

- 1) Give a single intensity transformation function T for spreading the intensities of an image so the lowest intensity is 0 and the highest is $L - 1$.

Let the image be represented as a function f_0 and let the maximum and minimum be represented by the notation f_{max} and f_{min} , respectively. The new minimum and maximum of the transformed image, denoted as f_t will be 0 and $L - 1$, respectively. Thus, we can see that the transformation function is:

$$f_t(x, y) = \frac{L - 1}{f_{max} - f_{min}}[f_0 - f_{min}] \quad (1)$$

- 2) (Histogram equalization in continuous variables) An image has the gray-level PDF

$$p_r(r) = \begin{cases} \frac{6r+2}{3(L-1)^2+2(L-1)} \\ 0, & \text{otherwise} \end{cases}$$

if $0 \leq r \leq L - 1$ with $L - 1 > 0$.

- (a) Verify some of the properties that a PDF has to satisfy: $p_r(r) \geq 0$ for all $r \in (-\infty, \infty)$ and $\int_{-\infty}^{\infty} p_r(r)dr = 1$.

By definition of probability density function (PDF), $p_r(r)$ is such that

$$\int_a^b p_r(r)dr = P(X \in [a, b]) \quad (2)$$

for any interval $[a, b]$. Since it is impossible for probabilities to be negative, we can write $P(X \in [a, b]) \geq 0$, and thus, we know that

$$\int_a^b p_r(r)dr \geq 0 \quad (3)$$

$$\Rightarrow p_r(r) \geq 0 \quad (4)$$

for any interval $[a, b]$. But, the above integral can only be non-negative for all intervals $[a, b]$ only if the integrand function itself is non-negative, that is, $p_r(r) \geq 0$ for all x . Thus, the first property is true.

Next, by the definition of probability, the sum of all of the probability of the events must be 1. Thus, we can write

$$1 = P(X \in (-\infty, \infty)) = \int_{-\infty}^{\infty} p_r(r)dr = 1. \quad (5)$$

- (b) Find the transformation function $s = T(r)$ obtained through histogram equalization in continuous variables.

$$s = T(r) = (L - 1) \int_0^r \frac{6r + 2}{3(L - 1)^2 + 2(L - 1)} dr \quad (6)$$

$$\Rightarrow \frac{2}{3(L - 1) + 2} \int_0^r (3r + 1)dr \quad (7)$$

$$\Rightarrow \frac{2}{3(L - 1) + 2} \times \left(\frac{3r^2}{2} + r \right) \quad (8)$$

$$\Rightarrow s = T(r) = \frac{3r^2 + 2r}{3(L - 1) + 2} \quad (9)$$

(c) Verify that $p_s(s)$ is a uniform "flat" distribution for $s \in [0, 1]$ (recall the formula $p_s(s) = p_r(r) \left| \frac{dr}{ds} \right|$).

$$p_s(s) = p_r(r) \left| \frac{dr}{ds} \right| = \frac{6r+2}{3(L-1)^2 + 2(L-1)} \left| \left[\frac{d}{dr} \frac{3r^2+2r}{3(L-1)+2} \right]^{-1} \right| \quad (10)$$

$$\Rightarrow \frac{6r+2}{3(L-1)^2 + 2(L-1)} \left| \left[\frac{6r+2}{3(L-1)+2} \right]^{-1} \right| \quad (11)$$

$$\Rightarrow \frac{1}{L-1} \quad (12)$$

Since r is nonnegative and we assume that $L > 1$, we can see that the result is a uniform PDF.

- 3) (Histogram matching in continuous variables) An image has the gray-level PDF $p_r(r) = -2r + 2$, with $0 \leq r \leq 1$. It is desired to transform the gray levels of this image so that they will have the specified $p_z(z) = 2z$, $0 \leq z \leq 1$. Assume continuous quantities and find the transformation (in terms of r and z) that will accomplish this (here $L - 1 = 1$).

First, we find the the histogram equalization transformation:

$$s = T(r) = (L-1) \int_0^r (-2r+2)dr = (L-1)(-r^2+2r). \quad (13)$$

Now, we look for the image with a specified histogram:

$$G(z) = (L-1) \int_0^z 2zdz = (L-1)z^2. \quad (14)$$

Now, we require $G(z) = s$, but

$$G(z) = (L-1)z^2, \quad (15)$$

so, now we can write

$$s = (L-1)z^2. \quad (16)$$

Next, we will solve for z :

$$z = \sqrt{\frac{s}{(L-1)}} \quad (17)$$

Now, we can generate the z 's directly from the intensities, r , of the input image:

$$z = \sqrt{(-r^2+2r)} \quad (18)$$

- 4) A linear spatial filter of size $(2a+1) \times (2b+1)$ defined by the transformation H , $g = H[f]$, is given by

$$g(x, y) = \sum_{s=-a}^a \sum_{t=-b}^b w(s, t) f(x+s, y+t), \quad (19)$$

where $f(x, y)$ is a given input image, a, b are positive integers, and $w(s, t)$ are weights for $-a \leq s \leq a$, $-b \leq t \leq b$.

- (a) Give the definition of a linear transformation $H : V \rightarrow V$, where V is a vector space.

Looking back at some linear algebra material (Math 115A), we know that a transformation H is linear if for all $m, n \in V$ and $c \in F$, we have that:

1. $H(m + n) = H(m) + H(n)$ (vector-addition)
 2. $H(cm) = cH(m)$ (scalar-multiplication)
- (b) Show that H defined above is indeed a linear transformation (assume images defined on the entire plane, or ignore border effects).

To prove that H is indeed a linear transformation, we need to prove that the two properties above hold throughout the transformation.

$$H(cm + n) = \sum_{s=-a}^a \sum_{t=-b}^b w(s, t)(cm + n)(x + s, y + t) \quad (20)$$

$$\Rightarrow \sum_{s=-a}^a \sum_{t=-b}^b [w(s, t)(cm)(x + s, y + t) + w(s, t)(n)(x + s, y + t)] \quad (21)$$

$$\Rightarrow \sum_{s=-a}^a \sum_{t=-b}^b w(s, t)(cm)(x + s, y + t) + \sum_{s=-a}^a \sum_{t=-b}^b w(s, t)(n)(x + s, y + t) \quad (22)$$

$$\Rightarrow c \sum_{s=-a}^a \sum_{t=-b}^b w(s, t)m(x + s, y + t) + \sum_{s=-a}^a \sum_{t=-b}^b w(s, t)n(x + s, y + t) \quad (23)$$

$$\Rightarrow cH(m) + H(n) \quad (24)$$