1a) Show in discrete variables that

$$\mathcal{F}\Big(f(x,y)e^{2\pi i(u_0\frac{x}{M}+v_0\frac{y}{N})}\Big) = F(u-u_0, v-v_0),$$

where $F = \mathcal{F}(f)$.

Answer: We start by writing the 2-D discrete Fourier transformation:

$$F(x,y) = \mathcal{F}[f(x,y)] = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \quad .$$

For the given equation $f(x,y)e^{j2\pi(u_0x+v_0y)}$, we can then write:

$$\mathcal{F}[f(x,y)e^{j2\pi(u_0x+v_0y)}] = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\Big[f(x,y)e^{j2\pi(u_0x+v_0y)}\Big]\,e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{1}$$

$$\Rightarrow \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi[\frac{(u-u_o)x}{M}+\frac{(v-v_o)y}{N}]} \tag{2}$$

$$\Rightarrow F(u-u_0, v-v_0) \quad . \tag{3}$$

1b) Using a), deduce the formula used in shifting the center of the transform by multiplication with $(-1)^{x+y}$, when $u_0 = M/2$ and $v_0 = N/2$, with M and N even positive integers.

Answer: We set $u_0 = M/2$ and $v_0 = N/2$ into our given equation:

$$F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left[\frac{\left(u-\frac{M}{2}\right)x}{M}+\frac{\left(v-\frac{N}{2}\right)y}{N}\right]} \quad . \tag{4}$$

We then can expand our exponential to get the following results:

$$\Rightarrow \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi u\frac{x}{M}}e^{-j2\pi v\frac{y}{N}}e^{j\pi(x+y)} \quad . \tag{5}$$

By applying Euler's formula, we know that $e^{j\pi} = cos(\pi) + jsin(\pi) = -1$. Thus, we can write:

$$\Rightarrow \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi u\frac{x}{M}}e^{-j2\pi v\frac{y}{N}}(-1)^{x+y} \tag{6}$$

$$\Rightarrow \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi[\frac{ux}{M}+\frac{vy}{N}]}(-1)^{x+y} \tag{7}$$

$$\Rightarrow \mathcal{F}\left[f(x,y)(-1)^{x+y}\right] \tag{8}$$

Thus, the formula for shifting the center of the transform is

$$\Rightarrow F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) = f(x,y)(-1)^{x+y} \quad . \tag{9}$$

2a) Show the translation property

$$\mathcal{F}\Big(f(x-x_0, y-y_0)\Big) = F(u,v)e^{-2\pi i(x_0 u/M + y_0 v/N)},$$

where $F(u,v) = \mathcal{F}(f(x,y))$.

Answer: We begin with the 2D inverse discrete Fourier transform:

$$f(x,y) = \mathcal{F}\left[F(u,v)\right] = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

We then can write:

$$\mathcal{F}\left[F(u,v)\right] = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[F(u,v)e^{j2\pi(ux_0+vy_0)}\right]e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{10}$$

$$\Rightarrow \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)e^{-j2\pi\left[\frac{(x-x_0)}{M}+\frac{(y-y_0)}{N}\right]} \tag{11}$$

$$\Rightarrow \mathcal{F}\Big(f(x-x_0, y-y_0)\Big) \tag{12}$$

2b) Consider the linear difference operator $g(x,y) = f(x+1,y) - f(x,y)$. Obtain the filter transfer function, $H(u,v)$, for performing the equivalent process in the frequency domain.

Answer: We first start by applying the Fourier transform to the given equation:

$$\mathcal{F}\left[f(x+1,y) - f(x,y)\right] \quad .$$

We then solve for the filter transfer function:

$$\mathcal{F}\left[f(x+1,y) - f(x,y)\right] \quad = \mathcal{F}\left[f(x-(-1), y-0) - f(x,y)\right] \tag{13}$$

$$\Rightarrow \mathcal{F}\left[f(x-(-1), y-0)\right] - \mathcal{F}\left[f(x,y)\right] \tag{14}$$

$$\Rightarrow F(u,v) \times e^{-j2\pi(\frac{-u}{M})} - F(u,v) \tag{15}$$

$$\Rightarrow F(u,v)\left[e^{-j2\pi(\frac{-u}{M})} - 1\right] \tag{16}$$

Thus, we can see that:

$$H(u,v) = e^{-j2\pi(\frac{-u}{M})} - 1$$

3) Prove the validity of the discrete convolution theorem in one variable (you may need to use the translation properties).

Answer: We first write the expression for discrete convolution of two functions:

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m) \quad .$$

We then take the Fourier transform on both sides of the equation:

$$\mathcal{F}\left[f(x) \star h(x)\right] = \sum_{x=0}^{M-1} \left[\sum_{m=0}^{M-1} f(m)h(x-m) \right] e^{-j2\pi\left(\frac{ux}{M}\right)} \tag{17}$$

$$\Rightarrow \sum_{m=0}^{M-1} f(m) \left[\sum_{x=0}^{M-1} h(x-m) e^{-j2\pi\left(\frac{ux}{M}\right)} \right] \tag{18}$$

Using the translation property, we can write:

$$\sum_{m=0}^{M-1} f(m) \left[\sum_{x=0}^{M-1} h(x-m) e^{-j2\pi\left(\frac{ux}{M}\right)} \right] = \sum_{m=0}^{M-1} f(m)H(u)e^{-j2\pi\left(\frac{um}{M}\right)} \tag{19}$$

$$\Rightarrow H(u) \sum_{m=0}^{M-1} f(m) e^{-j2\pi\left(\frac{um}{M}\right)} \tag{20}$$

$$\Rightarrow F(u)H(u) \tag{21}$$

4) Assume that $f(x)$ is given by the discrete IFT formula in one dimension. Show the periodicity property $f(x) = f(x+kM)$, where k is an integer.

Answer: We start with the equation $f(x+kM)$ and apply the Fourier transformation to it:

$$\mathcal{F}[f(x+kM)] = \sum_{u=0}^{M-1} F(u) e^{-j2\pi\left[\frac{((u+kM)x}{M}\right]}$$

We then get:

$$\Rightarrow \sum_{u=0}^{M-1} F(u) e^{j2\pi\left[\frac{ux}{M}\right]} e^{j2\pi uk} \tag{22}$$

Since $e^{j2\pi uk} = 1$, we can then write:

$$\Rightarrow \sum_{u=0}^{M-1} F(u) e^{j2\pi\left[\frac{ux}{M}\right]} \tag{23}$$

$$\Rightarrow f(x) \tag{24}$$

Therefore, the periodicity peropertiy holds.

5a) Implement the Gaussian lowpass filter in Eq. (4.3-8), using a radius $D_0 = 25$, and apply the algorithm to Fig4.11(a).

Please see attached pages for images.

```
A = imread('image.jpg');

[M N] = size(A);

B=double(A);

% multiply f by (-1)^(x+y) to shift the center
for i = 1:M
   for j = 1:N
         d = (i - 1) + (j - 1);
         C(i,j) = B(i,j)*(-1)^d;
   end
end

% compute the DFT of f*(-1)^{x+y}
D=fft2(C);

%Create filter
for u = 1:M
    for v = 1:N
        P = (u - ((2 * M - 1) / 2))^2;
        Q = (v - ((2 * N - 1) / 2))^2;
        H(u,v) = exp(-(P + Q) / 1250);
    end
end

%Apply filter to image
for i = 1:M
    for j = 1:N
        E(i,j) = D(i,j).*H(u,v);
    end
end

%Use inverse Fourier transformation
F = ifft2(E);

imshow(F);
```

5b) Highpass the input image used in (a), using a highpass Gaussian filter of radius $D_0 = 25$ (see eq. (4.4-4)).

Please see attached pages for images.

```
A = imread('image.jpg');

[M N] = size(A);

B=double(A);

% multiply f by (-1)^(x+y) to shift the center
for i = 1:M
   for j = 1:N
         d = (i - 1) + (j - 1);
         C(i,j) = B(i,j)*(-1)^d;
   end
end

% compute the DFT of f*(-1)^{x+y}
D=fft2(C);

%Create filter
for u = 1:M
    for v = 1:N
        P = (u - ((2 * M - 1) / 2))^2;
        Q = (v - ((2 * N - 1) / 2))^2;
        H(u,v) = 1 - exp(-(P + Q) / 1250);
    end
end

%Apply filter to image
for i = 1:M
    for j = 1:N
        E(i,j) = D(i,j).*H(u,v);
    end
end

%Use inverse Fourier transformation
F = ifft2(E);

imshow(F);
```