Theorem 3.3.10

Each of the following is equivalent to the AP:

a. $\forall$ z $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st n $>$ z

b. $\forall$ x $>$ 0, y $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st nx $>$ y

c. $\forall$ x $>$ 0, $\exists$ n $\in \mathbb{N}$ st $0 < \frac{1}{n} < x$

Proof.

We shall prove:

i) AP $\Rightarrow$ a

ii) a $\Rightarrow$ b

iii) b $\Rightarrow$ c

iv) c $\Rightarrow$ AP

In other words, they all imply each other.

a. AP $\Rightarrow$ a

Suppose: a is false.
So, $\forall$ z $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$, P(z, n) (st n $\leq$ z) **???**

Side Note

$\neg[\exists\ x_1\ \forall\ x_2$ st $p(x_1, x_2)] =$
$\forall\ x_1, \exists\ x_2$ st $\neg p(x_1, x_2)$

$\exists\ z_0 \in \mathbb{R}$ st $\forall$ n $\in \mathbb{N}$, n $\leq z_0$
This indicates that the AP is false.
Thus, AP $\Rightarrow$ a.

b. a $\Rightarrow$ b

Assume: a) is true.
Let: $z = \frac{y}{x} \in \mathbb{R}$
By **(a)**, $\exists$ n $\in \mathbb{N}$ st
$n > \frac{y}{x}$
nx $>$ y
Hence, a $\Rightarrow$ b is true.

c. b $\Rightarrow$ c

Assume: b) is true.
$\forall$ x $>$ 0, if y = 1,
we see from **(b)** that $\exists$ n $\in \mathbb{N}$ st nx $>$ 1
Then,
$x > \frac{1}{n} > 0$.
Hence, b $\Rightarrow$ c.

d. c $\Rightarrow$ AP

Reminder of c: $\forall$ x where $0 < x \in \mathbb{R}$, $\exists$ $n \in \mathbb{N}$ st. $0 < \frac{1}{n} < x$
Suppose: $\mathbb{N}$ is bounded above. (In other words, that the AP is false.
Thus, $\exists$ $z_0 \in \mathbb{R}$ st $0 < n \leq z_0$, $\forall$ $n \in \mathbb{N}$
$0 < n \leq z_0$
$\frac{1}{n} \geq \frac{1}{z_0}$
This contradicts c with $x = \frac{1}{z_0}$ where $0 < \frac{1}{z_0} \in \mathbb{R}$
Hence, result.

$\square$

Theorems 3.3.13 and 3.3.15

Let: $x, y \in \mathbb{R}$ st $x < y$
Then:

a. $\exists$ $r \in \mathbb{Q}$ st $x < r < y$

b. $\exists$ $z \in \mathbb{R} \setminus \mathbb{Q}$ st $x < z < y$

a

Case:
(i): $y > 0$
$y = 0.a_1a_2...a_n$ i.e. $0.141 = \frac{141}{1000}$
(ii): $y \leq 0$
$-y \geq 0$, $-y < -x$, $0 \leq -y < -x$
By case **(i)**, $\exists$ $r \in \mathbb{Q}$ st
$-y < r < -x$
$y > -r > x$
$x < -r < y$

b

$\exists$ $z \in \mathbb{R} \setminus \mathbb{Q}$ st $x < z < y$
Apply **(a)** to $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$ to find $r \in \mathbb{Q}$ st
$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$
$x < r\sqrt{2} < y$
Let: $r\sqrt{2} = z$
$x < z < y$

Hence, result.

Section 3.4: Topology of $\mathbb{R}$

Definitions 3.4.1 and 3.4.2

Let x $\in \mathbb{R}$ and $\epsilon > 0$.

(a)
An ϵ -neighborhood of x is:
N(x, ϵ) = {y $\in \mathbb{R}$: $|y - x| < \epsilon$ }

(b)
A deleted ϵ -neighborhood of x is:
N*(x, ϵ) = {y $\in \mathbb{R}$: $0 < |y - x| < \epsilon$ }

Open Set Topology: Definition 3.4.3 (interior / boundary point)

Let: S $\subset \mathbb{R}$
A point x $\in \mathbb{R}$ is an **interior point** of S if $\exists\ \epsilon > 0$ st N(x, ϵ) $\subset$ S.

If, $\forall\ \epsilon > 0$,
N(x, ϵ) $\cap$ S $\neq \emptyset$
and
N(x, ϵ) $\cap$ $\mathbb{R}$ $\setminus$ S $\neq \emptyset$
Then x is a **boundary point** of S.
The set of all interior points is denoted by **int S**.
The set of all boundary points is denoted by **bd S**.
Nota Bene (N.B.):
int S $\subset$ S and bd S = bd ($\mathbb{R}$ $\setminus$ S)

Side Note

Let: x $\in$ int S
Then $\exists\ \epsilon > 0$ st N(x, ϵ) $\subset$ S
In particular, x $\in$ S. Thus, int S $\subset$ S.
Let: S^C = $\mathbb{R}$ $\setminus$ S, and $\mathbb{R}$ $\setminus$ S^C = S
Then s $\in$ bd S^C if $\forall\ \epsilon > 0$,
N(x, ϵ) $\cap$ S^C $\neq \emptyset$
N(x, ϵ) $\cap$ $\mathbb{R}$ $\setminus$ S^C $\neq \emptyset$
Thus, N(x, ϵ) $\cap$ ($\mathbb{R}$ $\setminus$ S) $\neq \emptyset$, and N(x, ϵ) $\cap$ S $\neq \emptyset$
So, x $\in$ bd S

Theorem 1

Let: x $\in$ S $\subset \mathbb{R}$
Then either x $\in$ int S, or x $\in$ bd S.

Proof.

Let: $x \in S \subset \mathbb{R}$

i) $\exists\ \epsilon > 0$ st $N(x, \epsilon) \subset S$. Then, by def, $x \in \text{int } S$

ii) $\forall\ \epsilon > 0$, $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

However, since $x \in S$, then $N(x, \epsilon) \cap S \neq \emptyset$.

By definition, $x \in \text{bd } S$.

Hence, result.

$\square$

Section 3.4.4 Examples

a. **Let:** $S = (0, 5)$

Here, $\text{int } S = (0, 5)$ and $\text{bd } S = \{0, 5\}$