Reading Notes Chapter 4 3/14/25

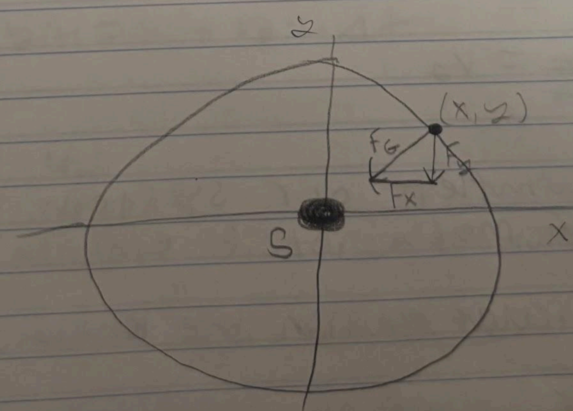
4.1 Kepler's Laws

- According to Newton's law of gravitation the magnitude of this force is:

$$F_G = \frac{G M_S M_E}{r^2}$$

- Where M_S and M_E are the masses of the Sun and Earth, r is the distance between them, and G is the grav. constant.

- Our goal is to calculate the pos. of Earth as a function of time.



$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_E} \qquad \frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_E}$$

From fig 1. we have

$$F_{Gx} = -\frac{GM_S M_E}{r^2}\cos\theta = -\frac{GM_S M_E x}{r^3}$$

-We now follow usual approach and write each of the second-order diff. eq:

$$\frac{dv_x}{dt} = -\frac{GM_S x}{r^3}$$

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_y}{dt} = -\frac{GM_S y}{r^3}$$

$$\frac{dy}{dt} = v_y$$

-To complete our system, we need units of mass.

-For circular motion we know force equals $\frac{M_E v^2}{r}$.

$$\frac{M_E v^2}{r} = F_G = \frac{GM_S M_E}{r^2}$$

- V = velocity of Earth and rearranging we get:

$$GM_S = v^2 r = 4\pi^2 AU^3/yr^2$$

- We convert equations of motions into diff. eq.

$$v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_i^3}\Delta t$$

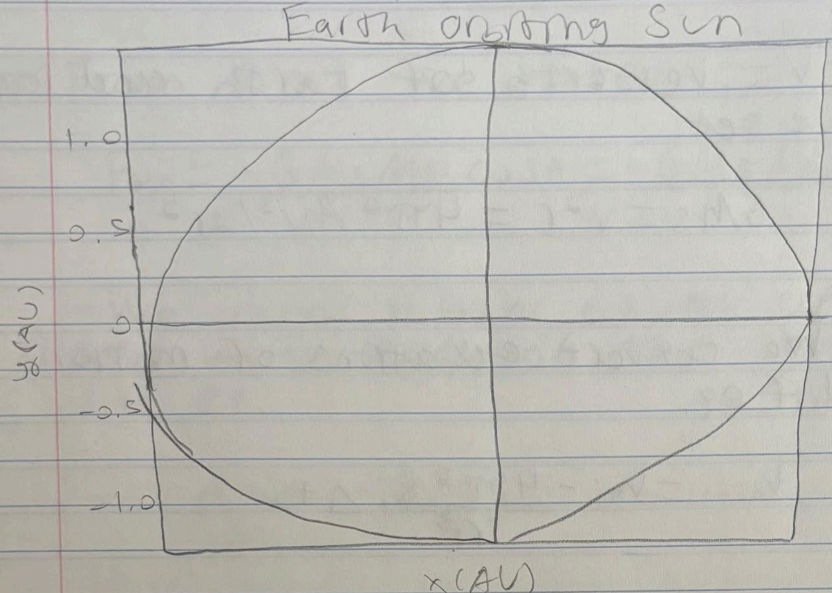
$$x_{i+1} = x_i + v_{x,i+1}\Delta t$$

$$v_{y,i+1} = v_{y,i} - \frac{4\pi^2 y_i}{r_i^3}\Delta t$$

$$y_{i+1} = y_i + v_{y,i+1}\Delta t$$

Ex. 4.1

- Calculate distance r_i from Sun: $r_i = (x_i^2 + y_i^2)^{1/2}$
- Compute $v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_i^3}\Delta t$ and $v_{y,i+1} =$
- Euler-Cromer: calc. x_{i+1}/y_{i+1} using $v_{x,i+1}/v_{y,i+1}$
- Record new pos. or plot as available
- Repeat for desired number



4.2 Inverse Square Law & Stability

- We consider a two-body system in which the interaction force depends on separation r.

- Relative motion can be studied as if it were one system.

- The moving body has a mass equal to the reduced mass:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

- The position is given by the relative displacement $\vec{r} \equiv \vec{r}_2 + \vec{r}_1$

- The orbital trajectory for a body of reduced mass is given in polar coords.

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L^2} F(r)$$

- $L = \mu r^2 \dot{\theta}$: angular momentum

- $F(r)$: force acting on the body

$$F(r) = -\frac{G M_s M_p}{r^2}$$

- Since $F(r)$ has the inverse square form $F(r) \propto 1/r^2$ can be solved.

$$\frac{1}{r} = \left(\frac{\mu G M_s M_p}{L^2}\right)\left[1 - e\cos(\theta + \theta_0)\right]$$

or choosing $\theta_0 = 0$

$$r = \left(\frac{L^2}{\mu G M_s M_p}\right)\frac{1}{1 - e\cos\theta}$$

- Since angular momentum $L = \sqrt{\mu G M_s M_p a(1-e^2)}$ is conserved, we may set this equal to $\mu r_{min} v_{max}$ and to $\mu r_{max} v_{min}$.

$$V_{max} = \sqrt{GM_s}\sqrt{\frac{(1+e)}{a(1-e)}\left(1+\frac{M_p}{M_s}\right)}$$

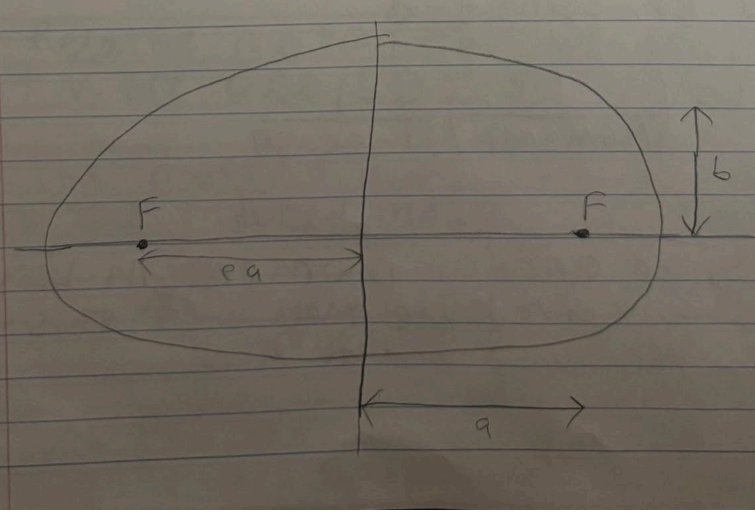
$$V_{min} = \sqrt{GM_s}\sqrt{\frac{(1-e)}{a(1+e)}\left(1+\frac{M_p}{M_s}\right)}$$

- Orbital Period T can be obtained by dividing the area enclosed by elliptical orbit by constant rate area is swept out

- Kepler's Third Law: T

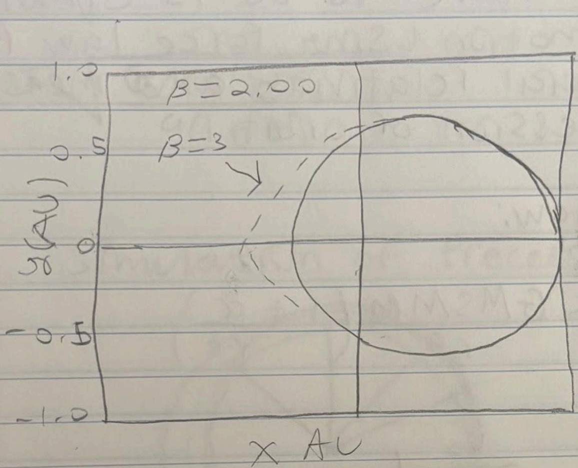
$$T^2/a^3 = \frac{4\pi^2}{G(M_s M_p)}$$

$$= \frac{4\pi^2}{GM_s}$$



- Figure 4.3: Hypothetical elliptical orbit.

- As β is closer to 2 orbits:



4.3 Precession of the Perihelion (Mercury)

- We have noted that most planets have orbits that are very nearly circular.

- The planets whose orbits deviate from circular are Mercury and Pluto.

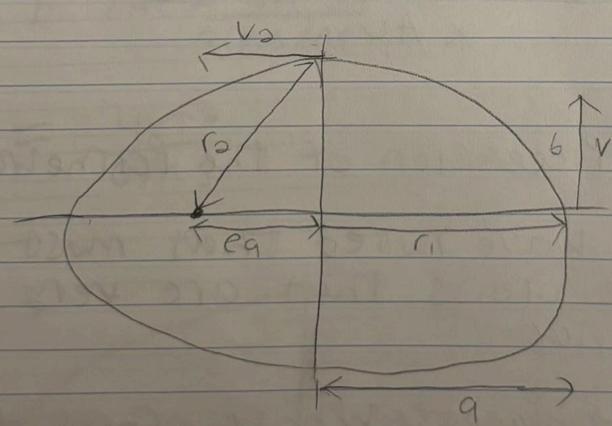
- The precession of perihelion (point nearest to the sun in orbit); the mas. is approx. 566 arcseconds per century;

- The precession due to general relativity can be calculated analytically, although it is complicated.

- All we have to do is simulate the orbital motion using force law predicted by general relativity and measure rate of precession of orbit.

- Force law:

$$F_G = \frac{G M_S M_M}{r^2}\left(1 + \frac{\alpha}{r^2}\right)$$



- We need to know initial conditions and then we get:

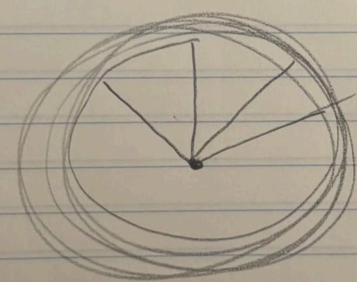
$$-\frac{G M_S M_M}{r_1} + \frac{1}{2} M_M v_1^2 = -\frac{G M_S M_M}{r_2} + \frac{1}{2} M_M v_2^2$$

$-r_1 v_1 = b v_2$

$$-v_1 = \sqrt{2GM_s\left[\frac{b^2}{a^2(1+e)^2-b^2}\right]\left[\frac{1}{\sqrt{e^2a^2+b^2}} - \frac{1}{a+ea}\right]}$$

$$= \sqrt{\frac{GM_s(1-e)}{a(1+e)}}$$

Simulation of Precession



4.4 Three-body Problem

- Midterm Project, what we did in class.