Homework 7: pages 184 - 185 numbers 1, 21(a)(b), 3(e), 4, 10, 13, 14 $\longleftarrow$ 14 is difficult, but not impossible!
(want to show that $\lim (1 + \frac{1}{n})^n$ exists)
Hint:
$(1 + b)^n = 1 + nb + \frac{n(n-1)}{2!}b^n + ... + \frac{n(n-1)...(n-(r-1))}{r!}b^r + ... + b^n$
In our problem, $b = \frac{1}{n}$
Look at it as $1 + \sum_{r=1}^{n} \frac{n(n-1)...(n-(r-1))}{r!} \frac{1}{n^r}$
$(1 + \frac{1}{n})^n$ goes in there somewhere somehow.
About the last homework (HW 6), problem 9:
If $s_n \leq t_n \ \forall \ n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = \infty$,
then $\lim_{n \to \infty} t_n = \infty$
So, $\forall \ M \in \mathbb{R}$, $\exists \ N \in \mathbb{N}$ st
$s_n > M, \forall \ n \geq N$
Notice that:
$t_n \geq s_n > M, \forall \ n \geq N$
So by definition, $\lim_{n \to \infty} t_n = \infty$

Section 4.3: Monotone Sequences and Cauchy Sequences

Definition 4.3.1

A sequence (s_n) is **increasing** (or **decreasing**) if $s_n \leq s_{n+1}$ (or $s_{n+1} \leq s_n$) $\forall \ n \in \mathbb{N}$. A sequence is
monotonic if it is increasing or decreasing.

Example 4.3.2

a. $a_n = n, \forall \ n \in \mathbb{N}$

   increasing

b. $b_n = 2^n, \forall \ n \in \mathbb{N}$

   increasing

c. $c_n = 2 - \frac{1}{n}, \forall \ n \in \mathbb{N}$

   increasing

d. $(d_n) = 1, 1, 2, 2, 3, 3...$

   increasing

e. $e_n = \frac{2}{n}, \forall \ ...$

   decreasing

f. $f_n = -3n$

   decreasing

g. $(g_n) = 1, 1, 1, ...$ $(g_n = 1, \forall \ n \in \mathbb{N}$)

   increasing and decreasing

h. $h_n = -1^n, \forall \ n \in \mathbb{N}$

   not monotonic

i. $i_n = \cos(\frac{n\pi}{3}) \ \forall \ n \in \mathbb{N}$

   not monotonic

Theorem 4.3.3 (Monotone Convergence Theorem)

A **monotonic sequence** is convergent iff it is bounded.

Proof.
Let: $\{s_n\}$ be a monotonically increasing sequence
$\longrightarrow$
Assume $\{s_n\}$ is convergent.
By Theorem 4.1.13, $\{s_n\}$ is bounded.
$\longleftarrow$
Conversely, assume $\{s_n\}$ is bounded.
Want to show: $\{s_n\}$ converges
Let the range of $\{s_n\}$ be denoted by $S = \{s_n : n \in \mathbb{N}\}$
Since $\{s_n\}$ is bounded, S is bounded above.
Thus, sup S exists.
Want to show: $\{s_n\}$ converges to sup S
Recall: The supremum is the least upper bound.
Thus,
$s_n \leq \sup S, \forall\, n \in \mathbb{N}$ **(1)**
and for $\epsilon > 0, \exists\, N(\epsilon) \in \mathbb{N}$ st $\forall\, n \geq N$,
$\sup S - \epsilon < s_n$
$\sup S - \epsilon < s_N \leq s_n \leq \sup S < \sup S + \epsilon$ **(2)**
Since $\{s_n\}$ is increasing and, using **(1)**,
From **(2)**, we see that
$-\epsilon < s_n - \sup S < \epsilon$, $\forall\, n \geq N$
Hence,
$|s_n - \sup S| < \epsilon$, $\forall\, n \geq N$,
which is equivalent to $\lim\limits_{n \to \infty} s_n = \sup S$
(since $|x| < a$ iff $-a < x < a$)

$\square$

The difficult homework problem is going to come from here.
Additional help:
$s_n = (1 + \frac{1}{n})^n$
First thing, show that it's increasing:
$a < s_n \leq s_{n+1}$
$(1 + \frac{1}{n})^n \leq (1 + \frac{1}{n+1})^{n+1}$
Second thing, show this:
(mini hint: $|s_n - s| < \epsilon\ \forall\, n \geq W$ (written on board, maybe he means M?)
$s_n < 3\ \forall\, n \in \mathbb{N}$
Turns up naturally:
$\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n = e$

Example 4.3.4

Let: $s_1 = 1$, $s_{n+1} = \sqrt{1 + s_n}$ $\forall$ n $\in \mathbb{N}$ with n ≥ 2
Prove that $\{s_n\}$ converges and find its limit.

$s_1 = 1$, $s_2 = \sqrt{2}$, $s_3 = \sqrt{1 + \sqrt{2}}$, $s_4 = \sqrt{1 + \sqrt{1 + \sqrt{2}}}$ ...

Conjecture

$\{s_n\}$ is increasing and $1 \leq s_n \leq 2$, $\forall$ n $\in \mathbb{N}$
Proposition as a function of n [P(n)]:
$s_n \leq s_{n+1}$, $\forall$ n $\in \mathbb{N}$
$s_1 = 1 < \sqrt{2} = s_2$
Suppose that, $\forall$ k $\in \mathbb{N}$,

$$\sqrt{1 + s_k} \leq \sqrt{1 + s_{k+1}}$$

Now,

$$s_{k+1} = \sqrt{1 + s_k} \leq \sqrt{1 + s_{k+1}} = s_{k+2}$$

So,

$$s_k \leq s_{k+1}$$

Hence, by induction, P(n): $s_n \leq s_{n+1}$ is true $\forall$ n $\in \mathbb{N}$
Q(n): $s_n \leq 2$ $\forall$ n $\in \mathbb{N}$
$s_1 = 1 < 2$
Assume for k $\in \mathbb{N}$ that $s_k < 2$
Consider:

$$s_{k+1} = \sqrt{1 + s_k} < \sqrt{1 + 3} = \sqrt{2 + 2} = 2$$

Hence, by induction, Q(n): $s_n < 2$ is true $\forall$ n $\in \mathbb{N}$
By the Montone Convergence Theorem,
$\exists$ s $\in \mathbb{R}$ st
$\lim\limits_{n \to \infty} s_n = s$
By HW problem 11, page 170.
Thus,
$\lim\limits_{n \to \infty} s_{n+1} = \lim\limits_{n \to \infty} s_n = s$

─────────────────────────Side Note─────────────────────────

$\{s_n\} \longrightarrow s$

$\{t_n\}$: $t_n = s_{n+k}$, $k \in \mathbb{N}$

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So, we claim that $\lim\limits_{n\to\infty} s_{n+1} = s = \lim\limits_{n\to\infty} \sqrt{1+s_n} = \sqrt{1+s}$

From Example 4.2.6,

$\lim\limits_{n\to\infty} \sqrt{t_n} = \sqrt{t}$ if $\lim\limits_{n\to\infty} t_n = t$

Also, by Theorem 4.2.1 (b), $\lim\limits_{n\to\infty} \sqrt{1+s_n} = \sqrt{1+s}$

(which is like saying $\lim\limits_{n\to\infty} t_n = t$)

Hence,

$$s = \sqrt{1+s}$$
$$s^2 = 1 + s$$
$$s^2 - s - 1 = 0$$
$$s = \frac{1(+/-)\sqrt{1-(-4)}}{2}$$
$$= \frac{1(+/-)\sqrt{5}}{2}$$

But one of those limits can't be true since limits are unique.

Since $s_n \geq 0$, $\forall\, n \in \mathbb{N}$,

then $\lim\limits_{n\to\infty} s_n = s \geq 0$, $\forall\, n \in \mathbb{N}$

(By Corollary 4.2.5)

Hence,

$s = \frac{1+\sqrt{5}}{2}$

$\{s_n\}$ is Cauchy if for $\epsilon > 0$, $\exists\, N \in \mathbb{N}$

st

$|s_n - s_m| < \epsilon \ \forall\, m,\, n \geq N$