Homework Due 10/5/17: (7 problems) Section 4.1 pages 169 - 170; 1, 6(b), 7(f), 9(a), 11, 12, 15
Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18

Theorem 4.1.13

Every convergent sequence is bounded.

Proof.

Assume: $S_n \longrightarrow S$ as r $\longrightarrow \infty$
Then for $\epsilon = 1$, $\exists$ N(ϵ) $\in \mathbb{N}$ st, $\forall$ n $\geq$ N,
$|S_n| - |S| \leq |\,|S_n| - |S|\,| \leq |S_n - S| < 1$ (page 121, Ex 61(a))

Side Note

$x \leq |x| \; \forall \; x \in \mathbb{R}$

So $|S_n| < 1 + |S|$, $\forall$ n $\geq$ N
$|S_n| \leq |\,|S_n - S + S| \leq |S_n - S| + |S| < 1 + |S| \; \forall \; n \geq N$
Then,
Let: m = max $\{|S_1|, |S_2|, \ldots |S_{N-1}|, |S|\}$
Then $|S_n| \leq m$, $\forall$ n $\in \mathbb{N}$
Hence, $\{S_n\}$ is bounded.

$\square$

Theorem 4.1.14

If a sequence converges, then its limit is unique.

Proof.

Assume: $\{S_n\}$ is a sequence and $S_n \longrightarrow S$ as n $\longrightarrow \infty$ and $S_n \longrightarrow t$ as n $\longrightarrow \infty$
Then $\forall \; \epsilon > 0$, $\exists$ N,(ϵ) st
$|S_n - S| < \frac{\epsilon}{2}$, $\forall$ n $\geq$ N **(1)**
Also,
$\exists \; N_2(\epsilon) \in \mathbb{N}$ st
$|S_n - t| < \frac{\epsilon}{2}$, $\forall \; n \geq N_2$ **(2)**
Set N = max $\{N_1, N_2\}$
From **(1)**, **(2)**
$|S - t| = |(S - S_n) + (S_n - t)| \leq |S - S_n| + |S_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, $\forall$ n $\geq$ N
Hence,
s = t

$\square$

4.2 Limit Theorems

Theorem 4.2.1

Suppose that $\{S_n\}$ and $\{t_n\}$ are convergent sequences with $\lim\limits_{n \to \infty} S_n = S$ and $\lim\limits_{n \to \infty} t_n = t$.

Then,

a. $\lim\limits_{n\to\infty}$ $(\mathrm{S}_n + \mathrm{t}_n) = \mathrm{s} + \mathrm{t}$

b. $\lim\limits_{n\to\infty}$ $\mathrm{kS}_n = \mathrm{ks}$ and $\lim\limits_{n\to\infty} (\mathrm{k} + \mathrm{S}_n) = \mathrm{k} + \mathrm{s}$, for any $\mathrm{k} \in \mathbb{R}$

c. $\lim\limits_{n\to\infty} (\mathrm{S}_n\mathrm{t}_n) = \mathrm{st}$

d. $\lim\limits_{n\to\infty} (\frac{S_n}{t_n}) = \frac{s}{t}$, provided that $\mathrm{t}_n \neq 0 \ \forall \ \mathrm{n} \in \mathbb{N}$ and $\mathrm{t} \neq 0$

Proof.

(a)
$|\mathrm{t} + \mathrm{s} - (\mathrm{s}_n + \mathrm{t}_n)| =$
$|(\mathrm{t} - \mathrm{t}_n) + (\mathrm{s} - \mathrm{s}_n)| \leq |\mathrm{t} - \mathrm{t}_n| + |\mathrm{s} - \mathrm{s}_n|$ **(1)**
$\forall \ \epsilon > 0, \exists \ \mathrm{N}_1(\epsilon), \mathrm{N}_2(\epsilon)$ st

$$|t - t_n| < \frac{\epsilon}{2} \ \forall n \geq N_1 \tag{2}$$

and

$$|S - S_n| < \frac{\epsilon}{2}, \ \forall n \geq N_2 \tag{3}$$

Let: $\mathrm{N} = \max \{\mathrm{N}_1, \mathrm{N}_2\}$
From **(1) - (3)**,
$|\mathrm{S} + \mathrm{t} - (\mathrm{s}_n + \mathrm{t}_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ \forall \ \mathrm{n} \geq \mathrm{N}$
Hence, result.

(c)
$|\mathrm{st} - \mathrm{s}_n\mathrm{t}_n| = |(\mathrm{st} - \mathrm{s}_n\mathrm{t}) + (\mathrm{s}_n\mathrm{t} - \mathrm{s}_n\mathrm{t}_n)| \leq |\mathrm{st} - \mathrm{s}_n\mathrm{t}| + |\mathrm{s}_n\mathrm{t} - \mathrm{s}_n\mathrm{t}_n| = |\mathrm{s} - \mathrm{s}_n||\mathrm{t}| + |\mathrm{s}_n||\mathrm{t} - \mathrm{t}_n|$
By theorem 4.1.3, $\exists \ \mathrm{m} > 0$ st $|\mathrm{s}_n| \leq \mathrm{M}_1 \ \forall \ \mathrm{n} \in \mathbb{N}$
So,
$|\mathrm{st} - \mathrm{s}_n\mathrm{t}_n| \leq |\mathrm{t}||\mathrm{s} - \mathrm{s}_n| + \mathrm{M}|\mathrm{t} - \mathrm{t}_n|$
Let: $\epsilon > 0$
Then $\exists \ \mathrm{N}_1(\epsilon), \mathrm{N}_2(\epsilon) \in \mathbb{N}$ st
$\forall \ |\mathrm{s} - \mathrm{s}_n| < \frac{|t|\epsilon}{|t|+M}, \forall \ \mathrm{n} \geq \mathrm{N}_1$
and
$\mathrm{M} \ |\mathrm{t} - \mathrm{t}_n| < \frac{M\epsilon}{|t|+M}, \forall \ \mathrm{n} \geq \mathrm{N}_2$
Set $\mathrm{N} = \max \{\mathrm{N}_1, \mathrm{N}_2\}$
Then
$|\mathrm{st} - \mathrm{s}_n\mathrm{t}_n| < |\mathrm{t}| \ \frac{\epsilon}{|t|+M} + \frac{M\epsilon}{|t|+M} = \epsilon \ (\frac{|t|+M}{|t|+M}) = \epsilon \ \forall \ \mathrm{n} \geq \mathrm{N}$
Hence, result.

(d)
Since $\frac{s_n}{t_n} = (\frac{1}{t_n})(\mathrm{s}_n)$, the proof follows from **(c)** if we can prove that $\lim\limits_{n\to\infty} \frac{1}{t_n} = \frac{1}{t}$
Now:
$|\frac{1}{t} + \frac{1}{t_n}| = |\frac{t_n - t}{tt_n}| = \frac{|t_n - t|}{|t||t_n|}$ **(1)**

Side Note

$|\mathrm{t}_n| > 1$
$|\mathrm{t}_n| \geq \mathrm{M}$

Recall that:
$|t| - |t_n| \leq |\ |t| - |t_n|\ | \leq |t - t_n|$ from page 121, example 6(a)
For $\epsilon = |t| > 0$, $\exists$ $N_1(\epsilon) \in \mathbb{N}$ st
$|t - t_n| < \frac{|t|}{2}$, $\forall$ n $\geq N_1$
Now $|t| - |t_n| \leq |\ |t| - |t_n|\ | \leq |t - t_n| < \frac{|t|}{2}$ $\forall$ n $\geq N_1$
So $|t_n| > \frac{|t|}{2}$
Equivalently, $\frac{|t-t_n|}{|t||t_n|} < \frac{2|t-t_n|}{|t||t|}$ **(2)**
From **(1)** and **(2)**
$|\frac{1}{n} - \frac{1}{t_n}| < \frac{2|t_n-t|}{|t|^2}$, $\forall$ n $\geq N_1$ **(3)**
Also,
$\exists$ $N_2(\epsilon) \in \mathbb{N}$ st
$|t_n - t| < \frac{\epsilon|t|^2}{2}$, $\forall$ n $\geq N_2$ **(4)**
Let: $N = \max\{N_1, N_2\}$
Then from **(3)** and **(4)**,
$|\frac{1}{t} - \frac{1}{t_n}| < \frac{2}{|t|^2}\ \frac{\epsilon|t|^2}{2} = \epsilon$ $\forall$ n $\geq$ N
Hence, result.

□

Example 4.2.2

Find
$\lim\limits_{n\to\infty} \frac{(4n^2-3)}{(5n^2-2n)}$
=
$\lim\limits_{n\to\infty} \frac{n^2(4-\frac{3}{n^2})}{n^2(5-\frac{2}{n})}$
Now,
$\lim\limits_{n\to\infty} \frac{3}{n^2} = 0 = \lim\limits_{n\to\infty} \frac{2}{n}$
By Theorem 4.2.1, **(b)**
$\lim\limits_{n\to\infty} (4 - \frac{3}{n^2}) = 4$
and
$\lim\limits_{n\to\infty} (5 - \frac{2}{n}) = 5$
By Theorem 4.2.11 (d),
$\lim\limits_{n\to\infty} \frac{(4n^2-3)}{(5n^2-2n)} = \frac{4}{5}$

Theorem 4.2.4

Assume that
$\lim\limits_{n\to\infty} s_n = s$
and
$\lim\limits_{n\to\infty} t_n = t$
If $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$
then $s \leq t$

Proof.

Assume $s > t$
Then $s - t = 0$
$\exists$ $N_1(s - t)$, $N_2(s - t) \in \mathbb{N}$ st

$|s - s_n|$
$|s_n - s| < \frac{s-t}{2}$, $\forall$ n $\geq N_1$ **(1)**
and
$|t_n - t| < \frac{s-t}{2}$, $\forall$ n $\geq N_2$ **(2)**
Let: $N = \max\{N_1, N_2\}$
From **(1)**
From **(2)**

□