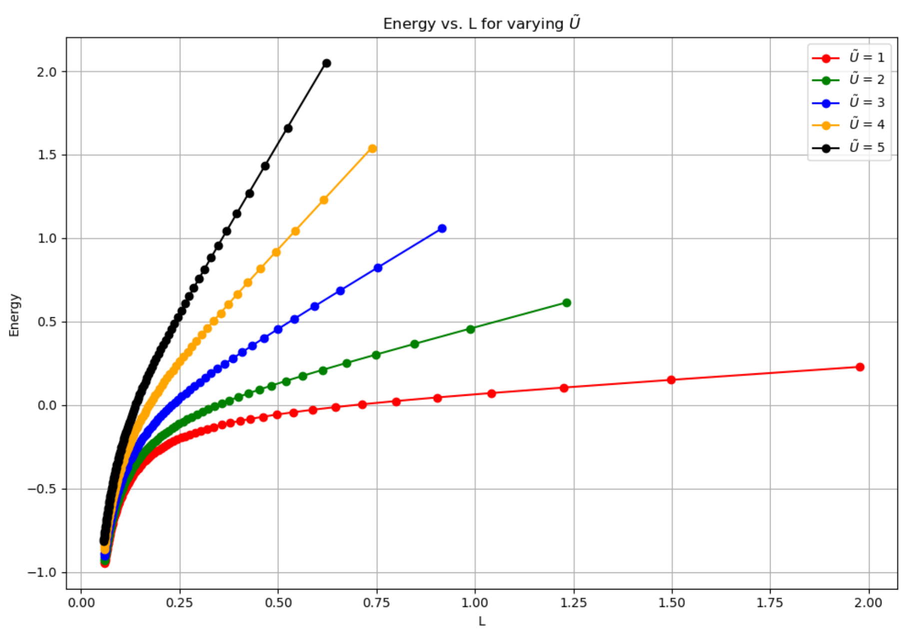

Figure 7: Potential Energy E against Quark Separation L. $\tilde{U}$ varied from 1 to 5. Gradients of linear regime correspond to $\frac{\tilde{U}^2}{2\pi}$.

The shape of the potential in Figure 7 is as expected.

In addition to plotting E against L, we can also plot E/L against L. This enables us to verify that the gradients in the linear regime in Figure 7 tend towards the string tension:

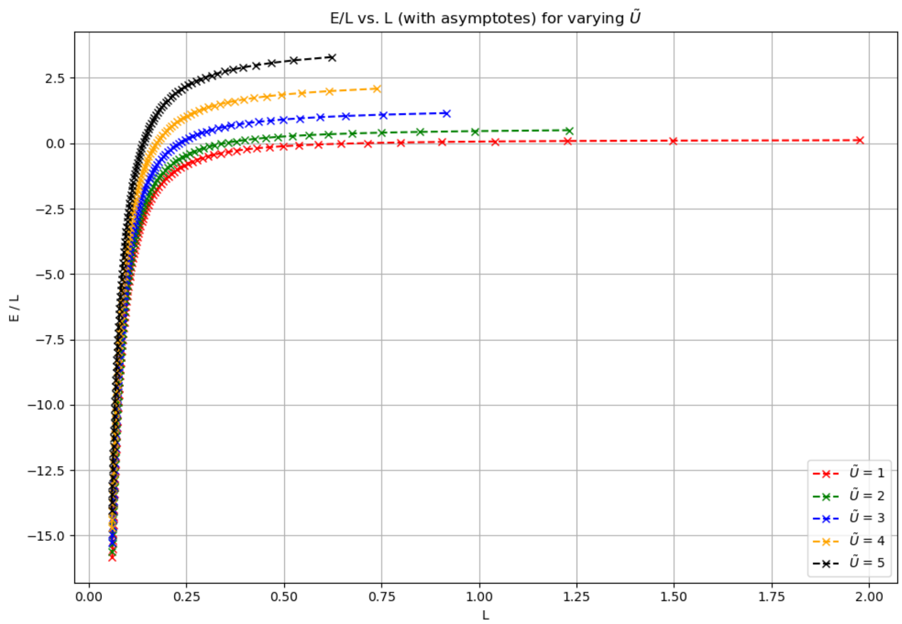

Figure 8: A plot of E/L against L

Again, this is the expected result. Figure 9 (below) also has the theoretical values of the string tension, $\tilde{U}^2/2\pi$, plotted for better visualization:

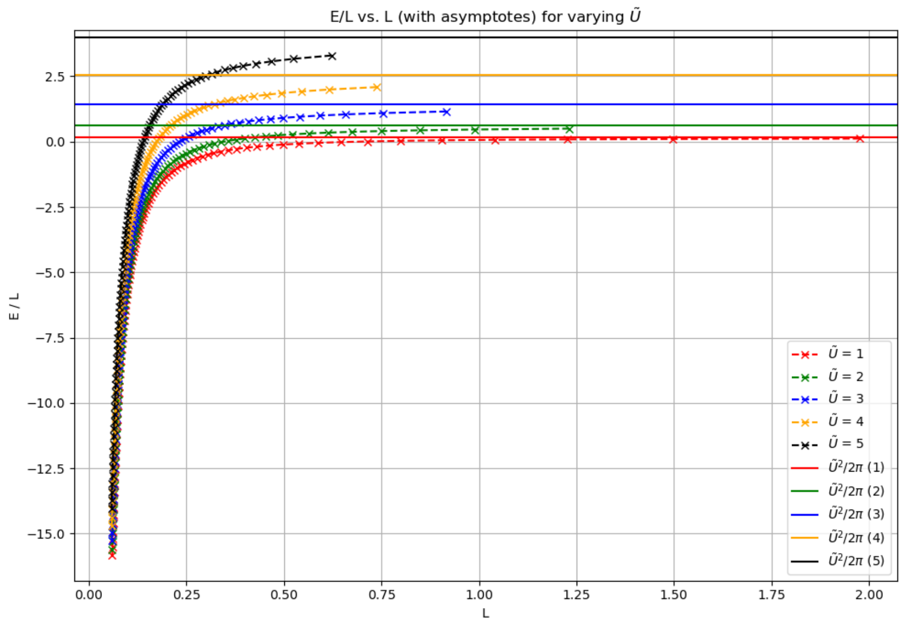

Figure 9: A plot of E/L against L, with asymptotic lines

As expected, the plots tend towards the string tension. Below is a table comparing the measured gradients from Figure 7 with the string tension:

$\tilde{U}$	Gradient of Linear Regime	Theoretical string tension ($\frac{\tilde{U}^2}{2\pi}$)
1	0.16633	0.15915
2	0.64800	0.63661
3	1.43776	1.43239
4	2.55385	2.54648
5	3.97600	3.97887

Table 1: Gradient extracted from data along with the theoretical expectation.

5 Analysis

In Figure 7, the quark-antiquark potential is plotted against L for varying values of $\tilde{U}$. The potential behaves Coulomb-like for small L, and transitions to a confining potential for large L. This result agrees with the general result provided in [13].

In Figures 8 and 9, the plot of E/L against L is shown for varying values of $\tilde{U}$, and in the latter plot the values of the string tension $\tilde{U}^2/2\pi$ are also shown. For small L, $E/L \to -\infty$, and for large L, $E/L \to \tilde{U}^2/2\pi$

Table 1 compares the gradient of the linear regime to the theoretically derived gradient (string tension). It is clear to see that they match quite well, which is a very promising result. There is also a clear indication that the results become more accurate as $\tilde{U}$ increases.

5.1 Transition Length

In Figure 7, we see that there is a transition between the non-confining and confining regimes. The transition length L_t can be a useful quantity to derive, as this can lead to further research. In order to find L_t, we can equate the potentials for small and large L. For small L, the quark-antiquark potential takes the form

$$E = -\frac{\eta}{L} \tag{5.1.1}$$

where η is some constant. For large L, the potential is

$$E = \frac{\tilde{U}^2}{2\pi} L \tag{5.1.2}$$

To find the transition length, we are interested only in comparing the magnitude of the potential in both regimes. So, taking $L = L_t$, and equating the magnitudes of the two equations:

$$\left| -\frac{\eta}{L_t} \right| = \left| \frac{\tilde{U}^2}{2\pi} L_t \right| \tag{5.1.3}$$

we find that

$$L_t = \sqrt{\frac{2\pi\eta}{\tilde{U}^2}} \tag{5.1.4}$$

combining this from the result from [12],

$$E = -\frac{2\sqrt{2}\pi^{3/2}R^2}{\Gamma(1/4)^4 L} \tag{5.1.5}$$

where $R = (4\pi g N)^{1/4}$ is the radius in string units, we can conclude η must satisfy the condition:

$$\eta = \frac{4\pi^2(2g^2N)^{1/2}}{\Gamma(1/4)^4} \tag{5.1.6}$$

6 Conclusion

The goals for this project were to derive an equation for the quark-antiquark potential in the $AdS_5 \times S^5$ background of $\mathcal{N} = 4$ SYM, and to prove analytically that the potential is of the form $E = \sigma L$ for large L, and $E \sim -1/L$ for small L.

It has been shown that, in the case of non-confinement (see Section 3.2), the result agrees with [12]. In the case of confinement, the results have shown that confinement occurs for large L, agreeing with [13]. The transition length between the Coulomb and linear regimes of the potential in Figure 7 was also determined.

Some limitations were encountered during this project: Firstly, the numerical integration of the potential (3.3.17) may lack precision due to the choice of integration bounds, or the method of numerical integration used. Secondly, a more comprehensive analysis of the transition length may have been outside the scope of this project, and as such this is recommended as future research in the area.

In summary, it has been demonstrated that the method of using Wilson Loop Operators to compute the quark-antiquark potential from the area of the string world-sheet has been successful in numerically computing the string tension for large L. Further work in this area could include using this same method to calculate the potential between 3 quarks, although that would entail more more complicated string dynamics than what was shown here.

Overall, it would be reasonable to conclude that the aims set forth at the beginning of this project have been met successfully, and the results have been most promising.

7 Acknowledgments

I would like to thank Professor Adi Armoni for his support and advice throughout this project, and for interesting conversations on general physics and philosophy. I am also grateful to Professor Carlos Nunez for his feedback and advice on my work and presentations. I would also like to thank the Department of Physics as a whole, for creating such a stimulating and attractive place to study.

A Beta and Gamma Functions

Here, we derive the numerical form of (3.2.14).

Taking

$$\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} \tag{A.0.1}$$

let $y^2 = 1/\sqrt{t}$, so $dy = -\frac{1}{4}t^{-5/4}dt$, therefore:

$$
\begin{aligned}
\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} &= -\frac{1}{4}\int_0^1 \frac{t^{-5/4}dt}{t^{-1/2}\sqrt{\frac{1}{t}-1}} \\
&= -\frac{1}{4}\int_0^1 dt\frac{t^{-5/4}}{t^{-1/2}t^{-1/2}\sqrt{1-t}} \\
&= -\frac{1}{4}\int_0^1 dt t^{-1/4}(1-t)^{-1/2}
\end{aligned} \tag{A.0.2}
$$

This is now in the form of the Beta function:

$$
B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{A.0.3}
$$

where $x = 3/4$ and $y = 1/2$. The Beta function can be expressed as a product of Gamma functions:

$$
B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{A.0.4}
$$

where the Gamma function is defined as

$$
\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}dt \tag{A.0.5}
$$

So, we now have

$$
\begin{aligned}
-\frac{1}{4}\int_0^1 dt t^{-1/4}(1-t)^{-1/2} &= -\frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right) \\
&= -\frac{1}{4}\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)}
\end{aligned} \tag{A.0.6}
$$

The following identities will now come in handy:

$$\begin{aligned}\Gamma(1/2) &= \sqrt{\pi} \\ \Gamma(z+1) &= z\Gamma(z) \\ \Gamma(1-z)\Gamma(z) &= \frac{\pi}{\sin(\pi z)}\end{aligned} \tag{A.0.7}$$

We can immediately see that, from (A.0.7),

$$\Gamma(5/4) = \frac{1}{4}\Gamma(1/4) \tag{A.0.8}$$

and,

$$\Gamma\left(1-\frac{1}{4}\right)\Gamma(1/4) = \Gamma(3/4)\Gamma(1/4) = \frac{\pi}{\sin(\pi/4)} \tag{A.0.9}$$

So,

$$\Gamma(3/4) = \frac{\pi\sqrt{2}}{\Gamma(1/4)} \tag{A.0.10}$$

Therefore,

$$\begin{aligned}-\frac{1}{4}\int_0^1 dt t^{-1/4}(1-t)^{-1/2} &= -\frac{1}{4}\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)} \\ &= -\frac{1}{4}\left(\frac{\pi\sqrt{2}}{\Gamma(1/4)}\right)\sqrt{\pi}\frac{4}{\Gamma(1/4)} \\ &= -\frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2}\end{aligned} \tag{A.0.11}$$

Thus, we have derived (3.2.14):

$$\frac{L}{2} = \frac{R^2}{U_0}\int_0^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \frac{R^2}{U_0}\frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} \tag{A.0.12}$$

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