A Method for Finding the Power Sum Formulas and Two Corresponding Methods for Calculating the Bernoulli Numbers

Joshua H. Droubay

1 Introduction

Consider the function $S_\phi : \mathbb{N} \to \mathbb{N}$ given by:

$$S_\phi(n) = \sum_{x=0}^{n} x^\phi = 0^\phi + 1^\phi + 2^\phi + 3^\phi + ... + n^\phi$$

where ϕ is a natural number. Though the above function can be evaluated for any $n \in \mathbb{N}$ by simply expanding the sum as shown above, such a representation can be computationally inefficient for large n, as well as theoretically fruitless; and so it is often useful to rewrite S_ϕ in an explicit form. Here we explore a method for devising explicit formulas for each S_ϕ, which take the form of polynomials of degree $\phi+1$ applied to n. This method involves solving a matrix equation of the form $\Lambda d = r$ for d, where Λ is an $\infty \times \infty$ triangular matrix, and d and r are vectors of infinite length (note that, though the system is infinite, the triangularity of Λ permits us to sequentially solve for the entries in d, which is all that is necessary for constructing explicit formulas for S_ϕ).

After exploring this method, we will then show that both the vector d and the matrix Λ bear a strong connection to the Bernoulli numbers; in particular, we will prove that the Bernoulli numbers (with $B_1 = \frac{1}{2}$) can be directly expressed in terms the entries of the matrix d, and then prove that the entries of inverse of the matrix Λ also produce the Bernoulli numbers (with $B_1 = -\frac{1}{2}$).

2 The Method for Finding the Power-Sum Formulas

Consider the values $d_{-1}, d_0, d_1, d_2, d_3 ...$, which can be found by sequentially solving the following system of equations:

$$\begin{bmatrix} 1/1! & 0 & 0 & 0 & 0 & ... \\ 1/2! & 1/1! & 0 & 0 & 0 & ... \\ 1/3! & 1/2! & 1/1! & 0 & 0 & ... \\ 1/4! & 1/3! & 1/2! & 1/1! & 0 & ... \\ 1/5! & 1/4! & 1/3! & 1/2! & 1/1! & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} d_{-1} \\ d_0 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1/0! \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix}$$

Then we have that, for $\phi \in \mathbb{N}$:

$$\sum_{x=0}^{n} x^{\phi} = S_{\phi}(n) = \sum_{i=-1}^{\phi-1} d_i \frac{\phi!}{(\phi - i)!} n^{\phi-i}$$

or

$$\sum_{x=0}^{n} x^{\phi} = S_{\phi}(n) = -\phi! d_{\phi} + \sum_{i=-1}^{\phi} d_i \frac{\phi!}{(\phi - i)!} n^{\phi-i}$$

where $0^{\phi-\phi} = 1$.

2.1 A Proof by Induction: Base Case

Let n = 0.

Then

$$\sum_{x=0}^{n} x^{\phi} = 0^{\phi} = \sum_{i=-1}^{\phi-1} 0 = \sum_{i=-1}^{\phi-1} d_i \frac{\phi!}{(\phi - i)!} (0)^{\phi-i} = \sum_{i=-1}^{\phi-1} d_i \frac{\phi!}{(\phi - i)!} (n)^{\phi-i}$$

as desired.

2.2 Inductive Step

Our intention is to show that

$$S_{\phi}(n+1) = S_{\phi}(n) + (n+1)^{\phi}$$

For if this is true, then it holds that $S_{\phi}(0) = 0 \Rightarrow S(1) = 0 + 1^{\phi} \Rightarrow S(2) = 0 + 1^{\phi} + 2^{\phi} \Rightarrow ... \Rightarrow S(n) = 0 + 1^{\phi} + 2^{\phi} + ... + n^{\phi}$.

We begin by making the following substitution:

$$k_{\phi-i} = d_i \frac{\phi!}{(\phi-i)!}$$

so that

$$\sum_{i=-1}^{\phi} d_i \frac{\phi!}{(\phi-i)!} n^{\phi-i} = \sum_{i=-1}^{\phi} k_{\phi-i} n^{\phi-i}$$

and

$$\begin{bmatrix} 1/1! & 0 & 0 & 0 & 0 & \dots \\ 1/2! & 1/1! & 0 & 0 & 0 & \dots \\ 1/3! & 1/2! & 1/1! & 0 & 0 & \dots \\ 1/4! & 1/3! & 1/2! & 1/1! & 0 & \dots \\ 1/5! & 1/4! & 1/3! & 1/2! & 1/1! & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} k_{\phi+1}\frac{(\phi+1)!}{\phi!} \\ k_{\phi}\frac{\phi!}{\phi!} \\ k_{\phi-1}\frac{(\phi-1)!}{\phi!} \\ k_{\phi-2}\frac{(\phi-2)!}{\phi!} \\ k_{\phi-3}\frac{(\phi-3)!}{\phi!} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1/0! \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix}$$

Rewriting the matrix equation above as a system of equations, we have:

$$\frac{(\phi+1)!}{1!\phi!} k_{\phi+1} = \frac{1}{0!}$$

$$\frac{(\phi+1)!}{2!\phi!} k_{\phi+1} + \frac{\phi!}{1!\phi!} k_{\phi} = \frac{1}{1!}$$

$$\frac{(\phi+1)!}{3!\phi!} k_{\phi+1} + \frac{\phi!}{2!\phi!} k_{\phi} + \frac{(\phi-1)!}{1!\phi!} k_{\phi-1} = \frac{1}{2!}$$

$$\frac{(\phi+1)!}{4!\phi!} k_{\phi+1} + \frac{\phi!}{3!\phi!} k_{\phi} + \frac{(\phi-1)!}{2!\phi!} k_{\phi-1} + \frac{(\phi-2)!}{1!\phi!} k_{\phi-2} = \frac{1}{3!}$$

$$\vdots$$

Call the equation at the top of the list the 0^{th} equation, the following equation the 1^{st}, and so on. Multiplying both sides of the i^{th} equation by $\frac{\phi!}{(\phi-i)!}$ yields:

$$\frac{(\phi+1)!}{1!\phi!} k_{\phi+1} = \frac{\phi!}{0!\phi!}$$

$$\frac{(\phi+1)!}{2!(\phi-1)!} k_{\phi+1} + \frac{\phi!}{1!(\phi-1)!} k_{\phi} = \frac{\phi!}{1!(\phi-1)!}$$

$$\frac{(\phi+1)!}{3!(\phi-2)!} k_{\phi+1} + \frac{\phi!}{2!(\phi-2)!} k_{\phi} + \frac{(\phi-1)!}{1!(\phi-2)!} k_{\phi-1} = \frac{\phi!}{2!(\phi-2)!}$$

$$\frac{(\phi+1)!}{4!(\phi-3)!}k_{\phi+1}+\frac{\phi!}{3!(\phi-3)!}k_{\phi}+\frac{(\phi-1)!}{2!(\phi-3)!}k_{\phi-1}+\frac{(\phi-2)!}{1!(\phi-3)!}k_{\phi-2}=\frac{\phi!}{3!(\phi-3)!}$$

$$\vdots$$

which can be rewritten as:

$$\binom{\phi+1}{1}k_{\phi+1}=\binom{\phi}{0}$$

$$\binom{\phi+1}{2}k_{\phi+1}+\binom{\phi}{1}k_{\phi}=\binom{\phi}{1}$$

$$\binom{\phi+1}{3}k_{\phi+1}+\binom{\phi}{2}k_{\phi}+\binom{\phi-1}{1}k_{\phi-1}=\binom{\phi}{2}$$

$$\binom{\phi+1}{4}k_{\phi+1}+\binom{\phi}{3}k_{\phi}+\binom{\phi-1}{2}k_{\phi-1}+\binom{\phi-2}{1}k_{\phi-2}=\binom{\phi}{3}$$

$$\vdots$$

where $\binom{x}{y}=\frac{x!}{y!(x-y)!}$. Because $\binom{x}{0}=1$ for all $x\in\mathbb{C}$, we have that:

$$\binom{\phi+1}{0}k_{\phi+1}=k_{\phi+1}$$

$$\binom{\phi+1}{1}k_{\phi+1}+\binom{\phi}{0}k_{\phi}=\binom{\phi}{0}+k_{\phi}$$

$$\binom{\phi+1}{2}k_{\phi+1}+\binom{\phi}{1}k_{\phi}+\binom{\phi-1}{0}k_{\phi-1}=\binom{\phi}{1}+k_{\phi-1}$$

$$\binom{\phi+1}{3}k_{\phi+1}+\binom{\phi}{2}k_{\phi}+\binom{\phi-1}{1}k_{\phi-1}+\binom{\phi-2}{0}k_{\phi-2}=\binom{\phi}{2}+k_{\phi-2}$$

$$\binom{\phi+1}{4}k_{\phi+1}+\binom{\phi}{3}k_{\phi}+\binom{\phi-1}{2}k_{\phi-1}+\binom{\phi-2}{1}k_{\phi-2}+\binom{\phi-3}{0}k_{\phi-3}=\binom{\phi}{3}+k_{\phi-3}$$

$$\vdots$$

Call the equation listed at the top the -1^{st} equation, the following the 0^{th} equation and so on. Multiplying both sides of the i^{th} equation by $n^{\phi-i}$ yields:

$$(\binom{\phi+1}{0}k_{\phi+1})n^{\phi+1}=k_{\phi+1}n^{\phi+1}$$

$$(\binom{\phi+1}{1}k_{\phi+1}+\binom{\phi}{0}k_{\phi})n^{\phi}=(\binom{\phi}{0}+k_{\phi})n^{\phi}$$

$$(\binom{\phi+1}{2}k_{\phi+1}+\binom{\phi}{1}k_{\phi}+\binom{\phi-1}{0}k_{\phi-1})n^{\phi-1}=(\binom{\phi}{1}+k_{\phi-1})n^{\phi-1}$$

$$(\binom{\phi+1}{3}k_{\phi+1}+\binom{\phi}{2}k_{\phi}+\binom{\phi-1}{1}k_{\phi-1}+\binom{\phi-2}{0}k_{\phi-2})n^{\phi-2}=(\binom{\phi}{2}+k_{\phi-2})n^{\phi-2}$$

$$\vdots$$

Distributing $n^{\phi-i}$ and adding together equations -1 through ϕ gives us the following equality:

$$\begin{array}{llll}
\binom{\phi+1}{0}k_{\phi+1}n^{\phi+1} & & & \\
+\binom{\phi+1}{1}k_{\phi+1}n^{\phi} & +\binom{\phi}{0}k_{\phi}n^{\phi} & & \\
+\binom{\phi+1}{2}k_{\phi+1}n^{\phi-1} & +\binom{\phi}{1}k_{\phi}n^{\phi-1} & +\binom{\phi-1}{0}k_{\phi-1}n^{\phi-1} & \\
+\binom{\phi+1}{3}k_{\phi+1}n^{\phi-2} & +\binom{\phi}{2}k_{\phi}n^{\phi-2} & +\binom{\phi-1}{1}k_{\phi-1}n^{\phi-2} & +\binom{\phi-2}{0}k_{\phi-2}n^{\phi-2} \\
\vdots & \vdots & \vdots & \vdots \\
+\binom{\phi+1}{\phi+1}k_{\phi+1}n^{\phi-\phi} & +\binom{\phi}{\phi}k_{\phi}n^{\phi-\phi} & +\binom{\phi-1}{\phi-1}k_{\phi-1}n^{\phi-\phi} & +\binom{\phi-2}{\phi-2}k_{\phi-2}n^{\phi-\phi}
\end{array}$$

$$=$$

$$k_{\phi+1}n^{\phi+1}+\binom{\phi}{0}n^{\phi}+k_{\phi}n^{\phi}+\binom{\phi}{1}n^{\phi-1}+k_{\phi-1}n^{\phi-1}+\binom{\phi}{2}n^{\phi-2}+k_{\phi-2}n^{\phi-2}+...+\binom{\phi}{\phi}n^{\phi-\phi}+k_{\phi-\phi}n^{\phi-\phi}$$

Rearranging the terms on each side of the equation gives us the following:

$$\binom{\phi+1}{0}k_{\phi+1}n^{\phi+1}+\binom{\phi+1}{1}k_{\phi+1}n^{\phi}+\binom{\phi+1}{2}k_{\phi+1}n^{\phi-1}+...+\binom{\phi+1}{\phi+1}k_{\phi+1}n^{\phi-\phi}$$

$$+\binom{\phi}{0}k_{\phi}n^{\phi}+\binom{\phi}{1}k_{\phi}n^{\phi-1}+\binom{\phi}{2}k_{\phi}n^{\phi-2}+...+\binom{\phi}{\phi}k_{\phi}n^{\phi-\phi}$$

$$+\binom{\phi-1}{0}k_{\phi-1}n^{\phi-1}+\binom{\phi-1}{1}k_{\phi-1}n^{\phi-2}+\binom{\phi-1}{2}k_{\phi-1}n^{\phi-3}+...+\binom{\phi-1}{\phi-1}k_{\phi-1}n^{\phi-\phi}$$

$$+\binom{\phi-2}{0}k_{\phi-2}n^{\phi-2}+\binom{\phi-2}{1}k_{\phi-2}n^{\phi-3}+\binom{\phi-2}{2}k_{\phi-2}n^{\phi-4}+...+\binom{\phi-2}{\phi-2}k_{\phi-2}n^{\phi-\phi}$$

$$\vdots$$

$$+\binom{0}{0}k_{0}n^{\phi-\phi}$$

$$=$$

$$k_{\phi+1}(\binom{\phi+1}{0}n^{\phi+1}+\binom{\phi+1}{1}n^{\phi}+\binom{\phi+1}{2}n^{\phi-1}+...+\binom{\phi+1}{\phi+1}n^{\phi-\phi})$$

$$+k_{\phi}(\binom{\phi}{0}n^{\phi}+\binom{\phi}{1}n^{\phi-1}+\binom{\phi}{2}n^{\phi-2}+...+\binom{\phi}{\phi}n^{\phi-\phi})$$

$$+k_{\phi-1}(\binom{\phi-1}{0}n^{\phi-1}+\binom{\phi-1}{1}n^{\phi-2}+\binom{\phi-1}{2}n^{\phi-3}+...+\binom{\phi-1}{\phi-1}n^{\phi-\phi})$$

$$+k_{\phi-2}(\binom{\phi-2}{0}n^{\phi-2}+\binom{\phi-2}{1}n^{\phi-3}+\binom{\phi-2}{2}n^{\phi-4}+...+\binom{\phi-2}{\phi-2}n^{\phi-\phi})$$

$$\vdots$$

$$+\binom{0}{0}k_0n^{\phi-\phi}$$

$$=$$

$$k_{\phi+1}n^{\phi+1}+k_{\phi}n^{\phi}+k_{\phi-1}n^{\phi-1}+k_{\phi-2}n^{\phi-2}+...+k_0n^{\phi-\phi}$$

$$+\binom{\phi}{0}n^{\phi}+\binom{\phi}{1}n^{\phi-1}+\binom{\phi}{2}n^{\phi-2}+...+\binom{\phi}{\phi}n^{\phi-\phi}$$

By the binomial theorem, we have that

$$\binom{x}{0}n^{x}+\binom{x}{1}n^{x-1}+\binom{x}{2}n^{x-2}+...=(n+1)^x$$

where $0^{x-x}=1$ satisfies the relationship in the case that $n=0$. Thus, we can simplify our equality as follows:

$$k_{\phi+1}(n+1)^{\phi+1}+k_{\phi}(n+1)^{\phi}+k_{\phi-1}(n+1)^{\phi-1}+...+k_0(n+1)^0$$

$$=$$

$$k_{\phi+1}n^{\phi+1}+k_{\phi}n^{\phi}+k_{\phi-1}n^{\phi-1}+...+k_0n^0+(n+1)^{\phi}$$

$$\Downarrow$$

$$\sum_{i=-1}^{\infty}k_{\phi-i}(n+1)^{\phi-i}=(n+1)^{\phi}+\sum_{i=-1}^{\infty}k_{\phi-i}n^{\phi-i}$$

Substituting $d_i\frac{\phi!}{(\phi-i)!}$ back in for $k_{\phi-i}$, and adding $-\phi!d_{\phi}$ to both sides of the equation, we finally arrive at the following equality:

$$-\phi!d_{\phi}+\sum_{i=-1}^{\infty}d_i\frac{\phi!}{(\phi-i)!}(n+1)^{\phi-i}=(n+1)^{\phi}-\phi!d_{\phi}+\sum_{i=-1}^{\infty}d_i\frac{\phi!}{(\phi-i)!}n^{\phi-i}$$

$$S_{\phi}(n+1)=S_{\phi}(n)+(n+1)^{\phi}$$

which concludes the proof.

3 Calculating the Bernoulli Numbers

Theorem: Let B_i denote the i^{th} Bernoulli number (with $B_1 = \frac{1}{2}$). Given the system of equations

$$\begin{bmatrix} 1/1! & 0 & 0 & 0 & 0 & ... \\ 1/2! & 1/1! & 0 & 0 & 0 & ... \\ 1/3! & 1/2! & 1/1! & 0 & 0 & ... \\ 1/4! & 1/3! & 1/2! & 1/1! & 0 & ... \\ 1/5! & 1/4! & 1/3! & 1/2! & 1/1! & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} d_{-1} \\ d_0 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1/0! \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix}$$

which produce a list of values for $d_{-1}, d_0, d_1, d_2...$, we have that:

$$B_{m+1} = (m+1)!d_m$$

3.1 Proof by Induction: Base case

Consider $m = -1$.

Faulhaber's formula gives that the power sum formulas can be written in terms of the Bernoulli numbers. In particular, the formula states that

$$\sum_{x=0}^{n} x^{\phi} = \frac{1}{\phi+1} \sum_{i=-1}^{\phi-1} \frac{(\phi+1)!}{(i+1)!(\phi-i)!} B_{i+1} n^{\phi-i}$$

for all $\phi \in \mathbb{N}$.

If we let $\phi = 0$ and $n = 1$, we have that:

$$\sum_{x=0}^{1} x^0 = \frac{1}{1}\frac{1!}{1!1!} B_0 1^1 = B_0$$

From section 2, we have that

$$\sum_{x=0}^{n} x^{\phi} = \sum_{i=-1}^{\phi-1} d_i \frac{\phi!}{(\phi-i)!} n^{\phi-i}$$

$$\sum_{x=0}^{1} x^0 = d_{-1}\frac{1!}{1!}1^1 = d_{-1}$$

Thus:

$$B_0 = \sum_{x=0}^{1} x^0 = d_1 = (-1+1)!d_1$$

$$B_{m+1} = (m+1)!d_m$$

as desired.

3.2 Inductive Step

Let y be some natural number. Assume that the theorem holds for $m = -1$ or m equal to any natural number less than or equal to y. Consider $m = y+1$. Let $\phi = y+2$. We have that

$$\sum_{i=-1}^{y+1} \frac{1}{y+3} \frac{(y+3)!}{(i+1)!(y+2-i)!} B_{i+1} n^{y+2-i} = \sum_{x=0}^{n} x^{y+2} = \sum_{i=-1}^{y+1} d_i \frac{(y+2)!}{(y+2-i)!} n^{y+2-i}$$

$$\sum_{i=-1}^{y+1} \frac{(y+2)!}{(i+1)!(y+2-i)!} B_{i+1} 1^{y+2-i} = \sum_{x=0}^{1} x^{y+2} = \sum_{i=-1}^{y+1} d_i \frac{(y+2)!}{(y+2-i)!} 1^{y+2-i}$$

$$\frac{(y+2)!}{(y+2)!1!} B_{y+2} + \sum_{i=-1}^{y} \frac{(y+2)!}{(i+1)!(y+2-i)!} B_{i+1} = \frac{(y+2)!}{1!} d_{y+1} + \sum_{i=-1}^{y} d_i \frac{(y+2)!}{(y+2-i)!}$$

Since $B_{i+1} = (i+1)!d_i$ for all natural numbers up to y and for -1, we can substitute $(i+1)d_i$ for B_{i+1} as follows:

$$B_{y+2} + \sum_{i=-1}^{y} \frac{(y+2)!}{(i+1)!(y+2-i)!} (i+1)!d_i = (y+2)!d_{y+1} + \sum_{i=-1}^{y} d_i \frac{(y+2)!}{(y+2-i)!}$$

$$B_{y+2} + \sum_{i=-1}^{y} \frac{(y+2)!}{(y+2-i)!} d_i = (y+2)!d_{y+1} + \sum_{i=-1}^{y} d_i \frac{(y+2)!}{(y+2-i)!}$$

With this, we can reduce the above equation, yielding:

$$B_{y+2} = (y+2)!d_{y+1}$$

$$B_{(y+1)+1} = ((y+1)+1)!d_{y+1}$$

$$B_{m+1} = (m+1)!d_m$$

as desired.

4 The Inverse of Λ

Define Λ to be the matrix:

$$\Lambda = \begin{bmatrix} 1/1! & 0 & 0 & 0 & 0 & ... \\ 1/2! & 1/1! & 0 & 0 & 0 & ... \\ 1/3! & 1/2! & 1/1! & 0 & 0 & ... \\ 1/4! & 1/3! & 1/2! & 1/1! & 0 & ... \\ 1/5! & 1/4! & 1/3! & 1/2! & 1/1! & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Define the vector $D = [D_{-1}, D_0, D_1, D_2, D_3, D_4, ...] = [d_{-1}, -d_0, d_1, d_2, d_3, d_4, ...]$, noting that $B_{m+1} = (m+1)!D_m$, under the convention that $B_1 = -1/2$ rather than $1/2$. Then we have that the inverse of the matrix Λ: Λ^{-1}, is given by:

$$\Lambda^{-1} = \begin{bmatrix} D & 0 & 0 & 0 & 0 & ... \\ & D & 0 & 0 & 0 & ... \\ & & D & 0 & 0 & ... \\ & & & D & 0 & ... \\ & & & & D & ... \end{bmatrix} = \begin{bmatrix} D_{-1} & 0 & 0 & 0 & 0 & ... \\ D_0 & D_{-1} & 0 & 0 & 0 & ... \\ D_1 & D_0 & D_{-1} & 0 & 0 & ... \\ D_2 & D_1 & D_0 & D_{-1} & 0 & ... \\ D_3 & D_2 & D_1 & D_0 & D_{-1} & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

4.1 Proof:

Define $\lambda_i = [0, 0, ..., D_{-1}, D_0, D_1 ...] = [0, 0, ..., D]$, where D begins in the i^{th} entry of λ_i. We will show here that $\Lambda\lambda_i = [0, ..., 0, 1, 0, ...]$, where 1 appears in the i^{th} position.

We begin by evaluating the expression ΛD. Let $d = [d_{-1}, d_0, d_1, d_2 ...]$. We have (from sections 2 and 3) that $d_0 = 1/2$ and $D_0 = -1/2$; therefore:

$$D = \begin{bmatrix} D_{-1} \\ -1/2 \\ D_1 \\ D_2 \\ D_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} d_{-1} \\ 1/2 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = d - I_2$$

where I_2 denotes the second column of the $\infty \times \infty$ identity matrix. Thus, we can rewrite the expression ΛD as $\Lambda(d - I_2) = \Lambda d - \Lambda I_2$. By the definition we assigned to d in section

2, we already know that

$$\Lambda d = \begin{bmatrix} 1/0! \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix}$$

Additionally, ΛI_2 simply evaluates to the second column in Λ, so that:

$$\Lambda I_2 = \begin{bmatrix} 0 \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix}$$

Thus:

$$\Lambda D = \Lambda d - \Lambda I_2 = \begin{bmatrix} 1/0! \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix} - \begin{bmatrix} 0 \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

With this information, we can now evaluate $\Lambda\lambda_i$. Because of Λ's symmetric properties, it can be rewritten as follows:

$$\Lambda = \begin{bmatrix} A & 0_m \\ B & \Lambda \end{bmatrix}$$

where A is an $(i-1) \times (i-1)$ square matrix, 0_m is an $(i-1) \times \infty$ matrix of zeroes, and B is an $\infty \times (i-1)$ matrix. Additionally, we can rewrite λ_i as:

$$\lambda_i = \begin{bmatrix} 0_v \\ D \end{bmatrix}$$

where 0_v is a vector of length $i-1$ whose components are all 0. We see that:

$$\Lambda\lambda_i = \begin{bmatrix} A & 0_m \\ B & \Lambda \end{bmatrix} \begin{bmatrix} 0_v \\ D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A & 0_m \end{bmatrix} \begin{bmatrix} 0_v \\ D \end{bmatrix} \\ \\ \begin{bmatrix} B & \Lambda \end{bmatrix} \begin{bmatrix} 0_v \\ D \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0_v \\ \Lambda D \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

where 1 appears in the i^{th} position of the vector $\Lambda\lambda_i$ as desired.

Thus:

$$\Lambda[\lambda_1, \lambda_2, \lambda_3 ...] = \begin{bmatrix} 1/1! & 0 & 0 & 0 & 0 & ... \\ 1/2! & 1/1! & 0 & 0 & 0 & ... \\ 1/3! & 1/2! & 1/1! & 0 & 0 & ... \\ 1/4! & 1/3! & 1/2! & 1/1! & 0 & ... \\ 1/5! & 1/4! & 1/3! & 1/2! & 1/1! & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} D & 0 & 0 & 0 & 0 & 0 & ... \\ & D & 0 & 0 & 0 & 0 & ... \\ & & D & 0 & 0 & 0 & ... \\ & & & D & 0 & 0 & ... \\ & & & & D & 0 & ... \\ & & & & & D & ... \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & ... \\ 0 & 1 & 0 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and

$$\Lambda^{-1} = \begin{bmatrix} D & 0 & 0 & 0 & 0 & ... \\ & D & 0 & 0 & 0 & ... \\ & & D & 0 & 0 & ... \\ & & & D & 0 & ... \\ & & & & D & ... \end{bmatrix}$$

5 An Interesting Corollary

Using the previous theorem, we will now prove a corollary concerning the Bernoulli numbers:

Let $B_0, B_1, B_2 ...$ denote the Bernoulli numbers, with $B_1 = -1/2$; then:

$$\sum_{k=0}^{i-1} \binom{i}{k} B_k = 0$$

for all $i \in \mathbb{N}, i > 1$

5.1 Proof:

We have that, for any indices $i, j,\ i \neq j,\ (\Lambda\Lambda^{-1})_{i,j} = 0$. By section 4, any entry $(\Lambda\Lambda^{-1})_{i,1}, i \neq 1$ (note that we have set $j = 0$) is given by:

$$(\Lambda\Lambda^{-1})_{i,0} = \left(\tfrac{1}{i!}, \tfrac{1}{(i-1)!}, \tfrac{1}{(i-2)!} ..., \tfrac{1}{1!}, 0, 0, ...\right) \cdot \left(D_{-1}, D_0, D_1, D_2 ...\right)$$

$$= \frac{1}{i!}D_{-1} + \frac{1}{(i-1)!}D_0 + \frac{1}{(i-2)!}D_1 + ... + \frac{1}{1!}D_{i-2} = \sum_{k=0}^{i-1} \frac{1}{(i-k)!}D_{k-1} = 0$$

We have that $D_{k-1} = B_k/k!$; so

$$\sum_{k=0}^{i-1} \frac{1}{(i-k)!}D_{k-1} = \sum_{k=0}^{i-1} \frac{1}{k!(i-k)!}B_k = 0$$

From this, we can conclude that:

$$i! \sum_{k=0}^{i-1} \frac{1}{k!(i-k)!}B_k = (i!)0$$

$$\sum_{k=0}^{i-1} \frac{i!}{k!(i-k)!}B_k = 0$$

$$\sum_{k=0}^{i-1} \binom{i}{k} B_k = 0$$