Derivation of Itô's Lemma

Brownian Motion W_t $\{W_t : t \geq 0\}$

$\boxed{W_0 = 0 \text{ (a.s.)}}$ $\boxed{W_{t_2} - W_{t_1} \perp W_{t_3} - W_{t_2}}$ $\boxed{W_{t+s} - W_t \sim N(0, s)}$

$\boxed{\lim_{h \to 0} W_{t+h}(\omega) = W_t(\omega) \text{ for all } t}$ Prove to be useful later

$\boxed{\langle W \rangle_t = t \text{ (Quadratic Variation)}}$ & $\boxed{\mathbb{E}[W_{t+s} | F_t] = W_t \text{ (Martingale)}}$

Itô Process: $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$

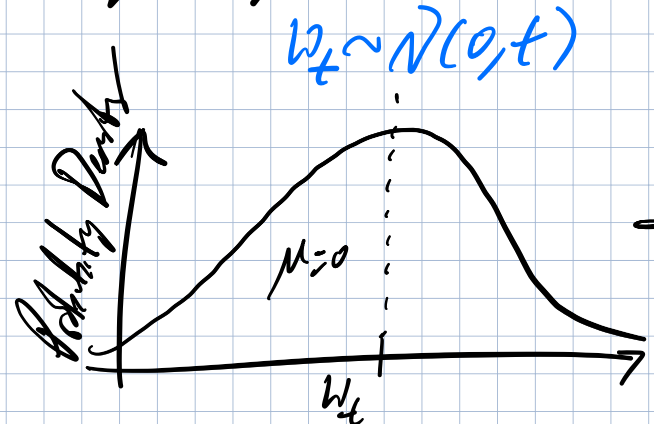
where $dW_t \sim N(0, dt)$

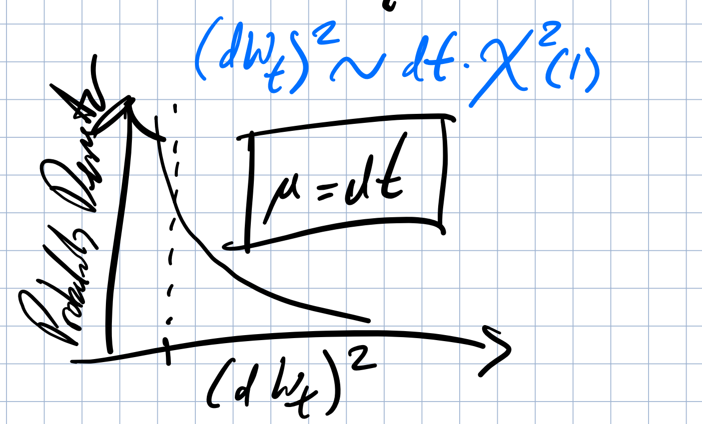
Itô Table: $\underline{(dt)^2 = 0}$ & $\underline{dt\,dW_t = 0}$ & $\underline{(dW_t)^2 = dt}$!!

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$Var(dW_t) = \mathbb{E}[(dW_t)^2] - (\cancel{\mathbb{E}[dW_t]})^2 \Rightarrow \boxed{dt = \mathbb{E}[(dW_t)^2]}$$

Alternatively we can use the χ^2 distribution ~~function~~

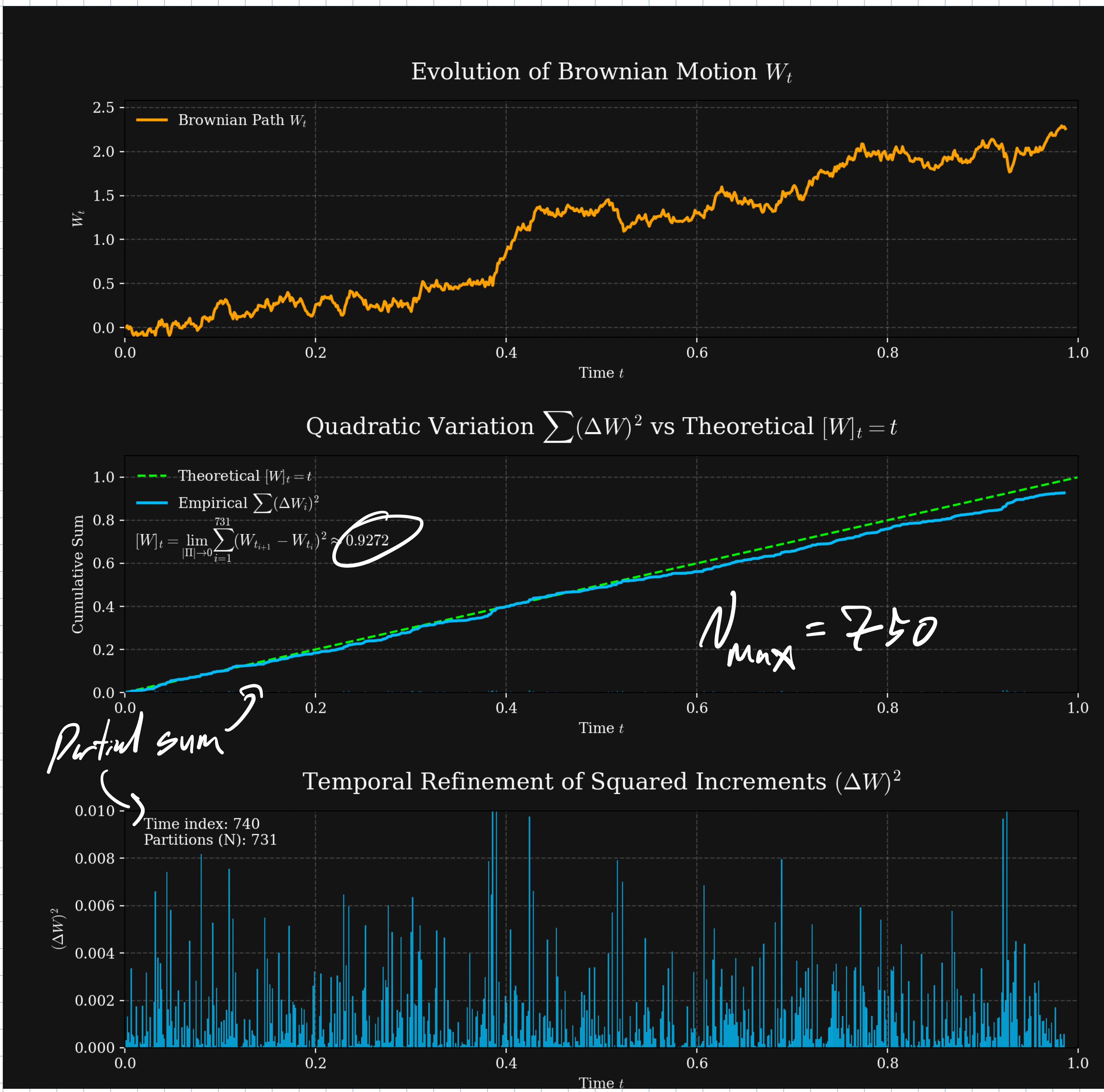




To go from $dt = \mathbb{E}[(dW_t)^2]$ to $dt = (dW_t)^2$

Quadratic Variation

$$\boxed{\langle W\rangle_t = \lim_{\|\Pi\|\to 0} \sum_k \left(W_{t_k} - W_{t_{k-1}}\right)^2 = t}$$



$\therefore$ Infinitesimally $(dW_t)^2 = dt$ // Expand via 2nd order Taylor Series

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2} + \cancel{\frac{\partial^2 f}{\partial t \partial x}dt\,dX_t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

$\rightarrow$ Recall $dX_t = \mu dt + \sigma dW_t$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\left[\mu dt + \sigma dW_t\right] + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left[\mu dt + \sigma dW_t\right]^2$$

$$= \frac{\partial f}{\partial t}dt + \mu\frac{\partial f}{\partial x}dt + \sigma\frac{\partial f}{\partial x}W_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left[\cancel{\mu^2 dt^2} + \cancel{2\mu\sigma\,dt\,dW_t} + \sigma^2(dW_t)^2\right]$$

$$df = \frac{\partial f}{\partial t}dt + \mu\frac{\partial f}{\partial x}dt + \sigma\frac{\partial f}{\partial x}W_t + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}dt$$

$$\boxed{df(t, X_t) = \left[\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right]dt + \sigma\frac{\partial f}{\partial x}dW_t}$$