Streaming - 2

Bloom Filters, Distinct Item counting, Computing moments

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    - Number of distinct elements in the last ***k*** elements of the stream
  - **(3) Estimating moments: AMS method**
    - Estimate std. dev. of last ***k*** elements

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    - **what is the probability that a bin does not get a ball ?**

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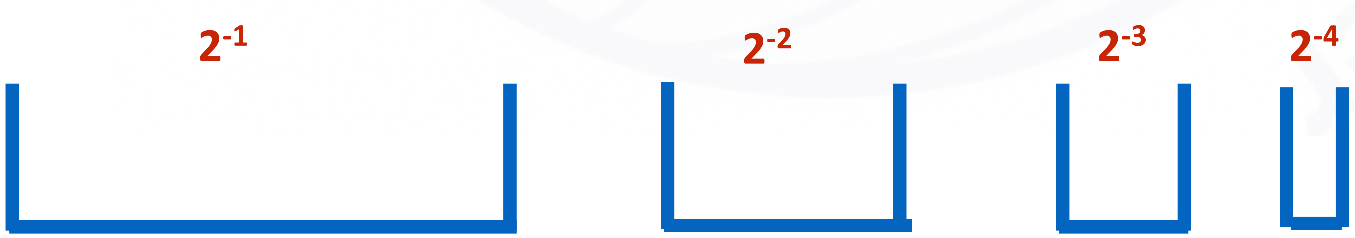

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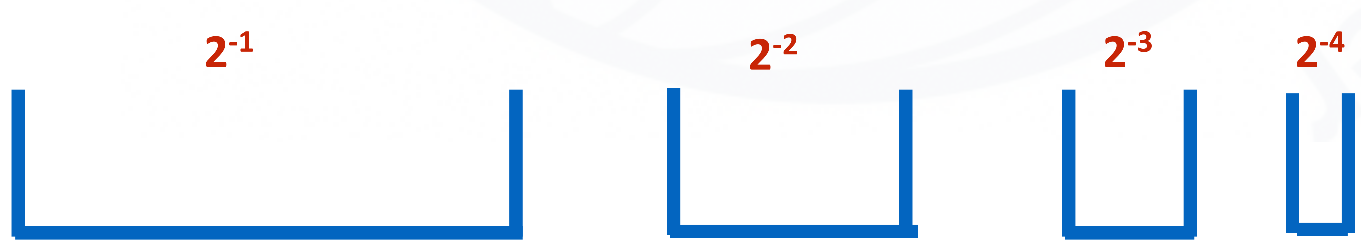

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- **Determine which tuples of stream are in *S***
- **Obvious solution: Hash table**
  - But suppose we **do not have enough memory** to store all of ***S*** in a hash table
    - E.g., we might be processing millions of filters on the same stream

Applications

- **Example: Email spam filtering**
  - We know 1 billion “good” email addresses
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- **Publish-subscribe systems**
  - You are collecting lots of messages (news articles)
  - People express interest in certain sets of keywords
  - Determine whether each message matches user’s interest

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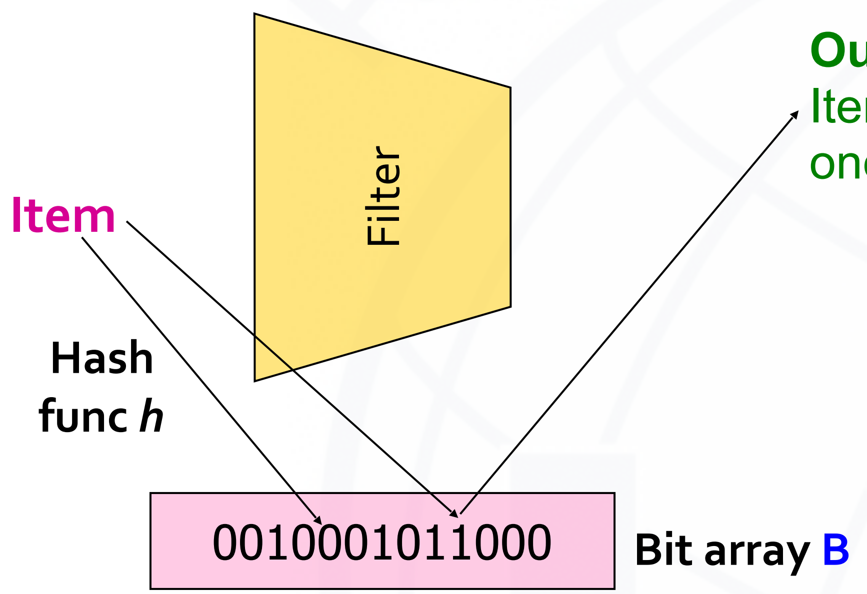
- Hash each member of ***s*** $\in$ ***S*** to one of ***n*** buckets, and set that bit to **1**, i.e., ***B[h(s)]=1***
- Hash each element ***a*** of the stream and output only those that hash to bit that was set to **1**
    - **Output *a* if B[h(a)] == 1**

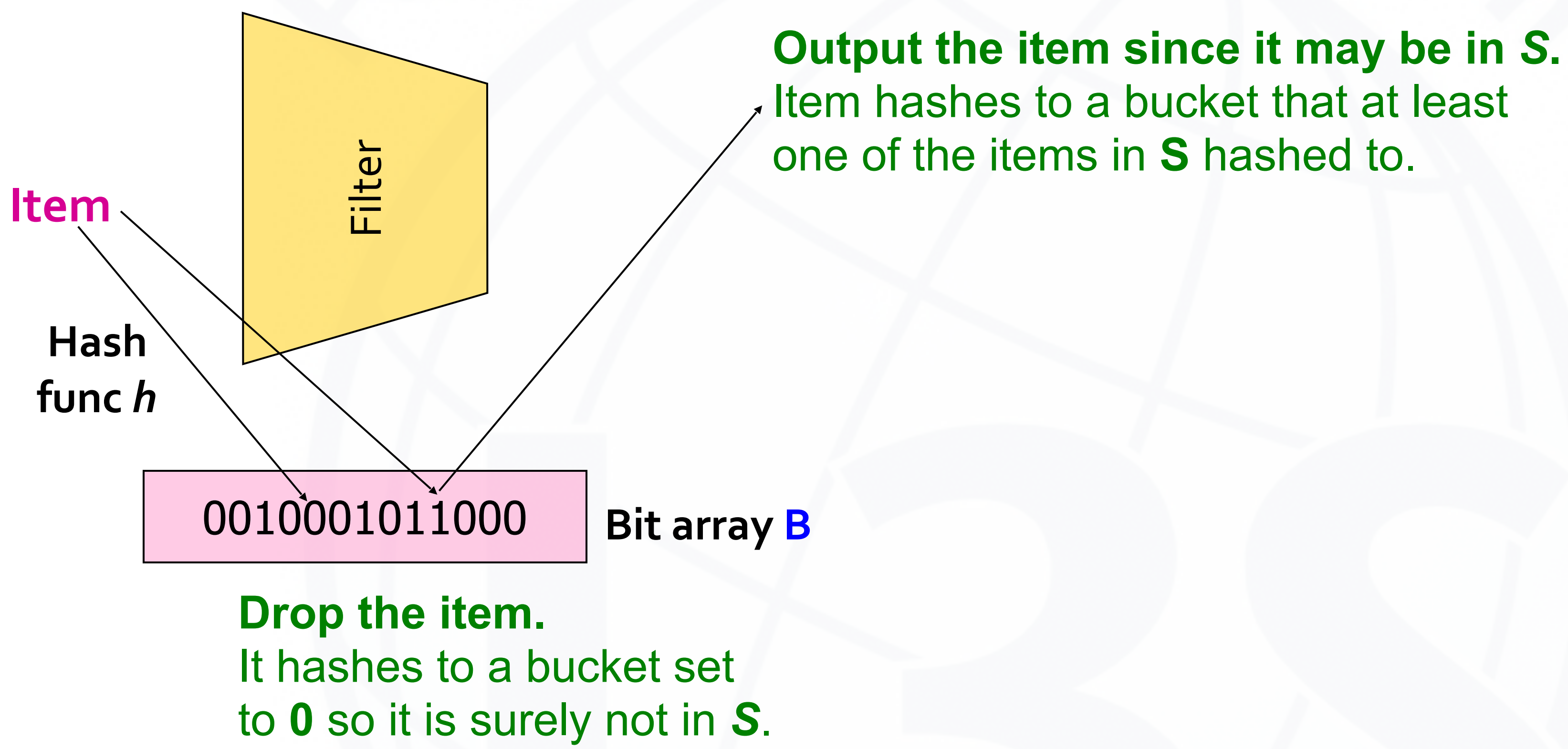
First Cut Solution (2)



Output the item since it may be in *S*.
Item hashes to a bucket that at least one of the items in **S** hashed to.

Drop the item.
It hashes to a bucket set to **0** so it is surely not in ***S***.

First Cut Solution (2)



- **Creates false positives but no false negatives**
  - If the item is in ***S*** we surely output it, if not we may still output it

First Cut Solution (3)

- **|S| = 1 billion email addresses**
  |B|= 1GB = 8 billion bits

- If the email address is in ***S***, then it surely hashes to a bucket that has the bit set to **1**,
  so it always gets through (***no false negatives***)

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- If the email address is in ***S***, then it surely hashes to a bucket that has the bit set to **1**,
  so it always gets through (***no false negatives***)

- Approximately **1/8** of the bits are set to **1**, so about **1/8th** of the addresses not in ***S*** get through to the output (***false positives***)
  - Actually, less than **1/8th**, because more than one address might hash to the same bit

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- **In our case:**
  - **Targets** = bits/bins
  - **balls** = hash values of items

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$$\underline{(1 - 1/n)}$$

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- **Example:** **10^9** balls, **$8 \cdot 10^9$** bins
  - Fraction of **1s** in **B** = **$1 - e^{-1/8}$ = 0.1175**
    - Compare with our earlier estimate: **1/8 = 0.125**

Multiple Hash Functions

H1	0	1	0	0
H2	0	0	0	1
H3	0	1	0	0
	0	**1**	**0**	**1**

Final Array is the union of all bins

Multiple Hash Functions

0	1	0	0
0	0	0	1
0	1	0	0
0	**1**	**0**	**1**

H2 H3

H1

Only allow if all bits are set

discarded

Bloom Filter

- Consider: **|S| =** $\boldsymbol{m}$**, |B| =** $\boldsymbol{n}$
- Use $\boldsymbol{k}$ independent hash functions $\boldsymbol{h_1,\ldots, h_k}$

(**note:** we have a single array B!)

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- **Run-time:**
  - When a stream element with key ***x*** arrives
    - If $B[h_i(x)] = 1$ for all $i = 1, \ldots, k$ then declare that ***x*** is in ***S***
      - That is, ***x*** hashes to a bucket set to **1** for every hash function $h_i(x)$
    - Otherwise discard the element ***x***

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- But we have k independent hash functions and we only let the element x through **if all** k hash element x to a bin of value **1**
- So, false **positive probability** = $(1 - e^{-km/n})^k$

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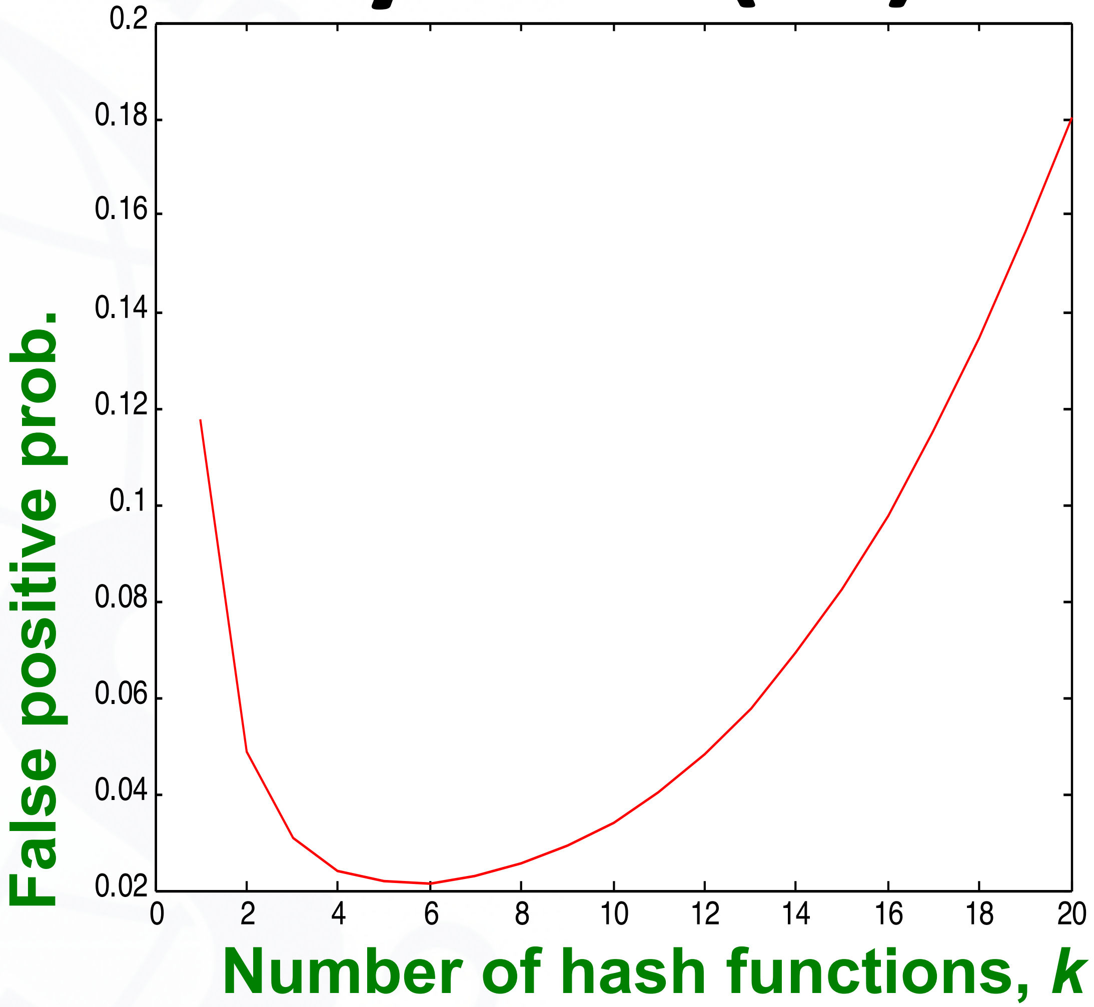

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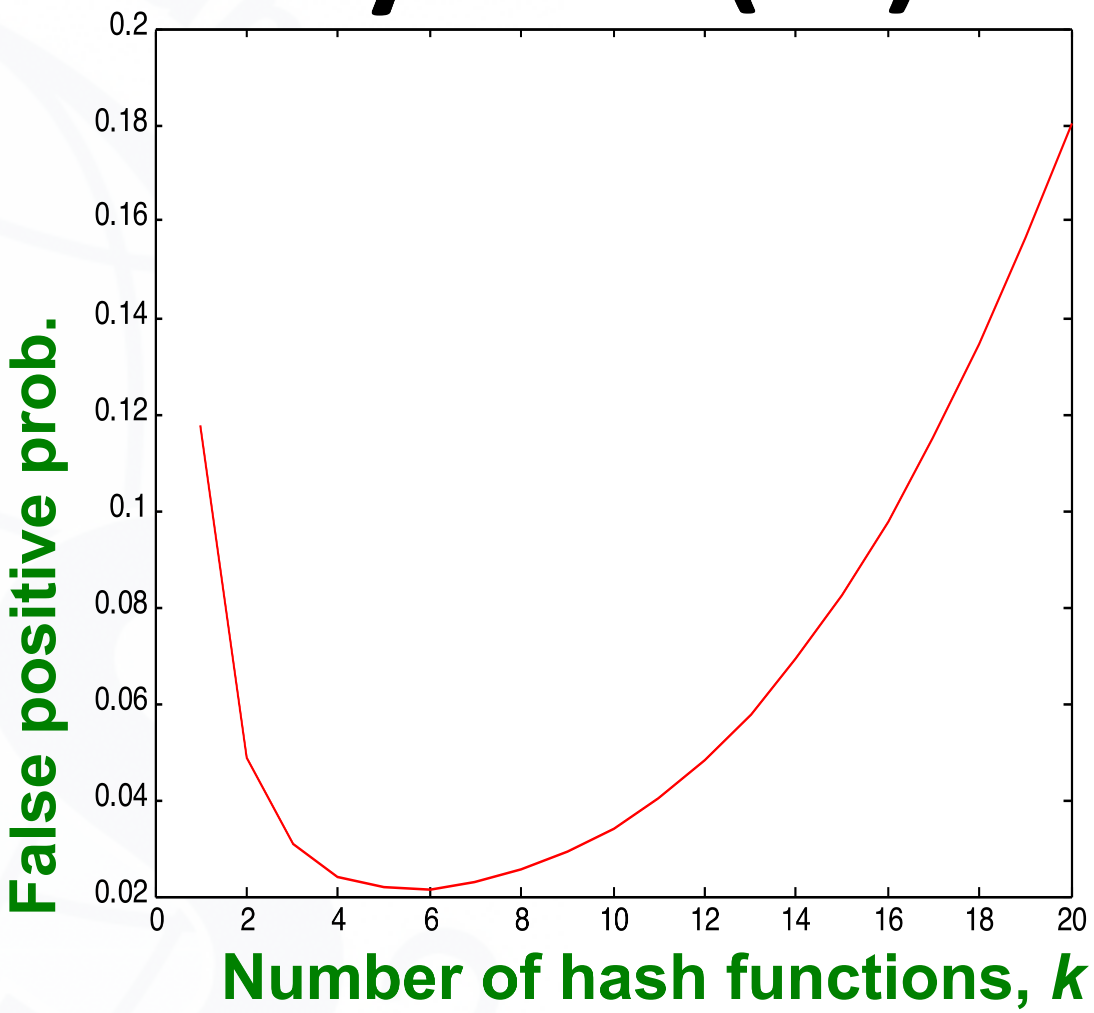

- **What happens as we keep increasing *k*?**

- "Optimal" value of ***k***: ***n/m* ln(2)**
  - **In our case:** Optimal **k = 8 ln(2) = 5.54 ≈ 6**
    - **Error at k = 6**: $(1 - e^{-1/6})^2$ = **0.0235**

Bloom Filter: Wrap-up

- **Bloom filters guarantee no false negatives, and use limited memory**
  - Great for pre-processing before more expensive checks

- **Suitable for hardware implementation**
  - Hash function computations can be parallelized

- Is it better to have **1** big **B** or ***k*** small **B**s?
  - **It is the same:** $(1 - e^{-km/n})^k$ vs. $(1 - e^{-m/(n/k)})^k$
  - But keeping **1 big B** is simpler

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- **Obvious approach:**
  Maintain the set of elements seen so far
  - That is, keep a hash table of all the distinct elements seen so far

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- **How many different Web pages does each customer request in a week?**

- **How many distinct products have we sold in the last week?**

Using Small Storage

- **Real problem: What if we do not have space to maintain the set of elements seen so far?**

- **Estimate the count in an unbiased way**

- **Accept that the count may have a little error, but limit the probability that the error is large**

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- **Estimated number of distinct elements = 2^R**

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    - So, if we saw the longest tail of ***r=2*** (i.e., item hash ending ***100**) then we have probably seen **about *4*** distinct items so far
  - **So, it takes to hash about 2^r items before we see one with zero-suffix of length *r***

Why It Works: More formally

- Now we show why Flajolet-Martin works
- Formally, we will show that probability of finding a tail of r zeros:
  - Goes to 1 if $m \gg 2^r$
  - Goes to 0 if $m \ll 2^r$

  where m is the number of distinct elements seen so far in the stream
- Thus, 2^R will almost always be around m!

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- **What is the probability that a given *h*(*a*) ends in at least *r* zeros is 2^{-r}**
  - **h(a)** hashes elements uniformly at random
  - Probability that a random number ends in at least ***r*** zeros is $\mathbf{2}^{-r}$
- Then, the probability of **NOT** seeing a tail of length ***r*** among ***m*** elements:

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  - If ***m << 2^r***, then prob. tends to **1**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 1$ as **$m/2^r \rightarrow 0$**
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    - So, the probability of finding a tail of length ***r*** tends to **1**
- **Thus, 2^R will almost always be around *m!***

Why It Doesn't Work

Why It Doesn't Work

- **$E[2^R]$ is actually infinite**
  - Probability halves when $R \rightarrow R+1$, but value doubles
- **Workaround involves using many hash functions h_i and getting many samples of R_i**
- **How are samples R_i combined?**
  - **Average?** What if one very large value 2^{R_i}?
  - **Median?** All estimates are a power of **2**
  - **Solution:**
    - Partition your samples into small groups
    - Take the median of groups
    - Then take the average of the medians

Generalization: Moments

- **Suppose a stream has elements chosen from a set *A* of *N* values**

- **Let m_i be the number of times value *i* occurs in the stream**

- **The k^{th} *moment* is** $\sum_{i \in A} (m_i)^k$

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$$\sum_{i \in A} (m_i)^k$$

- **0th moment =** number of distinct elements
  - The problem just considered
- **1st moment =** count of the numbers of elements = length of the stream
  - Easy to compute
- **2nd moment =** ***surprise number S*** =
  a measure of how uneven the distribution is

Example: Surprise Number

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- Item counts: **90, 1, 1, 1, 1, 1, 1, 1 ,1, 1, 1** **Surprise *S* = 8,110**

AMS Method

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- **AMS method works for all moments**
- **Gives an unbiased estimate**
- **We will just concentrate on the 2nd moment *S***
- **We pick and keep track of many variables *X*:**
  - For each variable ***X*** we store ***X.el*** and ***X.val***
    - ***X.el*** corresponds to the item ***i***
    - ***X.val*** corresponds to the **count** of item ***i***
  - Note this requires a count in main memory, so number of ***X***s is limited
- **Our goal is to compute** $S = \sum_i m_i^2$

One Random Variable (X)

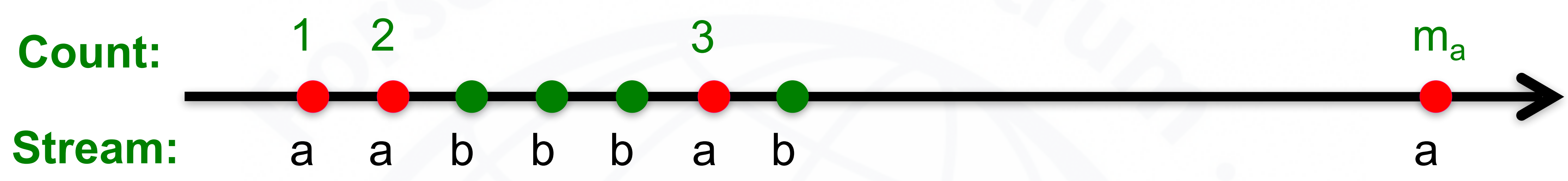
One Random Variable (X)

- **How to set *X.val* and *X.el*?**
  - Assume stream has length ***n*** (we relax this later)
  - Pick some random time ***t*** (***t<n***) to start, so that any time is equally likely
  - Let at time ***t*** the stream have item ***i***. ***We set X.el = i***
  - Then we maintain count ***c*** (***X.val = c***) of the number of ***i*s** in the stream starting from the chosen time ***t***
- **Then the estimate of the 2nd moment ($\sum_i m_i^2$) is:**

$$S = f(X) = n\,(2 \cdot c - 1)$$

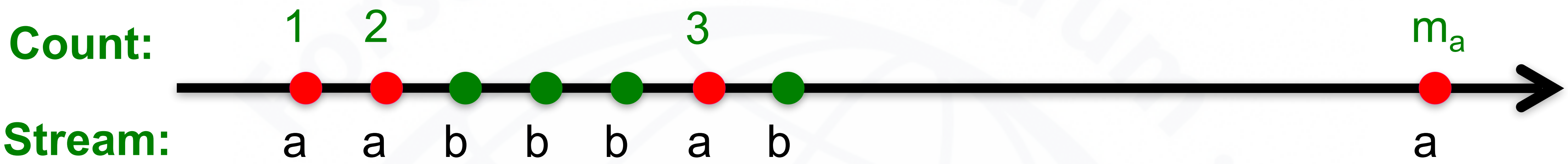
  - Note, we will keep track of multiple **X**s, (**X_1, X_2,… X_k**) and our final estimate will be $S = 1/k \sum_j^k f(X_j)$

Expectation Analysis



m_i … total count of item i in the stream (we are assuming stream has length n)

Expectation Analysis



- **2^{nd} moment is $S = \sum_i m_i^2$**
- c_t … number of times item at time t appears from time **t** onwards ($c_1 = m_a$, $c_2 = m_a - 1$, $c_3 = m_b$)
- $E[f(X)] = \frac{1}{n} \sum_{t=1}^{n} n(2c_t - 1)$

$$= \frac{1}{n} \sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$$

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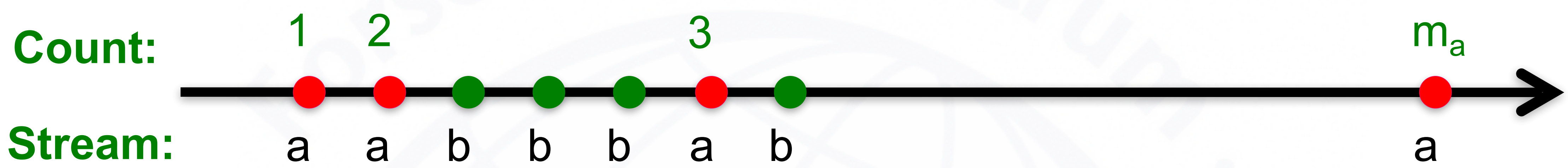

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Group times by the value seen

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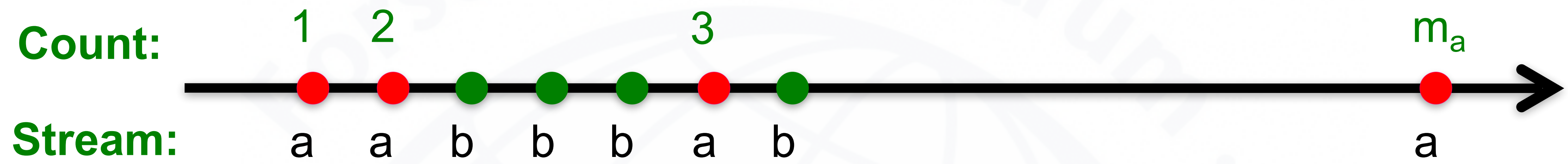

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Time t when the last i is seen ($c_t = 1$)

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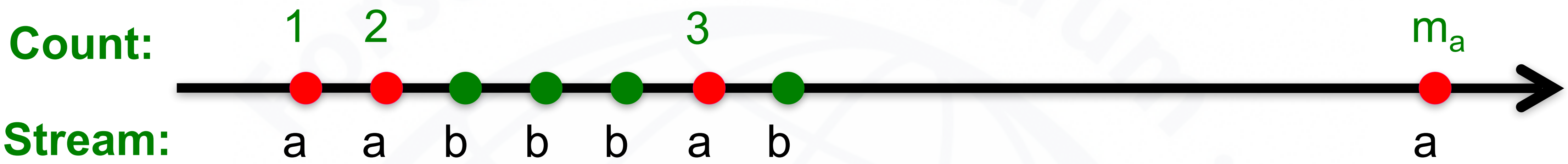

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Time t when the penultimate i is seen (c_t=2)

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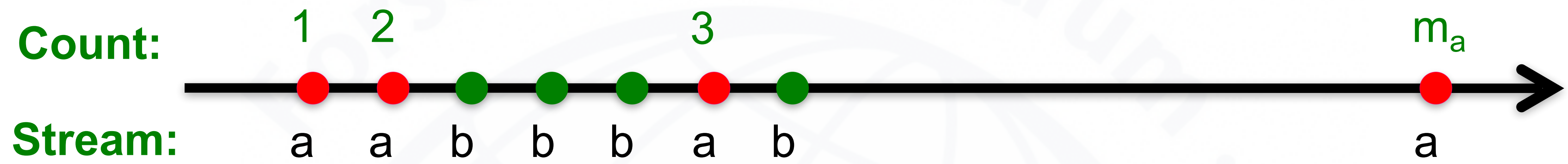

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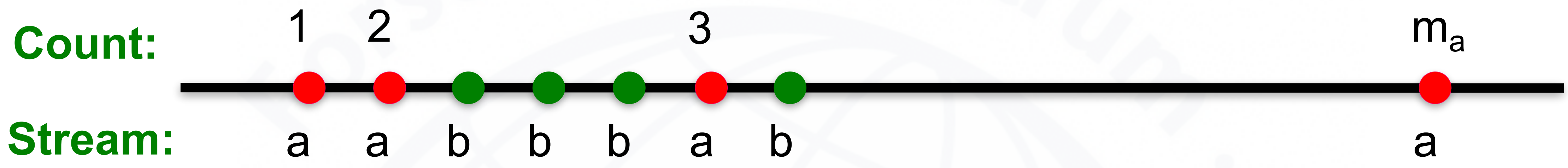
Time t when the penultimate i is seen (c_t=2)

Time t when the first i is seen (c_t=m_i)

Expectation Analysis



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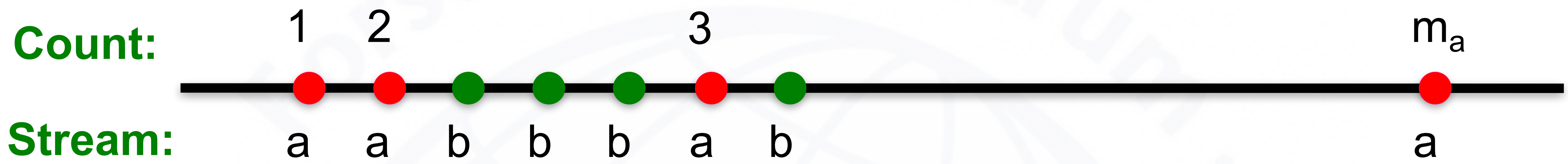

- $E[f(X)] = \frac{1}{n}\sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$
  - Little side calculation: $(1 + 3 + 5 + \cdots + 2m_i - 1) = \sum_{i=1}^{m_i}(2i - 1) = 2\frac{m_i(m_i+1)}{2} - m_i = (m_i)^2$
- **Then $\boldsymbol{E[f(X)] = \frac{1}{n}\sum_i n\,(m_i)^2}$**
- **So, $\mathbf{E[f(X)]} = \sum_i (m_i)^2 = S$**
- **We have the second moment (in expectation)!**

Higher-Order Moments

Higher-Order Moments

- **For estimating k[th] moment we essentially use the same algorithm but change the estimate:**
  - For **k=2** we used ***n* (2·c – 1)**
  - For **k=3** we use: ***n* (3·c^2 – 3c + 1)** (where **c=X.val**)
- **Why?**
  - **For k=2:** Remember we had $(1+3+5+\cdots+2m_i-1)$ and we showed terms ***2c-1*** (for **c=1,…,m**) sum to ***m*^{2}**
    - $\sum_{c=1}^{m} 2c-1 = \sum_{c=1}^{m} c^2 - \sum_{c=1}^{m} (c-1)^2 = m^2$
    - **So:** $\mathbf{2c-1 = c^2 - (c-1)^2}$
  - **For k=3:** **c^3 - (c-1)^3 = 3c^2 - 3c + 1**
- **Generally:** Estimate $= n\,(c^k - (c-1)^k)$

Combining Samples

Combining Samples

- **In practice:**
  - Compute $f(X) = n(2c - 1)$ for as many variables ***X*** as you can fit in memory
  - Average them in groups
  - Take median of averages
- **Problem: Streams never end**
  - We assumed there was a number ***n***, the number of positions in the stream
  - But real streams go on forever, so ***n*** is a variable – the number of inputs seen so far

Streams Never End: Fixups

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  keep ***n*** separately; just hold the count in ***X***

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Streams Never End: Fixups

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  We must throw some ***X***s out as time goes on:
  - **Objective:** Each starting time ***t*** is selected with probability ***k/n***
  - **Solution: (fixed-size sampling!)**
    - Choose the first ***k*** times for ***k*** variables
    - When the n^{th} element arrives (***n*** > ***k***), choose it with probability ***k/n***
    - If you choose it, throw one of the previously stored variables **X** out, with equal probability

Thats it !!