Joshua Mitchell; MATH 5374; HW 3; 4.4 and 5.4

4.4

Two matrices A, B $\in \mathbb{C}^{m \times m}$ are unitarily equivalent if A = QBQ* for some unitary Q $\in \mathbb{C}^{m \times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values?

$\Rightarrow$
Assume that A and B are unitarily equivalent. Specifically,
A = QBQ* for some unitary Q $\in \mathbb{C}^{m \times m}$
Notice:

$$\begin{aligned}
A &= QBQ^* \\
U_A \Sigma_A V_A^* &= Q U_B \Sigma_B V_B^* Q^* \\
U_A \Sigma_A V_A^* &= Q U_B \qquad \Sigma_B \qquad V_B^* Q^* \\
\Sigma_A &= U_A^* Q U_B \qquad \Sigma_B \qquad V_B^* Q^* V_A
\end{aligned}$$

Since Q, U$_B$, V$_B{}^*$, Q*, U$_A$, and V$_A{}^*$ are all unitary, any matrix that is a product of these 6 matrices is also unitary. Thus, no scaling is done by any product subset of these matrices.

Since both Σ_A and Σ_B are diagonal matrices composed of singular values of A and B, respectively, and can be written as non-scaling transformations of each other, they must have the same set of singular values.

$\Leftarrow$
Assume A and B have the same set of singular values. Specifically,

$$A = U_A \Sigma V_A^* \qquad B = U_B \Sigma V_B^*$$

Then,

$$\begin{aligned}
A &= U_A \Sigma V_A^* \\
U_A^* A V_A &= \Sigma
\end{aligned}$$

and

$$
\begin{aligned}
B &= U_B U_A^* A V_A V_B^* \\
&= U_B U_A^* \qquad A \qquad V_A V_B^*
\end{aligned}
$$

Now, we wish to show:

$$U_B U_A^* = (V_A V_B^*)^{-1} = (V_A V_B^*)^* = V_B V_A^*$$

5.4

Suppose A $\in \mathbb{C}^{\,m\times m}$ has an SVD A = UΣV*. Find an eigenvalue decomposition (5.1) of the 2m x 2m hermitian matrix B:

$$B = \underset{2m\times 2m}{\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}}$$

An eigenvalue decomposition of B will be of the form:

$$B = XVX^{-1}$$

Notice:

$$
\begin{aligned}
B &= \underset{2m\times 2m}{\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}} \\
&= \underset{2m\times 2m}{\begin{bmatrix} 0 & (U\Sigma V^*)^* \\ U\Sigma V^* & 0 \end{bmatrix}} \\
&= \underset{2m\times 2m}{\begin{bmatrix} 0 & V\Sigma U^* \\ U\Sigma V^* & 0 \end{bmatrix}} \\
&= \underset{2m\times 2m}{\begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}} \underset{2m\times 2m}{\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}} \underset{2m\times 2m}{\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}} \\
&= \underset{2m\times 2m}{\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}} \underset{2m\times 2m}{\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}} \underset{2m\times 2m}{\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}}
\end{aligned}
$$

The last decomposition has the singular values in the right place, but the before and after aren't inverses. The 2nd to last has them as inverses, but no decomposition in the right place.
If you have two matrices on each side of a diagonal matrix that are inverses, does this mean that the left matrix is a matrix of the eigenvectors of the resulting matrix?