Section 9-5 : Solving The Heat Equation

Okay, it is finally time to completely solve a partial differential equation. In the previous section we applied separation of variables to several partial differential equations and reduced the problem down to needing to solve two ordinary differential equations. In this section we will now solve those ordinary differential equations and use the results to get a solution to the partial differential equation. We will be concentrating on the heat equation in this section and will do the wave equation and Laplace's equation in later sections.

The first problem that we're going to look at will be the temperature distribution in a bar with zero temperature boundaries. We are going to do the work in a couple of steps so we can take our time and see how everything works.

The first thing that we need to do is find a solution that will satisfy the partial differential equation and the boundary conditions. At this point we will not worry about the initial condition. The solution we'll get first will not satisfy the vast majority of initial conditions but as we'll see it can be used to find a solution that will satisfy a sufficiently nice initial condition.

Example 1 Find a solution to the following partial differential equation that will also satisfy the boundary conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Show Solution

So, there we have it. The function above will satisfy the heat equation and the boundary condition of zero temperature on the ends of the bar.

The problem with this solution is that it simply will not satisfy almost every possible initial condition we could possibly want to use. That does not mean however, that there aren't at least a few that it will satisfy as the next example illustrates.

Example 2 Solve the following heat problem for the given initial conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

(a) $f(x) = 6\sin\left(\frac{\pi x}{L}\right)$

(b) $f(x) = 12\sin\left(\frac{9\pi x}{L}\right) - 7\sin\left(\frac{4\pi x}{L}\right)$

(a) $f(x) = 6\sin\left(\frac{\pi x}{L}\right)$ ***Show Solution*** ▸

(b) $f(x) = 12\sin\left(\frac{9\pi x}{L}\right) - 7\sin\left(\frac{4\pi x}{L}\right)$ ***Show Solution*** ▸

So, we've seen that our solution from the first example will satisfy at least a small number of highly specific initial conditions.

Now, let's extend the idea out that we used in the second part of the previous example a little to see how we can get a solution that will satisfy any sufficiently nice initial condition. The Principle of Superposition is, of course, not restricted to only two solutions. For instance, the following is also a solution to the partial differential equation.

$$u(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and notice that this solution will not only satisfy the boundary conditions but it will also satisfy the initial condition,

$$u(x,0) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Let's extend this out even further and take the limit as $M \to \infty$. Doing this our solution now becomes,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

This solution will satisfy any initial condition that can be written in the form,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This may still seem to be very restrictive, but the series on the right should look awful familiar to you after the previous chapter. The series on the left is exactly the **Fourier sine series** we looked at in that chapter. Also recall that when we can write down the Fourier sine series for any **piecewise smooth** function on $0 \leq x \leq L$.

So, provided our initial condition is piecewise smooth after applying the initial condition to our solution we can determine the B_n as if we were finding the Fourier sine series of initial condition. So we can either proceed as we did in that section and use the orthogonality of the sines to derive them or we can acknowledge that we've already done that work and know that coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

So, we finally can completely solve a partial differential equation.

Example 3 Solve the following BVP.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = 20 \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Show Solution ▸

That almost seems anti-climactic. This was a very short problem. Of course, some of that came about because we had a really simple constant initial condition and so the integral was very simple. However, don't forget all the work that we had to put into discussing Fourier sine series, solving boundary value problems, applying separation of variables and then putting all of that together to reach this point.

While the example itself was very simple, it was only simple because of all the work that we had to put into developing the ideas that even allowed us to do this. Because of how "simple" it will often be to actually get these solutions we're not actually going to do anymore with specific initial conditions. We will instead concentrate on simply developing the formulas that we'd be required to evaluate in order to get an actual solution.

So, having said that let's move onto the next example. In this case we're going to again look at the temperature distribution in a bar with **perfectly insulated** boundaries. We are also no longer going to go in steps. We will do the full solution as a single example and end up with a solution that will satisfy any piecewise smooth initial condition.

Example 4 Find a solution to the following partial differential equation.

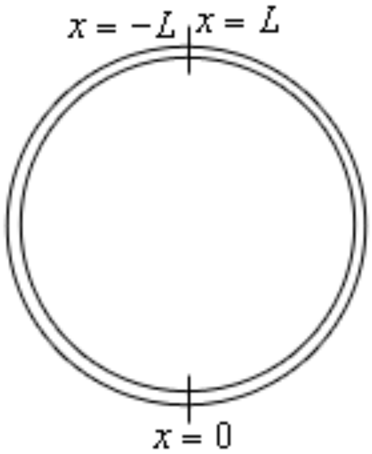
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

Show Solution

The last example that we're going to work in this section is a little different from the first two. We are going to consider the temperature distribution in a thin circular ring. We will consider the lateral surfaces to be perfectly insulated and we are also going to assume that the ring is thin enough so that the temperature does not vary with distance from the center of the ring.

So, what does that leave us with? Let's set $x = 0$ as shown below and then let x be the arc length of the ring as measured from this point.



We will measure x as positive if we move to the right and negative if we move to the left of $x = 0$. This means that at the top of the ring we'll meet where $x = L$ (if we move to the right) and $x = -L$ (if we move to the left). By doing this we can consider this ring to be a bar of length $2L$ and the heat equation that we **developed** earlier in this chapter will still hold.

At the point of the ring we consider the two "ends" to be in **perfect thermal contact**. This means that at the two ends both the temperature and the heat flux must be equal. In other words we must have,

$$u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

If you recall from the **section** in which we derived the heat equation we called these periodic boundary conditions. So, the problem we need to solve to get the temperature distribution in this case is,

Example 5 Find a solution to the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Show Solution

Okay, we've now seen three heat equation problems solved and so we'll leave this section. You might want to go through and do the two cases where we have a zero temperature on one boundary and a perfectly insulated boundary on the other to see if you've got this process down.

