Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18 (for 5(i) define tn to be 1 over sm, and then show that 1 over sm goes to 0)

Problem 1

Mark each statement True or False. Justify each answer.

a. If (s_n) and (t_n) are convergent sequences with $s_n \longrightarrow s$ and $t_n \longrightarrow t$, then $\lim (s_n + t_n) = s + t$ and $\lim (s_n t_n) = st$.

   True. By Theorem 4.2.1 (a) and (c).

b. If (s_n) converges to s and $s_n > 0 \; \forall \; n \in \mathbb{N}$, then $s > 0$.

   False. Counter example: $(s_n) = \frac{1}{n}$ ($s = 0$, but the moment you define n, $s_n > 0$)

c. The sequence (s_n) converges to s iff $\lim s_n = s$.

   False. The sequence converges to s iff s exists **as a real number**. If $s = +\infty$ then it can't converge.

d. $\lim s_n = +\infty$ iff $\lim (\frac{1}{s_n}) = 0$.

   False. If $\lim (\frac{1}{s_n}) = 0$ but $(s_n) = -1, -2, -3, ...$ then s_n does not diverge to $+\infty$

Problem 2

Mark each statement True or False. Justify each answer.

a. If $s_n = s$ and $\lim t_n = t$, then $\lim (s_n t_n) = st$.

   False. We don't know s_n's limit (which could be, for example, $(s_n) = n$, which diverges)

b. If $\lim s_n = +\infty$, then (s_n) is said to converge to $+\infty$.

   False. You can only converge to a finite number.

c. Given sequences (s_n) and (t_n) with $s_n \leq t_n \; \forall \; n \in \mathbb{N}$, if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

   True.

   Suppose $\exists$ sequences (s_n) and (t_n) st $s_n \leq t_n \; \forall \; n \in \mathbb{N}$ where $\lim s_n = +\infty$ and $\lim t_n$ is NOT $+\infty$.

   t_n diverges to $+\infty$ if $\forall \; M \in \mathbb{R}$, $\exists \; N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

   Let: $M \in \mathbb{R}$

   We know that

   $\exists \; N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

   Since $s_n \leq t_n \; \forall \; n \in \mathbb{N}$

   $\exists \; N \in \mathbb{N}$ st $n \geq N$ implies $s_n \geq t_n > M$

   $\exists \; N \in \mathbb{N}$ st $n \geq N$ implies $s_n > M$

   This is the definition of diverging to $+\infty$, a contradiction.

   Hence, result.

d. Suppose (s_n) is a sequence st the sequence of ratios $(\frac{s_{n+1}}{s_n})$ converges to L. If $L < 1$, then $\lim s_n = 0$.

   False.

   Let: $s_n = n(1)^{-n} \longrightarrow (\frac{s_{n+1}}{s_n}) = \frac{(n+1)(1)^{-(n+1)}}{n(1)^{-n}}$

   which converges to -1 which is less than 1 but does not have a limit of 0.

Problem 4

a. Prove Theorem 4.2.1(b):

Suppose that (s_n) and (t_n) are convergent sequences with lim $s_n = s$ and lim $t_n = t$. Then

(b) lim $(ks_n) = ks$ and lim $(k + s_n) = k + s$, for any $k \in \mathbb{R}$

We know that since s_n and t_n are convergent sequences with limits s and t, respectively.

So,

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - t| < \epsilon$

Want to show: $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|ks_n - ks| < \epsilon$

$|ks_n - ks| = |k(s_n - s)| = |k||s_n - s|$

So,

$|ks_n - ks| = |k||s_n - s| < \epsilon$

$|s_n - s| < |k|\epsilon = \epsilon_1(\epsilon)$

Since

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$

thus,

$\forall\ \epsilon_1 > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon_1$

and

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|ks_n - ks| < \epsilon$

Hence, lim $(ks_n) = ks$

Want to show: $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|k + s_n - (k + s)| < \epsilon$

We know:

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$

So,

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n + k - s - k| < \epsilon$

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n + k - (s + k)| < \epsilon$

Since this is true,

lim $(s_n + k) = k + s$

b. Prove Corollary 4.2.5:

If (t_n) converges to t and $t_n \geq 0\ \forall\ n \in \mathbb{N}$, then $t \geq 0$.

We know that

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - t| < \epsilon$

Suppose: $t < 0$

Let: $\epsilon = |t|$

$\forall\ n \in \mathbb{N}, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - t| < |t|$

Since t is negative,

$\forall\ n \in \mathbb{N}, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n + |t|| < |t|$

Since $t_n \geq 0$,

$\forall$ n $\in \mathbb{N}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n + |t| < |t|$

So,

$\forall$ n $\in \mathbb{N}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n < 0$

but $t_n \geq 0$, a contradiction.

Hence, result.

Problem 5

For s_n given by the following formulas, determine the convergence or divergence of the sequence (s_n). Find any limits that exist.

a. $s_n = \frac{3-2n}{1+n} \longrightarrow \frac{1}{2}$

b. $s_n = \frac{(-1)^n}{n+3} \longrightarrow 0$

c. $s_n = \frac{(-1)^n}{2n-1} \longrightarrow 0$

d. $s_n = \frac{2^{3n}}{3^{2n}} = \frac{8^n}{9^n} \longrightarrow 0$

e. $s_n = \frac{n^2-2}{n+1} \longrightarrow \infty$

f. $s_n = \frac{3+n-n^2}{1+2n} \longrightarrow -\infty$

g. $s_n = \frac{1-n}{2^n} \longrightarrow 0$

h. $s_n = \frac{3^n}{n^3+5} \longrightarrow \infty$

i. $s_n = \frac{n!}{2^n} \longrightarrow \infty$

j. $s_n = \frac{n!}{n^n} = \frac{1*2*3*4*5}{5*5*5*5*5}$ where n = 5 $\longrightarrow 0$

k. $s_n = \frac{n^2}{2^n} \longrightarrow 0$

l. $s_n = \frac{n^2}{n!} \longrightarrow 0$

Problem 9

Prove Theorem 4.2.12:
Suppose that (s_n) and (t_n) are sequences st $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$

a. If $\lim s_n = +\infty$ then $\lim t_n = +\infty$

Suppose $\exists$ sequences (s_n) and (t_n) st $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$ where $\lim s_n = +\infty$.

t_n diverges to $+\infty$ if $\forall$ M $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n > M$

Let: M $\in \mathbb{R}$

We know that

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n > M$

Since $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$,

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n \geq s_n > M$

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n > M$

This is the definition of diverging to $+\infty$.

Hence, t_n diverges to $+\infty$.

b. If lim $t_n = -\infty$ then lim $s_n = -\infty$

Suppose $\exists$ sequences (s_n) and (t_n) st $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$ where lim $t_n = -\infty$.

t_n diverges to $-\infty$ if $\forall$ M $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n <$ M

Let: M $\in \mathbb{R}$

We know that

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $t_n <$ M

Since $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$,

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n \leq t_n <$ M

$\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n <$ M

This is the definition of diverging to $-\infty$.

Hence, s_n diverges to $-\infty$.

Problem 10

Prove the converse part of Theorem 4.2.13:
Let (s_n) be a sequence of positive numbers. Then, lim $s_n = +\infty$ iff lim $(\frac{1}{s_n}) = 0$.
$\longrightarrow$
Assume: lim $s_n = +\infty$
Given any $\epsilon > 0$, let M = $\frac{1}{\epsilon}$. Then there exists a natural number N st n $\geq$ N implies that $s_n >$ M = $\frac{1}{\epsilon}$.
Since each s_n is positive, we have:
$|\frac{1}{s_n} - 0| < \epsilon$, whenever n $\geq$ N
Thus, lim $(\frac{1}{s_n}) = 0$.
$\longleftarrow$
Assume: lim $(\frac{1}{s_n}) = 0$
Thus,
$\forall$ $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|\frac{1}{s_n} - 0| < \epsilon$
So,
$|\frac{1}{s_n}| < \epsilon$
Since (s_n) is a sequence of positive numbers,
$\frac{1}{s_n} < \epsilon$
$\frac{1}{\epsilon} < s_n$
Let: $\frac{1}{\epsilon}$ = M
So,
$\forall$ $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies M= $\frac{1}{\epsilon} < s_n$
Thus, lim $s_n = +\infty$

Problem 17

a. Show that $\lim\limits_{n\to\infty} \frac{k^n}{n!} = 0$ $\forall$ k $\in \mathbb{R}$

Let: $\epsilon > 0$, k $\in \mathbb{R} > 0$

Want to show: $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|\frac{k^n}{n!} - 0| < \epsilon$

Recall Theorem 4.2.7 - "The Ratio Test"

Assume $\{s_n\}$ is a sequence of **positive** terms (i.e. $s_n > 0$, $\forall$ n $\in \mathbb{N}$) and $\lim\limits_{n\to\infty} \frac{s_{n+1}}{s_n}$ = L.

If L < 1, then $\lim\limits_{n\to\infty} s_n = 0$

Let: $s_n = \frac{k^n}{n!}$

Want to show: $\lim\limits_{n\to\infty} \frac{\frac{k^{n+1}}{(n+1)!}}{\frac{k^n}{n!}} < 1$

$\frac{\frac{k^{n+1}}{(n+1)!}}{\frac{k^n}{n!}} = \frac{n!k^{n+1}}{(n+1)!k^n} = \frac{k}{n+1}$

$\lim\limits_{n\to\infty} \frac{k}{n+1} = 0 = \mathrm{L}$

$\mathrm{L} < 1$

Hence, $\lim\limits_{n\to\infty} s_n = 0$ if $k \in \mathbb{R} > 0$ (why does this not apply to $k \leq 0$ again?)

b. What can be said about $\lim\limits_{n\to\infty} \frac{n!}{k^n}$?

It diverges to $+\infty$

Problem 18

Assume that (s_n) is a convergent sequence with $a \leq s_n \leq b \; \forall \; n \in \mathbb{N}$.
Prove that $a \leq \lim s_n \leq b$.
Let: $\lim s_n = s$
Want to show: $a \leq s \leq b$
Suppose: $a > s$ or $s > b$
We know that
$\forall \; \epsilon > 0, \exists \; N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$
Let: $\epsilon = a - s$
So,

$$-(a-s) < s_n - s < a - s$$
$$-a + s < s_n - s < a - s$$
$$-a + 2s < s_n < a$$

but $a \leq s_n$, a contradiction.
Let: $\epsilon = s - b$
So,

$$-(s-b) < s_n - s < s - b$$
$$-s + b < s_n - s < s - b$$
$$b < s_n < 2s - b$$

but $s_n \leq b$, a contradiction.
Hence, $a \leq \lim s_n \leq b$