

Joshua Mitchell; MATH 5374; HW 3; 4.4 and 5.4

4.4

Two matrices $A, B \in \mathbb{C}^{m \times m}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m \times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values?

\Rightarrow

Assume that A and B are unitarily equivalent. Specifically,

$A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m \times m}$

Notice:

$$\begin{aligned} A &= QBQ^* \\ U_A \Sigma_A V_A^* &= QU_B \Sigma_B V_B^* Q^* \\ U_A \Sigma_A V_A^* &= QU_B \quad \Sigma_B \quad V_B^* Q^* \\ \Sigma_A &= U_A^* QU_B \quad \Sigma_B \quad V_B^* Q^* V_A \end{aligned}$$

Since Q , U_B , V_B^* , Q^* , U_A , and V_A^* are all unitary, any matrix that is a product of these 6 matrices is also unitary. Thus, no scaling is done by any product subset of these matrices.

Since both Σ_A and Σ_B are diagonal matrices composed of singular values of A and B , respectively, and can be written as non-scaling transformations of each other, they must have the same set of singular values.

\Leftarrow

Assume A and B have the same set of singular values. Specifically,

$$A = U_A \Sigma V_A^* \quad B = U_B \Sigma V_B^*$$

Then,

$$\begin{aligned} A &= U_A \Sigma V_A^* \\ U_A^* A V_A &= \Sigma \end{aligned}$$

and

$$\begin{aligned}
B &= U_B U_A^* A V_A V_B^* \\
&= U_B U_A^* \quad A \quad V_A V_B^*
\end{aligned}$$

Now, we wish to show:

$$U_B U_A^* = (V_A V_B^*)^{-1} = (V_A V_B^*)^* = V_B V_A^*$$

5.4

Suppose $A \in \mathbb{C}^{m \times m}$ has an SVD $A = U \Sigma V^*$. Find an eigenvalue decomposition (5.1) of the $2m \times 2m$ hermitian matrix B :

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}_{2m \times 2m}$$

An eigenvalue decomposition of B will be of the form:

$$B = X V X^{-1}$$

Notice:

$$\begin{aligned}
B &= \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}_{2m \times 2m} \\
&= \begin{bmatrix} 0 & (U \Sigma V^*)^* \\ U \Sigma V^* & 0 \end{bmatrix}_{2m \times 2m} \\
&= \begin{bmatrix} 0 & V \Sigma U^* \\ U \Sigma V^* & 0 \end{bmatrix}_{2m \times 2m} \\
&= \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}_{2m \times 2m} \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}_{2m \times 2m} \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}_{2m \times 2m} \\
&= \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}_{2m \times 2m} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}_{2m \times 2m} \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}_{2m \times 2m}
\end{aligned}$$

The last decomposition has the singular values in the right place, but the before and after aren't inverses. The 2nd to last has them as inverses, but no decomposition in the right place.