2.1

Show that if a matrix A is both triangular and unitary, then it is diagonal.

Let A $\in \mathbb{C}^{m \times n}$ be triangular and unitary. We wish to show that A is, therefore, diagonal.

** ATTEMPT 1 (This didn't work, but I'd like to know if it could) **

However, suppose A is also not diagonal.
Then, $\exists$ $a_j \in$ A and some $a_{i,j} \in a_j$ such that $a_{i,j} \neq 0$ and i $\neq$ j.
Without loss of generality, let's assume A is upper triangular (i.e. that i < j).
Notice:
$a_{i,i} \in a_i$ and $a_{i,i} \neq 0$ (since A is triangular).
Since A is unitary, $a_i{*} \cdot a_j = 0$. Specifically,

$$a_{1,i} * \cdot a_{1,j} + a_{2,i} * \cdot a_{2,j} + ... + a_{i,i} * \cdot a_{i,j} + ... + a_{j,i} * \cdot a_{j,j} + ... a_{m,i} * \cdot a_{m,j} = 0$$

Notice: $a_{i,i} \cdot a_{i,j} \neq 0$ and $a_{k,i} \cdot a_{k,j} = 0$ for k > i. So,

$$a_{i,i} * \cdot a_{i,j} + ... + a_{j,i} * \cdot a_{j,j} + ... a_{m,i} * \cdot a_{m,j} = 0$$

Now, if I could just show that

$$a_{i+1,i} * \cdot a_{i+1,j} + ... a_{m,i} * \cdot a_{m,j} = 0$$

That would complete the proof by contradiction. Not sure how to do that, though.

** ATTEMPT 2 **

Proof sketch (by induction), since I'm running out of time:
Base case: up to $a_1$, A is diagonal since there's only one non-zero entry and it's at the top.
Inductive step: A is diagonal up to $a_{k-1}$.
Prove: A is diagonal up to k $-$ 1 $\Rightarrow$ A is diagonal up to k.
Idea: since $a_{k-1}$ is diagonal, if $a_k$ were to have any non-zero entries above $a_{k,k}$, then the vector $a_k$ would lean into the space occupied by one of the

vectors before it (i.e. if $a_k$ had a non zero entry at i < k, then it'd be leaning into the space occupied by $a_i$.

2.3

Let A $\in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector x $\in \mathbb{C}^{m}$ such that Ax = $\lambda$x for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

1. Prove that all eigenvalues of A are real.

   Let $\lambda$ be the eigenvector of x such that Ax = $\lambda$x. Notice:

$$\begin{aligned} Ax &= \lambda x \\ (Ax)^* &= (\lambda x)^* \\ x^*A^* &= x^*\lambda^* \\ x^*A^*x &= x^*\lambda^*x \\ x^*Ax &= x^*\lambda^*x \\ x^*Ax &= \lambda^*x^*x \\ x^*Ax &= \lambda^*\|x\|^2 \\ \frac{x^*Ax}{\|x\|^2} &= \lambda^* \end{aligned}$$

   and

$$\begin{aligned} Ax &= \lambda x \\ x^*Ax &= x^*\lambda x \\ &= \lambda x^*x \\ &= \lambda\|x\|^2 \\ \frac{x^*Ax}{\|x\|^2} &= \lambda \end{aligned}$$

   Hence, $\lambda^* = \lambda$

2. Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Let v1 and v2 be eigenvectors of A with corresponding eigenvalues $\lambda_1$ and $\lambda_2$ such that $\lambda_1 \neq \lambda_2$. So,

$$Av_1 = \lambda_1 v_1 \qquad Av_2 = \lambda_2 v_2$$

Notice:

$$\begin{aligned} Av_2 &= \lambda_2 v_2 \\ v_1^* A v_2 &= v_1^* \lambda_2 v_2 \\ &= \lambda_2 v_1^* v_2 \end{aligned}$$

and

$$\begin{aligned} Av_2 &= \lambda_2 v_2 \\ v_1^* A v_2 &= v_1^* A v_2 \\ &= ((v_1^* A v_2)^*)^* \\ &= (v_2^* A^* v_1)^* \\ &= (v_2^* A v_1)^* \\ &= (v_2^* \lambda_1 v_1)^* \\ &= v_1^* \lambda_1^* v_2 \\ &= \lambda_1 v_1^* v_2 \end{aligned}$$

Since

$$\lambda_2 v_1^* v_2 = \lambda_1 v_1^* v_2$$

and $\lambda_1 \neq \lambda_2$, v1 and v2 must be orthogonal.

3.2

Let $\|.\|$ denote a norm on $\mathbb{C}^{\,m}$ and a matrix norm on A. Show that p(A) $\leq$ $\|A\|$, where p(A) is the spectral radius of A (i.e. the largest absolute value of an eigenvalue of A).

Recall the definition of a induced matrix norm:

$$\|A\|_{(m,n)} = \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} = \sup_{x \in \mathbb{C}^m, \|x\|_{(m)}=1} \|Ax\|_{(m)}$$

Let v be the eigenvector of A corresponding to the eigenvalue $\lambda$ of largest magnitude (i.e. the eigenvalue with the largest absolute value).
By definition,

$$\|A\|_{(m,n)} = \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}}$$

And since eigenvectors of a are members of $\mathbb{C}^{\ m}$,

$$\begin{aligned} \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(m)}} &\geq \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|\lambda x\|_{(m)}}{\|x\|_{(m)}} \\ &= \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{|\lambda| \|x\|_{(m)}}{\|x\|_{(m)}} \\ &= \sup_{x \in \mathbb{C}^m, x \neq 0} |\lambda| \\ &= |\lambda| \\ &= p(A) \end{aligned}$$

Hence, p(A) $\leq \|A\|$