1 Orthogonal Vectors and Matrices

Adjoint:
The complex conjugate of a scalar, z, is denoted by either z*, and is obtained by negating the imaginary part.
The hermitian conjugate, or adjoint, of A $\in \mathbb{C}^{\,m\times n}$ is denoted by A*.
We mean to say that A* $\in \mathbb{C}^{\,n\times m}$, with entries a$^*{}_{i,j}$ equal to the complex conjugate of a$_{i,j}$.
If A $\in \mathbb{R}^{\,m\times n}$, then this is denoted by A^T.
If A = A^T, then A is symmetric.

1.1 Inner Product

Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^{\,m}$.
Take the convention that every vector is a column vector, and only deviate that if you specifically say it's a row vector.
The inner product is given by:

$$\mathbf{x}^*\mathbf{y} = \sum_{i=1}^{m} x_i y_i$$

The Euclidean length of x may be written norm(x), defined by

$$\begin{aligned} norm(x) &= \sqrt{x^* x} \\ &= (\sum_{i=1}^{m} |x_i|^2)^{\frac{1}{2}} \\ hi \end{aligned}$$

Recall:

$$x_i = a + b_i$$

$$|x_i| = \sqrt{a^2 + b^2}$$

We can also define an angle between two vectors as:

$$\cos\alpha = \frac{x^* y}{||x|| \times ||y||}$$

Inner products are bilinear, or linear in each vector.

$$
\begin{aligned}
(x_1 + x_2)^* y &= x_1^* y + x_2^* y \\
x^*(y_1 + y_2) &= x^* y_1 + x^* y_2 \\
&= (\alpha x)^*(\beta y) = \alpha\beta x^* y
\end{aligned}
$$

Note:

$$(AB)^* = B^* A^*$$

and

$$(AB)^{-1} = B^{-1} A^{-1}$$

1.2 Orthogonal Vectors

x and **y** are orthogonal if

$$\mathbf{x}^* \mathbf{y} = 0$$

Two sets of vectors X and Y are orthogonal if every $\mathbf{x} \in$ X is orthogonal to every $\mathbf{y} \in$ Y.
A set of nonzero vectors S is orthogonal if its elements are pairwise orthogonal.
If x, y $\in$ S, then x*y = or x = y.
A set S of vectors is orthonormal if S is orthogonal and all $\mathbf{x} \in$ S have the property:

$$||x|| = 1$$

Theorem:

The vectors in an orthogonal set S are linearly independent.

Proof.

If the vectors in S are not linearly independent, then some vector $\mathbf{v}_k \in$ S can be written as:

$$v_k = \sum_{i=1, i \neq k}^{n} c_i v_i$$

Since $\mathbf{v}_k \neq 0$,

$$\mathbf{v}_k^* \mathbf{v}_k = ||\mathbf{v}_k||^2 > 0$$

However,

$$\mathbf{v}_k^* \mathbf{v}_k = \sum_{i=1, i \neq k}^{n} c_i \mathbf{v}_k^* \mathbf{v}_i = 0$$

However, we said that this would be positive, which is a contradiction.
Hence, they're linearly independent.

$\square$

We may easily expand a vector into its orthogonal components.
Suppose

$$\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n, \}$$

is an orthogonal set, and let v be an arbitrary vector.

$$r = v - (q_1^* v) q_1 - (q_2^* v) q_2 - ... - (q_n^* v) q_n$$

So r, what's left of v, is orthogonal to all the q's.
To prove that r is orthogonal to all the q's, look at:

$$q_i^* r = q_i^* v - (q_1^* v) q_i^* q_1 - ... - (q_i^* v) q_i^* q_i - ... - (q_n^* v) q_i^* q_n = 0$$

1.3 Unitary Matrices

Q $\in \mathbb{C}^{\,m \times m}$ is unitary (or orthogonal if real) if Q* = Q^{-1}, or Q*Q = I

Note, bad terminology:
A matrix is orthogonal mean it forms an orthonormal.
If a matrix is orthogonal but not unit length vectors, then we say the matrix has an orthogonal set.

$$\underset{m\times m}{\begin{bmatrix} — & q_1^* & — \\ — & q_2^* & — \\ — & ... & — \\ — & q_m^* & — \end{bmatrix}} \underset{m\times m}{\begin{bmatrix} | & | & | & | \\ q_1 & q_2 & ... & q_m \\ | & | & | & | \end{bmatrix}} = \underset{m\times m}{\begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}}$$

i.e.

$$q_i^* q_j = \delta_{i,j} =$$

1 if i = j, 0 if i $\neq$ j.
kronecker delta
So Q^*b is the vector of coefficients of the expansion of b in the basis of the columns of Q.
So if you're trying to solve Ax = b, and A is unitary, then you could write Qx = b, and you could multiply by Q^* on both sides, so x = Q^*b.

If Q is unitary, then

$$(Qx)^*(Qy) = x^*y$$

and

$$\begin{aligned} ||x||^2 &= x \times x \\ &= (Qx)^*(Qx) \\ &= ||Qx||^2 \end{aligned}$$

Hence, the length of Qx is the same as x.
NORMS NORMS
A norm is a function:

$$ll.ll : \mathbb{C}^m \longrightarrow \mathbb{R}$$

that assigns a real-valued length to each vector that has three properties:

1. norm(x) $\geq$ 0 and norm(x) = 0 $\Leftarrow\ \Rightarrow$ x = $\mathbf{0}$
2. norm(x + y) $\leq$ norm(x) + norm(y)
3. norm(α x) = norm(α) * norm(x)

Other norms:

$$norm(x)_1 = \sum_{i=1}^{m} |x_i|$$

A diamond

$$norm(x)_2 = (\sum_{i=1}^{m} |x_i|^2)^{1/2}$$

A circle

$$norm(x)_p = (\sum_{i=1}^{m} |x_i|^p)^{1/p}$$

A squared off circle

$$norm(x)_\infty = max_{1 \leq i \leq m} |x_i|$$

A square

$$norm(x)_w = norm(Wx)$$

with some kind of diagonal matrix W, where $w_i \neq 0$. This will give you something like an ellipse.
You can build a matrix norm as induced by vector norms.
Given some norms n and m on the domain and range of some matrix A $\in \mathbb{C}^{m \times n}$, then the induced matrix norm, ll A ll sub (m, n) is the smallest number C such that ll Ax ll sub m $\leq$ C ll x ll sub n
or
ll A ll sub m, n = sup x $\in \mathbb{C}^{n}$ $\frac{llAxll_m}{llxll_(n)}$
Problems 2.1 and 2.3 are due Monday 2/4 or something, Problem 1.3 is due Wednesday 1/30.