Joshua Mitchell; MATH 5374; HW 5; 7.5

7.5

Let A be an m $\times$ n matrix (m $\geq$ n), and let A = $\hat{Q}\hat{R}$ be a reduced QR factorization.

1. Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are non-zero.

   $\Rightarrow$

   Suppose that A has rank n.

   Want to show: A has rank equal to the number of non-zero diagonal entries in $\hat{R}$ by induction.

   Base case: Column 1 of A spans 1 dimension. Since q_1 is non-zero and a_1 is non-zero, that implies that $r_{1,1}$ is non-zero.

   Inductive step: Assume that A has non-zero diagonal entries up to column k $-$ 1, k $<$ n.

   Now, we want to show that if A has non-zero diagonal entries up to column k $-$ 1, the kth diagonal entry of $\hat{R}$ must also be non-zero.

   However, suppose that the kth diagonal is 0.

   Since A is full rank up to column k $-$ 1, that implies that its $\hat{Q}\hat{R}$ decomposition up to k $-$ 1 is also full rank.

   This also means that any vector in $\mathbb{C}^{\,k-1}$ can be written as a linear combination of the first k $-$ 1 columns of A and of $\hat{Q}\hat{R}$.

   Since $\hat{R}_{i,k}$ for i:= k ... n is 0 and $\hat{Q}$ is orthonormal, $A_k = \hat{Q}\hat{R}_k$ is a vector that lies in the span of the first k $-$ 1 columns of A.

   This is a contradiction, since A is full rank. Hence, $\hat{R}_{k,k}$ must be non-zero.

   Thus, by induction, all diagonal entries of $\hat{R}$ must be non-zero.

$\Leftarrow$

Assume that all diagonal entries of $\hat{R}$ are non-zero.

Then

$$\begin{aligned} a_1 &= r_{11}q_1 \\ a_2 &= r_{12}q_1 + r_{22}q_2 \\ &\dots \\ a_n &= r_{1n}q_1 + r_{2n}q_2 + \dots + r_{nn}q_n \end{aligned}$$

where $r_{i,i} \neq 0$, for i:= 1 to n.

Notice: A up to a_1 is rank 1 as a base case.

Assume A up to A_{k-1} is rank k $-$ 1.

Then

$$a_k = r_{1k}q_1 + r_{2k}q_2 + \dots + r_{kk}q_k$$

Since A up to k $-$ 1 is full rank, $r_{k,k}$ is nonzero, and $\hat{Q}$ is orthonormal, the $r_{k,k}q_k$ vector is not a linear combination of $\hat{Q}\hat{R}_i$ for i:= 1 ... k $-$ 1. Therefore, a_k is not a linear combination of the first k $-$ 1 columns of A, and A up to column k has rank k.

Hence, by induction, A has rank n.

2. Suppose that $\hat{R}$ has k nonzero diagonal entries for some k with 0 $\leq$ k $<$ n. What does this imply about the rank of A? Exactly k? At least k? At most k? Give a precise answer, and prove it.

   Well, if $\hat{R}$ has 0 non-zero diagonal entries (i.e. all zeros down the diagonal) but has non-zero entries everywhere else in the upper part, then it still has rank n $-$ 1. So it can't be exactly k nor at most k. So if the only other option is at least k, then that has to be it.

   Proof:

   Suppose $\hat{R}$ has k non-zero diagonal entries. Then each corresponding column of A is a linear combination of orthonormal vectors from $\hat{Q}$. Since each successive linear combination adds a new dimension to the rank of A due to the non-zero entry on the diagonal, that means A has at least rank k.