Mobile Communications
Problem Set 5

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1. Does ALOHA suffer from the "hidden and exposed station" problem?

   Solution:

   ALOHA does not suffer from the exposed station problem since the stations do not listen to the medium before starting to send. The hidden station problem may naturally occur.

2. Consider light bulbs that fail after some iid random time that is exponentially distributed. A single bulb is used at a time and replaced immediately if it fails. The time between failures X is exponentially distributed with mean of E[X] = 200 hours. Given that a light bulb has been running for 30 hours what is the probability that the light bulb does not fail before overall 230 hours ?

   (Hint: Regard this process as a renewal process. The exponential distribution has a remarkable property.)

   Solution:

   We regard the bulb lifetime process as a renewal process. The time between two failures X is exponentially distributed with parameter λ. Next we exploit the memorylessness of the exponential distribution that governs the time between two renewals. Since for the exponential distribution it is known that $E[X] = 1/\lambda$ we get $\lambda = 1/(200\text{hrs})$. We are interested in the probability that the light bulb does not fail before overall 230 hours given that the bulb has been running for 30 hours. Since the exponential distribution is memoryless we get

$$\begin{aligned} P(X > 230 | X > 30) =& P(X > 200) \\ =& 1 - F_X(200) \\ =& e^{-\lambda 200} \\ =& \frac{1}{e} \end{aligned}$$

3. We decide to use the following car replacement strategy. Cars possess an iid lifetime X that is distributed according to some distribution F. A new car costs C. Dumping a car costs D.

   We start with a new car (car number 0) at time $t = 0$. If the car is working after time T we give it away for free and buy a new car for C. If the car is broken before T we pay $C + D$ for dumping the old car and buying a new one immediately.

- What is the long term cost of this policy?
- Given that the lifetime of a car X is distributed uniformly between 0 and W. What is the optimal value for the time T after which we give away the car?

Solution:

The process at hand is a renewal process with interarrival times given by $Z_i = \min\{T, X_i\}$ where X_i is the lifetime of car i. The car lifetime X is given by the distribution F such that $F(x) = \mathsf{P}[X \leq x]$ and $\bar{F}(x) = 1 - F(x)$.

The reward (cost) at renewal i is given by

$$R_i = C + D \cdot 1_{\mathsf{X_i<T}},$$

which captures the costs for both possibilities of buying a new car depending on the lifetime X_i.

The long term average reward (cost) is given (in the lecture) as

$$\frac{\mathsf{E}[R]}{\mathsf{E}[Z]}$$

The expected value of R is given by

$$\begin{aligned}\mathsf{E}[R] &= C + D \cdot \mathsf{P}[X < T]\\ &= C + D \cdot F(T)\end{aligned}$$

Since Z is a non-negative random variable given by $Z = \min\{T, X\}$ and T is a constant we calculate

$$\mathsf{P}[Z > z] = \mathsf{P}[X > z, T > z] = \mathsf{P}[X > z] \cdot 1_{\mathsf{z<T}}$$

The expected value of Z is given as

$$\begin{aligned}\mathsf{E}[Z] &= \int_0^\infty \mathsf{P}[Z > z]\, dz\\ &= \int_0^\infty \mathsf{P}[X > z] \cdot 1_{\mathsf{z<T}} dz\\ &= \int_0^T \mathsf{P}[X > z]\, dz\\ &= \int_0^T \bar{F}(z) dz\end{aligned}$$

The long term cost is given by

$$\frac{\mathsf{E}[R]}{\mathsf{E}[Z]} = \frac{C + D \cdot F(T)}{\int_0^T \bar{F}(z)dz} := h(T)$$

The long term cost is dependent on the distribution of the car lifetime and on the time T at which we give the car away. Generally, if T is too large the car will essentially break down and we have a cost of $C + D$. If we choose T too small we give away good cars and buy new ones too often causing unnecessary costs of C. The optimal value of T depends on the distribution of the car lifetimes.

Next, we consider car lifetimes that are uniformly distributed between 0 and W, i.e. $F(z) = z/W$ for $z \leq W$, $F(z) = 0$ for negative z and $F(z) = 1$ for $z \geq W$. We seek T that minimizes $h(T)$ above. Hence, we differentiate $h(T)$ w.r.t T and set the derivative equal to zero.

We find that

$$\begin{aligned} h(T) &= \frac{C + D \cdot \frac{T}{W}}{\int_0^T 1 - \frac{z}{W} dz} \\ &= \frac{2W\left(C + D\frac{T}{W}\right)}{2WT - T^2} \\ &= \frac{2WC + 2DT}{2WT - T^2} \end{aligned}$$

and

$$\frac{dh(T)}{dT} = \frac{4CW(T - W) + 2DT^2}{T^2(T - 2W)^2}$$

Setting the derivative equal to zero we find an optimal value for T as

$$T = \frac{\pm\sqrt{CW^2(C + 2D)} - CW}{D}$$

If $C = D$ we find $T = (\sqrt{3} - 1)W \approx 0.73W$.