

HW 9: page 203 - 205, #1, 2, 3(a)(c)(e)(g), 7(c), 13, 16, 18, 19

Exercise 1

Let: $f : D \rightarrow \mathbb{R}$ and $c \in D'$

Mark each statement True or False. Justify each answer.

- a. $\lim_{x \rightarrow c} f(x) = L$ iff $\forall \epsilon > 0, \exists \delta > 0$ st $|f(x) - L| < \epsilon$ whenever $x \in D$ and $|x - c| < \delta$

False. Let $f(c) \neq L$ as a counter example.

- b. $\lim_{x \rightarrow c} f(x) = L$ iff for every deleted neighborhood U of c , there exists a neighborhood V of L st $f(U \cap D) \subset V$

True, by Theorem 5.1.2 (since it's iff).

- c. $\lim_{x \rightarrow c} f(x) = L$ iff for every sequence $\{s_n\}$ in D that converges to c with $s_n \neq c \forall n$, the sequence $\{f(s_n)\}$ converges to L .

True, by Theorem 5.1.8.

- d. If f does not have a limit at c , then \exists a sequence $\{s_n\}$ in D with each $s_n \neq c$ st $\{s_n\}$ converges to c , but $\{f(s_n)\}$ is divergent.

True by Theorem 5.1.10(b).

Exercise 2

Let: $f : D \rightarrow \mathbb{R}$ and $c \in D'$

Mark each statement True or False. Justify each answer.

- a. For any polynomial P and any $c \in \mathbb{R}$, $\lim_{x \rightarrow c} P(x) = P(c)$

True.

$$\lim_{x \rightarrow c} P(x) = P(c) \text{ iff } \forall \epsilon > 0, \exists \delta > 0 \text{ st } x \in D \text{ and } 0 < |x - c| < \delta \Rightarrow |P(x) - P(c)| < \epsilon$$

From lecture: $\lim_{x \rightarrow c} P(x) = P(c)$ where P is a polynomial.

- b. For any polynomials P and Q , and any $c \in \mathbb{R}$,

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

False. This is not the case if $Q(c)$ is 0.

- c. In evaluating $\lim_{x \rightarrow a-} f(x)$ we only consider points that are greater than a .

False, by the definition of One Sided Limits.

- d. If f is defined in a deleted neighborhood of c , then $\lim_{x \rightarrow c} f(x) = L$ iff $\lim_{x \rightarrow c+} f(x) = \lim_{x \rightarrow c-} f(x) = L$

True. If f is defined in a deleted neighborhood of c , then by Theorem 5.1.2,

$$\lim_{x \rightarrow c} f(x) = L \text{ iff } \lim_{x \rightarrow c+} f(x) = \lim_{x \rightarrow c-} f(x) = L$$

Exercise 3(a)(c)(e)(g)

Determine the following limits:

a. $\lim_{x \rightarrow 1} \frac{3x^2+5}{x^3+1}$

$$\lim_{x \rightarrow 1} \frac{3x^2+5}{x^3+1} = \frac{3(1)^2+5}{1^3+1} = \frac{8}{2} = 4$$

b. $\lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1}$

$$\lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x}+1} = \frac{1}{1+1} = \frac{1}{2}$$

c. $\lim_{x \rightarrow 0} \frac{x^2+5x}{x^2-2}$

$$\lim_{x \rightarrow 0} \frac{x^2+5x}{x^2-2} = \frac{0^2+5(0)}{0^2-2} = \frac{0}{-2} = 0$$

d. $\lim_{x \rightarrow 0^-} \frac{4x}{|x|}$

Since we're only taking $x < 0$, $\frac{4x}{|x|}$ will always be -4 , so the limit from the left side is -4 .

Exercise 7(c)

Find the following limit and prove your answer.

$$\lim_{x \rightarrow c} \sqrt{x}, \text{ where } c \geq 0$$

Let: $f(x) = \sqrt{x}$

I think the limit is just \sqrt{c} .

$$\lim_{x \rightarrow c} \sqrt{x} = \sqrt{c} \text{ iff}$$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } x \in D \text{ and } 0 < |x - c| < \delta \Rightarrow |\sqrt{x} - \sqrt{c}| < \epsilon$$

We know that since the domain for \sqrt{x} is $x \geq 0$, D is $\{x : x \geq 0\}$

Exercise 13

Let f , g , and h be functions from D into \mathbb{R} , and let $c \in D'$

Assume: $f(x) \leq g(x) \leq h(x) \forall x \in D$ with $x \neq c$

Assume: $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$

Prove that $\lim_{x \rightarrow c} g(x) = L$

We know that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} h(x) = L$, so:

$$\forall \epsilon > 0, \exists \delta_f > 0 \text{ st } x \in D \text{ and } 0 < |x - c| < \delta_f \Rightarrow |f(x) - L| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta_h > 0 \text{ st } x \in D \text{ and } 0 < |x - c| < \delta_h \Rightarrow |h(x) - L| < \epsilon$$

Let: $\delta = \min \{\delta_h, \delta_f\}$

Now:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } x \in D \text{ and } 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon \text{ and } |h(x) - L| < \epsilon$$

$$|f(x) - L| < \epsilon \text{ and } |h(x) - L| < \epsilon$$

$$-\epsilon < f(x) - L < \epsilon \text{ and } -\epsilon < h(x) - L < \epsilon$$

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

$$-\epsilon < g(x) - L < \epsilon$$

$$|g(x) - L| < \epsilon$$

Hence,

$$\lim_{x \rightarrow c} g(x) = L$$

Exercise 16

Let: $f : D \rightarrow \mathbb{R}$ and $c \in D'$

Assume: $\lim_{x \rightarrow c} f(x) > 0$

Prove that \exists a deleted neighborhood U of c st $f(x) > 0 \forall x \in (U \cap D)$

Let: $\lim_{x \rightarrow c} f(x) = L$, where $L > 0$

So,

$\forall \epsilon > 0, \exists \delta > 0$ st $x \in D$ and $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$

Define ϵ st $L - 2\epsilon = 0$

Notice that $f(x) > 0, \forall f(x) \in V = N(L, \epsilon)$.

By Theorem 5.1.2, $\exists \delta > 0$ st $U = N^*(c, \delta)$ and $f(U \cap D) \subset V$

Hence, result.

Exercise 18

Let: $f : D \rightarrow \mathbb{R}$ and $c \in D'$

Assume: f has a limit at c (i.e. $\lim_{x \rightarrow c} f(x) = L$)

Prove that f is bounded on a neighborhood of c .

That is, prove that \exists a neighborhood U of c and a real number M st $|f(x)| \leq M \forall x \in (U \cap D)$

Since $\lim_{x \rightarrow c} f(x) = L$,

$\forall \epsilon > 0, \exists \delta > 0$ st $x \in D$ and $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$

and,

for each neighborhood V of L , \exists a deleted neighborhood U of c st $f(U \cap D) \subset V$.

Exercise 19

Assume: $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function st $f(x + y) = f(x) + f(y) \forall x, y \in \mathbb{R}$

Prove that f has a limit at 0 iff f has a limit at every point $c \in \mathbb{R}$

\longrightarrow

f has a limit at 0, so,

$\forall \epsilon > 0, \exists \delta > 0$ st $m \in D$ and $0 < |m| < \delta \Rightarrow |f(m) - L| < \epsilon$

Let: $m = x + y$ where $x, y \in \mathbb{R}$

Then:

$\forall \epsilon > 0, \exists \delta > 0$ st $(x + y) \in D$ and $0 < |x + y| < \delta \Rightarrow |f(x + y) - L| < \epsilon$

$|f(x) + f(y) - L| < \epsilon$