Last Lecture.

Final is over 5.1 through 5.4 and 6.1

Good homework problems: The true/false questions are always great.

6.1: 6*, 8*, 17

# Theorem 6.1.10: The Chain Rule

Recall:

$f(x) = x^n$ implies $f'(x) = nx^{n-1} \ \forall \ n \in \mathbb{Z} \setminus \{0\}$

**Let:** I, J be intervals in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ , where $f(I) \subset J$

Also,

**Let:** $c \in I$

If $f'(c)$ exists and $g'(f(c))$ exists, then $(g \ o \ f)'(c) = g'(f(c)) * f'(c)$

*Proof.*

Since $g'(f(c))$ exists,

$$\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c))$$

Define the function $h : J \longrightarrow \mathbb{R}$ where

$$h(y) = \frac{g(y) - g(f(c))}{y - f(c)}$$

if $y \neq f(c)$, and

$$h(y) = g'(f(c))$$

if $y = f(c)$.

Notice that $\lim\limits_{y \to f(c)} h(y) = g'(f(c)) = h(f(c))$ **(1)**

Since $f(c) \in J$, which is an interval, $f(c)$ is a limit/accumulation point of J.

Recall: I is an interval if where ever $x_1, x_2 \in I$, $x_1 < x_2$, and $x_1 < x < x_2$ then $x \in I$

Then, by Theorem 5.2.2(a)(d),

**(a)** h is continuous at f(c)

**(d)** $\lim\limits_{y \to f(c)} h(y) = h(f(c))$

h is continuous at f(c).

Also, since f is differentiable at c, it follows by Theorem 6.1.6 that f is continuous at c.

Also, h is continuous at f(c).

By Theorem 5.2.12, h o f is continuous at c.

By Theorem 5.2.2 (a)(d), we see that

$$\lim_{x \to c}(h \ o \ f)(x) = \lim_{x \to c} h(f(x)) = h(f(c)) = g'(f(c)) \text{ from (1)}$$

Now, let $x \in I$ with $x \neq c$.

Then,

$$(g \ o \ f)(c) = \lim_{x \to c} \frac{(g \ o \ f)(x) - (g \ o \ f)(c)}{x - c} = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \tag{2}$$

Notice that $h(y)(y - f(c)) = g(y) - g(f(c))$, $\forall \ y \in J$.

Thus,

$$h(f(x))[f(x) - f(c)] = g(f(x)) - g(f(c)) \tag{3}$$

Substituting **(3)** into **(2)** yields that

$$(g \circ f)'(c) = \lim_{x \to c} \frac{h(f(x))[f(x) - f(c)]}{x - c} = \lim_{x \to c}[h(f(x))]\lim_{x \to c}\big[\frac{f(x) - f(c)}{x - c}\big] = g'(f(c)) * f'(c)$$

Hence, result.

You might be wondering, though, why can't you do this?

$$(g \circ f)'(c) = \lim_{x \to c}\big[\frac{g(f(x)) - g(f(c))}{x - c} * \frac{f(x) - f(c)}{f(x) - f(c)}\big] = \lim_{x \to c}\big[\frac{g(f(x)) - g(f(c))}{f(x) - f(c)}\big] * \lim_{x \to c}\big[\frac{f(x) - f(c)}{x - c}\big]$$

You can't let f(x) − f(c) = 0. It might be 0.

$\square$