1) Recall that the 1D DFT is

$$F(u) = \sum_{x=0}^{M-1} f(x)e^{-2\pi iux/M}.$$

a) Assuming the formula above, show the identity

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{2\pi iux/M},$$

using the following orthogonality of exponentials

$$\sum_{u=0}^{M-1} e^{-2\pi iuy/M}e^{2\pi iux/M} = \begin{cases} M & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

Answer: We start with our given equation:

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{2\pi iux/M}$$

which we then substitute the one dimension discrete Fourier transformation, which gives us:

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1}\sum_{y=0}^{M-1} f(y)e^{-2\pi iuy/M}e^{2\pi iux/M} \quad . \tag{1}$$

Given the orthogonality of exponentials, we know that for the summation of

$$\sum_{u=0}^{M-1} e^{-2\pi iuy/M}e^{2\pi iux/M} \quad ,$$

we know that all cases except one will end up being 0, with the other case becoming M. Thus, we can write:

$$f(x) = \frac{1}{M} \times M \times \sum_{y=0}^{M-1} f(y) \tag{2}$$

$$\Rightarrow f(x) = f(y) \tag{3}$$

And thus, the identity holds.

b) Show now the converse of a): assume given $f(x)$ function of $F(u)$ in the discrete case, and show the identity for $F(u)$ (use the same orthogonality of exponentials).

Answer: We start with our given equation:

$$F(u) = \sum_{y=0}^{M-1} f(y)e^{-2\pi iuy/M}$$

which we then substitute the one dimension discrete Fourier transformation identity, which gives us:

$$F(u) = \sum_{y=0}^{M-1}\frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{2\pi iux/M}e^{-2\pi iuy/M} \tag{4}$$

$$\Rightarrow F(u) = \frac{1}{M} \sum_{y=0}^{M-1} \sum_{u=0}^{M-1} F(u) e^{2\pi iux/M} e^{-2\pi iuy/M}. \tag{5}$$

Using the orthogonality of exponentials, we know that all cases except one will end up being 0, with the other case becoming M. Thus, we can write:

$$\Rightarrow F(u) = \frac{1}{M} \times M \sum_{y=0}^{M-1} F(u) \tag{6}$$

$$\Rightarrow F(u) = F(u) \tag{7}$$

Thus, the identity holds.

2) Show that the continuous 2D Fourier transform is a linear process.

Answer: Let f, g be functions and let $a \in R$. From lecture, we can see that

$$F(\mu, v) = \mathcal{F}[f(t,z)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,z) e^{-j2\pi(\mu t + vz)} \; dtdz \tag{8}$$

and therefore, we can write:

$$\mathcal{F}[af(t,z) + g(t,z)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [af(t,z) + g(t,z)] \times e^{-j2\pi(\mu t + vz)} \; dtdz \tag{9}$$

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [af(t,z) \times e^{-j2\pi(\mu t + vz)}] + [g(t,z) \times e^{-j2\pi(\mu t + vz)}] \; dtdz \tag{10}$$

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [af(t,z) \times e^{-j2\pi(\mu t + vz)}] \; dtdz + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [g(t,z) \times e^{-j2\pi(\mu t + vz)}] \; dtdz \tag{11}$$

$$\Rightarrow a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(t,z) \times e^{-j2\pi(\mu t + vz)}] \; dtdz + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [g(t,z) \times e^{-j2\pi(\mu t + vz)}] \; dtdz \tag{12}$$

$$\Rightarrow a\mathcal{F}[f(t,z)] + \mathcal{F}[g(t,z)] \tag{13}$$

3) Compute in continuous variables the Fourier transform of the function

$$f(x) = \begin{cases} A, & \text{if } 0 \le x \le K, \\ 0, & \text{otherwise,} \end{cases}$$

where A and K are positive constants. Evaluate $F(0)$.

Answer: We start with the given equation and substitute in our function and its bounds of integration:

$$F(\mu) = \int_{0}^{K} A e^{-j2\pi\mu t} dt$$

We then solve for the equation:

$$\Rightarrow \frac{-A}{j2\pi\mu} \left[e^{-j2\pi\mu K} - e^{0}\right] \tag{14}$$

$$\Rightarrow \frac{-A}{j2\pi\mu} \left[e^{-j2\pi\mu K} - 1\right] \tag{15}$$

$$\Rightarrow \frac{A}{j2\pi\mu}\left[e^{j\pi\mu K} - e^{-j\pi\mu K}\right]e^{-j\pi\mu K} \quad . \tag{16}$$

Evaluating at $F(0)$, we get:

$$F(0) = \int_0^K f(x)e^{-j2\pi\mu x}dx = \int_0^K f(x)dx = \int_0^K Adx = AK \tag{17}$$

4) Consider again the 2D continuous Fourier transform and its inverse (denote by $H(\mu, v)$ the 2D Fourier transform of the spatial filter $h(x, y)$). Show that if the transform $H(\mu, v)$ is real and symmetric, i.e. if

$$H(\mu, v) = \overline{H(\mu, v)} = \overline{H(-\mu, -v)} = H(-\mu, -v),$$

then the corresponding spatial domain filter $h(x, y)$ is also real and symmetric.

Answer: First, we will prove that

$$h(x, y) = h(-x, -y) \quad .$$

We first write the 2D continuous Fourier transformation and its conjugate:

$$h(x, y) = \iint H(\mu, v)e^{j2\pi(\mu x + vy)} \quad \text{and} \quad h(-x, -y) = \iint H(\mu, v)e^{-j2\pi(\mu x + vy)} \tag{18}$$

We then can write that:

$$h(x, y) = \iint \overline{H(\mu, v)e^{-j2\pi(\mu x + vy)}} \tag{19}$$

$$\Rightarrow \overline{h(-x, -y)} \tag{20}$$

We then need to prove that

$$h(x, y) = \overline{h(x, y)}$$

First, we can write that:

$$h(x, y) = \iint \overline{H(\mu, v)}e^{j2\pi(\mu x + vy)} \quad \text{and} \quad \overline{h(x, y)} = \iint H(\mu, v)e^{-j2\pi(\mu x + vy)} \quad . \tag{21}$$

Since we know that $\overline{H(\mu, v)} = H(\mu, v)$, we can see that we will need to do a simple change of variables, where $s = -\mu$ and $t = -v$. Thus, we can write:

$$\overline{h(x, y)} = \iint H(s, t)e^{j2\pi(sx + ty)}dsdt \tag{22}$$

And after doing some calculus, we can see that $h(x, y) = \overline{h(x, y)}$. Since we were able to prove these two equalities, logically, the other one follows. Therefore, the spatial domain filter is real and symmetric.

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5) (Computational Project) **Fourier Spectrum and Average Value**

a) Use in Matlab "help fft" and "help fft2" to learn the commands for computing discrete Fourier transforms. Sample codes using the Fourier transform in 1D and 2D are posted on the class webpage.

Answer: The function 'fft' will return the discrete Fourier transformation, while the 'fft2' function will return the two-dimensional Fourier transformation.

b) Download Fig5.26a and compute its (centered) Fourier spectrum.

```
A = imread('image.jpg');

[M N] = size(A);

B = double(A);

for i = 1:M
    for j = 1:N
        d = (i - 1) + (j - 1);
        C(i,j) = B(i,j) * (-1)^d;
    end
end

fourier_transformation = fft2(C);

D = abs(fourier_transformation);

c=5;
for i = 1:M
    for j = 1:N
        E(i,j) = c * log(1 + D(i,j));
    end
end

figure
image(E); colormap(gray);
```

c) Display the spectrum.

Please see attached pages for images.

d) Using your algorithm, obtain the average value of the input image.

```
composite_value = 0;
for i = 1:M
    for j = 1:N
        composite_value = composite_value + B(i,j);
    end
end

average_value = composite_value/(M * N);
```

The average value of the input image is 138.004.