The Riemann Zeta Function and Primes

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Abstract

In this paper, we will be discussing the Riemann Zeta Function $\zeta(s)$ and how it's surprisingly related to prime numbers. We will first discuss why this topic is important to begin with, then we will transition into a little bit of history. In the history portion we will briefly cover a timeline from Leonhard Euler to Bernhard Riemann and discuss some of the math involved. Subsequently, we will begin to discuss more specific details of the Riemann Zeta function like the Zeta Zeros. It would also be important to understand the prime counting function $\pi(x)$ and its approximations. Only then, will we be able to touch on the idea as to how the Riemann Zeta function and the prime numbers are related.

1 Introduction

To begin, whether you have or haven't yet heard of the Riemann Zeta function $\zeta(s)$ is not important since we will be going over its most basic properties. But before we get into what the Riemann Zeta function actually is, it would be important to familiarize ourselves with the motivation behind writing this paper.

(a) Motivation

The reader should note that a large portion of the motivation of this paper is mathematical applications in computer science. More specifically, asymmetric encryption which utilizes the creation of keys using large prime numbers. These primes used are astronomical, and quite impossible, to calculate with classical computing.

The reason behind using prime numbers to encrypt our data is due to the original intention of computer networking as never being designed to keep secret information. Mathematicians found ways to hide private data utilizing prime numbers, which is used today to keep our banking information and social security numbers private.

The important note here is if we were to discover a way to predict or find any and all prime numbers, this would threaten privacy as we know it. This is where the Riemann Zeta function and the Zeta zeros bring much importance in the field of cryptography.

(b) $\zeta(s)$ and why its significant.

So what is the Riemann Zeta function anyways?

Simply put, its a simple summation that takes on complex values s. More specifically, the summation is defined by the following mathematical notation.

$$\zeta(s) = \sum \frac{1}{n^s}$$

This formula is so simple, that even an elementary school student could likely recognize the pattern when written out like so.

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + ...$$

The only portion that is slightly more interesting than its simple pattern, is the values it takes. We define s as a complex variable of the form $s = x + iy$, where x and y are real numbers and i is the imaginary part of s.

Other than its interesting property of taking in and outputting complex numbers, the Riemann Zeta function is purely just a simple summation. That's all it is.

So why is the Riemann Zeta function significant?

There is currently a **1 Million Dollar prize** for anyone who can solve the Riemann Hypothesis.[11] This Hypotheses, which we will be going over later in this paper, has remained unsolved for over well over 100 years.

If the Riemann Hypothesis is solved, this would lead to possible new techniques of finding all non-trivial Zeta Zeros, which would then lead to finding all prime numbers.

As mentioned previously, this would be a turning point in how we as a society secure our personal data which is pretty significant.

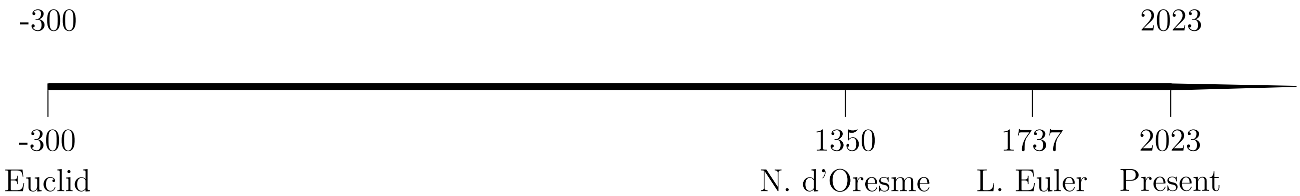
2 History of $\zeta(s)$

Before going further into the Riemann Zeta function, its important to understand the history of how this function was created to begin with. We will be going over initial concepts, that were over time built on in order to get to our function $\zeta(s)$ that we know and love today.

(a) The early beginnings

We begin by noting the following timeline containing the year some notable mathematicians found some important proofs.

[11] Parker, Matt. "Win a Million Dollars with Maths, No. 1: The Riemann Hypothesis." The Guardian, 3 Nov. 2010, www.theguardian.com/science/blog/2010/nov/03/million-dollars-maths-riemann-hypothesis.



300 B.C. - Euclid proved there were an infinite number of primes.[1]

We begin with the year 300BC, when Euler proved there are infinitely many primes. The proof's general outline is as follows.[3]

Theorem (Fundamental Theorem of Arithmetic)**.** *Every positive integer (except the number 1) can be represented in exactly one way apart from rearrangement as a product of one or more primes. Wolfram MathWorld* [12]

Proof. By The Fundamental Theorem of Arithmetic, we know all composite numbers can be made of some product of primes.

Say we have a finite number of primes. We will allow the number $N \in \mathbb{N}$ to have the prime decomposition of all primes. We have the following.

$$N = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot p_4^{e_4} \cdot \ldots \cdot p_n^{e_n}$$
$$p_i \in \{\text{unique primes}\} \text{ and } e_j \in \mathbb{N} \cup \{0\}$$

We will now add 1 to N to get the following.

$$N + 1 = (p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot p_4^{e_4} \cdot \ldots \cdot p_n^{e_n}) + 1$$

But we see that no prime number is a factor of N+1. We have a contradiction with our theorem.

Thus, there are an infinite number of primes. □

1350 - Nicole d'Oresme proved the harmonic series ($\sum \frac{1}{n}$) diverges.[1] [2]

It's worthy to note that for many years this problem stumped mathematicians until this discovery.

1737 - Leonhard Euler proves the sum of reciprocals of primes diverges.[1]

We will discuss more about Euler's discovery, as well as a surprising fact, below.

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[1] Edwards, Harold M. Riemann's Zeta Function. Dover Publications, 1974.

[3] Harrison, Lindsey. "From Euclid to Present: A Collection of Proofs regarding the In- finitude of Primes." GEORGIA'S DESIGNATED PUBLIC LIBERAL ARTS UNIVER- SITY, 14 Dec. 2013, www.gcsu.edu/sites/files/page-assets/node-808/attachments/harrison.pdf.

[12] Weisstein, Eric W. "Fundamental Theorem of Arithmetic." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/FundamentalTheoremofArithmetic.html

[2] Weisstein, Eric W. "Harmonic Series." From MathWorld–A Wolfram Web Resource.

(b) More about Euler's discovery

As mentioned, Euler discovered the following.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+...=\infty$$

Because of this discovery, Euler's proof gives a strong implication that the primes are rather dense in the integers.[1] We see this if we were to compare the above summation to the sum of the reciprocal of squares below, which sums to $\frac{\pi^2}{6}$.[4]

$$\text{(Diverges)} \sum_{p \text{ prime}}^{\infty} \frac{1}{p} >> \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ (Converges)}$$

Because of this, it can be implied that the prime numbers are much more dense in the integers than the squares.

Euler dove deeper into looking into the sums of the reciprocals of all powers. He then found an interesting pattern, which will be demonstrated below.[1]

Note: the $\star$ symbol denotes side work needed to understand the mathematical manipulation being done.

$$\text{Let } g(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+...=\sum_{n=1}^{\infty}\frac{1}{n^s},\ s\in\mathbb{R}$$

$$\star\frac{1}{2^s}g(s)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+...$$

$$(1-\frac{1}{2^s})g(s)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+...$$

$$\star\frac{1}{3^s}g(s)=\frac{1}{3^s}+\frac{1}{6^s}+\frac{1}{9^s}+\frac{1}{12^s}+\frac{1}{15^s}+...$$

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})g(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+...$$

$$(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})g(s)=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}...$$

$$(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})g(s)=1+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}...$$

[1] Edwards, Harold M. Riemann's Zeta Function. Dover Publications, 1974.
[4] Eremenko, A. "How Euler Found the Sum of Reciprocal Squares." Purdue University, 5 Nov. 2013, www.math.purdue.edu/eremenko/dvi/euler.pdf

We begin to notice a pattern ...

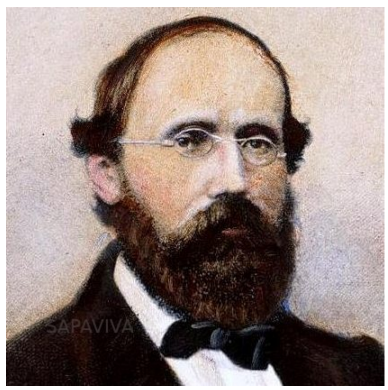
We begin to see that the primes show up in Euler's work. This then becomes the Euler product formula below.[1]

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \; n \in \mathbb{N}, s \in \mathbb{R}$$

Further more, we begin to see that $\sum \frac{1}{n^s}$ contains information about the prime numbers.

(c) Contributions of Riemann

Bernhard Riemann was a great mathematician, where his discovery of the Riemann Zeta function and its million dollar prize still troubles many in solving.



A huge contribution of Riemann is in his methods, from his published 1859 paper. His methods include the following.[1]

- The study of $\zeta(s)$ as a function of a complex variable. $\star$
- The study of complex zeros. $\star$
- Fourier inversion
- Möbius inversion.
- Approximation of functions like $\pi(x)$. $\star$

Note: of this list above, we will only be looking at the items with the $\star$ symbols.

Riemann expanded on Euler's work of his product formula below.[1]

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}, \; s \in \mathbb{R}$$

[1] Edwards, Harold M. Riemann's Zeta Function. Dover Publications, 1974.

Riemann decided to expand this function to the complex plane, where s is a complex input, and the output of the function is also complex.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ s \in \mathbb{C}$$

This function, shown above, is what's known as the **Riemann Zeta function**.[1]

3 $\zeta(s)$ and its zeros

Armed with his newly found formula, Riemann wanted to know one decievingly simple question.

Riemann asked: **When does** $\zeta(s) = 0$**?** (Otherwise known as *"Zeta Zeros"*)

In order to answer this question, we will go through what we know about these zeros of the Riemann Zeta function.

First, we must understand that there are two types of zeta zeros. These are **trivial** and **non-trivial** zeros.[10]

The **trivial zeros** are located at the negative even integers as follows.

$$\zeta(s) = 0 \impliedby s \in \{(-2+i0), (-4+i0), (-6+i0), (-8+i0), ...\}$$

This is found using techniques in analytic continuation, which will not be discussed in this paper. We can just note to ourselves that analytic continuation is used to extend our domain of a function. In this case $\zeta(s)$ is defined only when $s > 1$ so by Analytic Continuation it could be expanded to where $s \leq 1$.

The **non-trivial zeros** are values that have been shown to be located between $0 < \mathrm{Re}s < 1$, which is called the **critical strip**. This critical strip is illustrated on the imaginary plane below.[5]

[10]Frenkel, Edward. "Riemann Hypothesis - Numberphile." Youtube, 11 Mar. 2014, www.youtube.com/watch?v=d6c6uIyieoo
[5]Weisstein, Eric W. "Critical Strip." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/CriticalStrip.html

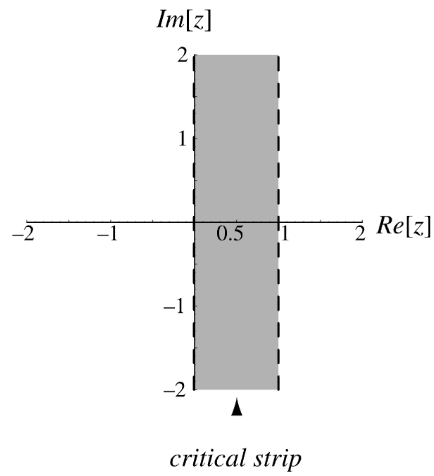
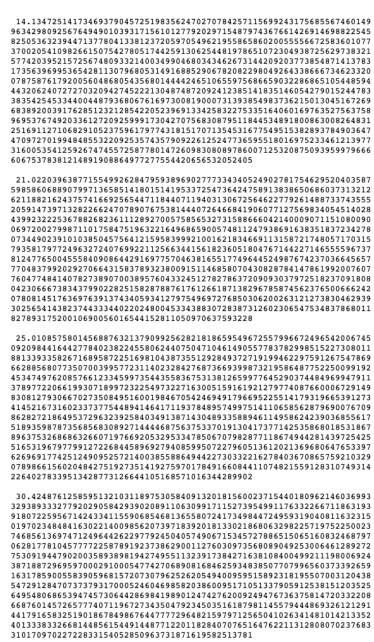


Image from Wolfram MathWorld

This leads us to the famous **Riemann Hypothesis**, which Riemann hypothesised that all non-trivial zeros contained in this strip lie on the line where Res is $\frac{1}{2}$.[10]

Countless attempts have been made to prove (or disprove) the Riemann Hypothesis, with none being successful to this date.

Of the trivial zeros, the following is a snippet of the imaginary parts of the first four zeta zeros.[6]



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Numbers provided by AT&T Labs

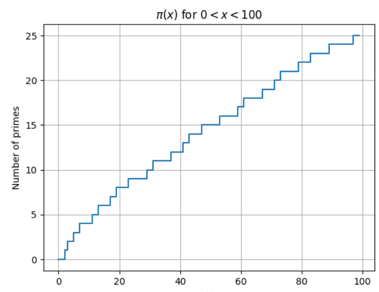
[6]Odlyzko, Andrew M. "The First 100 (Non Trivial) Zeros of the Riemann Zeta Function." Plouffe.Fr, www.plouffe.fr/simon/constants/zeta100.html

4 The Prime Counting Function $\pi(x)$

The prime counting function, denoted $\pi(x)$, is a function that counts the prime numbers.[9] More specifically, the output increments by 1 every time a number in the input is prime. A few inputs and outputs to the function is shown below.

$$\pi(1) = 0$$
$$\pi(2) = 1$$
$$\pi(3) = 2$$
$$\pi(4) = 2$$
$$\pi(5) = 3$$

The following is a graph of $\pi(x)$ that I've created in Python using the matplotlib library.



Graph programmed by Joshua Barsky

Its important to note that there is no current written formula for the prime counting function, but rather a concept of how it works. Since the inception of $\pi(x)$, finding an explicit formula for this function has been highly sought after.

5 Approximating $\pi(x)$

(a) Gauss approximating $\pi(x)$

Intrigued by the prime counting function, in 1792, Carl Friedrich Gauss discovered that the density of prime numbers on average appears to be $\frac{1}{\log x}$.[1] Gauss then proceeded to calculate, by hand, the number of primes under a specified value and compare it to his integral function using his discovery of the density of prime numbers.

The following table shows some of Gauss's hand calculated values for the number of primes under a specific value chosen for x.[1]

[9]Unknown author. "The "Encoding" of the Distribution of Prime Numbers by the Nontrivial Zeros of the Riemann Zeta Function [Common Approach]." University of Exeter, empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/encoding1.htm.
[1]Edwards, Harold M. Riemann's Zeta Function. Dover Publications, 1974.

x	Count of primes	$\int \frac{dn}{\log n}$	Difference
500,000	41,556	41,606.4	50.4
1,000,000	78,501	78,627.5	126.5
3,000,000	216,745	216,970.6	225.6

At the time, it wasn't clear what Gauss exactly meant by the integral formula in the table. But when compared to D.N. Lehmer's later work, we can guess Gauss meant to integrate a continuous variable n from 2 to x. Below is the work done by D.N. Lehmer.[1]

x	Count of primes	$\int_2^x \frac{dt}{\log t}$	Difference
500,000	41,538	41,606	68
1,000,000	78,498	78,628	130
3,000,000	216,816	216,971	155

Today, the integral $\int_2^x \frac{dt}{\log t}$ is now known as the **Offset Logarithmic Integral**.[1]

In order to get the **Offset Logarithmic Integral** approximation for $\pi(x)$, the following calculation can be made.

$$\text{Li}(x) = \int_2^x \frac{dt}{\ln t} = \int_0^x \frac{dt}{\ln t} - \int_0^2 \frac{dt}{\ln t} = \text{li}(x) - \text{li}(2)$$

$$\text{Li}(x) = \text{li}(x) - \text{li}(2)$$

Where $\text{Li}(x)$ is the proper notation for the *Offset Logarithmic Integral*, and $\text{li}(x)$ is the proper notation for the regular *Logarithmic Integral* defined below.

$$\text{li}(x) = \int_0^x \frac{dt}{\ln t}$$

(b) Riemann approximating $\pi(x)$

Riemann's goal was to obtain a formula for $\pi(x)$. He began by defining a different kind of prime counting function $J(x)$. The difference between $\pi(x)$ and $J(x)$ is instead of adding 1 at each prime, $J(x)$ adds $\frac{1}{n}$ at each nth prime power $p^n \leq x$.[1]

Using this $J(x)$, Riemann was then able to show the **connection** between ζ and J.

$$\frac{\log \zeta(s)}{s} = \int_0^\infty J(x) x^{-s-1} dx$$

[1] Edwards, Harold M. Riemann's Zeta Function. Dover Publications, 1974

We notice that the left hand side of the equation contains $\zeta(s)$, while the right hand side of the equation contains $J(x)$. Hence, the connection being shown.

He then was able to find the following formula using the zeta zeros for each p in the summation portion of the equation.[7]

$$J(x) = \mathrm{li}(x) - \sum_{p} \mathrm{li}(x^{p}) - \ln(2) + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\ln(t)}$$

[Cantors Paradise Website]

Working his way from $J(x)$, he then used a tool in Number Theory called Möbius inversion to obtain the following formula for $\pi(x)$.

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}})$$

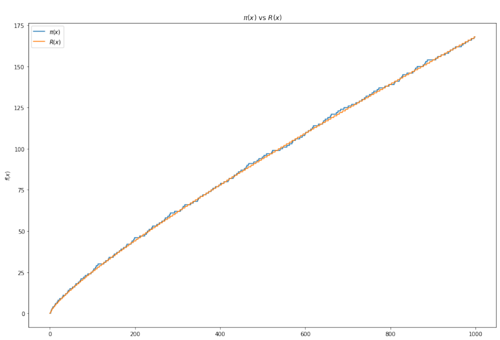
Which, at first glance, may look to be a formula for our prime counting function. Upon further inspection, we can conclude that the equation above is merely a translation between $\pi(x)$ and $J(x)$. In other words, we are restating the same prime counting function in two different ways using a formula.

Riemann was then able to create the following formula for an approximation of $\pi(x)$. It's also nice to see that Riemann's approximation is identical to the *Gram Series* shown on the very right hand side. Notice how the *Gram Series* utilizes the Riemann Zeta function.[7] [8]

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{\frac{1}{n}}) = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k! k \zeta(k+1)}$$

[Cantors Paradise Website]

We will now look at how $R(x)$ compares to $\pi(x)$ for the first one thousand values. The following is a graph I've created in Python using the matplotlib library.

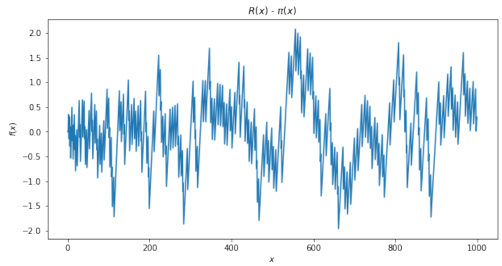


Graph programmed by Joshua Barsky

[7] Müller, Kasper. "Riemann'S Explicit Formula — A Beautiful Expression for the Prime Counting Function." Cantors Paradise, 24 Aug. 2022, www.cantorsparadise.com/riemanns- explicit-formula-a-beautiful-expression-for-the-prime-counting-function-1c83f5b65dfd

[8] Weisstein, Eric W. "Riemann Prime Counting Function." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/RiemannPrimeCountingFunction.html

We can also take a look to see how $R(x)$ differs from $\pi(x)$ for their first one thousand values. The following is a graph I've created in Python using the matplotlib library.



Graph programmed by Joshua Barsky

Clearly, we see the values remain close for this large-scale of values with a maximum difference of two. This is quite remarkable.

6 Riemann Zeta and the Primes

Let us look again at Riemann's $R(x)$ approximation.

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{\frac{1}{n}})$$

We can use this approximation to define our prime counting function $\pi(x)$ as follows.[7]

$$\pi(x) = R(x) - \sum_{p} R(x^{p}) - \sum_{m=1}^{\infty} R(x^{-2m})$$

So do we, after all, have a prime counting function which we can define and use to create havoc in the world of today's encryption? Well, not exactly.

The problem is the term $\sum_p R(x^p)$ is a summation using all **non-trivial zeros p**. We can recall that we cannot find all of the non-trivial zeros. That, of course, is the basis of the Riemann Hypothesis which hasn't been solved.

7 An interesting relationship

As we discovered, the Riemann Zeta function has this unexpected and surprising relationship to primes. Of course we defined a function $\pi(x)$ assuming we had all zeta zeros, but we could also see another interesting result even if we were to graph this defined formula using just a handful of these zeta zeros. We begin to see our formulated graph gets closer and closer to true $\pi(x)$ as we add zeta zeros into the equation. Below is an illustration when eighteen zeta zeros are added into our formulated $\pi(x)$.

[7] Müller, Kasper. "Riemann'S Explicit Formula — A Beautiful Expression for the Prime Counting Function." Cantors Paradise, 24 Aug. 2022, www.cantorsparadise.com/riemanns- explicit-formula-a-beautiful-expression-for-the-prime-counting-function-1c83f5b65dfd

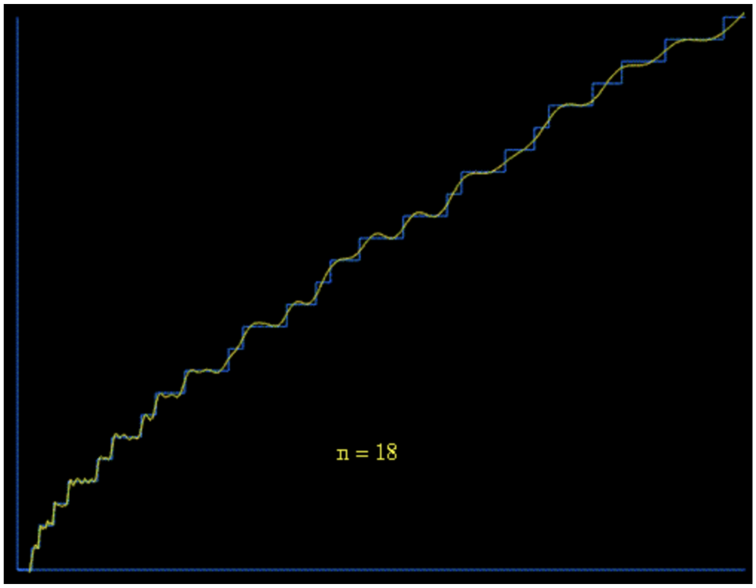


Image from empslocal.ex.ac.uk

As we add more zeta zeros, the closer the formulated graph resembles true $\pi(x)$.

8 Conclusion

We see that $\zeta(s)$ and the prime numbers have an interesting relationship. Showing that despite its seeming unpredictability, that the prime numbers may infact have some type of order or pattern we have yet to uncover. The Riemann Zeta function provides us with the vehicle into this insight, that we would have never had otherwise. Maybe one day we can solve the Riemann Hypothesis and finally be able to understand primes in ways we could never have thought possible. Hopefully if that's done we have already transitioned our encryption methods to other mathematical models not involving prime numbers (there is talk of encryption using vector spaces).

Hopefully you've found this paper helpful in your understanding of the Riemann Zeta function and how closely it is related to the prime numbers. It's exciting to look forward to what in our world may change as more and more complex equations become solved (like maybe the Riemann Hypothesis one day).

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