Assignment Set: 6, 7, 15, 17, 19, 21 from pages 141 - 142

6)

Find the closure of each set:

a. $\{\ \frac{1}{n}$: n $\in \mathbb{N}\ \}$
   Answer: $\emptyset$

b. $\mathbb{N}$
   Answer: $\mathbb{N}$

c. $\mathbb{Q}$
   Answer: $\mathbb{R}$

d. $\bigcap_{n=1}^{\infty}\ (0, \frac{1}{n})$
   Answer: $\emptyset$

e. { x : $|x-5| \leq \frac{1}{2}$}
   $[4.5, 5.5]$
   Answer: $[4.5, 5.5]$

f. { x : $x^2 > 0$}
   $(0, \infty)$
   Answer: $[0, \infty)$

7)

Let S, T $\subset \mathbb{R}$. Find a counterexample of each of the following:

a. If P is the set of all isolated points of S, then P is a closed set.
   Answer: Let S = $\mathbb{N}$

b. Every open set contains at least two points.
   Answer: $\emptyset$

c. If S is closed, then cl(int S) = S.
   Answer: Let S = $\mathbb{Q}$

d. If S is open, then int (cl S) = S.
   Answer: Let S = $(-1, 0) \cup (0, 1)$

e. bd (cl S) = bd S
   Answer: Let S = $(-1, 0) \cup (0, 1)$

f. bd (bd S) = bd S
   Answer: Let S = $\mathbb{Q}$. Then bd S is $\mathbb{R}$, and bd (bd S) = $\emptyset \neq \mathbb{R}$.

g. bd (S $\cup$ T) = (bd S) $\cup$ (bd T)
   Answer: Let S = $\mathbb{R}$, T = $(0, 1)$. bd (S $\cup$ T) = $\emptyset$, but bd S $\cup$ bd T = $\emptyset \cup \{0, 1\}$

h. bd (S $\cap$ T) = (bd S) $\cap$ (bd T)
   Answer: Let S = $(0, 1)$, T = $(1, 2)$. bd (S $\cap$ T) = $\emptyset$, but bd S $\cap$ bd T = 1.

15)

Prove: If x is an accumulation point of the set S, then every neighborhood of x contains infinitely many points of S.

Proof.
Suppose that $\exists$ a deleted neighborhood of x, called N, that contains n points x_1, x_2, ... x_n of S where n is a finite amount and $x_1 \leq x_2, \leq ... x_n$
x is an accumulation point on S if $\forall \epsilon > 0$, N*(x, ϵ) $\cap$ S $\neq \emptyset$.
N is a deleted neighborhood of S if $\forall$ x $\in \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$, x $\in$ N.
Let $\hat{\epsilon} = \epsilon + \epsilon$, and $x_0 = x_1 - \hat{\epsilon}$.
By definition, $x_0 \in$ N, since N is a neighborhood $\forall \epsilon > 0$.
However, N only has n elements. A contradiction.
So, N can't be a deleted neighborhood since it has a finite number of elements, which means x can't be an accumulation point.

□

17)

Prove: S$'$ is a closed set.

Proof.

By definition, $\forall s \in S'$, $\epsilon > 0$, N*(s, ϵ) $\cap$ S $\neq \emptyset$
Notice that if S$'$ is empty or S$'$ is $\mathbb{R}$, then S$'$ is a closed set and we are done.
If S$'$ is not empty, $\exists$ at least one element.
Let: $\mathbb{R}$ $\setminus$ S' $\subset \mathbb{R}$, x $\in \mathbb{R}$ $\setminus$ S'
Want to show: $\mathbb{R}$ $\setminus$ S$'$ is open.
$\mathbb{R}$ $\setminus$ S$'$ is open iff $\mathbb{R}$ $\setminus$ S$'$ = int ($\mathbb{R}$ $\setminus$ S$'$)
int $\mathbb{R}$ $\setminus$ S$'$ = {s: N(s, ϵ) $\subset \mathbb{R}$ $\setminus$ S$'$ }

□

19)

Suppose S is a nonempty bounded set and let m = sup S. Prove or give a counter example: m is a boundary point of S.

Proof.

By definition,
s $\leq$ m, $\forall$ s $\in$ S, and,
$\forall$ $\epsilon > 0$, $\exists$ s' $\in$ S st m $-$ $\epsilon <$ s'
By the second part of the definition of the supremum of S, N(m, ϵ) $\cap$ S $\neq \emptyset$.
Notice also that, by the first part of the definition of the supremum of S, (m + ϵ) $\notin$ S. This means that N(m, ϵ) $\cap$ $\mathbb{R}$ $\setminus$ S $\neq \emptyset$.
By definition, m is a boundary point.

□

21)

Let A be a nonempty open subset of $\mathbb{R}$ and let Q $\subset \mathbb{Q}$. Prove: A $\cap$ Q $\neq \emptyset$.

Proof.
Notice that Q $\subset \mathbb{Q} \subset \mathbb{R}$.
Since A is nonempty, $\exists$ at least one element a $\in \mathbb{R}$.
Since A is nonempty and open, a + $\epsilon \in$ A.
If a $\in \mathbb{Q}$, then result.
If a $+\epsilon \in \mathbb{Q}$, then result.
If a $\notin \mathbb{Q}$ and (a + ϵ) $\notin \mathbb{Q}$, then:
Let x = a, y = a + ϵ, z = y $-$ x.
By Archimedes' axiom, $\exists$ n st n > $\frac{1}{z}$
nz > 1
ny $-$ nx > 1
Since the difference between ny and nx is bigger than 1,
$\exists m \in \mathbb{Z}$ st nx < m < ny.
See that since x < $\frac{m}{n} < y$, $\frac{m}{n}$ is a rational number, and $\frac{m}{n} \in$ A.
Hence, result. ☐