In particular, for every countable $X \subseteq M$ there is a countable submodel $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$ and $\mathcal{N} \models T$.

Proof Let $T = \{\varphi_0, \varphi_1, \ldots\}$. By Proposition 6.22 player **II** has a winning strategy in $G_{\text{cub}}(\mathcal{C}_{\varphi_n})$. By Lemma 6.14, player **II** has a winning strategy in $G_{\text{cub}}(\bigcap_{n=0}^{\infty} \mathcal{C}_{\varphi_n})$. If $X \in \bigcap_{n=0}^{\infty} \mathcal{C}_{\varphi_n}$, then $[X]_{\mathcal{M}} \models T$. $\square$

6.5 The Semantic Game

The truth of a first-order sentence in a structure can be defined by means of a simple game called the Semantic Game. We examine this game in detail and give some applications of it.

Definition 6.24 Suppose L is a vocabulary, $\mathcal{M}$ is an L-structure, φ^* is an L-formula, and s^* is an assignment for M. The game $\text{SG}^{\text{sym}}(\mathcal{M}, \varphi^*)$ is defined as follows. In the beginning player **II** holds (φ^*, s^*). The rules of the game are as follows:

1. If φ is atomic, and s satisfies it in $\mathcal{M}$, then the player who holds (φ, s) wins the game, otherwise the other player wins.
2. If $\varphi = \neg\psi$, then the player who holds (φ, s), gives (ψ, s) to the other player.
3. If $\varphi = \psi \wedge \theta$, then the player who holds (φ, s), switches to hold (ψ, s) or (θ, s), and the other player decides which.
4. If $\varphi = \psi \vee \theta$, then the player who holds (φ, s), switches to hold (ψ, s) or (θ, s), and can himself or herself decide which.
5. If $\varphi = \forall x\psi$, then the player who holds (φ, s), switches to hold $(\psi, s[a/x])$ for some a, and the other player decides for which.
6. If $\varphi = \exists x\psi$, then the player who holds (φ, s), switches to hold $(\psi, s[a/x])$ for some a, and can himself or herself decide for which.

As was pointed out in Section 4.2, $\mathcal{M} \models_s \varphi$ if and only if player **II** has a winning strategy in the above game, starting with (φ, s). Why? If $\mathcal{M} \models_s \varphi$, then the winning strategy of player **II** is to play so that if she holds (φ', s'), then $\mathcal{M} \models_{s'} \varphi'$, and if player **I** holds (φ', s'), then $\mathcal{M} \not\models_{s'} \varphi'$.

For practical purposes it is useful to consider a simpler game which presupposes that the formula is in negation normal form. In this game, as in the Ehrenfeucht–Fraïssé Game, player **I** assumes the role of a doubter and player **II** the role of confirmer. This makes the game easier to use than the full game $\text{SG}^{\text{sym}}(\mathcal{M}, \varphi)$.

I	**II**
x_0	
x_1	y_0
	y_1
$\vdots$	$\vdots$

Figure 6.11 The game $G_\omega(W)$.

x_n	y_n	Explanation	Rule
$(\varphi, \emptyset)$		**I** enquires about $\varphi \in T$.	
	$(\varphi, \emptyset)$	**II** confirms.	Axiom rule
(φ_i, s)		**I** tests a played $(\varphi_0 \wedge \varphi_1, s)$ by choosing $i \in \{0, 1\}$.	
	(φ_i, s)	**II** confirms.	$\wedge$-rule
$(\varphi_0 \vee \varphi_1, s)$		**I** enquires about a played disjunction.	
	(φ_i, s)	**II** makes a choice of $i \in \{0, 1\}$.	$\vee$-rule
$(\varphi, s[a/x])$		**I** tests a played $(\forall x\varphi, s)$ by choosing $a \in M$.	
	$(\varphi, s[a/x])$	**II** confirms.	$\forall$-rule
$(\exists x\varphi, s)$		**I** enquires about a played existential statement.	
	$(\varphi, s[a/x])$	**II** makes a choice of $a \in M$.	$\exists$-rule

Figure 6.12 The game $\text{SG}(\mathcal{M}, T)$.

Definition 6.25 The *Semantic Game* $\text{SG}(\mathcal{M}, T)$ of the set T of L-sentences in NNF is the game (see Figure 6.11) $G_\omega(W)$, where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 6.12 and if player **II** plays the pair (φ, s), where φ is a basic formula, then $\mathcal{M} \models_s \varphi$.

In the game $\text{SG}(\mathcal{M}, T)$ player **II** claims that every sentence of T is true in $\mathcal{M}$. Player **I** doubts this and challenges player **II**. He may doubt whether a certain $\varphi \in T$ is true in $\mathcal{M}$, so he plays $x_0 = (\varphi, \emptyset)$. In this round, as in some other rounds too, player **II** just confirms and plays the same pair as player **I**. This may seem odd and unnecessary, but it is for book-keeping purposes only. Player **I** in a sense gathers a finite set of formulas confirmed by player **II** and tries to end up with a basic formula which cannot be true.

Theorem 6.26 *Suppose L is a vocabulary, T is a set of L-sentences, and $\mathcal{M}$ is an L-structure. Then the following are equivalent:*

1. $\mathcal{M} \models T$.
*2. Player **II** has a winning strategy in SG$(\mathcal{M}, T)$.*

Proof Suppose $\mathcal{M} \models T$. The winning strategy of player **II** in SG$(\mathcal{M}, T)$ is to maintain the condition $\mathcal{M} \models_{s_i} \psi_i$ for all $y_i = (\psi_i, s_i)$, $i \in \mathbb{N}$, played by her. It is easy to see that this is possible. On the other hand, suppose $\mathcal{M} \not\models T$, say $\mathcal{M} \not\models \varphi$, where $\varphi \in T$. The winning strategy of player **I** in SG$(\mathcal{M}, T)$ is to start with $x_0 = (\varphi, \emptyset)$, and then maintain the condition $\mathcal{M} \not\models_{s_i} \psi_i$ for all $y_i = (\psi_i, s_i)$, $i \in \mathbb{N}$, played by **II**:

1. If $y_i = (\psi_i, s_i)$, where ψ_i is basic, then player **I** has won the game, because $\mathcal{M} \not\models_{s_i} \psi_i$.
2. If $y_i = (\psi_i, s_i)$, where $\psi_i = \theta_0 \wedge \theta_1$, then player **I** can use the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ to find $k < 2$ such that $\mathcal{M} \not\models_{s_i} \theta_k$. Then he plays $x_{i+1} = (\theta_k, s_i)$.
3. If $y_i = (\psi_i, s_i)$, where $\psi_i = \theta_0 \vee \theta_1$, then player **I** knows from the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ that whether **II** plays (θ_k, s_i) for $k = 0$ or $k = 1$, the condition $\mathcal{M} \not\models_{s_i} \theta_k$ still holds. So player **I** can play $x_{i+1} = (\psi_i, s_i)$ and keep his winning criterion in force.
4. If $y_i = (\psi_i, s_i)$, where $\psi_i = \forall x \varphi$, then player **I** can use the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ to find $a \in M$ such that $\mathcal{M} \not\models_{s_i[a/x]} \varphi$. Then he plays $x_{i+1} = (\varphi, s_i[a/x])$.
5. If $y_i = (\psi_i, s_i)$, where $\psi_i = \exists x \varphi$, then player **I** knows from the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ that whatever $(\varphi, s_i[a/x])$ player **II** chooses to play, the condition $\mathcal{M} \not\models_{s_i[a/x]} \varphi$ still holds. So player **I** can play $(\exists x \varphi, s_i)$ and keep his winning criterion in force.

□

Example 6.27 Let $L = \{f\}$ and $\mathcal{M} = (\mathbb{N}, f^{\mathcal{M}})$, where $f(n) = n + 1$. Let

$$\varphi = \forall x \exists y \approx fxy.$$

Clearly, $\mathcal{M} \models \varphi$. Thus player **II** has, by Theorem 6.26, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.13 shows how the game might proceed. On

I	**II**	Rule
$(\forall x \exists y \approx fxy, \emptyset)$	$(\forall x \exists y \approx fxy, \emptyset)$	Axiom rule
$(\exists y \approx fxy, \{(x, 25)\})$	$(\exists y \approx fxy, \{(x, 25)\})$	$\forall$-rule
$(\exists y \approx fxy, \{(x, 25)\})$	$(\approx fxy, \{(x, 25), (y, 26)\})$	$\exists$-rule
$\vdots$	$\vdots$	

Figure 6.13 Player **II** has a winning strategy in SG$(\mathcal{M}, \{\varphi\})$.

I	**II**	Rule
$(\forall x \exists y \approx fyx, \emptyset)$	$(\forall x \exists y \approx fyx, \emptyset)$	Axiom rule
$(\exists y \approx fyx, \{(x, 0)\})$	$(\exists y \approx fyx, \{(x, 0)\})$	$\forall$-rule
$(\exists y \approx fyx, \{(x, 0)\})$	$(\approx fyx, \{(x, 0), (y, 2)\})$	$\exists$-rule
	(**II** has no good move)	

Figure 6.14 Player **I** wins the game SG$(\mathcal{M}, \{\psi\})$.

the other hand, suppose

$$\psi = \forall x \exists y \approx fyx.$$

Clearly, $\mathcal{M} \not\models \varphi$. Thus player **I** has, by Theorem 6.26 and Theorem 3.12, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.14 shows how the game might proceed.

Example 6.28 Let $\mathcal{M}$ be the graph of Figure 6.15.
and

$$\varphi = \forall x (\exists y \neg x E y \wedge \exists y x E y).$$

Clearly, $\mathcal{M} \models \varphi$. Thus player **II** has, by Theorem 6.26, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.16 shows how the game might proceed. On the other hand, suppose

$$\psi = \exists x (\forall y \neg x E y \vee \forall y x E y).$$

Clearly, $\mathcal{M} \not\models \varphi$. Thus player **I** has, by Theorem 6.26 and Theorem 3.12, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.17 shows how the game might proceed.

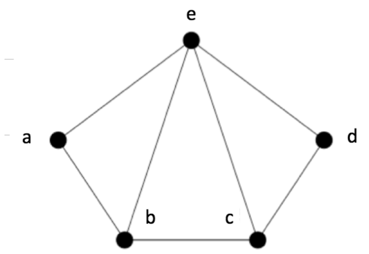

Figure 6.15 The graph $\mathcal{M}$.

I	**II**	Rule
$(\forall x(\exists y \neg xEy \wedge \exists y xEy), \emptyset)$	$(\forall x(\exists y \neg xEy \wedge \exists y xEy), \emptyset)$	Axiom rule
$(\exists y \neg xEy \wedge \exists y xEy, \{(x,d)\})$	$(\exists y \neg xEy \wedge \exists y xEy, \{(x,d)\})$	$\forall$-rule
$(\exists y xEy, \{(x,d)\})$	$(\exists y xEy, \{(x,d)\})$	$\wedge$-rule
$(\exists y xEy, \{(x,d)\})$	$(xEy, \{(x,d),(y,c)\})$	$\exists$-rule
$\vdots$	$\vdots$	

Figure 6.16 Player **II** has a winning strategy in SG($\mathcal{M}, \{\varphi\}$).

I	**II**	Rule
$(\exists x(\forall y \neg xEy \vee \forall y xEy), \emptyset)$	$(\exists x(\forall y \neg xEy \vee \forall y xEy), \emptyset)$	Axiom rule
$(\exists x(\forall y \neg xEy \vee \forall y xEy), \emptyset)$	$(\forall y \neg xEy \vee \forall y xEy), \{(x,a)\})$	$\exists$-rule
$(\forall y \neg xEy \vee \forall y xEy, \{(x,a)\})$	$(\forall y \neg xEy, \{(x,a)\})$	$\vee$-rule
$(\neg xEy, \{(x,a),(y,d)\})$	$(\neg xEy, \{(x,a),(y,d)\})$	$\forall$-rule

Figure 6.17 Player **I** wins the game SG($\mathcal{M}, \{\psi\}$).

$$(\forall x_0 P(x_0) \vee \exists x_1 Q(x_1)). \tag{6.4}$$

Note that formula (6.3) of quantifier rank 2 is logically equivalent to the formula (6.4) which has quantifier rank 1. So the nesting can sometimes be eliminated. In formulas (6.1) and (6.2) nesting cannot be so eliminated.

Proposition 6.3 *Suppose L is a finite vocabulary without function symbols. For every n and for every set $\{x_1, \ldots, x_n\}$ of variables, there are only finitely many logically non-equivalent first-order L-formulas of quantifier rank $< n$ with the free variables $\{x_1, \ldots, x_n\}$.*

Proof The proof is exactly like that of Proposition 4.15. □

Note that Proposition 6.3 is not true for infinite vocabularies, as there would be infinitely many logically non-equivalent atomic formulas, and also not true for vocabularies with function symbols, as there would be infinitely many logically non-equivalent equations obtained by iterating the function symbols.

6.3 Characterizing Elementary Equivalence

We now show that the concept of a back-and-forth sequence provides an alternative characterization of elementary equivalence

$$\mathcal{A} \equiv \mathcal{B} \quad \text{i.e.} \quad \forall \varphi \in FO(\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi).$$

This is the original motivation for the concepts of a back-and-forth set, back-and-forth sequence, and Ehrenfeucht–Fraïssé Game. To this end, let

$$\mathcal{A} \equiv_n \mathcal{B}$$

mean that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of FO of quantifier rank $\leq n$.

We now prove an important leg of the Strategic Balance of Logic, namely the marriage of truth and separation:

Proposition 6.4 *Suppose L is an arbitrary vocabulary. Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures and $n \in \mathbb{N}$. Consider the conditions:*

(i) $\mathcal{A} \equiv_n \mathcal{B}$.
(ii) $\mathcal{A}\restriction_{L'} \simeq_p^n \mathcal{B}\restriction_{L'}$ *for all finite* $L' \subseteq L$.

We have always $(ii) \to (i)$ and if L has no function symbols, then $(ii) \leftrightarrow (i)$.

Proof (ii)$\to$(i). If $\mathcal{A} \not\equiv_n \mathcal{B}$, then there is a sentence φ of quantifier rank $\leq n$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$. Since φ has only finitely many symbols, there is a finite $L' \subseteq L$ such that $\mathcal{A}\restriction_{L'} \not\equiv_n \mathcal{B}\restriction_{L'}$. Suppose $(P_i : i \leq n)$ is a back-and-forth sequence for $\mathcal{A}\restriction_{L'}$ and $\mathcal{B}\restriction_{L'}$. We use induction on $i \leq n$ to prove the following

Claim If $f \in P_i$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$, then

$$(\mathcal{A}\restriction_{L'}, a_1, \ldots, a_k) \equiv_i (\mathcal{B}\restriction_{L'}, fa_1, \ldots, fa_k).$$

If $i = 0$, the claim follows from $P_0 \subseteq \mathrm{Part}(\mathcal{A}\restriction_{L'}, \mathcal{B}\restriction_{L'})$. Suppose then $f \in P_{i+1}$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$. Let $\varphi(x_0, x_1, \ldots, x_k)$ be an L'-formula of FO of quantifier rank $\leq i$ such that

$$\mathcal{A}\restriction_{L'} \models \exists x_0 \varphi(x_0, a_1, \ldots, a_k).$$

Let $a \in A$ so that $\mathcal{A}\restriction_{L'} \models \varphi(a, a_1, \ldots, a_k)$ and $g \in P_i$ such that $a \in \mathrm{dom}(g)$ and $f \subseteq g$. By the induction hypothesis, $\mathcal{B}\restriction_{L'} \models \varphi(ga, ga_1, \ldots, ga_k)$. Hence

$$\mathcal{B}\restriction_{L'} \models \exists x_0 \varphi(x_0, fa_1, \ldots, fa_k).$$

The claim is proved. Putting $i = n$ and using the assumption $P_n \neq \emptyset$, gives a contradiction with $\mathcal{A}\restriction_{L'} \not\equiv_n \mathcal{B}\restriction_{L'}$.

(i)→ (ii). Assume L has no function symbols. Fix $L' \subseteq L$ finite. Let P_i consist of $f : A \to B$ such that $\mathrm{dom}(f) = \{a_0, \ldots, a_{n-i-1}\}$ and

$$(\mathcal{A}\restriction_{L'}, a_0, \ldots, a_{n-i-1}) \equiv_i (\mathcal{B}\restriction_{L'}, fa_0, \ldots, fa_{n-i-1}).$$

We show that $(P_i : i \leq n)$ is a back-and-forth sequence for $\mathcal{A}\restriction_{L'}$ and $\mathcal{B}\restriction_{L'}$. By (i), $\emptyset \in P_n$ so $P_n \neq \emptyset$. Suppose $f \in P_i, i > 0$, as above, and $a \in A$. By Proposition 6.3 there are only finitely many pairwise non-equivalent L'-formulas of quantifier rank $i - 1$ of the form $\varphi(x, x_0, \ldots, x_{n-i-1})$ in FO. Let them be $\varphi_j(x, x_0, \ldots, x_{n-i-1})$, $j \in J$. Let

$$J_0 = \{j \in J : \mathcal{A}\restriction_{L'} \models \varphi_j(a, a_0, \ldots, a_{n-i-1})\}.$$

Let

$$\psi(x, x_0, \ldots, x_{n-i-1}) = \bigwedge_{j \in J_0} \varphi_j(x, x_0, \ldots, x_{n-i-1}) \wedge \bigwedge_{j \in J \setminus J_0} \neg\varphi_j(x, x_0, \ldots, x_{n-i-1}).$$

Now $\mathcal{A}\restriction_{L'} \models \exists x \psi(x, a_0, \ldots, a_{n-i-1})$, so as we have assumed $f \in P_i$, we have $\mathcal{B}\restriction_{L'} \models \exists x \psi(x, fa_0, \ldots, fa_{n-i-1})$. Thus there is some $b \in B$ with $\mathcal{B}\restriction_{L'} \models \psi(b, fa_0, \ldots, fa_{n-i-1})$. Now $f \cup \{(a, b)\} \in P_{i-1}$. The other condition (5.15) is proved similarly. □

The above proposition is the standard method for proving models elementary equivalent in FO. For example, Proposition 6.4 and Example 5.26 together give $(Z, <) \equiv (Z + Z, <)$. The exercises give more examples of partially isomorphic pairs – and hence elementary equivalent – structures. The restriction on function symbols can be circumvented by first using quantifiers to eliminate nesting of function symbols and then replacing the unnested equations $f(x_1, \ldots, x_{n-1}) = x_n$ by new predicate symbols $R(x_1, \ldots, x_n)$.

Let $\mathrm{Str}(L)$ denote the class of all L-structures. We can draw the following important conclusion from Proposition 6.4 (see Figure 6.1):

Corollary *Suppose L is a vocabulary without function symbols. Then for all $n \in \mathbb{N}$ the equivalence relation*

$$\mathcal{A} \equiv_n \mathcal{B}$$

divides $\mathrm{Str}(L)$ into finitely many equivalence classes C_i^n, $i = 1, \ldots, m_n$, such that for each C_i^n there is a sentence φ_i^n of FO with the properties:

1. *For all L-structures $\mathcal{A}$: $\mathcal{A} \in C_i^n \iff \mathcal{A} \models \varphi_i^n$.*
2. *If φ is an L-sentence of quantifier rank $\leq n$, then there are $i_1, \ldots, i_k$ such that $\models \varphi \leftrightarrow (\varphi_{i_1}^n \vee \ldots \vee \varphi_{i_k}^n)$.*

Proof Let φ_i^n be the conjunction of all the finitely many L-sentences of quantifier rank $\leq n$ that are true in some (every) model in C_i^n (to make the conjunction finite we do not repeat logically equivalent formulas). For the second claim, let $\varphi_{i_1}^n, \ldots, \varphi_{i_k}^n$ be the finite set of all L-sentences of quantifier rank $\leq n$ that are consistent with φ. If now $\mathcal{A} \models \varphi$, and $\mathcal{A} \in C_i^n$, then $\mathcal{A} \models \varphi_i^n$. On the other hand, if $\mathcal{A} \models \varphi_i^n$ and there is $\mathcal{B} \models \varphi_i^n$ such that $\mathcal{B} \models \varphi$, then $\mathcal{A} \equiv_n \mathcal{B}$, whence $\mathcal{A} \models \varphi$. □

We can actually read from the proof of Proposition 6.4 a more accurate description for the sentences φ_i. This leads to the theory of so-called *Scott formulas* (see Section 7.4).

Theorem 6.5 *Suppose K is a class of L-structures. Then the following are equivalent (see Figure 6.2):*

1. *K is FO-definable, i.e. there is an L-sentence φ of FO such that for all L-structures $\mathcal{M}$ we have $\mathcal{M} \in K \iff \mathcal{M} \models \varphi$.*
2. *There is $n \in \mathbb{N}$ such that K is closed under $\simeq_p^n$.*

As in the case of graphs, Theorem 6.5 can be used to demonstrate that certain properties of models are not definable in FO:

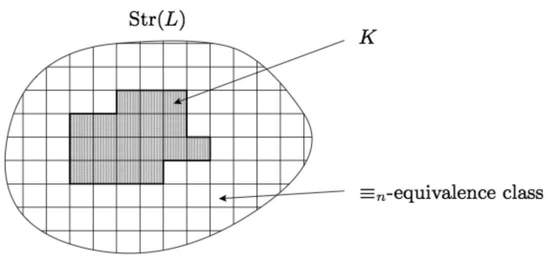

Figure 6.1 First-order definable model class K.

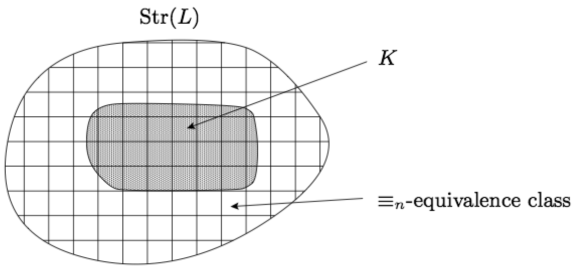

Figure 6.2 Not first-order definable model class K.

Example 6.6 Let $L = \emptyset$. The following properties of L-structures $\mathcal{M}$ are not expressible in FO:

1. M is infinite.
2. M is finite and even.

In both cases it is easy to find, for each $n \in \mathbb{N}$, two models $\mathcal{M}_n$ and $\mathcal{N}_n$ such that $\mathcal{M}_n \simeq_p^n \mathcal{N}_n$, $\mathcal{M}$ has the property, but $\mathcal{N}$ does not.

Example 6.7 Let $L = \{P\}$ be a unary vocabulary. The following properties of L-structures (M, A) are not expressible in FO:

1. $|A| = |M|$.
2. $|A| = |M \setminus A|$.

3. $|A| \leq |M \setminus A|$.

This is demonstrated by the models $(\mathbb{N}, \{1, \ldots, n\})$, $(\mathbb{N}, \mathbb{N} \setminus \{1, \ldots, n\})$, and $(\{1, \ldots, 2n\}, \{1, \ldots, n\})$.

Example 6.8 Let $L = \{<\}$ be a binary vocabulary. The following properties of L-structures $\mathcal{M} = (M, <)$ are not expressible in FO:

1. $\mathcal{M} \cong (\mathbb{Z}, <)$.
2. All closed intervals of $\mathcal{M}$ are finite.
3. Every bounded subset of $\mathcal{M}$ has a supremum.

This is demonstrated in the first two cases by the models $\mathcal{M}_n = (\mathbb{Z}, <)$ and $\mathcal{N}_n = (\mathbb{Z} + \mathbb{Z}, <)$ (see Example 5.26), and in the third case by the partially isomorphic models: $\mathcal{M} = (\mathbb{R}, <)$ and $\mathcal{N} = (\mathbb{R} \setminus \{0\}, <)$.

6.4 The Löwenheim–Skolem Theorem

In this section we show that if a first-order sentence φ is true in a structure $\mathcal{M}$, it is true in a countable substructure of $\mathcal{M}$, and even more, there are countable substructures of $\mathcal{M}$ in a sense "everywhere" satisfying φ. To make this statement precise we introduce a new game from Kueker (1977) called the Cub Game.

Definition 6.9 Suppose A is an arbitrary set. $\mathcal{P}_\omega(A)$ is defined as the set of all countable subsets of A.

The set $\mathcal{P}_\omega(A)$ is an auxiliary concept useful for the general investigation of countable substructures of a model with universe A. One should note that if A is infinite, the set $\mathcal{P}_\omega(A)$ is uncountable.[1] For example, $|\mathcal{P}_\omega(\mathbb{N})| = |\mathbb{R}|$. The set $\mathcal{P}_\omega(A)$ is closed under intersections and countable unions but not necessarily under complements, so it is a (distributive) lattice under the partial order $\subseteq$, but not a Boolean algebra. The sets in $\mathcal{P}_\omega(A)$ cover the set A entirely, but so do many proper subsets of $\mathcal{P}_\omega(A)$ such as the set of all singletons in $\mathcal{P}_\omega(A)$ and the set of all finite sets in $\mathcal{P}_\omega(A)$.

Definition 6.10 Suppose A is an arbitrary set and $\mathcal{C}$ a subset of $\mathcal{P}_\omega(A)$. The *Cub Game of* $\mathcal{C}$ is the game $G_{\text{cub}}(\mathcal{C}) = G_\omega(A, W)$, where W consists of sequences $(a_1, a_2, \ldots)$ with the property that $\{a_1, a_2, \ldots\} \in \mathcal{C}$.

[1] Its cardinality is $|A|^\omega$.

Figure 7.2

Proof Let **I** play first $x_0 \in \mathcal{G}'$ which is not a torsion element. The response $y_o \in \mathcal{G}$ of **II** is a torsion element, so if we use algebraic notation, we have $z_1, \ldots, z_n$ such that

$$\begin{aligned} y_0 + y_0 &= z_1. \\ z_1 + y_0 &= z_2. \\ &\vdots \\ z_n + y_0 &= 0. \end{aligned}$$

Now **I** declares there are $n+2$ moves left, and plays $x_i = z_i$ for $i = 1, \ldots, n$. Let the responses of **II** be $y_1, \ldots, y_n$. Next **I** plays $x_{n+1} = 0_{\mathcal{G}}$, and **II** plays $y_{n+1} \in \mathcal{G}'$. Since $x_0 \in \mathcal{G}'$ is not a torsion element, **II** cannot have played $y_{n+1} = 0_{\mathcal{G}'}$ or else she loses. So there is x_{n+2} in $\mathcal{G}'$ with $x_{n+2} + y_{n+1} \neq x_{n+2}$. Now finally **I** plays this x_{n+2}, and **II** plays y_{n+2}. As $y_{n+2} + x_{n+1} = y_{n+2}$, **II** has now lost.

□

7.3 The Dynamic Ehrenfeucht–Fraïssé Game

From $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$ we could go on to define a game $\mathrm{EFD}_{\omega+\omega}(\mathcal{M}, \mathcal{M}')$ in which player **I** starts by choosing a natural number n and declaring that we are going to play the game $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$. But what is the general form of such games? We can have a situation where player **I** wants to decide that after n_0 moves he decides how many moves are left. At that point he decides that after n_1 moves he will decide how many moves now are left. At that point he decides that after n_2 moves he ... until finally he decides that the game lasts n_k more moves. A natural way of making this decision process of player **I** exact is to say that player **I** moves down an ordinal. For example, if he moves down the ordinal $\omega + \omega + 1$, he can move as in Figure 7.2.

So first he wants n_0 moves and after they have been played he decides on n_1. If he moves down on the ordinal $\omega \cdot \omega + 1$, he first chooses k and wants

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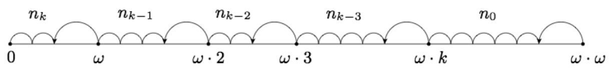

Figure 7.3

n_0 moves and after they have been played he can still make k changes of mind about the length of the rest of the game (see Figure 7.3).

Definition 7.13 Let L be a relational vocabulary and $\mathcal{M}, \mathcal{M}'$ L-structures such that $\mathcal{M} \cap \mathcal{M}' = \emptyset$. Let α be an ordinal. The Dynamic Ehrenfeucht–Fraïssé Game $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is the game $G_\omega(M \cup M' \cup \alpha, W_{\omega,\alpha}(\mathcal{M}, \mathcal{M}'))$, where $W_{\omega,\alpha}(\mathcal{M}, \mathcal{M}')$ is the set of

$$p = (x_0, \alpha_0, y_0, \ldots, x_{n-1}, \alpha_{n-1}, y_{n-1})$$

such that

(D1) For all $i < n : x_i \in M \leftrightarrow y_i \in M'$.
(D2) $\alpha > \alpha_0 > \ldots > \alpha_{n-1} = 0$.
(D3) If we denote

$$v_i = \begin{cases} x_i \text{ if } x_i \in M \\ y_i \text{ if } y_i \in M \end{cases} \quad \text{and } v_i' = \begin{cases} x_i \text{ if } x_i \in M' \\ y_i \text{ if } y_i \in M' \end{cases}$$

then

$$f_p = \{(v_0, v_0'), \cdots, (v_{n-1}, v_{n-1}')\}$$

is a partial isomorphism $\mathcal{M} \to \mathcal{M}'$.

Note that $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is *not* a game of length α. Every play in the game $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is finite, it is just how the length of the game is determined during the game where the ordinal α is used. Compared to $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, the only new feature in $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is condition (D2). Thus $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is more difficult for **I** to play than $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, but – if $\alpha \geq \omega$ – easier than any $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$.

Lemma 7.14 ***(1)*** *If* **II** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *and* $\beta \leq \alpha$, *then* **II** *has a winning strategy in* $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$.

(2) *If* **I** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *and* $\alpha \leq \beta$, *then* **I** *has a winning strategy in* $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$.

Proof (1) Any move of **I** in $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$ is as it is a legal move of **I** in $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$. Thus if **II** can beat **I** in EFD_α she can beat him in EFD_β.

(2) If **I** knows how to beat **II** in EFD_α, he can use the very same moves to beat **II** in EFD_β. □

Lemma 7.15 *If α is a limit ordinal $\neq 0$ and* **II** *has a winning strategy in the game $\mathrm{EFD}_\beta(\mathcal{M}, \mathcal{M}')$ for each $\beta < \alpha$, then* **II** *has a winning strategy in the game $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$.*

Proof In his opening move **I** plays $\alpha_0 < \alpha$. Now **II** can pretend we are actually playing the game $\mathrm{EFD}_{\alpha_0+1}(\mathcal{M}, \mathcal{M}')$. And she has a winning strategy for that game! □

Back-and-forth sequences are a way of representing a winning strategy of player **II** in the game EFD_α.

Definition 7.16 A *back-and-forth sequence* $(P_\beta : \beta \leq \alpha)$ is defined by the conditions

$$\emptyset \neq P_\alpha \subseteq \ldots \subseteq P_0 \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B}) \tag{7.1}$$

$$\forall f \in P_{\beta+1} \forall a \in A \exists b \in B \exists g \in P_\beta (f \cup \{(a, b)\} \subseteq g) \text{ for } \beta < \alpha \tag{7.2}$$

$$\forall f \in P_{\beta+1} \forall b \in B \exists a \in A \exists g \in P_\beta (f \cup \{(a, b)\} \subseteq g) \text{ for } \beta < \alpha. \tag{7.3}$$

We write

$$\mathcal{A} \simeq_p^\alpha \mathcal{B}$$

if there is a back-and-forth sequence of length α for $\mathcal{A}$ and $\mathcal{B}$.

The following proposition shows that back-and-forth sequences indeed capture the winning strategies of player **II** in $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$:

Proposition 7.17 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are two L-structures. The following are equivalent:*

1. $\mathcal{A} \cong_p^\alpha \mathcal{B}$.
2. **II** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$.

Proof Let us assume $A \cap B = \emptyset$. Let $(P_i : i \leq \alpha)$ be a back-and-forth sequence for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i \in \mathbb{N})$ for **II**. Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \alpha_0, \ldots, x_{j-1}, \alpha_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P_{\alpha_i}$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_i)$. Now player **I** plays x_j and $\alpha_j < \alpha_{j-1}$. Note that $f_{j-1} \in P_{\alpha_j+1}$. By assumption there is $f_j \in P_{\alpha_j}$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$ and if $x_j \in B$, then $x_j \in \mathrm{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$, and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_n : n \in \mathbb{N})$ is a winning strategy of **II**. Let Q consist of all plays of $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P_β consist of all possible f_p where $p = (x_0, \alpha_0, y_0, \ldots, x_{i-1}, \alpha_{i-1}, y_{i-1})$ is a position in the game $\mathrm{EFD}_\alpha(\mathcal{A}, \mathcal{B})$ with an extension in Q and $\alpha_{i-1} \geq \beta$. It is clear that $(P_\beta : \beta \leq \alpha)$ has the properties (7.1) and (7.2). □

We have already learnt in Lemma 7.14 that the bigger the ordinal α in $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ is, the harder it is for player **II** to win and eventually, in a typical case, her luck turns and player **I** starts to win. From that point on it is easier for **I** to win the bigger α is. Lemma 7.15, combined with the fact that the game is determined, tells us that there is a first ordinal where player **I** starts to win. So all the excitement concentrates around just one ordinal up to which player **II** has a winning strategy and starting from which player **I** has a winning strategy. It is clear that this ordinal tells us something important about the two models. This motivates the following:

Definition 7.18 An ordinal α such that player **II** has a winning strategy in $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ and player **I** has a winning strategy in $\mathrm{EFD}_{\alpha+1}(\mathcal{M}, \mathcal{M}')$ is called the *Scott watershed of $\mathcal{M}$ and $\mathcal{M}'$*.

By Lemma 7.14 the Scott watershed is uniquely determined, if it exists. In two extreme cases the Scott watershed does not exist. First, maybe **I** has a winning strategy even in $\mathrm{EF}_0(\mathcal{M}, \mathcal{M}')$. Here $\mathrm{Part}(\mathcal{M}, \mathcal{M}') = \emptyset$. Secondly, player **II** may have a winning strategy even in $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, so **I** has no chance in any $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$, and there is no Scott watershed. In any other case the Scott watershed exists. The bigger it is, the closer $\mathcal{M}$ and $\mathcal{M}'$ are to being isomorphic. Respectively, the smaller it is, the farther $\mathcal{M}$ and $\mathcal{M}'$ are from being isomorphic. If the watershed is so small that it is finite, the structures $\mathcal{M}$ and $\mathcal{M}'$ are not even elementary equivalent.

General problem: Given $\mathcal{M}$ and $\mathcal{M}'$, find the Scott watershed!

How far afield do we have to go to find the Scott watershed? It is very natural to try first some small ordinals. But if we try big ordinals, it would be nice to know how high we have to go. There is a simple answer given by the next proposition: If the models have infinite cardinality κ, and the Scott watershed exists, then it is $< \kappa^+$. Thus for countable models we only need to check

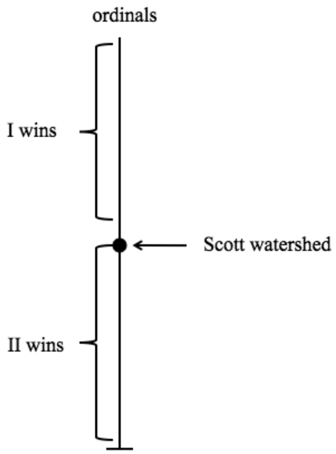

Figure 7.4

countable ordinals. For finite models this is not very interesting: if the models have at most n elements, and there is a watershed, then it is at most n.

Proposition 7.19 *If* **II** *has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *for all* $\alpha < (|M| + |M'|)^+$ *then* **II** *has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$.

Proof Let $\kappa = |M| + |M'|$. The idea of **II** is to make sure that

($\star$) If the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ has reached a position

$$p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \text{ with } f_p = \{(v_0, v_0'), \ldots, (v_{n-1}, v_{n-1}')\}$$

then **II** has a winning strategy in

$$\mathrm{EFD}_{\alpha+1}((\mathcal{M}, v_0, \ldots, v_{n-1}), (\mathcal{M}', v_0', \ldots, v_{n-1}')) \tag{7.4}$$

for all $\alpha < \kappa^+$.

In the beginning $n = 0$ and condition ($\star$) holds. Let us suppose **II** has been able to maintain ($\star$) and then **I** plays x_n in $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$. Let us look at the possibilities of **II**: She has to play some y_n and there are $\leq \kappa$ possibilities. Let Ψ be the set of them. Assume none of them works. Then for each legal move y_n there is $\alpha_{y_n} < \kappa^+$ such that

Incomplete version for students of easllc2012 only.

(1) **II** does not have a winning strategy in

$$\mathrm{EFD}_{\alpha_{y_n}}((\mathcal{M}, v_0, \ldots, v_n), (\mathcal{M}', v_0', \ldots, v_n'))$$

where

$$v_n = \begin{cases} x_n \text{ if } x_n \in M \\ y_n \text{ if } x_n \in M' \end{cases} \text{ and } v_n' = \begin{cases} y_n \text{ if } y_n \in M' \\ x_n \text{ if } y_n \in M. \end{cases}$$

Let $\alpha = \sup_{y_n \in \Psi} \alpha_{y_n}$. As $|\Psi| \leq \kappa$, we have $\alpha < \kappa^+$. By the induction hypothesis, **II** has a winning strategy in the game (7.4). So, let us play this game. We let **I** play x_n and α. The winning strategy of **II** gives $y_n \in \Psi$. Let v_n and v_n' be determined as above. Now

(2) **II** has a winning strategy in $\mathrm{EFD}_\alpha((\mathcal{M}, v_0, \ldots, v_n), (\mathcal{M}', v_0', \ldots, v_n'))$.

We have a contradiction between (1), (2), $\alpha_{y_n} < \alpha$ and Lemma 7.14. □

The above theorem is particularly important for countable models since countable partially isomorphic structures are isomorphic. Thus the countable ordinals provide a complete hierarchy of thresholds all the way from not being even elementary equivalent to being actually isomorphic. For uncountable models the hierarchy of thresholds reaches only to partial isomorphism which may be far from actual isomorphism.

We list here two structural properties of $\simeq_p^\alpha$, which are very easy to prove. There are many others and we will meet them later.

Lemma 7.20 *(Transitivity) If* $\mathcal{M} \simeq_p^\alpha \mathcal{M}'$ *and* $\mathcal{M}' \simeq_p^\alpha \mathcal{M}''$, *then* $\mathcal{M} \simeq_p^\alpha \mathcal{M}''$.

Proof Exercise 7.14. □

Lemma 7.21 *(Projection) If* $\mathcal{M} \simeq_p^\alpha \mathcal{M}'$, *then* $\mathcal{M} \restriction L \simeq_p^\alpha \mathcal{M}' \restriction L$.

Proof Exercise 7.15. □

We shall now introduce one of the most important concepts in infinitary logic, namely that of a Scott height of a structure. It is an invariant which sheds light on numerous aspects of the model.

Definition 7.22 The *Scott height* $\mathrm{SH}(\mathcal{M})$ of a model $\mathcal{M}$ is the supremum of all ordinals $\alpha + 1$, where α is the Scott watershed of a pair

$$(\mathcal{M}, a_1, \ldots, a_n) \not\simeq_p (\mathcal{M}, b_1, \ldots, b_n)$$

and $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$.

Lemma 7.23 $\mathrm{SH}(\mathcal{M})$ *is the least* α *such that if* $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$ *and*

$$(\mathcal{M}, a_1, \ldots, a_n) \simeq_p^{\alpha} (\mathcal{M}, b_1, \ldots, b_n)$$

then

$$(\mathcal{M}, a_1, \ldots, a_n) \simeq_p^{\alpha+1} (\mathcal{M}, b_1, \ldots, b_n).$$

Proof Exercise 7.16. □

Theorem 7.24 *If* $\mathcal{M} \simeq_p^{\mathrm{SH}(\mathcal{M})+\omega} \mathcal{M}'$*, then* $\mathcal{M} \simeq_p \mathcal{M}'$*.*

Proof Let $\mathrm{SH}(\mathcal{M}) = \alpha$. The strategy of **II** in $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is to make sure that if the position is

$$(1) \qquad p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

then

$$(2) \qquad (\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\alpha} (\mathcal{M}', v'_0, \ldots, v'_{n-1}).$$

In the beginning of the game (2) holds by assumption. Let us then assume we are in the middle of the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$, say in position p, and (2) holds. Now player **I** moves x_n, say $x_n = v_n \in M$. We want to find a move $y_n = v'_n \in M'$ of **II** which would yield

$$(3) \qquad (\mathcal{M}, v_0, \ldots, v_n) \simeq_p^{\alpha} (\mathcal{M}', v'_0, \ldots, v'_n).$$

Now we use the assumption $\mathcal{M} \simeq_p^{\alpha+\omega} \mathcal{M}'$. We play a sequence of rounds of an auxiliary game $G = \mathrm{EFD}_{\alpha+n+1}(\mathcal{M}, \mathcal{M}')$ in which player **II** has a winning strategy τ. First player **I** moves the elements $v'_0, \ldots, v'_{n-1}$. Let the responses of player **II** according to τ be $u_0, \ldots, u_{n-1}$. We get

$$(4) \qquad (\mathcal{M}', v'_0, \ldots, v'_{n-1}) \simeq_p^{\alpha+1} (\mathcal{M}, u_0, \ldots, u_{n-1}).$$

By transitivity,

$$(\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\alpha} (\mathcal{M}, u_0, \ldots, u_{n-1}).$$

See Figure 7.5.

By Lemma 7.23,

$$(\mathcal{M}, v_0, \ldots, v_{n-1}) \simeq_p^{\alpha+1} (\mathcal{M}, u_0, \ldots, u_{n-1}).$$

Now we apply the definition of $\simeq_p^{\alpha+1}$ and find $a \in M$ such that

$$(5) \qquad (\mathcal{M}, v_0, \ldots, v_{n-1}, v_n) \simeq_p^{\alpha} (\mathcal{M}, u_0, \ldots, u_{n-1}, a).$$

Finally we play one more round of the auxiliary game G using (4) so that

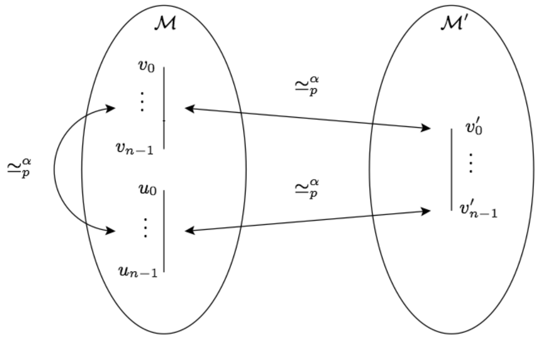

Figure 7.5

player **I** moves $a \in M$ and **II** moves according to τ an element $y_n = v'_n \in M'$. Again

$$(\mathcal{M}', v'_0, \ldots, v'_{n-1}, v'_n) \simeq_p^{\alpha} (\mathcal{M}, u_0, \ldots, u_{n-1}, a),$$

which together with (5) gives (3). □

Note, that for countable models we obtain the interesting corollary:

Corollary *If* $\mathcal{M}$ *is countable, then for any other countable* $\mathcal{M}'$ *we have*

$$\mathcal{M} \simeq_p^{\mathrm{SH}(\mathcal{M})+\omega} \mathcal{M}' \iff \mathcal{M} \cong \mathcal{M}'.$$

The *Scott spectrum* $\mathrm{ss}(T)$ of a first-order theory is the class of Scott heights of its models:

$$\mathrm{ss}(T) = \{\mathrm{SH}(\mathcal{M}) : \mathcal{M} \models T\}.$$

It is in general quite difficult to determine what the Scott spectrum of a given theory is. For some theories the Scott spectrum is bounded from above. An extreme case is the case of the empty vocabulary, where the Scott height of any model is zero. It follows from Example 7.29 below that the Scott spectrum of the theory of linear order is unbounded in the class of all ordinals. A *gap* in a Scott spectrum $\mathrm{ss}(T)$ is an ordinal which is missing from $\mathrm{ss}(T)$.

Vaught's Conjecture: If T is a countable first-order theory, then T has, up to isomorphism, either $\leq \aleph_0$ or exactly $2^{\aleph_0}$ countable models.