QBF: Solving and Proofs
Exercise Sheet 1

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1. Working from the definitions of resolution and propositional entailment, prove formally that resolution is refutationally sound and complete: For all CNFs F,

$$F \vDash \bot \quad \Leftrightarrow \quad F \vdash \bot$$

2. Sketch an argument that resolution is implicationally sound and complete: For all CNFs F and clauses C,

$$F \vDash C \quad \Leftrightarrow \quad F \vdash C$$

   Hint: implicational completeness can be shown by construction using refutational completeness and weakening.

3. Define an operation for restricting resolution refutations ($\Pi \mapsto \Pi[\alpha]$) and use it to prove closure under restrictions:

   If Π is a derivation of a clause C from a CNF F, and α is a partial assingment to $\text{vars}(F)$, then $\Pi[\alpha]$ is a derivation of $C[\alpha]$ from $F[\alpha]$ no larger or wider than Π.

   Hint: define and prove this first for a singleton assignment $x \mapsto b$, then extend the definition to arbitrary partial assignments as successive applications of singletons.

4. Working from the definitions of Q-resolution, QBF models and countermodels, prove formally that Q-resolution is refutationally sound and complete:

   A QBF has a Q-resolution refutation if, and only if, it is false.

   Highlight exactly where the Folklore Theorem is used in your argument.

5. By defining an operation similar to (3), prove formally that Q-resolution is closed under existential restrictions:

   If Π is a derivation of a clause C from a QBF F, and α is a partial assingment to $\mathrm{vars}_{\exists}(F)$, then $\Pi[\alpha]$ is a derivation of $C[\alpha]$ from $F[\alpha]$ no larger or wider than Π.