Derivation of the Black-Scholes Equation from Itô's Lemma

There are only 2 assets in this simplified universe because we don't want arbitrage $\rightarrow$ Risk free bank account

$$dB_t = rB_t\,dt \quad \& \quad B_0 = 1$$

Risky stock $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ (Geometric Brownian Motion)

$\rightarrow$ Apply Itô's Lemma to the option price $V(S,t)$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2 \quad \text{Plug in GBM}$$

$$dV = \frac{\partial V}{\partial t}dt + \mu S_t\frac{\partial V}{\partial S}dt + \sigma S_t\frac{\partial V}{\partial S}dW_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\Big[\cancel{\mu^2 S_t^2\, dt^2} + \cancel{2\mu S_t^2\, dt\, \sigma\, dW_t} + \sigma^2 S_t^2\, \underbrace{dW_t^2}_{\rightarrow dt}\Big]$$

Recall Itô's Table: $(dt)^2 = 0 \quad dt\,dW_t = 0 \quad (dW_t)^2 = dt$

$$\Rightarrow dV = \frac{\partial V}{\partial t}dt + \mu S_t\frac{\partial V}{\partial S}dt + \sigma S_t\frac{\partial V}{\partial S}dW_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}dt$$

$$\boxed{dV = \left(\frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S_t\frac{\partial V}{\partial S}dW_t}$$

Final Itô's Lemma for $V(S,t)$ Geometric Brownian Motion

- We must eliminate risk to avoid arbitrage $\Rightarrow$ Remove dW_t

$$\Pi_t = \underbrace{V(S,t)}_{\text{long option}} - \underbrace{\Delta S_t}_{\Delta \text{ shares of stock - hedging needed to cancel randomness}} \Rightarrow \text{Itô} \Rightarrow d\Pi_t = dV - \Delta\, dS_t$$

$$\Rightarrow \Delta\, dS_t = \Delta\mu S_t\,dt + \Delta\sigma S_t\,dW_t$$

$$\Rightarrow d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S_t\frac{\partial V}{\partial S}dW_t - \Delta\mu S_t\,dt + \Delta\sigma S_t\,dW_t$$

$$\Rightarrow d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} - \Delta \mu S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \underbrace{\left(\sigma S_t \frac{\partial V}{\partial S} - \Delta \sigma S_t\right)}_{\text{need to set to } 0} dW_t$$

$$\sigma S_t \left(\frac{\partial V}{\partial S} - \Delta\right) = 0$$

$$\boxed{\frac{\partial V}{\partial S} = \Delta}$$

$$\Rightarrow d\Pi_t = \left(\frac{\partial V}{\partial t} + \cancel{\mu S_t \frac{\partial V}{\partial S}} - \cancel{\mu S_t \frac{\partial V}{\partial S}} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

$$\boxed{d\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt} \quad \& \quad \boxed{\Pi_t = V - S_t \frac{\partial V}{\partial S}}$$

Our portfolio must grow w/ risk free rate $r \Rightarrow d\Pi_t = r\Pi_t \, dt$

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) \cancel{dt} = r\left(V - S_t \frac{\partial V}{\partial S}\right) \cancel{dt}$$

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S} - rV = 0}$$

Black-Scholes PDE (Backward-in-time)

Terminal Conditions $\rightarrow V_{call}(S,T) = \max(S-K, 0)$

$\rightarrow V_{put}(S,T) = \max(K-S, 0)$

Define new variables $\rightarrow$ $x = \ln\left(\frac{S}{K}\right)$ & $\tau = T - t$

Rewrite as a function of x & τ $\Rightarrow$ $\boxed{V(S,t) = Ku(x,\tau)}$

$$\Rightarrow \frac{\partial}{\partial t}\left(Ku(x,\tau)\right) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S^2}\left(Ku(x,\tau)\right) + r S_t \frac{\partial}{\partial S}\left(Ku(x,\tau)\right) - r\left(Ku(x,\tau)\right) = 0$$

$$\Rightarrow K\left[\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru\right] = 0$$

Calculate derivatives $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2}$ in terms of $Ku(x,\tau)$

$$\frac{\partial V}{\partial t} = K\frac{\partial}{\partial t}\left[u(x,\tau)\right] = K\frac{\partial u}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial t}}_{=-1} + \cancel{K\frac{\partial u}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{=0}} = \boxed{-K\frac{\partial u}{\partial \tau}}$$

$$\frac{\partial V}{\partial S} = K\frac{\partial}{\partial S}\left[u(x,\tau)\right] = K\frac{\partial u}{\partial x}\frac{\partial x}{\partial S} + \cancel{K\frac{\partial u}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial S}}_{=0}} = \boxed{\frac{K}{S}\frac{\partial u}{\partial x}}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left[\frac{K}{S}\frac{\partial u}{\partial x}\right] = -\frac{K}{S^2}\frac{\partial u}{\partial x} + \frac{K}{S^2}\frac{\partial^2 u}{\partial x^2} = \boxed{\frac{K}{S^2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)}$$

Plug 3 derivatives back into Black-Scholes

$V = \cancel{K}u(x,\tau)$

$$\Rightarrow -\cancel{K}\frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2 \cancel{S^2}\frac{\cancel{K}}{\cancel{S^2}}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) + r\cancel{S}\frac{\cancel{K}}{\cancel{S}}\frac{\partial u}{\partial x} - rV = 0$$

$$\boxed{\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} - ru = \frac{\partial u}{\partial \tau}}$$

Black-Scholes PDE in (x,τ) coordinates

We must now turn this into the standard heat equation $\boxed{\frac{\partial \tilde{u}}{\partial \tau} = \frac{\partial^2 \tilde{u}}{\partial y^2}}$

Define a new function $\tilde{u}(x,\tau)$ such that:

$$\boxed{u(x,\tau) = e^{Ax+B\tau}\cdot\tilde{u}(x,\tau)}$$

→ We must choose constants A & B carefully

→ We want to cancel out terms so that $\tilde{u}$ satisfies the heat equation

Calculate $\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \tau}\left[e^{Ax+B\tau}\cdot\tilde{u}\right] = Be^{Ax+B\tau}\cdot\tilde{u} + e^{Ax+B\tau}\cdot\frac{\partial \tilde{u}}{\partial \tau} = e^{Ax+B\tau}\left(\frac{\partial \tilde{u}}{\partial \tau} + B\tilde{u}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[e^{Ax+B\tau}\cdot\tilde{u}\right] = Ae^{Ax+B\tau}\cdot\tilde{u} + e^{Ax+B\tau}\cdot\frac{\partial \tilde{u}}{\partial x} = e^{Ax+B\tau}\left(\frac{\partial \tilde{u}}{\partial x} + A\tilde{u}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[e^{Ax+B\tau}\left(\frac{\partial \tilde{u}}{\partial x} + A\tilde{u}\right)\right] = Ae^{Ax+B\tau}\left(\frac{\partial \tilde{u}}{\partial x} + A\tilde{u}\right) + e^{Ax+B\tau}\cdot\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + A\frac{\partial \tilde{u}}{\partial x}\right) = e^{Ax+B\tau}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + 2A\frac{\partial \tilde{u}}{\partial x} + A^2\tilde{u}\right)$$

Plug derivatives back into Black-Scholes PDE in (x,τ) coordinates

$$\cancel{e^{Ax+B\tau}}\left(\frac{\partial \tilde{u}}{\partial \tau}+B\tilde{u}\right)=\frac{1}{2}\sigma^2\cancel{e^{Ax+B\tau}}\left(\frac{\partial^2\tilde{u}}{\partial x^2}+2A\frac{\partial \tilde{u}}{\partial x}+A\tilde{u}\right)+\left(r-\frac{1}{2}\sigma^2\right)\cdot$$
$$\cancel{e^{Ax+B\tau}}\left(\frac{\partial \tilde{u}}{\partial x}+A\tilde{u}\right)-\tilde{u}re^{Ax+B\tau}$$

$$\boxed{\frac{\partial \tilde{u}}{\partial \tau}+B\tilde{u}=\frac{1}{2}\sigma^2\frac{\partial^2\tilde{u}}{\partial x^2}+\left(\sigma^2A+r-\frac{1}{2}\sigma^2\right)\frac{\partial \tilde{u}}{\partial x}+\left(\frac{1}{2}\sigma^2A^2+\left(r-\frac{1}{2}\sigma^2\right)A-r\right)\tilde{u}}$$

Intermediate-form Black-Scholes PDE w/ exponential gauge substitution

We must solve for A & B so that:

$$\underline{\left(\sigma^2A+r-\frac{1}{2}\sigma^2\right)\frac{\partial \tilde{u}}{\partial x}=0} \quad \& \quad \underline{B\tilde{u}=\left(\frac{1}{2}\sigma^2A^2+\left(r-\frac{1}{2}\sigma^2\right)A-r\right)\tilde{u}}$$

$$\boxed{A=\frac{1}{2}-\frac{r}{\sigma^2}}$$

Plug in

$$B=\frac{1}{2}\sigma^2\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)^2+\left(r-\frac{1}{2}\sigma^2\right)\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)-r$$

$$\boxed{B=-\frac{1}{2}r-\frac{1}{2}\frac{r^2}{\sigma^2}-\frac{1}{8}\sigma^2} \Rightarrow \text{Setting } A \,\&\, B \text{ into the Black-Scholes}$$

$$\boxed{\frac{\partial \tilde{u}}{\partial \tau}=\frac{1}{2}\sigma^2\frac{\partial^2\tilde{u}}{\partial x^2}}$$

Standard Heat Equation $\Rightarrow$ set $a^2=\frac{1}{2}\sigma^2$

$\Rightarrow \dfrac{\partial \tilde{u}}{\partial \tau}=a^2\dfrac{\partial^2\tilde{u}}{\partial x^2} \rightarrow$ Set initial condition $(\tau=0)$ option expiry

Recall $Ku(x,\tau)=V(s,t) \Rightarrow u(x,0)=\max\left(\frac{S}{K}-1,0\right)$

Recall $x=\ln\left(\frac{S}{K}\right) \Rightarrow \frac{S}{K}=e^x$ substitute

Normalized option payoff (initial condition) $\rightarrow$ $\boxed{u(x,0)=\max(e^x-1,0)}$

Solve for $\tilde{u}(x,\tau)$ when $\tau=0$

Recall $u(x,\tau)=e^{Ax+B\tau}\cdot\tilde{u}(x,\tau) \Rightarrow \tilde{u}(x,\tau)=e^{-Ax-B\tau}\cdot u(x,\tau)$ $\quad \tau=0$

$\Rightarrow \boxed{\tilde{u}(x,0)=e^{-Ax}\cdot\max(e^x-1,0)}$ $\rightarrow$ Initial condition for Heat equation $\rightarrow$ Transformed Scaled version of the original option payoff

Now we solve the heat equation

$$\tilde{u}(x,\tau)=\frac{1}{\sqrt{4\pi a^2\tau}}\int_{-\infty}^{\infty}f(y)\cdot\exp\left(-\frac{(x-y)^2}{4a^2\tau}\right)dy \rightarrow \text{Convolution with the heat kernel}$$

Standard result: $\boxed{\tilde{u}(x,\tau) = \Phi(d_1) - e^{x}\Phi(d_2)}$

where: $\boxed{d_1 = \dfrac{x + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}$ & $\boxed{d_2 = d_1 - \sigma\sqrt{\tau}}$

Reverse from $\tilde{u}(x,\tau) \rightarrow u(x,\tau)$ (Recall $u(x,\tau) = e^{Ax+B\tau}\cdot\tilde{u}(x,\tau)$)

$\Rightarrow u(x,\tau) = e^{Ax+B\tau}\Phi(d_1) - e^{x(A+1)+B\tau}\Phi(d_2)$

Recall $V(S,t) = Ku(x,\tau) \rightarrow$ Replace $u(x,\tau)$ for $V(S,t)$

$\Rightarrow V(S,t) = Ke^{Ax+B\tau}\Phi(d_1) - Ke^{x(A+1)+B\tau}\Phi(d_2) \rightarrow$ Replace $x = \ln(\frac{S}{K})$

$\Rightarrow V(S,t) = Ke^{A\ln(\frac{S}{K})}e^{B\tau}\Phi(d_1) - Ke^{\ln(\frac{S}{K})(A+1)}e^{B\tau}\Phi(d_2)$

$$V(S,t) = e^{B\tau}\left[S^A K^{1-A}\Phi(d_1) - S^{A+1}K^{-A}\Phi(d_2)\right]$$

$\rightarrow S$ $\qquad \rightarrow Ke^{-r(T-t)}$

We want to demonstrate that $e^{B\tau}S^A K^{1-A}\Phi(d_1) = \underline{S\Phi(d_1)}$

$e^{B\tau}S^{A+1}K^{-A}\Phi(d_2) = \underline{Ke^{-r(T-t)}}$

First Term

$e^{B\tau}S^A K^{1-A} \rightarrow S\left(\dfrac{K^{1-A}e^{B\tau}}{S^{1-A}}\right) \rightarrow$ Plug in $= S$

$A = \frac{1}{2} - \frac{r}{\sigma^2}$, $\tau = T - t$,

$B = -\frac{1}{2}r - \frac{r^2}{2\sigma^2} - \frac{\sigma^2}{8}$

$\Rightarrow V(S,t) = \underline{S\Phi(d_1)} - e^{B\tau}S^{A+1}K^{-A}\Phi(d_2)$

Second Term

$e^{B\tau}S^{A+1}K^{-A} \rightarrow S\left(\dfrac{S^A e^{B\tau}}{K^A}\right) \rightarrow$ Plug in $= Ke^{-r(T-t)}$

$\Rightarrow V(S,t) = S\Phi(d_1) - \underline{Ke^{-r(T-t)}\Phi(d_2)}$

$$\boxed{\therefore V(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \quad \text{where } d_1 = \frac{\ln(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \;\&\; d_2 = d_1 - \sigma\sqrt{T-t}}$$

Q.E.D.