# Lecture 2

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# Now from games to logic

# Vocabulary

A vocabulary is a set L of

predicate symbols P,Q,R,...

function symbols f,g,h,...

constant symbols c,d,e,...

Arity function:

$$\boxed{\#_L : L \to \mathbb{N}}$$

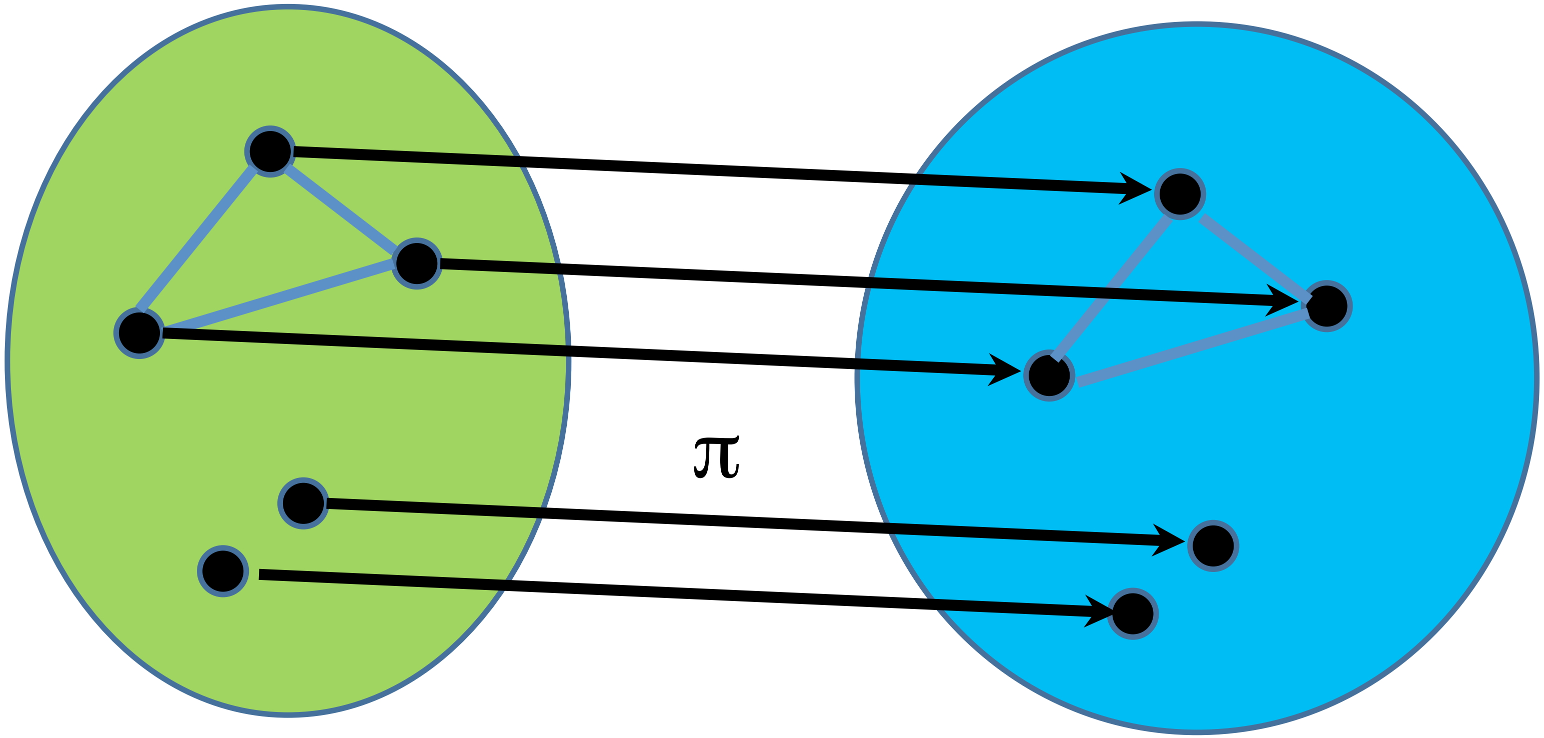
# Model

- A model (or structure) $\mathcal{M}$, for a vocabulary L is a non-empty set M, called the universe of $\mathcal{M}$, and:
  - A subset $P^{\mathcal{M}}$ of $M^n$ for every unary predicate symbol P in L of arity n
  - A function $f^{\mathcal{M}}$ of $M^n$ into M for every function symbol f in L of arity n
  - An element $c^{\mathcal{M}}$ of M for every constant symbol c in L.

# Examples

- Graphs
- Groups
- Unary structures
- Ordered sets
- Equivalence relations
- Fields

# Isomorphism



# Isomorphism defined

**Definition 4.1.2.** *$L$–structures $\mathcal{M}$ and $\mathcal{M}'$ are* isomorphic, *if there is a bijection*

$$\pi : M \to M'$$

*such that*

1. *For all $a_1, \ldots, a_{\#_L(R)} \in M$:*

$$\boxed{(a_1, \ldots, a_{\#_L(R)}) \in R^{\mathcal{M}} \iff (\pi(a_1), \ldots, \pi(a_{\#_L(R)})) \in R^{\mathcal{M}'}}$$

2. *For all $a_1, \ldots, a_{\#_L(f)} \in M$:*

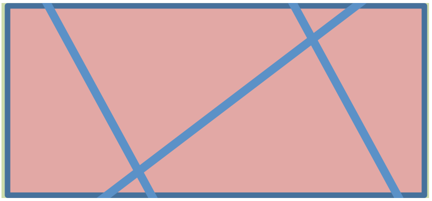
$$\boxed{f^{\mathcal{M}'}(\pi(a_1), \ldots, \pi(a_{\#_L(f)})) = \pi(f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)})).}$$

3. $\pi(c^{\mathcal{M}}) = c^{\mathcal{M}'}$.

*In this case we say that $\pi$ is an* isomorphism *$\mathcal{M} \to \mathcal{M}'$*

$$\pi : \mathcal{M} \cong \mathcal{M}'.$$

# Substructure



**Definition 4.2.1.** *An $L$-structure $\mathcal{M}$ is a* substructure *of another $L$-structure $\mathcal{M}'$, in symbols $\mathcal{M} \subseteq \mathcal{M}'$, if:*

1. $M \subseteq M'$
2. $R^{\mathcal{M}} = R^{\mathcal{M}'} \cap M^n$ *if $R \in L$ is an $n$-ary predicate symbol.*
3. $f^{\mathcal{M}} = f^{\mathcal{M}'} \restriction M^n$ *if $f \in L$ is an $n$-ary function symbol.*
4. $c^{\mathcal{M}} = c^{\mathcal{M}'}$ *if $c \in L$ is a constant symbol.*

# Generated substructure

**Lemma 4.2.1.** *Suppose $L$ is a vocabulary, $\mathcal{M}$ an $L$-structure and $X \subseteq M$. Suppose furthermore that either $L$ contains constant symbols or $X \neq \emptyset$. There is a unique $L$-structure $\mathcal{N}$ such that:*

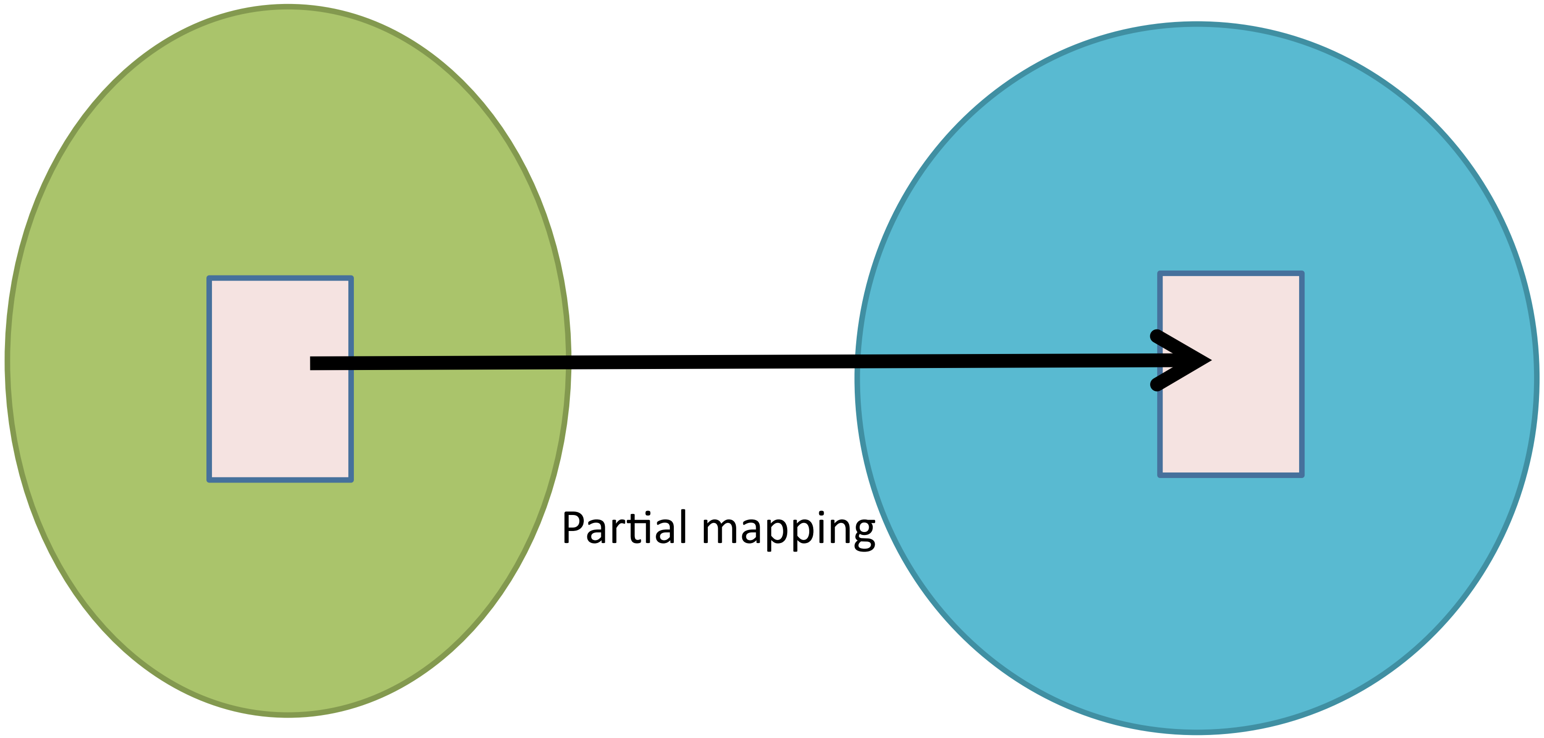
*1. $\mathcal{N} \subseteq \mathcal{M}$.*
*2. $X \subseteq N$.*
*3. If $\mathcal{N}' \subseteq \mathcal{M}$ and $X \subseteq N'$, then $\mathcal{N} \subseteq \mathcal{N}'$.*

*Proof.* Let $X_0 = X \cup \{c^{\mathcal{M}} : c \in L\}$ and inductively

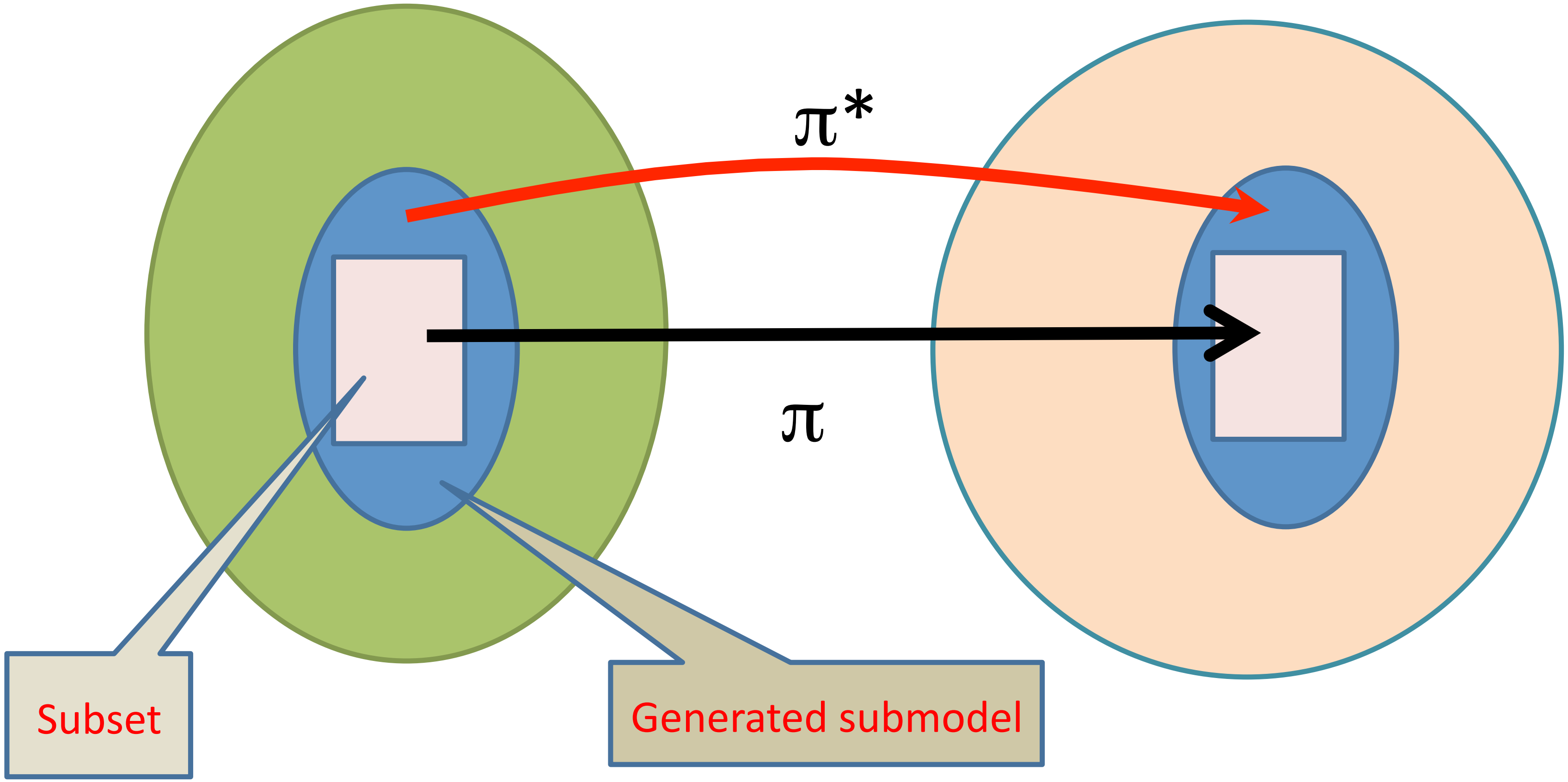
$$X_{n+1} = \{f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)}) : a_1, \ldots, a_{\#_L(f)} \in X_n, f \in L\}.$$

It is easy to see that the set $N = \bigcup_{n \in \mathbb{N}} X_n$ is the universe of the unique structure $\mathcal{N}$ claimed to exist in the lemma. $\square$

# Partial mappings



# Lifting



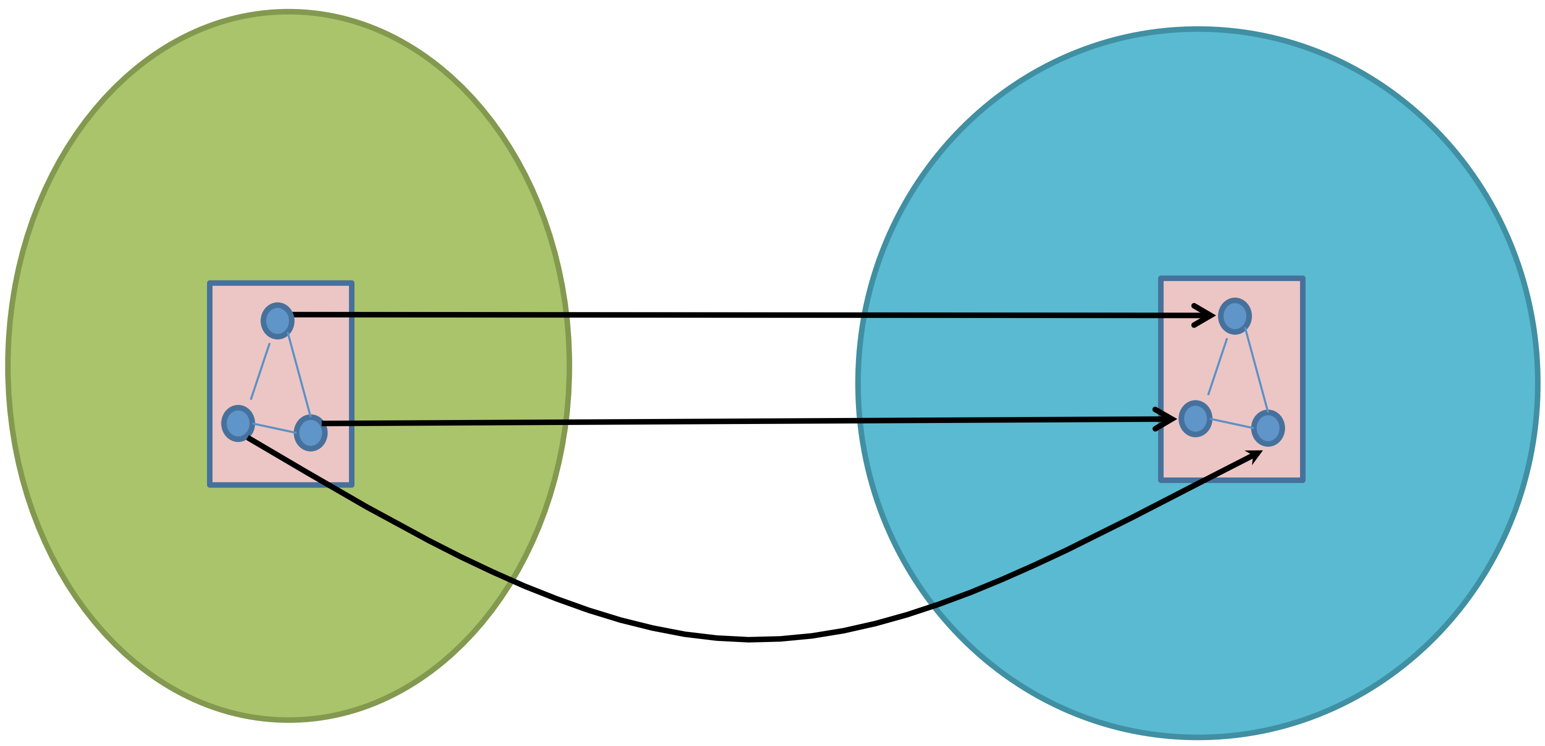
# Lifting defined

Notation for the submodel generated by X

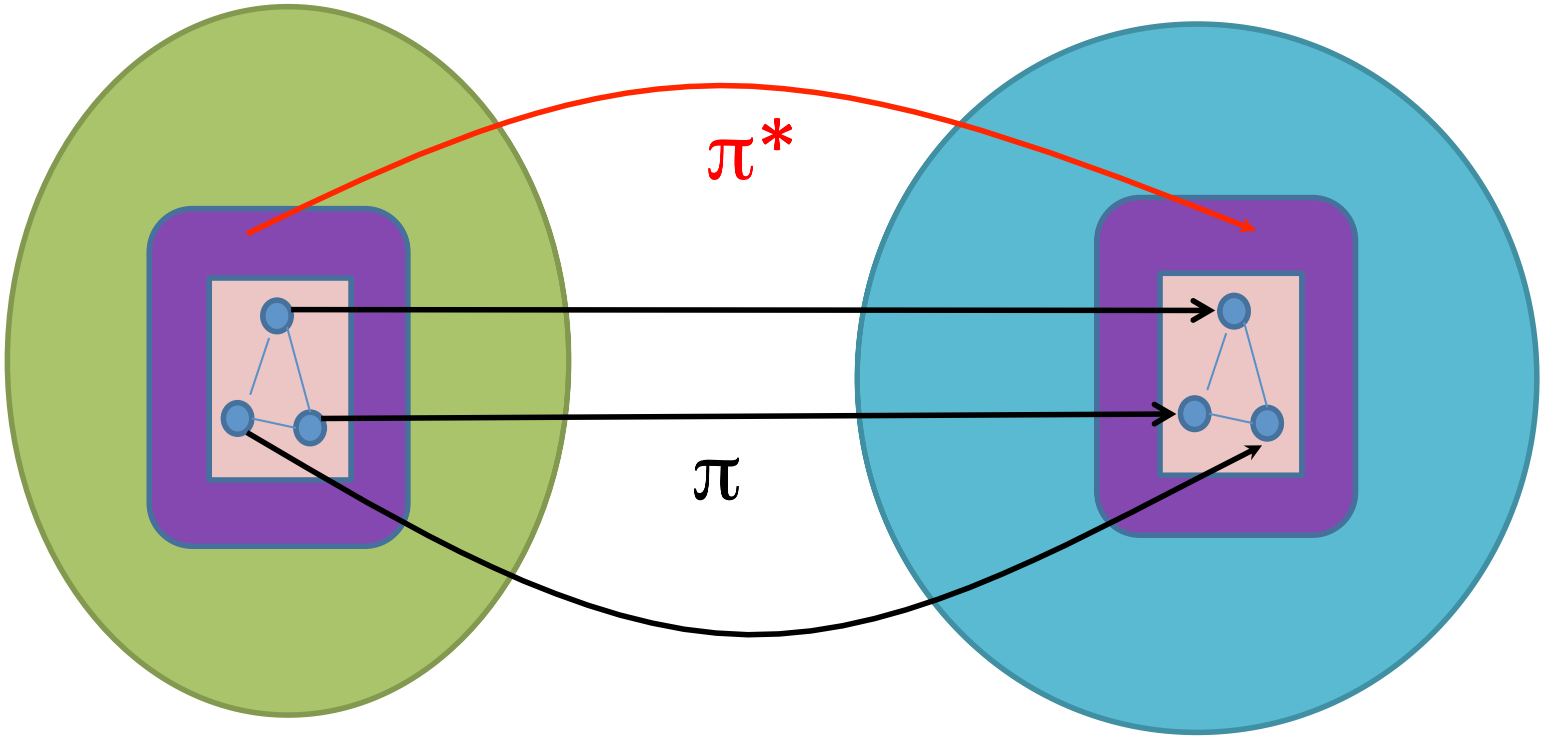
$$[X]_{\mathcal{M}}$$

**Lemma 4.2.2.** *Suppose $L$ is a vocabulary. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $L$-structures and $\pi : M \to N$ is a partial mapping. There is at most one isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{N}}$ extending $\pi$.*

# Partial isomorphism



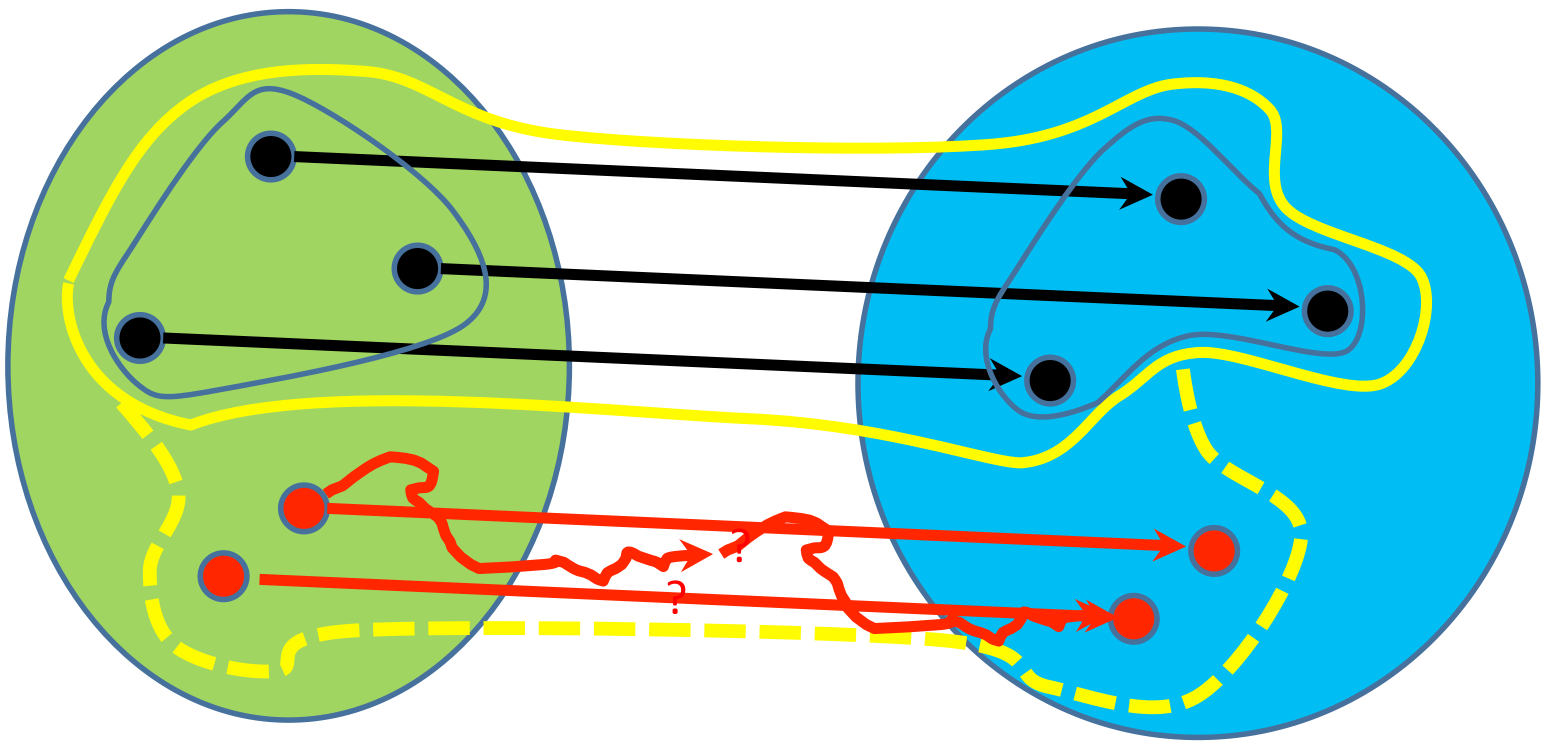
# When there are functions



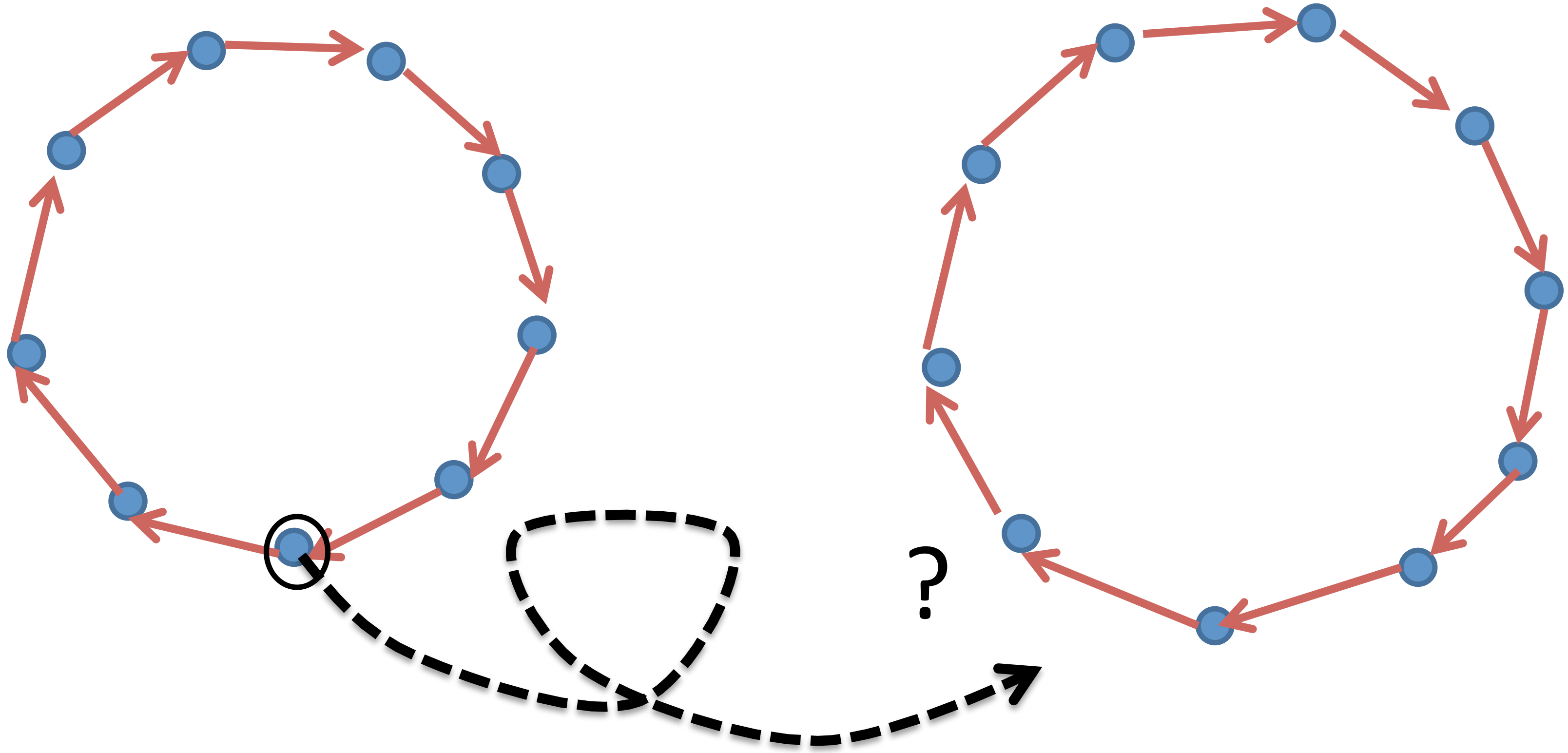
# Partial isomorphism

**Definition 4.3.1.** *Suppose $L$ is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are $L$-structures. A partial mapping $\pi : M \to M'$ is a* partial isomorphism *$\mathcal{M} \to \mathcal{M}'$ if there is an isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{M}'}$ extending $\pi$. We use $\mathrm{Part}(\mathcal{M}, \mathcal{M}')$ to denote the set of partial isomorphisms $\mathcal{M} \to \mathcal{M}'$. If $\mathcal{M} = \mathcal{M}'$ we call $\pi$ a* partial automorphism.

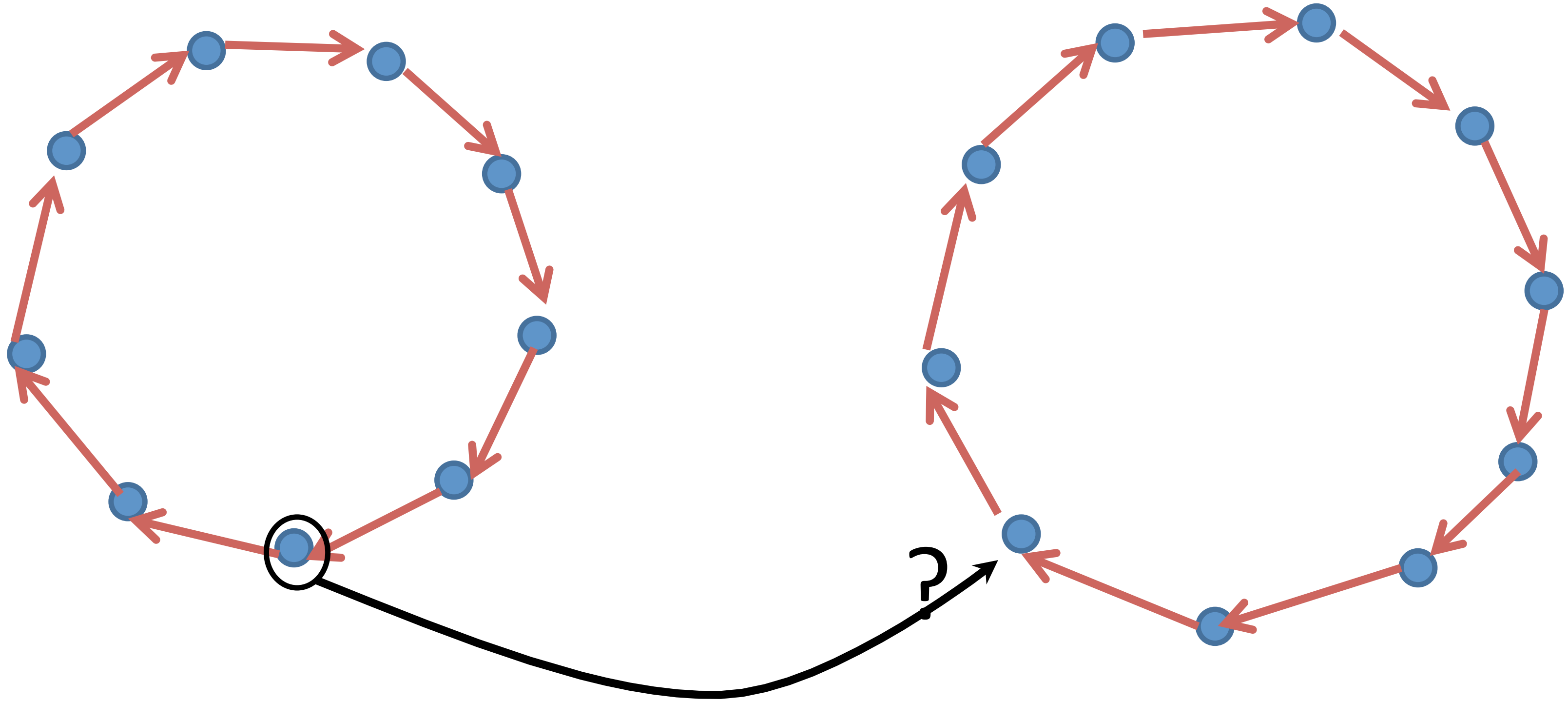
# <u>Extending</u> a partial isomorphism



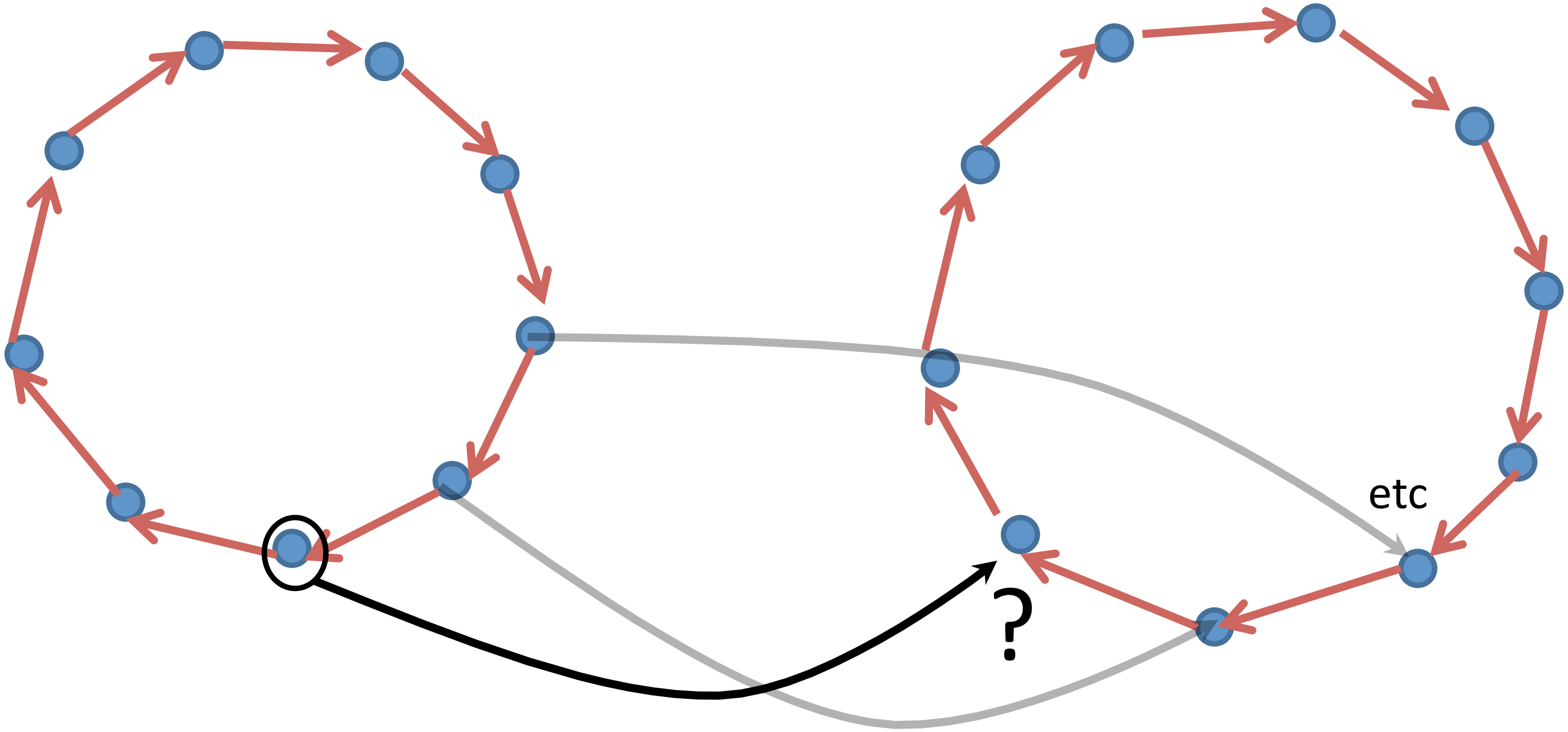
# Two unary functions



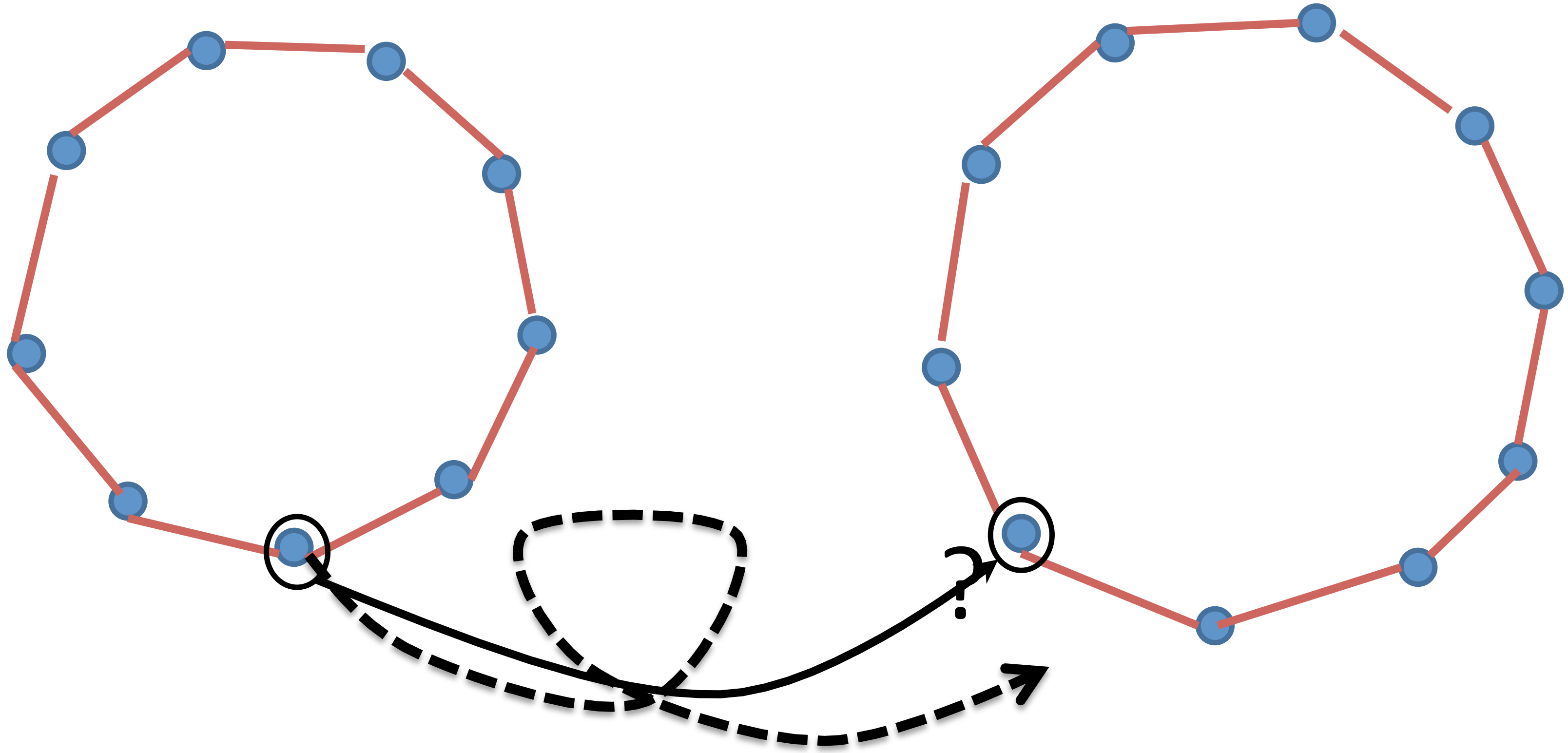
# Two unary functions



# Two unary functions

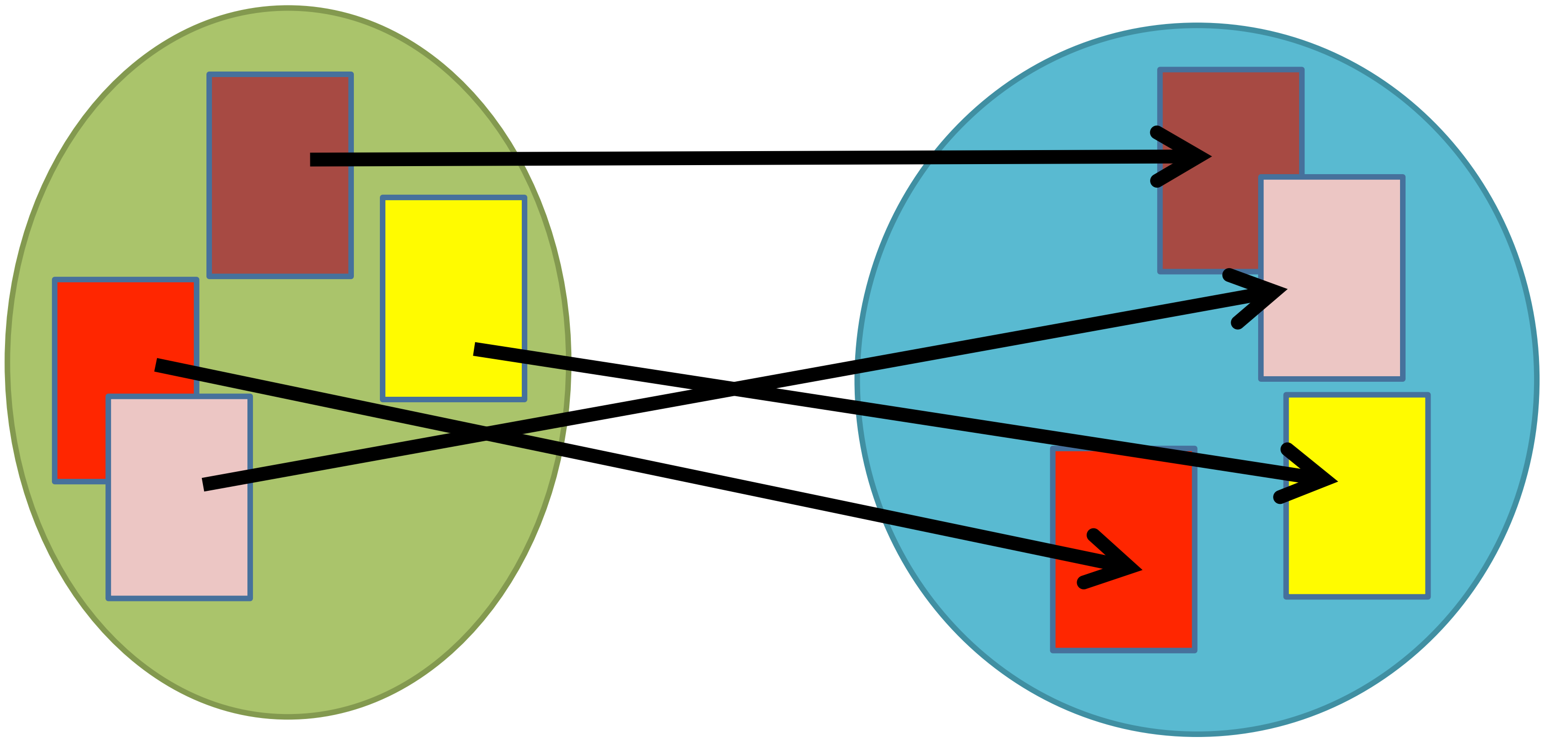


# Two graphs

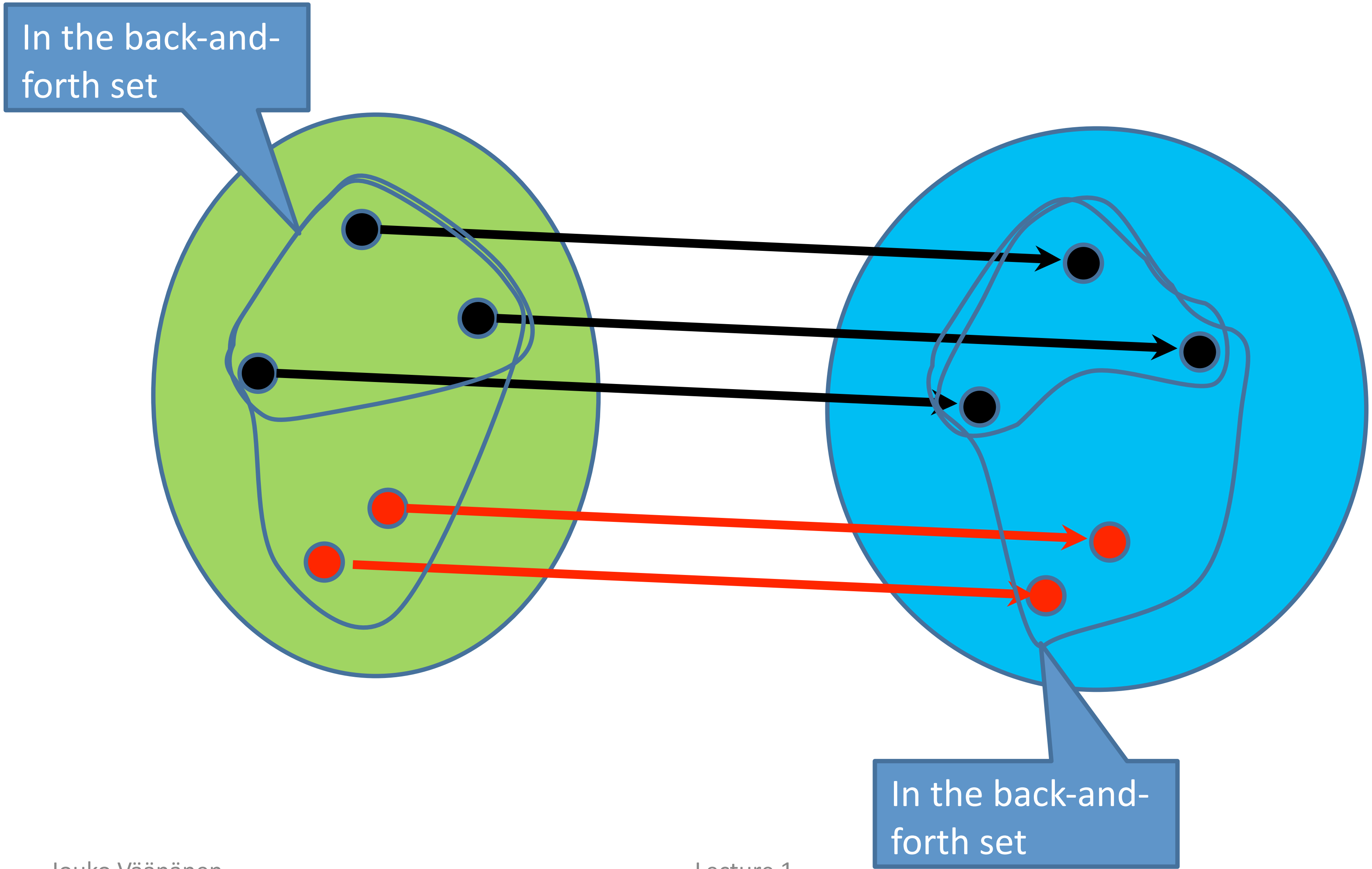


# Back-and-forth set

A set of partial isomorphisms satisfying the sc. back-and-forth condition.



# Back-and-forth condition



# Back and forth set defined

**Definition 2.3.2.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are $L$-structures. A* back-and-forth set *for $\mathcal{A}$ and $\mathcal{B}$ is any non-empty set $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B})$ such that*

$$\forall f \in P \forall a \in A \exists g \in P (f \subseteq g \text{ and } a \in \mathrm{dom}(g)) \tag{2.8}$$

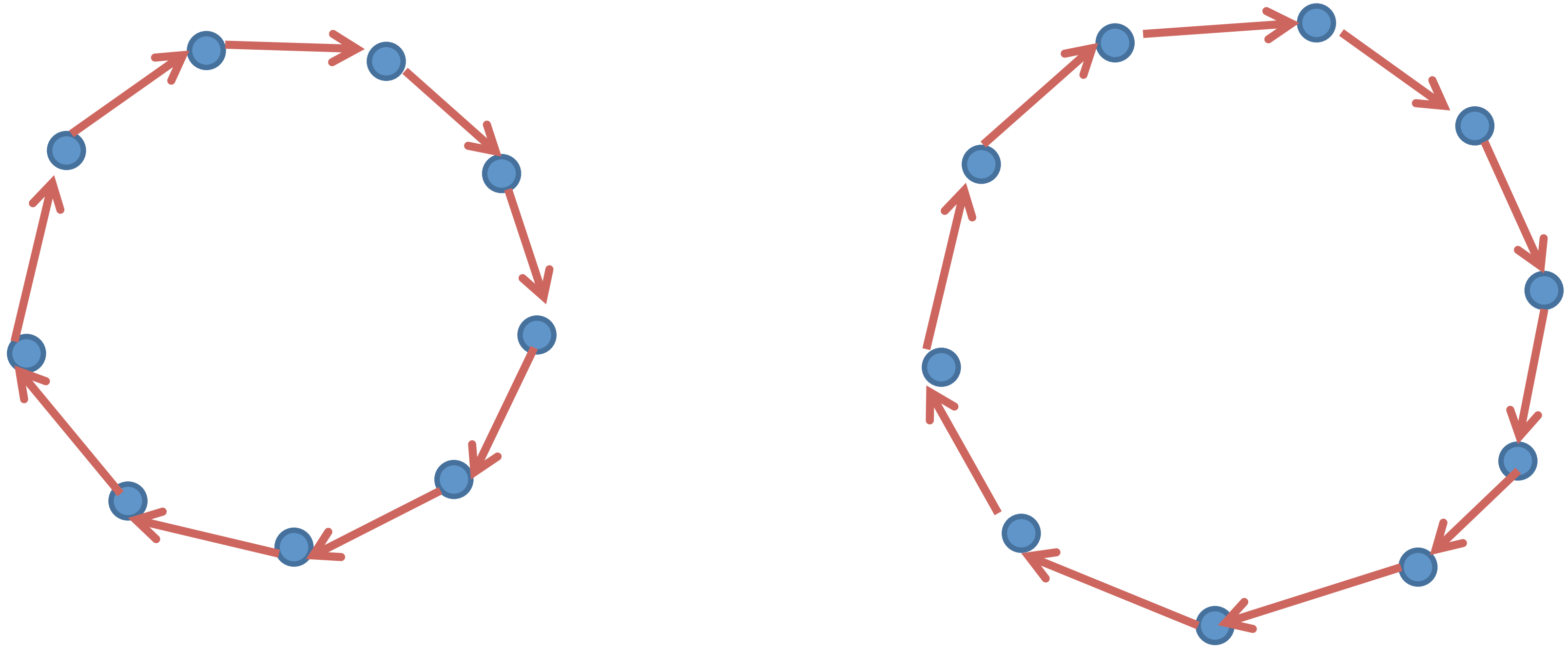
$$\forall f \in P \forall b \in B \exists g \in P (f \subseteq g \text{ and } b \in \mathrm{rng}(g)) \tag{2.9}$$

# Partially isomorphic models

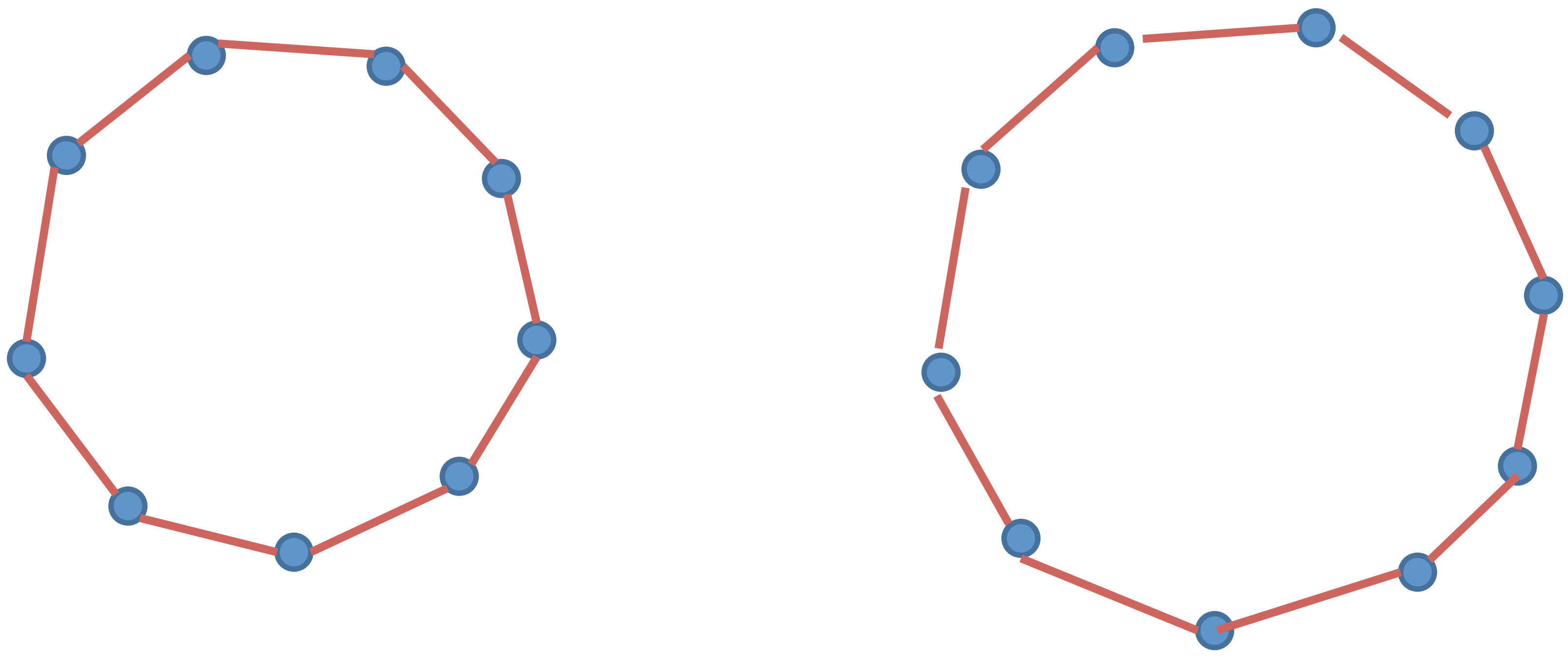
$$\mathcal{A} \simeq_p \mathcal{B}$$

Models are partially isomorphic if there is a back-and-forth set of partial isomorphisms between them.

# Not partially isomorphic



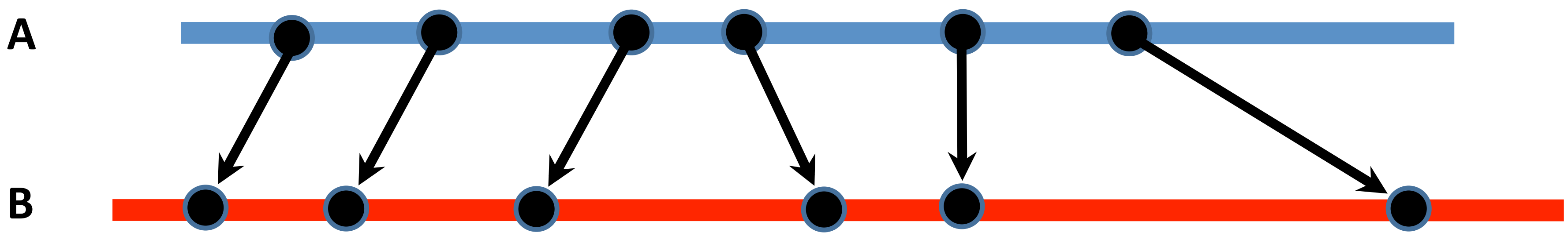
# Not partially isomorphic



# Dense total orders

- All dense total orders without endpoints are partially isomorphic.

$$P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ is finite}\}$$

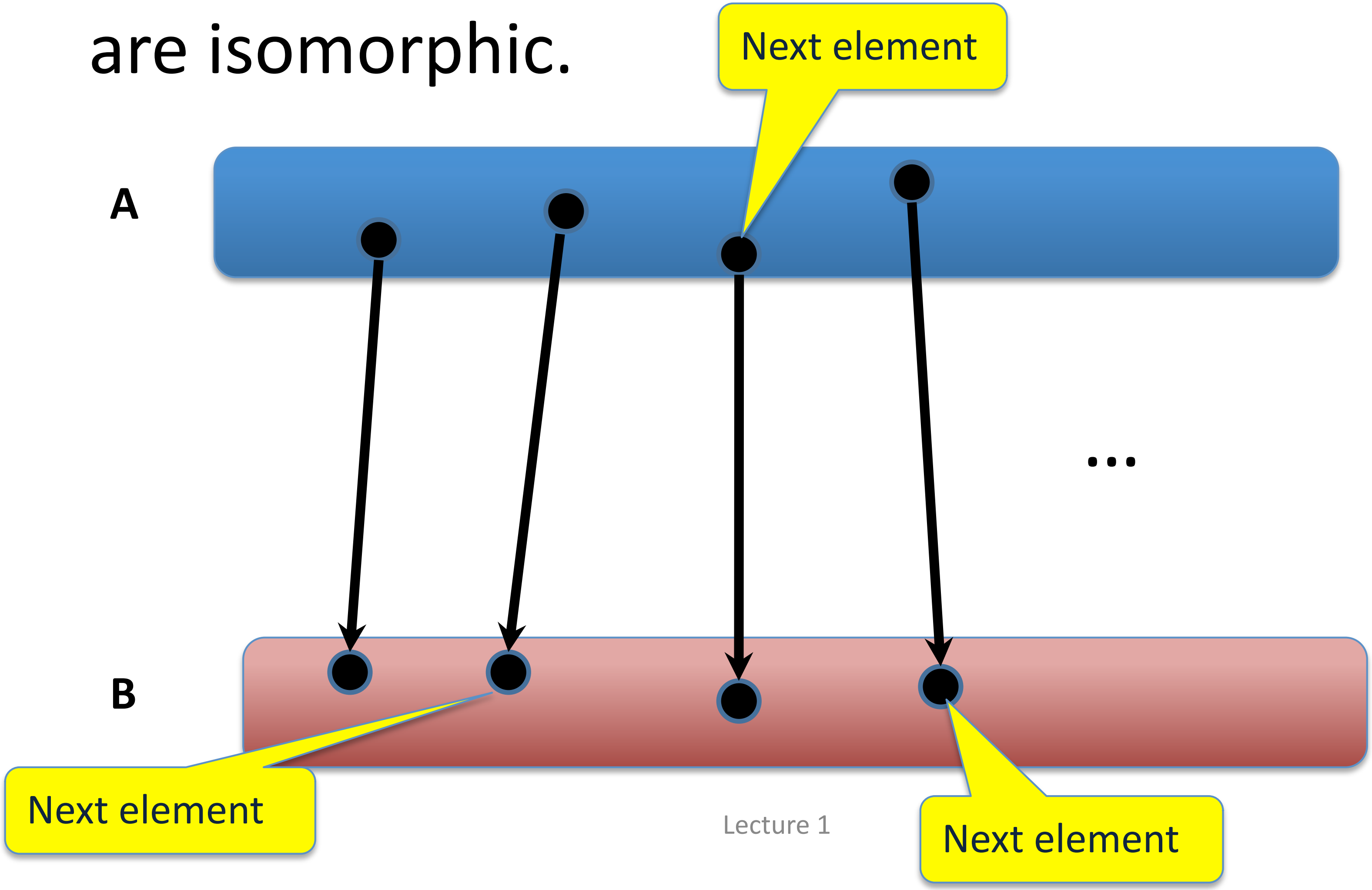


**Proposition 4.3.2.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are dense linear orders without endpoints. Then $\mathcal{A} \simeq_p \mathcal{B}$.*

*Proof.* Let $P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ finite}\}$. It turns out that this straightforward choice works. Clearly, $P \neq \emptyset$. Suppose then $f \in P$ and $a \in A$. Let us enumerate $f$ as $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ where $a_1 < \ldots < a_n$. Since $f$ is a partial isomorphism, also $b_1 < \ldots < b_n$. Now we consider different cases. If $a < a_1$, we choose $b < b_1$ and then $f \cup \{(a, b)\} \in P$. If $a_i < a < a_{i+1}$, we choose $b \in B$ so that $b_i < b < b_{i+1}$ and then $f \cup \{(a, b)\} \in P$. If $a_n < a$, we choose $b > b_n$ and again $f \cup \{(a, b)\} \in P$. Finally, if $a = a_i$, we let $b = b_i$ and then $f \cup \{(a, b)\} = f \in P$. We have proved (4.8). Condition (4.9) is proved similarly. $\square$

# Countable models

- All countable partially isomorphic structures are isomorphic.



**Proposition 2.3.1.** *If $\mathcal{A} \simeq_p \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are countable, then $\mathcal{A} \cong \mathcal{B}$.*

*Proof.* Let us enumerate $A$ as $(a_n : n < \omega)$ and $B$ as $(b_n : n < \omega)$. Let $P$ be a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. Since $P \neq \emptyset$, there is some $f_0 \in P$. We define a sequence $(f_n : n < \omega)$ of elements of $P$ as follows: Suppose $f_n \in P$ is defined. If $n$ is even, say $n = 2m$, let $y \in B$ and $f_{n+1} \in P$ such that $f_n \cup \{(a_m, y)\} \subseteq f_{n+1}$. If $n$ is odd, say $n = 2m + 1$, let $x \in A$ and $f_{n+1} \in P$ such that $f_n \cup \{(x, b_m)\} \subseteq f_{n+1}$. Finally, let

$$f = \bigcup_{n=0}^{\infty} f_n.$$

Clearly, $f : \mathcal{A} \cong \mathcal{B}$. $\square$

# Zero-one law

- Extension axioms
- Countable categoricity
- Glebskii et al, Fagin, zero-one law
  - Application of back-and-forth

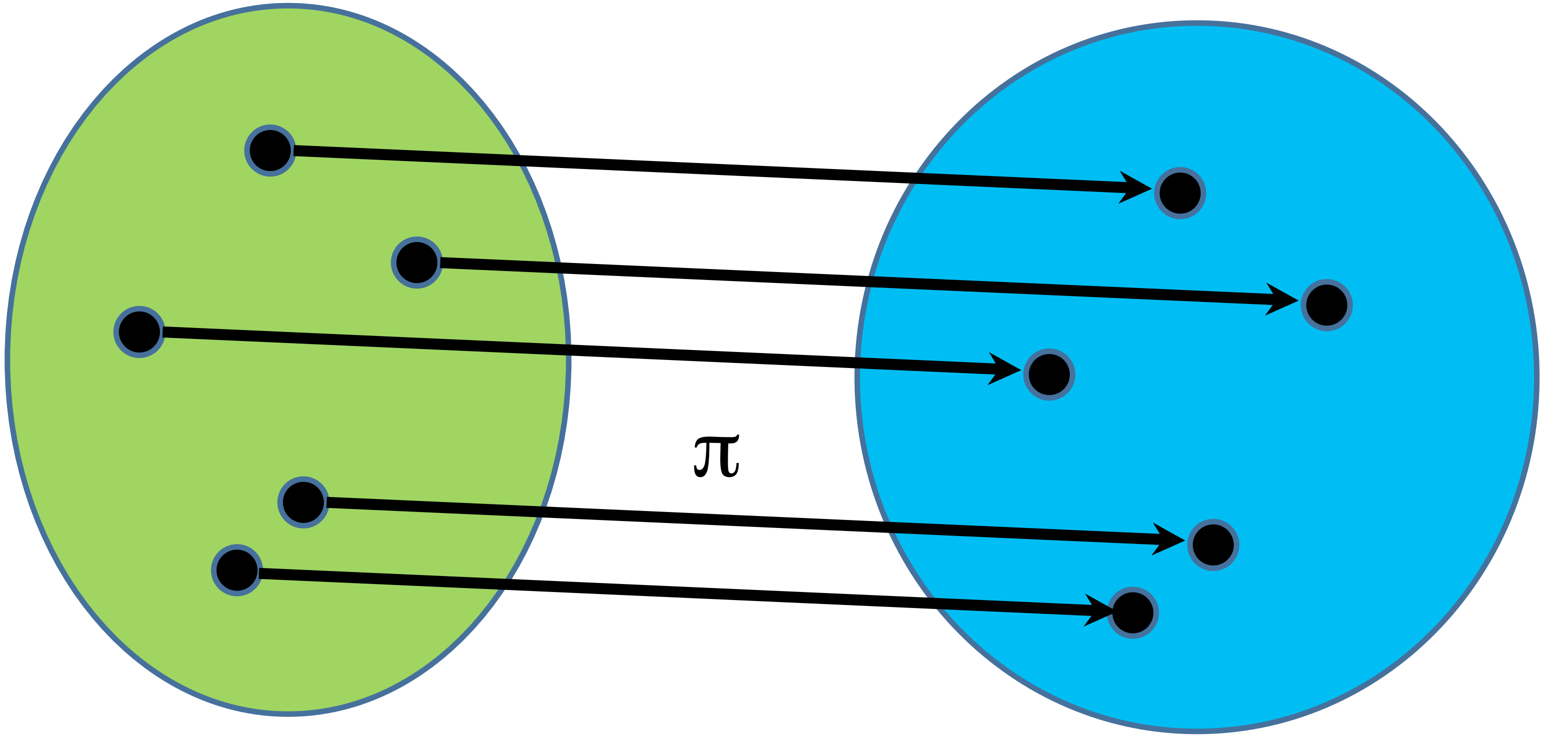
# Ehrenfeucht-Fraïssé game

| I | II |
|---|---|
| $x_0$ | |
| | $y_0$ |
| $x_1$ | |
| | $y_1$ |
| $\vdots$ | |
| | $\vdots$ |
| $x_{n-1}$ | |
| | $y_{n-1}$ |

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

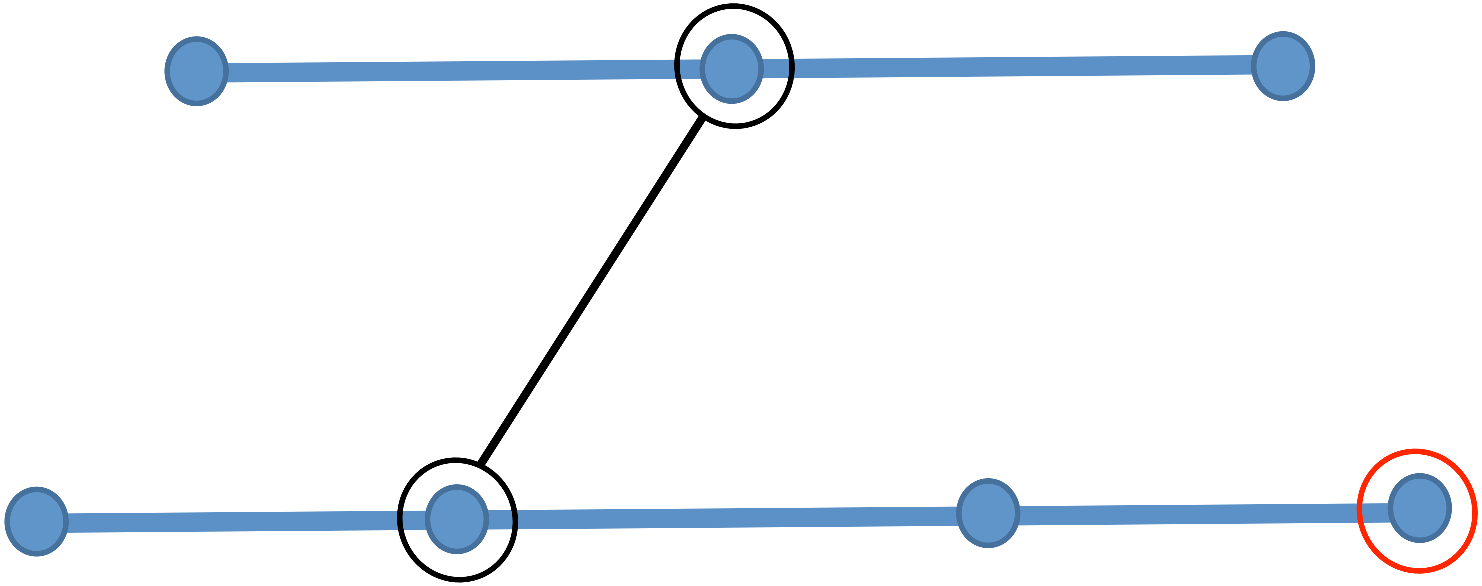
$$W \subseteq A^{2n}$$

# Ehrenfeucht-Fraïssé game



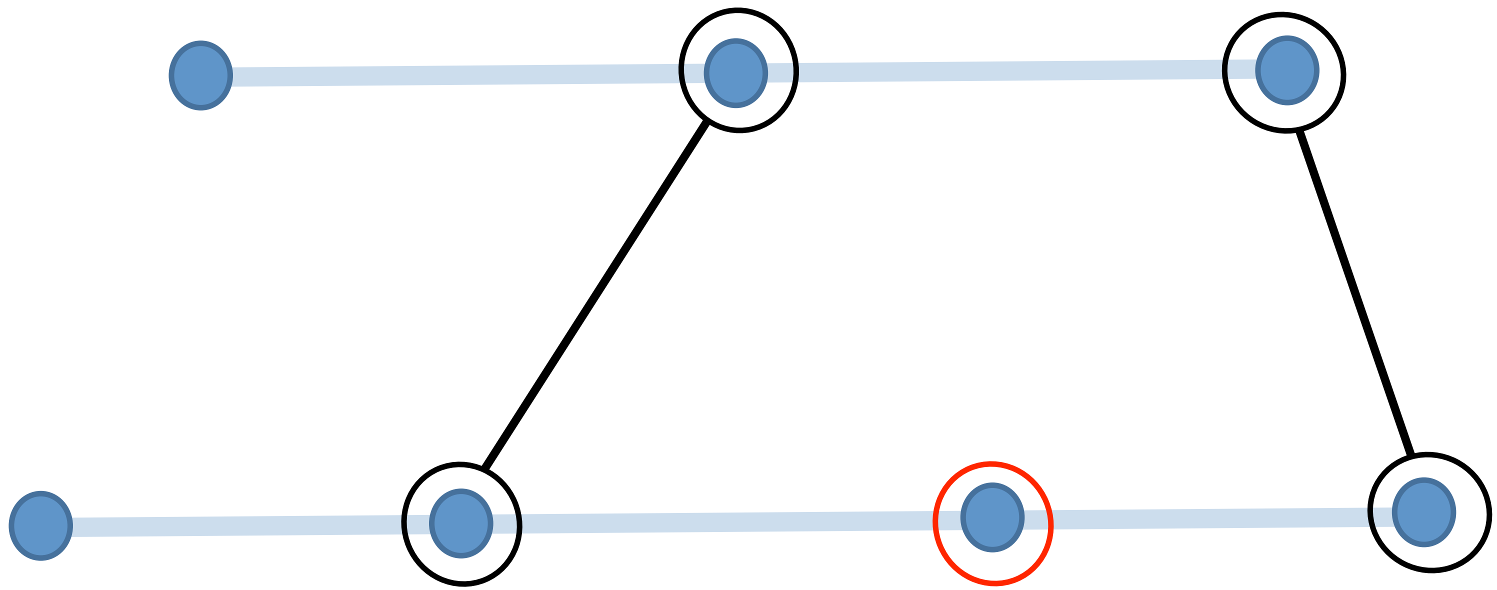

Player II wins if $\pi$ is a partial isomorphism

# Two graphs, two moves



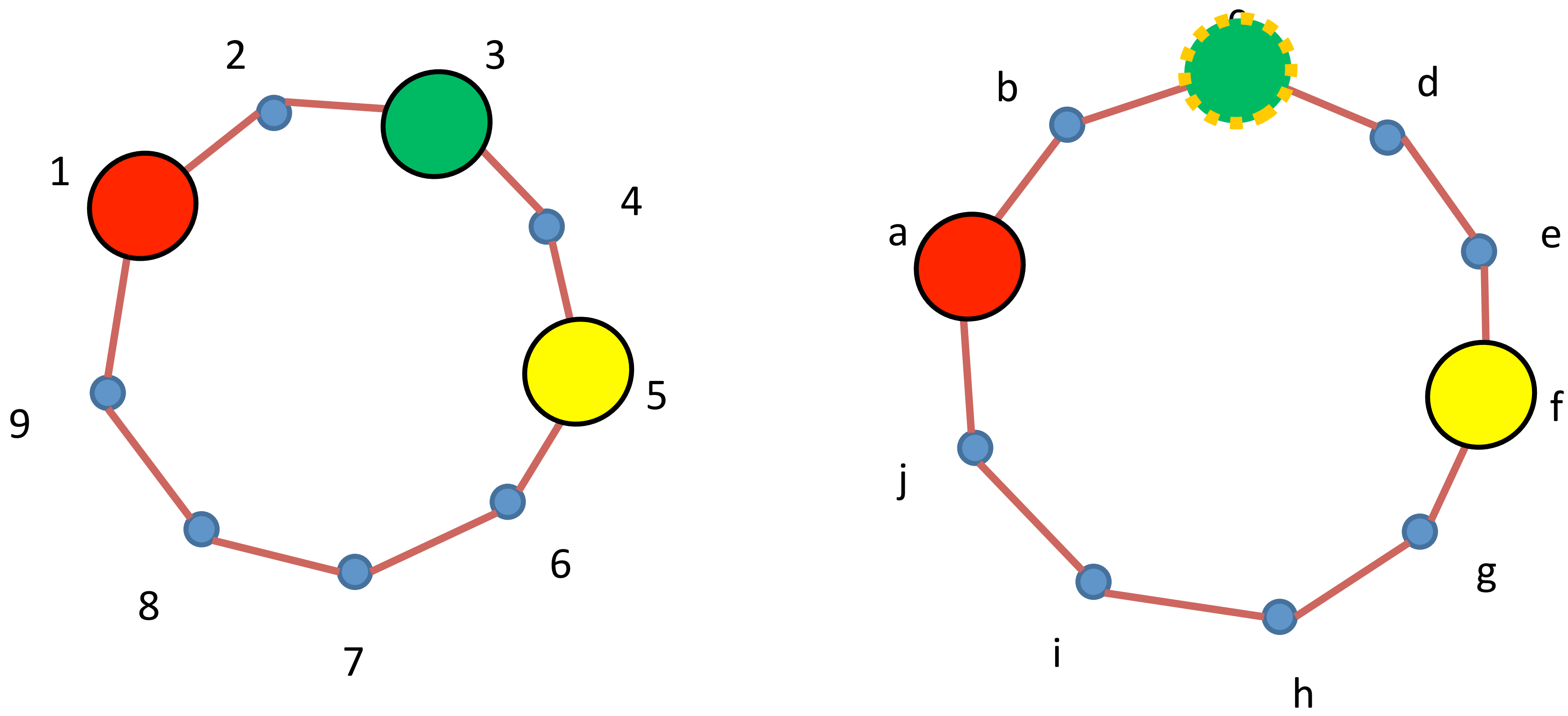
I has a winning strategy

# Two orders, three moves

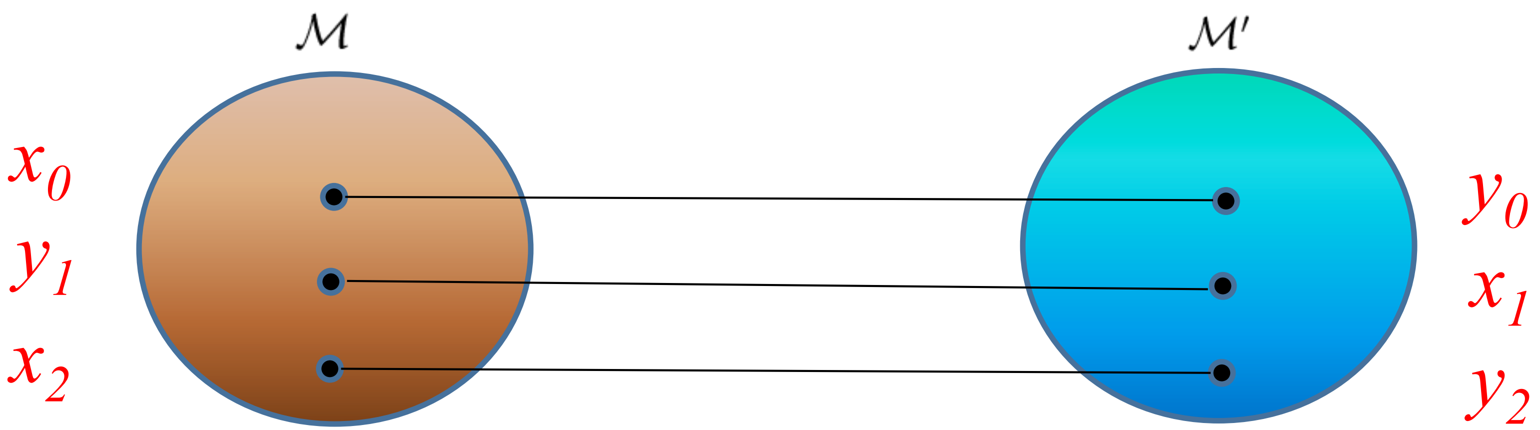


I has a winning strategy

# Two cycles – four moves




I has a winning strategy

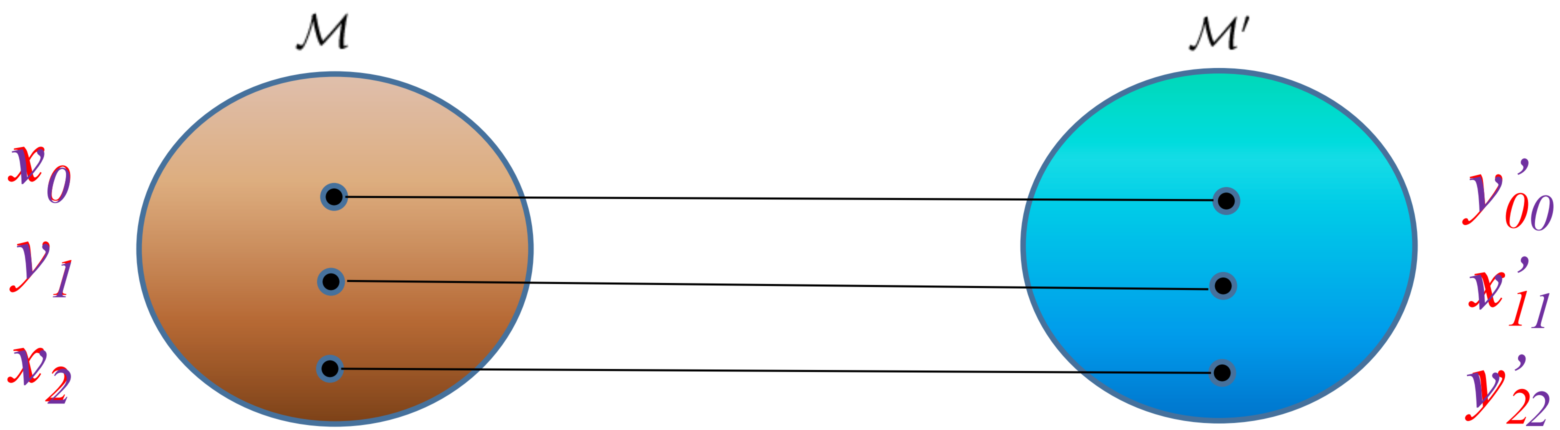



**Definition 4.4.1.** *Suppose $L$ is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are $L$-structures such that $M \cap M' = \emptyset$. The* Ehrenfeucht-Fraïssé game $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$ *is the game*

$$G_n(M \cup M', W_n(\mathcal{M}, \mathcal{M}'))$$

*where $W_n(\mathcal{M}, \mathcal{M}') \subseteq (M \cup M')^{2n}$ is the set of $p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ such that:*

*(G1) For all $i < n$: $x_i \in M \iff y_i \in M'$.*




*(G2) If we denote*

$$v_i = \begin{cases} x_i & \text{if } x_i \in M \\ y_i & \text{if } y_i \in M \end{cases}$$

$$v'_i = \begin{cases} x_i & \text{if } x_i \in M' \\ y_i & \text{if } y_i \in M', \end{cases}$$

*then*

$$f_p = \{(v_0, v'_0), \ldots, (v_{n-1}, v'_{n-1})\}$$

*is a partial isomorphism* $\mathcal{M} \to \mathcal{M}'$.

# The infinite game

*We call* $v_i$ *and* $v'_i$ *above* corresponding *elements. The infinite game* $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ *is defined quite similarly, that is, it is the game* $G_\omega(M \cup M', W_\omega(\mathcal{M}, \mathcal{M}'))$*, where* $W_\omega(\mathcal{M}, \mathcal{M}')$ *is the set of* $p = (x_0, y_0, x_1, y_1, \ldots)$ *such that for all* $n \in \mathbb{N}$ *we have* $(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W_n(\mathcal{M}, \mathcal{M}')$.

Note that the game $\mathrm{EF}_\omega$ is a closed game.

# Back-and-forth set = winning strategy of II

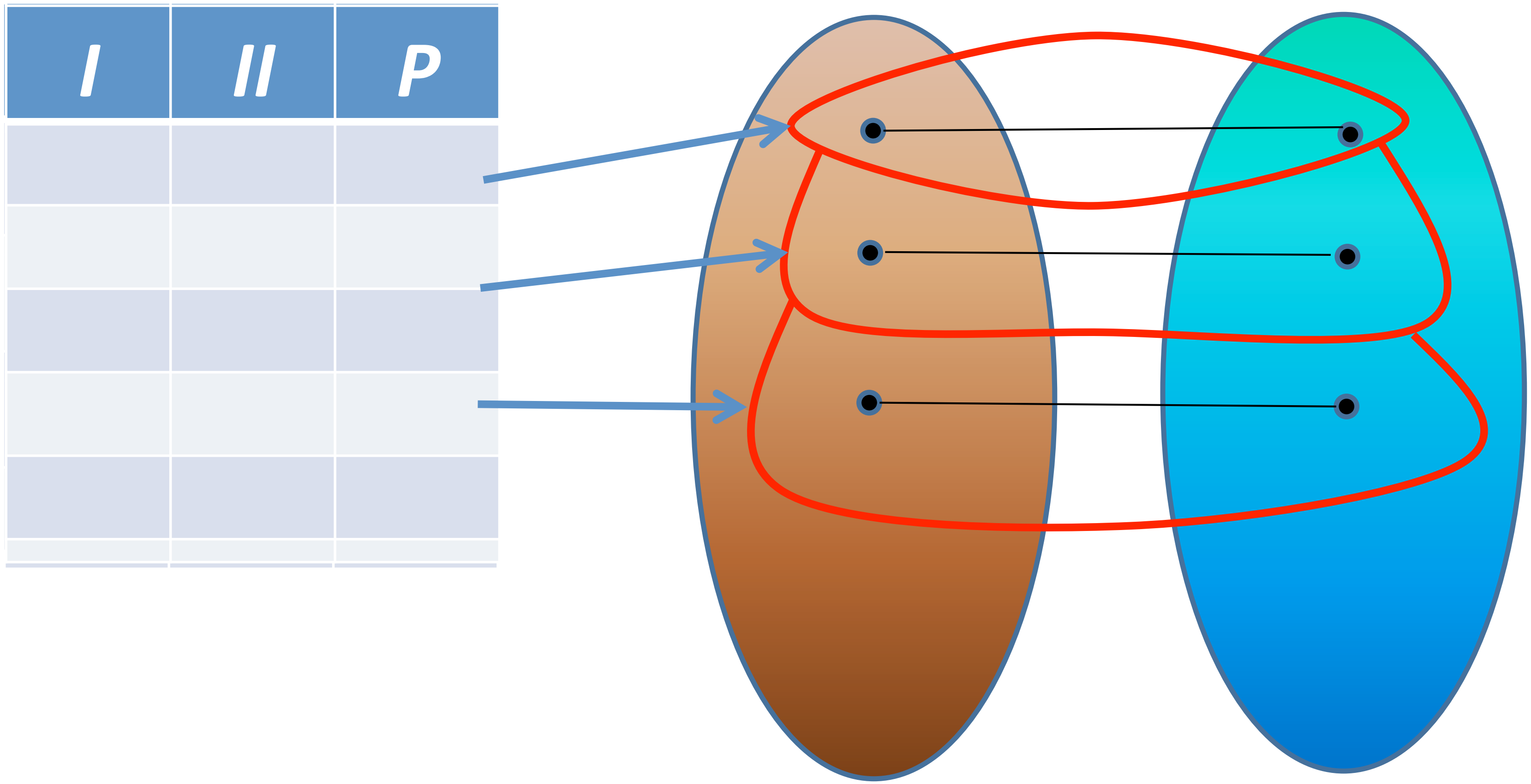
**Proposition 4.4.1.** *Suppose $L$ is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are two $L$-structures. The following conditions are equivalent:*

*1.* $\mathcal{A} \cong_p \mathcal{B}$

i.e. there is a back-and-forth set for A and B

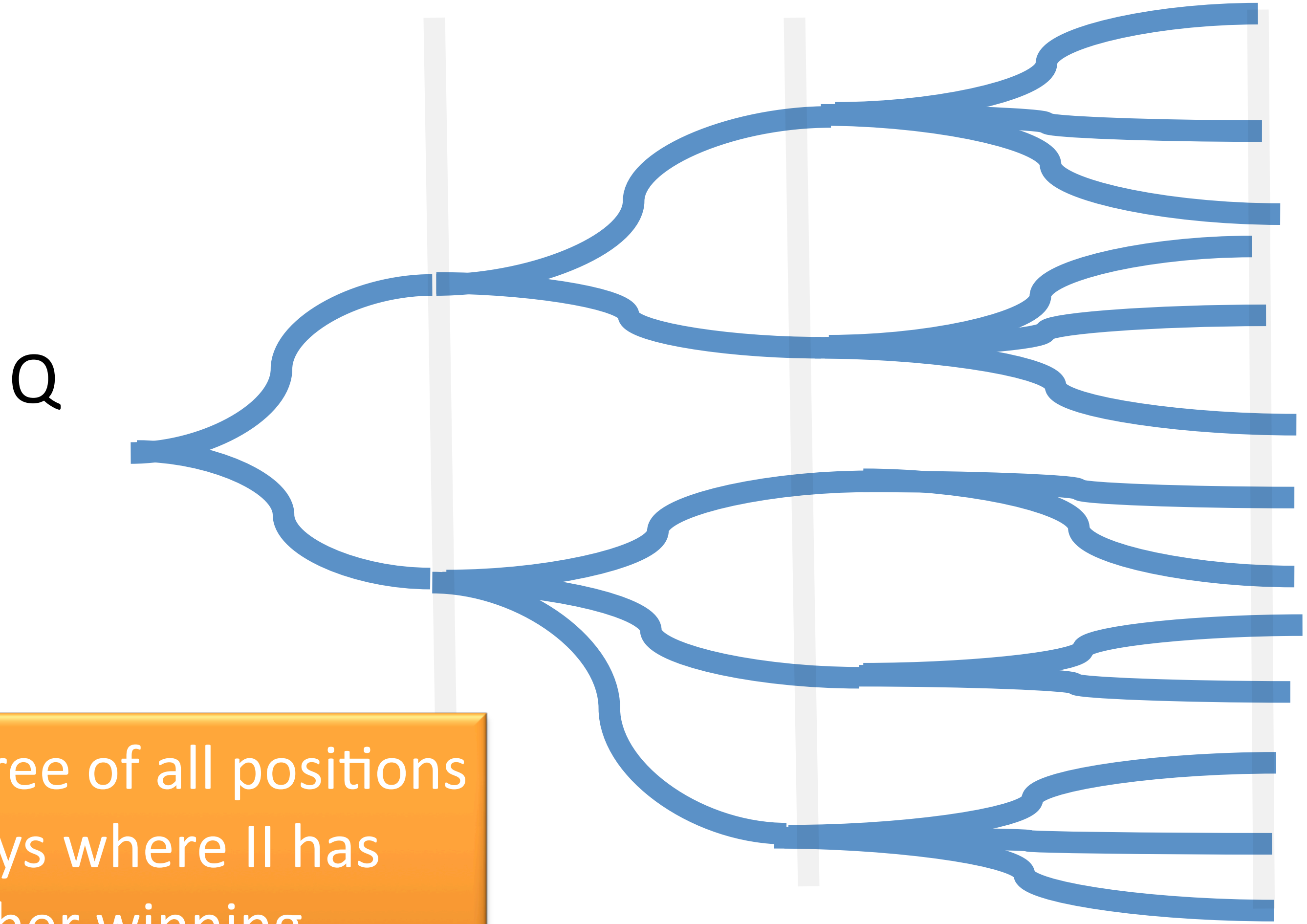
*2. II has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

# From a b-a-f set to a strategy




The strategy of II is to play so that after each move she knows, and remembers, a big enough function in P.

# From a strategy to a b-a-f set




The tree of all positions in plays where II has used her winning strategy

# From a strategy to a b-a-f set

Suppose II has a winning strategy $\tau$.

We form the back-and-forth set from all functions of the form

$$f_p = \{(v_0, v_0'), \ldots, (v_{n-1}, v_{n-1}')\}$$

where

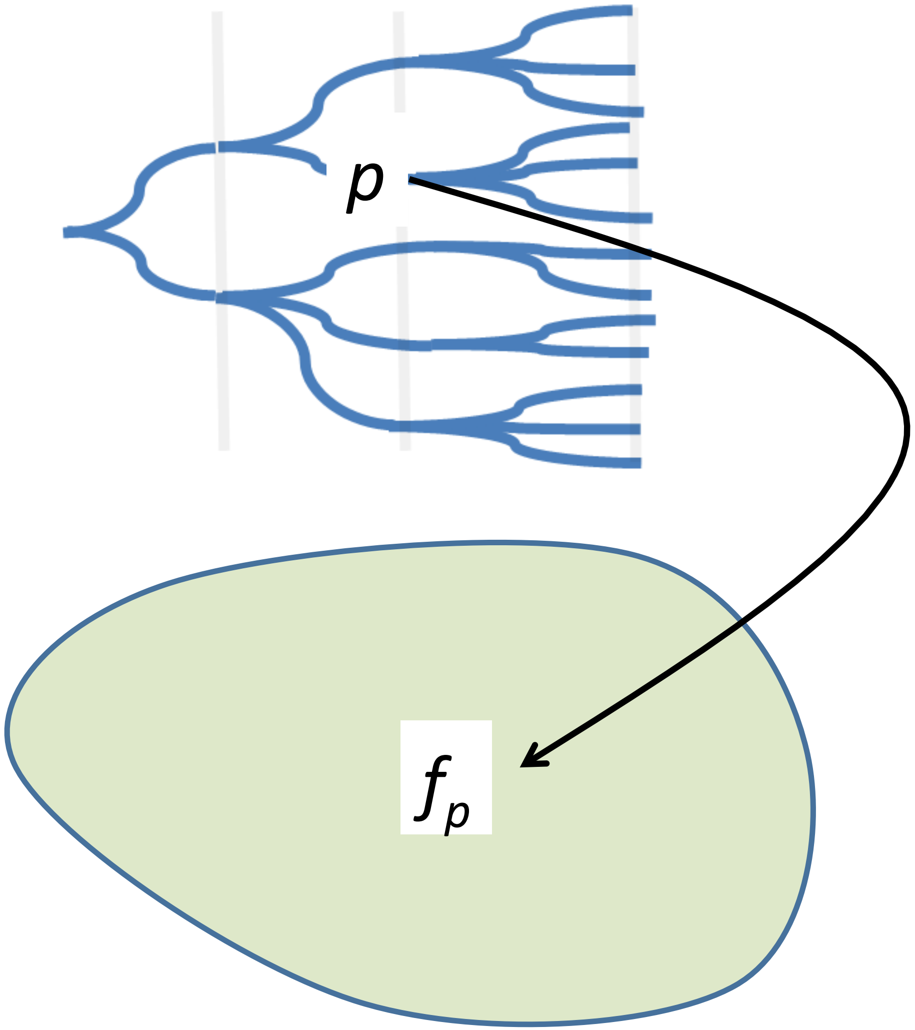
$$p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

is a position in a play where II has used her winning strategy $\tau$.

See slides 7,8

# The back-and-forth set as a projection

From a tree




to a po set

Can we prove a similar result for the finite EF game?

# Back-and-forth **sequence**

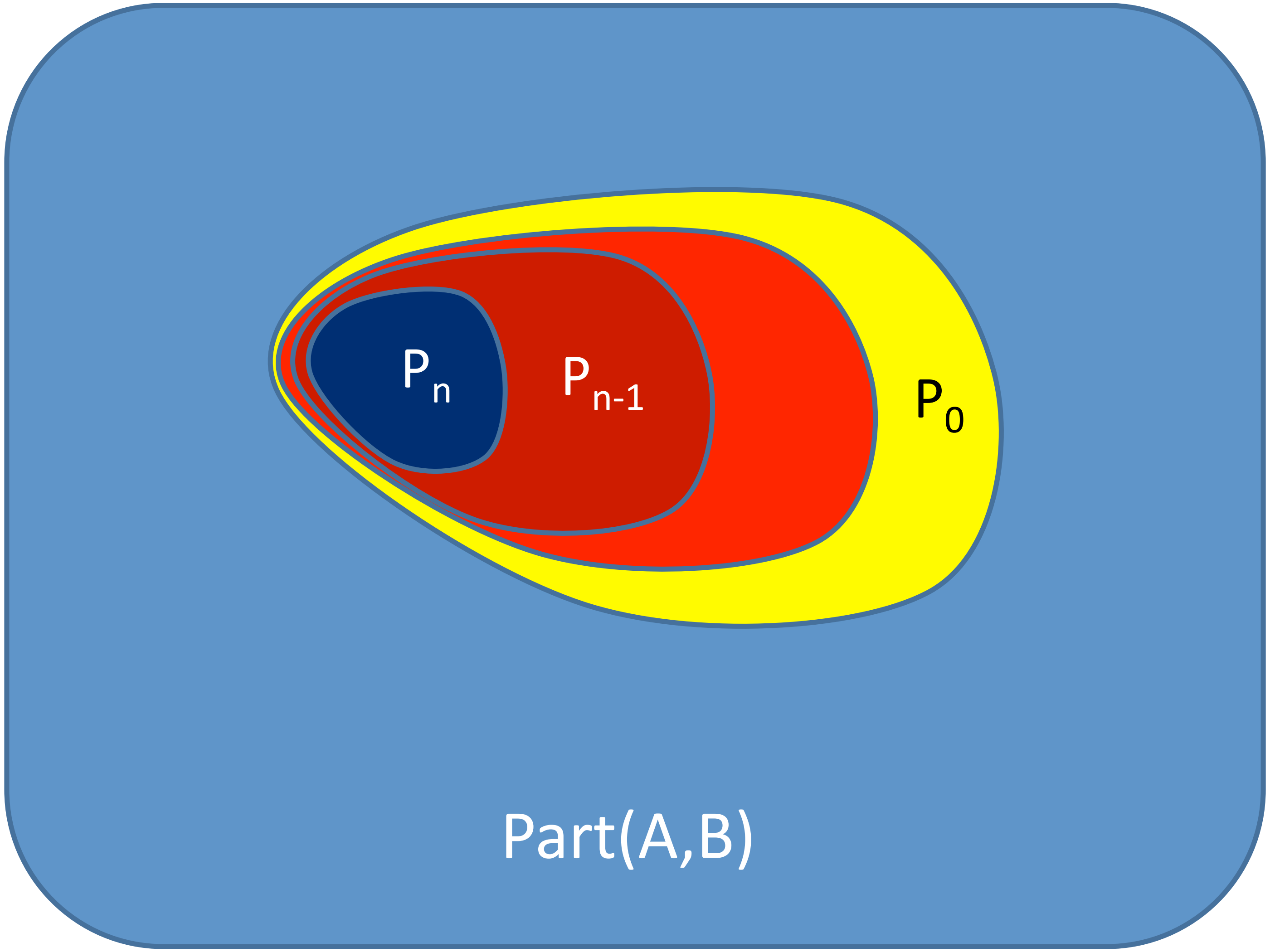
A back-and-forth set: can go back-and-forth.

A back-and-forth sequence tells how many **times** we can go back-and-forth.

**Definition 4.5.1.** *A* back-and-forth sequence $(P_i : i \leq n)$ *is characterized by the conditions*

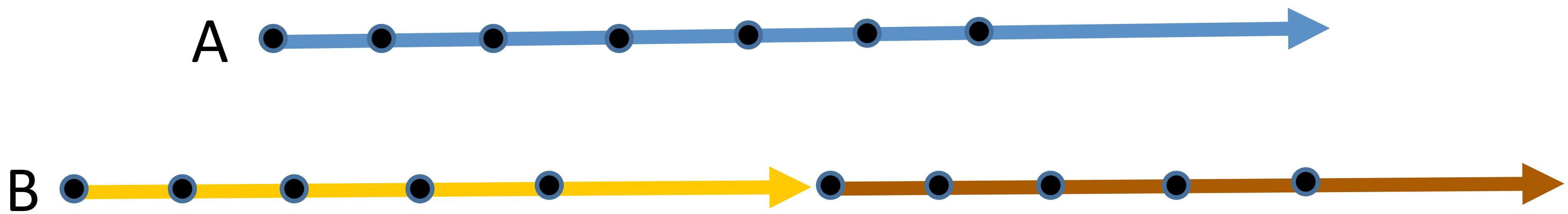
$$\emptyset \neq P_n \subseteq \ldots \subseteq P_0 \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B})$$
$$\forall f \in P_{i+1} \forall a \in A \exists b \in B \exists g \in P_i (f \cup \{(a,b)\} \subseteq g) \text{ for } i < n.$$
$$\forall f \in P_{i+1} \forall b \in B \exists a \in A \exists g \in P_i (f \cup \{(a,b)\} \subseteq g) \text{ for } i < n.$$

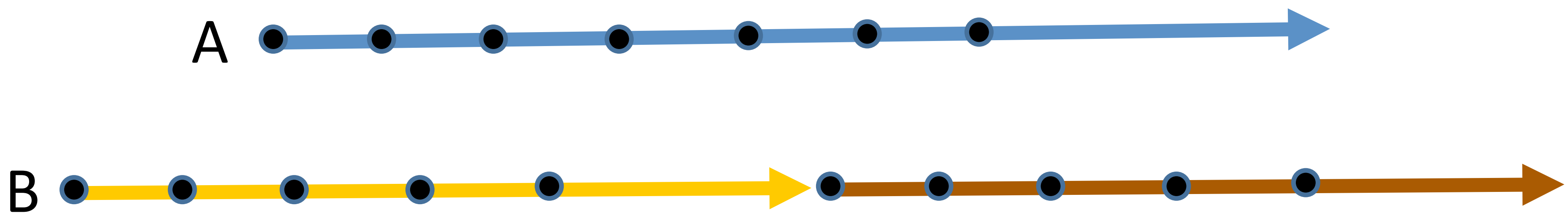
$$\boxed{\mathcal{A} \simeq_p^n \mathcal{B}}$$



$P_n$
$P_{n-1}$
$P_0$
Part(A,B)

# Two ordered sets





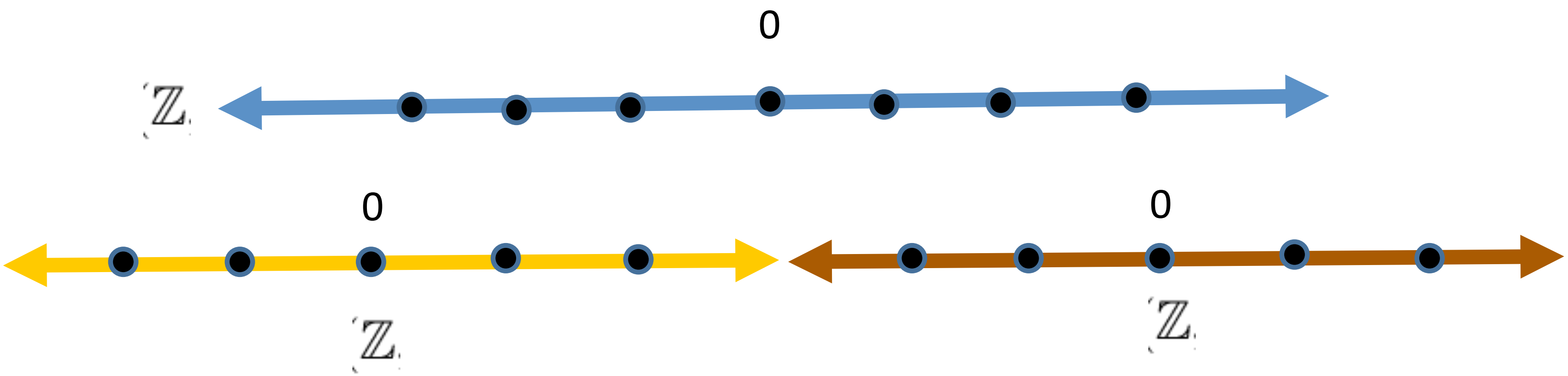

$$P_2 = \{\emptyset\}.$$

$$P_1 = \{\{(a,b)\} : 0 < a \in \mathbb{N},\ 0 < b \in \mathbb{N}+\mathbb{N}\} \cup \{(0,0)\} \cup P_2.$$

$$P_0 = \{\{(a_0,b_0),(a_1,b_1)\} : a_0 < a_1 \in \mathbb{N},\ b_0 < b_1 \in \mathbb{N}+\mathbb{N}\} \cup P_1.$$

$$(\mathbb{N},<) \simeq_p^2 (\mathbb{N}+\mathbb{N},<)$$

$$(\mathbb{N},<) \not\simeq_p^3 (\mathbb{N}+\mathbb{N},<).$$




$$(\mathbb{Z},<) \simeq_p^n (\mathbb{Z}+\mathbb{Z},<) \text{ for all } n \in \mathbb{N}$$

$$(\mathbb{Z},<) \not\simeq_p (\mathbb{Z}+\mathbb{Z},<).$$

# B-a-f sequence = II's strategy

**Proposition 4.5.2.** *Suppose $L$ is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are two $L$-structures. The following conditions are equivalent:*

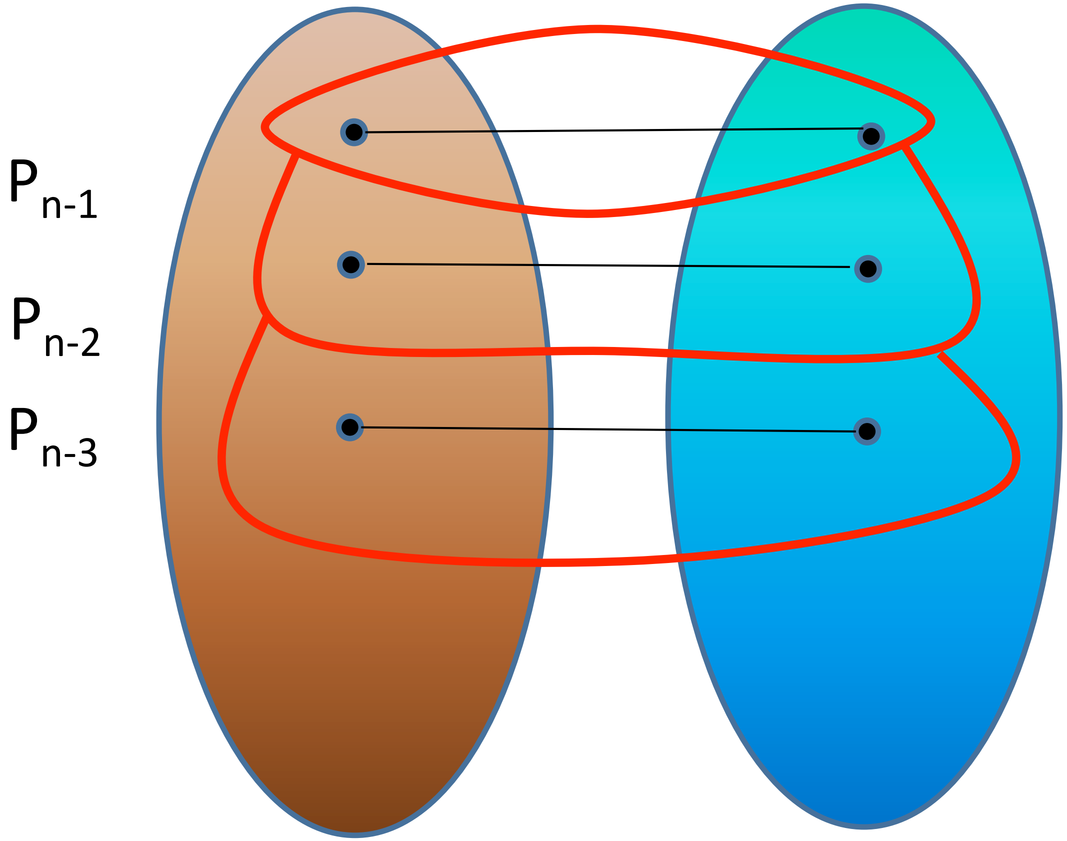
*1.* $\mathcal{A} \cong_p^n \mathcal{B}$

i.e. there is a back-and-forth sequence of length n for A and B

*2. II has a winning strategy in* $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.

# The n-move strategy

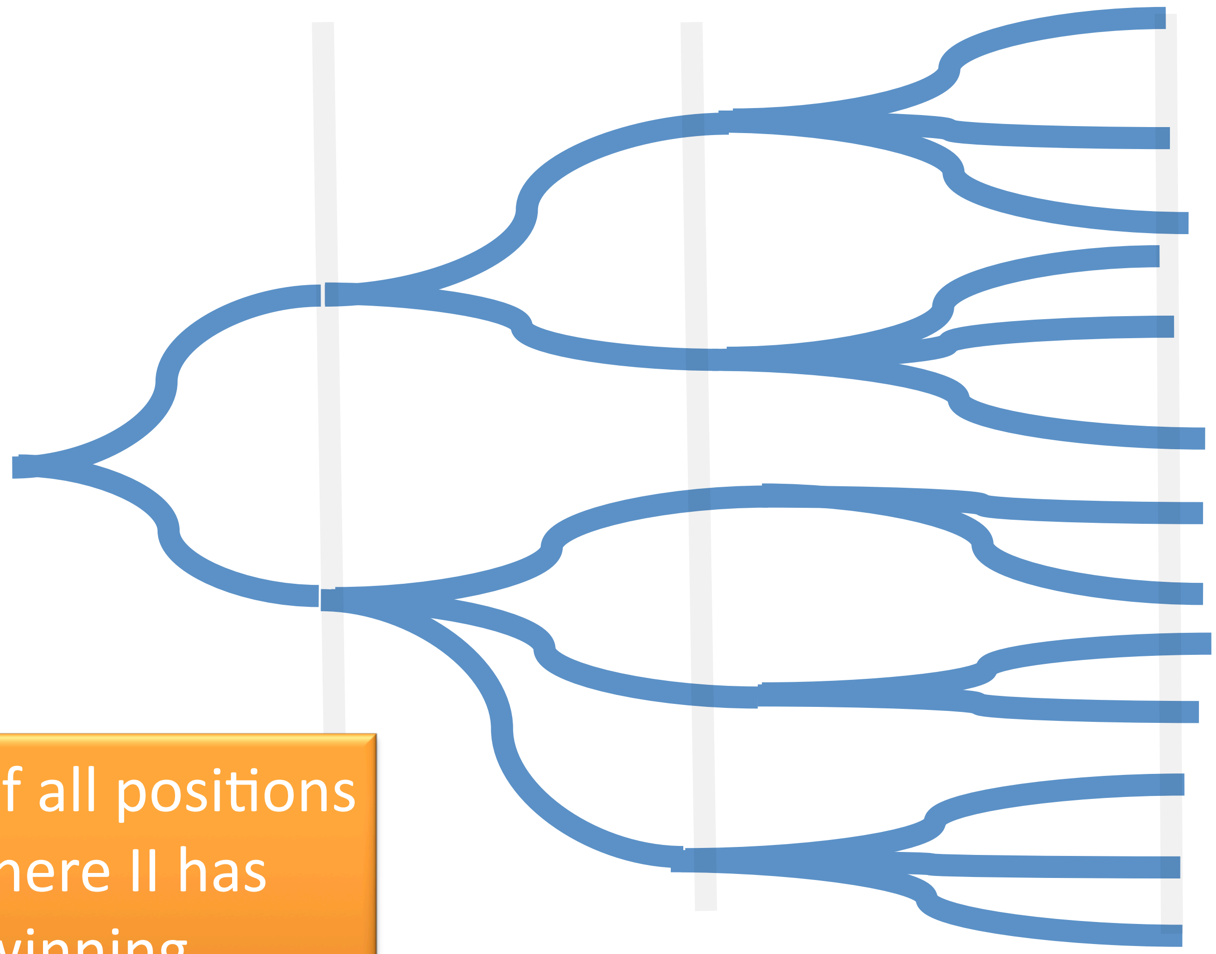
| I | II | |
|---|---|---|
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |




The strategy of II is to play so that after each move she knows, and remembers, a big enough function in the appropriate member of the b-a-f sequence..

# From strategy to b-a-f sequence

Q

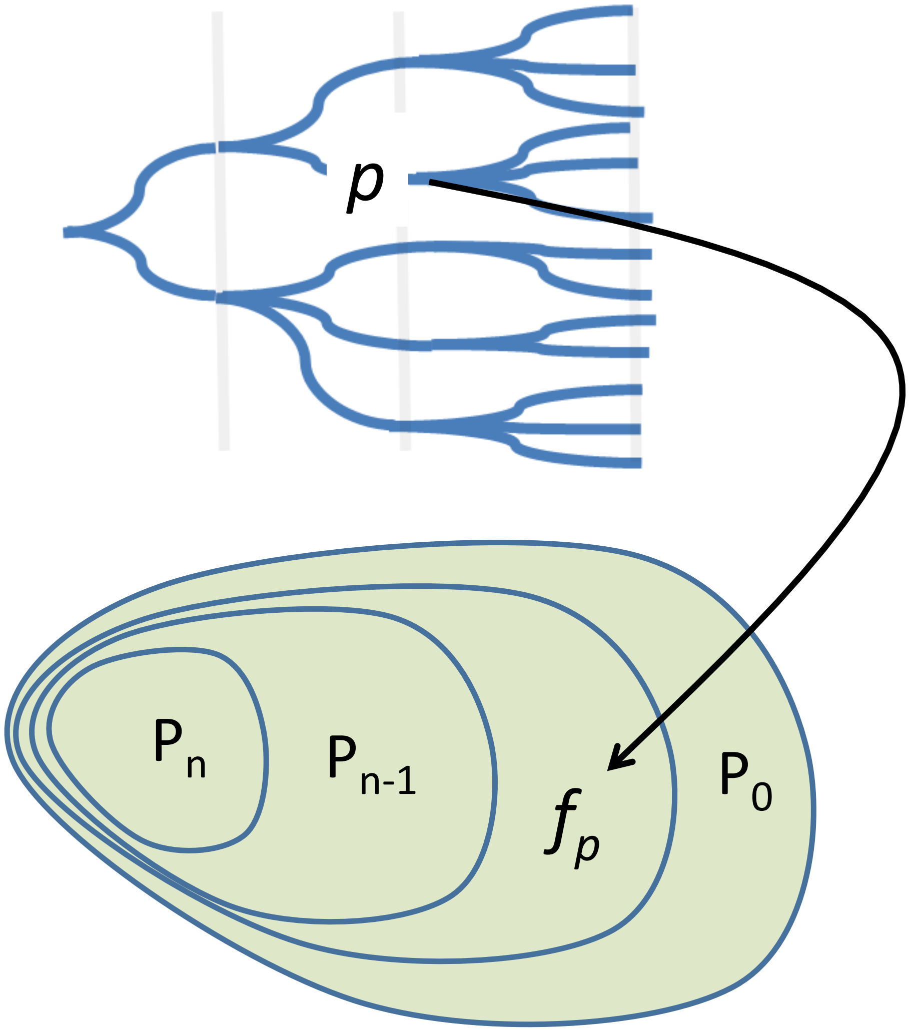


The tree of all positions in plays where II has used her winning strategy

# The back-and-forth set as a projection

From a tree

to po sets



# The back-and-forth sequence

Suppose II has a winning strategy $\tau$ in the game of length *m*.

We form the back-and-forth sequence by putting to $P_{m-n}$ all functions of the form

$$f_p = \{(v_0, v_0'), \ldots, (v_{n-1}, v_{n-1}')\}$$

where

$$p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

is a position in a play where II has used her winning strategy $\tau$.

- $P_m = \{\emptyset\}$
- $P_{m-1} = \{\emptyset\} \quad \cup \{\{(x_0,\tau_0(x_0))\}: \ x_0 \text{ in } A\}$
  $\cup \{\{(\tau_0(x_0), x_0)\}: x_0 \text{ in } B\}$
- $P_{m-2} = \{\emptyset\} \quad \cup \{\{(x_0,\tau_0(x_0))\}: \ x_0 \text{ in } A\}$
  $\cup \{\{(\tau_0(x_0), x_0)\}: x_0 \text{ in } B\}$
  $\cup \{\{(x_0,\tau_0(x_0)), (x_1,\tau_1(x_0,x_1))\}: \ x_0 \in A, \ x_1 \in A\}$
  $\cup \{\{(x_0,\tau_0(x_0)), (\tau_1(x_0,x_1),x_1)\}: \ x_0 \in A, x_1 \in B\}$
  $\cup \{\{(\tau_0(x_0), x_0), (x_1,\tau_1(x_0,x_1))\}: x_0 \in B, \ x_1 \in A\}$
  $\cup \{\{(\tau_0(x_0), x_0), (\tau_1(x_0,x_1), x_1)\}: x_0 \in B, x_1 \in B\}$