Homework 7: pages 184 - 185 numbers 1, 21(a)(b), 3(e), 4, 10, 13, 14 $\longleftarrow$ 14 is difficult, but not impossible!
(want to show that lim $(1 + \frac{1}{n})^n$ exists)
Hint:
$(1 + b)^n = 1 + nb + \frac{n(n-1}{2!}b^n + ... + \frac{n(n-1)...(n-(r-1))}{r!}b^r + ... + b^n$
In our problem, b = $\frac{1}{n}$
Look at it as $1 + \sum_{r=1}^{n} \frac{n(n-1)...(n-(r-1))}{r!}\frac{1}{n^r}$
$(1 + \frac{1}{n})^n$ goes in there somewhere somehow.
About the last homework (HW 6):
(9)
If $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$ and $\lim\limits_{n\to\infty} s_n = \infty$,
then $\lim\limits_{n\to\infty} t_n = \infty$
So,
$\forall$ M $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st
$s_n > M$, $\forall$ n $\geq$ N
Notice that:
$t_n \geq s_n > M$, $\forall$ n $\geq$ N
So by definition, $\lim\limits_{n\to\infty} t_n = \infty$

Section 4.3: Monotone Sequences and Cauchy Sequences

Definition 4.3.1

A sequence (s_n) is **increasing** (or **decreasing**) if $s_n \leq s_{n+1}$ (or $s_{n+1} \leq s_n$) $\forall$ n $\in \mathbb{N}$. A sequence is **monotonic** if it is increasing or decreasing.

Example 4.3.2

a. $a_n = n$, $\forall$ n $\in \mathbb{N}$
   increasing
b. $b_n = 2^n$, $\forall$ n $\in \mathbb{N}$
   increasing
c. $c_n = 2 - \frac{1}{n}$, $\forall$ n $\in \mathbb{N}$
   increasing
d. (d_n) = 1, 1, 2, 2, 3, 3...
   increasing
e. $e_n = \frac{2}{n}$, $\forall$ ...
   decreasing
f. $f_n = -3n$
   decreasing
g. (g_n) = 1, 1, 1, ... ($g_n = 1$, $\forall$ n $\in \mathbb{N}$)
   increasing and decreasing
h. $h_n = \text{-}1^n$, $\forall$ n $\in \mathbb{N}$
   not monotonic

i. $i_n = \cos(\frac{n\pi}{3})\ \forall$ n $\in \mathbb{N}$
   not monotonic

Theorem 4.3.3 (Monotone Convergence Theorem)

A **monotonic sequence** is convergent iff it is bounded.

Proof.
Let: $\{s_n\}$ be a monotonically increasing sequence
$\longrightarrow$
Assume $\{s_n\}$ is convergent.
By Theorem 4.1.13, $\{s_n\}$ is bounded.
$\longleftarrow$
Conversely, assume $\{s_n\}$ is bounded.
Want to show: $\{s_n\}$ converges
Let the range of $\{s_n\}$ be denoted by $S = \{s_n : n \in \mathbb{N}\}$
Since $\{s_n\}$ is bounded, S is bounded above.
Thus, sup S exists.
Want to show: $\{s_n\}$ converges to sup S
Recall: The supremum is the least upper bound.
Thus,
$s_n \leq \sup S, \forall\ n \in \mathbb{N}$ **(1)**
and for $\epsilon > 0$, $\exists\ N(\epsilon) \in \mathbb{N}$ st $\forall\ n \geq N$,
$\sup S - \epsilon < s_n$
$\sup S - \epsilon < s_N \leq s_n \leq \sup S < \sup S + \epsilon$ **(2)**
Since $\{s_n\}$ is increasing and, using **(1)**,
From **(2)**, we see that
$-\epsilon < s_n - \sup S < \epsilon, \forall\ n \geq N$
Hence,
$|s_n - \sup S| < \epsilon, \forall\ n \geq N$,
which is equivalent to $\lim\limits_{n\to\infty} s_n = \sup S$
(since $|x| < a$ iff $-a < x < a$)

∎

The difficult homework problem is going to come from here.
Additional help:
$s_n = (1 + \frac{1}{n})^n$
First thing, show that it's increasing:
$a < s_n \leq s_{n+1}$
$(1 + \frac{1}{n})^n \leq (1 + \frac{1}{n+1})^{n+1}$
Second thing, show this:
(mini hint: $|s_n - s| < \epsilon\ \forall\ n \geq W$ (written on board, maybe he means M?)
$s_n < 3\ \forall\ n \in \mathbb{N}$
Turns up naturally:
$\lim\limits_{n\to\infty} (1 + \frac{1}{n})^n = e$

Example 4.3.4

Let: $s_1 = 1$, $s_{n+1} = \sqrt{1+s_n}$ $\forall$ n $\in \mathbb{N}$ with n $\geq$ 2
Prove that $\{s_n\}$ converges and find its limit.
$s_1 = 1$, $s_2 = \sqrt{2}$, $s_3 = \sqrt{1+\sqrt{2}}$, $s_4 = \sqrt{1+\sqrt{1+\sqrt{2}}}$ ...

Conjecture

$\{s_n\}$ is increasing and $1 \leq s_n \leq 2$, $\forall$ n $\in \mathbb{N}$
Proposition as a function of n [P(n)]:
$s_n \leq s_{n+1}$, $\forall$ n $\in \mathbb{N}$
$s_1 = 1 < \sqrt{2} = s_2$
Suppose that, $\forall$ k $\in \mathbb{N}$,

$$\sqrt{1+s_k} \leq \sqrt{1+s_{k+1}}$$

Now,

$$s_{k+1} = \sqrt{1+s_k} \leq \sqrt{1+s_{k+1}} = s_{k+2}$$

So,

$$s_k \leq s_{k+1}$$

Hence, by induction, P(n): $s_n \leq s_{n+1}$ is true $\forall$ n $\in \mathbb{N}$
Q(n): $s_n \leq 2$ $\forall$ n $\in \mathbb{N}$
$s_1 = 1 < 2$
Assume for k $\in \mathbb{N}$ that $s_k < 2$
Consider:

$$s_{k+1} = \sqrt{1+s_k} < \sqrt{1+3} = \sqrt{2+2} = 2$$

Hence, by induction, Q(n): $s_n < 2$ is true $\forall$ n $\in \mathbb{N}$
By the Montone Convergence Theorem,
$\exists$ s $\in \mathbb{R}$ st
$\lim_{n\to\infty} s_n = s$
By HW problem 11, page 170.
Thus,
$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} s_n = s$

Side Note

$\{s_n\} \longrightarrow s$
$\{t_n\}$: $t_n = s_{n+k}$, k $\in \mathbb{N}$

So, we claim that $\lim_{n\to\infty} s_{n+1} = s = \lim_{n\to\infty} \sqrt{1+s_n} = \sqrt{1+s}$
From Example 4.2.6,
$\lim_{n\to\infty} \sqrt{t_n} = \sqrt{t}$ if $\lim_{n\to\infty} t_n = t$
Also, by Theorem 4.2.1 (b), $\lim_{n\to\infty} \sqrt{1+s_n} = \sqrt{1+s}$
(which is like saying $\lim_{n\to\infty} t_n = t$)
Hence,

$$\begin{aligned} s &= \sqrt{1+s} \\ s^2 &= 1+s \\ s^2 - s - 1 &= 0 \\ s &= \frac{1(+/-)\sqrt{1-(-4)}}{2} \\ &= \frac{1(+/-)\sqrt{5}}{2} \end{aligned}$$

But one of those limits can't be true since limits are unique.
Since $s_n \geq 0$, $\forall$ n $\in \mathbb{N}$,
then $\lim\limits_{n\to\infty}$ s_n = s $\geq$ 0, $\forall$ n $\in \mathbb{N}$
(By Corollary 4.2.5)
Hence,
s = $\frac{1+\sqrt{5}}{2}$
$\{s_n\}$ is Cauchy if for $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$
st
$|s_n$ - $s_m| < \epsilon$ $\forall$ m, n $\geq$ N