Formal Concept Analysis

I Contexts, Concepts, and Concept Lattices

Robert Jäschke
Asmelash Teka Hadgu

FG Wissensbasierte Systeme/L3S Research Center
Leibniz Universität Hannover

slides based on a lecture by Prof. Gerd Stumme

Agenda

2. Many-valued Contexts and Conceptual Scaling
   - Many-valued Contexts
   - Conceptual Scaling
   - Elementary Scales

Many-valued Contexts

- so far: *one-valued attributes*
- in language, the word “attribute” refers not only to properties which an object may have or not
- attributes like “color”, “weight”, “sex”, or “grade” have *values*
- now: *many-valued attributes*
- (DIN 2330 calls many-valued attributes *Merkmalarten*.)

Many-valued Contexts: Definition

Def.: A *many-valued context* (G, M, W, I) consists of sets G, M and W and a ternary relation I between G, M and W (i.e., $I \subseteq G \times M \times W$) for which it holds that

$$(g, m, w) \in I \text{ and } (g, m, v) \in I \quad \text{always implies} \quad w = v.$$

The elements of

- G are called *objects*, those of
- M *(many-valued) attributes* and those of
- W *attribute values*.

$(g, m, w) \in I$ is read as "the attribute m has the value w for the object g".

Many-valued Contexts: Properties

- Many-valued attributes can be regarded as partial maps from G in W.
- We can write $m(g) = w$ instead of $(g, m, w) \in I$.
- example: *maintainability(mid-engine) = very poor*

representation as a table

$$\begin{array}{c|c} & \overbrace{\qquad\qquad\qquad m\quad}^{M} \\ \hline G\left\{\; g \right. & \qquad\qquad\qquad m(g) \end{array}$$

The entry in row g and column m represents the attribute value $m(g)$.

(If the attribute m does not have a value for the object g, there will be no entry.)

Many-valued Contexts

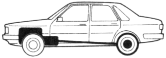
- *domain* of an attribute m:

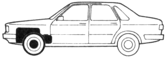
$$\mathrm{dom}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}$$

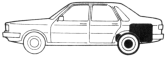
- An attribute m is called *complete*, if $\mathrm{dom}(m) = G$.
- A many-valued context is *complete*, if all its attributes are complete.

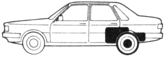
Many-valued Contexts: “Drive Concepts for Motorcars”

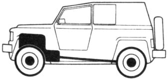
A comparison of the different options to arrange the engine and the drive mechanism of a motorcar.[1]


conventional


front-wheel


rear-wheel


mid-engine


all-wheel

	De	Dl	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
all-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

De := drive efficiency empty
Dl := drive efficiency loaded
R := road holding/handling properties;
S := self-steering efficiency

E := economy of space
C := cost of construction
M := maintainability

[1] Schlag nach! 100 000 Tatsachen aus allen Wissensgebieten. BI Verlag, 1982

Conceptual Scaling

How can we compute concepts for a many-valued context?

- By *transforming* it into a one-valued context:
  - each many-valued attribute is interpreted by means of a context
  - this context is called *conceptual scale*
- The concepts of this *derived* context are *interpreted* as concepts of the many-valued context.
- This process is called *conceptual scaling*
  - conceptual scales are not uniquely determined
  - result depends on the chosen scales

Conceptual Scaling

Def.: A *scale* for the attribute m of a many-valued context is a (one-valued) context $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. The objects of a scale are called *scale values*, the attributes are called *scale attributes*.

$\mathbb{S}_\mathsf{R} :=$

	$++$	$+$	$--$
excellent	×	×	
good		×	
very poor			×

- Every context can be used as a scale.
- Formally, there is no difference between a scale and a context.
- We will use the term "scale" only for contexts which have a clear conceptual structure and which bear meaning.

Conceptual Scaling

Def.: If (G, M, W, I) is a many-valued context and $\mathbb{S}_m, m \in M$ are scale contexts, then the *derived context with respect to plain scaling* is the context (G, N, J) with

$$N := \bigcup_{m \in M} \dot{M}_m,$$

and

$$gJ(m, n) :\Longleftrightarrow m(g) = w \text{ and } wI_m n.$$

($\dot{M}_m := \{m\} \times M_m$, to ensure that the attributes are disjoint)

Conceptual Scaling: "Drive Concepts for Motorcars"

	De	Dl	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
all-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

Using those scales:

$\mathbb{S}_{\mathsf{De}} := \mathbb{S}_{\mathsf{Dl}} :=$

	++	+	−
excellent	×	×	
good		×	
poor			×

$\mathbb{S}_{\mathsf{R}} :=$

	++	+	−−
excellent	×	×	
good		×	
very poor			×

$\mathbb{S}_{\mathsf{S}} :=$

	u	o	n	u/n
understeering	×			
oversteering		×		
neutral			×	
understeering/neutral				×

$\mathbb{S}_{\mathsf{C}} :=$

	vl	l	m	h
very low	×	×		
low		×		
medium			×	
high				×

$\mathbb{S}_{\mathsf{E}} := \mathbb{S}_{\mathsf{M}} :=$

	++	+	−	−−
excellent	×	×		
good		×		
poor			×	
very poor			×	×

. . . we get the following context:

Conceptual Scaling: “Drive Concepts for Motorcars”

	De			Dl			R			S				E				C				M			
	++	+	−	++	+	−	++	+	−−	u	o	n	u/n	++	+	−	−−	vl	l	m	h	++	+	−	−−
conventional			×		×			×		×					×					×		×	×		
front-wheel		×				×	×	×		×				×	×			×	×				×		
rear-wheel	×	×		×	×				×		×					×			×				×		
mid-engine	×	×		×	×			×				×				×	×		×					×	×
all-wheel	×	×		×	×			×					×		×						×			×	

(If we had used the scale $\mathbb{S}_E$ for the attributes De, Dl, and R as well, the derived context would have only turned out slightly different.)

Conceptual Scaling

The derived one-valued context is obtained from the many-valued context (G, M, W, I) and the scale contexts $\mathbb{S}_m, m \in M$ as follows:

- the object set G remains unchanged
- every many-valued attribute m is replaced by the attributes of $\mathbb{S}_m$

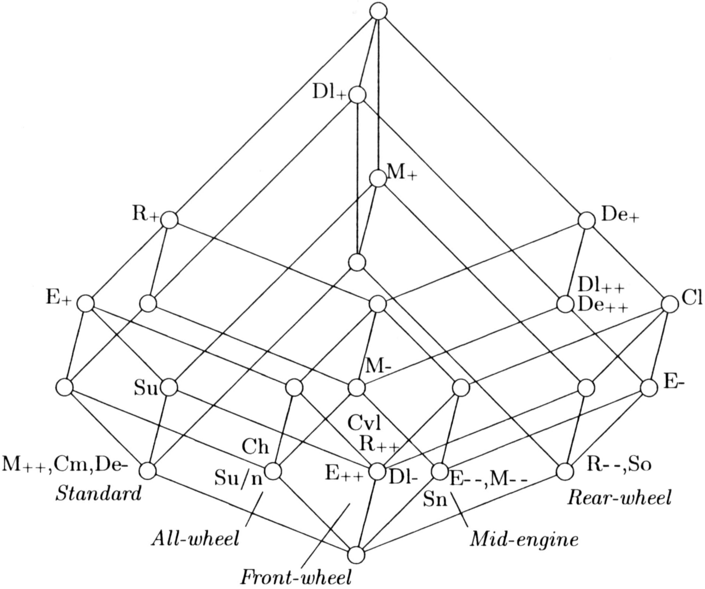
In the table representation, we can visualize scaling as follows: Every attribute value $m(g)$ is replaced by the row of $\mathbb{S}_m$ which belongs to $m(g)$:

	De	Dl	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good				very poor
all-wheel	excellent	excellent	good	under			poor

	++	+	−
excellent	×	×	
good		×	
poor			×

	De			Dl			R												C			M			
	++	+	−	++	+	−	++	+	−−										l	m	h	++	+	−	−−
conventional			×		×			×		×					×					×		×	×		
front-wheel		×				×	×	×		×				×	×			×	×				×		
rear-wheel	×	×		×	×				×		×					×			×				×		
mid-engine	×	×		×	×			×				×				×	×		×					×	×
all-wheel	×	×		×	×			×					×		×						×			×	

Conceptual Scaling: “Drive Concepts for Motorcars”



De := drive efficiency empty
Dl := drive efficiency loaded
R := road holding/handling properties
S := self-steering efficiency
E := economy of space
C := cost of construction
M := maintainability

Conceptual Scaling

Which contexts can we use for scaling?

- formally, any binary relation can be regarded as a context

→ interesting contexts from mathematics

  - have structural properties occurring very rarely with empirical data
  - great importance for data analysis
  - "ideal structures"
  - can be used as scales

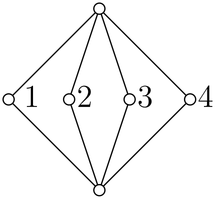
We introduce so-called *elementary scales*.

Elementary Scales

Nominal Scales: $\mathbb{N}_n := (\mathbf{n}, \mathbf{n}, =)$[2]

- to scale attributes, the values of which *mutually exclude* each other
- example: attribute with values {*masculine, feminine, neuter*}
- we obtain a *partition* of the objects into extents
- the partitions correspond to the values of the attribute

	1	2	3	4
1	×			
2		×		
3			×	
4				×



The Nominal Scale $\mathbb{N}_4$.

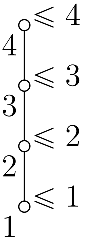
[2] $\mathbf{n} := \{1, \ldots, n\}$

Elementary Scales

Ordinal Scales: $\mathbb{O}_n := (\mathbf{n}, \mathbf{n}, \leqslant)$

- to scale attributes, the values of which are *ordered* and each value implies the weaker ones
- example: attribute with values $\{$*loud, very loud, extremely loud*$\}$
- attribute values result in a chain of extents, interpreted as a *hierarchy*

	1	2	3	4
1	×	×	×	×
2		×	×	×
3			×	×
4				×



The Ordinal Scale $\mathbb{O}_4$.

Elementary Scales

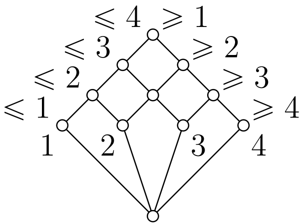
Interordinal Scales: $\mathbb{I}_n := (\mathbf{n}, \mathbf{n}, \leqslant) \mid (\mathbf{n}, \mathbf{n}, \geqslant)$

- questionnaires often offer opposite pairs as possible answers allowing a choice of intermediate values
- for example *active–passive*, *talkative–taciturn*, etc.

→ *bipolar* ordering of the values

- extents of the interordinal scale are precisely the intervals of values
- the *betweenness relation* is reflected conceptually

	$\leqslant 1$	$\leqslant 2$	$\leqslant 3$	$\leqslant 4$	$\geqslant 1$	$\geqslant 2$	$\geqslant 3$	$\geqslant 4$
1	×	×	×	×	×			
2		×	×	×	×	×		
3			×	×	×	×	×	
4				×	×	×	×	×



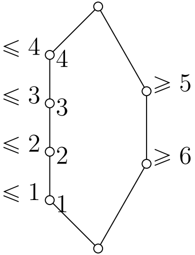
The Interordinal Scale $\mathbb{I}_4$.

Elementary Scales

Biordinal Scales: $\mathbb{M}_{n,m} := (\mathbf{n}, \mathbf{n}, \leqslant) \dot{\cup} (\mathbf{m}, \mathbf{m}, \geqslant)$

- often opposite pairs are used simpler: each object is assigned one of the two poles, allowing graduations – "*partition with a hierarchy*"
- example: $\{$*very low*, *low*, *loud*, *very loud*$\}$ (➞ suggests *loud* and *low* mutually exclude each other, *very loud* implies *loud*, *very low* implies *low*)
- example: school mark *excellent* is also *very good*, *good*, and *satisfactory*, but not *unsatisfactory* or a *fail*

	⩽1	⩽2	⩽3	⩽4	⩾5	⩾6
1	×	×	×	×		
2		×	×	×		
3			×	×		
4				×		
5					×	
6					×	×



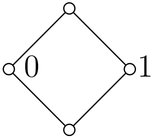
The Biordinal Scale $\mathbb{M}_{4,2}$.

Elementary Scales

The **Dichotomic Scale:** $\mathbb{D} := (\{0,1\},\{0,1\},=)$

- special case: isomorphic to $\mathbb{N}_2$ amd $\mathbb{M}_{1,1}$
- closely related to $\mathbb{I}_2$
- frequently used to scale attributes with values like $\{$*yes*, *no*$\}$

	0	1
0	×	
1		×



The Dichotomic Scale $\mathbb{D}$.

Elementary Scales

- frequently, all many-valued attributes can be interpreted with respect to the same scale or family of scales
- *nominally scaled context*: if all scales $\mathbb{S}_m$ are nominal scales, etc.
- a many-valued context is called *nominal*, if the nature of the data suggests nominal scaling
- a many-valued context is called an *ordinal context* if for each attribute the set of values is ordered in a natural way

Elementary Scales: Example "Forum Romanum"

	Forum Romanum	B	GB	M	P
1	Arch of Septimus Severus	*	*	**	*
2	Arch of Titus	*	**	**	
3	Basilica Julia			*	
4	Basilica of Maxentius	*			
5	Phocas column		*	**	
6	Curia				*
7	House of the Vestals			*	
8	Portico of Twelve Gods		*	*	*
9	Temple of Antonius and Fausta	*	*	***	*
10	Temple of Castor and Pollux	*	**	***	*
11	Temple of Romulus		*		
12	Temple of Saturn			**	*
13	Temple of Vespasian			**	
14	Temple of Vesta		**	**	*

Example of an ordinal context: Ratings of monuments on the Forum Romanum in different travel guides (B = Baedecker, GB = Les Guides Bleus, M = Michelin, P = Polyglott). The context becomes ordinal through the number of stars awarded. If no star has been awarded, this is rated zero.

Elementary Scales: Example "Forum Romanum"

	Forum Romanum	B	GB		M			P
		*	*	**	*	**	***	*
1	Arch of Septimus Severus	×	×		×	×		×
2	Arch of Titus	×	×	×	×	×		
3	Basilica Julia				×			
4	Basilica of Maxentius	×						
5	Phocas column		×		×	×		
6	Curia							×
7	House of the Vestals				×			
8	Portico of Twelve Gods		×		×			×
9	Temple of Antonius and Fausta	×	×		×	×	×	×
10	Temple of Castor and Pollux	×	×	×	×	×	×	×
11	Temple of Romulus		×					
12	Temple of Saturn				×	×		×
13	Temple of Vespasian				×	×		
14	Temple of Vesta		×	×	×	×		×

Elementary Scales: Example “Forum Romanum”