Exercise 1

Let f : D $\longrightarrow \mathbb{R}$ and let c $\in$ D. Mark each statement True or False. Justify each answer.

a. f is continuous at c iff $\forall\ \epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in D$

   True. By definition of continuous.

b. if f(D) is a bounded set, then f is continuous on D

   False.

   Let: f : D $\longrightarrow \mathbb{R}$ be defined by D = $\{\mathbb{R}\}$ and f = {1 if $x \neq 0$, 0 otherwise.}

   Pick $\epsilon = 0.5$. Notice that there is no δ such that $|f(x) - f(0)| < 0.5$

c. if c is an isolated point of D, then f is continuous at c

   True. If you just pick a δ st only x = c fits in $|x - c| < \delta$ (which is possible since it's an isolated point), then that works for any $\epsilon > 0$ since $|f(x) - f(c)|$ will always be 0.

d. if f is continuous at c and (x_n) is a sequence in D, then $x_n \longrightarrow c$ whenever $f(x_n) \longrightarrow f(c)$

   True.

   So, what we are asking is:

   (1) f is continuous at c, **(2)** (x_n) is a sequence in D, and **(3)** $f(x_n) \longrightarrow f(c)$ implies $x_n \longrightarrow c$

   True or false?

   Want to show: $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st $n \geq N$ implies $|x_n - c| < \epsilon$

   $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ st $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$

   So, if we let $x = x_n$ (since $x_n \in D$),

   For $\epsilon > 0$, $\exists\ \delta > 0$ st $x_n \in D$ and $|x_n - c| < \delta$ implies $|f(x_n) - f(c)| < \epsilon$ **(4)**

   We also know that, by **(3)**,

   $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st $n \geq N$ implies $|f(x_n) - f(c)| < \epsilon$

   So, by **(1)** and **(3)**,

   For $\epsilon > 0$, $\exists\ \delta > 0$ st $|x_n - c| < \delta$ implies $|f(x_n) - f(c)| < \epsilon$

   and for this same ϵ , there is an N $\in \mathbb{N}$ st $n \geq N$ implies the same.

e. if f is continuous at c, then for every neighborhood V of f(c), there exists a neighborhood U of c such that $f(U \cap D) = V$

   True. By Theorem 5.2.2 (c).

Exercise 2 (omit d)

Let f : D $\longrightarrow$ R and let c $\in$ D. Mark each statement True or False. Justify each answer.

a. if f is continuous at c and c is an accumulation point of D, then $\lim\limits_{x \to c} f(x) = f(c)$

b. Every polynomial is continuous at each point in $\mathbb{R}$

c. if $\{x_n)\}$ is a Cauchy sequence in D, then $\{f(x_n\}$ is convergent.

d. if f : $\mathbb{R} \longrightarrow \mathbb{R}$ and g : $\mathbb{R} \longrightarrow \mathbb{R}$ are both continuous on $\mathbb{R}$, then f o g and g o f are both continuous on $\mathbb{R}$

Exercise 3

Let: $f(x) = (x^2 + 4x - 21)/(x - 3)$ for $x \neq 3$.
How should f(3) be defined so that f will be continuous at 3?

Exercise 5 (prove the result)

Find an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous at exactly one point.

Exercise 10

a. Let $f : D \longrightarrow \mathbb{R}$ and define $|f| : D \longrightarrow \mathbb{R}$ by $|f|(x) = |f(x)|$. Suppose that f(x) is continuous at $c \in D$. Prove that $|f|$ is continuous at c.

b. if $|f|$ is continuous at c, does it follow that f is continuous at c? Justify your answer.

Exercise 11 (just prove the "max" result)

Define max(f, g) and min(f, g) as in Example 2.11.
Example 2.11
$\max(f, g)(x) = \max\{f(x), g(x)\}$

Show that:
$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$

Exercise 13

Let: $f : D \longrightarrow \mathbb{R}$ be continuous at $c \in D$ **Assume:** $f(c) > 0$
Prove that $\exists\, \alpha > 0$ and a neighborhood U of c st $f(x) > \alpha$, $\forall\, x \in U \cap D$

Exercise 16

(First prove that for any $H \subset \mathbb{R}$, $f^{-1}(R \setminus H) = \mathbb{R} \setminus f^{-1}(H)$, use this in conjunction with Theorem 5.2.14)
Let: $f : \mathbb{R} \longrightarrow \mathbb{R}$
Prove that f is continuous on $\mathbb{R}$ iff $f^{-1}(H)$ is a closed set whenever H is a closed set.