Newsvendor Problems: A New Way to Integrated Forecasting and Optimisation

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Abstract

Newsvendor problems form a classical and important family of stochastic optimisation problems. The standard solution approach decomposes the problem into two steps: estimation of the demand distribution, then determination of the optimal production quantity (or quantities) for the given distribution. We propose a new, integrated solution approach, which estimates the optimal production quantity directly from the data. Our approach can be used even when the demand distribution is not stationary. Some encouraging computational results are given.

Keywords: newsvendor problems, forecasting, data-driven optimisation, sales and operations planning

1 Introduction

Inventory control is a classical and important topic in Operations Research and Operations Management (see, e.g., the books [**Po02**, **SPP98**, **Zi00**]). In this paper, we focus on *Newsvendor Problem* (NVP), which referrs to a *single-period stochastic* inventory control problem.

In early works on NVP [**AHM51**, **MK51**], it is assumed that the demand in each time period comes from a known probability distribution. Of course, in practice, this is not the case — a fact already noted in 1958 by Scarf [**Sc58**]. Assuming that historical demand data is available, one can attempt to address this difficulty by decomposing the problem into an estimation / forecasting phase and an optimisation phase. In the first phase, one makes some assumption (e.g., normality) regarding the underlying data generating process, and uses the past data to estimate the parameters of the process. In the second phase, one determines the order quantity (or quantities) based on the estimated parameter values.

Throughout this paper, we will call this two-phase approach the *disjoint* approach. An advantage of the disjoint approach is that forecasting and optimisation experts can operate independently within an organisation. This makes things easier to manage. On the other hand, as noticed by several authors [**BT06**, **BM12**, **Ka94**, **KT96**, **KTB20**], there are two disadvantages:

- The two phases use different objective functions. Indeed, in the first phase, the objective is to minimise a function of the forecasting errors, such as the root mean square error or mean absolute error. In the second phase, however, the goal is usually to maximise expected profit.
- If the forecasting model is misspecified, and/or there is substantial noise in the data, the effect on the optimisation phase is very hard to predict. In particular, upside and downside errors may have very different effects on expected profit.

An alternative to the disjoint approach is to use a single, *integrated* approach, in which the order quantities are determined directly from the data. A simple example of an integrated approach is *quantile regression* [**Br16**, **Hu19**]. A nice feature of quantile regression is that it makes no assumptions about the demand distribution. Unfortunately, it can only be applied to relatively simple NVPs, for which one can express the optimal order quantities in terms of quantiles of demand.

In this paper, we introduce a new integrated approach. It is very flexible, and can be applied to a wide variety of NVPs, with complex profit functions. Roughly speaking, it involves forecasting the optimal order quantities instead of the demand. Moreover, instead of determining parameter values

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that minimise some function of the forecasting errors, our approach attempts to maximise the expected profit directly.

After explaining our approach formally, we show that it reduces to quantile regression in the case of the simplest NVP (with only one product, stationary demand, and linear profit functions). We then perform extensive computational experiments, on several different NVPs. The results show that our method performs well, in comparison with the disjoint approach, according to several different measures of quality.

The rest of the paper is organized as follows. Section **??** provides a brief review of the relevant literature. Section **??** presents the new method and shows that it is a generalisation of quantile regression. In Section **??**, we present and discuss the computational results. Finally, Section **??** gives some insights, remarks and suggestions.

2 Literature Review

?⟨se:lit⟩? Since the literature on NVPs is vast, we mention here only works of direct relevance. The reader looking for more information is directed to the books [**Ch12**, **Po02**, **SPP98**, **Zi00**].

2.1 The classical newsvendor problem

In the simplest NVP, found in textbooks [**Ch12**], a company purchases goods at the beginning of a time period at a cost of v per unit, and aims to sell them by the end of the period at a price p per unit. The demand during the period is a random variable Y with known probability density function f and cumulative distribution function F. At the end of the period, any surplus goods will lead to a *holding cost* of c_h per unit. On the other hand, shortage of goods during the period will lead to a *shortage cost* of c_s per unit. The goal is to determine an *order quantity* Q, prior to the period, that maximises the expected profit.

For a given Q and a given realisation y of Y, the profit over the period is:

$$\pi(Q,y) = \begin{cases} py - vQ - c_h(Q-y), & \text{if } Q \geq y \\ pQ - vQ - c_s(y-Q), & \text{if } Q < y. \end{cases}$$

The expected value of $\pi(Q,y)$ is:

$$\Pi(Q) = \int_0^Q \big[py - vQ - c_h(Q-y)\big] f(y)dy + \int_Q^\infty \big[pQ - vQ - c_s(y-Q)\big] f(y)dy.$$

It is common to call $c_u = p - v + c_s$ the 'underage' cost and $c_o = v + c_h$ the 'overage' cost. Some calculus then shows that the order quantity that maximises $\Pi(Q)$ is:

$$Q^* = F^{-1}\left(\frac{c_u}{c_o + c_u}\right),$$

where F^{-1} is the inverse function of F. Thus, Q^* is the τ^{th} quantile of f, with $\tau = {c_u}/{(c_o+c_u)}$.

2.2 More complex newsvendor problems

Since the NVP was introduced in the 1950s [**AHM51**, **MK51**], researchers have considered several extensions of the problem, including variants with multiple product types [**HW63**, **LL96**, **MS00**], quantity discounts [**Kh95**], different risk measures [**EGS95**], product substitution [**BAA99**], nonlinear cost functions [**HOS12**], non-stationary demand [**KWH15**], and price setting [**KC62**, **Mi59**, **PD99**].

For the purpose of what follows, we now explain one variant, the 'Nonlinear Newsvendor Problem' (NNVP), in detail (see also [**BT06**, **HOS12**, **HN16**, **KC62**, **Kh95**, **KK18**, **Mi59**, **PSC15**, **PD99**]). In the NNVP, the profit function takes the form:

$$\pi(Q,y) = \begin{cases} P(Q,y) - V(Q) - C_h(Q,y), & \text{for } Q \geq y \\ P(Q,y) - V(Q) - C_s(Q,y), & \text{for } Q < y, \end{cases}$$

where V, P, C_h and C_s are now *functions* rather than constants.

If $\pi(Q, y)$ has a particularly simple form (e.g., if it is piecewise-linear), then it may be possible to use calculus to express the optimal order quantity as a quantile. In general, however, a closed-form expression as a quantile is unlikely to exist.

We now review one particular NNVP, taken from [**KK18**, **PD99**, **RK02**], that we are going to use in our numerical experiments. The purchase cost v and selling price p are constants, but C_h and C_s are functions. Overstock items incur a fixed unit penalty $\alpha > 0$, but they can be sold in a salvage market with fixed unit sales price β, with $0 < \beta < v$. The demand in the salvage market is itself a random variable, with known distribution, which we denote by u. That is, we have:

$$C_h(Q, y) \; = \; \alpha[Q - y]^+ - \beta\, \mathbb{E}\left[\min\left\{[Q - y]^+, u\right\}\right].$$

Moreover, the shortage penalty is proportional to the shortage quantity. That is:

$$C_s(Q, y) = \zeta\left([y - Q]^+\right)^2$$

for some constant $\zeta > 0$.

2.3 Quantile regression

Returning to the classical NVP, we now consider the (more realistic) case in which the demand distribution is unknown, but we have historical demands $y_1, y_2, \ldots, y_s$. For this case, *quantile regression* has proven to perform rather well. The basic idea is as follows [**BT06**, **Br16**, **CS19**, **HNS15**, **Hu19**]:

1. Compute the value of τ that maximises expected profit;
2. Use quantile regression to compute an estimate of the τ^{th} quantile of the demand in the next time period, which we denote by $\hat{y}^{(\tau)}_{s+1}$;
3. Set the order quantity $\hat{Q}_{s+1}$ to $\hat{y}^{(\tau)}_{s+1}$.

Unfortunately, quantile regression is efficient only on large samples [**Hu19**, **RV19**]. Another drawback is that the performance of this approach depends crucially on the underlying target service level. In the experiments of Huber [**Hu19**] and Rudin [**RV19**], the benefit of using the quantile regression method is limited to target service levels smaller than 0.8.

2.4 Other integrated approaches

There exist other integrated methods. One approach is to use machine learning techniques, such as neural networks, to estimate the optimal order quantity from historical data. The application of machine learning to NVPs can be seen in [**CS19**, **OST20**, **RV19**]. Another integrated approach is based on so-called *one-shot decision theory* [**Guo11**, **GM14**, **Ma19**]. A numerical example was shown in [**Guo11**], and extensions can be found in [**Ma19**]. For the sake of brevity, we do not describe these alternative integrated methods in detail.

3 The Proposed Estimator for NVPs

?⟨se:new⟩? We have seen that, while quantile regression can be an attractive integrated approach, it has some drawbacks. In particular, for most non-trivial NVPs, it is very hard to express the optimal order quantity as a demand quantile *a priori*. In order to overcome this limitation, we propose an alternative integrated method, based on a regression model with a specialised loss function.

For each historical period $t \in [1, s]$, we assume that the observed demand y_t was a realisation of a random variable Y_t. Then, in principle, there exists an order quantity, say Q^*_t, that maximises the expected profit given Y_t and Π. Thus, if we had set Q_t to Q^*_t prior to observing the true demand y_t, we would have maximised our expected profit in period t. Putting it another way, if we could somehow uncover the hidden structure of the time series $\left\{Q^*_1, \ldots, Q^*_s\right\}$, we would be able to estimate Q^*_{s+1} directly.

Of course, in practice, the distributions Y_t are unknown, and the values Q^*_t are not observable. So we approximate the Q^*_t values using a regression model. For each t, the regression model yields an estimate of Q^*_t, which we denote by $\hat{Q}_t$. The estimate $\hat{Q}_{s+1}$ can then be used as the order quantity in the next time period.

The crucial feature of our approach is that, instead of using the standard least-squares loss function to estimates the regression parameters, we choose the parameters that maximise the expected profit function $\sum_{t=1}^{s}\pi\big(\hat{Q}_t, y_t\big)$. In more detail, we modify the estimator of the model into:

$$\hat{\boldsymbol{\beta}} = \text{argmax}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}} \sum_{t=1}^{s} \pi(\hat{Q}_t, y_t),$$

where we assume a linear relationship,

$$\hat{\mathbf{Q}} = \mathbf{X}\boldsymbol{\beta}$$

and

$$\hat{\mathbf{Q}} = \begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ \vdots \\ \hat{Q}_s \end{pmatrix}, \mathbf{X} = \begin{pmatrix} \mathbf{x}_1^{\mathsf{T}} \\ \mathbf{x}_2^{\mathsf{T}} \\ \vdots \\ \mathbf{x}_s^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{s1} & \cdots & x_{sp} \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix},$$

where where $\mathbf{x}_t^{\mathsf{T}}$ is the tth row of matrix $\mathbf{X}$, p is the number of explanatory variables. While we focus on the linear regression model in this paper, the dynamic models, such as ARIMA or ETS [**HKO08**], can be used instead as efficiently. After estimating the model based on the maximum of the profit, we can have $\hat{Q}_{s+1}$.

In the case of NVP, the proposed estimator has following useful statistical properties:

- **Quantile regression transformation**
  We have (see Appendix **??**):

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \text{argmax}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}} \sum_{t=1}^{s} \pi(\mathbf{x}_t^{\mathsf{T}}\boldsymbol{\beta}, y_t) \\ &= \text{argmin}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}} \sum_{t=1}^{s} \{c_o[\mathbf{x}_t^{\mathsf{T}}\boldsymbol{\beta} - y_t]^+ + c_u[y_t - \mathbf{x}_t^{\mathsf{T}}\boldsymbol{\beta}]^+\}. \end{aligned}$$

  By setting $\tau = {c_u}/{(c_o+c_u)}$ and $\rho_\tau(u) = u(\tau - \mathbb{I}_{(u<0)})$, we can transform the estimating function into quantile regression:

$$\hat{\boldsymbol{\beta}} = \text{argmin}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}} \sum_{t=1}^{s} \rho_\tau(y_t - \mathbf{x}_t^{\mathsf{T}}\boldsymbol{\beta}).$$

  Therefore, the estimator inherts the consistency [**Koe05**], efficiency [**KM99**] and the asymptotically normality properties [**KHM05**] of quantile regression.

- **Scale and shift equivalence**
  For any $a > 0$, $\gamma \in \mathbb{R}^{p+1}$ and $\tau \in [0, 1]$ (see Appendix **??**),

$$\hat{\boldsymbol{\beta}}(a\mathbf{y}, \mathbf{X}) = a\hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}).$$

$$\hat{\boldsymbol{\beta}}(\mathbf{y} + \mathbf{X}\gamma, \mathbf{X}) = \hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}) + \gamma.$$

- **Equivalence to reparameterization of design**
  Let A be any $(p+1)\times(p+1)$ non-singular matrix,

$$\hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}A) = A^{-1}\hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}).$$

  Proof:
  Define that

$$\mathbf{D} = \mathbf{X}A = \begin{pmatrix} \mathbf{d}_1^{\mathsf{T}} \\ \mathbf{d}_2^{\mathsf{T}} \\ \vdots \\ \mathbf{d}_s^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1^{\mathsf{T}}\mathbf{a}^1 & \mathbf{x}_1^{\mathsf{T}}\mathbf{a}^2 & \cdots & \mathbf{x}_1^{\mathsf{T}}\mathbf{a}^{p+1} \\ \mathbf{x}_2^{\mathsf{T}}\mathbf{a}^1 & \mathbf{x}_2^{\mathsf{T}}\mathbf{a}^2 & \cdots & \mathbf{x}_2^{\mathsf{T}}\mathbf{a}^{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_s^{\mathsf{T}}\mathbf{a}^1 & \mathbf{x}_s^{\mathsf{T}}\mathbf{a}^2 & \cdots & \mathbf{x}_s^{\mathsf{T}}\mathbf{a}^{p+1} \end{pmatrix},$$

where $\mathbf{a}^n$ is the nth column of matrix A.
We have

$$\begin{aligned}
&\hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}A) \\
&= \text{argmax}_{\boldsymbol{\beta} \in \mathbb{R}^{p+1}} \sum_{t=1}^{s} \pi(\mathbf{d}_t^\mathsf{T} \boldsymbol{\beta}, y_t) \\
&= \text{argmin}_{\boldsymbol{\beta} \in \mathbb{R}^{p+1}} \sum_{t=1}^{s} \{c_o[\mathbf{d}_t^\mathsf{T} \boldsymbol{\beta} - y_t]^+ + c_u[y_t - \mathbf{d}_t^\mathsf{T} \boldsymbol{\beta}]^+\}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathbf{d}_t^\mathsf{T} \boldsymbol{\beta} &= \sum_{n=1}^{p+1} \mathbf{x}_t^\mathsf{T} \mathbf{a}^n \beta_n \\
&= \sum_{n=1}^{p+1} \sum_{m=1}^{p+1} x_{tm} a_{mn} \beta_n \\
&= \sum_{m=1}^{p+1} x_{tm} \mathbf{a}_m^\mathsf{T} \boldsymbol{\beta} \\
&= \mathbf{x}_t^\mathsf{T} A \boldsymbol{\beta},
\end{aligned}$$

where a_{mn} is the element at mth row and nth column of A.
We can derive:

$$\begin{aligned}
&\hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}A) \\
&= \text{argmin}_{\boldsymbol{\beta} \in \mathbb{R}^{p+1}} \sum_{t=1}^{s} \left\{ c_o \left[\mathbf{x}_t^\mathsf{T} A \boldsymbol{\beta} - y_t\right]^+ + c_u \left[y_t - \mathbf{x}_t^\mathsf{T} A \boldsymbol{\beta}\right]^+ \right\} \\
&= \text{argmax}_{\boldsymbol{\beta} \in \mathbb{R}^{p+1}} \sum_{t=1}^{s} \pi(\mathbf{x}_t^\mathsf{T} A \boldsymbol{\beta}, y_t) \\
&= A^{-1} \hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}).
\end{aligned}$$

4 Computational Experiments

?⟨se:results⟩? 4.1 Simulation design

In this section, we conduct the simulation experiment in order to demonstrate the performance of the proposed method. The data are simulated from ARIMA $(1,0,0)(1,0,0)_4$ with $\theta = 0.3, \Theta = 0.5$, and the simulation will be performed base on 20,000 iterations. We consider the simplest scenario in the beginning, where the underlying profit function is linear , the error term of the demand follows normal distribution, and we know the model of the data generating process. To further explore the method, we consider five data lengths (40,120,480,1200,4800) in order to see how does the proposed method performed when the learning period varies.

In the experiment, we consider two benchmark methods to compare performance with:

- Disjoint method that use ARIMA $(1,0,0)(1,0,0)_4$ to forecast in the first phase and determine the optimal quantile from the profit function in the second phase,
- Integrated method that use Quantile Regression with seasonal dummies and lag term 1 and 4 as independent variables.

Moreover, we use the exact model and parameters from data generating process to be our upper bound. We build the proposed method and benchmark methods based on the given data set, and compare their performance on the 1-step ahead forecast. The comparison of performance includes:

1. Inventory Error ($IE = \hat{Q}_{s+1} - y_{s+1}$),
2. Percentage Profit Loss ($PPL = \frac{\pi(y_{s+1}, y_{s+1}) - \pi(\hat{Q}_{s+1}, y_{s+1})}{\pi(y_{s+1}, y_{s+1})}$),
3. Service Level ($SL = \mathbb{I}_{(\hat{Q}_{s+1} > y_{s+1})}$),

4. Fill Rate ($MFR = \frac{\min\{\hat{Q}_{s+1}, y_{s+1}\}}{y_{s+1}}$).

We expect to see the performance of both the proposed method and benchmark methods converge to this upper bound when data length increases, and we also investigate the differences of their performance when data is insufficient. The results are presented in tables and figures. In Figure **??**, DGP denotes

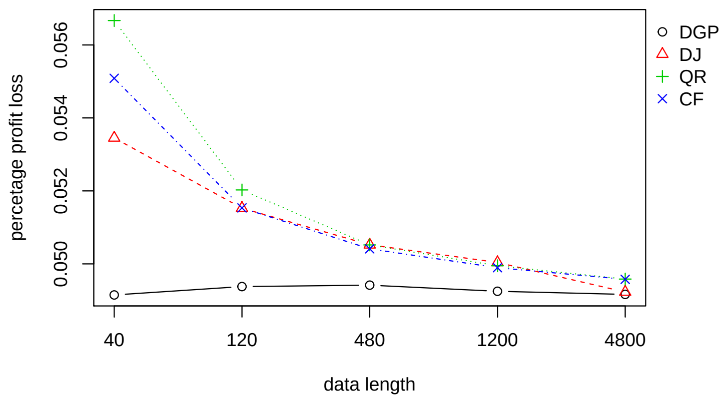

Figure 1: Percentage profit loss vs. data size at 0.5 target service level

?⟨fig:ppl0.5⟩?

the exact ARIMA model from data generating process as out upper bound, DJ denotes the disjoint method, QR denotes the quantile regression method, and CF denotes our proposed method. This figure represents the performance of methods regarding to percentage profit loss in the situation where the target service level = 0.5. We can see both the proposed method and the benchmark methods converge to the upper bound when data length grows. Besides, the performance of our proposed method and quantile regression is very similar. This is exactly what we expected base on the transformation property we proved in Section **??**. When the data size is small, the

Figure is the beanplot of absolute inventory error with data length equal to 480. We choose this particular data length since all methods are stabled when length equal or bigger than it as we can see from Figure. Actually, the plot looks similar in other data lengths, but we don't show them all in line. We can see all methods perform similarly on the metric of absolute inventory error either in mean value or distribution shape.

Regarding to service level, we can see that, in Figure, the trend is similar to the one we observed in 'percentage profit loss'. Both the proposed method and the quantile regression method become stable when data length grows to a 'normal' value. The disjoint method, on the other hand, seems not to be very sensitive to the data length. However, if we pay attention to the scale, the different is not too much, and we can say that all methods generated acceptable results.

However, the service level only present if the demand was meet or not, but it doesn't show how how big the gap is if the demand was not successfully meet. For this propose, we can use the beanplot of fill rate at data length equal to 480 to see that (see Figure). Once again, the plot looks similar in other data lengths. It is clear that the disjoint method has smaller mean fill rate, which is in line with the observation we have from Figure . Besides, the beanplot also demonstrates that given the circumstance demand was not meet, the proposed method is more likely to have a smaller gap than the disjoint method.

4.2 Nonlinear profit function

In this subsection, we consider the simulation where, as before, the error term of the demand follows normal distribution, and we know the model of the data generating process, but the underlying profit function is nonlinear[1] (solution see [**KK18**]). In this experiment, we pay specific attention to the percentage profit loss, and we explore how the performance changes on both the proposed method and the disjoint method when we alter the profit function to nonlinear. The data will still be generated from ARIMA process, and the five data lengths are maintained. The results can be seen in

[1] $\pi(Q,y) = \begin{cases} 20y - 8Q - 4(Q-y) + 5\,\mathbb{E}[\min\{(Q-y),u\}], & \text{if } Q \geq y \\ 20Q - 8Q - 7(y-Q)^2, & \text{if } Q < y, \end{cases}$ and $u \sim \mathcal{U}(0,50)$

5 Concluding Remarks

?⟨se:end⟩? BLAH...

Appendices

A Derivation of quantile regression transformation

?⟨app:A⟩? We have:

$$\min[a,b] = a - [a-b]^+,$$

and

$$a - b = [a-b]^+ - [b-a]^+.$$

We can transform:

$$\begin{aligned}
\pi(\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}, y_t) &= p\min[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}, y_t] - v\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} + c_h[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + c_s[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+ \\
&= p\{\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - [\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+\} - v\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} + c_h[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + c_s[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+ \\
&= (p-v)\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} + (c_h - p)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + c_s[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+.
\end{aligned}$$

Therefore, we have (since y_t is fixed):

$$\begin{aligned}
&\max\sum_{s=1}^{t}\pi(\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}, y_t) \\
&\quad= \max\sum_{t=1}^{s}\{(p-v)\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} + (c_h - p)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + c_s[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+\} \\
&\quad= \max\sum_{t=1}^{s}\{(p-v)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t] + (c_h - p)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + c_s[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+\} \\
&\quad= \max\sum_{t=1}^{s}\{(p-v)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ - (p-v)[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+ \\
&\qquad + (c_h - p)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + c_s[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+\} \\
&\quad= \min\sum_{t=1}^{s}\{(v - c_h)[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t]^+ + (p - v - c_s)[y_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+\}
\end{aligned}$$

B Derivation of scale and shift equivalence

?⟨app:B⟩? Scale equivalence:

$$\begin{aligned}
\hat{\boldsymbol{\beta}}(a\mathbf{y}, \mathbf{X}) &= \operatorname{argmax}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}}\sum_{t=1}^{s}\pi(\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}, ay_t) \\
&= \operatorname{argmin}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}}\sum_{t=1}^{s}\{c_o[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - ay_t]^+ + c_u[ay_t - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+\} \\
&= \operatorname{argmax}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}}\sum_{t=1}^{s}\pi\left(\mathbf{x}_t^\mathsf{T}\frac{\boldsymbol{\beta}}{a}, y_t\right) \\
&= a\hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}).
\end{aligned}$$

Shift equivalence:

$$\begin{aligned}
&\hat{\boldsymbol{\beta}}(\mathbf{y} + \mathbf{X}\gamma, \mathbf{X}) \\
&\quad= \operatorname{argmax}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}}\sum_{t=1}^{s}\pi(\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}, y_t + \mathbf{x}_t^\mathsf{T}\gamma) \\
&\quad= \operatorname{argmin}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}}\sum_{t=1}^{s}\{c_o[\mathbf{x}_t^\mathsf{T}\boldsymbol{\beta} - y_t - \mathbf{x}_t^\mathsf{T}\gamma]^+ + c_u[y_t + \mathbf{x}_t^\mathsf{T}\gamma - \mathbf{x}_t^\mathsf{T}\boldsymbol{\beta}]^+\} \\
&\quad= \operatorname{argmax}_{\boldsymbol{\beta}\in\mathbb{R}^{p+1}}\sum_{t=1}^{s}\pi[\mathbf{x}_t^\mathsf{T}(\boldsymbol{\beta} - \gamma), y_t] \\
&\quad= \hat{\boldsymbol{\beta}}(\mathbf{y}, \mathbf{X}) + \gamma.
\end{aligned}$$