

**Misc. Notes:**

Ex 3.4.8. e)

 $\mathbb{R}$  is both open and closed. $\text{int } \mathbb{R} = \mathbb{R}'$  $\emptyset = \text{bd } \mathbb{R} \subset \mathbb{R}$ 

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 $S \subset \mathbb{R}$  $s \in S'$  if,  $\forall \epsilon > 0$ ,  $N^*(x, \epsilon) \cap S \neq \emptyset$ 

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HW: pages 141 - 142, numbers 6, 7, 15, 17, 19, 21

**Theorem 3.4.17 - pg 118****Let:**  $S \subset \mathbb{R}$ 

Then

- a.  $S$  is closed iff  $S' \subset S$
- b.  $\text{cl } S$  is a closed set
- c.  $S$  is closed iff  $S = \text{cl } S$
- d.  $\text{cl } S = S \cup S' = S \cup \text{bd } S$

*Proof.***a)** $S$  is closed iff  $S' \subset S$  $\longrightarrow$ **Suppose:**  $S$  is closed.**Want to show:**  $S' \subset S$ **Let:**  $x \in S'$ Thus,  $\forall \epsilon > 0$ 

$$N^*(x, \epsilon) \cap S \neq \emptyset \quad (1)$$

**Want to show:**  $x \in S$ **Assume:**  $x \notin S$ 

Then, from (1),

$$N(x, \epsilon) \cap S \neq \emptyset \quad (2)$$

and

$$N(x, \epsilon) \cap \neg S \neq \emptyset \quad (3)$$

From (2) and (3),

 $x \in \text{bd } S \subset S$  by definition of a closed set. This is a contradiction.Hence,  $x \in S$ .

This proves:

$$S' \subset S$$

←

Conversely,

**Suppose:**  $S' \subset S$

**Want to show:**  $\mathbb{R} \setminus S$  is open  $\Rightarrow S$  is closed.

**Let:**  $x \in \mathbb{R} \setminus S$

**Want to show:**  $\exists \epsilon > 0$  st  $N(x, \epsilon) \subset \mathbb{R} \setminus S$

Since  $x \notin S$ , we see that  $x \notin S'$  because  $S' \subset S$ .

Thus,  $\exists \epsilon > 0$  st  $N^*(x, \epsilon) \cap S = \emptyset$

Since  $x \notin S$ , we have:

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

Hence,  $N(x, \epsilon) \subset \mathbb{R} \setminus S$ , which proves that  $\mathbb{R} \setminus S$  is open, or, equivalently, that  $S$  is closed.

This completes the proof of a).

**b)**

$\text{cl } S$  is a closed set

Recall that  $\text{cl } S = S \cup S'$ .

**Want to show:**  $\mathbb{R} \setminus \text{cl } S$  is open  $\Rightarrow \text{cl } S$  is closed

**Let:**  $x \in \text{cl } (\mathbb{R} \setminus S)$  (aka  $(S \cup S')$  Complement)

We must find an  $\epsilon > 0$  st  $N(x, \epsilon) \subset \mathbb{R} \setminus S \cup S'$

Now  $x \notin S$  and  $x \notin S'$ .

$\exists \epsilon > 0$  st  $N^*(x, \epsilon) \cap S = \emptyset$

However,  $x \notin S$ , so  $\exists \epsilon > 0$  st

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

Equivalently,  $N(x, \epsilon) \subset \mathbb{R} \setminus S$

We claim that  $N(x, \epsilon) \cap S' = \emptyset$

Since:

$$\neg[x \in S \cup S']$$

$$\neg[x \in S \text{ or } x \in S']$$

$$x \notin S \text{ and } x \notin S'$$

which is equivalent to  $N(x, \epsilon) \subset \mathbb{R} \setminus S'$

**Let:**  $y \in N(x, \epsilon)$

By Theorem 2(a), the set  $N(x, \epsilon)$  is open.

So  $\exists \hat{\epsilon} > 0$  st  $N(y, \hat{\epsilon}) \subset N(x, \epsilon)$ .

In particular,  $y \notin N(x, \epsilon)$ .

From (1):  $N^*(y, \hat{\epsilon}) \cap S = \emptyset$ .

So  $y \notin S'$  or, equivalently,  $y \in \mathbb{R} \setminus S'$ .

Thus,  $y \in N(x, \epsilon) \Rightarrow y \in \mathbb{R} \setminus S'$

This proves that  $N(x, \epsilon) \subset \mathbb{R} \setminus S'$  or, equivalently,

$$N(x, \epsilon) \cap S' = \emptyset \quad (2)$$

From (1) and (2),  $N(x, \epsilon) \cap (S \cup S') = \emptyset$ .

Hence,

$$N(x, \epsilon) \subset (S \cup S')^C = \text{cl } S^C \quad (3)$$

Thus, **(3)** and ??? prove that  $\text{cl } S^C$  is open.

Hence, by Theorem 3.4.7,  $\text{cl } S$  is closed.

**c)**

$S$  is closed iff  $S = \text{cl } S (= S \cup S')$

$\longrightarrow$

**Suppose:**  $S$  is closed.

**Want to show:**  $S = S \cup S'$ .

By definition,  $S \subset S \cup S'$ .

**Want to show:**  $S \cup S' \subset S$

Let  $x \in S$

Since  $\text{cl } S = S \cup S'$ ,  $x \in S \cup S'$ .

If  $x \in S$ , then we are finished.

If  $x \in S' \setminus S$

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Side Note

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Venn Diagram:  $(S \setminus S') \cap S' = \emptyset$  ???

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Then by a),  $S' \subset S$ , since  $S$  is closed.

Hence,  $x \in S$ , and we are finished.

$\longleftarrow$

Conversely,

**Suppose:**  $S = S \cup S'$

**Want to show:**  $S$  is closed.

By (b),  $\text{cl } S$  is closed.

Since,  $S = S \cup S' = \text{cl } S$ ,  $S$  is also closed.

**d)**

$\text{cl } S = S \cup S' = S \cup \text{bd } S$

**Let:**  $x \in S \cup S'$

If  $x \in S$ , then  $x \in S \cup \text{bd } S$ .

So,  $S \cup S \subset S \cup \text{bd } S$  in this case.

If  $x \in S' \setminus S$ , then  $\forall \epsilon > 0$ ,  $N(x, \epsilon) \cap S \neq \emptyset$ , which implies  $x \in \mathbb{R} \setminus S$  and  $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$

Thus,  $x \in \text{bd } S \subset S \cup \text{bd } S$ .

Hence,  $S \cup S' \subset S \cup \text{bd } S$ .

For the reverse conclusion, let  $x \in S \cup \text{bd } S$ .

If  $x \in S$ , then  $x \in S \cup S'$ . So, in this case,  $S \cup \text{bd } S \subset S \cup S' = \text{cl } S$ .

if  $x \in \text{bd } S \setminus S$ , then, in particular,

$\forall \epsilon > 0$ ,

$$N(x, \epsilon) \cap S \neq \emptyset$$

which implies that  $x \in S' \subset S \cup S'$ .

Hence,  $S \cup \text{bd } S \subset S \cup S'$ .

Hence, result.

□