

HW 2: page 140-141, #2-5 (Section 3.4)

Theorem 3.3.10

Each of the following is equivalent to the AP:

- a. $\forall z \in \mathbb{R}, \exists n \in \mathbb{N} \text{ st } n > z$
- b. $\forall x > 0, y \in \mathbb{R}, \exists n \in \mathbb{N} \text{ st } nx > y$
- c. $\forall x > 0, \exists n \in \mathbb{N} \text{ st } 0 < \frac{1}{n} < x$

Proof.

We shall prove:

- i) $\text{AP} \Rightarrow \text{a}$
- ii) $\text{a} \Rightarrow \text{b}$
- iii) $\text{b} \Rightarrow \text{c}$
- iv) $\text{c} \Rightarrow \text{AP}$

In other words, they all imply each other.

a. $\text{AP} \Rightarrow \text{a}$

Suppose: a is false.

So, $\forall z \in \mathbb{R}, \exists n \in \mathbb{N}, P(z, n) \text{ (st } n \leq z) \text{ ???}$

Side Note

$$\neg[\exists x_1 \forall x_2 \text{ st } p(x_1, x_2)] =$$

$$\forall x_1, \exists x_2 \text{ st } \neg p(x_1, x_2)$$

$$\exists z_0 \in \mathbb{R} \text{ st } \forall n \in \mathbb{N}, n \leq z_0$$

This indicates that the AP is false.

Thus, $\text{AP} \Rightarrow \text{a}$.

b. $\text{a} \Rightarrow \text{b}$

Assume: a) is true.

Let: $z = \frac{y}{x} \in \mathbb{R}$

By (a), $\exists n \in \mathbb{N} \text{ st}$

$$n > \frac{y}{x}$$

$$nx > y$$

Hence, $\text{a} \Rightarrow \text{b}$ is true.

c. $b \Rightarrow c$

Assume: b is true.

$\forall x > 0$, if $y = 1$,

we see from **(b)** that $\exists n \in \mathbb{N}$ st $nx > 1$

Then,

$x > \frac{1}{n} > 0$.

Hence, $b \Rightarrow c$.

d. $c \Rightarrow AP$

Reminder of c : $\forall x$ where $0 < x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ st. $0 < \frac{1}{n} < x$

Suppose: \mathbb{N} is bounded above. (In other words, that the AP is false.

Thus, $\exists z_0 \in \mathbb{R}$ st $0 < n \leq z_0, \forall n \in \mathbb{N}$

$0 < n \leq z_0$

$\frac{1}{n} \geq \frac{1}{z_0}$

This contradicts c with $x = \frac{1}{z_0}$ where $0 < \frac{1}{z_0} \in \mathbb{R}$

Hence, result.

□

Theorems 3.3.13 and 3.3.15

Let: $x, y \in \mathbb{R}$ st $x < y$

Then:

a. $\exists r \in \mathbb{Q}$ st $x < r < y$

b. $\exists z \in \mathbb{R} \setminus \mathbb{Q}$ st $x < z < y$

a

Case:

(i): $y > 0$

$y = 0.a_1a_2\dots a_n$ i.e. $0.141 = \frac{141}{1000}$

(ii): $y \leq 0$

$-y \geq 0, -y < -x, 0 \leq -y < -x$

By case **(i)**, $\exists r \in \mathbb{Q}$ st

$-y < r < -x$

$y > -r > x$

$x < -r < y$

b

$\exists z \in \mathbb{R} \setminus \mathbb{Q}$ st $x < z < y$

Apply **(a)** to $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$ to find $r \in \mathbb{Q}$ st

$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$

$x < r\sqrt{2} < y$

Let: $r\sqrt{2} = z$

$x < z < y$

Hence, result.

Section 3.4: Topology of \mathbb{R}

Definitions 3.4.1 and 3.4.2

Let $x \in \mathbb{R}$ and $\epsilon > 0$.

(a)

An ϵ -neighborhood of x is:

$$N(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon\}$$

(b)

A deleted ϵ -neighborhood of x is:

$$N^*(x, \epsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$$

Open Set Topology: Definition 3.4.3 (interior / boundary point)

Let: $S \subset \mathbb{R}$

A point $x \in \mathbb{R}$ is an **interior point** of S if $\exists \epsilon > 0$ st $N(x, \epsilon) \subset S$.

If, $\forall \epsilon > 0$,

$$N(x, \epsilon) \cap S \neq \emptyset$$

and

$$N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$$

Then x is a **boundary point** of S .

The set of all interior points is denoted by **int** S .

The set of all boundary points is denoted by **bd** S .

Nota Bene (N.B.):

$$\text{int } S \subset S \text{ and } \text{bd } S = \text{bd } (\mathbb{R} \setminus S)$$

Side Note

Let: $x \in \text{int } S$

Then $\exists \epsilon > 0$ st $N(x, \epsilon) \subset S$

In particular, $x \in S$. Thus, $\text{int } S \subset S$.

Let: $S^C = \mathbb{R} \setminus S$, and $\mathbb{R} \setminus S^C = S$

Then $s \in \text{bd } S^C$ if $\forall \epsilon > 0$,

$$N(x, \epsilon) \cap S^C \neq \emptyset$$

$$N(x, \epsilon) \cap \mathbb{R} \setminus S^C \neq \emptyset$$

Thus, $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$, and $N(x, \epsilon) \cap S \neq \emptyset$

So, $x \in \text{bd } S$

Theorem 1

Let: $x \in S \subset \mathbb{R}$

Then either $x \in \text{int } S$, or $x \in \text{bd } S$.

Proof.

Let: $x \in S \subset \mathbb{R}$

i) $\exists \epsilon > 0$ st $N(x, \epsilon) \subset S$. Then, by def, $x \in \text{int } S$

ii) $\forall \epsilon > 0, N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

However, since $x \in S$, then $N(x, \epsilon) \cap S \neq \emptyset$.

By definition, $x \in \text{bd } S$.

Hence, result. □

Section 3.4.4 Examples

a. **Let:** $S = (0, 5)$

Here, $\text{int } S = (0, 5)$ and $\text{bd } S = \{0, 5\}$

To see this****,

Let: $x \in (0, 5), \epsilon = \min \{x, 5-x\}$

Then $N(x, \epsilon) \subset (0, 5)$

To see this, let $y \in N(x, \epsilon)$.

Want to show: $0 < y < 5$

Since $y \in N(x, \epsilon)$, we have

$$x - \epsilon < y < x + \epsilon \tag{1}$$

Notice that

$$\epsilon \leq x \tag{2}$$

and

$$\epsilon \leq 5 - x \tag{3}$$

From (3),

$$x + \epsilon \leq x + (5 - x) = 5 \tag{4}$$

From (2),

$$x - \epsilon \leq x - \epsilon, \quad 0 \leq x - \epsilon \tag{5}$$

From (1), (4), (5),

$$0 \leq x - \epsilon < y \leq x + \epsilon < 5,$$

We see that $y \in (0, 5)$.

Hence, $0 < y < 5$.

Since $N(x, \epsilon) \subset (0, 5)$, we see that

$$\text{int } S = (0, 5) = S$$

Want to show: $0 \in \text{bd } S$

Let: $0 < \epsilon < 5$

Notice that:

$$N(0, \epsilon) \cap (0, 5) \neq \emptyset \text{ and } N(0, \epsilon) \cap (\mathbb{R} \setminus (0, 5)) \neq \emptyset$$

Using $y = (+/-) \frac{\epsilon}{2}$, notice that:

$$y \in N(0, \epsilon) \text{ since } |(+/-) \frac{\epsilon}{2}| < \epsilon$$

****2****

b. **Let:** $S = [0, 5]$

Here, $\text{int } S = (0, 5)$, $\text{bd } S = \{0, 5\}$

Notice that $\text{bd } S \subset S$

c. **Let:** $S = [0, 5)$

Here, $\text{int } S = (0, 5)$, $\text{bd } S = 0, 5$

Notice that some bd points are in S , but some aren't.

d. **Let:** $S = [2, \infty)$

Here, $\text{int } S = (2, \infty)$, $\text{bd } S = \{2\}$

e. **Let:** $S = \mathbb{R}$

$\text{int } S = \mathbb{R} = S$, $\text{bd } S = \emptyset$

Here, $\text{bd } S \subset S$

Side Note

—(—(—)——)—

0, $x - \epsilon$, y , x , $x + \epsilon$, 5

from 0 to x is x , from x to 5 is $5 - x$

****2****

Side Note

—(——(——)—)—

$0 - \epsilon$, $-\epsilon/2$, 0, $\epsilon/2$, $0 + \epsilon$, 5