Assignment Set: 1, 2, 3, 4, 6, 8 from pages 148 - 149

1)

Mark each statement as true or false. Justify each answer.

a. A set S is compact iff every open cover of S contains a finite subcover.

   True.

   The original definition is:

   A set S $\subset \mathbb{R}$ is said to be compact if every open cover has a finite subcover.

   $\longrightarrow$ (every open cover $\Rightarrow$ compact)

   Let: S $\subset \mathbb{R}$ be a set st every open cover has a finite subcover.

   Then, by definition, S is compact.

   $\longleftarrow$ (every open cover $\Leftarrow$ compact)

   Let: S $\subset \mathbb{R}$ be compact

   Then, by definition, S is a set st every open cover has a finite subcover.

   Hence, result.

b. Every finite set is compact.

   True.

   Let: S be a finite set

   So, $|S|$ = n $\in \mathbb{N}$

   Let: G = S $\cup \emptyset$

   Notice that $\emptyset$ is open and that G covers S.

   Notice also that G = S.

   Since every open cover of S has to contain S, every open cover of S contains G, a finite subcover of S.

   Hence, result.

c. No infinite set is compact.

   Not true.

   If S = (0, 1), there are infinite values between 0 and 1.

   However, (0, 1) is compact.

d. If a set is compact, then it has a maximum and a minimum.

   Not true.

   If S = (0, 1), then S is compact, but the upper and lower bounds of S are not members of S, so S has no maximum and minimum.

e. If a set has a maximum and a minimum, then it is compact.

   Let: G = S $\cup \emptyset$

   Notice that $\emptyset$ is open and that G covers S.

   Notice also that G = S.

   Since every open cover of S has to contain S, every open cover of S contains G, a finite subcover of S.

   Hence, result.

2)

Mark each statement as true or false. Justify each answer.

a. Some unbounded sets are compact.

   False. By Heine-Borel, S is compact only if closed and bounded.

b. If S $\subset \mathbb{R}$ is compact, then $\exists$ x $\in \mathbb{R}$ st s $\in$ S$'$

   False. The empty set is compact and contains no elements.

c. If S is compact and s $\in$ S$'$, then s $\in$ S.

   True. By Heine-Borel, if S is compact, then S is closed and bounded. If S is closed, then S = int S. By Theorem 3.4.17, if S is closed, then S$' \subset$ S. Since s $\in$ S$'$, s $\in$ S.

d. If S is unbounded, then S has at least one accumulation point.

   False. $\mathbb{N}$ is a counter example.

e. **Let:** F = $\{$A$_i$, i $\in \mathbb{N}$ $\}$. Suppose that the intersection of any finite subfamily of F is nonempty. If $\cap$ F = $\emptyset$, then, for some k $\in \mathbb{N}$, A$_k$ is not compact.

   ???

3)

Show that each subset of $\mathbb{R}$ is not compact by describing an open cover for it that has no finite subcover.

a. ‘[1, 3) $\{$n $\in \mathbb{N}$: $\bigcup_{i=1}^{n}$ (0, 2 + $\sum_{k=1}^{i} \frac{1}{2^k}$)$\}$

b. ‘[1,2) $\{$n $\in \mathbb{N}$: $\bigcup_{i=1}^{n}$ ((2 − $\sum_{i=1}^{n} \frac{1}{2^i}$), (3 + $\sum_{i=1}^{n} \frac{1}{2^i}$))$\}$

c. $\mathbb{N}$ $\{$n $\in \mathbb{N}$: $\bigcup_{i=1}^{n+1}$ (0, n)$\}$

d. $\{\frac{1}{n}$: n $\in \mathbb{N}$ $\}$ $\{$ n $\in \mathbb{N}$: $\bigcup_{i=1}^{n}$ (0, $\sum_{k=1}^{i} \frac{1}{2^k}$)$\}$

e. $\{$x $\in \mathbb{Q}$: 0 $\leq$ x $\leq$ 2$\}$ - but wait, isn’t this a closed and bounded set?

4)

Prove that the intersection of any collection of compact sets is compact.
Let: S be $\bigcap_{\alpha \in I}$ G$_\alpha$ where I is an index set
Let: $\bigcap_{\alpha \in I}$ G$_\alpha$ be nonempty (since if it’s empty, then it’s compact anyhow)
By Heine-Borel, G$_\alpha$ is both closed and bounded $\forall$ α .
Let: U be the set of all least upper bounds $\forall$ G$_\alpha$’s, and L be the set of all greatest lower bounds $\forall$ G$_\alpha$’s
Since G$_\alpha$ is closed $\forall$ G$_\alpha$, each element in U is a max, and each element in L is a min.
Since $\bigcap_{\alpha \in I}$ G$_\alpha$ is an intersection, its minimum will be max L, and its maximum will be min U, which we know exists because $\bigcap_{\alpha \in I}$ G$_\alpha \neq \emptyset$
Since $\bigcap_{\alpha \in I}$ G$_\alpha$ has a min and a max, and is the intersection of compact sets, it is both closed and bounded, and is therefore compact.

6)

Show that compactedness is necessary in Corollary 3.5.8. That is, find a family of intervals $\{A_n : n \in \mathbb{N}\}$ with $A_{n+1} \subset A_n \ \forall$ n, $\bigcup_{n=1}^{\infty} A_n = \emptyset$, and such that:

a. The sets A_n are all closed. $\{n \in \mathbb{N} : \emptyset\}$

b. The sets A_n are all bounded. $\{n \in \mathbb{N} : (0, 0)\}$

8)

If $S \subset \mathbb{R}$ is compact and $T \subset S$ is closed, then T is compact.

a. Prove this using the definition of compactness.

b. Prove this using the Heine-Borel theorem.