Summary of the book[1]

A First Course in Quantitative Finance

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[1] [Maz18]

Notation

$\emptyset$	The empty set.
$A \subset S$	Given a set S, $A \subset S$ denotes that A is a subset of S. Following the convention of the book, there's no notational difference between proper and improper subsets.
S^{c}	Given a set S, S^{c} denotes the complement of S.
$\#S$	Given a set S, $\#S$ denotes the cardinality (number of elements) of S.
$\mathbb{R}$	$\mathbb{R}$ denotes the real numbers.
$\triangleq$	Definition symbol.

Part I
Technical Basics

1 A Primer on Probability

1.1 Probability and Measure

D. 1: Sample space

A set $\Omega = \{\omega_1, \omega_2, \dots\}$ with elementary states $\omega_1, \omega_2, \dots$ which may or may not realize is called a sample space. It is the set of all possible outcomes of an experiment.

D. 2: Event

An event is a set of elementary states of the world, for each of which we can tell with certainty whether or not it has realized after the random experiment is over.
Any subset E of the sample space, $E \subset \Omega$, is known as an event. In other words, an event is a set consisting of possible outcomes of the experiment.

D. 3: Complement

Let U be the set of all elements under study (the "universe") and let $A \subset U$. Then $A^{\complement}$ is called the complement of A. $A^{\complement}$ is the set of of elements that are not in A.

$$A^{\complement} = \{x \in U : x \notin A\}.$$

D. 4: σ-algebra

A family $\mathcal{F}$ of sets (events) $A, A_1, A_2, \dots$ is called a σ-algebra, if it satisfies the following conditions

(i) $\mathcal{F}$ is nonempty, i.e., $\mathcal{F} \neq \emptyset$,

(ii) if $A \in \mathcal{F}$ then $A^{\complement} \in \mathcal{F}$,

(iii) if $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

T. 1: De Morgan

The De Morgan's rule:

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^{\complement}\right)^{\complement}. \tag{1}$$

From this, the subsequent two laws, known as the De Morgan's laws, can be derived[1].

$$\bigcup_{n=1}^{\infty} A_n^{\complement} = \left(\bigcap_{n=1}^{\infty} A_n\right)^{\complement} \tag{2}$$

$$\bigcap_{n=1}^{\infty} A_n^{\complement} = \left(\bigcup_{n=1}^{\infty} A_n\right)^{\complement} \tag{3}$$

D. 5: Measurable space

Given a sample space Ω and a σ-algebra $\mathcal{F}$, the pair $(\Omega, \mathcal{F})$ is called a measurable space.

D. 6: Power set

The power set[2] of a set S, denoted as 2^S, is the set of all subsets of S, including $\emptyset$ and S itself.

[1] See section 2.1 for derivations.

[2] The power set is often (like in this book) denoted as 2^S. The reason for this is that a power set of S has $2^{\#S}$ elements (subsets of S). Intuitively, one can either include an element of S in a subset or not, i.e., for each element of S there are two choices, leading to $2^{\#S}$ possible subsets.

D. 7: Borel-σ-algebra on $\mathbb{R}$

The Borel-σ-algebra on $\mathbb{R}$, denoted as $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by all open sets (a, b), where $a, b \in \mathbb{R}$ and $a \leq b$.

D. 8: Generated σ-algebra

The σ-algebra generated[3] by the event A is $\mathcal{F} = \{\emptyset, A, A^{\complement}, \Omega\}$, denoted as $\sigma(A)$.

D. 9: Measure

A function $\mu : \mathcal{F} \to \mathbb{R}_0^+$, with the properties

(i) $\mu(\emptyset) = 0$,

(ii) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$, for $A_1, A_2, \ldots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$,

is called a measure on the measurable space $(\Omega, \mathcal{F})$.

D. 10: Measure space

Given a measureable space $(\Omega, \mathcal{F})$ and a measure μ on $(\Omega, \mathcal{F})$, the triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

D. 11: Probability space

A measure space $(\Omega, \mathcal{F}, \mu)$ where the measure satisfies $\mu(\Omega) = 1$ is called a probability space. The associated measure μ is then called probability and is abbreviated as $P(A)$ for $A \in \mathcal{F}$. Therefore, the probability space triple is written as $(\Omega, \mathcal{F}, P)$.

1.2 Filtrations and the Flow of Information

D. 12: Filtration

The ascending sequence of σ-algebras $\mathcal{F}_t$, with $\mathcal{F}_0 \subset \mathcal{F}_t \subset \mathcal{F}$, is called a filtration.
If a filtration is generated by successively observing the particular outcomes of a process (like a coin toss), it is called the natural filtration of that process.

1.3 Conditional Probability and Independence

L. 1

Given a probability space $(\Omega, \mathcal{F}, P)$ and an event $A \in \mathcal{F}$ with $P(A) > 0$. Now define

$$\mathcal{F}_A = \{A \cap B \ : \ B \in \mathcal{F}\},$$

the family of all intersections of A with every event in $\mathcal{F}$. Then $\mathcal{F}_A$ is itself a σ-algebra on A and the pair $(A, \mathcal{F}_A)$ is a measurable space.

D. 13: Conditional probability

Given a probability space $(\Omega, \mathcal{F}, P)$ and two events $A, B \in \mathcal{F}$. For $P(A) > 0$, the probability measure $P(B \mid A)$ is called the conditional probability of B given A, and is defined as

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}.$$

L. 2

Given a probability space $(\Omega, \mathcal{F}, P)$ and an event $A \in \mathcal{F}$ such that $P(A) > 0$. Then the triple $(A, \mathcal{F}_A, P(\cdot \mid A))$ forms a new probability space.

[3] It can be shown that $\sigma(A)$ is the smallest σ-algebra containing A.

T. 2: Bayes' rule

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^{\complement})P(B^{\complement})}$$

D. 14: Independence

Two events A and B are said to be independent, if

$$P(A \cap B) = P(A)P(B)$$

A direct consequence of indepence is that if events A and B are independent, then the conditional probability of A given B collapses to the unconditional one:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

2 Derivations and Proofs

2.1 De Morgan's Laws

The first law (Equation 2) can be derived by taking the complement on both sides of Equation 1:

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^{\complement} = \left(\left(\bigcup_{n=1}^{\infty} A_n^{\complement}\right)^{\complement}\right)^{\complement} = \bigcup_{n=1}^{\infty} A_n^{\complement}.$$

The second law (Equation 3) can be found by replacing A_n with $B_n \triangleq A_n^{\complement}$:

$$\bigcap_{n=1}^{\infty} B_n^{\complement} \triangleq \bigcap_{n=1}^{\infty} A_n \stackrel{1}{=} \left(\bigcup_{n=1}^{\infty} A_n^{\complement}\right)^{\complement} \triangleq \left(\bigcup_{n=1}^{\infty} B_n\right)^{\complement}.$$

Bibliography

[Maz18] Thomas Mazzoni. *A First Course in Quantitative Finance*. Cambridge University Press, 2018. DOI: 10.1017/9781108303606.