Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18 (for 5(i) define tn to be 1 over sm, and then show that 1 over sm goes to 0)

Problem 1

Mark each statement True or False. Justify each answer.

a. If (s_n) and (t_n) are convergent sequences with $s_n \longrightarrow s$ and $t_n \longrightarrow t$, then lim $(s_n + t_n) = s + t$ and lim $(s_n t_n) = st$.

   True. By Theorem 4.2.1 (a) and (c).

b. If (s_n) converges to s and $s_n > 0 \ \forall \ n \in \mathbb{N}$, then $s > 0$.

   False. Counter example: $(s_n) = \frac{1}{n}$ ($s = 0$, but the moment you define n, $s_n > 0$)

c. The sequence (s_n) converges to s iff lim $s_n = s$.

   False. The sequence converges to s iff s exists **as a real number**. If $s = +\infty$ then it can't converge.

d. lim $s_n = +\infty$ iff lim $(\frac{1}{s_n}) = 0$.

   False. If lim $(\frac{1}{s_n}) = 0$ but $(s_n) = -1, -2, -3, ...$ then s_n does not diverge to $+\infty$

Problem 2

Mark each statement True or False. Justify each answer.

a. If $s_n = s$ and lim $t_n = t$, then lim $(s_n t_n) = st$.

   False. We don't know s_n's limit (which could be, for example, $(s_n) = n$, which diverges)

b. If lim $s_n = +\infty$, then (s_n) is said to converge to $+\infty$.

   False. You can only converge to a finite number.

c. Given sequences (s_n) and (t_n) with $s_n \leq t_n \ \forall \ n \in \mathbb{N}$, if lim $s_n = +\infty$, then lim $t_n = +\infty$.

   True.

   Suppose $\exists$ sequences (s_n) and (t_n) st $s_n \leq t_n \ \forall \ n \in \mathbb{N}$ where lim $s_n = +\infty$ and lim t_n is NOT $+\infty$.

   t_n diverges to $+\infty$ if $\forall$ M $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

   Let: M $\in \mathbb{R}$

   We know that

   $\exists$ N $\in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

   Since $s_n \leq t_n \ \forall \ n \in \mathbb{N}$

   $\exists$ N $\in \mathbb{N}$ st $n \geq N$ implies $s_n \geq t_n > M$

   $\exists$ N $\in \mathbb{N}$ st $n \geq N$ implies $s_n > M$

   This is the definition of diverging to $+\infty$, a contradiction.

   Hence, result.

d. Suppose (s_n) is a sequence st the sequence of ratios $(\frac{s_{n+1}}{s_n})$ converges to L. If $L < 1$, then lim $s_n = 0$.

   False.

   Let: $s_n = n(1)^{-n} \longrightarrow (\frac{s_{n+1}}{s_n}) = \frac{(n+1)(1)^{-(n+1)}}{n(1)^{-n}}$

   which converges to -1 which is less than 1 but does not have a limit of 0.

Problem 4

a. Prove Theorem 4.2.1(b):

   Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. Then

   (b) $\lim\,(ks_n) = ks$ and $\lim\,(k + s_n) = k + s$, for any $k \in \mathbb{R}$

b. Prove Corollary 4.2.5

   Theorem 4.2.4:

   Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. If $s_n \leq t_n \ \forall \ n \in \mathbb{N}$, then $s \leq t$.

   Corollary 4.2.5:

   If (t_n) converges to t and $t_n \geq 0 \ \forall \ n \in \mathbb{N}$, then $t \geq 0$.

Problem 5

For s_n given by the following formulas, determine the convergence or divergence of the sequence (s_n). Find any limits that exist.

a. $s_n = \frac{3-2n}{1+n} \longrightarrow \frac{1}{2}$

b. $s_n = \frac{(-1)^n}{n+3} \longrightarrow 0$

c. $s_n = \frac{(-1)^n}{2n-1} \longrightarrow 0$

d. $s_n = \frac{2^{3n}}{3^{2n}} = \frac{8^n}{9^n} \longrightarrow 0$

e. $s_n = \frac{n^2-2}{n+1} \longrightarrow \infty$

f. $s_n = \frac{3+n-n^2}{1+2n} \longrightarrow -\infty$

g. $s_n = \frac{1-n}{2^n} \longrightarrow 0$

h. $s_n = \frac{3^n}{n^3+5} \longrightarrow \infty$

i. $s_n = \frac{n!}{2^n} \longrightarrow \infty$

j. $s_n = \frac{n!}{n^n} = \frac{1*2*3*4*5}{5*5*5*5*5}$ where $n = 5 \longrightarrow 0$

k. $s_n = \frac{n^2}{2^n} \longrightarrow 0$

l. $s_n = \frac{n^2}{n!} \longrightarrow 0$

Problem 9

Prove Theorem 4.2.12:
Suppose that (s_n) and (t_n) are sequences st $s_n \leq t_n \ \forall \ n \in \mathbb{N}$

a. If $\lim s_n = +\infty$ then $\lim t_n = +\infty$

   Suppose $\exists$ sequences (s_n) and (t_n) st $s_n \leq t_n \ \forall \ n \in \mathbb{N}$ where $\lim s_n = +\infty$ and $\lim t_n$ is NOT $+\infty$.

   t_n diverges to $+\infty$ if $\forall \ M \in \mathbb{R}$, $\exists \ N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

   Let: $M \in \mathbb{R}$

We know that

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n >$ M

Since $s_n \leq t_n \ \forall$ n $\in \mathbb{N}$,

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n \geq t_n >$ M

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n >$ M

This is the definition of diverging to $+\infty$, a contradiction.

Hence, s_n diverges to $+\infty$.

b. If lim $t_n = -\infty$ then lim $s_n = -\infty$

Suppose $\exists$ sequences (s_n) and (t_n) st $s_n \leq t_n \ \forall$ n $\in \mathbb{N}$ where lim $s_n = -\infty$ and lim t_n is NOT $-\infty$.

t_n diverges to $-\infty$ if $\forall$ M $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n <$ M

Let: M $\in \mathbb{R}$

We know that

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n <$ M

Since $s_n \leq t_n \ \forall$ n $\in \mathbb{N}$,

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n \leq t_n <$ M

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n <$ M

This is the definition of diverging to $-\infty$, a contradiction.

Hence, s_n diverges to $-\infty$.

Problem 10

Prove the converse part of Theorem 4.2.13:
Let (s_n) be a sequence of positive numbers. Then, lim $s_n = +\infty$ iff lim $(\frac{1}{s_n}) = 0$.
$\longrightarrow$
Assume: lim $s_n = +\infty$
Given any $\epsilon > 0$, let M $= \frac{1}{\epsilon}$. Then there exists a natural number N st n $\geq$ N implies that $s_n >$ M $= \frac{1}{\epsilon}$.
Since each s_n is positive, we have:
$|\frac{1}{s_n} - 0| < \epsilon$, whenever n $\geq$ N
Thus, lim $(\frac{1}{s_n}) = 0$.
$\longleftarrow$

Problem 17

a. Show that $\lim\limits_{n\to\infty} \frac{k^n}{n!} = 0 \ \forall$ k $\in \mathbb{R}$

b. What can be said about $\lim\limits_{n\to\infty} \frac{n!}{k^n}$?

Problem 18

Assume that (s_n) is a convergent sequence with a $\geq s_n \geq$ b $\forall$ n $\in \mathbb{N}$.
Prove that a $\leq$ lim $s_n \leq$ b.