Due 4/25 (Wednesday):
All (turn in): Chapter 10, page 206, 14-18, 20, 24

Chapter 10

Recall:
A homomorphism ϕ from a group G to a group G′ is a mapping from G into G′ that preserves the group operation; that is, ϕ(ab) = ϕ(a)ϕ(b) for a, b ∈ G.
The kernel of a homomorphism ϕ from a group G to a group G′ with identity e′ is the set {x ∈ G : ϕ(x) = e′ }.

Exercise 14

Explain why the correspondence x $\longrightarrow$ 3x from Z_{12} to Z_{10} is not a homomorphism.

Because ϕ is not OP:
$\phi(3 * 4) = \phi(12) = \phi(0) = (3 * (0 \bmod 12)) \bmod 10 = e$, and
$\phi(3)\phi(4) = (3 * (3 \bmod 12) * 3 * (4 \bmod 12)) \bmod 10 = (9 * 3 * 4) \bmod 10 = (108) \bmod 10 = 8$

Exercise 15

Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and Ker ϕ = {0, 10, 20}.
If ϕ(23) = 9, determine all elements that map to 9.

$\phi(ab \bmod 30) = \phi(a \bmod 30)\phi(b \bmod 30)$
$\phi(23) = 9$.
$\phi(0) = \phi(10) = \phi(20) = 0$
It looks like it's $\phi(x) = 3x$:
$\phi(23) = 3 * 23 \bmod 30 = 69 \bmod 30 = 9$.
Thus,
$\phi(3)$, $\phi(13)$, and $\phi(23)$ all map to 9.

Exercise 16

Prove that there is no homomorphism from $Z_8 \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Suppose ∃ ϕ: $Z_8 \bigoplus Z_2 \longrightarrow Z_4 \bigoplus Z_4$, such that ϕ is a homomorphism.
Because Z_8 is of order 8, and $|Z_2|$ divides 8, there is an element of order 8 in $Z_8 \bigoplus Z_2$, let's call it z_8.
Thus, $z_8 \in Z_8 \bigoplus Z_2$.
Because ϕ is OP, ∃ z ∈ $Z_4 \bigoplus Z_4$ such that $\phi(z_8)$ = z and |z| = 8.
However, there is no element of order 8 in $Z_4 \bigoplus Z_4$. A contradiction.
Hence, no homomorphism exists.

Exercise 17

Prove that there is no homomorphism from $Z_{16} \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Suppose $\exists\ \phi$: $Z_{16} \bigoplus Z_2 \longrightarrow Z_4 \bigoplus Z_4$, such that ϕ is a homomorphism.
Because Z_{16} is of order 16, and $|Z_2|$ divides 8, there is an element of order 16 in $Z_{16} \bigoplus Z_2$, let's call it z_{16}.
Thus, $z_{16} \in Z_{16} \bigoplus Z_2$.
Because ϕ is OP, $\exists$ z $\in Z_4 \bigoplus Z_4$ such that $\phi(z_{16})$ = z and |z| = 16.
However, there is no element of order 16 in $Z_4 \bigoplus Z_4$. A contradiction.
Hence, no homomorphism exists.

Exercise 18

Can there be a homomorphism from $Z_4 \bigoplus Z_4$ onto Z_8? Can there be a homomorphism from Z_{16} onto $Z_2 \bigoplus Z_2$? Explain your answers.

Exercise 20

How many homomorphisms are there from Z_{20} onto Z_8? How many are there to Z_8?

Exercise 24

Suppose that ϕ: $Z_{50} \longrightarrow Z_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine ϕ(x).

b. Determine the image of ϕ.

c. Determine the kernel of ϕ.

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.