01/23's HW:

Exercise 1.3: Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is upper-triangular if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a nonsingular m × m upper-triangular matrix, then R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices.)

Proof.

Let A be a nonsingular m × m upper-triangular matrix:

$$A = \underbrace{\begin{bmatrix} a_{11} & ... & ... & a_{1m} \\ 0 & ... & ... & ... \\ ... & 0 & ... & ... \\ 0 & ... & 0 & a_{mm} \end{bmatrix}}_{m\times m}$$

Let $\mathbf{e}_j$ be the jth unit vector in the vector space $\mathbb{C}^m$, i.e.

$$I = \underbrace{\begin{bmatrix} | & | & | & | \\ e_1 & e_2 & ... & e_m \\ | & | & | & | \end{bmatrix}}_{m\times m}$$

We want to show that $Z = A^{-1}$ is also upper-triangular. Let's look at $\mathbf{a}_1$, the first column of A.

$$\mathbf{e}_1 = Z\mathbf{r}_1$$

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{m\times 1} = \underbrace{\begin{bmatrix} z_{1,1} & ... & ... & z_{1,m} \\ z_{2,1} & ... & ... & z_{2,m} \\ ... & ... & ... & z_{1,m} \\ z_{m,1} & ... & z_{m,m-1} & z_{m,m} \end{bmatrix}}_{m\times m} \underbrace{\begin{bmatrix} a_1 \\ 0 \\ ... \\ 0 \end{bmatrix}}_{m\times 1}$$

Notice:
If we look at entries 2 through m of $\mathbf{e}_1$, they're all zero.
This means that $\mathbf{r}_1 \cdot \mathbf{z}_i = 0$ for Z rows i = 2 through m.
Since entries 2 through m of $\mathbf{r}_1$ are 0,

$$0 = z_{i,1} \times r_{11}$$

for i = 2 through m.

Since $r_{11} \neq 0$,

$$z_{i,1} = 0$$

for i = 2 through m. Thus, we have:

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{m \times 1} = \underbrace{\begin{bmatrix} z_{1,1} & z_{1,2} & \dots & z_{1,m} \\ 0 & z_{2,2} & \dots & z_{2,m} \\ \dots & \dots & \dots & z_{1,m} \\ 0 & z_{m,2}. & z_{m,m-1} & z_{m,m} \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} a_1 \\ 0 \\ \dots \\ 0 \end{bmatrix}}_{m \times 1}$$

In the above case, Z is upper-triangular in at least the first column. We establish this as the base case, and desire to show that Z is upper triangular in general.
Inductive step:

Assume every entry in $\mathbf{z}_j$ past entry j is 0 for j = 1, 2, ..., i − 1. We want to show that it is also true for j = i.
Let j = i where i < m. Notice:

$$\underbrace{\begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}}_{m \times 1} = A\mathbf{z}_j = \underbrace{\begin{bmatrix} z_{1,1} & \dots & \dots & z_{1,j} & \dots & z_{1,m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & z_{j-1,j-1} & z_{j-1,j} & \dots & z_{j-1,m} \\ 0 & \dots & 0 & z_{j,j} & \dots & z_{j,m} \\ 0 & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & z_{m,j} & \dots & z_{m,m} \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} a_1 \\ \dots \\ a_{j-1} \\ a_j \\ 0 \\ \dots \\ 0 \end{bmatrix}}_{m \times 1}$$

So,

$$A\mathbf{z}_j = \underset{m\times m}{\begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & z_{j,j} & \ldots & z_{j,m} \\ 0 & \ldots & 0 & z_{j+1,j} & \ldots & z_{j+1,m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & z_{m,j} & \ldots & z_{m,m} \end{bmatrix}} \underset{m\times 1}{\begin{bmatrix} a_1 \\ \ldots \\ a_{j-1} \\ a_j \\ 0 \\ \ldots \\ 0 \end{bmatrix}}$$

Notice:
To the left of the jth column of Z is all zeros, and the bottom of the a column (below a_j) is also all zeros. Thus, the dot product of Z's row with $\mathbf{a}_j$ is determined solely by the $z_{k,j}$ entry times the a_j entry (for k = j + 1 to m).

Since $0 = z_{k,j} \times a_j$ for k = j + 1 to m and a_j is non-zero by definition, $z_{k,j}$ must be zero for k = j + 1 to m.
Hence, there is all zeros below $z_{j,j}$ for all j.
Hence, $Z = A^{-1}$ is upper-triangular.

$\square$