Lecture 3

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Course plan

- Monday: Games, models
- Tuesday: Finite and infinite EF game.
- Wednesday: Semantic game.
- Thursday: Model existence game.
- Friday: Dependence logic.

The infinite game

We call v_i *and* v'_i *above* corresponding *elements. The infinite game* $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ *is defined quite similarly, that is, it is the game* $G_\omega(M \cup M', W_\omega(\mathcal{M}, \mathcal{M}'))$, *where* $W_\omega(\mathcal{M}, \mathcal{M}')$ *is the set of* $p = (x_0, y_0, x_1, y_1, \ldots)$ *such that for all* $n \in \mathbb{N}$ *we have* $(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W_n(\mathcal{M}, \mathcal{M}')$.

Note that the game EF_ω is a closed game.

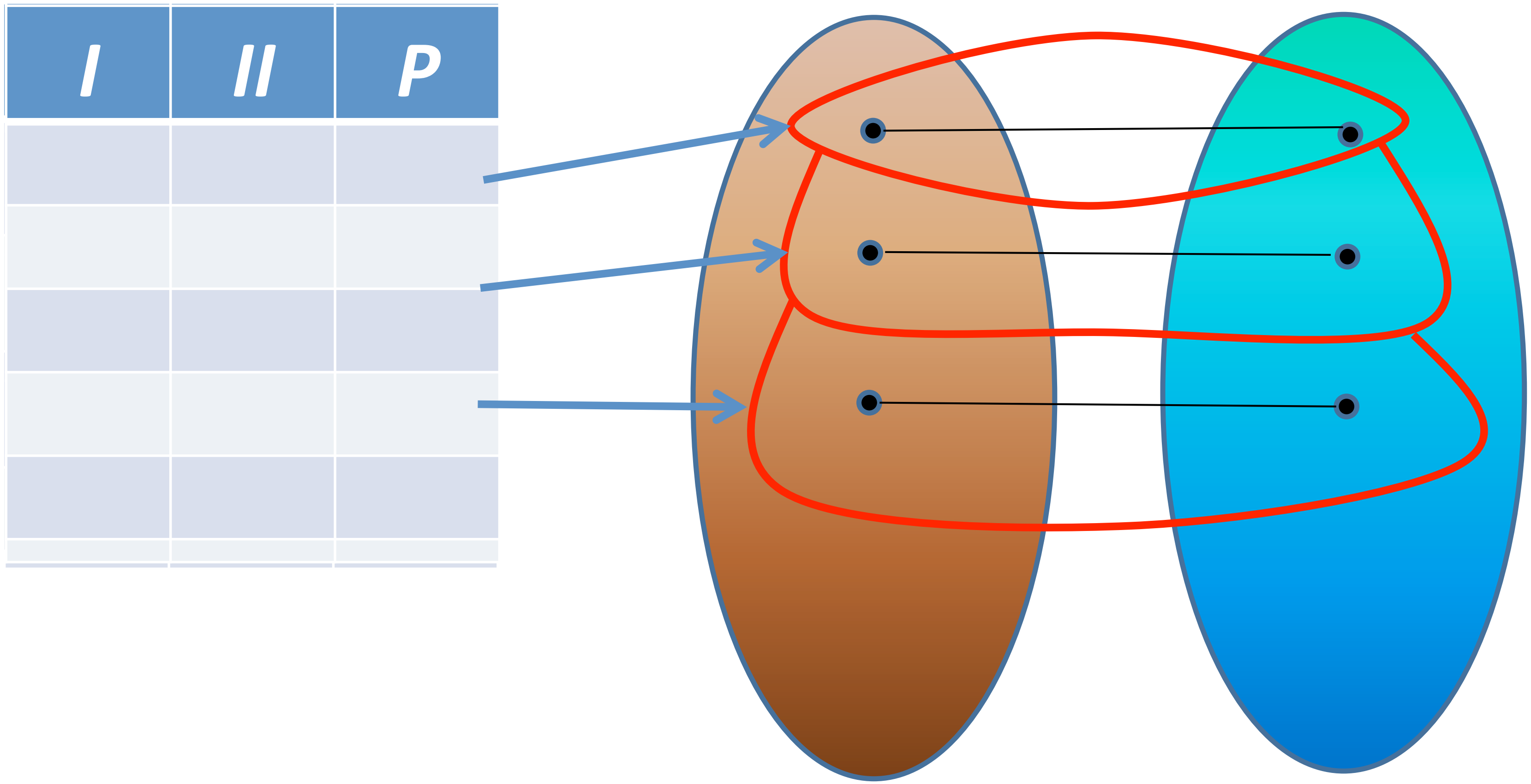
Back-and-forth set = winning strategy of II

Proposition 4.4.1. *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are two L-structures. The following conditions are equivalent:*

i.e. there is a back-and-forth set for A and B

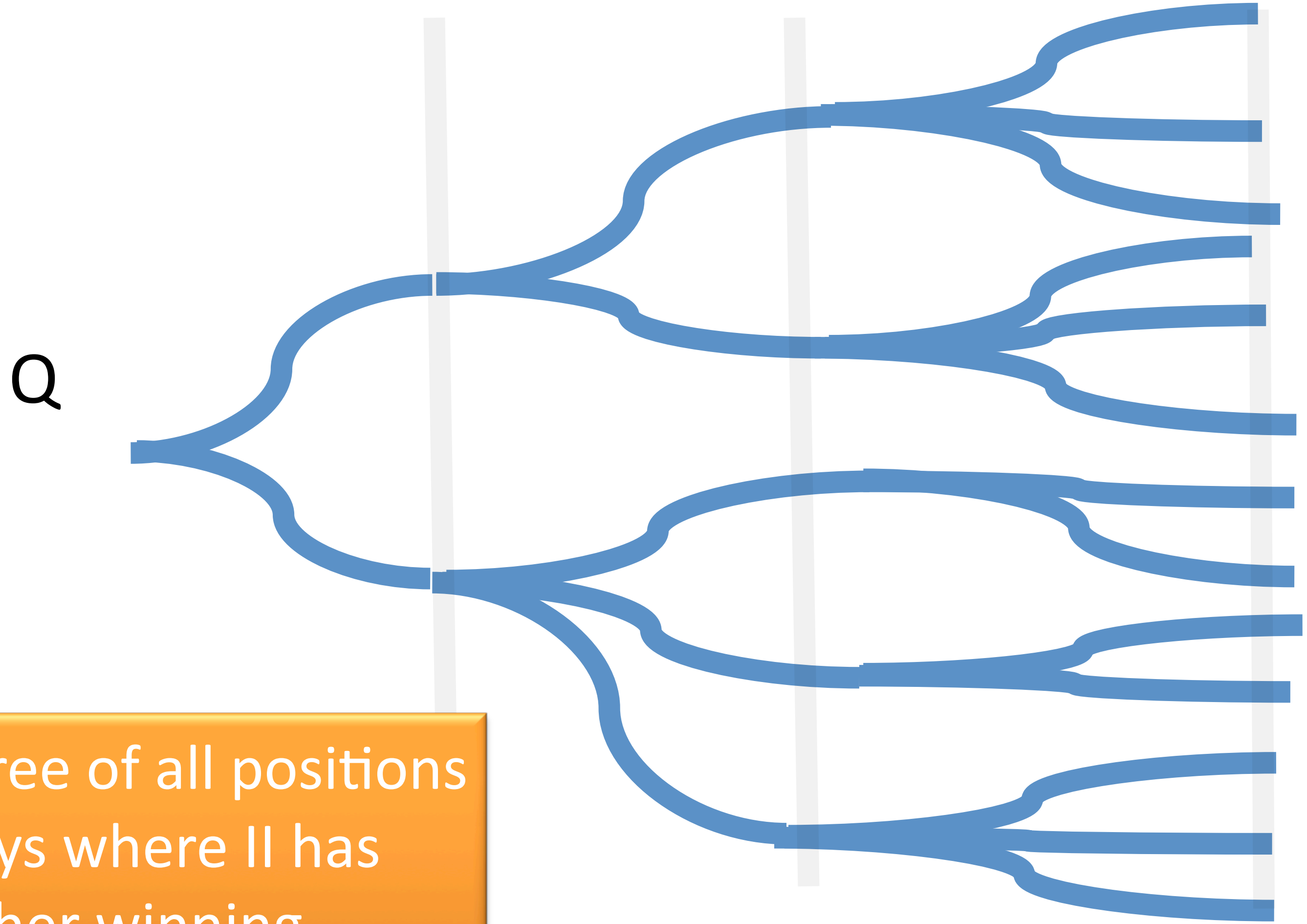
1. $\mathcal{A} \cong_p \mathcal{B}$
2. *II has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

From a b-a-f set to a strategy



The strategy of II is to play so that after each move she knows, and remembers, a big enough function in P.

From a strategy to a b-a-f set



The tree of all positions in plays where II has used her winning strategy

From a strategy to a b-a-f set

Suppose II has a winning strategy τ.

We form the back-and-forth set from all functions of the form

$$f_p = \{(v_0, v'_0), \ldots, (v_{n-1}, v'_{n-1})\}$$

where

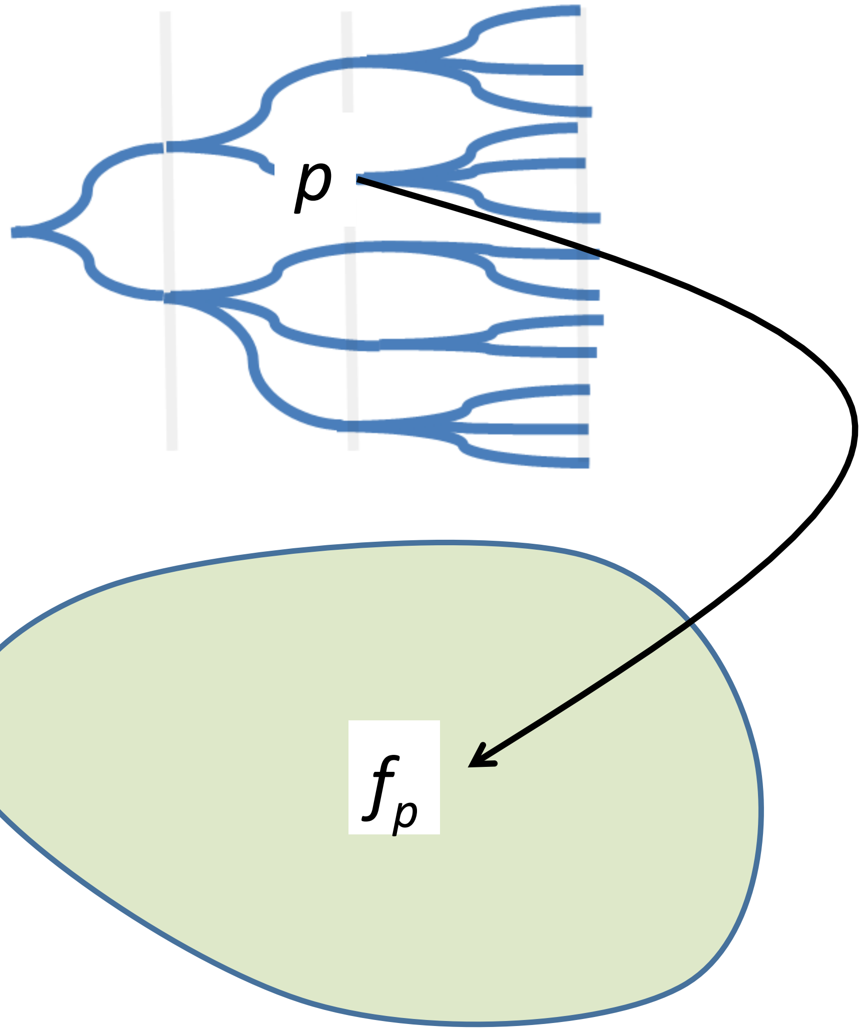
$$p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

is a position in a play where II has used her winning strategy τ.

See slides 7,8

The back-and-forth set as a projection

From a tree



to a po set

Semantic game: Is φ true in $\mathcal{M}$?

Max

Susan

Players hold a formula, one player at a time. Each thinks that if he or she holds the formula, it is true.

To account for free variables, they actually hold a pair (φ,s), where s is an assignment.

Beginning of the game

Max

Susan

(φ, s)

In the beginning *s* is
the empty assignment.

Disjunction move

Max

Susan

$(\vee_i \varphi_i, s)$ $(\vee_i \varphi_i, s)$

(φ_i, s) (φ_i, s)

Conjunction move

Max

Susan

$(\wedge_i \varphi_i, s)$

$(\wedge_i \varphi_i, s)$

(φ_i, s)

(φ_i, s)

Existential quantifier move

Max

Susan

$(\exists x\varphi, s)$

$(\exists x\varphi, s)$

$(\varphi, s(a/x))$

$(\varphi, s(a/x))$

Universal quantifier move

Max

Susan

($\forall x \varphi$,s)

($\forall x \varphi$,s)

(φ,s(a/x))

(φ,s(a/x))

Negation move

Max

Susan

$(\neg\varphi,s)$ → (φ,s)

(φ,s) ← $(\neg\varphi,s)$

End of the game, φ atomic

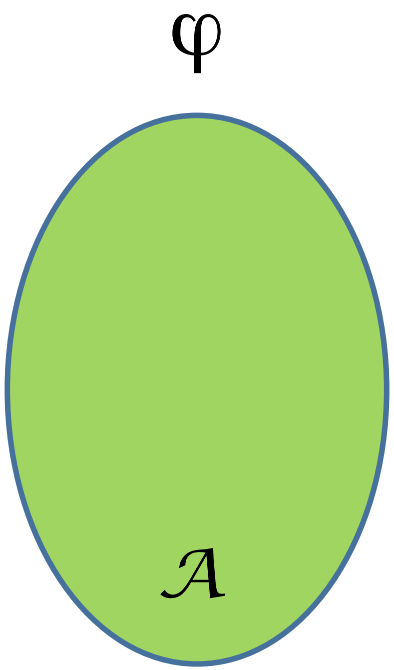
Max

Susan

(φ, s) true false

(φ, s) true false

Game theoretic semantics



If φ is true in $\mathcal{M}$ then Susan has a winning strategy

- Susan holds only true formulas and makes sure Max gets to hold only false ones

If φ is false in $\mathcal{M}$ then Max has a winning strategy

- Max makes sure Susan gets to hold only false ones, while he himself holds only true formulas

Both implications are equivalences!

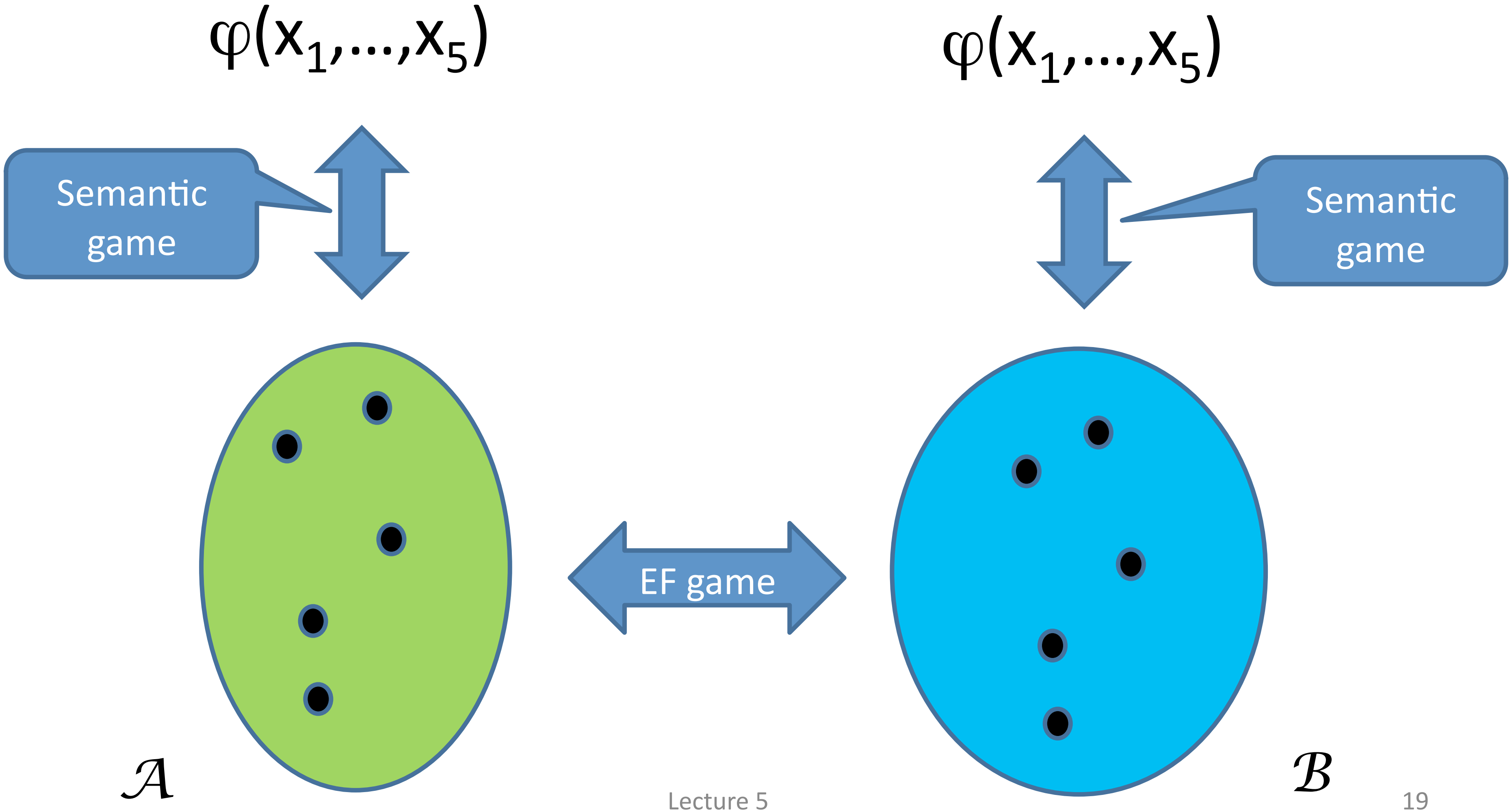
Theorem

The following are equivalent:

1. Models *A* and *B* satisfy the same formulas of quantifier rank at most k.
2. Player II has a winning strategy in the k-move EF-game on *A* and *B*.

Asynchronous parallel games

(Strategic Balance of Logic)



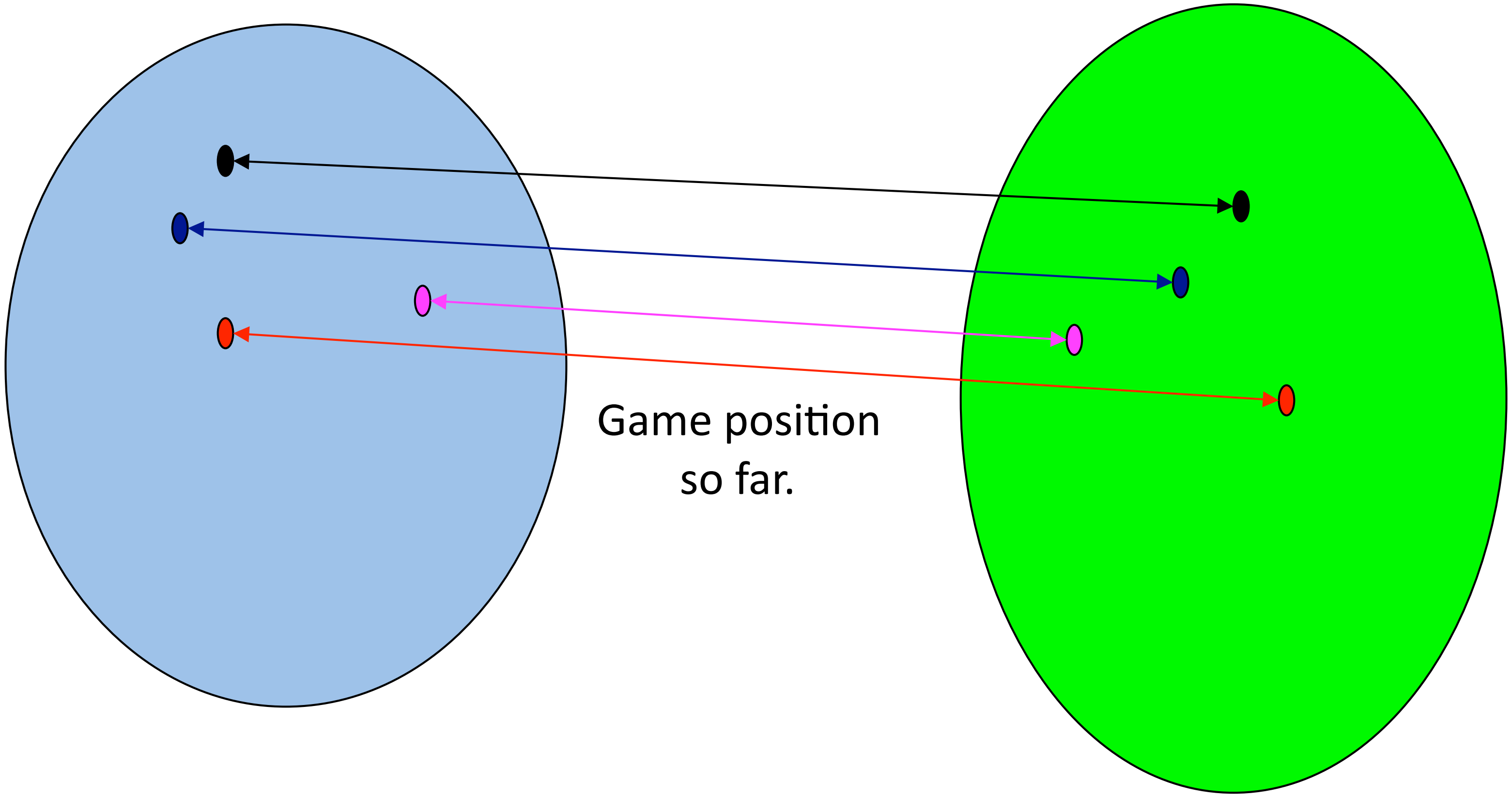
Asynchronous parallel games
(Strategic Balance of Logic)

Susan: $(\varphi \wedge \psi, s)$

Susan: (φ, s)

Susan: $(\varphi \wedge \psi, s')$

Susan: (φ, s')



Game position
so far.

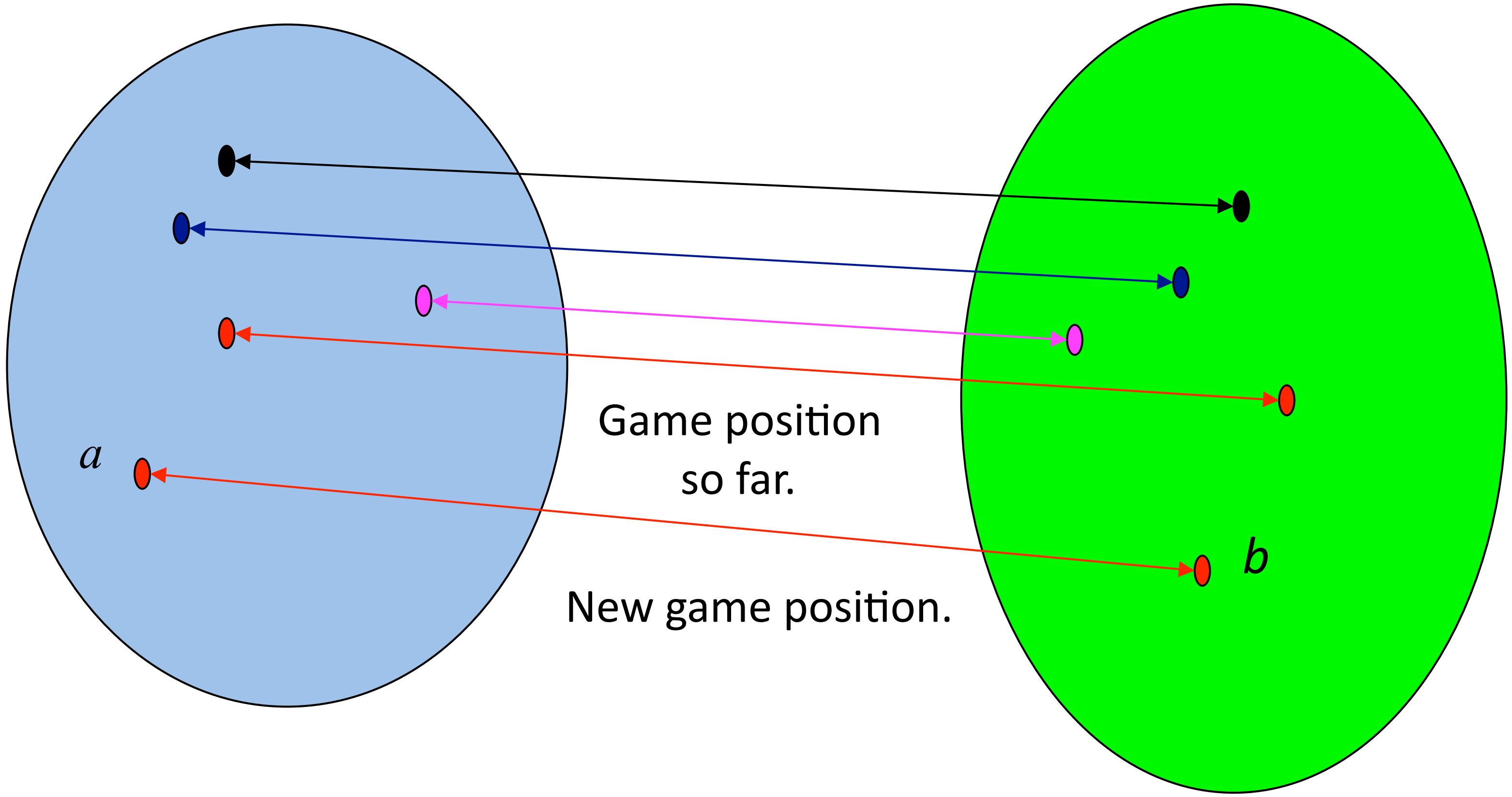
Asynchronous parallel games
(Strategic Balance of Logic)

Susan: $(\exists x_5\varphi(x_1,\ldots,x_5),s)$

Susan: $(\varphi(x_1,\ldots,x_5),s(a/x_5))$

Susan: $(\exists x_5\varphi(x_1,\ldots,x_5),s')$

Susan: $(\varphi(x_1,\ldots,x_5),s'(b/x_5))$



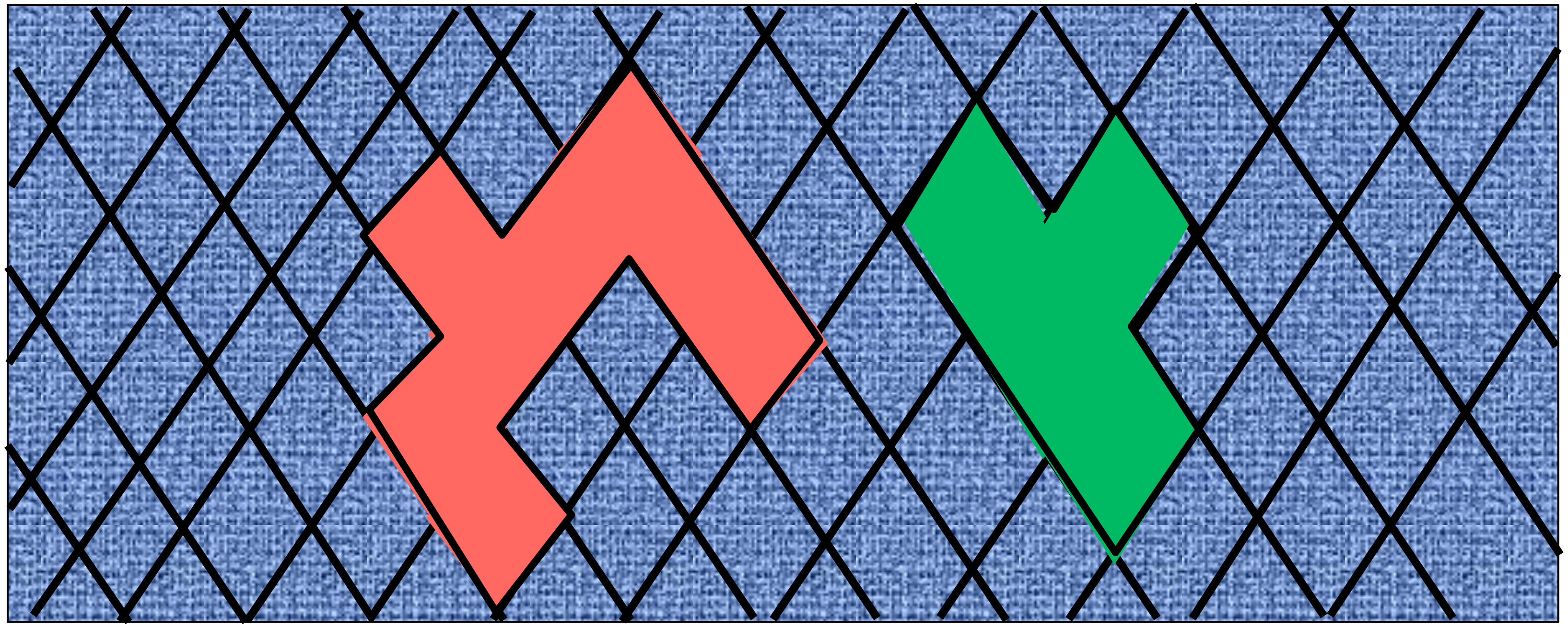
Application

$$\langle Z, <\rangle \equiv \langle Z + Z, <\rangle$$

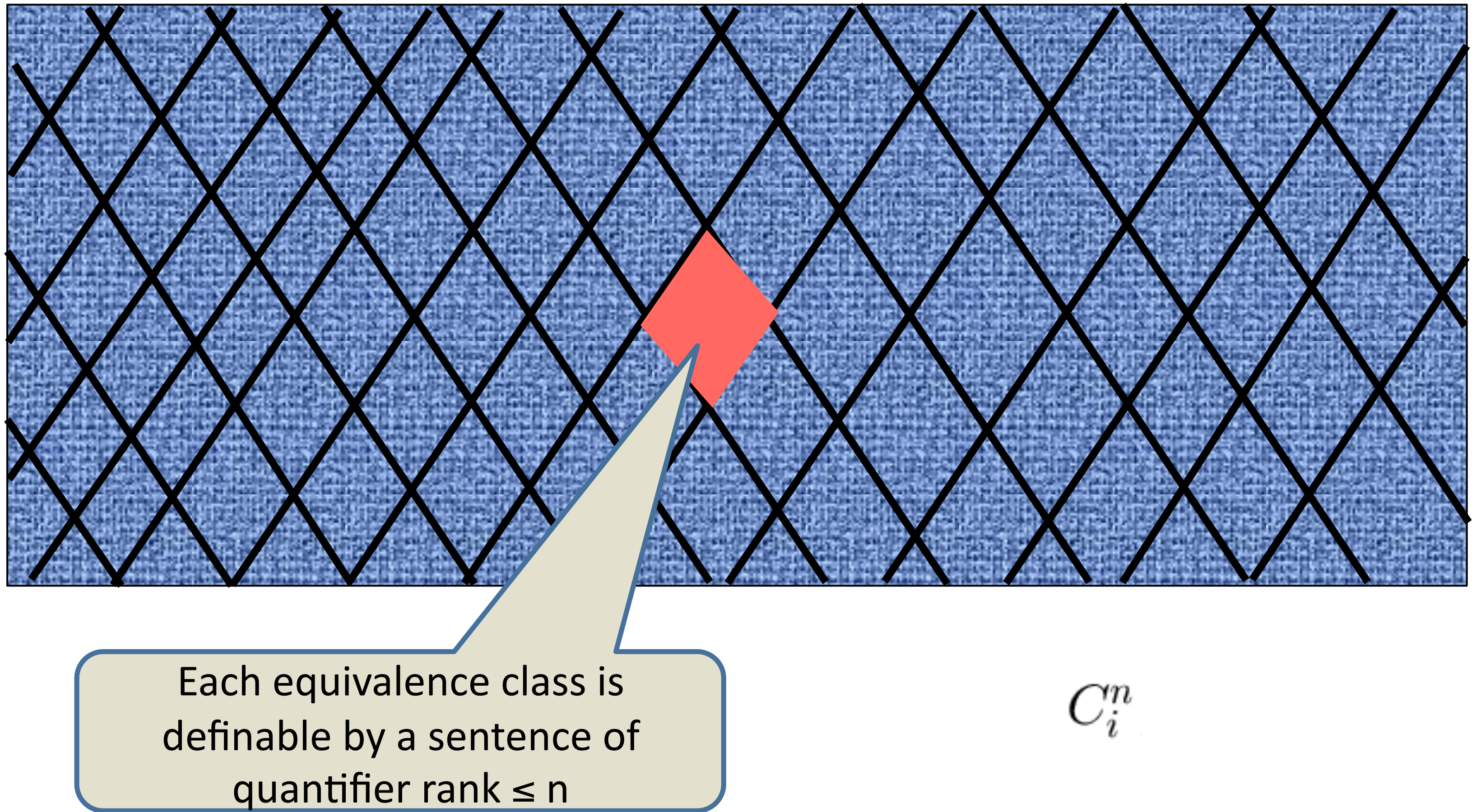
Definable model class

- A model class K is defined by the sentence φ if K is the class of models of φ.
- A model class is (first order) definable if it is defined by some first order sentence φ.

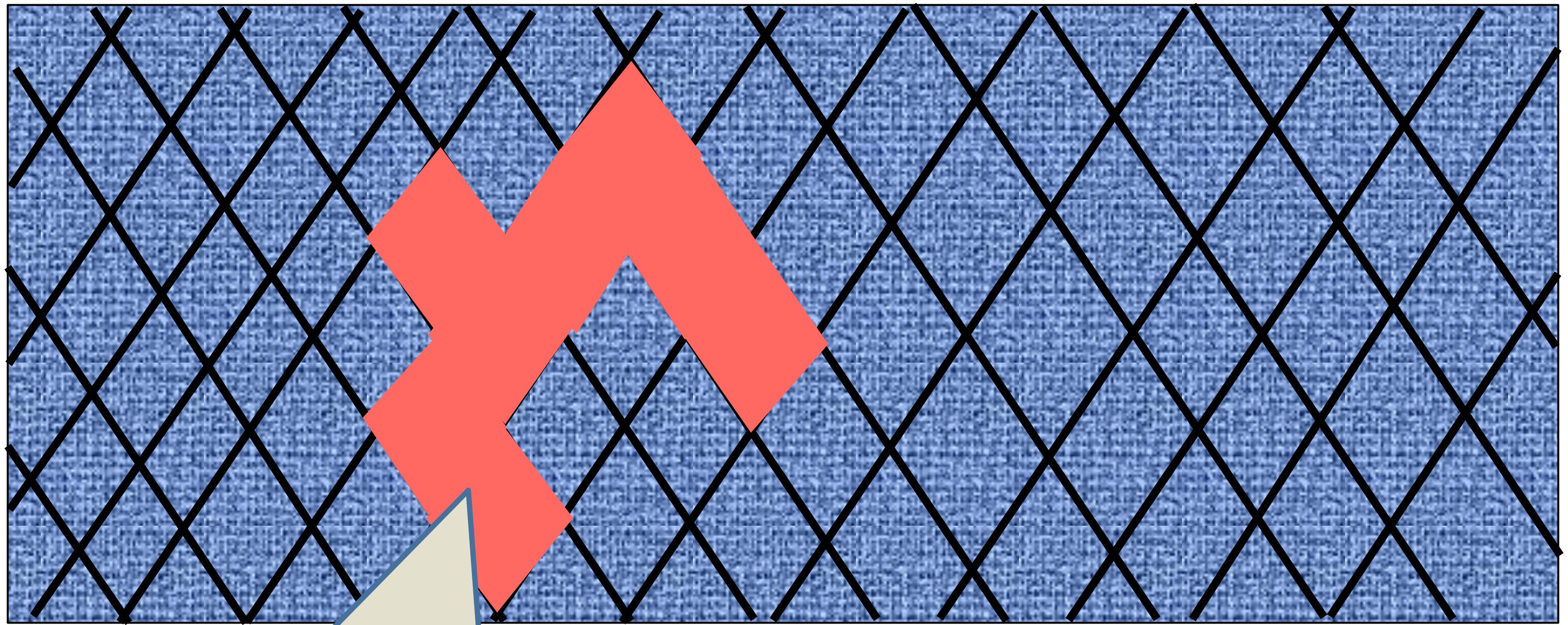
Definable model classes



$\equiv_n$ - classes



$\equiv_n$ - classes



Every model class which is definable by a sentence of quantifier rank ≤ n, is a union of equivalence classes

A model class is definable in first order logic

if and only if

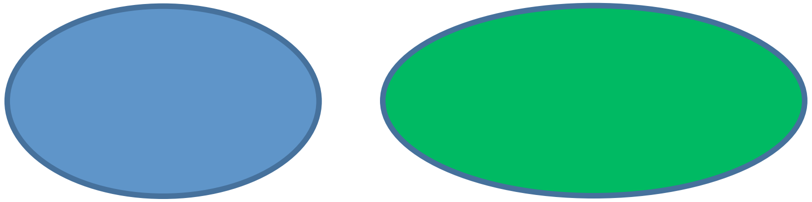
For some *n*: if *A* is in the class and *B* is not in the class, then I has a winning strategy in $EF_n(A,B)$.

Not first order definable

$L = \emptyset$

M is infinite.

M is finite and even.



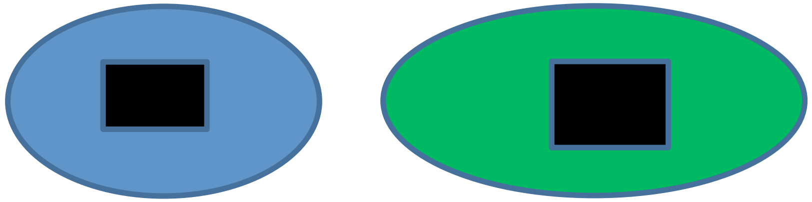
$L = \{P\}$

(M, A)

$|A| = |M|$.

$|A| = |M \setminus A|$.

$|A| \leq |M \setminus A|$.



$L = \{<\}$

$\mathcal{M} \cong (\mathbb{Z}, <)$.

All closed intervals of $\mathcal{M}$ are finite.

Every bounded subset of $\mathcal{M}$ has a supremum.

Dynamic game

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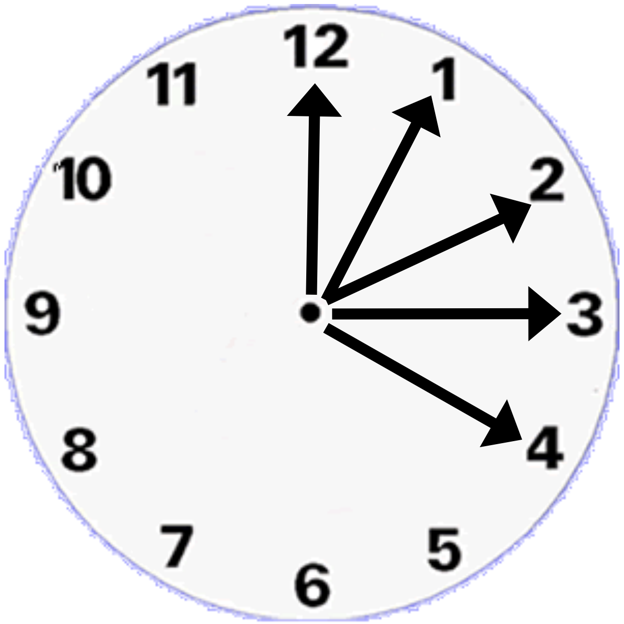
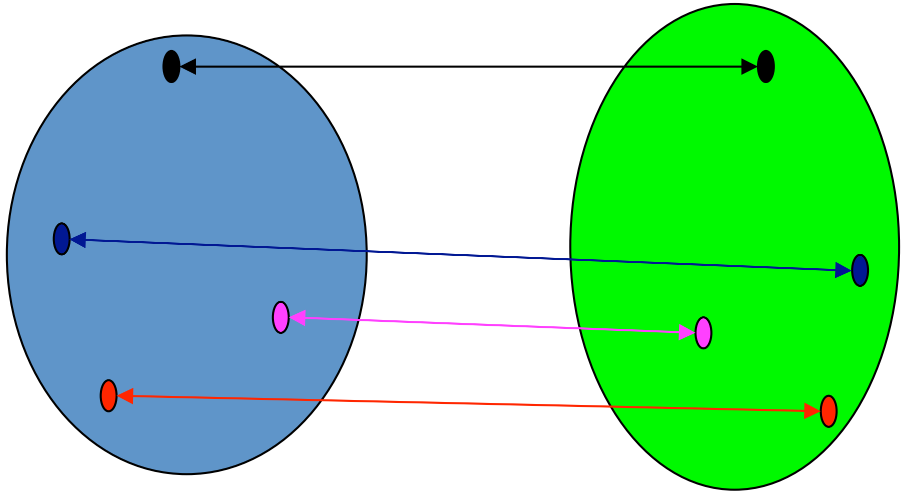
Player I has a clock and controls the length of the game

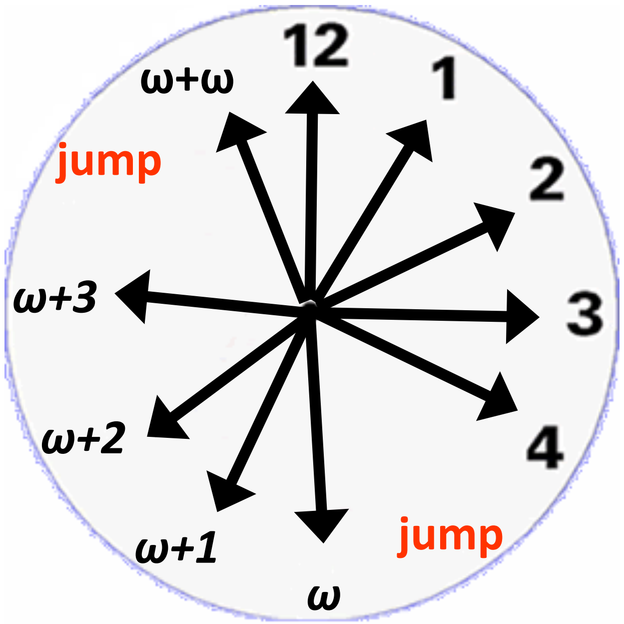
Clock for $EF_4(A,B)$





Game clock is a way of making sense of
finite but potentially infinite games

A finite but potentially infinite game



"Midnight" = ω+ω+1

The decision pattern of Player I can be quite complicated

$$\omega^{\omega^{3}} + \omega \cdot 5 + 1$$

Still the game is always finite!

The Dynamic Ehrenfeucht-Fraïssé game

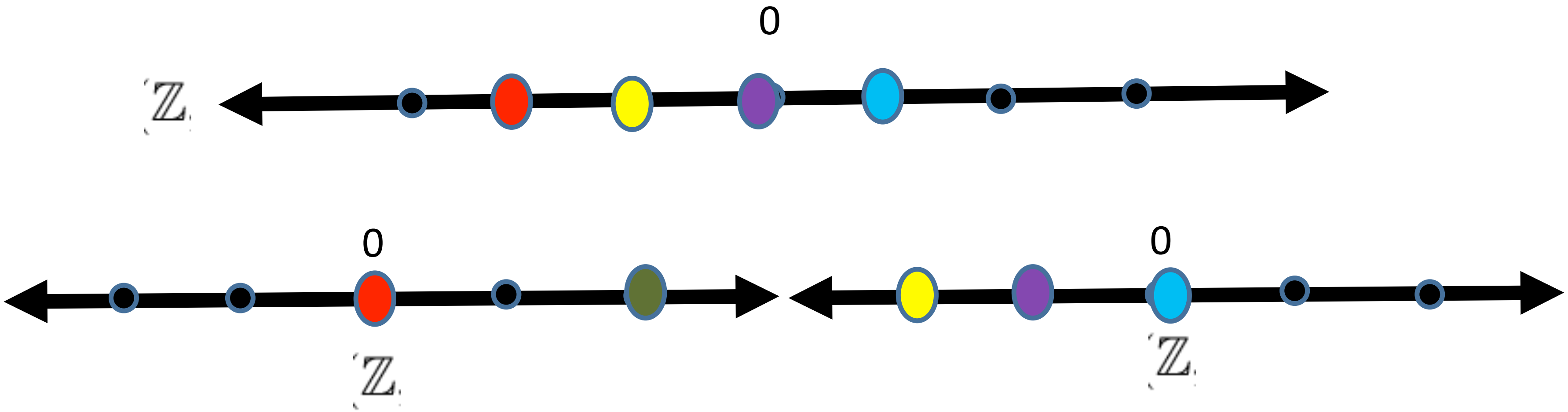
$$EFD_{\alpha}(A,B)$$

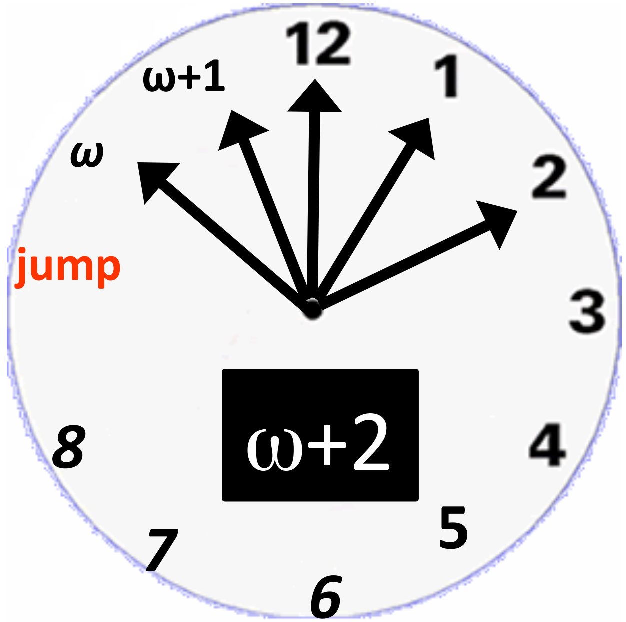
$= EF_{\omega}(A,B)$ with game clock α

$A \simeq_p^{\alpha} B$

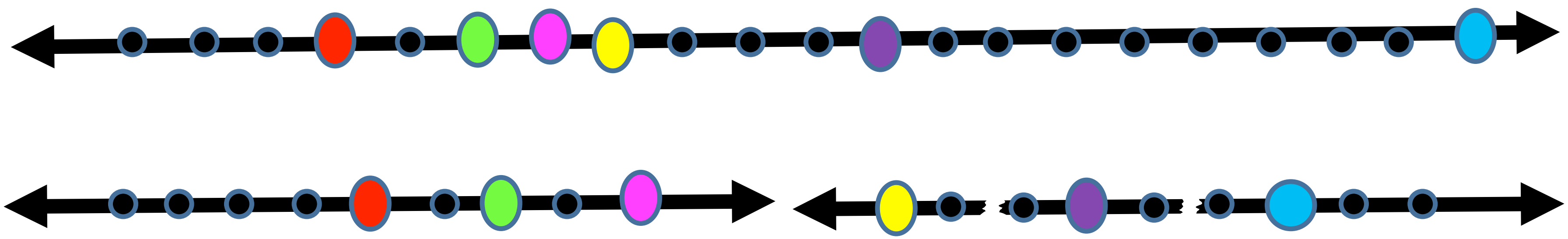
if II has a winning strategy

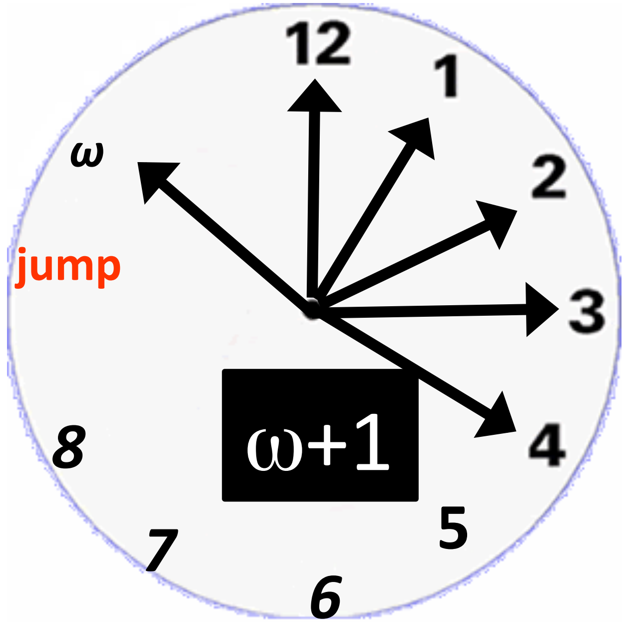
in $EFD_{\alpha}(A,B)$





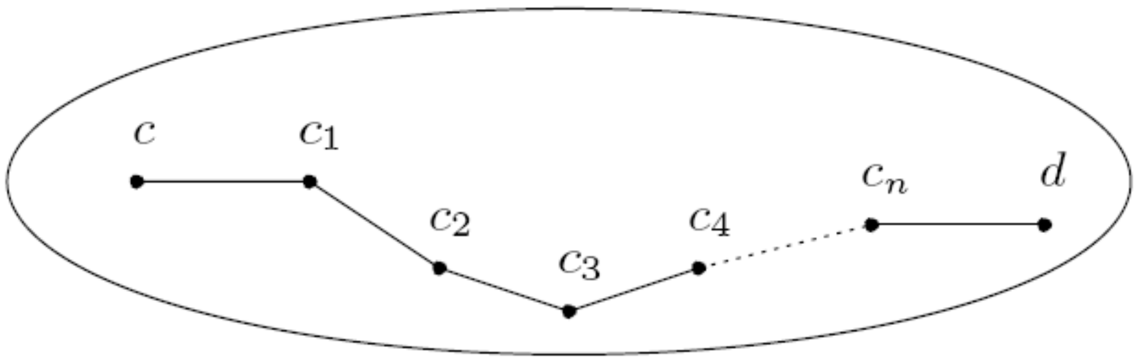
Player I wins

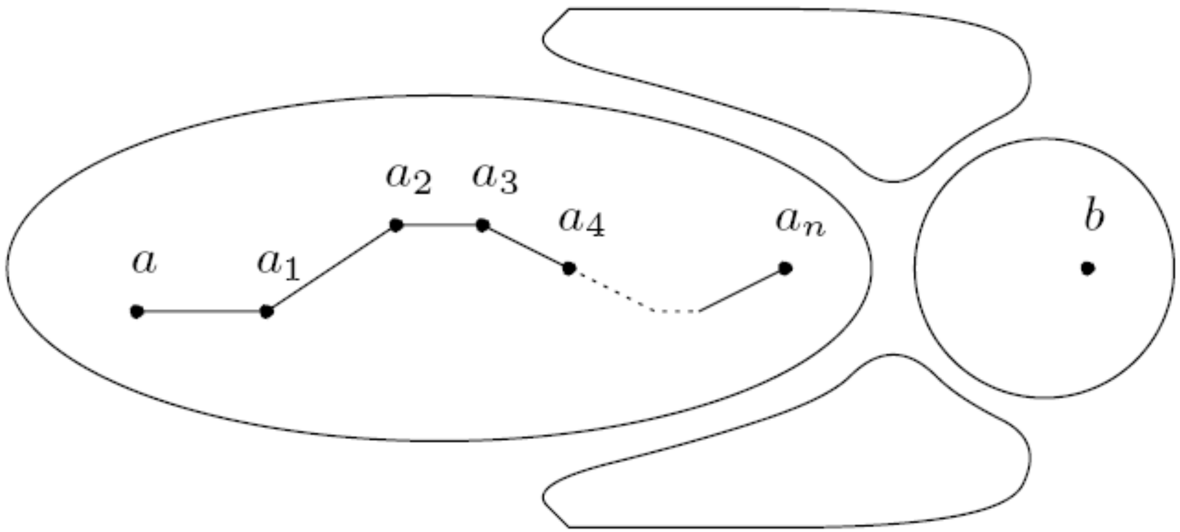




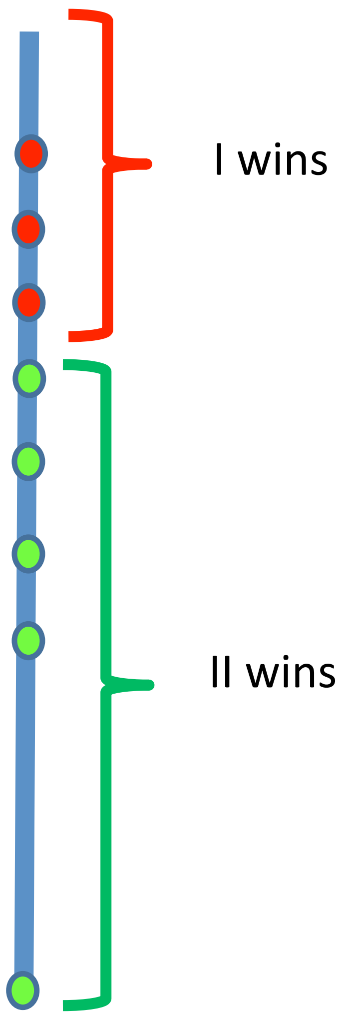
Player II wins

Connectedness, ω+2 in the clock





Scott watershed



Dana Scott

Examples

The Scott watershed of

$(\mathbb{Z}, <)$ and $(\mathbb{Z} + \mathbb{Z}, <)$ is ω+1.

(ω,<) and (ω+ ω,<) is ω.

($\mathbb{Q}$,<) and ($\mathbb{R}$,<) does not exist.

Scott watershed of countable models is countable

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Proposition 6.1.3. *If II has a winning strategy in* $\mathrm{EFD}_\alpha(\mathcal{M}, \mathcal{M}')$ *for all* $\alpha < (|M| + |M'|)^+$ *then II has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$.

Thus the Scott watershed of M and M' is always an ordinal $< (|M|+|M'|)^+$.

Scott height of a countable model is countable

Corollary 6.1.1. *(Scott's Theorem I) If $\mathcal{M}$ is countable, then for any other countable $\mathcal{M}'$ we have*

$$\mathcal{M} \simeq_p^{\mathrm{SH}(\mathcal{M})+\omega} \mathcal{M}' \iff \mathcal{M} \cong \mathcal{M}'.$$

Hierarchy of countable models