Due 4/9:
G1 (present): page 150: 1, 7, 8
G2 (present): page 150: 3, 6, 9, 12, 14 (me: 3, 14)
All (turn in): page 150: 17, 19, 29, 36 (me)
Due 4/11:
Present: page 167: 20
All (turn in): page 167: 1, 22

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Exercise 3

Let H = {0, ±3, ±6, ±9...}. Rewrite the condition $a^{-1}b \in H$ given in property 6 of the lemma on page 139 in additive notation. Assume that the group is Abelian. Use this to decide whether or not the following cosets of H are the same.

Property 6: aH = bH iff $a^{-1}b \in H$
Rewritten: a + H = b + H iff $a^{-1} + b \in H$

a. **11 + H and 17 + H**: $-11 + 17 = 6 \in H$, so yes.

b. **−1 + H and 5 + H**: 1 + 5 = 6 mem H, so yes.

c. **7 + H and 23 + H**: $-7 + 23 = 16 \notin H$, so no.

Exercise 14

Let C^* be the group of nonzero complex numbers under multiplication and let $H = \{a + bi \in C^* : a^2 + b^2 = 1\}$. Give a geometric description of the cosets (3 + 4i)H and (c + di)H.
Well,
$(3 + 4i)H = \{(3 + 4i)h : h \in H\}$
$(3 + 4i)H = \{(3 + 4i)(a + bi) : a + bi \in C^*, a^2 + b^2 = 1\}$
$(3 + 4i)H = \{3a + 4ai + 3bi - 4b: a + bi \in C^*, a^2 + b^2 = 1\}$
$(3 + 4i)H = \{3a + (4a + 3b)i - 4b: a + bi \in C^*, a^2 + b^2 = 1\}$
thus,
$(c + di)H = \{ca + (da + cb)i - db: a + bi \in C^*, a^2 + b^2 = 1\}$
$(c + di)H = \{(ca - db) + (da + cb)i: a + bi \in C^*, a^2 + b^2 = 1\}$
It looks like the subset H just indicates the elements that create a unit circle.
When we multiply by some real constant > 1, we just get a coset that represents a bigger circle.
When we multiply by some complex constant (e.g. 2i), we just get a coset that represents a flipped circle (where x, y becomes y, x), and if the complex constant has a scaling factor (e.g. 2), then the circle grows by that factor.
As far as the description of a coset with a positive real and positive complex part, I think it transforms it into an ellipse.

Exercise 17

Let G be a group with |G| = pq: p, q are prime. Prove that every proper subgroup of G is cyclic.
Let H be a proper subgroup of G.
Since G is finite, |H| divides |G|.

Case:

i) |H| = 1: Then H is cyclic by default.

ii) |H| $\neq$ 1: Then by the fundamental theorem of arithmetic, |H| = t: t $\in$ {p, q}
Notice: |H| > 1.
Let h $\in$ H: h $\neq$ e.
Then $1 < |<h>| \leq$ |H|.
Since H is finite, $|<h>|$ divides |H|.
Since |H| is prime, its factors are only 1 and |H|.
Since $|<h>| \neq 1$, this implies that $|<h>|$ = |H|.
Hence, H must be cyclic.

Exercise 19

Compute 5^{15} mod 7 and 7^{13} mod 11.

Fermat's Little Theorem: For every integer a and prime p, a^p mod p = a mod p

$$\begin{aligned} 5^{15} \mod 7 &= 5^{15*7} \mod 7 \\ &= (5^{15})^7 \mod 7 \end{aligned}$$

Exercise 29

Let |G| = 33. What are the possible orders for the elements of G? Show that G must have an element of order 3.

Exercise 36

Let G be a group and |G| = 21. If g $\in$ G and g^{14} = e, what are the possibilities for |g|?

Well, since g is a generator for H, a cyclic subgroup of G, that means that |H| must be a factor of |G|. Since |G| = 21 and 14 doesn't divide 21, |H| must be some factor of both 21 and 14, but lower than 14. Those possibilities are: 1, 7

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Exercise 1

Prove that the external direct product of any finite number of groups is a group.

Exercise 20

Find a subgroup of $\mathbf{Z}_{12}$ (+) $\mathbf{Z}_{18}$ that is isomorphic to $\mathbf{Z}_9$ (+) $\mathbf{Z}_4$.

Exercise 22

Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $\mathbf{Z}_{30}$ (+) $\mathbf{Z}_{20}$.