Quantified Boolean Formulas: Solving and Proofs

Lower Bounds

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Universal Expansion

Overview of Universal Expansion

- A method of deleting universal variables from a QBF
- After expansion we have only existential variables, i.e. a propositional formula
- This propositional formula is called the expansion of Φ, written $\exp(\Phi)$
- Semantics preserved: $\exp(\Phi)$ is satisfiable if, and only if, Φ is true
- Expansion introduces new variables and increases the formula size (exponentially in the worst case)

Expansion of One Universal Variable

- Consider a QBF with a single universal variable
$$\Phi = \exists x \forall u \exists y \cdot F(x, u, y)$$
- Eliminate variable u by expansion
$$\exists x \exists y_0 \exists y_1 \cdot F(x, 0, y_0) \wedge F(x, 1, y_1)$$
- We use two copies of y and two copies of F to respect the dependence of y on the expanded variable u
- Expanding u does not change the truth value
- The resulting formula has only existential variables, so it is essentially a propositional formula
$$\exp(\Phi) = F(x, 0, y_0) \wedge F(x, 1, y_1)$$
- $\exp(\Phi)$ is satisfiable if, and only if, Φ is true

Expansion of Two Universal Variables

- Consider a QBF with two universal variables

$$\Phi = \exists x \forall u \exists y \forall v \exists z \cdot F(x, u, y, v, z)$$

- Eliminate variable u by expansion

$$\exists x \exists y_0 \exists y_1 \forall v \exists z_0 \exists z_1 \cdot F(x, 0, y_0, v, z_0) \wedge F(x, 1, y_1, v, z_1)$$

- Eliminate variable v by expansion

$$\exists x \exists y_0 \exists y_1 \exists z_{00} \exists z_{01} \exists z_{10} \exists z_{11} \cdot F(x, 0, y_0, 0, z_{00}) \wedge$$
$$F(x, 0, y_0, 1, z_{01}) \wedge F(x, 1, y_1, 0, z_{10}) \wedge F(x, 1, y_1, 1, z_{11})$$

- Neither expansion changes the truth value

$$\exp(\Phi) = F(x, 0, y_0, 0, z_{00}) \wedge F(x, 0, y_0, 1, z_{01}) \wedge$$
$$F(x, 1, y_1, 0, z_{10}) \wedge F(x, 1, y_1, 1, z_{11})$$

- $\exp(\Phi)$ is satisfiable if, and only if, Φ is true

Annotating with assignments

- In general, if there are n universal variables, the expansion is conjunction of 2^n substitution instances of the matrix
- Each substitution instance corresponds to one of the 2^n universal assingments
- To respect dependencies, variables must be copied
- In a substitution instance corresponding to $\alpha \in \langle \text{vars}_\forall(\Phi) \rangle$, a variable x is annotated with the restriction of α to its dependency set $L(x)$

$$\Phi = \exists x \forall u \exists y \forall v \exists z \cdot F(x, u, y, v, z)$$

$$\exp(\Phi) = \cdots \land F(x, 0, y_0, 1, z_{01}) \land \ldots$$

$$\exp(\Phi) = \cdots \land F(x, 0, y_{u \mapsto 0}, 1, z_{u \mapsto 0, v \mapsto 1}) \land \ldots$$

Universal Expansion in General

- The expansion of a QBF $\Phi = \mathcal{Q} \cdot F$ is the CNF

$$\exp(\Phi) := \bigcup_{\alpha \in \langle \mathsf{vars}_{\forall}(\Phi) \rangle} F\left[\alpha \cup \left\{ x \mapsto x_{\alpha \restriction L(x)} : x \in \mathsf{vars}_{\exists}(\Phi) \right\}\right]$$

- Proposition: For any QBF Φ, $\exp(\Phi)$ is satisfiable if, and only if, Φ is true.

- In fact, there is a natural one-one correspondence between satisfying assignments of $\exp(\Phi)$ and models of Φ.

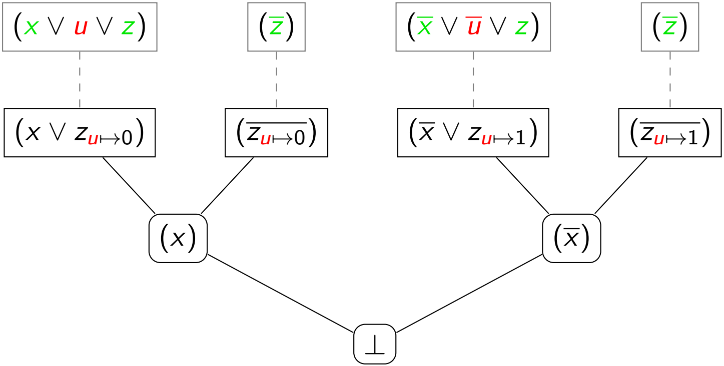
The QBF Proof System ∀Exp+Res

Definition: A ∀Exp+Res refutation of a QBF Φ is a Resolution refutation of $\exp(\Phi)$.

Example ∀Exp+Res Refutation

$$\Phi = \exists x \forall u \exists z \cdot (x \vee u \vee z) \wedge (\overline{x} \vee \overline{u} \vee z) \wedge (\overline{z})$$

$$\exp(\Phi) = (x \vee z_{u\mapsto 0}) \wedge (\overline{z_{u\mapsto 0}}) \wedge (\overline{x} \vee z_{u\mapsto 1}) \wedge (\overline{z_{u\mapsto 1}})$$



Which lower bound techniques apply?

Techniques for propositional proof systems

- size-width relation [Ben-Sasson & Wigderson 01]
- feasible interpolation [Krajíček 97]
- game-theoretic techniques [Pudlák, Buss, Impagliazzo, ...]

In QBF proof systems

- size-width relations fail for QBF resolution systems
- feasible interpolation holds for QBF resolution systems
- game-theoretic techniques work for weak tree-like systems [Beyersdorff et. al 16, 17, Chen 16]

We need new techniques

- not derived from propositional proof complexity

Lower Bounds via Semantic Measures

- Many QBF proof systems have strategy extraction
- From a refutation, we effeciently compute a countermodel
- Hence, if the countermodel is ‘large’, so is the refutation
- Gives rise to lower bound techniques based on semantic measures: definitions of countermodel ‘size’
- For example, minimal range of a countermodel

Definitions

- The partial expansion of a QBF $\Phi = \mathcal{Q} \cdot F$ w.r.t. a set of universal assignments $R \subseteq \langle \text{vars}_\forall(\Phi) \rangle$ is the CNF

$$\exp(\Phi, R) := \bigcup_{\alpha \in R} F\left[\alpha \cup \left\{x \mapsto x_{\alpha \restriction L(x)} : x \in \text{vars}_\exists(\Phi)\right\}\right]$$

- A countermodel for a QBF F is a function $f : \langle \text{vars}_\exists(F) \rangle \to \langle \text{vars}_\forall(F) \rangle$ such that
  - dependency: for each $u \in \text{vars}_{\forall(F)}$ and $\alpha, \beta \in \langle \text{vars}_\exists(F) \rangle$, if α, β agree on $L(u)$, then $f(\alpha), f(\beta)$ agree on u.
  - semantic for each $\alpha \in \langle \text{vars}_\exists(F) \rangle$, $\alpha \cup f(\alpha)$ falsifies the matrix of F.

A Lower Bound Technique

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \text{vars}_{\forall}(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \text{rng}(h)$ for some countermodel h of Φ.

Definition: We define $\sigma(\Phi)$ as the minumum cardinality of the range of a countermodel for a false QBF Q:

$$\sigma(\Phi) := \min\{|\text{rng}(h)| : h \text{ is a countermodel for } \Phi\}$$

Corollary: Any $\forall$Exp+Res refutation of a false QBF Φ has size at least $\sigma(\Phi)$.

- If a partial expansion of Φ is unsatisfiable, it contains at least $\sigma(\Phi)$ non-trivial conjuncts
- So a $\forall$Exp+Res refutation of Φ requires at least $\sigma(\Phi)$ axioms

Application to the Equality Formulas

$$EQ_n := \exists x_1 \cdots x_n \forall u_1 \cdots u_n \exists z_1 \cdots z_n \cdot$$
$$\left(\bigwedge_{i\in[n]} (x_i \vee u_i \vee z_i) \right) \wedge \left(\bigwedge_{i\in[n]} (\overline{x_i} \vee \overline{u_i} \vee z_i) \right) \wedge \left(\bigvee_{i\in[n]} \overline{z_i} \right)$$

- The countermodel is unique:

$$\begin{aligned} h \quad : \quad \langle \text{vars}_\exists(EQ_n) \rangle \quad &\to \quad \langle \text{vars}_\forall(EQ_n) \rangle \\ \alpha \quad &\mapsto \quad \{u_1 \mapsto \alpha(x_1), \ldots, u_n \mapsto \alpha(x_n)\} \end{aligned}$$

- Clear: $\text{rng}(h) = \langle \text{vars}_\forall(EQ_n) \rangle$ and $|\text{rng}(h)| = 2^n$
- So $\sigma(EQ_n) = 2^n$, and we apply the technique
- Any $\forall$Exp+Res refutation of EQ_n has size at least 2^n

Theorem: The equality formulas require exponential size $\forall$Exp+Res refutations.

Proof (only if direction)

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \text{vars}_\forall(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \text{rng}(h)$ for some countermodel h of Φ.

- Proof by induction on the quantifier depth d of Φ
- Quantifier depth is the number of blocks:

  $$\exists X_1 \forall U_1 \exists X_2 \forall U_2 \exists X_3 \cdot \phi(X_1, U_1, X_2, U_2, X_3)$$

  here the quantifier depth is $d = 5$
- Base case $d = 0$ is trivial: $\Phi = \phi(\emptyset)$
- We have $R = \langle \text{vars}_\forall(\Phi) \rangle = \emptyset$
- $\exp(\Phi, R) = \phi$
- Note that ϕ is either $\top$ or the empty clause $\bot$
- Suppose $\exp(\Phi)$ is unsatisfiable; then ϕ is the empty clause
- Then Φ has a trivial countermodel h with $\text{rng}(h) = \emptyset \subseteq R$

Proof (only if direction)

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \mathrm{vars}_{\forall}(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \mathrm{rng}(h)$ for some countermodel h of Φ.

- Inductive step $d \geq 1$, universal case: $\Phi = \forall U \mathcal{Q} \cdot \phi(U, \mathrm{vars}(\mathcal{Q}))$
- Suppose that $\exp(\Phi, R)$ is unsatisfiable
- Idea: partition $\exp(\Phi, R)$ into conjucts corresponding to assignments to U
- $R' := \{\alpha\restriction_U : \alpha \in R\}$
- For each $\beta \in R'$, define $R_\beta := \{\alpha \in R : \beta \subseteq \alpha\}$
- Observe that $R = \bigcup_{\beta \in R'} R_\beta$
- Hence $\exp(\Phi, R) = \bigwedge_{\beta \in R'} \exp(\Phi, R_\beta)$
- The conjucts of $\bigwedge_{\beta \in R'} \exp(\Phi, R_\beta)$ are pairwise variable-disjoint
- Hence there exists $\beta \in R'$ such that $\exp(\Phi, R_\beta)$ is unsatisfiable

Proof (only if direction)

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \text{vars}_{\forall}(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \text{rng}(h)$ for some countermodel h of Φ.

- Fix $\beta \in R'$ such that $\exp(\Phi, R_\beta)$ is unsatisfiable
- Define $S_\beta := \{\alpha \setminus \beta : \alpha \in R_\beta\}$
- Observe that $\exp(\Phi, R_\beta)$ is syntactically equivalent to $\exp(\Phi[\beta], S_\beta)$; just delete β from the annotations
- Hence $\exp(\Phi[\beta], S_\beta)$ is unsatisfiable
- $\Phi[\beta]$ has quantifier depth $d - 1$; by inductive hypothesis:
- $S_\beta \supseteq \text{rng}(h)$ for some countermodel h of $\Phi[\beta]$.
- Form a countermodel h' for Φ simply by adding β to every element of the range of h (check this satisfies the definition)
- $\text{rng}(h') \subseteq R_\beta \subseteq R$

Proof (only if direction)

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \mathsf{vars}_{\forall}(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \mathsf{rng}(h)$ for some countermodel h of Φ.

- Inductive step $d \geq 1$, existential case:
  $\Phi = \exists X \mathcal{Q} \cdot \phi(X, \mathsf{vars}(\mathcal{Q}))$
- Suppose that $\exp(\Phi, R)$ is unsatisfiable
- Idea: restrict by all assignments to X
- let $\alpha \in \langle X \rangle$; observe that $\exp(\Phi, R)[\alpha]$ is unsatisfiable
- observe that $\exp(\Phi, R)[\alpha] = \exp(\Phi[\alpha], R)$
- $\Phi[\alpha]$ has quantifier depth $d - 1$; by inductive hypothesis:
- $R \supseteq \mathsf{rng}(h_\alpha)$ for some countermodel h_α of $\Phi[\alpha]$.

Proof (only if direction)

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \mathsf{vars}_{\forall}(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \mathsf{rng}(h)$ for some countermodel h of Φ.

- For each $\alpha \in \langle X \rangle$, $R \supseteq \mathsf{rng}(h_\alpha)$ for some countermodel h_α of $\Phi[\alpha]$.

- Construct a countermodel for Φ:

$$\begin{array}{rcccc} h & : & \langle \mathsf{vars}_{\exists}(\Phi) \rangle & \rightarrow & \langle \mathsf{vars}_{\forall}(\Phi) \rangle \\ & & \beta & \mapsto & h_{\beta \restriction X}(\beta) \end{array}$$

  and check it satisfies the countermodel definition

- Observe that $\mathsf{rng}(h) = \bigcup_{\alpha \in \langle X \rangle} \mathsf{rng}(h_\alpha) \subseteq R$

A Very Easy $\forall$Exp+Res Lower Bound

For $n \geq 1$, the pigeonhole formula PHP_n is the conjunction of

- $(x_{i,1} \vee \cdots \vee x_{i,n})$ for $1 \leq i \leq n+1$, and
- $(\overline{x_{i,j}} \vee \overline{x_{i',j}})$ for $1 \leq i < i' \leq n+1$ and $1 \leq j \leq n$.

- Pigeonhole formulas require exponential size resolution refutations
- Form a prenex QBF by quantifying all variables exisentially
- No universal variables - so the expansion is exactly the pigeonhole formula: $PHP_n = \exp(PHP_n)$
- Hence these 'QBFs' require exponential size $\forall$Exp+Res refutations

Propositional hardness transfers to QBF

- If $\phi_n(X)$ is hard for resolution, then $\exists X\, \phi_n(X)$ is hard for $\forall$Exp+Res.
- propositional hardness (Σ_1): not what we want to study
- we want QBF systems with proof size modulo propositional hardness
- so we need a method of elimating Σ_1 (propositional) hardness from the proof size measure

Oracles

- We borrow an idea from complexity theory
- An oracle is a Turing Machine with a black box that can decide a specified decision problem in a single time step
- For example, a TM with a SAT oracle can solve any NP problem in polynomial time
- Allows us to study complexity modulo NP (or any complexity class)

Genuine QBF hardness

- can be modelled precisely by allowing NP oracles in QBF proofs
- informally: all 'essentially propositional' derivations are allowed in a single inference
- in line with QBF solvers using embedded SAT solvers

Genuine Lower Bounds in $\forall$Exp+Res

- Replace resolution and weakening with the following rule

oracle: $$\frac{C_1, \ldots, C_k}{C} \qquad C_1 \wedge \cdots \wedge C_k \models C$$

Genuine Lower Bounds in $\forall$Exp+Res

oracle: $$\frac{C_1, \ldots, C_k}{C} \qquad C_1 \wedge \cdots \wedge C_k \vDash C$$

- The resolution phase is given for free (exactly one inference to refute the expansion)
- Therefore, proof size is effectively defined by the number of axioms
- precisely: given a CNF F, the minimal size of a refutation is the minimal cardinality of an unsatisfiable subset of clauses $(+1)$
- Therefore genuine hardness is characterised by large expansion

Characterisation of Genuine Lower Bounds in Expansion

Theorem: Let Φ be a QBF, and let $R \subseteq \langle \text{vars}_\forall(\Phi) \rangle$. The partial expansion $\exp(\Phi, R)$ is unsatisfiable if, and only if, $R \supseteq \text{rng}(h)$ for some countermodel h of Φ.

- This theorem completely characterises genuine lower bounds in $\forall$Exp+Res
- By characterising expansion size in terms of countermodel range
- There are short (oracle) refutations of Φ if, and only if, $\sigma(\Phi)$ is small

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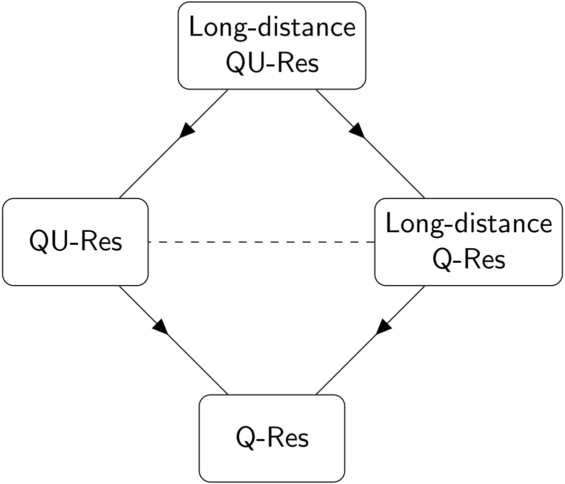
Q-Resolution Lower Bounds

QBF Proof Systems - The Small Picture



Q-Resolution

- Consider a QBF $\mathcal{Q} \cdot F$

axiom:	$\dfrac{}{C}$	C is a clause in F
resolution:	$\dfrac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$	C and D are clauses x is an existential variable $C \vee D$ is non-tautologous
weakening:	$\dfrac{C}{C \vee D}$	C and D are clauses $C \vee D$ is non-tautologous
universal reduction:	$\dfrac{C \vee a}{C}$	C is a clause a is a universal literal $\text{vars}_{\exists}(C) \subseteq L(\text{var}(a))$

QU-Resolution

- Resolution is allowed over universal variables as well

axiom:	$\overline{\quad C \quad}$	C is a clause in F
resolution:	$\dfrac{C \vee p \qquad D \vee \overline{p}}{C \vee D}$	C and D are clauses p is a variable $C \vee D$ is non-tautologous
weakening:	$\dfrac{C}{C \vee D}$	C and D are clauses $C \vee D$ is non-tautologous
universal reduction:	$\dfrac{C \vee a}{C}$	C is a clause a is a universal literal $\mathrm{vars}_\exists(C) \subseteq L(\mathrm{var}(a))$

QU-resolution

- Completeness: QU-Res clearly simulates Q-Res
- Soundness: exactly as for Q-Res; resolution remains propositionally sound even with universal pivots
- QU-Res is exponentially stronger than Q-Res
- Separation via the Kleine Büning formulas

QU-Resolution with an NP Oracle

- Resolution and weakening replaced by arbitrary propositional inferences

axiom:	$\dfrac{}{C}$	C is a clause in F
oracle:	$\dfrac{C_1, \ldots, C_k}{C}$	$C_1 \wedge \cdots \wedge C_k \vDash C$ C is not tautological
universal reduction:	$\dfrac{C \vee a}{C}$	C is a clause a is a universal literal $\mathrm{vars}_{\exists}(C) \subseteq L(\mathrm{var}(a))$

QU-Res Refutations of the Equality Formulas

$$EQ_n \quad := \quad \exists x_1 \cdots x_n \forall u_1 \cdots u_n \exists z_1 \cdots z_n .$$
$$\left(\bigwedge_{i \in [n]} (x_i \vee u_i \vee z_i) \right) \wedge \left(\bigwedge_{i \in [n]} (\overline{x_i} \vee \overline{u_i} \vee z_i) \right) \wedge \left(\bigvee_{i \in [n]} \overline{z_i} \right)$$

- There is essentially only one way to refute them:
    - Resolve over all z_i to obtain $(x_1 \vee \cdots \vee x_n \vee u_1 \vee \cdots \vee u_n)$
    - Perform n universal reductions to obtain $(x_1 \vee \cdots \vee x_n)$
    - Repeat to obtain all 2^n clauses in the x_i
    - Resolve all 2^n clauses to get the empty clause
- With an NP oracle:
    - Derive $(x_1 \vee \cdots \vee x_n \vee u_1 \vee \cdots \vee u_n)$ immediately
    - Perform n universal reductions to obtain $(x_1 \vee \cdots \vee x_n)$
    - Repeat to obtain all 2^n clauses in the x_i
    - Derive the empty clause immediately
- Key: we need all 2^n universal clauses as subclauses

Equality is Hard in QU-Res with an Oracle

Theorem: EQ_n requires refutations of size 2^n in NP-QU-Res

- Argument: example of the strategy extraction technique
- Proof ingredients:
  - Closure under existential restrictions (with subclause property)
  - Properties of refutations of QBFs of the form $\forall U \mathcal{Q} \cdot F$
  - Semantic properties of EQ_n
- Without loss of generality: refutations contain no redundant clauses (every clause has a path to the unique empty clause)

Lemmata

Closure under existential restrictions

Lemma: Let Π be an NP-QU-Res refutation of a QBF Φ, let α be a partial assignment to $\text{vars}_{\exists}(\Phi)$. Then $\Pi[\alpha]$ is an NP-QU-Res refutation of $\Phi[\alpha]$ whose every clause is a subclause of one in Π.

First universal clause

Lemma: Let C be the first fully universal clause in an NP-QU-Res refutation of a QBF $\Phi = \forall U \mathcal{Q} \cdot F$. If $\beta \in \langle U \rangle$ falsifies $C\restriction_U$, then $\Phi[\beta]$ is false.

Uniqueness of EQ_n countermodel

Lemma: For any $\alpha \in \langle x_1, \ldots, x_n \rangle$, the only $\beta \in \langle u_1, \ldots, u_n \rangle$ for which $\mathrm{EQ}_n[\alpha][\beta]$ is false is $\beta_\alpha := \{u_i \mapsto \alpha(x_i)\}_{i \in [n]}$.

Proof of Hardness

Theorem: EQ_n requires refutations of size 2^n in NP-QU-Res

- Let Π be a ref. of EQ_n, $X = \{x_1, \ldots, x_n\}$, $U = \{u_1, \ldots, u_n\}$.
- Idea: for all $\beta \in \langle U \rangle$, $\overline{\beta}$ appears as a subclause in Π.
- Hence there are 2^n distinct clauses in Π (we have no tautological clauses).
- Let $\alpha \in \langle x_1, \ldots, x_n \rangle$. By "closure under $\exists$-restrictions", $\Pi[\alpha]$ is a refutation of $\text{EQ}[\alpha]$, whose first block is U.
- Let C_α be the first fully universal clause in $\Pi[\alpha]$. By "first block universal literals" and "uniqueness of EQ_n countermodel", the only assignment in $\langle U \rangle$ falsifying C is β_α.
- Hence C_α is $\overline{\beta_\alpha}$.
- $\{\beta_\alpha : \alpha \in \langle X \rangle\} = \langle U \rangle$. Hence $|\Pi| \geq 2^n$

Lemmata Proofs 1

Closure under existential restrictions

Lemma: Let Π be an NP-QU-Res refutation of a QBF Φ, let α be a partial assignment to $\mathsf{vars}_\exists(\Phi)$. Then $\Pi[\alpha]$ is an NP-QU-Res refutation of $\Phi[\alpha]$ whose every clause is a subclause of one in Π.

- Let $\Pi = C_1, \ldots, C_k$, define $\Pi[\alpha] := C_1[\alpha], \ldots, C_k[\alpha]$
- Clearly, each $C_i[\alpha]$ is a subclause of C_i
- By induction on $i \in [k]$, show that $C_i[\alpha]$ is valid in Π'
- Axiom: if $C_i \in F$, then $C_i[\alpha] \in F[\alpha]$
- Oracle: if $C_{i_1} \wedge \cdots \wedge C_{i_r} \vDash C_i$ then $C_{i_1}[\alpha] \wedge \cdots \wedge C_{i_r}[\alpha] \vDash C_i[\alpha]$
- $\forall$-reduction: if C_i was derived from $(C_i \vee a)$, we have $\mathsf{vars}_\exists(C_i) \subseteq L(\mathsf{var}(a))$. Hence $\mathsf{vars}_\exists(C_i[\alpha]) \subseteq L(\mathsf{var}(a))$, and $C_i[\alpha]$ can be derived from $(C_i \vee a)[\alpha] = (C_i[\alpha] \vee a)$

Lemmata Proofs 2

First universal clause

Lemma: Let C be the first fully universal clause in an NP-QU-Res refutation of a QBF $\Phi = \forall U \mathcal{Q} \cdot F$. If $\beta \in \langle U \rangle$ falsifies $C\restriction_U$, then $\Phi[\beta]$ is false.

- Let Π be any refutation of Φ.
- Consider the refutation Π' obtained from Π by reducing all universal literals from C to derive the empty clause, then deleting all clauses which have no path to this empty clause.
- Let $\beta \in \langle U \rangle$ falsify $C\restriction_U$
- Show that $\Pi'[\beta]$ is a refutation of $\Phi[\beta]$, which is therefore false
- Crux - restriction by universal assignments is not closed in general, since one can satisfy a reduced literal
- But here, the only reduced U-literals in Π' are falsified by β.

Lemmata Proofs 3

Uniqueness of EQ_n countermodel

Lemma: For any $\alpha \in \langle x_1, \ldots, x_n \rangle$, the only $\beta \in \langle u_1, \ldots, u_n \rangle$ for which $\mathrm{EQ}_n[\alpha][\beta]$ is false is $\beta_\alpha := \{u_i \mapsto \alpha(x_i)\}_{i \in [n]}$.

- Easily verified by inspection
- Let $\alpha \in \langle X \rangle$
- $\mathrm{EQ}_n[\alpha]$ is the QBF

$$\forall u_1 \cdots \forall u_n \exists z_1 \cdots \exists z_n \cdot (a_1 \vee z_1) \wedge \cdots \wedge (a_n \vee z_n) \wedge (\overline{z_1} \vee \cdots \vee \overline{z_n})$$

where the only assignment to U falsifying all the literals $a_1, \ldots, a_n$ is β_α

General Technique for Σ_3 QBFs

Definition: We define $\sigma(\Phi)$ as the minumum cardinality of the range of a countermodel for a false QBF Q:

$$\sigma(\Phi) := \min\{|\text{rng}(h)| : h \text{ is a countermodel for } \Phi\}$$

Theorem: Let Φ be a QBF of the form $\exists X \forall U \exists Z \cdot F$. Φ requires NP-QU-Res refutations of size $\sigma(\Phi)$.

- Proof idea: a form of strategy extraction
- Restrict by each assignment to X, choose an assignment to U falsifying the first universal clause
- This defines a countermodel for Φ
- Therefore we meet at least $\sigma(\Phi)$ first universal clauses
- Each is a subclause of the original proof

General Technique for Σ_3 QBFs

Theorem: Let Φ be a QBF of the form $\exists X \forall U \exists Z \cdot F$. Φ requires NP-QU-Res refutations of size $\sigma(\Phi)$.

- Subtlety: these first universal clauses are not necessarily wide: they may omit variables in U, they may even be empty
- Therefore they are not necessarily subclauses of distinct clauses in the original refutation
- Not actually a problem, but the argument is a little messy
- Easy fix: assume without loss of generality that universal reduction applies a total assignment to U
- Instead of the subclause property for existential restrictions, we have the following: any reduction assignment in the restriction is a reduction assignment in the original refutation

General Technique for All QBFs

- We look at countermodel range per universal block
- Compute the minimum of each range over all countermodels
- The take the maximum

Definition: For each universal block U of a QBF Φ, we define

$$\sigma_U(\Phi) := \min\{|\mathrm{rng}(h)\restriction_U| : h \text{ is a countermodel for } \Phi\}$$

Theorem: A QBF Φ requires NP-QU-Res refutations of size

$$\max\{\sigma_U(\Phi) : U \text{ is a universal block in } \Phi\}\,.$$

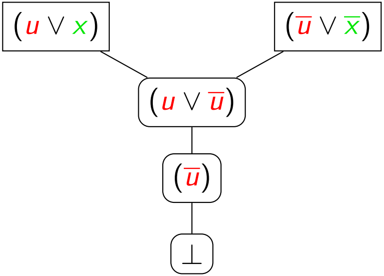
3

Long-Distance Q-Resolution

Bad tautologies

- Allowing tautologies is not sound
- Some true QBFs become refutable

$$\forall u \exists x \cdot (u \vee x) \wedge (\bar{u} \vee \bar{x})$$

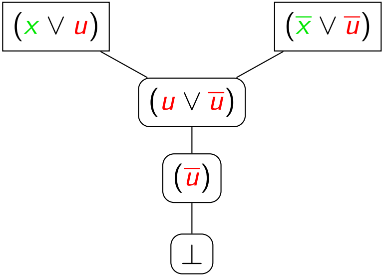


- Problem: u is left of the pivot x

Good tautologies

- No problem if u is right of the pivot x
- Swap variable order - now QBF is false

$$\exists x \forall u \cdot (x \vee u) \wedge (\bar{x} \vee \bar{u})$$



- *These* tautologies cause no problems [Zhang and Malik 2002]

Long-distance Q-Resolution

- Allows some tautologies [Balabanov and Jiang 2012]
- Consider a QBF $\Phi = \mathcal{Q} \cdot \Phi$

axiom: $\dfrac{}{C}$ — C is a clause in F

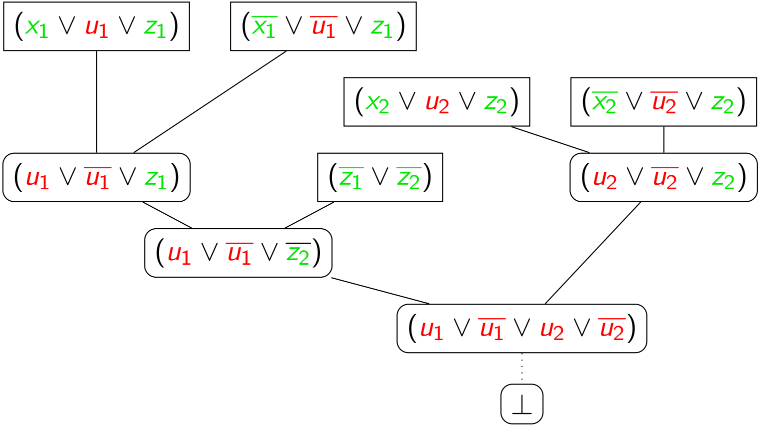
resolution: $\dfrac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$ — C and D are clauses; x is an existential variable

$$a \in C,\ \overline{a} \in D,\ \mathrm{var}(a) \in \mathrm{vars}_{\forall}(\Phi) \Rightarrow \mathrm{var}(a) \notin L(x)$$

universal reduction: $\dfrac{C \vee a}{C}$ — C is a clause; a is a universal literal; $\mathrm{vars}_{\exists}(C) \subseteq L(\mathrm{var}(a))$

Example Long-Distance Refutation

$$\exists x_1 \exists x_2 \forall u_1 \forall u_2 \exists z_1 \exists z_2 \cdot (x_1 \vee u_1 \vee z_1) \wedge (\overline{x_1} \vee \overline{u_1} \vee z_1) \wedge (x_2 \vee u_2 \vee z_2) \wedge (\overline{x_2} \vee \overline{u_2} \vee z_2) \wedge (\overline{z_1} \vee \overline{z_2})$$



About Long-Distance Q-Resolution

- Completeness: LD-Q-Res clearly simulates Q-Res
- Soundness: not so simple
  - syntactic use of tautologies obfuscates semantics
  - tautologies represent Boolean functions of existential pivots
  - this is fine, because the (universal) tautology variable always depends on the (existential) pivot
- Long-distance resolution adds strength:
- Linear-size LD-Q-Res refutations of the equality formulas
- Therefore LD-Q-Res is exponentially separated from Q-Res
- This is important, because QCDCL solvers use long-distance resolution