Definition 3.4.6 - Def of Open/Closed Set

Let: $S \subset \mathbb{R}$

if bd $S \subset S$, then S is closed.
if bd $S \subset (\mathbb{R} \setminus S)$, then S is open.

Theorem 3.4.7

a. A set S is open iff S = int S; i.e. iff $\forall$ s $\in$ S, s is an **interior point**.

b. A set S is closed iff its compliment, $\mathbb{R} \setminus S$ is open.

   Equivalently, a set s is open iff $\mathbb{R} \setminus S$ is closed.

Proof.

(a):
$\longrightarrow$
Assume: S is open
Want to show: S = int S
By definition, int $S \subset S$.
Want to show: $S \subset$ int S
Let: $x \in S$ **(1)**
Want to show: $x \in$ int S
Since S is open, bd $S \subset \mathbb{R} \setminus S$
So, $x \notin$ bd S.
Thus, $\exists\ \epsilon > 0$ st $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$
$\forall\ \epsilon > 0$, $N(x, \epsilon) \cap S \neq \emptyset$
$x \in$ bd S if $\forall\ \epsilon > 0$,
$N(x, \epsilon) \cap S \neq \emptyset$ and $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$
Thus, $N(x, \epsilon) \subset S$.
So, $x \in$ int S.
This proves that $S \subset$ int S
$\longleftarrow$
Assume: S = int S
Want to show: S is open
Let: $x \in$ bd S
Want to show: $x \in \mathbb{R} \setminus S$
Since $x \in$ bd S, we conclude that $x \notin$ int S.

Side Note

$x \in$ bd S if, $\forall\ \epsilon > 0$,
$N(x, \epsilon) \cap S \neq \emptyset$ and $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$

Thus, $x \in \mathbb{R} \setminus S$.
So, bd $S \subset \mathbb{R} \setminus S$.
So, by definition,
S is open.

(b): S is closed iff $\mathbb{R} \setminus$ S is open.
So, x $\notin$ bd S.
Thus, $\exists\ \epsilon > 0$ st N$(x, \epsilon)\ \cap$ S $\neq \emptyset$
Hence, N(x, ϵ) $\subset \mathbb{R} \setminus$ S
So, $\mathbb{R} \setminus$ S is open from **(a)**.
$\longleftarrow$
Assume: $\mathbb{R} \setminus$ S is open
Want to show: S is closed
Let: x $\in$ bd S
Want to show: x $\in$ S
Since x $\in$ bd S, $\forall\ \epsilon > 0$,

$$\text{N}(x, \epsilon)\ \cap\ \text{S} \neq \emptyset \tag{1}$$

and

$$\text{N}(x, \epsilon)\ \cap\ (\mathbb{R} \setminus \text{S}) \neq \emptyset \tag{2}$$

Since $\mathbb{R} \setminus$ S is open, $\forall$ s $\in \mathbb{R} \setminus$ S, s is an **interior point** of $\mathbb{R} \setminus$ S.
Thus, x $\in$ S.
We have shown that bd S $\subset$ S.
By definition, S is closed.

$\square$

Example 3.4.8

a. ‘[0, 5] is a closed set. ($\mathbb{R} \setminus$ [0, 5] = ($-\infty$, 0) $\cup$ (5, ∞))

b. (0, 5) is an open set.

c. ‘[0, 5) is neither open nor closed.

d. ‘[2, ∞) is a closed set.

e. $\mathbb{R}$ is both open and closed.

   bd $\mathbb{R} = \emptyset \subset \mathbb{R}$

   Also, int $\mathbb{R} = \mathbb{R}$

   Also, $\emptyset$ is both open and closed.

Theorem 2 (not in book)

Let: x $\in \mathbb{R}$, $\epsilon > 0$
Then:

a. N(x, ϵ) is an open set

b. N*(x, ϵ) is an open set

(a)

Proof.

N(x, ϵ) = {y : $|y - x| < \epsilon$ } i.e. $-\epsilon < y - x < \epsilon$
So, y $\in$ N(x, ϵ) iff x $-$ ϵ $<$ y $<$ x + ϵ
Let: y $\in$ N(x, ϵ)
We shall find $\hat{\epsilon} > 0$ st
N(y, $\hat{\epsilon}$) $\subset$ N(x, ϵ), which will show that
N(x, ϵ) is open.
————————————————————Side Note————————————————————-
—-(—————-————-)—
x-ep, y-ephat, y, yplusEphat, x, xplusEp

Let: $\hat{\epsilon}$ = min $\{y - (x - \epsilon), x + \epsilon - y\}$ **(1)**
Want to show: N(y, $\hat{\epsilon}$) $\subset$ N(x, ϵ)
Let: z $\in$ N(y, $\hat{\epsilon}$)
Then, y $-$ $\hat{\epsilon}$ $<$ z $<$ y + $\hat{\epsilon}$ **(2)**
From **(1)**, $\hat{\epsilon} \leq y - (x - \epsilon)$ **(3)**
and
$\hat{\epsilon}$ $\leq$ x + ϵ $-$ y **(4)**
So from **(4)**,
y + $\hat{\epsilon}$ $\leq$ y + x + ϵ $-$ y_i
y + $\hat{\epsilon}$ $\leq$ x + ϵ
From **(3)**, (x $-$ ϵ) $-$ y $\leq$ $-\hat{\epsilon}$ **(5)**
Then,
y + (x $-$ ϵ) $-$ y $\leq$ y $-$ $\hat{\epsilon}$
x $-$ ϵ $\leq$ y $-$ $\hat{\epsilon}$ **(6)**
From **(2)**, **(5)**, **(6)**,
x $-$ ϵ $\leq$ y $-$ $\hat{\epsilon}$ $<$ z $<$ y + $\hat{\epsilon}$ $\leq$ x + ϵ
Therefore,
x $-$ ϵ $<$ z $<$ x + ϵ
Thus, z $\in$ N(x, ϵ).
Hence,
N(y, $\hat{\epsilon}$) $\subset$ N(x, ϵ)
Which proves that
N(x, ϵ) is open.

(b): N*(x, ϵ) is an open set. Similar to **(a)**.

□

Theorem 3.4.10

Let: I be an index set. I $\subset$ $\mathbb{N}$ $\subset$ $\mathbb{R}$
Suppose: $G_\alpha \subset \mathbb{R}$ is an open set $\forall$ $\alpha \in$ I
Then,

a. $\bigcup_{\alpha \in I}$ G_α is an open set.

b. If $G_i \subset \mathbb{R}$ is open $\forall$ i = 1, 2, ... n where $n \in \mathbb{N}$

   Then $\bigcap_{i=1}^{n} G_i$ is open.

Proof.

(a):

Let: $x \in \bigcup_{\alpha \in I} G_i$
Thus, $\exists\ \alpha_0 \in I$ st $x \in G_{\alpha_0}$.
Since G_{α_0} is open, $\exists\ \epsilon_0 > 0$ st $N(x, \epsilon_0) \subset G_{\alpha_0}$
Thus, $N(x, \epsilon_0) \subset \bigcup_{\alpha \in I} G_\alpha$
This proves that $x \in \text{int}\ (\bigcup_{\alpha \in I} G_\alpha)$
By Theorem 3.4.7 a),
$\bigcup_{\alpha \in I} G_\alpha$ is open.

(b):

Let: $x \in \bigcap_{i=1}^{n} G_i$
Thus, $x \in G_i\ \forall$ i = 1, 2, ... n
Since G_i is open $\forall$ i = 1, 2, ... n
$\exists\ \epsilon_i > 0$ st $N(x, \epsilon_i) \subset G_i\ \forall$ i from 1 to n.
Choose $\epsilon = \min\ \{\epsilon_1, \epsilon_2, \ldots \epsilon_n\} > 0$
Then $N(x, \epsilon) \subset N(x, \epsilon_i)\ \forall$ i from 1 to n.
Hence, $N(x, \epsilon) \subset \bigcap_{i=1}^{n} G_i$
Hence, $\bigcap_{i=1}^{n} G_i$ is open. $\square$

---Side Note---

—-(—(———)—)—-
x-epi, x-ep, x, xplusEp, xplusEpi

Corollary 3.4.11

a. Let F_α be closed $\forall\ \alpha \in I$, I is an index set.

   Then $\bigcap_{\alpha \in I} F_\alpha$ is closed.

b. Let F_i be closed $\forall$ i from 1 to n.

   Then $(\ \bigcup_{i=1}^{n} F_i\ )$ is closed.

(a):

Notice by de Moivre's theorem:
$\mathbb{R} \setminus (\bigcap_{\alpha \in I} F_\alpha) = \bigcup_{\alpha \in I} (\mathbb{R} \setminus F_\alpha)$
Which is open by Theorem 3.4.101 a), since
$\mathbb{R} \setminus F_\alpha$ is open by Theorem 3.4.71 b).
Hence, $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(b): Similar.

Example 3.4.12

Let: $G_n = (\frac{-1}{n}, \frac{1}{n})\ \forall\ n \in \mathbb{N}$
Then $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is closed.
Compare with Theorem 3.4.101 b):
$(-\infty, 0) \cup (0, -\infty)$

Accumulation (or Limit) Points; Definition 3.4.14

Let: $S \subset \mathbb{R}$
If $\forall\ \epsilon > 0$, $N^*(x, \epsilon)\ \cap\ S \neq \emptyset$,
Then $x \in \mathbb{R}$ is an **accumulation** or **limit** point.
The set of all accumulation points of S is denoted by $\mathbf{S'}$.
If $x \in S \setminus S'$,
then x is an **isolated point**,
in which case, $\exists\ \epsilon > 0$ st $N(x, \epsilon) \cap S = \{x\}$

Definition 3.4.16 - Closures

Let: $S \subset \mathbb{R}$
Then the **closure** of S, denoted by **cl S**, is defined to be:
$\text{cl } S = S \cup S'$

For example:
$S = (0, 1) \cup \{2\}$
S' = [0, 1]
$\text{bd } S = \{0, 1, 2\}$