Joshua Mitchell; MATH 5374; HW 3; 4.4 and 5.4

4.4

Two matrices A, B $\in \mathbb{C}^{m\times m}$ are unitarily equivalent if A = QBQ* for some unitary Q $\in \mathbb{C}^{m\times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values?

$\Rightarrow$

Assume that A and B are unitarily equivalent. Specifically,
A = QBQ* for some unitary Q $\in \mathbb{C}^{m\times m}$
Notice:

$$\begin{aligned} A &= QBQ^* \\ U_A\Sigma_A V_A^* &= QU_B\Sigma_B V_B^* Q^* \\ U_A\Sigma_A V_A^* &= QU_B \qquad \Sigma_B \qquad V_B^* Q^* \\ \Sigma_A &= U_A^* Q U_B \qquad \Sigma_B \qquad V_B^* Q^* V_A \end{aligned}$$

Since Q, U$_B$, V$_B{}^*$, Q*, U$_A$, and V$_A{}^*$ are all unitary, any matrix that is a product of these 6 matrices is also unitary. Thus, no scaling is done by any product subset of these matrices.

Since both Σ_A and Σ_B are diagonal matrices composed of singular values of A and B, respectively, and can be written as non-scaling transformations of each other, they must have the same set of singular values.

$\Leftarrow$

Assume A and B have the same set of singular values. Specifically,

$$A = U_A\Sigma V_A^* \qquad B = U_B\Sigma V_B^*$$

Then,

$$\begin{aligned} A &= U_A\Sigma V_A^* \\ U_A^* A V_A &= \Sigma \end{aligned}$$

and

$$
\begin{aligned}
B &= U_B U_A^* A V_A V_B^* \\
&= U_B U_A^* \qquad A \qquad V_A V_B^*
\end{aligned}
$$

Now, we wish to show:

$$U_B U_A^* = (V_A V_B^*)^{-1} = (V_A V_B^*)^* = V_B V_A^*$$

Alas, I'm not sure how to do that.

5.4

Suppose A $\in \mathbb{C}^{m \times m}$ has an SVD A = UΣV*. Find an eigenvalue decomposition (5.1) of the 2m x 2m hermitian matrix B:

$$B = \underset{2m \times 2m}{\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}}$$

An eigenvalue decomposition of B will be of the form:

$$B = XVX^{-1}$$

Recall: Theorems 6.4.3 and 6.4.4 of Linear Algebra with Applications, edition 4:

Theorem 6.4.3 (Schur's Theorem):

For each n $\times$ n matrix A, there exists a unitary matrix U such that U*AU is upper triangular.

and

Theorem 6.4.4 (Spectral Theorem):

If A is Hermitian, then there exists a unitary matrix U that diagonalizes A (i.e. that U*AU = D, a diagonal matrix).

Since B is hermitian,

$$U^*BU = D$$

where U is a unitary matrix and D is a diagonal matrix. With some algebra,

$$B = UDU^*$$

Since U diagonalizes B, it follows that the diagonal elements of D are the eigenvalues of A and the column vectors of U are eigenvectors of A.