

2.1

Show that if a matrix A is both triangular and unitary, then it is diagonal.

Let $A \in \mathbb{C}^{m \times n}$ be triangular and unitary. We wish to show that A is, therefore, diagonal.

** ATTEMPT 1 (This didn't work, but I'd like to know if it could) **

However, suppose A is also not diagonal.

Then, $\exists a_j \in A$ and some $a_{i,j} \in a_j$ such that $a_{i,j} \neq 0$ and $i \neq j$.

Without loss of generality, let's assume A is upper triangular (i.e. that $i < j$).

Notice:

$a_{i,i} \in a_i$ and $a_{i,i} \neq 0$ (since A is triangular).

Since A is unitary, $a_i^* \cdot a_j = 0$. Specifically,

$$a_{1,i}^* \cdot a_{1,j} + a_{2,i}^* \cdot a_{2,j} + \dots + a_{i,i}^* \cdot a_{i,j} + \dots + a_{j,i}^* \cdot a_{j,j} + \dots a_{m,i}^* \cdot a_{m,j} = 0$$

Notice: $a_{i,i} \cdot a_{i,j} \neq 0$ and $a_{k,i} \cdot a_{k,j} = 0$ for $k > i$. So,

$$a_{i,i}^* \cdot a_{i,j} + \dots + a_{j,i}^* \cdot a_{j,j} + \dots a_{m,i}^* \cdot a_{m,j} = 0$$

Now, if I could just show that

$$a_{i+1,i}^* \cdot a_{i+1,j} + \dots a_{m,i}^* \cdot a_{m,j} = 0$$

That would complete the proof by contradiction. Not sure how to do that, though.

** ATTEMPT 2 **

Proof sketch (by induction), since I'm running out of time:

Base case: up to a_1 , A is diagonal since there's only one non-zero entry and it's at the top.

Inductive step: A is diagonal up to a_{k-1} .

Prove: A is diagonal up to $k-1 \Rightarrow A$ is diagonal up to k .

Idea: since a_{k-1} is diagonal, if a_k were to have any non-zero entries above $a_{k,k}$, then the vector a_k would lean into the space occupied by one of the

vectors before it (i.e. if a_k had a non zero entry at $i < k$, then it'd be leaning into the space occupied by a_i).

2.3

Let $A \in \mathbb{C}^{m \times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

1. Prove that all eigenvalues of A are real.

Let λ be the eigenvalue of x such that $Ax = \lambda x$. Notice:

$$\begin{aligned} Ax &= \lambda x \\ (Ax)^* &= (\lambda x)^* \\ x^* A^* &= x^* \lambda^* \\ x^* A^* x &= x^* \lambda^* x \\ x^* Ax &= x^* \lambda^* x \\ x^* Ax &= \lambda^* x^* x \\ x^* Ax &= \lambda^* \|x\|^2 \\ \frac{x^* Ax}{\|x\|^2} &= \lambda^* \end{aligned}$$

and

$$\begin{aligned} Ax &= \lambda x \\ x^* Ax &= x^* \lambda x \\ &= \lambda x^* x \\ &= \lambda \|x\|^2 \\ \frac{x^* Ax}{\|x\|^2} &= \lambda \end{aligned}$$

Hence, $\lambda^* = \lambda$

2. Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Let v_1 and v_2 be eigenvectors of A with corresponding eigenvalues λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$. So,

$$Av_1 = \lambda_1 v_1 \quad Av_2 = \lambda_2 v_2$$

Notice:

$$\begin{aligned} Av_2 &= \lambda_2 v_2 \\ v_1^* Av_2 &= v_1^* \lambda_2 v_2 \\ &= \lambda_2 v_1^* v_2 \end{aligned}$$

and

$$\begin{aligned} Av_2 &= \lambda_2 v_2 \\ v_1^* Av_2 &= v_1^* \lambda_2 v_2 \\ &= ((v_1^* Av_2)^*)^* \\ &= (v_2^* A^* v_1)^* \\ &= (v_2^* \lambda_1 v_1)^* \\ &= \lambda_1 v_2^* v_1 \\ &= \lambda_1 v_1^* v_2 \end{aligned}$$

Since

$$\lambda_2 v_1^* v_2 = \lambda_1 v_1^* v_2$$

and $\lambda_1 \neq \lambda_2$, v_1 and v_2 must be orthogonal.

3.2

Let $\|\cdot\|$ denote a norm on \mathbb{C}^m and a matrix norm on A . Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of A (i.e. the largest absolute value of an eigenvalue of A).

Recall the definition of a induced matrix norm:

$$\|A\|_{(m,n)} = \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} = \sup_{x \in \mathbb{C}^m, \|x\|_{(m)}=1} \|Ax\|_{(m)}$$

Let v be the eigenvector of A corresponding to the eigenvalue λ of largest magnitude (i.e. the eigenvalue with the largest absolute value).

By definition,

$$\|A\|_{(m,n)} = \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}}$$

And since eigenvectors of A are members of \mathbb{C}^m ,

$$\begin{aligned} \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(m)}} &\geq \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{\|\lambda x\|_{(m)}}{\|x\|_{(m)}} \\ &= \sup_{x \in \mathbb{C}^m, x \neq 0} \frac{|\lambda| \|x\|_{(m)}}{\|x\|_{(m)}} \\ &= \sup_{x \in \mathbb{C}^m, x \neq 0} |\lambda| \\ &= |\lambda| \\ &= p(A) \end{aligned}$$

Hence, $p(A) \leq \|A\|$