1) Give a single intensity transformation function T for spreading the intensities of an image so the lowest intensity is 0 and the highest is $L-1$.

Let the image be represented as a function f_0 and let the maximum and minimum be represented by the notation f_{max} and f_{min}, respectively. The new minimum and maximum of the transformed image, denoted as f_t will be 0 and $L-1$, respectively. Thus, we can see that the transformation function is:

$$f_t(x,y) = \frac{L-1}{f_{max}-f_{min}}[f_0 - f_{min}] \tag{1}$$

2) (Histogram equalization in continuous variables) An image has the gray-level PDF

$$p_r(r) = \begin{cases} \frac{6r+2}{3(L-1)^2+2(L-1)} \\ 0, \text{ otherwise} \end{cases}$$

if $0 \leq r \leq L-1$ with $L-1 > 0$.

(a) Verify some of the properties that a PDF has to satisfy: $p_r(r) \geq 0$ for all $r \in (-\infty, \infty)$ and $\int_{-\infty}^{\infty} p_r(r)dr = 1$.
By defintion of probability density function (PDF), $p_r(r)$ is such that

$$\int_a^b p_r(r)dr = P(X \in [a,b]) \tag{2}$$

for any interval $[a,b]$. Since it impossible for probabilites to be negative, we can write $P(X \in [a,b]) \geq 0$, and thus, we know that

$$\int_a^b p_r(r)dr \geq 0 \tag{3}$$

$$\Rightarrow p_r(r) \geq 0 \tag{4}$$

for any interval $[a,b]$. But, the above integral can only be non-negative for all intervals $[a,b]$ only if the integrand function itself is non-negative, that is, $p_r(r) \geq 0$ for all x. Thus, the first property is true.
Next, by the definition of probability, the sum of all of the probability of the events must be 1. Thus, we can write

$$1 = P(X \in (-\infty, \infty)) = \int_{-\infty}^{\infty} p_r(r)dr = 1. \tag{5}$$

(b) Find the tranformation function $s = T(r)$ obtained through histogram equalization in continuous variables.

$$s = T(r) = (L-1)\int_0^r \frac{6r+2}{3(L-1)^2+2(L-1)}dr \tag{6}$$

$$\Rightarrow \frac{2}{3(L-1)+2}\int_0^r (3r+1)dr \tag{7}$$

$$\Rightarrow \frac{2}{3(L-1)+2} \times (\frac{3r^2}{2}+r) \tag{8}$$

$$\Rightarrow s = T(r) = \frac{3r^2+2r}{3(L-1)+2} \tag{9}$$

(c) Verify that $p_s(s)$ is a uniform "flat" distribution for $s \in [0,1]$ (recall the formula $p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$).

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = \frac{6r+2}{3(L-1)^2+2(L-1)}\left|\left[\frac{d}{dr}\frac{3r^2+2r}{3(L-1)+2}\right]^{-1}\right| \tag{10}$$

$$\Rightarrow \frac{6r+2}{3(L-1)^2+2(L-1)}\left|\left[\frac{6r+2}{3(L-1)+2}\right]^{-1}\right| \tag{11}$$

$$\Rightarrow \frac{1}{L-1} \tag{12}$$

Since r is nonnegative and we assume that $L > 1$, we can see that the result is a uniform PDF.

3) (Histogram matching in continuous variables) An image has the gray-level PDF $p_r(r) = -2r + 2$, with $0 \le r \le 1$. It is desired to transform the gray levels of this image so that they will have the specified $p_z(z) = 2z$, $0 \le z \le 1$. Assume continuous quantities and find the transformation (in terms of r and z) that will accomplish this (here $L - 1 = 1$).

First, we find the the histogram equalization transformation:

$$s = T(r) = (L-1)\int_0^r (-2r+2)dr = (L-1)(-r^2+2r). \tag{13}$$

Now, we look for the image with a specified histogram:

$$G(z) = (L-1)\int_0^z z^2 dz = (L-1)\frac{z^3}{3}. \tag{14}$$

Now, we require $G(z) = s$, but

$$G(z) = (L-1)\frac{z^3}{3}, \tag{15}$$

so, now we can write

$$s = (L-1)\frac{z^3}{3}. \tag{16}$$

Next, we will solve for z:

$$z = \left[\frac{3s}{(L-1)}\right]^{1/3} \tag{17}$$

Now, we can generate the z's directly from the intensities, r, of the input image:

$$z = \left[\frac{3s}{(L-1)}\right]^{1/3} = \left[\frac{3(-r^2+2r)(L-1)}{(L-1)}\right]^{1/3} \tag{18}$$

$$\Rightarrow \left[3(-r^2+2r)\right]^{1/3} \tag{19}$$

4) A linear spatial filter of size $(2a+1) \times (2b+1)$ defined by the transformation H, $g = H[f]$, is given by

$$g(x,y) = \sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t) f(x+s, y+t), \tag{20}$$

where $f(x,y)$ is a given input image, a, b are positive integers, and $w(s,t)$ are weights for $-a \le s \le a$, $-b \le t \le b$.

(a) Give the definition of a linear transformation $H : V \to V$, where V is a vector space.

Looking back at some linear algebra materal (Math 115A), we know that a transformation H is linear if for all $m, n \in V$ and $c \in F$, we have that:

1. $H(m + n) = H(m) + H(n)$ (vector-addition)
2. $H(cm) = cH(m)$ (scalar-multiplication)

(b) Show that H defined above is indeed a linear transformation (assume images defined on the entire plane, or ignore border effects).

To prove that H is indeed a linear transformation, we need to prove that the two properties above hold throughout the transformation.

$$H(cm + n) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t)(cm + n)(x + s, y + t) \tag{21}$$

$$\Rightarrow \sum_{s=-a}^{a} \sum_{t=-b}^{b} [w(s,t)(cm)(x + s, y + t) + w(s,t)(n)(x + s, y + t)] \tag{22}$$

$$\Rightarrow \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t)(cm)(x + s, y + t) + \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t)(n)(x + s, y + t) \tag{23}$$

$$\Rightarrow c \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t)m(x + s, y + t) + \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t)n(x + s, y + t) \tag{24}$$

$$\Rightarrow cH(m) + H(n) \tag{25}$$