Homework Due 10/5/17: (7 problems) Section 4.1 pages 169 - 170; 1, 6(b), 7(f), 9(a), 11, 12, 15
Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18

Theorem 4.1.13

Every convergent sequence is bounded.

Proof.

Assume: $S_n \longrightarrow S$ as r $\longrightarrow \infty$
Then for $\epsilon = 1$, $\exists$ N(ϵ) $\in \mathbb{N}$ st, $\forall$ n $\geq$ N,
$|S_n| - |S| \leq | \; |S_n| - |S| \; | \leq |S_n - S| < 1$ (page 121, Ex 61(a))

Side Note

$x \leq |x| \; \forall \; x \in \mathbb{R}$

So $|S_n| < 1 + |S|$, $\forall$ n $\geq$ N
$|S_n| \leq | \; |S_n - S + S| \leq |S_n - S| + |S| < 1 + |S| \; \forall$ n $\geq$ N
Then,
Let: m = max $\{|S_1|, |S_2|, \ldots |S_{N-1}|, |S|\}$
Then $|S_n| \leq$ m, $\forall$ n $\in \mathbb{N}$
Hence, $\{S_n\}$ is bounded.

$\square$

Theorem 4.1.14

If a sequence converges, then its limit is unique.

Proof.

Assume: $\{S_n\}$ is a sequence and $S_n \longrightarrow S$ as n $\longrightarrow \infty$ and $S_n \longrightarrow t$ as n $\longrightarrow \infty$
Then $\forall \; \epsilon > 0$, $\exists$ N,(ϵ) st
$|S_n - S| < \frac{\epsilon}{2}$, $\forall$ n $\geq$ N **(1)**
Also,
$\exists \; N_2(\epsilon) \in \mathbb{N}$ st
$|S_n - t| < \frac{\epsilon}{2}$, $\forall$ n $\geq N_2$ **(2)**
Set N = max $\{N_1, N_2\}$
From **(1)**, **(2)**
$|S - t| = |(S - S_n) + (S_n - t)| \leq |S - S_n| + |S_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, $\forall$ n $\geq$ N
Hence,
s = t

$\square$

4.2 Limit Theorems

Theorem 4.2.1

Suppose that $\{S_n\}$ and $\{t_n\}$ are convergent sequences with $\lim\limits_{n\to\infty} S_n = S$ and $\lim\limits_{n\to\infty} t_n = t$.

Then,

a. $\lim_{n\to\infty}$ (S_n + t_n) = s + t

b. $\lim_{n\to\infty}$ kS_n = ks and $\lim_{n\to\infty}$(k + S_n) = k + s, for any k $\in \mathbb{R}$

c. $\lim_{n\to\infty}$($S_n t_n$) = st

d. $\lim_{n\to\infty}$ $(\frac{S_n}{t_n})$ = $\frac{s}{t}$, provided that $t_n \neq 0$ $\forall$ n $\in \mathbb{N}$ and t $\neq 0$

Proof.

(a)
|t + s − (s_n + t_n)| =
|(t − t_n) + (s − s_n)| ≤ |t − t_n| + |s - s_n| **(1)**
$\forall$ $\epsilon > 0$, $\exists$ $N_1(\epsilon)$, $N_2(\epsilon)$ st

$$|t - t_n| < \frac{\epsilon}{2} \; \forall n \geq N_1 \tag{2}$$

and

$$|S - S_n| < \frac{\epsilon}{2}, \; \forall n \geq N_2 \tag{3}$$

Let: N = max {N_1, N_2}
From **(1) - (3)**,
|S + t - (s_n + t_n)| < $\frac{\epsilon}{2}$ + $\frac{\epsilon}{2}$ = ϵ $\forall$ n ≥ N
Hence, result.

(c)
|st − $s_n t_n$| = |(st − s_nt) + (s_nt − $s_n t_n$)| ≤ |st − s_nt| + |s_nt − $s_n t_n$| = |s − s_n||t| + |s_n||t − t_n|
By theorem 4.1.3, $\exists$ m > 0 st |s_n| ≤ M_1 $\forall$ n $\in \mathbb{N}$
So,
|st − $s_n t_n$| ≤ |t||s − s_n| + M|t − t_n|
Let: $\epsilon > 0$
Then $\exists$ $N_1(\epsilon)$, $N_2(\epsilon)$ $\in \mathbb{N}$ st
$\forall$ |s - s_n| < $\frac{|t|\epsilon}{|t|+M}$, $\forall$ n ≥ N_1
and
M |t - t_n| < $\frac{M\epsilon}{|t|+M}$, $\forall$ n ≥ N_2
Set N = max {N_1, N_2}
Then
|st − $s_n t_n$| < |t| $\frac{\epsilon}{|t|+M}$ + $\frac{M\epsilon}{|t|+M}$ = ϵ $(\frac{|t|+M}{|t|+M})$ = ϵ $\forall$ n ≥ N
Hence, result.

(d)
Since $\frac{s_n}{t_n}$ = $(\frac{1}{t_n})$(s_n), the proof follows from **(c)** if we can prove that $\lim_{n\to\infty}$ $\frac{1}{t_n}$ = $\frac{1}{t}$
Now:
|$\frac{1}{t}$ + $\frac{1}{t_n}$| = |$\frac{t_n - t}{t t_n}$| = $\frac{|t_n - t|}{|t||t_n|}$ **(1)**

Side Note

|t_n| > 1
|t_n| ≥ M

Recall that:
$|t| - |t_n| \leq |\ |t| - |t_n|\ | \leq |t - t_n|$ from page 121, example 6(a)
For $\epsilon = |t| > 0$, $\exists$ $N_1(\epsilon) \in \mathbb{N}$ st
$|t - t_n| < \frac{|t|}{2}$, $\forall$ n $\geq N_1$
Now $|t| - |t_n| \leq |\ |t| - |t_n|\ | \leq |t - t_n| < \frac{|t|}{2}$ $\forall$ n $\geq N_1$
So $|t_n| > \frac{|t|}{2}$
Equivalently, $\frac{|t-t_n|}{|t||t_n|} < \frac{2|t-t_n|}{|t||t|}$ **(2)**
From **(1)** and **(2)**
$|\frac{1}{n} - \frac{1}{t_n}| < \frac{2|t_n-t|}{|t|^2}$, $\forall$ n $\geq N_1$ **(3)**
Also,
$\exists$ $N_2(\epsilon) \in \mathbb{N}$ st
$|t_n - t| < \frac{\epsilon|t|^2}{2}$, $\forall$ n $\geq N_2$ **(4)**
Let: $N = \max\{N_1, N_2\}$
Then from **(3)** and **(4)**,
$|\frac{1}{t} - \frac{1}{t_n}| < \frac{2}{|t|^2}\ \frac{\epsilon|t|^2}{2} = \epsilon$ $\forall$ n $\geq$ N
Hence, result.

□

Example 4.2.2

Find
$\lim_{n\to\infty} \frac{(4n^2-3)}{(5n^2-2n)}$
=
$\lim_{n\to\infty} \frac{n^2(4-\frac{3}{n^2})}{n^2(5-\frac{2}{n})}$
Now,
$\lim_{n\to\infty} \frac{3}{n^2} = 0 = \lim_{n\to\infty} \frac{2}{n}$
By Theorem 4.2.1, **(b)**
$\lim_{n\to\infty}(4 - \frac{3}{n^2}) = 4$
and
$\lim_{n\to\infty} (5 - \frac{2}{n}) = 5$
By Theorem 4.2.11 (d),
$\lim_{n\to\infty} \frac{(4n^2-3)}{(5n^2-2n)} = \frac{4}{5}$

Theorem 4.2.4

Assume that
$\lim_{n\to\infty} s_n = s$
and
$\lim_{n\to\infty} t_n = t$
If $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$
then s $\leq$ t

Proof.

Assume s > t
Then s − t = 0
$\exists$ $N_1(s - t)$, $N_2(s - t) \in \mathbb{N}$ st

$|s - s_n|$
$|s_n - s| < \frac{s-t}{2}$, $\forall$ n $\geq N_1$ **(1)**
and
$|t_n - t| < \frac{s-t}{2}$, $\forall$ n $\geq N_2$ **(2)**
Let: $N = \max\{N_1, N_2\}$
From **(1)**
$\frac{-(s-t)}{2} < s_n - s < \frac{s-t}{2}$, $\forall$ n $\geq$ N **(3)**
From **(2)**
$\frac{-(s-t)}{2} < y_n - s < \frac{s-t}{2}$, $\forall$ n $\geq$ N **(4)**
Setting n = N:
(3) $\longrightarrow s_n > s - \frac{s-t}{2} = s - \frac{s}{2} + \frac{t}{2} = \frac{s+t}{2}$ **(5)**
(4) $\longrightarrow t_n < \frac{s-t}{2} + t = \frac{s+t}{2}$ **(6)**
(5) and **(6)** yield to the contradiction that $t_n < \frac{s+t}{2} < s_n$
Hence, result.

$\square$