

**Exercise 1**

Let  $A = \{0, 1, 2, 3, 4\}$  and  $B = \{0, 1, 2, 3\}$ . For each of the relations  $R$  from  $A$  to  $B$  listed below list all pairs  $(a, b) \in \mathbb{R}$  and write the corresponding  $\{0, 1\}$ -indicator-matrix.

a.  $a = b : (0, 0), (1, 1), (2, 2), (3, 3)$

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	0	0	0

b.  $a + b = 4 : (1, 3), (2, 2), (3, 1), (4, 0)$

0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	1
1	0	0	0

c.  $a > b : (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)$

0	0	0	0
1	1	0	0
1	0	0	0
1	1	1	0
1	1	1	1

d.  $a$  divides  $b : (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1, 2), (2, 2), (1, 3)$

0	0	0	0
1	1	1	1
1	0	1	0
1	0	0	0
1	0	0	0

**Exercise 2**

For each of these relations on the set  $\{1, 2, 3, 4\}$  decide whether or not it is reflexive, symmetric, antisymmetric, and transitive.

- $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
- $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
- $\{(2, 4), (4, 2)\}$
- $\{(1, 2), (2, 3), (3, 4)\}$
- $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

<i>Relation</i>	<i>R</i>	<i>S</i>	<i>A</i>	<i>T</i>
<i>a</i>	0	0	0	1
<i>b</i>	1	1	0	1
<i>c</i>	0	1	0	1
<i>d</i>	0	0	1	0
<i>e</i>	1	1	1	1
<i>f</i>	0	0	0	1

**Exercise 3**

Let  $R$  be the relation  $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$ , and let  $S$  be the relation  $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$  on the set  $A = \{1, 2, 3, 4\}$

- Find  $R \cup S$   
 $\{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- Find  $R \cap S$   
 $\{(3, 1)\}$
- Find  $R \circ S$   
 $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

**Exercise 4**

Let  $R$  be the relation  $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$  on the set  $A = \{1, 2, 3, 4\}$ .

- Find the reflexive closure of  $R$ .  
 $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 1), (3, 3), (4, 4)\}$
- Find the symmetric closure of  $R$ .  
 $\{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- Find the transitive closure of  $R$ .  
 $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (1, 4)\}$

**Exercise 5**

Prove the following:

- a. A relation  $R$  is reflexive iff  $R^{-1}$  is reflexive (where  $R^{-1}$  is the inverse relation that just reverses the order).

→

Assume  $R$  is reflexive.

Let  $(a, a) \in R$

Then  $(a, a) \in R^{-1}$

Hence,  $R^{-1}$  is reflexive.

←

Assume  $R^{-1}$  is reflexive.

Let  $(a, a) \in R^{-1}$

Then  $(a, a) \in R$

Hence,  $R$  is reflexive.

- b. A relation  $R$  is symmetric iff  $R = R^{-1}$ .

→

Assume  $R$  is symmetric.

Let  $(a, b) \in R$ .

Want to show:  $(a, b) \in R^{-1}$ .

Notice:  $(b, a) \in R$ .

Thus,  $(a, b) \in R^{-1}$ .

Hence,  $R = R^{-1}$ .

←

Assume  $R = R^{-1}$ .

Let  $(a, b) \in R$ .

Then  $(a, b) \in R^{-1}$ .

$(a, b) \in R \Rightarrow (b, a) \in R^{-1}$ .

But since  $R^{-1} = R$ ,  $(b, a) \in R$ .

So,  $(a, b) \in R \Rightarrow (b, a) \in R$ .

Hence,  $R$  is symmetric..

- c. A relation  $R$  is anti-symmetric iff  $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$

→

Assume  $R$  is anti-symmetric.

Then  $(a, b), (b, a) \in R \Rightarrow a = b$ .

So,  $R \cap R^{-1}$  will only contain tuples such that  $a = b$ .

←

Assume  $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$ .

Let  $(a, b) \in R$ . If  $a \neq b$ , then  $(a, b) \notin R \cap R^{-1}$ . Thus,  $(a, b) \notin R^{-1}$ .

Hence,  $R$  is anti-symmetric.

**Exercise 6**

Let  $R$  be the relation represented by the matrix  $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ . Find the matrices for the relations:

a.  $R^2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b.  $R^3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c.  $R^4$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**Exercise 7**

Which of these relations on  $\{0, 1, 2, 3\}$  are equivalence relations? If they are not, why?

a.  $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Yes.

b.  $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

No,  $(1, 1)$  isn't in there.

c.  $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

Yes.

d.  $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

No,  $(1, 2)$  isn't in there.

e.  $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Yes.

**Exercise 8**

List the ordered pairs in the equivalence relations produced by these partitions of  $\{0, 1, 2, 3, 4, 5\}$ .

a.  $\{0\}, \{1, 2\}, \{3, 4, 5\}$

$(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (3, 4), (4, 5), (3, 5), (5, 3), (4, 3)...$

b.  $\{0, 1\}, \{2, 3\}, \{4, 5\}$

c.  $\{0, 1, 2\}, \{3, 4, 5\}$

d.  $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

**Exercise 9**

Which of these relations on  $\{0, 1, 2, 3\}$  are partial orderings? If they are not, why?

- a.  $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Yes.

- b.  $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

No:  $(0, 2)$  and  $(2, 0)$  are both in there.

- c.  $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

No:  $(1, 2)$  and  $(2, 1)$  are both in there.

- d.  $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

No:  $(1, 3)$  and  $(3, 1)$  are both in there.

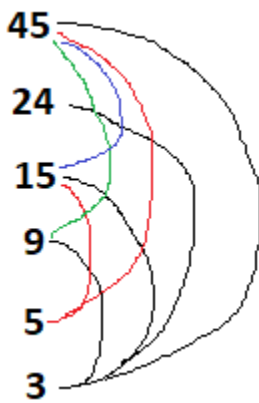
- e.  $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

No:  $(0, 1)$  and  $(1, 0)$  are both in there.

**Exercise 10**

Answer these questions for the divides poset  $(\{3, 5, 9, 15, 24, 45\}; |)$ .

- a. Draw the Hasse diagram



- b. List the maximal and minimal elements.

Maximal:  $\{45, 24\}$ . Minimal:  $\{3, 5\}$

- c. Is there a greatest element? A least element?

There is no element greater than nor less than all others.

- d. Find all upper bounds of  $\{3, 5\}$ . Find the least upper bound of  $\{3, 5\}$ , if it exists.

$\text{UB}(\{3, 5\})$ :  $\{15, 45\}$ .  $\text{LUB}(\{3, 5\})$ :  $\{15\}$

- e. Find all the lower bounds of  $\{15, 45\}$ . Find the greatest lower bound of  $\{15, 45\}$ , if it exists.

$\text{LB}(\{15, 45\})$ :  $\{3, 5, 15\}$ .  $\text{GLB}(\{15, 45\})$ :  $\{15\}$

**Exercise 11**

Prove the following:

- a. There is exactly one greatest element of a poset, if such an element exists.

Suppose  $\exists a, b \in P$  such that  $a$  and  $b$  are the greatest elements of  $P$ .

Then  $a \geq x$  and  $b \geq x \forall x \in P$ .

So  $a \geq b$  and  $b \geq a$ .

Thus,  $a = b$ .

- b. There is exactly one maximal element in a poset with a greatest element.

Let  $P$  be a poset and let  $a$  be the greatest element in  $P$ .

Let  $b \in P$  such that  $b \neq a$ .

Then, by definition,  $a \leq b$ .

Thus,  $a$  is the only maximal element in  $P$ .

- c. The least upper bound of a set in a poset is unique if it exists.

Let  $P$  be a poset and  $a \in P$ .

Suppose  $\exists U_1$  and  $U_2 \in P$  such that  $U_1$  and  $U_2$  are least upper bounds for  $a$  and  $U_1 \neq U_2$ .

Then, by definition,  $U_1 \leq U_2$  and  $U_2 \leq U_1$ .

Hence,  $U_1 = U_2$ .

**Exercise 12**

Determine whether these posets are lattices.

- a.  $(\{1, 3, 6, 9, 12\}; |)$

No, 9 join 6 doesn't have a LUB.

- b.  $(\{1, 5, 25, 125\}; |)$

Yes.

- c.  $(\mathbb{Z}; \geq)$

Yes, but it's not a complete lattice.

- d.  $(\mathcal{P}(S), \subset)$ , where  $\mathcal{P}(S)$  is the power set of a set  $S$ .

Yes.

**Exercise 13**

Show that every totally ordered set is a lattice.

Let  $T$  be a totally ordered set, and let  $a, b \in T$ .

Since  $T$  is totally ordered, either  $a \leq b$  or  $b \leq a$ .

Case:

- i)  $a \leq b$

Then  $a \text{ meet } b = a$ , and  $a \text{ join } b = b$ .

- ii)  $b \leq a$

Then  $b \text{ meet } a = b$ , and  $b \text{ join } a = a$ .

Hence, any two elements have a LUB and GLB.

**Exercise 14**

Show that every finite lattice has a least element and a greatest element.

Let  $L$  be a finite lattice.

Suppose there are two least elements in  $L$ :  $l_1, l_2$  such that  $l_1 \neq l_2$

Let  $l = l_1 \text{ meet } l_2$  (which exists because  $L$  is a lattice)

Case:

- i)  $l = l_1$ : a contradiction, since  $l_2$  is the least element.
- ii)  $l = l_2$ : a contradiction, since  $l_1$  is the least element.
- iii)  $l \neq l_1$  and  $l \neq l_2$ : a contradiction, since  $l_1$  and  $l_2$  are the least elements.

Thus, the least element in  $L$  is unique, if it exists.

Let  $A = a_1 \text{ meet } a_2 \text{ meet } \dots \text{ meet } a_n$  where  $n = |L|$  and  $a_i \in L$

Since  $A$  exists and is the least possible element,  $L$  has a least element.

WLOG, the same is true for a greatest element. (can I do this?)

**Exercise 15**

Give an example of an infinite lattice with

- a. neither a least nor a greatest element.

$$(\mathbb{Z}, \leq)$$

- b. a least but not a greatest element.

$$(\mathbb{Z}^+, \leq)$$

- c. a greatest but not a least element.

$$(\mathbb{Z}^-, \leq)$$

- d. both a least and a greatest element.

$$(\mathbb{Q}^{[0,1]}, \leq)$$

**Exercise 16**

Show that in any lattice  $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ . Note:  $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$  was shown in class.)

Proof of  $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$ :

$$Z \leq Z$$

$$(X \wedge Y) \wedge Z \leq Z \text{ (1)}$$

We also know:

$$(X \wedge Y) \wedge Z \leq X \wedge Y \leq X \text{ (2)}$$

$$(X \wedge Y) \wedge Z \leq X \wedge Y \leq Y \text{ (3)}$$

And:

$$(X \wedge Y) \wedge Z \leq X \wedge Z \text{ by (1) and (2)}$$

And:

$$(X \wedge Y) \wedge Z \leq X \wedge (Y \wedge Z) \text{ by (1), (2) and (3)}$$

Proof of  $(x \wedge y) \wedge z \geq x \wedge (y \wedge z)$ :

$$X \geq X$$

$$X \geq X \wedge (Y \wedge Z)$$

$$Y \wedge Z \geq X \wedge (Y \wedge Z)$$

$$Y \geq Y \wedge Z \geq X \wedge (Y \wedge Z)$$

$$Z \geq Y \wedge Z \geq X \wedge (Y \wedge Z)$$

Thus,

$$(X \wedge Y) \wedge Z \geq X \wedge (Y \wedge Z)$$

### Exercise 17

Show that in any lattice  $x \vee (x \wedge y) = x$ . Note: the dual absorption law was shown in class.

$$X \vee (X \wedge Y) \geq X \text{ (1)}$$

$$X \wedge Y \leq X$$

$$X \vee (X \wedge Y) \leq X \vee X = X$$

$$X \vee (X \wedge Y) \leq X \text{ (2)}$$

By (1), (2), and antisymmetry,

$$X \vee (X \wedge Y) = X$$

### Exercise 18

Show that any lattice  $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ . Note: the dual distributive inequality was shown in class.

$$X \vee Y \geq X$$

$$X \vee Y \geq Y \geq Y \wedge Z$$

$$X \vee Y \geq X \vee (Y \wedge Z)$$

$$X \vee Z \geq X \vee (Y \wedge Z)$$

$$(X \vee Y) \wedge (X \vee Z) \geq X \vee (Y \wedge Z)$$

### Exercise 19

Show that the two distributive equalities are equivalent. That is,  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$  if, and only if,  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ .

→

$$\text{WTS: } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$(x \wedge y) \vee (x \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$$

←

$$\text{WTS: } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$$

$$(x \wedge y) \vee (x \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$



**Exercise 20**

Show that the distributive law implies the modular law. That is, if a lattice satisfies one (hence both, from problem 19), then  $(x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z)$ .

WTS:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ or } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \Rightarrow (x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z)$$

**Exercise 21**

Check if the lattice  $N_5$  is distributive.

I checked. It isn't. Just kidding, here's why:

Let's look at  $Z \wedge (Y \vee X) = (Z \wedge Y) \vee (Z \wedge X)$

$$Z \wedge (Y \vee X)$$

$$Z \wedge 1$$

$$Z$$

And:

$$(Z \wedge Y) \vee (Z \wedge X)$$

$$0 \vee X$$

$$X$$

This is saying  $Z = X$ , which is not true.