2.1

Show that if a matrix A is both triangular and unitary, then it is diagonal.

Let A $\in \mathbb{C}^{m \times n}$ be triangular and unitary. We wish to show that A is, therefore, diagonal.
However, suppose A is also not diagonal.
Then, $\exists$ $a_j \in$ A and some $a_{i,j} \in a_j$ such that $a_{i,j} \neq 0$ and i $\neq$ j.
Without loss of generality, let's assume A is upper triangular (i.e. that i < j).
Notice:
$a_{i,i} \in a_i$ and $a_{i,i} \neq 0$ (since A is triangular).
Since A is unitary, $a_i{}^* \cdot a_j = 0$. Specifically,

$$a_{1,i} * \cdot a_{1,j} + a_{2,i} * \cdot a_{2,j} + ... + a_{i,i} * \cdot a_{i,j} + ... + a_{j,i} * \cdot a_{j,j} + ... a_{m,i} * \cdot a_{m,j} = 0$$

Notice: $a_{i,i} \cdot a_{i,j} \neq 0$ and $a_{k,i} \cdot a_{k,j} = 0$ for k > i. So,

$$a_{i,i} * \cdot a_{i,j} + ... + a_{j,i} * \cdot a_{j,j} + ... a_{m,i} * \cdot a_{m,j} = 0$$

2.3

Let A $\in \mathbb{C}^{m \times m}$ be hermitian. An eigenvector of A is a nonzero vector x $\in \mathbb{C}^{m}$ such that Ax = $\lambda$x for some $\lambda \in \mathbb{C}$ , the corresponding eigenvalue.

1. Prove that all eigenvalues of A are real.

   Let $\lambda$ be the eigenvector of x such that Ax = $\lambda$x. Notice:

$$\begin{aligned}
Ax &= \lambda x \\
(Ax)^* &= (\lambda x)^* \\
x^* A^* &= x^* \lambda^* \\
x^* A^* x &= x^* \lambda^* x \\
x^* A x &= x^* \lambda^* x \\
x^* A x &= \lambda^* x^* x \\
x^* A x &= \lambda^* \|x\|^2 \\
\frac{x^* A x}{\|x\|^2} &= \lambda^*
\end{aligned}$$

and

$$\begin{aligned}
Ax &= \lambda x \\
x^* Ax &= x^* \lambda x \\
&= \lambda x^* x \\
&= \lambda \|x\|^2 \\
\frac{x^* Ax}{\|x\|^2} &= \lambda
\end{aligned}$$

Hence, $\lambda^* = \lambda$

2. Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

   Let v1 and v2 be eigenvectors of A with corresponding eigenvalues $\lambda_1$ and $\lambda_2$ such that $\lambda_1 \neq \lambda_2$. So,

$$Av_1 = \lambda_1 v_1 \qquad Av_2 = \lambda_2 v_2$$

   Notice:

$$\begin{aligned}
Av_2 &= \lambda_2 v_2 \\
v_1^* Av_2 &= v_1^* \lambda_2 v_2 \\
&= \lambda_2 v_1^* v_2
\end{aligned}$$

   and

$$\begin{aligned}
Av_2 &= \lambda_2 v_2 \\
v_1^* Av_2 &= v_1^* Av_2 \\
&= ((v_1^* Av_2)^*)^* \\
&= (v_2^* A^* v_1)^* \\
&= (v_2^* Av_1)^* \\
&= (v_2^* \lambda_1 v_1)^* \\
&= v_1^* \lambda_1^* v_2 \\
&= \lambda_1 v_1^* v_2
\end{aligned}$$

Since

$$\lambda_2 v_1^* v_2 = \lambda_1 v_1^* v_2$$

and $\lambda_1 \neq \lambda_2$, v1 and v2 must be orthogonal.

3.2

Let $\|.\|$ denote a norm on $\mathbb{C}^{\ m}$ and a matrix norm on A. Show that p(A) $\leq$ $\|A\|$, where p(A) is the spectral radius of A (i.e. the largest absolute value of an eigenvalue of A).

Recall the definition of a induced matrix norm:

$$\|A\|_{(m,n)} = \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} = \sup_{x \in \mathbb{C}^m, \|x\|_{(m)}=1} \|Ax\|_{(m)}$$

Let v be the eigenvector of A corresponding to the eigenvalue $\lambda$ of largest magnitude (i.e. the eigenvalue with the largest absolute value).
By definition,

$$\|A\|_{(m,n)} = \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}}$$

And since eigenvectors of a are members of $\mathbb{C}^{\ m}$,

$$\begin{aligned}
\sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(m)}} &\geq \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|\lambda x\|_{(m)}}{\|x\|_{(m)}} \\
&= \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{|\lambda| \|x\|_{(m)}}{\|x\|_{(m)}} \\
&= \sup_{x \in \mathbb{C}^m, x \neq 0} |\lambda| \\
&= |\lambda| \\
&= p(A)
\end{aligned}$$

Hence, p(A) $\leq \|A\|$