Joshua Mitchell; MATH 5374; HW 4; 6.1, 6.3,

6.1

If P is an orthogonal projector, then I − 2P is unitary. Prove this algebraically, and give a geometric interpretation.

In other words, we want to show:

$$(I-2P)(I-2P)^* = I$$

Recall that if P is an orthogonal projector, then P = P^*.

$$\begin{aligned}
&(I-2P)(I-2P)^* \\
&\qquad = (I-2P)(I-2P^*) \\
&\qquad = I^2 - 2P - 2P^* + 4PP^* \\
&\qquad = I^2 - 4P + 4PP \\
&\qquad = I^2 - 4P + 4P \\
&\qquad = I^2 \\
&\qquad = I
\end{aligned}$$

Hence, I − 2P is unitary.

The geometric interpretation is this:
I − P is a comp projector to P. Since they project onto opposite spaces and I − P is already unitary, scaling the vectors subtracted from I doesn't change the direction, which still makes it orthogonal.

6.3

Give A ∈ $\mathbb{C}^{m\times n}$ with m ≥ n, show that A^*A is nonsingular if and only if A has full rank.

⇒ (beginning with A^*A is non-singular)
A^*A is non-singular implies that

$$\sigma_n \geq \sigma_{n-1} \geq ... \geq \sigma_1 > 0$$

By Theorem 5.4, σ's of A are the square roots of A*A.
Thus,

$$\sigma_i = \sqrt{\lambda_i}$$

and

$$\sigma_i^2 = \lambda_i > 0$$

for i := 1 ... n
Since all n eigenvalues are greater than 0 and m $\geq$ n, A has full rank.

$\Leftarrow$ (beginning with A has full rank)

A has full rank implies that

$$\lambda_n \geq \lambda_{n-1} \geq ... \geq \lambda_1 > 0$$

displaymath

Notice:

$$A^*A = A^2 = (XVX^{-1})^2 = XV^2X^{-1}$$

Since all eigenvalues of A are nonzero, that means all entries of V^2 are nonzero.
Since all entries of V^2 are non-zero, that means that A^2 is full rank.
Since A^2 is full rank, that implies A^2 and A^*A are non-singular.