Exam Tuesday, 31st of October (Halloween)
Covers: Section 4.2 (4.2.5 through end of section), 4.3, 4.4

Limit Superior & Limit Inferior

Definition 4.4.9

Let $\{s_n\}$ be a bounded sequence.
A **subsequential limit** of $\{s_n\}$ is a real number s such that $s = \lim\limits_{k\to\infty} s_{n_k}$ for some subsequence $\{s_{n_k}\}$.
If $S = \{s \in \mathbb{R} : \lim\limits_{k\to\infty} s_{n_k} = s$ for some $\{s_{n_k}\}$ of $\{s_n\}\}$, then

a. the **limit superior** (or **upper limit**) of $\{s_n\}$ is given by $\limsup s_n = \sup S$

b. the **limit inferior** (or **lower limit**) of $\{s_n\}$ is given by $\liminf s_n = \inf S$

c. Clearly, $\liminf s_n \leq \limsup s_n$. If it happens that $\liminf s_n < \limsup s_n$, then we say that $\{s_n\}$ **oscillates**.

Side Note

$|s_n| \leq M, \forall\ n \in \mathbb{N}$
$-M < s_n < M$
If $\lim\limits_{k\to\infty} s_{n_k} = s \in S$, then
$-M < s_{n_k} < M$, so
$-M < s < M$
#18, page 179

Theorem 1

A bounded sequence $\{s_n\}$ converges to s iff $\liminf s_n = \limsup s_n = s$

Proof.
$\longrightarrow$ Assume $\{s_n\}$ converges to s.
By Theorem 4.4.4, $S = \{s\}$ (contains only one element).
Then,
$\liminf s_n = \inf S = s$
$\limsup s_n = \sup S = s$
So,
$\liminf s_n = \limsup s_n = s$
$\longleftarrow$
(see HW 8, Exercise 9, page 194)

□

Example 4.4.10

Let: $s_n = (-1)^n + \frac{1}{n}$
Show that
lim inf $s_n = -1$,
lim sup $s_n = 1$
Notice that if
n is even $\Rightarrow s_n = 1 + \frac{1}{n}$
n is odd $\Rightarrow s_n = -1 + \frac{1}{n}$
Thus,
$\lim_{k\to\infty} s_{2k} = 1$
$\lim_{k\to\infty} s_{2k+1} = -1$
Thus,
$S = \{-1, 1\}$
Hence,
lim sup $s_n = 1$
lim inf $s_n = -1$

Theorem 4.4.11

Let $\{s_n\}$ be a bounded sequence and let
$s^* =$ lim sup s_n
$s_* =$ lim inf s_n

a. $\forall\ \epsilon > 0, \exists\ N(\epsilon\ ) \in \mathbb{N}$ st

$$s_n < s^* + \epsilon \text{ for n} \geq \text{N}$$

b. $\forall\ \epsilon > 0$ and $i \in \mathbb{N}$, $\exists\ j > i$ st

$$s_j > s^* - \epsilon$$

   i.e. there are an infinite number of terms of $\{s_n\}$ that are greater than $s^* - \epsilon$

   i.e. in the interval $(s^* - \epsilon\ , s^* + \epsilon\ )$, there are an infinite number of terms of s_n.

   Outside of that interval, there are a finite number of terms of s_n.

c. $\forall\ \epsilon > 0, \exists\ N(\epsilon\ ) \in \mathbb{N}$ st

$$s_n > s_* - \epsilon\ \forall \text{ n} \geq \text{N}$$

d. $\forall\ \epsilon > 0$ and $i \in \mathbb{N}$, $\exists\ j > i$ st

$$s_j < s_* + \epsilon$$

Proof.

We shall prove a and b. c and d are similar.
(a)
Suppose it's false. i.e.:
Suppose: $\exists\ \epsilon > 0$ st $\forall$ N $\in \mathbb{N}$, $\exists$ n $\geq$ N st

$$s_n \geq s^* + \epsilon$$

———————————Side Note———————————

In other words, suppose: $\{s_{n_k} \geq s^* + \epsilon\}$
By Theorem 4.4.4, every bounded sequence has a convergent subsequence.
If we let $\{s_{n_k}\}$ be a subsequence of itself and label it differently:
$\{s_{n_l}\}_{i=1}^{\infty}$,
then
$s_{n_l} \longrightarrow$ s as l $\longrightarrow \infty$

So, for N = 1, $\exists$ $n_1 \geq$ N st

$$s_{n_1} \geq s^* + \epsilon$$

Then,
for N = n_1 + 1, $\exists$ $n_2 \geq n_1 + 1 > n_1$ st

$$s_{n_2} \geq s^* + \epsilon$$

So, inductively, we find a subsequence $\{s_{n_k}\}$ st

$$s_{n_k} \geq s^* + \epsilon \ \forall \ \mathrm{k} \in \mathbb{N}$$

Since $\{s_{n_k}\}$ is itself a bounded sequence, there is a subsequence of $\{s_{n_k}\}$ that we refer to by:
$\{s_{n_l}\}_{l=1}^{\infty}$
st
$\lim\limits_{l\to\infty} s_{n_l}$ = s $\in$ k (Theorem 4.4.7)
where s $\geq s^* + \epsilon$
Since s $\in$ S, we see that lim sup $s_n = s^* \geq s^* + \epsilon$, which is a contradiction.
Hence, **(a)** is true.
(b)
Suppose it's false. i.e.:
Suppose: $\exists\ \epsilon > 0$ and $\exists$ i $\in \mathbb{N}$ st $\forall$ j > i,

$$s_j \leq s^* - \epsilon$$

Thus, if $\{s_{n_k}\}$ is a subsequence st $\lim\limits_{k\to\infty} s_{n_k}$ = s, then

$$s \leq s^* - \epsilon$$

which is like saying:
$s^* \leq s^* - \epsilon$
(a contradiction)
For further clarification, notice that $s^* - \epsilon$ is an upper bound for all s $\in$ S, which says: $s^* \leq s^* - \epsilon$
(a contradiction)
Summary:
In **(a)**, we said $\exists$ $N_1 \in \mathbb{N}$ st $s_n < s + \epsilon \ \forall$ n $\geq N_1$
In **(b)**, we said $\exists$ $N_2 \in \mathbb{N}$ st $s - \epsilon < s_n \ \forall$ n $\geq N_2$

At the bottom of page 190:
Furthermore,
if $s^* \in \mathbb{R}$ satisfying **(a)** and **(b)**,
then $s^* = \limsup s_n$
Also,
if $s_* \in \mathbb{R}$ satisfying **(c)** and **(d)**,
then $s_* = \liminf s_n$
We shall complete the proof by proving the result for s^*

Let: $s^* \in \mathbb{R}$ satisfy **(a)** and **(b)**
We claim that $s^* = \limsup s_n$, and will prove it by contradiction.
Suppose: $s^* \neq \limsup s_n$
Case:

i) $s^* > \limsup s_n$

   So, $s^* - \epsilon$ is between $\limsup s_n$ and s^*

   Let:
$$\epsilon = \frac{s^* - \limsup s_n}{2}$$

   Version One:

   By **(b)**, for this $\epsilon > 0$, and for $i \in \mathbb{N}$, $\exists$ $j \in \mathbb{N}$ st

   $j > i$ and
$$s_j > s^* - \epsilon$$

   Since there are an infinite number of possible values of j, there is a subsequence $\{s_{n_k}\}$ st
$$s_{n_k} > s^* - \epsilon$$

   $\forall$ $k \in \mathbb{N}$

   This contradicts the definition of $\limsup s_n$.

   Thus, there is a further subsequence converging to a limit s st
$$s \geq s^* - \epsilon \geq s^*$$

   Which is also a contradiction.

   Version Two:

   By **(b)**, for $i = 1$, $\exists$ $j = n_1 > 1$ st
$$s_{n_1} > s^* - \epsilon$$

   Then, for $i = n_1$, $\exists$ $j - n_2 > n_1$ st
$$s_{n_2} > s^* - \epsilon$$

   So, inductively, we find a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ st
$$s_{n_k} > s^* - \epsilon$$

   Since $\{s_{n_k}\}$ is a bounded sequence.

   So, there is a convergent subsequence $\{s_{n_l}\}$ of $\{s_{n_k}\}$ st $\lim\limits_{l\to\infty} s_{n_l} = s$ where $s \geq s^* - \epsilon$

   So, for $s \in S$, $\limsup s_n \geq s \geq s^* - \epsilon = \frac{\limsup s_n + s^*}{2} > \limsup s_n$, **a contradiction**.

   Hence, $s^* \ngtr \limsup s_n$.

ii) $s^* < \limsup s_n$

Let: $\epsilon = \frac{\limsup s_n - s^*}{2}$

By **(a)**, $\exists$ N(ϵ) $\in \mathbb{N}$ st

$$s_n < s^* + \epsilon \text{ for n} \geq \text{N}$$

Thus, $\exists$ s $\in$ S st

$$s \leq s^* + \epsilon$$

Thus, $\limsup s_n \leq s^* + \epsilon = \frac{\limsup s_n + s^*}{2} < \limsup s_n$, **a contradiction**.

Hence, $s^* < \limsup s_n$

□

Cases (i) and (ii) together yield the contradiction that $s^* = \limsup s_n$, another contradiction.
On page 195, problem # (a): Prove that $\limsup s_n = \lim\limits_{n \to \infty} (\sup \{s_{n+1}, s_{n+2}, s_{n+3} ...\})$

Side Note

If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ st $\lim\limits_{k \to \infty} s_{n_k} = s$.
Then s $\in$ S

Corollary 4.4.12

Let $\{s_n\}$ be a bounded sequence and let $s^* = \limsup s_n$, $s_* = \liminf s_n$.
Then, $s_*, s^* \in S$ (i.e. s_*, s^* are themselves subsequential limit points).

Proof.

For $\epsilon = 1$, by Theorem 4.4.11 **(a)**, $\exists$ $N_1 \in \mathbb{N}$ st

$$s_n < s^* + 1, \text{ for n} \geq \text{N}_1 \tag{1}$$

By Theorem 4.4.11, **(b)**, for $\epsilon = 1$, i = N_1, $\exists$ $n_1 >$ i = N_1 st

$$s_{n_1} > s^* - 1$$

and

$$s^* - 1 < s_{n_1} < s^* + 1 \text{ using (1)}$$

For $\epsilon = \frac{1}{2}$, $\exists$ $N_2 \in \mathbb{N}$ st

$$s_n < s^* + \frac{1}{2} \text{ for n} \geq \text{N}_2 \text{ using (a)} \tag{2}$$

Also, for i = $\max\{n_1, N_2\}$, $\exists$ j = $n_2 >$ i (i.e. $n_2 > n_1$ and $n_2 > N_2$) st

$$s^* - \frac{1}{2} < s_{n_2} < s^* + \frac{1}{2}$$

Inductively, we can construct a sequence $\{s_{n_k}\}$ of $\{s_n\}$ st

$$s^* - \frac{1}{k} < s_{n_k} < s^* + \frac{1}{k}$$

Hence, $|s_{n_k} - s^*| < \frac{1}{k} \longrightarrow 0$ as k $\longrightarrow \infty$
Hence, $s^* = \lim\limits_{k \to \infty} s_{n_k}$, which completes the proof. □

Theorem 4.4.14

Assume that $\{r_n\}$ converges to $r \in \mathbb{R}$ where $r > 0$ and $\{s_n\}$ is bounded.
Then $\limsup r_n s_n = r \limsup s_n$

Proof.

$\exists M_1, M_2 \in \mathbb{R}$ st

$$|s_n| \leq M_1 \text{ and } |r_n| \leq M_2, \forall n \in \mathbb{N}$$

So,

$$|r_n s_n| \leq M_1 M_2$$

Thus, the sequence $\{r_n s_n\}$ is bounded, which means $\limsup r_n s_n$ exists.
By Corollary 4.4.12, $\exists$ a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ st $\lim\limits_{k\to\infty} s_{n_k} = \limsup s_n$
Then, $\lim\limits_{k\to\infty} r_{n_k} s_{n_k} = (\lim\limits_{k\to\infty} r_{n_k})(\lim\limits_{k\to\infty} s_{n_k}) = r \limsup s_n$
(i.e. $r \limsup s_n$ is a subsequential limit point of the sequence $\{r_n s_n\}$)
Thus, $r \limsup s_n \leq \limsup r_n s_n$ **(1)**

Also, assume that $\{r_{n_l} s_{n_l}\}$ is a subsequence of $\{r_n s_n\}$ st

$$\lim_{l\to\infty} r_{n_l} s_{n_l} = t$$

Then,

$$\lim_{l\to\infty} s_{n_l} = \lim_{l\to\infty} s_{n_l} (\lim_{l\to\infty} \frac{r_{n_l}}{r}) = \lim_{l\to\infty} \frac{s_{n_l} r_{n_l}}{r} = \frac{t}{r} \leq \limsup s_n$$

So, $t \leq r \limsup s_n$ (which says $r \limsup s_n$ is an upper bound)
So, $\limsup r_n s_n \leq r \limsup s_n$ **(2)**
(1) and **(2)** imply that $\limsup r_n\, s_n = r \limsup s_n$ ☐

Unbounded Sequences

Case:

i) $\{s_n\}$ is unbounded above.

   By Theorem 4.4.8, there is a monotonic subsequence $\{s_{n_k}\}$ of $\{s_n\}$ st $\lim\limits_{k\to\infty} s_{n_k} = \infty$

   In this case, we define $\limsup s_n = \infty$

ii) $\{s_n\}$ is bounded above but unbounded below.

   subcase (i): $\exists$ a subsequence $\{s_{n_k}\}$ st $\lim\limits_{k\to\infty} s_{n_k} = s \in \mathbb{R}$

   In this case, $s \in S$, and, as before, $\limsup s_n = \sup S$

   subcase (ii): No subsequence of $\{s_n\}$ converges to a finite limit.

   In this case, $\lim\limits_{n\to\infty} s_n = -\infty$, and we define $\limsup s_n = -\infty$

   So for any $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ st

   $$s_n < M \text{ for } n \geq N$$

   and $S = \{-\infty\}$