Minimum Polynomial & Disturbance/Tracking Poles*

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1 Minimum Polynomial of A Square Matrix

Let $A \neq 0$ be an n–square matrix over $\mathcal{F}$[1]. Since $A \in \mathcal{M}_n(\mathcal{F})$[2], the set $\{I, A, A^2, \ldots, A^{n^2}\}$ is linearly dependent and there exist scalars $a_0, a_1, a_2, \ldots, a_{n^2}$ not all 0 such that

$$\phi(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n^2} A^{n^2} = 0$$

In this section we shall be concerned with that monic–polynomial[3] $m(\lambda) \in \mathcal{F}[\lambda]$[4] of minimum degree such that $m(A) = 0$. Clearly, either $m(\lambda) = \phi(\lambda)$ or $m(\lambda)$ is a proper divisor of $\phi(\lambda)$. In either case, $m(\lambda)$ will be called the *minimum polynomial* [1, p.177] of A.

The most elementary procedure for obtaining the minimum polynomial of $A \neq 0$ involves the following routine:

1. If $A = a_0 I$, $a_0 \in \mathcal{F}$, then $m(\lambda) = \lambda - a_0$.
2. If $A \neq aI$ for all $a \in \mathcal{F}$ but $A^2 = a_1 A + a_0 I$ with $a_0, a_1 \in \mathcal{F}$, then $m(\lambda) = \lambda^2 - a_1\lambda - a_0$.
3. If $A^2 \neq aA + bI$ for all $a, b \in \mathcal{F}$ but $A^3 = a_2 A^2 + a_1 A + A_0 I$ with $a_0, a_1, a_2 \in \mathcal{F}$, then $m(\lambda) = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$

and so on.

Example 1 Find the minimum polynomial of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Since $A - I = 0$, the answer is $m(\lambda) = \lambda - 1$.

Example 2 Find the minimum polynomial of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Since $A^2 = 0$, the answer is $m(\lambda) = \lambda^2$.

Example 3 Find the minimum polynomial of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ over Q.

Since $A \neq a_0 I$ for all $a_0 \in Q$, set

$$A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + a_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 + a_0 & 2a_1 & 2a_1 \\ 2a_1 & a_1 + a_0 & 2a_1 \\ 2a_1 & 2a_1 & a_1 + a_0 \end{bmatrix}$$

After checking every entry, we conclude that $A^2 = 4A + 5I$ and the minimum polynomial is $\lambda^2 - 4\lambda - 5$.

*This material is based on reference [1] and [2]

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[1] $\mathcal{F}$: Field- See Appendix.

[2] $\mathcal{M}_n(\mathcal{F})$: Total matrix algebra (the set of all n×n matrices over $\mathcal{F}$)

[3] Monic Polynomial - See Appendix.

[4] $\mathcal{F}[x]$: Set of all polynomials in x with coefficients in $\mathcal{F}$

Foregoing example 3 suggests that the constant term of the minimum polynomial of $A \neq 0$ is different from 0 if and only if A is non–singular. It can be used for computing the inverse of a non–singular matrix follows.

Example 4 For A of example 3, find the inverse A^{-1}.
Since $A^2 - 4A - 5I = 0$ we have, after multiplying by A^{-1}, $A - 4I - 5A^{-1} = 0$;

hence, $A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$.

2 Plant Model

Consider multivariable linear time–invariant (LTI) plants, i.e., it will be assumed that the following model can describe the plant to be controlled:

$$\begin{aligned} \dot{x} &= Ax + Bu + E\omega \\ y &= Cx + Du + F\omega \\ y_m &= C_m x + D_m u + F_m \omega \\ e &= y - y_{ref} \end{aligned} \tag{1}$$

where u are the control inputs, x is the state, y are the outputs to be regulated, y_m are the measurable outputs, ω are the disturbances, y_{ref} are the tracking signals and e are the errors in the system. It will be assumed that the plant has m control inputs, n states, r outputs to be regulated, r_m measurable outputs, Ω disturbances, r tracking signals and r errors. Thus here the variables $u, x, y, y_m, \omega, y_{ref}, e$ are actually column vectors with dimensions as follows:

$$\begin{aligned} u &\in R^m \\ x &\in R^n \\ y &\in R^r \\ y_m &\in R^{r_m} \\ \omega &\in R^\Omega \\ y_{ref} &\in R^r \\ e &\in R^r \end{aligned}$$

3 Class of controllers

In general, in order to modify the behavior of the plant (1) to solve the servomechanism problem, it is necessary to apply a controller to it. A controller, in general, is a dynamic device which is applied to the plant, and has as its input y_m and y_{ref} and for its output u. Many different types of controllers are possible to apply; in this lesson, we shall only consider LTI controllers. The most general LTI controller is described by:

$$\begin{aligned} \dot{\xi} &= \Lambda_0 \xi + \Lambda_1 y_m + \Lambda_2 y_{ref} \\ u &= K_0 \xi + K_1 y_m + K_2 y_{ref} \end{aligned} \tag{2}$$

which has as inputs the tracking signal y_{ref} and the measurable outputs y_m, and has as outputs the control signal u. Here ξ is a state variable of the controller and $K_0, K_1, K_2, \Lambda_0, \Lambda_1, \Lambda_2$ are constant gain matrices associated with the controller.

4 Class of Tracking/Disturbance Signals

In classical control, two classes of signals which are widely used to describe tracking signals y_{ref} and disturbance signals ω are the class of step and ramp type signals. Here a large class of signals will be considered, namely any signal which can arise from an unstable LTI system will be considered. In particular, it will be assumed that the class of tracking signals and disturbance signals is specified, and arise from:

$$\begin{aligned} \dot z_1 &= A_1 z_1\ ,\ z_1 \in R^{\overline{n_1}} \qquad & \dot z_2 &= A_2 z_2\ ,\ z_2 \in R^{\overline{n_2}} \\ \sigma_1 &= C_1 z_1 & \omega &= C_2 z_2 \\ y_{ref} &= G\,\sigma_1 \end{aligned} \tag{3}$$

for some initial conditions $z_1(0), z_2(0)$, where for non–redundancy we assume that (C_1, A_1) and (C_2, A_2) are observable, and that rank $G = \dim(\sigma_1)$. For non–triviality, we assume that the signals y_{ref}, ω do not decay to zero, i.e., we assume that $Re(\lambda_1^i) \geq 0\ ,\ i = 1, 2, \ldots, \overline{n_1}\ ,\ Re(\lambda_2^i) \geq 0\ ,\ i = 1, 2, \ldots, \overline{n_2}$ where $\{\lambda_1^i,\ i = 1, 2, \ldots, \overline{n_1}\}\ ,\ \{\lambda_2^i,\ i = 1, 2, \ldots, \overline{n_2}\}$ denote the eigenvalues of A_1, A_2 respectively. This class of tracking/disturbance signals includes most classes of signals which occur in industrial systems, e.g. constant, ramp, polynomial, sinusoidal, polynomial–sinusoidal, exponential–polynomial signals.

The above equations (1),(2), and (3) are very important for this lession since they define the plant model, the controller, and the tracking and disturbance signals. These models form the fundamental mathematical building blocks for the material which we will study in this lesson. Figure 1 illustrates the relationship between these mathematical models.

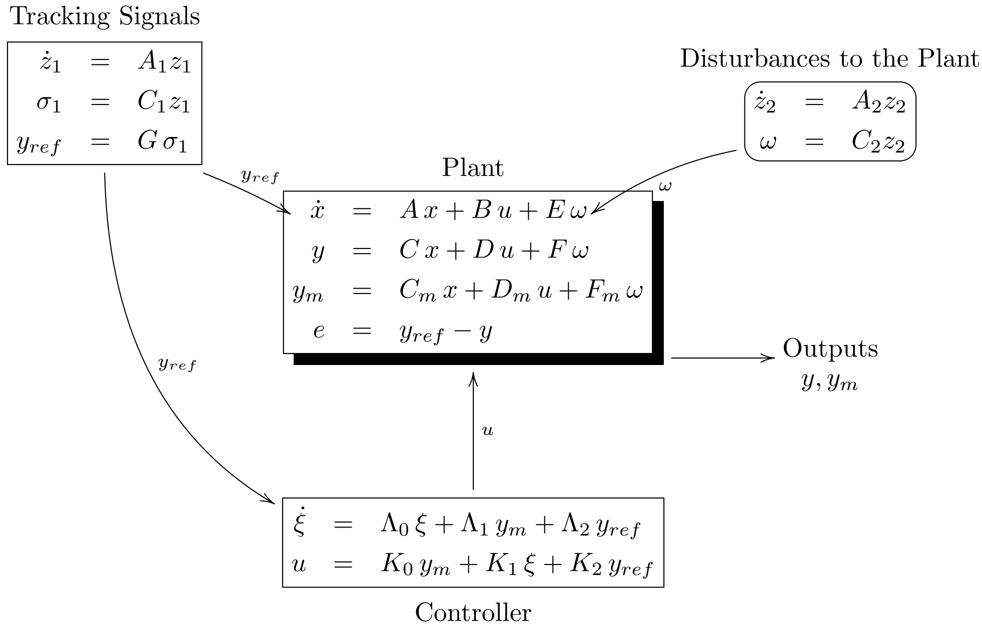

Figure 1: Mathematical model for linear system

Given (3), let $\{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ be the zeros of the least common multiple of the minimum polynomial of A_1 and minimum polynomial of A_2 (multiplicities repeated), and call

$$\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_q\} \tag{4}$$

the *disturbance/tracking* poles[2, pp.95–96] of (3).

Note: Often the class of disturbance/tracking poles $\{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ is used, rather than (3), to describe the class of disturbance/reference input signals being considered, e.g.

- the class of constant signals is denoted by $\Lambda = \{0\}$
- the class of ramp signals is denoted by $\Lambda = \{0, 0\}$
- the class of sinusoidal signals ($sin\,\omega t$) is denoted by $\Lambda = \{j\omega, -j\omega\}$
- the class of exponential signals ($e^{\lambda t}$) is denoted by $\Lambda = \{\lambda\}$

Example 5 Find the disturbance/tracking poles of the system (3) given that

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}, \ \theta \neq 0$$

$$C_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

First, find the minimum polynomial $m_1(\lambda)$, $m_2(\lambda)$ of each A_1, A_2 respectively.

$$\begin{array}{lcl} {A_1}^3 = {A_1}^2 & \rightarrow & m_1(\lambda) = \lambda^2(\lambda - 1) \\ {A_2}^2 = -\theta^2 I & \rightarrow & m_2(\lambda) = \lambda^2 + \theta^2 \end{array}$$

Thus the zeros of the least common multiple of the $m_1(\lambda)$ and $m_2(\lambda)$ is $\{0, 0, 1, j\theta, -j\theta\}$.

Appendix : Terminologies

A Group

A non-empty set $\mathcal{G}$ on which a binary operation $\circ$ is defined is said to form a group with respect to this operation provided, for arbitrary $a, b, c \in \mathcal{G}$, the following properties hold: [1, p.82]

1. $(a \circ b) \circ c = a \circ (b \circ c)$ (associative law)
2. There exists $u \in \mathcal{G}$ such that $a \circ u = u \circ a = a$ (existence of identity element)
3. For each $a \in \mathcal{G}$ there exists $a^{-1} \in \mathcal{G}$ such that $a \circ a^{-1} = a^{-1} \circ a = u$ (existence of inverse)

Note 1. The reader must not be confused by the use in 3. of a^{-1} to denote the inverse of a under the operation $\circ$. The notation is merely borrowed from that previously used in connection with multiplication. Whenever the group operation is addition, a^{-1} is to be interpreted as the additive inverse $-a$.

Note 2. A group is called *abelian* if the group operation is commutative.

B Ring

A non–empty set $\mathcal{R}$ is said to form a *ring* with respect to the binary operations addition (+) and multiplication ($\bullet$) provided, for arbitrary $a, b, c \in \mathcal{R}$, the following properties hold: [1, p.101]

1. $(a + b) + c = a + (b + c)$ (associative law of addition)
2. $a + b = b + a$ (commutative law of addition)
3. There exists $z \in \mathcal{R}$ such that $a + z = a$. (existence of an additive identity (zero))
4. For each $a \in \mathcal{R}$ there exists $-a \in \mathcal{R}$ such that $a + (-a) = z$. (existence of additive inverses)
5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative law of multiplication)
6. $a(b + c) = a \cdot b + a \cdot c$ (distributive laws)

C Field

A ring $\mathcal{F}$ whose non–zero elements form an abelian multiplicative group is called *field*. [1, p.118]

D Monic Polynomial

Let $\mathcal{R}$ be a ring with unity u. Any polynomial $\alpha(x)$ of degree m over $\mathcal{R}$ with leading coefficient u, the unity of $\mathcal{R}$, will be called *monic*.[1, p.126]

Example 6 The polynomials 1, $x + 3$, and $x^2 - 5x + 4$ are monic while $2x^2 - x + 5$ is not a monic polynomial over I.

References

[1] J. Frank Ayres, *Modern Algebra*. Schaum's Outline Series, McGraw-Hill, June 1965.

[2] Michael K. Masten *et al.*, *Modern Control System*. IEEE, 1995.