Homework Due 10/5/17: (7 problems) Section 4.1 pages 169 - 170; 1, 6(b), 7(f), 9(a), 11, 12, 15
Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18

Theorem 4.1.13

Every convergent sequence is bounded.

Proof.

Assume: $s_n \longrightarrow s$ as $r \longrightarrow \infty$
Then for $\epsilon = 1$, $\exists$ $N(\epsilon) \in \mathbb{N}$ st, $\forall$ $n \geq N$,
$|s_n| - |s| \leq |\, |s_n| - |s| \,| \leq |s_n - s| < 1$ (page 121, Ex 61(a))

Side Note

$x \leq |x|$ $\forall$ $x \in \mathbb{R}$

So $|s_n| < 1 + |s|$, $\forall$ $n \geq N$
$|s_n| \leq |\, |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s|$ $\forall$ $n \geq N$
Then,
Let: $m = \max \{|s_1|, |s_2|, \ldots |s_{N-1}|, |s|\}$
Then $|s_n| \leq m$, $\forall$ $n \in \mathbb{N}$
Hence, $\{s_n\}$ is bounded.

□

Theorem 4.1.14

If a sequence converges, then its limit is unique.

Proof.

Assume: $\{s_n\}$ is a sequence and $s_n \longrightarrow s$ as $n \longrightarrow \infty$ and $s_n \longrightarrow t$ as $n \longrightarrow \infty$
Then $\forall$ $\epsilon > 0$, $\exists$ $N,(\epsilon)$ st
$|s_n - s| < \frac{\epsilon}{2}$, $\forall$ $n \geq N$ **(1)**
Also,
$\exists$ $N_2(\epsilon) \in \mathbb{N}$ st
$|s_n - t| < \frac{\epsilon}{2}$, $\forall$ $n \geq N_2$ **(2)**
Set $N = \max \{N_1, N_2\}$
From **(1)**, **(2)**
$|s - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, $\forall$ $n \geq N$
Hence,
$s = t$

□

4.2 Limit Theorems

Theorem 4.2.1

Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$.
Then,

a. $\lim_{n\to\infty} (s_n + t_n) = s + t$

b. $\lim_{n\to\infty} ks_n = ks$ and $\lim_{n\to\infty} (k + s_n) = k + s$, for any $k \in \mathbb{R}$

c. $\lim_{n\to\infty} (s_n t_n) = st$

d. $\lim_{n\to\infty} (\frac{s_n}{t_n}) = \frac{s}{t}$, provided that $t_n \neq 0 \ \forall\ n \in \mathbb{N}$ and $t \neq 0$

Proof.

(a)
$|t + s - (s_n + t_n)| =$
$|(t - t_n) + (s - s_n)| \leq |t - t_n| + |s - s_n|$ **(1)**
$\forall\ \epsilon > 0, \exists\ N_1(\epsilon), N_2(\epsilon)$ st

$$|t - t_n| < \frac{\epsilon}{2}\ \forall n \geq N_1 \tag{2}$$

and

$$|s - s_n| < \frac{\epsilon}{2},\ \forall n \geq N_2 \tag{3}$$

Let: $N = \max\{N_1, N_2\}$
From **(1) - (3)**,
$|s + t - (s_n + t_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ \forall\ n \geq N$
Hence, result.

(c)
$|st - s_n t_n| = |(st - s_n t) + (s_n t - s_n t_n)| \leq |st - s_n t| + |s_n t - s_n t_n| = |s - s_n||t| + |s_n||t - t_n|$
By theorem 4.1.3, $\exists\ m > 0$ st $|s_n| \leq M_1\ \forall\ n \in \mathbb{N}$
So,
$|st - s_n t_n| \leq |t||s - s_n| + M|t - t_n|$
Let: $\epsilon > 0$
Then $\exists\ N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$ st
$\forall\ |s - s_n| < \frac{|t|\epsilon}{|t|+M}, \forall\ n \geq N_1$
and
$M\ |t - t_n| < \frac{M\epsilon}{|t|+M}, \forall\ n \geq N_2$
Set $N = \max\{N_1, N_2\}$
Then
$|st - s_n t_n| < |t|\ \frac{\epsilon}{|t|+M} + \frac{M\epsilon}{|t|+M} = \epsilon\ (\frac{|t|+M}{|t|+M}) = \epsilon\ \forall\ n \geq N$
Hence, result.

(d)
Since $\frac{s_n}{t_n} = (\frac{1}{t_n})(s_n)$, the proof follows from **(c)** if we can prove that $\lim\limits_{n\to\infty} \frac{1}{t_n} = \frac{1}{t}$
Now:
$|\frac{1}{t} + \frac{1}{t_n}| = |\frac{t_n - t}{tt_n}| = \frac{|t_n - t|}{|t||t_n|}$ **(1)**

Side Note

$|t_n| > 1$
$|t_n| \geq M$

Recall that:
$|t| - |t_n| \leq |\ |t| - |t_n|\ | \leq |t - t_n|$ from page 121, example 6(a)
For $\epsilon = |t| > 0$, $\exists\ N_1(\epsilon\ ) \in \mathbb{N}$ st
$|t - t_n| < \frac{|t|}{2}$, $\forall$ n $\geq N_1$
Now $|t| - |t_n| \leq |\ |t| - |t_n|\ | \leq |t - t_n| < \frac{|t|}{2}$ $\forall$ n $\geq N_1$
So $|t_n| > \frac{|t|}{2}$
Equivalently, $\frac{|t-t_n|}{|t||t_n|} < \frac{2|t-t_n|}{|t||t|}$ **(2)**
From **(1)** and **(2)**
$|\frac{1}{n} - \frac{1}{t_n}| < \frac{2|t_n - t|}{|t|^2}$, $\forall$ n $\geq N_1$ **(3)**
Also,
$\exists\ N_2(\epsilon\ ) \in \mathbb{N}$ st
$|t_n - t| < \frac{\epsilon |t|^2}{2}$, $\forall$ n $\geq N_2$ **(4)**
Let: $N = \max\ \{N_1, N_2\}$
Then from **(3)** and **(4)**,
$|\frac{1}{t} - \frac{1}{t_n}| < \frac{2}{|t|^2}\ \frac{\epsilon |t|^2}{2} = \epsilon$ $\forall$ n $\geq$ N
Hence, result.

□

Example 4.2.2

Find
$\lim\limits_{n\to\infty} \frac{(4n^2-3)}{(5n^2-2n)}$
=
$\lim\limits_{n\to\infty} \frac{n^2(4-\frac{3}{n^2}}{n^2(5-\frac{2}{n})}$
Now,
$\lim\limits_{n\to\infty} \frac{3}{n^2} = 0 = \lim\limits_{n\to\infty} \frac{2}{n}$
By Theorem 4.2.1, **(b)**
$\lim\limits_{n\to\infty} (4 - \frac{3}{n^2}) = 4$
and
$\lim\limits_{n\to\infty} (5 - \frac{2}{n}) = 5$
By Theorem 4.2.11 (d),
$\lim\limits_{n\to\infty} \frac{(4n^2-3)}{(5n^2-2n)} = \frac{4}{5}$

Theorem 4.2.4

Assume that
$\lim_{n\to\infty} s_n = s$
and
$\lim_{n\to\infty} t_n = t$
If $s_n \leq t_n \ \forall \ n \in \mathbb{N}$
then $s \leq t$

Proof.

Assume $s > t$
Then $s - t = 0$
$\exists \ N_1(s - t), N_2(s - t) \in \mathbb{N}$ st
$|s - s_n|$
$|s_n - s| < \frac{s-t}{2}, \forall \ n \geq N_1$ **(1)**
and
$|t_n - t| < \frac{s-t}{2}, \forall \ n \geq N_2$ **(2)**
Let: $N = \max \{N_1, N_2\}$
From **(1)**
$\frac{-(s-t)}{2} < s_n - s < \frac{s-t}{2}, \forall \ n \geq N$ **(3)**
From **(2)**
$\frac{-(s-t)}{2} < y_n - s < \frac{s-t}{2}, \forall \ n \geq N$ **(4)**
Setting $n = N$:
(3) $\longrightarrow s_n > s - \frac{s-t}{2} = s - \frac{s}{2} + \frac{t}{2} = \frac{s+t}{2}$ **(5)**
(4) $\longrightarrow t_n < \frac{s-t}{2} + t = \frac{s+t}{2}$ **(6)**
(5) and **(6)** yield to the contradiction that $t_n < \frac{s+t}{2} < s_n$
Hence, result.

$\square$