A Method for Obtaining the Power Summation Formulas

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1 Introduction

Consider the function S_ϕ, defined by:

$$S_\phi(n) = 1^\phi + 2^\phi + 3^\phi + 4^\phi + 5^\phi + ... + n^\phi = \sum_{k=1}^{n} k^\phi$$

where $\phi \in \mathbb{N}$. We will here explore a method of devising explicit formulas for each S_ϕ, where the formulas S_ϕ take the form of polynomials of degree $\phi + 1$. This method will involve finding the formula for each S_ϕ inductively, using the coefficients of all previous polynomials $S_{\phi-1}, S_{\phi-2}..., S_1$ to compute the coefficients of the polynomial S_ϕ.

2 The Method

Let S_ϕ be a function defined by:

$$S_\phi(n) = 1^\phi + 2^\phi + 3^\phi + 4^\phi + 5^\phi + ... + n^\phi = \sum_{k=1}^{n} k^\phi$$

where $\phi, n \in \mathbb{N}$. Firstly, $S_\phi(n)$ may be written as a polynomial of degree $\phi + 1$ with no n^0 term. Secondly, if all polynomial equations $S_{\phi-q}(n), q \in \mathbb{N}, q < \phi$ are written as:

$$S_{\phi-1}(n) = a_0 n^\phi + a_1 n^{\phi-1} + a_2 n^{\phi-2} + a_3 n^{\phi-3} ... + a_{\phi-1} n = \textstyle\sum_{k=1}^{n} k^{\phi-1}$$

$$S_{\phi-2}(n) = b_1 n^{\phi-1} + b_2 n^{\phi-2} + b_3 n^{\phi-3} + b_4 n^{\phi-4} ... + b_{\phi-1} n = \textstyle\sum_{k=1}^{n} k^{\phi-2}$$

$$S_{\phi-3}(n) = c_2 n^{\phi-2} + c_3 n^{\phi-3} + c_4 n^{\phi-4} + c_5 n^{\phi-5} ... + c_{\phi-1} n = \textstyle\sum_{k=1}^{n} k^{\phi-3}$$

$$\vdots$$

$$S_1(n) = z_{\phi-2} n^2 + z_{\phi-1} n = \textstyle\sum_{k=1}^{n} k$$

then

$$S_\phi(n) = -\frac{1}{a_0+1}\begin{bmatrix} -a_0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -a_1 & -a_0/a_0 & a_0 & 0 & 0 & 0 & \dots & 0 \\ -a_2 & -a_1/a_0 & a_1 & b_1 & 0 & 0 & \dots & 0 \\ -a_3 & -a_2/a_0 & a_2 & b_2 & c_2 & 0 & \dots & 0 \\ -a_4 & -a_3/a_0 & a_3 & b_3 & c_3 & d_3 & \dots & 0 \\ -a_5 & -a_4/a_0 & a_4 & b_4 & c_4 & d_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_{\phi-1} & -a_{\phi-2} & a_{\phi-2} & b_{\phi-2} & c_{\phi-2} & d_{\phi-2} & \dots & z_{\phi-2} \\ 0 & -a_{\phi-1} & a_{\phi-1} & b_{\phi-1} & c_{\phi-1} & d_{\phi-1} & \dots & z_{\phi-1} \end{bmatrix}\begin{bmatrix} 1 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{\phi-2} \\ a_{\phi-1} \end{bmatrix}\cdot\begin{bmatrix} n^{\phi+1} \\ n^{\phi} \\ n^{\phi-1} \\ n^{\phi-2} \\ n^{\phi-3} \\ n^{\phi-4} \\ \vdots \\ n^2 \\ n \end{bmatrix}$$

where the first column of the above matrix consists of all constants a_j multiplied by -1 and ends with 0, where the second column begins with 0, and is then followed by all constants a_j divided by $-a_0$, and where any other remaining entry $R_{i,j}$, where i is the row and j is the column, is the constant of the polynomial $S_{\phi-j+2}$ corresponding to the $n^{\phi-i+2}$ term (the constant may be 0).

For example:

$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + (0)n$$

So we have:

$$S_4(n) = -\frac{1}{\frac{1}{4}+1}\begin{bmatrix} -1/4 & 0 & 0 & 0 & 0 \\ -1/2 & -1 & 1/4 & 0 & 0 \\ -1/4 & -2 & 1/2 & 1/3 & 0 \\ 0 & -1 & 1/4 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/6 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1/4 \\ 1/2 \\ 1/4 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} n^5 \\ n^4 \\ n^3 \\ n^2 \\ n \end{bmatrix}$$

$$\Downarrow$$

$$S_4(n) = -\frac{4}{5}\begin{bmatrix} -1/4 \\ -1/2 + (-1)(1/4) + (1/4)(1/2) \\ -1/4 + (-2)(1/4) + (1/2)(1/2) + (1/3)(1/4) \\ (-1)(1/4) + (1/4)(1/2) + (1/2)(1/4) \\ (1/6)(1/4) \end{bmatrix}\cdot\begin{bmatrix} n^5 \\ n^4 \\ n^3 \\ n^2 \\ n \end{bmatrix}$$

$$\Downarrow$$

$$S_4(n) = -\frac{4}{5}\begin{bmatrix} -1/4 \\ -5/8 \\ -5/12 \\ 0 \\ 1/24 \end{bmatrix} \cdot \begin{bmatrix} n^5 \\ n^4 \\ n^3 \\ n^2 \\ n \end{bmatrix}$$

$$\Downarrow$$

$$S_4(n) = \begin{bmatrix} 1/5 \\ 1/2 \\ 1/3 \\ 0 \\ -1/30 \end{bmatrix} \cdot \begin{bmatrix} n^5 \\ n^4 \\ n^3 \\ n^2 \\ n \end{bmatrix}$$

$$\Downarrow$$

$$S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + (0)n^2 - \frac{1}{30}n$$

2.1 Proof by induction: base case

Preliminary note: for the proof, rather than writing the formulas $S_{\phi-q}(n)$ as:

$$S_{\phi-1}(n) = a_0n^{\phi} + a_1n^{\phi-1} + a_2n^{\phi-2} + a_3n^{\phi-3}... + a_{\phi-1}n$$

$$S_{\phi-2}(n) = b_1n^{\phi-1} + b_2n^{\phi-2} + b_3n^{\phi-3} + b_4n^{\phi-4}... + b_{\phi-1}n$$

$$S_{\phi-3}(n) = c_2n^{\phi-2} + c_3n^{\phi-3} + c_4n^{\phi-4} + c_5n^{\phi-5}... + c_{\phi-1}n$$

$$\vdots$$

$$S_1(n) = z_{\phi-2}n^2 + z_{\phi-1}n$$

we will instead have that:

$$S_{\phi-i} = \alpha_0^{\phi-i}n^{\phi} + \alpha_1^{\phi-i}n^{\phi-1} + \alpha_2^{\phi-i}n^{\phi-2} + \alpha_3^{\phi-i}n^{\phi-3} + ... + \alpha_{\phi-1}^{\phi-i}n$$

where any $\alpha_j^{\phi-i}$ is the constant of the function $S_{\phi-i}$ corresponding to $n^{\phi-j}$ (note firstly that the i in α_j^i is not a power but a superscript, and secondly that α_j^i may be zero), so that:

$$S_\phi(n) = -\frac{1}{\alpha_0^{\phi-1}+1}\begin{bmatrix} -\alpha_0^{\phi-1} & 0 & 0 & 0 & 0 & ... & 0 \\ -\alpha_1^{\phi-1} & -\alpha_0^{\phi-1}/\alpha_0^{\phi-1} & \alpha_0^{\phi-1} & 0 & 0 & ... & 0 \\ -\alpha_2^{\phi-1} & -\alpha_1^{\phi-1}/\alpha_0^{\phi-1} & \alpha_1^{\phi-1} & \alpha_1^{\phi-2} & 0 & ... & 0 \\ -\alpha_3^{\phi-1} & -\alpha_2^{\phi-1}/\alpha_0^{\phi-1} & \alpha_2^{\phi-1} & \alpha_2^{\phi-2} & \alpha_2^{\phi-3} & ... & 0 \\ -\alpha_4^{\phi-1} & -\alpha_3^{\phi-1}/\alpha_0^{\phi-1} & \alpha_3^{\phi-1} & \alpha_3^{\phi-2} & \alpha_3^{\phi-3} & ... & 0 \\ -\alpha_5^{\phi-1} & -\alpha_4^{\phi-1}/\alpha_0^{\phi-1} & \alpha_4^{\phi-1} & \alpha_4^{\phi-2} & \alpha_4^{\phi-3} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -\alpha_{\phi-1}^{\phi-1} & -\alpha_{\phi-2}^{\phi-1}/\alpha_0^{\phi-1} & \alpha_{\phi-2}^{\phi-1} & \alpha_{\phi-2}^{\phi-2} & \alpha_{\phi-2}^{\phi-3} & ... & \alpha_{\phi-2}^{1} \\ 0 & -\alpha_{\phi-1}^{\phi-1}/\alpha_0^{\phi-1} & \alpha_{\phi-1}^{\phi-1} & \alpha_{\phi-1}^{\phi-2} & \alpha_{\phi-1}^{\phi-3} & ... & \alpha_{\phi-1}^{1} \end{bmatrix}\begin{bmatrix} 1 \\ \alpha_0^{\phi-1} \\ \alpha_1^{\phi-1} \\ \alpha_2^{\phi-1} \\ \alpha_3^{\phi-1} \\ \alpha_4^{\phi-1} \\ \vdots \\ \alpha_{\phi-2}^{\phi-1} \\ \alpha_{\phi-1}^{\phi-1} \end{bmatrix} \cdot \begin{bmatrix} n^{\phi+1} \\ n^{\phi} \\ n^{\phi-1} \\ n^{\phi-2} \\ n^{\phi-3} \\ n^{\phi-4} \\ \vdots \\ n^2 \\ n \end{bmatrix}$$

Base case: consider $\phi = 1$. $S_0(n) = 1^0 + 2^0 + 3^0 + ... + n^0 = (1)n$, so we have from the theorem that:

$$S_1(n) = -\frac{1}{1+1}\begin{bmatrix} -n & 0 \\ 0 & -n \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} n^2 \\ n \end{bmatrix} = \frac{1}{2}n^2 + \frac{1}{2}n$$

We can verify this by rewriting $S_1(n)$ in the following way:

$$S_1(n) = 1 + 2 + 3 + 4 + 5 + 6... + n$$

$$= \begin{matrix} 1 & +1 & +1 & +1 & +1 & +1 & ... & +1 \\ & +1 & +1 & +1 & +1 & +1 & ... & +1 \\ & & +1 & +1 & +1 & +1 & ... & +1 \\ & & & +1 & +1 & +1 & ... & +1 \\ & & & & +1 & +1 & ... & +1 \\ & & & & & +1 & ... & +1 \\ & & & & & & & \vdots \\ & & & & & & & +1 \end{matrix} = \begin{matrix} n \\ +n-1 \\ +n-2 \\ +n-3 \\ +n-4 \\ +n-5 \\ \vdots \\ +n-(n-1) \end{matrix}$$

so we have:

$$S_1(n) = n * n - 1 - 2 - 3 - 4 - 5 - ... - (n-1) - n + n$$
$$S_1(n) = n^2 + n - S_1(n)$$
$$S_1(n) + S_1(n) = n^2 + n$$
$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

Thus, the theorem holds for $\phi = 1$.

2.2 Inductive step

Now let us consider the case where $\phi = x$, for some arbitrary $x > 1$, and where all $S_{x-q}, q \in \mathbb{N}, q < x$ can be written as polynomials of degree $x - q + 1$, satisfying the theorem.

We begin by writing $S_x(n)$ its original form:

$$S_x(n) = 1^x + 2^x + 3^x + 4^x + 5^x + 6^x + ... + n^x = \sum_{k=1}^{n} k^x$$

Rewriting each term "k_i^x" as "$k_i^{x-1} * k$", we have:

$$S_x(n) = 1^{x-1}(1) + 2^{x-1}(2) + 3^{x-1}(3) + 4^{x-1}(4) + 5^{x-1}(5) + 6^{x-1}(6) + ... + n^{x-1}(n)$$

Utilizing the properties of multiplication, we can further expand the formula as follows:

$$S_x(n) = 1^{x-1} + (2^{x-1} + 2^{x-1}) + (3^{x-1} + 3^{x-1} + 3^{x-1}) + (4^{x-1} + 4^{x-1} + 4^{x-1} + 4^{x-1}) + ... + n^{x-1}(n)$$

$$= \begin{array}{cccccccc}
1^{x-1} & +2^{x-1} & +3^{x-1} & +4^{x-1} & +5^{x-1} & +6^{x-1} & ... & +n^{x-1} \\
 & +2^{x-1} & +3^{x-1} & +4^{x-1} & +5^{x-1} & +6^{x-1} & ... & +n^{x-1} \\
 & & +3^{x-1} & +4^{x-1} & +5^{x-1} & +6^{x-1} & ... & +n^{x-1} \\
 & & & +4^{x-1} & +5^{x-1} & +6^{x-1} & ... & +n^{x-1} \\
 & & & & +5^{x-1} & +6^{x-1} & ... & +n^{x-1} \\
 & & & & & +6^{x-1} & ... & +n^{x-1} \\
 & & & & & & & \vdots \\
 & & & & & & & +n^{x-1}
\end{array}$$

Note each line in the above formula is of the form the definition of $S_{x-1}(n)$. We can use this to again rewrite $S_x(n)$ in terms of S_{x-1}:

$$= \begin{array}{ccccccc}
1^{x-1} & +2^{x-1} & +3^{x-1} & +4^{x-1} & +5^{x-1} & ... & +n^{x-1} \\
 & +2^{x-1} & +3^{x-1} & +4^{x-1} & +5^{x-1} & ... & +n^{x-1} \\
 & & +3^{x-1} & +4^{x-1} & +5^{x-1} & ... & +n^{x-1} \\
 & & & +4^{x-1} & +5^{x-1} & ... & +n^{x-1} \\
 & & & & +5^{x-1} & ... & +n^{x-1} \\
 & & & & & & \vdots \\
 & & & & & & +n^{x-1}
\end{array} = \begin{array}{c}
\sum_{k=1}^{n} k^{x-1} \\
+ \sum_{k=1}^{n} k^{x-1} - \sum_{k=1}^{1} k^{x-1} \\
+ \sum_{k=1}^{n} k^{x-1} - \sum_{k=1}^{2} k^{x-1} \\
+ \sum_{k=1}^{n} k^{x-1} - \sum_{k=1}^{3} k^{x-1} \\
+ \sum_{k=1}^{n} k^{x-1} - \sum_{k=1}^{4} k^{x-1} \\
\vdots \\
+ \sum_{k=1}^{n} k^{x-1} - \sum_{k=1}^{n-1} k^{x-1}
\end{array}$$

$$
= \begin{array}{c} S_{x-1}(n) \\ +S_{x-1}(n) - S_{x-1}(1) \\ +S_{x-1}(n) - S_{x-1}(2) \\ +S_{x-1}(n) - S_{x-1}(3) \\ +S_{x-1}(n) - S_{x-1}(4) \\ \vdots \\ +S_{x-1}(n) - S_{x-1}(n-1) - S_{x-1}(n) + S_{x-1}(n) \end{array}
$$

$$
= nS_{x-1}(n) + S_{x-1}(n) - \sum_{k=1}^{n} S_{x-1}(k)
$$

We have assumed that $S_{x-1}(n)$ is a polynomial that can be written as $S_{x-1}(n) = \alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2}...$. Using this assumption, we can expand the above formula for $S_x(n)$:

$$
S_x(n) = (n+1)S_{x-1}(n) - \sum_{k=1}^{n}(\alpha_0^{x-1}k^x + \alpha_1^{x-1}k^{x-1} + \alpha_2^{x-1}k^{x-2} + \alpha_3^{x-1}k^{x-3} + \alpha_4^{x-1}k^{x-4} + ... + \alpha_{x-1}^{x-1}k)
$$

With the properties of summations, we can again expand the argument inside of the sum:

$$
S_x(n) = (n+1)S_{x-1}(n) - \sum_{k=1}^{n} \alpha_0^{x-1}k^x - \sum_{k=1}^{n} \alpha_1^{x-1}k^{x-1} - \sum_{k=1}^{n} \alpha_2^{x-1}k^{x-2} - \sum_{k=1}^{n} \alpha_3^{x-1}k^{x-3} - ... - \sum_{k=1}^{n} \alpha_{x-1}^{x-1}k
$$

$$
= (n+1)S_{x-1}(n) - \alpha_0^{x-1}\sum_{k=1}^{n} k^x - \alpha_1^{x-1}\sum_{k=1}^{n} k^{x-1} - \alpha_2^{x-1}\sum_{k=1}^{n} k^{x-2} - \alpha_3^{x-1}\sum_{k=1}^{n} k^{x-3} - ... - \alpha_{x-1}^{x-1}\sum_{k=1}^{n} k
$$

We have that $S_j(n) = \sum_{k=1}^{n} k^j$. We can therefore substitute $S_j(n)$ into the above formula as follows:

$$
S_x(n) = (n+1)S_{x-1}(n) - \alpha_0^{x-1}S_x(n) - \alpha_1^{x-1}S_{x-1}(n) - \alpha_2^{x-1}S_{x-2}(n) - \alpha_3^{x-1}S_{x-3}(n) - ... - \alpha_{x-1}^{x-1}S_1(n)
$$

Finally, we can solve for $S_x(n)$, obtaining a formula written in terms of $S_{x-q}(n)$:

$$
S_x(n) + \alpha_0^{x-1}S_x(n) = (n+1)S_{x-1}(n) - \alpha_1^{x-1}S_{x-1}(n) - \alpha_2^{x-1}S_{x-2}(n) - \alpha_3^{x-1}S_{x-3}(n) - ... - \alpha_{x-1}^{x-1}S_1(n)
$$

$$
\Downarrow
$$

$$S_x(n) = \frac{1}{\alpha_0^{x-1}+1}((n+1)S_{x-1}(n) - \alpha_1^{x-1}S_{x-1}(n) - \alpha_2^{x-1}S_{x-2}(n) - \alpha_3^{x-1}S_{x-3}(n) - ... - \alpha_{x-1}^{x-1}S_1(n))$$

We have from our inductive assumption that all S_{x-q} can be written as polynomials of degree $x-q+1$. Substituting these polynomials in for $S_{q-x}(n)$, we can again rewrite the formula for $S_x(n)$:

$$S_x(n) = \frac{1}{\alpha_0^{x-1}+1}\begin{pmatrix} (n+1)(\alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2} + \alpha_3^{x-1}n^{x-3} + ... + \alpha_{x-1}^{x-1}n) \\ -\alpha_1^{x-1}(\alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2} + \alpha_3^{x-1}n^{x-3} + ... + \alpha_{x-1}^{x-1}n) \\ -\alpha_2^{x-1}(\alpha_1^{x-2}n^{x-1} + \alpha_2^{x-2}n^{x-2} + \alpha_3^{x-2}n^{x-3} + \alpha_4^{x-2}n^{x-4} + ... + \alpha_{x-1}^{x-2}n) \\ -\alpha_3^{x-1}(\alpha_2^{x-3}n^{x-2} + \alpha_3^{x-3}n^{x-3} + \alpha_4^{x-3}n^{x-4} + \alpha_5^{x-3}n^{x-5} + ... + \alpha_{x-1}^{x-3}n) \\ -\alpha_4^{x-1}(\alpha_3^{x-4}n^{x-3} + \alpha_4^{x-4}n^{x-4} + \alpha_5^{x-4}n^{x-5} + \alpha_6^{x-4}n^{x-6} + ... + \alpha_{x-1}^{x-4}n) \\ \vdots \\ -\alpha_{x-1}^{x-1}(\alpha_{x-2}^{1}n^2 + \alpha_{x-1}^{1}n) \end{pmatrix}$$

$$= \frac{1}{\alpha_0^{x-1}+1}\begin{pmatrix} (n+1)(\alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2} + \alpha_3^{x-1}n^{x-3} + ... + \alpha_{x-1}^{x-1}n) \\ -(\alpha_1^{x-1}\alpha_0^{x-1}n^x + \alpha_1^{x-1}\alpha_1^{x-1}n^{x-1} + \alpha_1^{x-1}\alpha_2^{x-1}n^{x-2} + ... + \alpha_1^{x-1}\alpha_{x-1}^{x-1}n) \\ -(\alpha_2^{x-1}\alpha_1^{x-2}n^{x-1} + \alpha_2^{x-1}\alpha_2^{x-2}n^{x-2} + \alpha_2^{x-1}\alpha_3^{x-2}n^{x-3} + ... + \alpha_2^{x-1}\alpha_{x-1}^{x-2}n) \\ -(\alpha_3^{x-1}\alpha_2^{x-3}n^{x-2} + \alpha_3^{x-1}\alpha_3^{x-3}n^{x-3} + \alpha_3^{x-1}\alpha_4^{x-3}n^{x-4} + ... + \alpha_3^{x-1}\alpha_{x-1}^{x-3}n) \\ -(\alpha_4^{x-1}\alpha_3^{x-4}n^{x-3} + \alpha_4^{x-1}\alpha_4^{x-4}n^{x-4} + \alpha_4^{x-1}\alpha_5^{x-4}n^{x-5} + ... + \alpha_4^{x-1}\alpha_{x-1}^{x-4}n) \\ \vdots \\ -(\alpha_{x-1}^{x-1}\alpha_{x-2}^{1}n^2 + \alpha_{x-1}^{x-1}\alpha_{x-1}^{1}n) \end{pmatrix}$$

$$= \frac{1}{\alpha_0^{x-1}+1}(n(\alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2} + \alpha_3^{x-1}n^{x-3} + \alpha_4^{x-1}n^{x-4} + ... + \alpha_{x-1}^{x-1}n)$$

$$+(\alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2} + \alpha_3^{x-1}n^{x-3} + \alpha_4^{x-1}n^{x-4} + ... + \alpha_{x-1}^{x-1}n)$$

$$-\sum_{k=1}^{x-1}(\alpha_k^{x-1}\alpha_0^{x-k}n^x + \alpha_k^{x-1}\alpha_1^{x-k}n^{x-1} + \alpha_k^{x-1}\alpha_2^{x-k}n^{x-2} + \alpha_k^{x-1}\alpha_3^{x-k}n^{x-3} + ... + \alpha_k^{x-1}\alpha_{x-1}^{x-k}n))$$

(Note again that some of the constants $\alpha_j^{\phi-i}$ inside of the sum's argument will be zero (for example, $S_1(n)$ is a polynomial of degree 2, so, if $x=3$, the constant "α_0^1", corresponding to the term "$\alpha_0^1 n^3$" will be zero)). Finally, we can split up the argument inside of the sum to rewrite S_x as a polynomial of degree $x+1$ as follows:

$$S_x(n) = \frac{1}{\alpha_0^{x-1}+1}(\alpha_0^{x-1}n^{x+1} + \alpha_1^{x-1}n^x + \alpha_2^{x-1}n^{x-1} + \alpha_3^{x-1}n^{x-2} + \alpha_4^{x-1}n^{x-3} + ... + \alpha_{x-1}^{x-1}n^2$$

$$+\alpha_0^{x-1}n^x + \alpha_1^{x-1}n^{x-1} + \alpha_2^{x-1}n^{x-2} + \alpha_3^{x-1}n^{x-3} + \alpha_4^{x-1}n^{x-4} + ... + \alpha_{x-1}^{x-1}n$$

$$-\sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_0^{x-k}n^x - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_1^{x-k}n^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_2^{x-k}n^{x-2} - ... - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_{x-1}^{x-k}n)$$

$$= \frac{1}{\alpha_0^{x-1}+1}\begin{pmatrix} (\alpha_0^{x-1} + \alpha_{-1}^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_{-1}^{x-k})n^{x+1} \\ +(\alpha_1^{x-1} + \alpha_0^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_0^{x-k})n^x \\ +(\alpha_2^{x-1} + \alpha_1^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_1^{x-k})n^{x-1} \\ +(\alpha_3^{x-1} + \alpha_2^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_2^{x-k})n^{x-2} \\ +(\alpha_4^{x-1} + \alpha_3^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_3^{x-k})n^{x-3} \\ \vdots \\ +(\alpha_{x-1}^{x-1} + \alpha_{x-2}^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_{x-2}^{x-k})n^2 \\ +(\alpha_x^{x-1} + \alpha_{x-1}^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_{x-1}^{x-k})n \end{pmatrix}$$

(Note that: 1: $\alpha_{-1}^{x-1} - \sum_{k=1}^{x-1}\alpha_k^{x-1}\alpha_{-1}^{x-k} = 0$, as there are no terms of the form "α_{-1}^i" since each polynomial S_{x-q} is of a degree less than $x+1$; and 2: $\alpha_x^{x-1} = 0$ since there are no constant terms in any polynomial S_{x-q})

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Though the above formula gives us a valid method of building the power summation formulas, it is a bit messy to look at, and certainly not memorable. Seeing this, we will rewrite the formula in terms of an organized matrix equation.

Let us call the information inside of the parentheses the polynomial function $\Psi(n)$, so that we have:

$$\Psi(n) = \begin{pmatrix} (\alpha_0^{x-1} + \alpha_{-1}^{x-1} - \sum_{k=1}^{x-1} \alpha_k^{x-1}\alpha_{-1}^{x-k})n^{x+1} \\ +(\alpha_1^{x-1} + \alpha_0^{x-1} - \sum_{k=1}^{x-1} \alpha_k^{x-1}\alpha_0^{x-k})n^{x} \\ +(\alpha_2^{x-1} + \alpha_1^{x-1} - \sum_{k=1}^{x-1} \alpha_k^{x-1}\alpha_1^{x-k})n^{x-1} \\ \vdots \\ +(\alpha_{x-1}^{x-1} + \alpha_{x-2}^{x-1} - \sum_{k=1}^{x-1} \alpha_k^{x-1}\alpha_{x-2}^{x-k})n^{2} \\ +(\alpha_{x}^{x-1} + \alpha_{x-1}^{x-1} - \sum_{k=1}^{x-1} \alpha_k^{x-1}\alpha_{x-1}^{x-k})n \end{pmatrix}$$

Now consider the constant of the polynomial $\Psi(n)$ corresponding to n^{x-j}: $\alpha_{j+1}^{x-1} + \alpha_j^{x-1} - \sum_{k=1}^{x-1} \alpha_k^{x-1}\alpha_j^{x-k}$, which we will call "$\beta_j$". Expanding the sum, we have:

$$\beta_j = \alpha_{j+1}^{x-1} + \alpha_j^{x-1} - \alpha_1^{x-1}\alpha_j^{x-1} - \alpha_2^{x-1}\alpha_j^{x-2} - \alpha_3^{x-1}\alpha_j^{x-3} - \alpha_4^{x-1}\alpha_j^{x-4} - ... - \alpha_{x-1}^{x-1}\alpha_j^{1}$$

$$\beta_j = \alpha_{j+1}^{x-1} + \alpha_j^{x-1}\frac{\alpha_0^{x-1}}{\alpha_0^{x-1}} - \alpha_1^{x-1}\alpha_j^{x-1} - \alpha_2^{x-1}\alpha_j^{x-2} - \alpha_3^{x-1}\alpha_j^{x-3} - \alpha_4^{x-1}\alpha_j^{x-4} - ... - \alpha_{x-1}^{x-1}\alpha_j^{1}$$

Notice that β_j can be written as the dot product of two vectors of the form:

$$\beta_j = \langle \alpha_{j+1}^{x-1}, \alpha_j^{x-1}/\alpha_0^{x-1}, -\alpha_j^{x-1}, -\alpha_j^{x-2}, -\alpha_j^{x-3}, ..., -\alpha_j^{1}\rangle \cdot \langle 1, \alpha_0^{x-1}, \alpha_1^{x-1}, \alpha_2^{x-1}, \alpha_3^{x-1}, ..., \alpha_{x-1}^{x-1}\rangle$$

which can be rewritten as the product of two vectors:

$$\beta_j = \begin{bmatrix} \alpha_{j+1}^{x-1} & \alpha_j^{x-1}/\alpha_0^{x-1} & -\alpha_j^{x-1} & -\alpha_j^{x-2} & -\alpha_j^{x-3} & ... & -\alpha_j^{1} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_0^{x-1} \\ \alpha_1^{x-1} \\ \alpha_2^{x-1} \\ \alpha_3^{x-1} \\ \vdots \\ \alpha_{x-1}^{x-1} \end{bmatrix}$$

If we call the horizontal vector $\begin{bmatrix} \alpha_{j+1}^{x-1} & \alpha_j^{x-1}/\alpha_0^{x-1} & -\alpha_j^{x-1} & -\alpha_j^{x-2} & -\alpha_j^{x-3} & ... & -\alpha_j^{1} \end{bmatrix}$ "u_j" and the constant vertical vector $\begin{bmatrix} 1 & \alpha_0^{x-1} & \alpha_1^{x-1} & \alpha_2^{x-1} & \alpha_3^{x-1} & ... & \alpha_{x-1}^{x-1} \end{bmatrix}$ "v", we have:

$$\Psi(n) = \beta_{-1}n^{x+1} + \beta_0 n^x + \beta_1 n^{x-1} + \beta_2 n^{x-2} + \beta_3 n^{x-3} + ... + \beta_{x-1} n$$

$$\Psi(n) = u_{-1}vn^{x+1} + u_0vn^x + u_1vn^{x-1} + u_2vn^{x-2} + u_3vn^{x-3} + ... + u_{x-1}vn$$

which we can again rewrite as a dot product of vectors:

$$\Psi(n) = \begin{bmatrix} u_{-1}v \\ u_0v \\ u_1v \\ u_2v \\ u_3v \\ \vdots \\ u_{x-1}v \end{bmatrix} \cdot \begin{bmatrix} n^{x+1} \\ n^x \\ n^{x-1} \\ n^{x-2} \\ n^{x-3} \\ \vdots \\ n \end{bmatrix}$$

$$= \begin{bmatrix} u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{x-1} \end{bmatrix} v \cdot \begin{bmatrix} n^{x+1} \\ n^x \\ n^{x-1} \\ n^{x-2} \\ n^{x-3} \\ \vdots \\ n \end{bmatrix}$$

Substituting the actual vectors for u_j and v yields the matrix formula:

$$\Psi(n) = \begin{bmatrix} \alpha_0^{x-1} & \alpha_{-1}^{x-1}/\alpha_0^{x-1} & -\alpha_{-1}^{x-1} & -\alpha_{-1}^{x-2} & -\alpha_{-1}^{x-3} & ... & -\alpha_{-1}^1 \\ \alpha_1^{x-1} & \alpha_0^{x-1}/\alpha_0^{x-1} & -\alpha_0^{x-1} & -\alpha_0^{x-2} & -\alpha_0^{x-3} & ... & -\alpha_0^1 \\ \alpha_2^{x-1} & \alpha_1^{x-1}/\alpha_0^{x-1} & -\alpha_1^{x-1} & -\alpha_1^{x-2} & -\alpha_1^{x-3} & ... & -\alpha_1^1 \\ \alpha_3^{x-1} & \alpha_2^{x-1}/\alpha_0^{x-1} & -\alpha_2^{x-1} & -\alpha_2^{x-2} & -\alpha_2^{x-3} & ... & -\alpha_2^1 \\ \alpha_4^{x-1} & \alpha_3^{x-1}/\alpha_0^{x-1} & -\alpha_3^{x-1} & -\alpha_3^{x-2} & -\alpha_3^{x-3} & ... & -\alpha_3^1 \\ \alpha_5^{x-1} & \alpha_4^{x-1}/\alpha_0^{x-1} & -\alpha_4^{x-1} & -\alpha_4^{x-2} & -\alpha_4^{x-3} & ... & -\alpha_4^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{x-1}^{x-1} & \alpha_{x-2}^{x-1}/\alpha_0^{x-1} & -\alpha_{x-2}^{x-1} & -\alpha_{x-2}^{x-2} & -\alpha_{x-2}^{x-3} & ... & -\alpha_{x-2}^1 \\ \alpha_x^{x-1} & \alpha_{x-1}^{x-1}/\alpha_0^{x-1} & -\alpha_{x-1}^{x-1} & -\alpha_{x-1}^{x-2} & -\alpha_{x-1}^{x-3} & ... & -\alpha_{x-1}^1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_0^{x-1} \\ \alpha_1^{x-1} \\ \alpha_2^{x-1} \\ \alpha_3^{x-1} \\ \alpha_4^{x-1} \\ \vdots \\ \alpha_{x-2}^{x-1} \\ \alpha_{x-1}^{x-1} \end{bmatrix} \cdot \begin{bmatrix} n^{x+1} \\ n^x \\ n^{x-1} \\ n^{x-2} \\ n^{x-3} \\ n^{x-4} \\ \vdots \\ n^2 \\ n \end{bmatrix}$$

and substituting $\Psi(n)$ back into the function $S_x(n)$, we have:

$$S_x(n) = \frac{1}{\alpha_0^{x-1}+1}\begin{bmatrix} \alpha_0^{x-1} & \alpha_{-1}^{x-1}/\alpha_0^{x-1} & -\alpha_{-1}^{x-1} & -\alpha_{-1}^{x-2} & -\alpha_{-1}^{x-3} & ... & -\alpha_{-1}^{1} \\ \alpha_1^{x-1} & \alpha_{0}^{x-1}/\alpha_0^{x-1} & -\alpha_{0}^{x-1} & -\alpha_{0}^{x-2} & -\alpha_{0}^{x-3} & ... & -\alpha_{0}^{1} \\ \alpha_2^{x-1} & \alpha_{1}^{x-1}/\alpha_0^{x-1} & -\alpha_{1}^{x-1} & -\alpha_{1}^{x-2} & -\alpha_{1}^{x-3} & ... & -\alpha_{1}^{1} \\ \alpha_3^{x-1} & \alpha_{2}^{x-1}/\alpha_0^{x-1} & -\alpha_{2}^{x-1} & -\alpha_{2}^{x-2} & -\alpha_{2}^{x-3} & ... & -\alpha_{2}^{1} \\ \alpha_4^{x-1} & \alpha_{3}^{x-1}/\alpha_0^{x-1} & -\alpha_{3}^{x-1} & -\alpha_{3}^{x-2} & -\alpha_{3}^{x-3} & ... & -\alpha_{3}^{1} \\ \alpha_5^{x-1} & \alpha_{4}^{x-1}/\alpha_0^{x-1} & -\alpha_{4}^{x-1} & -\alpha_{4}^{x-2} & -\alpha_{4}^{x-3} & ... & -\alpha_{4}^{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{x-1}^{x-1} & \alpha_{x-2}^{x-1}/\alpha_0^{x-1} & -\alpha_{x-2}^{x-1} & -\alpha_{x-2}^{x-2} & -\alpha_{x-2}^{x-3} & ... & -\alpha_{x-2}^{1} \\ \alpha_{x}^{x-1} & \alpha_{x-1}^{x-1}/\alpha_0^{x-1} & -\alpha_{x-1}^{x-1} & -\alpha_{x-1}^{x-2} & -\alpha_{x-1}^{x-3} & ... & -\alpha_{x-1}^{1} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_0^{x-1} \\ \alpha_1^{x-1} \\ \alpha_2^{x-1} \\ \alpha_3^{x-1} \\ \alpha_4^{x-1} \\ \vdots \\ \alpha_{x-2}^{x-1} \\ \alpha_{x-1}^{x-1} \end{bmatrix} \cdot \begin{bmatrix} n^{x+1} \\ n^x \\ n^{x-1} \\ n^{x-2} \\ n^{x-3} \\ n^{x-4} \\ \vdots \\ n^2 \\ n \end{bmatrix}$$

Finally, we can obtain our original function for $S_x(n)$ by substituting zeros into the matrix in any place we know there are zeros, and by substituting -1 out of the matrix:

$$S_x(n) = -\frac{1}{\alpha_0^{x-1}+1}\begin{bmatrix} -\alpha_0^{x-1} & 0 & 0 & 0 & 0 & ... & 0 \\ -\alpha_1^{x-1} & -\alpha_{0}^{x-1}/\alpha_0^{x-1} & \alpha_{0}^{x-1} & 0 & 0 & ... & 0 \\ -\alpha_2^{x-1} & -\alpha_{1}^{x-1}/\alpha_0^{x-1} & \alpha_{1}^{x-1} & \alpha_{1}^{x-2} & 0 & ... & 0 \\ -\alpha_3^{x-1} & -\alpha_{2}^{x-1}/\alpha_0^{x-1} & \alpha_{2}^{x-1} & \alpha_{2}^{x-2} & \alpha_{2}^{x-3} & ... & 0 \\ -\alpha_4^{x-1} & -\alpha_{3}^{x-1}/\alpha_0^{x-1} & \alpha_{3}^{x-1} & \alpha_{3}^{x-2} & \alpha_{3}^{x-3} & ... & 0 \\ -\alpha_5^{x-1} & -\alpha_{4}^{x-1}/\alpha_0^{x-1} & \alpha_{4}^{x-1} & \alpha_{4}^{x-2} & \alpha_{4}^{x-3} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -\alpha_{x-1}^{x-1} & -\alpha_{x-2}^{x-1}/\alpha_0^{x-1} & \alpha_{x-2}^{x-1} & \alpha_{x-2}^{x-2} & \alpha_{x-2}^{x-3} & ... & \alpha_{x-2}^{1} \\ 0 & -\alpha_{x-1}^{x-1}/\alpha_0^{x-1} & \alpha_{x-1}^{x-1} & \alpha_{x-1}^{x-2} & \alpha_{x-1}^{x-3} & ... & \alpha_{x-1}^{1} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_0^{x-1} \\ \alpha_1^{x-1} \\ \alpha_2^{x-1} \\ \alpha_3^{x-1} \\ \alpha_4^{x-1} \\ \vdots \\ \alpha_{x-2}^{x-1} \\ \alpha_{x-1}^{x-1} \end{bmatrix} \cdot \begin{bmatrix} n^{x+1} \\ n^x \\ n^{x-1} \\ n^{x-2} \\ n^{x-3} \\ n^{x-4} \\ \vdots \\ n^2 \\ n \end{bmatrix}$$

Thus we have shown the theorem to hold true for any $S_x(n), x > 1$, and we have that the theorem is true for any $\phi \in \mathbb{N}$.