Theorem 1 (infinum definition)

Let: $\emptyset \neq$ S $\subset \mathbb{R}$, S is bounded below.
Then S possesses a greatest lower bound denoted by **inf S** (the **infinum** of S), where inf S $\in \mathbb{R}$, satisfying:

i) inf S $\leq$ s $\forall$ s $\in$ S

ii) $\forall$ $\epsilon > 0$, $\exists$ s_1 st inf S + $\epsilon > s_1$

Proof.

Let: S be bounded below
Then

$$\exists m \in \mathbb{R} \text{ st } m \leq s \ \forall s \in S \tag{1}$$

Define the set -S to be $\{-s :$ s $\in$ S$\}$
So, $\emptyset \neq$ -S $\subset \mathbb{R}$
From **(1)**, -S $\leq -$m $\forall$ s $\in$ S.
Thus, $-$m is an upper bound for -S.
By the Axiom of Completeness of $\mathbb{R}$ (AoC), sup(-S) $\in \mathbb{R}$ exists.
By definition,

$$-s \leq sup(-S), \ \forall s \ \in S \tag{2}$$

and $\forall$ $\epsilon > 0$, $\exists$ -s_1 $\in$ S st

$$sup(-S) - \epsilon < -s_1 \text{ where } s_1 \in S \tag{3}$$

From **(2)**,

$$-sup(-S) \leq S \ \forall s \in S \tag{4}$$

Want to show: $-$sup(-S) = inf S
From **(3)**,

$$-sup(-S) + \epsilon > s_1 \text{ where } s_1 \in S \tag{5}$$

We see that from **(4)** and **(5)**,
inf S = $-$sup(-S).
Hence, result. ∎

Theorem 3.3.7

Given nonempty subsets of A, B (A, B $\subset \mathbb{R}$),
Let: C = {x + y: x $\in$ A, y $\in$ B}
If A and B have suprema, then C has a supremum: sup C = sup A + sup B

Proof.

Let: c $\in$ C
Then c = x + y for some x $\in$ A, y $\in$ B.
It follows that:
x $\leq$ sup A, y $\leq$ sup B
x + y $\leq$ sup A + sup B
c $\leq$ sup A + sup B

$$c \leq sup\ A + sup\ B\ \forall\ c \in C \tag{1}$$

By the AoC, sup C $\in \mathbb{R}$ exists.
For $\epsilon > 0$, $\exists$ $x_0 \in$ A, $y_0 \in$ B st

$$sup\ A - \frac{\epsilon}{2} < x_0 \tag{2}$$

$$sup\ B - \frac{\epsilon}{2} < y_0 \tag{3}$$

From **(2)** and **(3)**,
sup A - $\frac{\epsilon}{2}$ + sup B - $\frac{\epsilon}{2}$ < $x_0 + y_0 = c_0 \in$ C
So,

$$sup\ A + sup\ B - \epsilon < c_0 \tag{4}$$

From **(1)** and **(4)**, sup C = sup A + sup B
Hence, result.

□

Theorem 3.3.8

Suppose $\emptyset \neq \mathrm{D} \subset \mathbb{R}$ and
$\mathrm{f} : \mathrm{D} \longrightarrow \mathbb{R}$
$\mathrm{g} : \mathrm{D} \longrightarrow \mathbb{R}$
$\mathrm{f(D)} = \{\mathrm{f(x)} : \mathrm{x} \in \mathrm{D}\}$
If $\forall$ x, y $\in$ D, f(x) $\leq$ g(y), then
f(D) is bounded above and g(D) is bounded below.
Furthermore, sup(f(D)) $\leq$ inf(g(D))

Proof.

Let: $\mathrm{y_0} \in \mathrm{D}$
Then f(x) $\leq$ g($\mathrm{y_0}$) $\forall$ x $\in$ D
So, f(D) is bounded above by g($\mathrm{y_0}$).
By the AoC, sup(f(D)) exists and sup(f(D)) $\leq$ g($\mathrm{y_0}$)

Since $\mathrm{y_0} \in \mathrm{D}$ was arbitrary, we see that
sup(f(D)) $\leq$ g(y) $\forall$ y $\in$ D
Thus, sup(f(D)) is a lower bound for g(D)
$\mathrm{g(D)} = \{\mathrm{g(y)} : \mathrm{y} \in \mathrm{D}\}$
Hence, inf(g(D)) $\in \mathbb{R}$ exists and
sup(f(D)) $\leq$ inf(g(D))
Hence, result.

□

Theorem 3.3.9: Archimedian Property / Principle of $\mathbb{R}$ (AP)

The set $\mathbb{N} = \{1, 2, 3...\}$ is unbounded above in $\mathbb{R}$

Proof.

Suppose: $\mathbb{N}$ is bounded above.
By the AoC, sup $\mathbb{N} \in \mathbb{R}$ exists.
So,

i) n $\leq$ sup $\mathbb{N}$ $\forall$ n $\in \mathbb{N}$ **(1)**

ii) $\forall\ \epsilon > 0$, $\exists$ n $\in \mathbb{N}$ st sup $\mathbb{N} - \epsilon < \mathrm{n_0}$ **(2)**

Using **(2)** with $\epsilon = 1$, $\exists\ \mathrm{n_0}(1) \in \mathbb{N}$ st sup $\mathbb{N} - \epsilon < \mathrm{n_0}$
Then, sup $\mathbb{N} < 1 + \mathrm{n_0}$ **(3)**
See that **(3)** contradicts **(1)** with $\mathrm{n} = 1 + \mathrm{n_0} \in \mathbb{N}$
By contradiction, $\mathbb{N}$ is unbounded.

□