Reading Notes Appendix C

4/9/25

Goal: Our goal in this appendix is to give a general introduction to the Fourier Transform and how it is implemented.

The signal $y(t)$ can be written as:

$$y(t) = \sum_{j=1}^{5} y_j \sin(2\pi f_j t + \phi_j)$$

where y_j is the amplitude, f_j is the frequency, and ϕ_j the phase of the jith sine wave.

- Most signals will be more complicated, so the sum may involve a large (perhaps ∞) number of sine waves.

- We can now express $y(t)$ as:

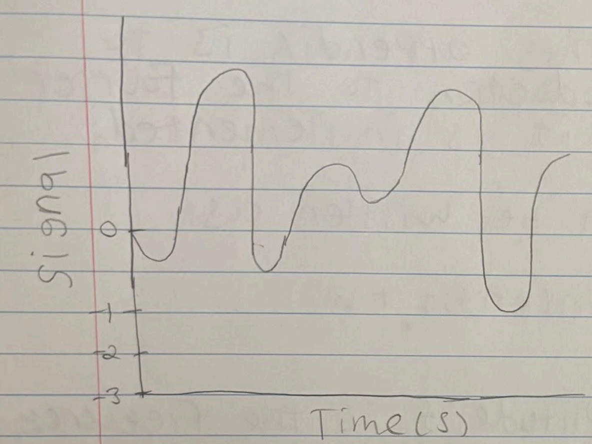
$$y(t) = \int_{-\infty}^{\infty} Y(f) e^{-2\pi i f t} \, df$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(w/2\pi) e^{-iwt} \, dw$$

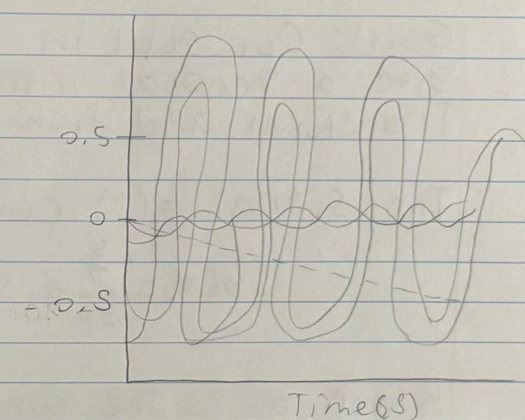
where $i = \sqrt{-1}$

- We can also plot the hypothetical and individual sine waves.

Fourier decomposition



Fourier components



- The frequency of these tones are the only frequencies present in the original signal and the Fourier transformation.

- Most sound signals are composed of many such frequencies.

C2. Discrete Fourier Transform

- The next issue is to compute this transformation.

- Ex. Pendulum

- We calculated the angular positions at times $t_m = m\Delta t$, where m was an integer and Δt was the time step.

$$z_m = \frac{1}{N}\sum_{n=0}^{N-1} Y_n e^{-2\pi i m n/N}$$

$$z_m = \sum_{m=0}^{N-1} z_m e^{2\pi i m n/N}$$

- Where the index of m on z corresponds to the discrete times $t_m = m\Delta t$ and the index n on Y corresponds to $f_n = n/(N\Delta t)$

- The forward and inverse discrete Fourier transforms are related via:

$$\sum_{n=0}^{N-1} e^{2\pi i n(m-m')/N} = N\delta_{m,m'}$$

Where $N\delta_{m,m'}$ is the Kronecker delta function.

- These are equivalent ways of describing the same collection of delta points.

- Since the Y_n are complex, it appears that we have 2N pieces of info. in the frequency domain.

- If Y_n are real we only have N pieces in the time domain.

C3 Fast Fourier Transform (FFT)

- Most scientific programmers are not likely to program an FFT themselves; they instead use preprogrammed routines.

- FFT algorithms are used in practice which are sufficiently complicated:

$$Y_n = Y_n^e + w^n Y_n^o$$

$$= \sum_{m'=0}^{3} y_{2m'} w^{2m'n} + w^n \sum_{m'=0}^{3} y_{2m'+1} w^{2m'n}$$

where $w = e^{2\pi i/N}$

- If we use binary representation $n = 4n_2 + 2n_1 + n_0$ we get:

$$w^{2m'n} = e^{2\pi i 2m'(4n_2 + 2n_1 + n_0)/8}$$

$$= w^{2m'(2n_1 + n_0)}$$

- Since $w^{2m'n_2} = 1$ for $n_2 = 1$ and 0.

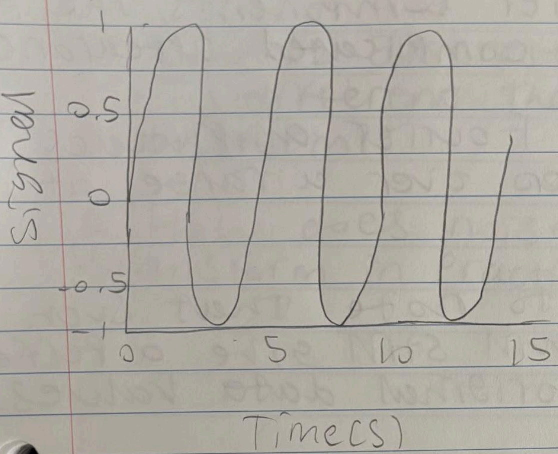
$$Y_n = Y^e_{n_1,n_0} + w^n Y^o_{n_1,n_0}$$

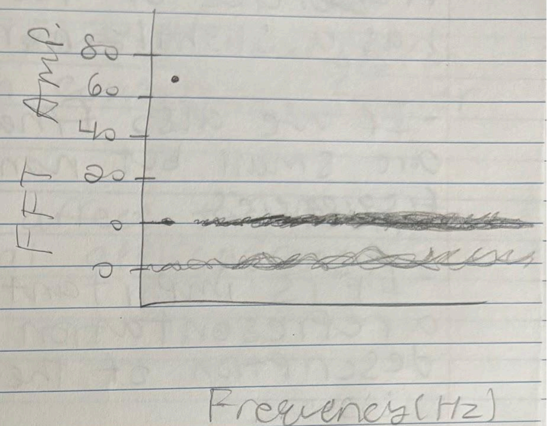
- We can repeat this for:

$$Y^e_{n_1,n_0} = Y^{ee}_{n_0} + w^{4n_1+2n_0} Y^{eo}_{n_0}$$

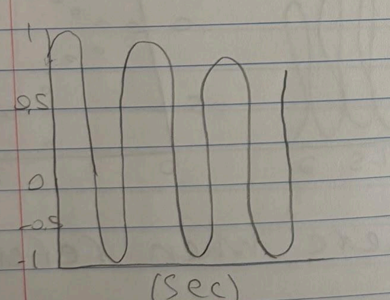
$$Y^o_{n_1} = Y^{oe}_{n_0} + w^{4n_1+2n_0} Y^{eo}_{n_0}$$

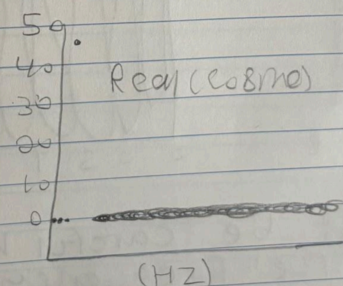
Example 1.

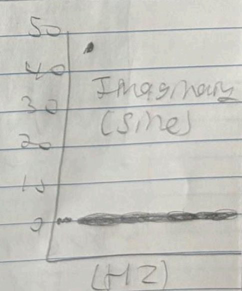




C4. Sampling Interval/Number of Data Points







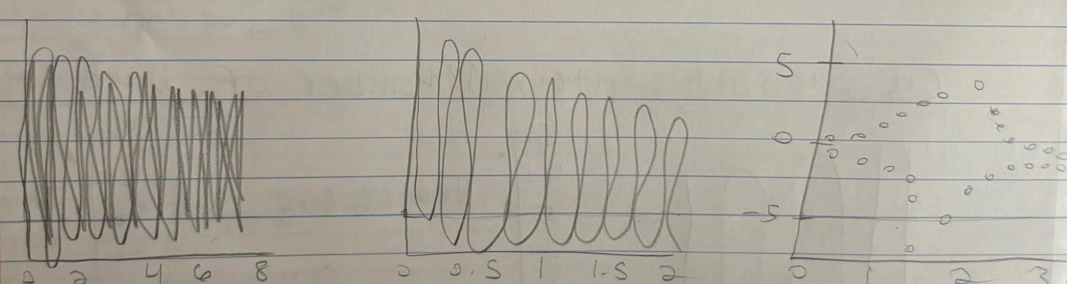
-Phase Shifted by π/4

- If sampling time does not match the frequencies of Fourier components, the FFT has a slightly more complicated appearance.

- If we also find Fourier amplitudes that are small but nonzero over a range of frequencies.

- It is important to note that such a representation will still give a perfect description of the original data values.

6.5 Aliasing



- We must be careful here when referring to the "true" frequency.

- When samples are recorded only at intervals of $\Delta t = 0.2\,s$, these two sine waves yield precisely the same signal.

Questions:

1. How is discrete fourier transform (DFT) different than continuous fourier transform?

2. How does noise affect the fourier transform in practical applications?