

Theorem 3.2.8 - pg 118

Let $x, y \in \mathbb{R}$

- a. If $x \leq y + \epsilon \forall \epsilon > 0$, then $x \leq y$
- b. If $|x - y| \leq \epsilon \forall \epsilon > 0$, then $|x - y| = 0$ or, evidently, $x = y$

a)

If $x \leq y + \epsilon \forall \epsilon > 0$, then $x \leq y$

Proof.

Suppose that:

$$x > y$$

$$x - y > 0$$

Let

$$\epsilon = \frac{x + y}{2} > 0$$

See that

$$\begin{aligned} y + \epsilon &= y + \frac{x + y}{2} \\ &= y + \frac{x}{2} - \frac{y}{2} \\ &= \frac{x}{2} + \frac{y}{2} \\ &< \frac{x}{2} + \frac{x}{2} \\ &< x \\ y + \epsilon &< x \quad \textbf{(1)} \end{aligned}$$

Thus, by contrapositive, the result is true.

□

b)

If $|x - y| \leq \epsilon \ \forall \ \epsilon > 0$, then $|x - y| = 0$ or, evidently, $x = y$

Proof.

Suppose that:

$$|x - y| > 0$$

Let

$$\epsilon = \frac{|x - y|}{2}$$

See that

$$1 > \frac{1}{2}$$

$$|x - y| > \frac{1}{2}|x - y|$$

$$|x - y| > \epsilon$$

Thus, by contrapositive, the result is true.

□

Definition 3.2.9

If $x \in \mathbb{R}$,

$$|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem 3.2.10

Let $x, y \in \mathbb{R}$ and $a \geq 0$

Then

- a. $|x| \geq 0$
- b. $|x| \leq a \text{ iff } -a \leq x \leq a$
- c. $|xy| = |x||y|$
- d. $|x + y| \leq |x| + |y|$

a)

$$|x| \geq 0$$

Proof.

Case:

- i) $x \geq 0$:
then $|x| = x \geq 0$
- ii) $x < 0 \Rightarrow -x > 0$
then $|x| = -x \geq 0$

Hence, result □

b)

$$|x| \leq a \text{ iff } -a \leq x \leq a$$

Since it's a biconditional, first we prove $p \Rightarrow q$, then $q \Rightarrow p$.

Proof.

Notice that:

$$-a \leq -|x|$$

Case:

- i) $x \geq 0$
then $0 \leq x = |x|$
and $\therefore |x| \leq a$
Also, since $x = |x| \geq 0$, $-a \leq x$ or $-a \leq 0 - a \leq x \leq a$

$$\text{ii) } x < 0 \quad |x| = -x \leq ax \leq -a \therefore -a \leq x - a \leq x \leq a$$

Hence, result.

←

Conversely, we shall prove that $q \Rightarrow p$

Suppose: $-a \leq x \leq a$

Then:

$$\text{i) } x \geq 0, \text{ then } |x| = x \leq a$$

$$\text{ii) } x < 0, \text{ then } |x| = -x \leq a$$

Hence, result.

c)

$$|xy| = |x||y|$$

Notice that if $x = 0$ (p) or $y = 0$ (q), then $|xy| = 0 = |x||y|$.

WLOG, assume that not $[p \text{ or } q] = \text{not } p \cap \text{not } q$.

$$\text{i) } x > 0 \text{ and } y > 0 \text{ then } |x| = x, |y| = y \text{ Also, } xy > 0 \text{ So, } |xy| = xy = |x||y|$$

$$\text{ii) } x < 0, y < 0 \text{ then } |x| = -x, |y| = -y, xy > 0 \text{ So, } |xy| = xy = -|x|(-|y|) = |x||y|$$

$$\text{iii) } x > 0, y < 0 \text{ OR } y > 0, x < 0 \text{ WLOG, let } x > 0, y < 0 \quad |xy| = |x||y| \quad |yx| = |y||x| \quad |x| = x, |y| = -y, \\ xy < 0 \text{ So, } |xy| = -(xy) = -[x|(-y)|] = -[-|x||y|] = |x||y|$$

d)

$$|x + y| \leq |x| + |y|$$

Let: $Z = x + y$, and $a = |x| + |y|$

If $a \geq 0$, then $|Z| \leq a$ iff $-a \leq Z \leq a$

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

From b), since $|x| + |y| \geq 0$,

Want to show: $-(|x| + |y|) \leq x + y \leq |x| + |y|$

Notice: $-|x| \leq x \leq |x|$

$$|x| = x \text{ or } |x| = -x \text{ or } -|x| = x$$

Then

$$-|x| - |y| \leq x + y \leq |x| + |y|$$

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

By b), this is equivalent to

$$|x + y| \leq |x| + |y|$$

□