

Misc. Notes:

Ex 3.4.8. e)

 \mathbb{R} is both open and closed. $\text{int } \mathbb{R} = \mathbb{R}'$ $\emptyset = \text{bd } \mathbb{R} \subset \mathbb{R}$

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 $S \subset \mathbb{R}$ $s \in S'$ if, $\forall \epsilon > 0$, $N^*(x, \epsilon) \cap S \neq \emptyset$

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HW: pages 141 - 142, numbers 6, 7, 15, 17, 19, 21

Theorem 3.4.17 - pg 118**Let:** $S \subset \mathbb{R}$

Then

- a. S is closed iff $S' \subset S$
- b. $\text{cl } S$ is a closed set
- c. S is closed iff $S = \text{cl } S$
- d. $\text{cl } S = S \cup S' = S \cup \text{bd } S$

*Proof.***a)** S is closed iff $S' \subset S$ \longrightarrow **Suppose:** S is closed.**Want to show:** $S' \subset S$ **Let:** $x \in S'$ Thus, $\forall \epsilon > 0$

$$N^*(x, \epsilon) \cap S \neq \emptyset \quad (1)$$

Want to show: $x \in S$ **Assume:** $x \notin S$

Then, from (1),

$$N(x, \epsilon) \cap S \neq \emptyset \quad (2)$$

and

$$N(x, \epsilon) \cap \neg S \neq \emptyset \quad (3)$$

From (2) and (3),

 $x \in \text{bd } S \subset S$ by definition of a closed set. This is a contradiction.Hence, $x \in S$.

This proves:

$$S' \subset S$$

←

Conversely,

Suppose: $S' \subset S$

Want to show: $\mathbb{R} \setminus S$ is open $\Rightarrow S$ is closed.

Let: $x \in \mathbb{R} \setminus S$

Want to show: $\exists \epsilon > 0$ st $N(x, \epsilon) \subset \mathbb{R} \setminus S$

Since $x \notin S$, we see that $x \notin S'$ because $S' \subset S$.

Thus, $\exists \epsilon > 0$ st $N^*(x, \epsilon) \cap S = \emptyset$

Since $x \notin S$, we have:

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

Hence, $N(x, \epsilon) \subset \mathbb{R} \setminus S$, which proves that $\mathbb{R} \setminus S$ is open, or, equivalently, that S is closed.

This completes the proof of a).

b)

$\text{cl } S$ is a closed set

Recall that $\text{cl } S = S \cup S'$.

Want to show: $\mathbb{R} \setminus \text{cl } S$ is open $\Rightarrow \text{cl } S$ is closed

Let: $x \in \text{cl } (\mathbb{R} \setminus S)$ (aka $(S \cup S')$ Complement)

We must find an $\epsilon > 0$ st $N(x, \epsilon) \subset \mathbb{R} \setminus S \cup S'$

Now $x \notin S$ and $x \notin S'$.

$\exists \epsilon > 0$ st $N^*(x, \epsilon) \cap S = \emptyset$

However, $x \notin S$, so $\exists \epsilon > 0$ st

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

Equivalently, $N(x, \epsilon) \subset \mathbb{R} \setminus S$

We claim that $N(x, \epsilon) \cap S' = \emptyset$

Since:

$$\neg[x \in S \cup S']$$

$$\neg[x \in S \text{ or } x \in S']$$

$$x \notin S \text{ and } x \notin S'$$

which is equivalent to $N(x, \epsilon) \subset \mathbb{R} \setminus S'$

Let: $y \in N(x, \epsilon)$

By Theorem 2(a), the set $N(x, \epsilon)$ is open.

So $\exists \hat{\epsilon} > 0$ st $N(y, \hat{\epsilon}) \subset N(x, \epsilon)$.

In particular, $y \notin N(x, \epsilon)$.

From (1): $N^*(y, \hat{\epsilon}) \cap S = \emptyset$.

And since $y \notin S$,

So $y \notin S'$ or, equivalently, $y \in \mathbb{R} \setminus S'$.

Thus, $y \in N(x, \epsilon) \Rightarrow y \in \mathbb{R} \setminus S'$

This proves that $N(x, \epsilon) \subset \mathbb{R} \setminus S'$ or, equivalently,

$$N(x, \epsilon) \cap S' = \emptyset \quad (2)$$

From (1) and (2), $N(x, \epsilon) \cap (S \cup S') = \emptyset$.

Hence,

$$N(x, \epsilon) \subset (S \cup S')^C = \text{cl } S^C \quad (3)$$

Thus, **(3)** and ??? prove that $\text{cl } S^C$ is open.

Hence, by Theorem 3.4.7, $\text{cl } S$ is closed.

c)

S is closed iff $S = \text{cl } S (= S \cup S')$

\longrightarrow

Suppose: S is closed.

Want to show: $S = S \cup S'$.

By definition, $S \subset S \cup S'$.

Want to show: $S \cup S' \subset S$

Let $x \in S$

Since $\text{cl } S = S \cup S'$, $x \in S \cup S'$.

If $x \in S$, then we are finished.

If $x \in S' \setminus S$

Side Note

Venn Diagram: $(S \setminus S') \cap S' = \emptyset$???

Then by a), $S' \subset S$, since S is closed.

Hence, $x \in S$, and we are finished.

\longleftarrow

Conversely,

Suppose: $S = S \cup S'$

Want to show: S is closed.

By (b), $\text{cl } S$ is closed.

Since, $S = S \cup S' = \text{cl } S$, S is also closed.

d)

$\text{cl } S = S \cup S' = S \cup \text{bd } S$

Let: $x \in S \cup S'$

If $x \in S$, then $x \in S \cup \text{bd } S$.

So, $S \cup S \subset S \cup \text{bd } S$ in this case.

If $x \in S' \setminus S$, then $\forall \epsilon > 0$, $N(x, \epsilon) \cap S \neq \emptyset$, which implies $x \in \mathbb{R} \setminus S$ and $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$

Thus, $x \in \text{bd } S \subset S \cup \text{bd } S$.

Hence, $S \cup S' \subset S \cup \text{bd } S$.

For the reverse conclusion, let $x \in S \cup \text{bd } S$.

If $x \in S$, then $x \in S \cup S'$. So, in this case, $S \cup \text{bd } S \subset S \cup S' = \text{cl } S$.

if $x \in \text{bd } S \setminus S$, then, in particular,

$\forall \epsilon > 0$,

$$N(x, \epsilon) \cap S \neq \emptyset$$

which implies that $x \in S' \subset S \cup S'$.

Hence, $S \cup \text{bd } S \subset S \cup S'$.

Hence, result.

□