# QBF: Solving and Proofs
# Exercise Sheet 2

Joshua Blinkhorn

Friedrich-Schilller-Universität Jena

| | | |
|---|---|---|
| axiom: | $\dfrac{}{C}$ | $C$ is a clause in $F$ |
| oracle: | $\dfrac{C_1, \ldots, C_k}{C}$ | $C_1 \wedge \cdots \wedge C_k \vDash C$<br>$C$ is not tautological |
| universal reduction: | $\dfrac{C \vee a}{C}$ | $C$ is a clause<br>$a$ is a universal literal<br>$\mathrm{vars}_\exists(C) \subseteq L(\mathrm{var}(a))$ |

**Fig. 1.** The inference rules of NP-QU-Resolution. $L(\mathrm{var}(a))$ denotes the existential variables quantified in $\mathcal{Q}$ before the universal variable $\mathrm{var}(a)$.

**Definition 1.** *An* NP-QU-resolution derivation *from a QBF $\Phi = \mathcal{Q} \cdot F$ is a sequence a clauses $\Pi = C_1, \ldots, C_k$ derived with the rules in Figure 1. The* size *of $\Pi$ is $k$. We call $\Pi$ a* refutation *when $C_k$ is the empty clause.*

| | | |
|---|---|---|
| axiom: | $\dfrac{}{C}$ | $C$ is a clause in $F$ |
| oracle: | $\dfrac{C_1, \ldots, C_k}{C}$ | $C_1 \wedge \cdots \wedge C_k \vDash C$ |
| universal reduction: | $\dfrac{C}{C[\alpha]}$ | $C$ is a clause<br>$\alpha$ is a total assignment to $U$<br>$U$ is a universal block of $\mathcal{Q}$<br>$\mathrm{vars}_\exists(C) \subseteq L(U)$ |

**Fig. 2.** The inference rules of P. $L(U)$ denotes the existential variables quantified in $\mathcal{Q}$ before the universal block $U$.

**Definition 2.** *A* P derivation *from a QBF $\Phi = \mathcal{Q} \cdot F$ is a sequence a clauses $\Pi = C_1, \dots, C_k$ derived with the rules in Figure 2. The* size *of $\Pi$ is $k$. We call $\Pi$ a* refutation *when $C_k$ is the empty clause.*

**Exercises**

1. The proof system P is NP-QU-Resolution with a stronger form of universal reduction. Show that P $p$-simulates NP-QU-Resolution (i.e. show that an NP-QU-Resolution refutation of a QBF $\Phi$ can be transformed efficiently into a P refutation of $\Phi$).

2. Explain why it is not necessary to forbid tautological clauses in P.

3. Let $\Phi = \forall U \mathcal{Q} \cdot F$ be a QBF, and let $\Pi$ be a $P$ refutation of $\Phi$. Suppose that $\Pi$ contains more than one universal reduction on the block $U$. Show that $\Pi$ can be transformed into a refutation with at most one universal reduction step on $U$ by deleting some of the clauses.

4. Let $\Phi = \forall U \mathcal{Q} \cdot F$ be a QBF, and let $\Pi$ be a $P$ refutation of $\Phi$ with exactly one universal reduction step on $U$. Let $\alpha \in \langle U \rangle$ be the assignment used in this step. Show that $\Phi[\alpha]$ is false.

5. Let $\Phi = \exists X \forall U \exists Z \cdot F$ be a QBF, and let $\Pi$ be a refutation of $\Phi$. Using the result of the previous exercise, and the fact that $P$ is closed under existential restrictions, demonstrate how to produce a countermodel for $\Phi$ from $\Pi$.

6. Let $\Phi = \exists X \forall U \exists Z \cdot F$ be a QBF. Prove that the shortest P refutation of $\Phi$ has size at least $\sigma(\Phi)$, where
$$\sigma(\Phi) := \min\{|\mathrm{rng}(h)| : h \text{ is a countermodel for } \Phi\}\,.$$

7. *Optional exercise.* Let $\Phi = \exists X_1 \forall U_1 \cdots \exists X_n \forall U_n \exists X_{n+1}$ be a QBF, and for each $i \in [n]$ let
$$\sigma_i(\Phi) := \min\{|\mathrm{rng}(h)\restriction_{U_i}| : h \text{ is a countermodel for } \Phi\}\,.$$
where $\mathrm{rng}(h)\restriction_{U_i}$ is obtained from $\mathrm{rng}(h)$ by deleting the assignments to all univeral blocks except $U_i$. Prove that the shortest P refutation of $\Phi$ has size at least $\max\{\sigma_i(\Phi) : i \in [n]\}$.

   *Hint: You will need to extend the countermodel extraction method from exercise 4 to the general case.*