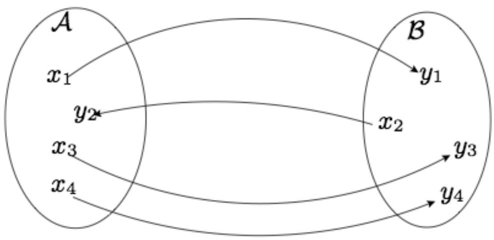

Figure 5.5 The Ehrenfeucht–Fraïssé Game.

at will. Then **II** is required to respond with some element b_1 of B so that

$$\{(a_1, b_1)\} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.10}$$

Alternatively, **I** might have chosen an element b_1 of B and then **II** would have been required to produce an element a_1 of A such that (5.10) holds. The one-element mapping $\{(a_1, b_1)\}$ is called the *position* in the game after the first move.

Now the game goes on. Again **I** asks what is the image of an element a_2 of A (or alternatively he can ask what is the pre-image of an element b_2 of B). Then **II** produces an element b_2 of B (or in the alternative case an element a_2 of A). In either case the choice of **II** has to satisfy

$$\{(a_1, b_1), (a_2, b_2)\} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.11}$$

Again, $\{(a_1, b_1), (a_2, b_2)\}$ is called the position after the second move.

We continue until the position

$$\{(a_1, b_1), \ldots, (a_n, b_n)\} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$$

after the n^{th} move has been produced. If **II** has been able to play all the moves according to the rules she is declared the winner. Let us call this game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$. Figure 5.5 pictures the situation after four moves. If **II** can win repeatedly whatever moves **I** plays, we say that **II** has a *winning strategy*.

Example 5.18 Suppose $\mathcal{A}$ and $\mathcal{B}$ are two L-structures and $L = \emptyset$. Thus the structures $\mathcal{A}$ and $\mathcal{B}$ consist merely of a universe with no structure on it. In this singular case any one-to-one mapping is a partial isomorphism. The only thing player **II** has to worry about, say in (5.11), is that $a_1 = a_2$ if and only if $b_1 = b_2$. Thus **II** has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ if A and B both have at least n elements. So **II** can have a winning strategy even if A and B have different cardinality and there could be no isomorphism between them for the trivial reason that there is no bijection. The intuition here is that by playing a finite number of elements, or even $\aleph_0$ many, it is not possible to get hold of the cardinality of the universe if it is infinite.

Example 5.19 Let $\mathcal{A}$ be a linear order of length 3 and $\mathcal{B}$ a linear order of length 4. How many moves does **I** need to beat **II**? Suppose $A = \{a_1, a_2, a_3\}$ in increasing order and $B = \{b_1, b_2, b_3, b_4\}$ in increasing order. Clearly, if **I** plays at any point the smallest element, also **II** has to play the smallest element or face defeat on the next move. Also, if **I** plays at any point the smallest but one element, also **II** has to play the smallest but one element or face defeat in two moves. Now in $\mathcal{A}$ the smallest but one element is the same as the largest but one element, while in $\mathcal{B}$ they are different. So if **I** starts with a_2, **II** has to play b_2 or b_3, or else she loses in one move. Suppose she plays b_2. Now **I** plays b_3 and **II** has no good moves left. To obey the rules, she must play a_3. That is how long she can play, for now when **I** plays b_4, **II** cannot make a legal move anymore. In fact **II** has a winning strategy in $\mathrm{EF}_2(\mathcal{A}, \mathcal{B})$ but **I** has a winning strategy in $\mathrm{EF}_3(\mathcal{A}, \mathcal{B})$.

We now proceed to a more exact definition of the Ehrenfeucht–Fraïssé Game.

Definition 5.20 Suppose L is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are L-structures such that $M \cap M' = \emptyset$. The *Ehrenfeucht–Fraïssé Game* $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$ is the game $\mathcal{G}_n(M \cup M', W_n(\mathcal{M}, \mathcal{M}'))$, where $W_n(\mathcal{M}, \mathcal{M}') \subseteq (M \cup M')^{2n}$ is the set of $p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ such that:

(G1) For all $i < n$: $x_i \in M \iff y_i \in M'$.
(G2) If we denote

$$v_i = \begin{cases} x_i & \text{if } x_i \in M \\ y_i & \text{if } y_i \in M \end{cases} \quad v_i' = \begin{cases} x_i & \text{if } x_i \in M' \\ y_i & \text{if } y_i \in M', \end{cases}$$

then

$$f_p = \{(v_0, v_0'), \ldots, (v_{n-1}, v_{n-1}')\}$$

is a partial isomorphism $\mathcal{M} \to \mathcal{M}'$.

We call v_i and v_i' *corresponding* elements. The infinite game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is defined quite similarly, that is, it is the game $\mathcal{G}_\omega(M \cup M', W_\omega(\mathcal{M}, \mathcal{M}'))$, where $W_\omega(\mathcal{M}, \mathcal{M}')$ is the set of $p = (x_0, y_0, x_1, y_1, \ldots)$ such that for all $n \in \mathbb{N}$ we have $(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W_n(\mathcal{M}, \mathcal{M}')$.

Note that the game EF_ω is a closed game.

Proposition 5.21 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are L-structures. The following are equivalent:*

1. $\mathcal{A} \simeq_p \mathcal{B}$.
2. **II** *has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

Proof Assume $A \cap B = \emptyset$. Let P be first a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i < \omega)$ for **II**. Since $P \neq \emptyset$ we can fix an element f of P. Condition (5.8) tells us that if $a_1 \in A$, then there are $b_1 \in B$ and g such that

$$f \cup \{(a_1, b_1)\} \subseteq g \in P. \tag{5.12}$$

Let $\tau_0(a_1)$ be one such b_1. Likewise, if $b_1 \in B$, then there are $a_1 \in A$ such that (5.12) holds and we can let $\tau_0(b_1)$ be some such a_1. We have defined $\tau_0(c_1)$ whatever c_1 is. To define $\tau_1(c_1, c_2)$, let us assume **I** played $c_1 = a_1 \in A$. Thus (5.12) holds with $b_1 = \tau_0(a_1)$. If $c_2 = a_2 \in A$ we can use (5.8) again to find $b_2 = \tau_1(a_1, a_2) \in B$ and h such that

$$f \cup \{(a_1, b_1), (a_2, b_2)\} \subseteq h \in P.$$

The pattern should now be clear. The back-and-forth set P guides **II** to always find a valid move. Let us then write the proof in more detail: Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \ldots, x_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_{i-1})$. Now player **I** plays x_j. By assumption there is $f_j \in P$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$ and if $x_j \in B$, then $x_j \in \mathrm{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$ and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_n : n < \omega)$ is a winning strategy of **II**. Let Q consist of all plays of $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P consist of all possible f_p where p is a position in the game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ with an extension in Q. It is clear that P is non-void and has the properties (5.8) and (5.9). □

To prove partial isomorphism of two structures we now have two alternative methods:

1. Construct a back-and-forth set.
2. Show that player **II** has a winning strategy in EF_ω.

By Proposition 5.21 these methods are equivalent. In practice one uses the game as a guide to intuition and then for a formal proof one usually uses a back-and-forth set.

5.6 Back-and-Forth Sequences

Back-and-forth sets and winning strategies of player **II** in the Ehrenfeucht–Fraïssé Game EF_ω correspond to each other. There is a more refined concept, called a back-and-forth sequence, which corresponds to a winning strategy of player **II** in the finite game EF_n.

Definition 5.22 A *back-and-forth sequence* $(P_i : i \leq n)$ is defined by the conditions

$$\emptyset \neq P_n \subseteq \ldots \subseteq P_0 \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.13}$$

$$\forall f \in P_{i+1} \forall a \in A \exists b \in B \exists g \in P_i (f \cup \{(a,b)\} \subseteq g) \text{ for } i < n. \tag{5.14}$$

$$\forall f \in P_{i+1} \forall b \in B \exists a \in A \exists g \in P_i (f \cup \{(a,b)\} \subseteq g) \text{ for } i < n. \tag{5.15}$$

If P is a back-and-forth set, we can get back-and-forth sequences $(P_i : i \leq n)$ of any length by choosing $P_i = P$ for all $i \leq n$. But the converse is not true: the sets P_i need by no means be themselves back-and-forth sets. Indeed, pairs of countable models may have long back-and-forth sequences without having any back-and-forth sets. Let us write

$$\mathcal{A} \simeq_p^n \mathcal{B}$$

if there is a back-and-forth sequence of length n for $\mathcal{A}$ and $\mathcal{B}$.

Lemma 5.23 *The relation $\simeq_p^n$ is an equivalence relation on $Str(L)$.*

Proof Exactly as Lemma 5.15. □

Example 5.24 We use $(\mathbb{N} + \mathbb{N}, <)$ to denote the linear order obtained by putting two copies of $(\mathbb{N}, <)$ one after the other. (The ordinal of this order is $\omega + \omega$.) Now $(\mathbb{N}, <) \simeq_p^2 (\mathbb{N} + \mathbb{N}, <)$, for we may take

$$P_2 = \{\emptyset\}.$$

$$P_1 = \{\{(a,b)\} : 0 < a \in \mathbb{N},\ 0 < b \in \mathbb{N} + \mathbb{N}\} \cup \{(0,0)\} \cup P_2.$$

$$P_0 = \{\{(a_0,b_0),(a_1,b_1)\} : a_0 < a_1 \in \mathbb{N},\ b_0 < b_1 \in \mathbb{N} + \mathbb{N}\} \cup P_1.$$

Note that $(\mathbb{N}, <) \not\simeq_p^3 (\mathbb{N} + \mathbb{N}, <)$.

Proposition 5.25 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are discrete linear orders (i.e. every element with a successor has an immediate successor and every element with a predecessor has an immediate predecessor) with no endpoints, and $n \in \mathbb{N}$. Then $\mathcal{A} \simeq_p^n \mathcal{B}$.*

Proof Let P_i consist of $f \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ with the following property: $f = \{(a_0, b_0), \ldots, (a_{n-i-1}, b_{n-i-1})\}$ where

$$a_0 \leq \ldots \leq a_{n-i-1},$$
$$b_0 \leq \ldots \leq b_{n-i-1},$$

and for all $0 \leq j < n-i-1$ if $|(a_j, a_{j+1})| < 2^i$ or $|(b_j, b_{j+1})| < 2^i$, then $|(a_j, a_{j+1})| = |(b_j, b_{j+1})|$. □

Example 5.26 $(\mathbb{Z}, <) \simeq_p^n (\mathbb{Z} + \mathbb{Z}, <)$ for all $n \in \mathbb{N}$, but note that $(\mathbb{Z}, <) \not\simeq_p (\mathbb{Z} + \mathbb{Z}, <)$.

Proposition 5.27 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are L-structures. The following are equivalent:*

1. $\mathcal{A} \simeq_p^n \mathcal{B}$.
2. **II** *has a winning strategy in* $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.

Proof Let us assume $A \cap B = \emptyset$. Let $(P_i : i \leq n)$ be a back-and-forth sequence for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i \leq n)$ for **II**. Since $P_n \neq \emptyset$ we can fix an element f of P_n. Condition (5.14) tells us that if $a_1 \in A$, then there are $b_1 \in B$ and g such that

$$f \cup \{(a_1, b_1)\} \subseteq g \in P_{n-1}. \tag{5.16}$$

Let $\tau_0(a_1)$ be one such b_1. Likewise, if $b_1 \in B$, then there are $a_1 \in A$ such that (5.16) holds and we can let $\tau_0(b_1)$ be some such a_1. We have defined $\tau_0(c_1)$ whatever c_1 is. To define $\tau_1(c_1, c_2)$, let us assume **I** played $c_1 = a_1 \in A$. Thus (5.16) holds with $b_1 = \tau_0(a_1)$. If $c_2 = a_2 \in A$ we can use (5.13) again to find $b_2 = \tau_1(a_1, a_2) \in B$ and h such that

$$f \cup \{(a_1, b_1), (a_2, b_2)\} \subseteq h \in P_{n-2}.$$

The pattern should be clear now. As before, the back-and-forth sequence guides **II** to always find a valid move. Let us then write the proof in more detail: Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \ldots, x_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P_{n-i}$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_i)$. Now player **I** plays x_j. By assumption there is $f_j \in P_{n-j}$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$ and if $x_j \in B$, then $x_j \in \mathrm{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$ and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_i : i \leq n)$ is a winning strategy of **II**. Let Q consist of all plays of $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P_{n-i} consist of all possible f_p where $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in the game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ with an extension in Q. It is clear that $(P_i : i \leq n)$ has the properties (5.13) and (5.14). Note that:

$$P_n = \{\emptyset\}$$

$$P_{n-1} = \{(x_0, \tau_0(x_0)) : x_0 \in A \cup B\}$$

$$P_{n-2} = \{(x_0, \tau_0(x_0), x_1, \tau_1(x_0, x_1)) : x_0, x_1 \in A \cup B\}$$

$$P_0 = \{(x_0, \tau_0(x_0), \ldots, x_{n-1}, \tau_{n-1}(x_0, \ldots, x_{n-1})) : x_0, \ldots, x_{n-1} \in A \cup B\}.$$

□

5.7 Historical Remarks and References

Back-and-forth sets are due to Fraïssé (1955). The Ehrenfeucht–Fraïssé Game was introduced in Ehrenfeucht (1957) and Ehrenfeucht (1960/1961). Back-and-forth sequences were introduced in Karp (1965). Exercise 5.40 is from Ellentuck (1976). Exercise 5.40 is from Ellentuck (1976). Exercise 5.54 is from Barwise (1975). Exercise 5.71 is from Rosenstein (1982).

Exercises

5.1 Show that isomorphism of structures is an equivalence relation in the sense that it is reflexive, symmetric, and transitive.

5.2 Suppose L is a finite vocabulary, $\mathcal{B}$ is a countable L-model, and $\{b_n : n < \omega\}$ is an enumeration of the domain B of $\mathcal{B}$. Suppose $\mathcal{A}$ is a countable L-model. Show that the following are equivalent:

(1) $\mathcal{A} \cong \mathcal{B}$.

(2) There is an enumeration $\{a_n : n < \omega\}$ of the domain of $\mathcal{A}$ so that for all atomic L-formulas $\theta(x_0, \ldots, x_n)$ and all $n < \omega$ we have

$$\mathcal{A} \models \theta(a_0, \ldots, a_n) \iff \mathcal{B} \models \theta(b_0, \ldots, b_n).$$

5.3 Suppose L is a vocabulary and $\mathcal{M}$ is an L-structure. Show that the set $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$ forms a group under the operation of composition of functions.

5.4 Give an example of $\mathcal{M}$ such that $\mathrm{Aut}(\mathcal{M})$ (see the previous exercise) is:

1. The trivial one-element group.
2. A non-trivial abelian group (e.g. the additive group of the integers).
3. A non-abelian group (e.g. the symmetric group S_3).

5.5 How many automorphisms do the following structures have.

1. A linear order of n elements.
2. $(\mathbb{N}, <)$.
3. $(\mathbb{Z}, <)$.
4. $(\mathbb{Q}, <)$.

5.6 Show that if $\mathcal{A}$ and $\mathcal{B}$ are unary structures, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$ we have $|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})|$. Easier version: Show that if $\mathcal{A}$ and $\mathcal{B}$ are unary structures with a finite universe of size n, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$ we have $|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})|$.

5.7 Suppose $\mathcal{M}$ is a unary structure in which every ϵ-constituent has exactly three elements. How many elements does $\mathcal{M}$ have? How many automorphisms does $\mathcal{M}$ have?

5.8 $L = \{P_1, \ldots, P_m\}$, where each P_i is unary. Show that the number of non-isomorphic L-structures on the universe $\{1, \ldots, n\}$ is $\binom{n+2^m-1}{2^m-1}$.

5.9 Describe the group of automorphisms of a finite unary structure.

5.10 Suppose $\mathcal{M}$ is an equivalence relation with a finite universe such that $EC_n(\mathcal{M}) = 2$ for each $n = 1, \ldots, 5$ and $EC_n(\mathcal{M}) = 0$ for other n. How many elements are there in the universe of $\mathcal{M}$? How many automorphisms does $\mathcal{M}$ have?

5.11 Show that for any $m \in \mathbb{N}$ there is $m^* \in \mathbb{N}$ such that if $n \geq m^*$ then there are more than n^m non-isomorphic equivalence relations on the universe $\{1, \ldots, n\}$. Conclude that for any $m \in \mathbb{N}$ there is $m^* \in \mathbb{N}$ such that if $n \geq m^*$ then there are more non-isomorphic equivalence relations on the universe $\{1, \ldots, n\}$ than non-isomorphic $\{P_1, \ldots, P_m\}$-structures, where each P_i is unary.

5.12 Show that if $\mathcal{A}$ and $\mathcal{B}$ are equivalence relations, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $\kappa \leq |A \cup B|$ we have $EC_\kappa(\mathcal{A}) = EC_\kappa(\mathcal{B})$. Easier version: Show that if $\mathcal{A}$ and $\mathcal{B}$ are equivalence relations with a finite universe of size n, then $\mathcal{A} \cong \mathcal{B}$ if and only if for all $m \leq n$ we have $EC_m(\mathcal{A}) = EC_m(\mathcal{B})$.

5.13 Describe the group of automorphisms of a finite equivalence relation.

5.14 Show that if $\mathcal{M}$ and $\mathcal{N}$ are countable dense linear orders, then $\mathcal{M} \cong \mathcal{N}$ if and only if $SG(\mathcal{M}) = SG(\mathcal{N})$. Demonstrate that this is not true for non-dense countable linear orders or for uncountable dense linear orders.

5.15 Show that two well-orders $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $o(\mathcal{M}) = o(\mathcal{N})$.

5.16 Prove that two well-founded trees $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $\mathrm{stp}_{\mathcal{M}} = \mathrm{stp}_{\mathcal{N}}$.

5.17 Prove that two successor structures $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $CC_a(\mathcal{M}) = CC_a(\mathcal{N})$ for all $a \in \mathbb{N} \cup \{\infty\}$. Easier version: Prove that two successor structures $\mathcal{M}$ and $\mathcal{N}$ both of which have only finitely many components are isomorphic if and only if $CC_a(\mathcal{M}) = CC_a(\mathcal{N})$ for all $a \in \mathbb{N} \cup \{\infty\}$.

5.18 Show that any uncountable collection of countable non-isomorphic successor structures has to contain a successor structure with infinitely many cycle components.

5.19 Describe the group of automorphisms of a successor structure with n $\mathbb{Z}$-components and m_i i-cycle components for $i = 1, \ldots, k$.

5.20 Give an example of an infinite structure $\mathcal{M}$ with no substructures $\mathcal{N} \neq \mathcal{M}$.

5.21 Consider $\mathcal{M} = (\mathbb{Z}, +)$. What is $[X]_{\mathcal{M}}$, if X is

1. $\{0\}$,
2. $\{1\}$,
3. $\{2, -2\}$.

5.22 Consider $\mathcal{M} = (\mathbb{Z}, +, -)$. What is $[X]_{\mathcal{M}}$, if X is $\{13, 17\}$?

5.23 Suppose $\mathcal{M}$ is a successor structure consisting of the standard component and two five-cycles. Show that there are exactly four possibilities for the set $[X]_{\mathcal{M}}$.

5.24 Show that the universe of $[X]_{\mathcal{M}}$ is the intersection of all universes of substructures $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$.

5.25 Prove Lemma 5.12.

5.26 Show that every Boolean algebra $\mathcal{M}$ is isomorphic to a substructure of $(\mathcal{P}(A), \subseteq)$, where A is the set of all ultrafilters of $\mathcal{M}$. (This is the so-called *Stone's Representation Theorem*.)

5.27 Show that every tree every element of which has height $< \omega$ is isomorphic to a substructure of the tree $(A^{<\omega}, \leq)$ for some set A.

5.28 Suppose $L = \emptyset$. Show that any two infinite L-structures are partially isomorphic.

5.29 Suppose $L = \{P_1, \ldots, P_n\}$ is a *unary* vocabulary. Suppose we have two L-structures $\mathcal{M}$ and $\mathcal{N}$ satisfying the following condition: For all $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$ and all $m \in \mathbb{N}$ it holds that

$$|C_\epsilon(\mathcal{M})| = m \iff |C_\epsilon(\mathcal{N})| = m.$$

Show that this is a necessary and sufficient condition for the two structures to be partially isomorphic.

5.30 Suppose that two equivalence relations $\mathcal{M}$ and $\mathcal{N}$ satisfy the following conditions for all $n, m < \omega$:

1. $EC_n(\mathcal{M}) = m \iff EC_n(\mathcal{N}) = m.$
2. If one has exactly m infinite classes, then so does the other. In symbols:

$$\sum_{\aleph_0 \leq \kappa \leq |M|} EC_\kappa(\mathcal{M}) = m \iff \sum_{\aleph_0 \leq \kappa \leq |N|} EC_\kappa(\mathcal{N}) = m.$$

Show that these are a necessary and sufficient condition for the two structures to be partially isomorphic.

5.31 For elements t of a well-founded tree $\mathcal{M}$ we can define

$$\mathrm{dom}(\mathrm{stp}'_{\mathcal{M},t}) = \{\mathrm{stp}'_{\mathcal{M},s} : s \in \mathrm{ImSuc}(t)\}$$

$$\mathrm{stp}'_{\mathcal{M},t}(\mathrm{stp}'_{\mathcal{M},s}) = \min(\aleph_0, |\{s' \in \mathrm{ImSuc}(t) : \mathrm{stp}'_{\mathcal{M},s} = \mathrm{stp}'_{\mathcal{M},s'}\}|).$$

Suppose $\mathcal{M}$ and $\mathcal{N}$ are well-founded trees such that $\mathrm{stp}'_{\mathcal{M}} = \mathrm{stp}'_{\mathcal{N}}$. Show that $\mathcal{M}$ and $\mathcal{N}$ are partially isomorphic. Give an example of two well-founded partially isomorphic trees that are not isomorphic.

5.32 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are successor structures, $f \in \mathrm{Part}(\mathcal{M}, \mathcal{N})$. Show:

1. f maps elements of the standard component of $\mathcal{M}$ to elements of the standard component of $\mathcal{N}$.
2. f maps elements of a cycle component of $\mathcal{M}$ of size n to elements of a cycle component of $\mathcal{N}$ of size n.
3. f maps elements of a $\mathbb{Z}$-component of $\mathcal{M}$ to elements of a $\mathbb{Z}$-component of $\mathcal{N}$.

5.33 Suppose that two successor structures $\mathcal{M}$ and $\mathcal{N}$ satisfy the following conditions for all $n, m < \omega$: