HW 8: pages 193, #1, 2, 3, 5, 9, 10, 17
For 2(c), see Theorem 1 and Example 9 below
Make sure when you do these problems, justify the answer by either writing down the theorem name or providing a counter example.

Section 4.3 Continued

Theorem 4.4.4

If a sequence s_n converges to $s \in \mathbb{R}$, then every subsequence of $\{s_n\}$ converges to s as well.

Proof.

Assume $\{s_n\}$ converges to s.
$\forall\ \epsilon > 0$, $\exists$ $N(\epsilon) \in \mathbb{N}$ st $|s_n - s| < \epsilon$ for $n \geq N$ **(1)**

Let: $\{s_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{s_n\}$

Side Note

$\forall$ $k \in \mathbb{N}$,
if $n_k \geq k$ and k diverges to ∞, then n_k diverges to ∞
So, for $N \in \mathbb{N}$, $\exists$ $k \in \mathbb{N}$ st $n_k > N$ for $k \geq K$

By practice 4.4.3,
$\lim\limits_{k\to\infty} n_k = \infty$
Thus,
$\exists$ $K \in \mathbb{N}$ st $n_k > N$ for $k \geq K$ **(2)**
From **(1)** and **(2)**,

$$|s_{n_k} - s| < \epsilon \text{ for } k \geq K$$

Hence,
$\lim\limits_{k\to\infty} s_{n_k} = s$ □

Example 4.4.5 (see page 170, Ex 7(f) for a similar example) (can use this for hw)

Prove that if $0 < x < 1$, then $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$

Proof.

Let: $x \in \mathbb{R}$, $0 < x < 1$
Define $s_n = x^{\frac{1}{n}}$ for $n \in \mathbb{N}$
We shall prove that $\{s_n\}$ is an increasing sequence that is bounded above.

Notice that for n $\in \mathbb{N}$,

$$\begin{aligned} x^{\frac{1}{n+1}} - x^{\frac{1}{n}} &= x^{\frac{1}{n+1}}(1 - x^{\frac{1}{n}-\frac{1}{n+1}}) \\ &= x^{\frac{1}{n+1}}(1 - x^{n(n+1)}) \\ &= \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{n+1-n}{n(n+1)} \\ &= \frac{1}{n(n+1)} > 0 \end{aligned}$$

So, $s_{n+1} \geq s_n$, $\forall$ n $\in \mathbb{N}$, i.e. $\{s_n\}$ is increasing and $s_n = x^{\frac{1}{n}} < 1$ $\forall$ n $\in \mathbb{N}$.
By the Monotone Convergence Theorem (4.3.3?),
$\exists$ s $\in \mathbb{R}$ st

$$\lim_{n\to\infty} s_n = s$$

Now $\{s_{2k}\}$ is a subsequence of s_n.
By Theorem 4.4.4,
$\lim_{k\to\infty} s_{2k} = s$

Side Note

This is $\{s_{n_k}\}$ where $n_k = 2k$ $\forall$ k $\in \mathbb{N}$

However,
$s_{2k} = x^{\frac{1}{2k}} = (x^{\frac{1}{k}})^{\frac{1}{2}} = \sqrt{s_k}$
By Exercise 4.2.6,
$\lim_{k\to\infty} s_{2k} = \sqrt{s}$
Thus, s = $\sqrt{s}$
But, if a sequence converges, then the limit is unique.
So,

$$s^2 = s$$

$$s(s-1) = 0$$

$$s = 0, 1$$

However, we know that one of those must be wrong.
Since $s_1 = x^{\frac{1}{1}} = x > 0$ and $s_n \geq s$ $\forall$ n $\in \mathbb{N}$,
we see that s $\neq$ 0.
Hence, s = 1

□

Exercise 4.4.6

If $s_n = (-1)^n$ $\forall$ n $\in \mathbb{N}$, prove that $\{s_n\}$ diverges.
Notice that
$s_{2k} = 1$ $\forall$ k $\in \mathbb{N}$
while
$s_{2k-1} = -1$, $\forall$ k $\in \mathbb{N}$
Thus, we have subsequences $\{s_{2k}\}$, $\{s_{2k-1}\}$ st

$$\lim_{k\to\infty} s_{2k} = 1 \text{ and } \lim_{k\to\infty} s_{2k-1} = -1$$

Hence, $\{s_n\}$ diverges.

Theorem 4.4.7

Every bounded sequence has a convergent subsequence.

Proof.

Let: $\{s_n\}$ be a bounded sequence
Denote S as the range of $\{s_n\}$: S = $\{s_n$: n $\in \mathbb{N}$ $\}$

i) S is finite.

$\exists$ k $\in \mathbb{N}$ st S = $\{s_1, s_2, ... s_k\}$

Then there is at least one element s $\in$ S st s is equal to an infinite number of terms of $\{s_n\}$. (i.e. if the range has a finite number of elements, then that means s_n jumps between each of those elements an infinite number of times. Think of 1, −1, 1, −1...)

Thus,

Choose n_1 such that s_{n_1} = s.

Then,

Choose $n_2 > n_1$ such that s_{n_2} = s.

Inductively, $\exists$ $s_{n_k} \in$ S such that s_{n_k} = s and $n_1 < n_2 < ... < n_k$

Hence, $\lim\limits_{k\to\infty} s_{n_k}$ = s

ii) S is infinite.

Since $\{s_n\}$ is bounded, S (our set described above) is also bounded.

By the Bolzano-Weierstrass Theorem, $\exists$ s $\in S'$ (i.e. an accumulation or limit point: s)

By HW Exercise 15, page 142 (section 3.4), if x $\in S'$, then N(x, ϵ) contains an infinite number of points in S.

Thus, $\exists$ $s_{n_1} \in$ S st $s_{n_1} \in$ N(s, 1) (i.e. (s − 1, s + 1)

$|s_{n_1} - s| < \frac{1}{1}$

Then,

$\exists$ $n_2 > n_1$ st

$|s_{n_2} - s| < \frac{1}{2}$

So, inductively, we can keep doing this (i.e. for N(s, $\frac{1}{3}$), N(s, $\frac{1}{4}$), etc)

Thus,

$$|s_{n_k} - s| < \frac{1}{k} \text{ and } n_1 < n_2 < ... < n_k$$

Hence, $\lim\limits_{k\to\infty} s_{n_k}$ = s

□

Theorem 4.4.8

Every unbounded sequence contains a monotonic sequence that diverges to ∞ or $-\infty$

Proof.

Let: $\{s_n\}$ be a sequence that is unbounded above
Then, for m $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st
$s_n >$ m if n $\geq$ N
Notice that this implies that there are an infinite number of terms of s_n that are strictly larger than m.
(If there were only a finite number of terms greater than m, then s_n wouldn't be unbounded above. There would be a largest term, which would make it have an upper bound.)
Thus,
$\exists$ $n_1 \in \mathbb{N}$ st
$s_{n_1} > 1$
Then,
$\exists$ $n_2 > n_1$ st
$s_{n_2} > 2$
So, inductively, for k $\in \mathbb{N}$, $\exists$ $n_k \in \mathbb{N}$ st

$$s_{n_k} > k \text{ where } n_1 < n_2 < ... < n_k$$

Hence, for m $\in \mathbb{R}$, the AP guarantees k $\in \mathbb{N}$ st $s_{n_k} >$ k $>$ m, $\forall$ k $\geq$ K.
This implies that $\lim\limits_{k\to\infty} s_{n_k} = \infty$
If S is bounded above, then S must be unbounded below, a similar method shows that there is a subsequence $\{s_{n_l}\}$ st $\lim\limits_{l\to\infty} s_{n_l} = -\infty$

$\square$