Definition of a king:
For any x, either King $\longrightarrow$ x, or King $\longrightarrow$ y $\longrightarrow$ x for some path y.
Prop 1.4.30 - Every tournament has a king
—

A graph is **acyclic** if it has no cycle.
A graph is a **forest** if it is acyclic.
A graph is a **tree** if it is a connected acyclic graph.
¡pictures of trees¿
A **leaf** is a pendant vertex (i.e. a vertex with degree 1)
A **star** is ***
¡picture of a star¿
The **distance**, d(u, v), is the length of the shortest path between two vertices u and v.

Lemma 2.1.3:

Every tree G st $|V(G)| \geq 2$ has ≥ 2 leaves.
Deleting a leaf results in a smaller tree on n - 1 vertices.

Proof.

¡picture of maximal path. i.e. dot-dot-dot-dot¿
No leaf is an internal vertex of a path.
We would use an induction method to prove this:
B *** A(n) $\Rightarrow$ B(n)
A(n): T is a tree on n vertices
B(n): T has n - 1 edges
Want to show: num edges = num vertices - 1
¡picture from top right of Method of Induction page¿
Induction on n:
Step 1:
T' = T - {a leaf}
$T'$is a tree on n - 1 vertex
Step 2:
T'has n - 2 edges (induction hypothesis)
Step 3:
T = T'+ {an edge}
T has n - 2 + 1 = n - 1 edges.

□

Theorem 2.1.A (or 4?)

a. connected, no cycle. n vertices (do I have n - 1 edges?)

b. connected, n - 1 edges

c. n - 1 edges, no cycle (not sure if connected)

d. For any u, v $\in$ V, $\exists$ exactly one u, v - path. No loops.

Proof.

We're going to say these three things are equivalent.
We did A $\Rightarrow$ B in previous slides. (induction on n)
For B $\Rightarrow$ C:
Want to show: G has no cycles
Suppose G has cycles (contradiction):
¡picture from Theorem 2.1.A (or 4)
G′ = G - {e_1, e_2, ...} is acyclic
acyclic, connected, n - 1 vertices = tree
G′is connected, (using any tree that has n vertices has n - 1 edges), G′has n - 1 edges
C $\Rightarrow$ A (if you have 3 and 2, then prove you have 1):
Suppose c(G) (number of components) = k (by contradiction).
¡pictures of n_1 vertices, n_2 vertices.. n_k vertices¿ ¡has n_1 - 1 edges, n_2 - 1 edges, etc...¿

$$n - 1 = e(G) = \sum_{i=1}^{k}(n_1 - 1) = \sum_{i=1}^{k}(n_i - k) = n - k$$

The only solution is that k = 1.

□

Corollary 2.1.5

a. Every edge of a tree is a cut-edge.

b. Adding one edge to a tree forms exactly one cycle.

c. Every connected graph contains a spanning tree.

A spanning subgraph of G is a subgraph of G that contains all the vertices of G.
A spanning tree is a spanning subgraph that is a tree.

Proposition 2.1.8 (or B)

Tree T has k edges, simple graph G has min(G) $\geq$ k (minimum degree bigger than or equal to k) $\longrightarrow$ T is a subgraph of G.
T′ = T - {a leaf} has k - 1 edges.
¡picture of G¿
To prove this, we would use induction on k.
min vertex(G) $\geq$ k $\geq$ k - 1
T′has k vertices.
Base: k = 1
If T has only 2 vertices, then T has 1 edge. This is a trivial case.
¡Missing some other stuff¿

Definition 2.1.9

eccentricity (for any connected graph) ϵ (u) = max{d(u, v) : v $\in$ V(G)}
¡picture below eccentricity¿ (where 4 is the radius, 7 is the diameter)
The radius, rad(G), is the minimum *** = min of ϵ (u) where u $\in$ V
The diameter, diam(G), is the maximum *** = max of ϵ (u) where u $\in$ V
ϵ (u) = d(u, v) for some leaf v

Theorem 2.1.13 (Jordan, 1869)

The center of a tree is always one edge or one vertex.

Proof.

We do induction on n.
Let: T′ = T - {all leaves}
$\epsilon_{T'}$(u) = ϵ_T(u) - 1
If G $\neq$ a line segment with a vertex at each end, then no leaf can be a center vertex.

$\square$

Theorem 2.1.10 [Not On Test]

Not on test

Theorem 2.1.11

G is simple, Diam(G) $\geq$ 3 $\longrightarrow$ Diam($\overline{G}$) $\leq$ 3

Proof.

Claim: x cannot be adjacent to both u and v, otherwise distance will be smaller than 3. So, at least one of them is not true.
Dotted lines signify non-adjacency.
In the case of neither x nor y being adjacent to u, then in $\overline{G}$, u is adjacent to both x and y.
In the case

$\square$

hi

How many simple graphs with vertex set [n] are there?
[n] = {1, 2, 3, ... n}
In other words, how many labelled graphs on n vertices are there?
Answer: $2^{\binom{n}{2}}$
How many trees with vertex set [n] are there?
Cayley's formula: n^{n-2} (proof is very complicated, only need to understand conclusion) (the number of labelled trees on n vertices)
Labelled trees: 1:1 (a_1, a_2, ... a_{n-2}) : $a_i \in$ [n]
Suppose I have 3 vertices. How many labeled trees can I get?
Answer: 3 possible labeled trees. If you have vertices 1, 2, and 3, then you can have 3 graphs: **(1)** 12, 23, **(2)** 23, 31, **(3)** 31, 12
Let's say I have a complete graph, K_n, with labelled vertices. How many labelled spanning trees does it have? Answer: n^{n-2} (Cayley's formula)
To generalize the problem:
Contraction of edge:
¡picture with edge uv and vertices A, B, C $\longrightarrow$ picture with multiple edges between w and each vertex in C¿
Example 2.2.9:
¡picture of a square with a diagonal edge, same picture minus diagonal edge (G - e), bird (G dot e))
Proposition 2.2.8

τ(G) (the number of spanning trees of G) $= \tau$(G - e) + τ(G dot e)
Let: T be a spanning tree of G
case i: e (the diagonal edge in the picture) is not in E(T)
Any spanning tree of the original graph without using the diagonal is still a spanning tree. If you don't use the diagonal edge, it's from G - e.
case ii: e (the diagonal edge in the picture) is in E(T)
You should prove Prop 2.2.8 for practice.
Remark 2.2.10 (Basis case for computing τ(G))
Suppose G has no cycle other than multiple edges. Then:
τ(G) = {product of edge multiplicities if G is connected, 0 if disconnected}
¡Remark picture¿
1 * 2 * 3 choices
You're encouraged to try an example. For example: K_4
For a general graph, the Cayley formula doesn't work. Here's a 3rd way:
Theorem 2.2.12
Let: G be a loopless n-graph (graph with n vertices)
Q $= (q_{ij})_{n\times n}$ is defined by:
q_{ij} = {d(v_i) if i = j, $-a_{ij}$ if i $\neq$ j}
a_{ij} : the number of edges joining v_i and v_j
Q^* : obtained by deleting any row s and column t of Q.
Then τ(G) $= (-1)^{s+t}$ det Q^*
¡picture 3¿
The sum of every row and every column is equal to 0. Why?
The diagonal talks about the degree, but the off diagonal takes off each edge.
Determining the determinant of the matrix Q doesn't give you any new information, but deleting any row or column and making a submatrix, Q_*, of the matrix, and then taking the determinant will.
The absolute value of det Q_* (or just multiplying by $(-1)^{s+t}$ will give you τ(G)
(we took both a row (1st row) and a column (1st column) to make it symmetric and make it easier to take the determinant)

Conjecture 2.2.13 (still open)

K_{2m+1} decomposes into 2m + 1 copies of T with m edges.
So K_{2m+1} has $\frac{(2m+1)(2m)}{2}$ edges
Therefore, there are 2m + 1 copies of T.
Graceful labeling: a 1-1 correspondence between the vertices and a set of numbers for each vertex (packed very closely from 0 to n - 1 (where n is the number of vertices))
f: V $\longrightarrow$ distinct number $\in$ {0, 1, ... m} (a bijection from a vertex set to "this" set)
For example. If T has m edges, how many vertices do you have? Answer: m + 1
You want every vertex to have a distinct number.
¡graceful labelling picture¿
f(uv): = |f(u) - f(v)|
{f(u, v): uv $\in$ E} = {1, 2, ... m}

Conjecture 2.2.15 (Kotzig, Ringel, 1964) (open, stronger than conjecture 2.2.13)

Every tree has a graceful labeling.

Theorem 2.2.16

T is graceful $\Rightarrow$ Conjecture 2.2.12 holds
¡picture 1, picture 2 - examples of this¿
Remember, we want all edges to receive different numbers (Graceful labelling), so when we do a rotation, all of the different edges are covered exactly once
displacement(i, j): number of unit moves from i to j
K_{2m+1}
end of 10/11 lecture