Final is **not** cumulative! Covers this material and onward.

Chapter 5 - Limits and Continuity

5.1.1: Definition (Limits of Functions)

Let: D $\subset \mathbb{R}$, f : D $\longrightarrow \mathbb{R}$, and c $\in$ D$'$ (i.e. c is an accumulation point)
We say that L $\in \mathbb{R}$ is a **limit** of f at c if, $\forall$ $\epsilon > 0$, $\exists$ $\delta(\epsilon) > 0$ st when x $\in$ D and 0 < |x - c| < δ, then

$$|f(x) - L| < \epsilon$$

(i.e. the limit as x goes to c of f(x) = L)

$$x \pm c$$

$$-\delta < x - c < \delta$$

$$c - \delta < x < c + \delta$$

Recall the definition of a limit:

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

where x is fixed.

Theorem 5.1.2

Let: f : D $\longrightarrow \mathbb{R}$, c $\in$ D$'$
Then:
The limit x $\longrightarrow$ c of f(x) = L exists iff for each neighborhood V of L, $\exists$ a deleted neighborhood U of c st f(U $\cap$ D) $\subset$ V.

Proof.
$\longrightarrow$
Suppose lim x $\longrightarrow$ c of f(x) = L.
Then,
for each neighborhood V of L (i.e. for each $\epsilon > 0$, V = N(L, ϵ)), $\exists$ a deleted neighborhood U of c
(i.e. $\exists$ $\delta(\epsilon) > 0$ st N*(c, δ) = U)
st f(U $\cap$ D) $\subset$ V
$\longleftarrow$
The converse is similar.
Remember: definitions are iff

$\square$

Example 5.1.3

Let: k $\in \mathbb{R}$
Define f : $\mathbb{R}$ $\longrightarrow \mathbb{R}$ by f(x) = k, $\forall$ x $\in \mathbb{R}$
Let c $\in \mathbb{R}$
Show that $\lim\limits_{x \to c}$ f(x) = k
Solution:
For each $\epsilon > 0$,

$$|f(x) - k| = |k - k| = 0 < \epsilon$$

whenever $0 < |x - c| < \epsilon$

Example 5.1.4

Confirm that $\lim_{x \to c} f(x) = c$ for the function $f(x) = x$, where $c \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$
Solution:
For each $\epsilon > 0$,

$$|f(x) - c| = |x - c| < \epsilon$$

whenever $0 < |x - c| < \delta = \epsilon$

Theorem 5.1.8

Let: $f : D \longrightarrow \mathbb{R}$, $c \in D'$
Then,
$\lim_{x \to c} f(x) = L$ iff for **every** sequence $\{s_n\}$ in D st $s_n \neq c$, $\forall\, n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = c$,
it follows that the sequence $\{f(s_n)\}$ converges to L.
(i.e. the values of s_n eventually get within a δ neighborhood of c)

Proof.
$\longrightarrow$
Assume: $\lim_{x \to c} f(x) = L$
Let: $\{s_n\}$ be a sequence in D st $s_n \neq c$ $\forall\, n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = c$
Want to show: $\lim_{n \to \infty} f(s_n) = L$
Now, $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ st

$$|f(x) - L| < \epsilon \tag{1}$$

whenever $0 < |x - c| < \delta$ **and** $\mathbf{x} \in \mathbf{D}$ (we need this part so that $|f(x) - L|$ makes sense).
I'd like to know that $|f(s_n) - L|$ gets close to 0, so:
Since $\lim_{n \to \infty} s_n = c$, $\exists\, N \in \mathbb{N}$ st

$$0 < |s_n - c| < \delta \tag{2}$$

for $n \geq N$
From **(1)** and **(2)**,

$$|f(s_n) - L| < \epsilon$$

for $n \geq N$
(if we think of $f(s_n)$ as our t_n, where $t_n \longrightarrow L$ as $n \longrightarrow \infty$)
By definition,
$\lim_{n \to \infty} f(s_n) = L$

$\longleftarrow$
Conversely, using the contrapositive,
Assume: $\lim_{x \to c}$ f(x) does not exist.

Side Note

Negating that:
$\exists$ L $\in \mathbb{R}$ st
$\lim_{x \to c}$ f(x) = L
$\forall\ \epsilon > 0$, $\exists\ \delta > 0$ st
$\forall$ x st $0 < |x - c| < \delta$,
$|f(x) - L| < \epsilon$

Thus,
for each L $\in \mathbb{R}$, $\exists\ \epsilon_0 > 0$ st
$\forall\ \delta > 0$, $\exists$ x st $0 < |x - c| < \delta$ st

$$|f(x) - L| \geq \epsilon_0$$

Side Note

First we proved p $\Rightarrow$ q.
Now we're going to prove q $\Rightarrow$ p by proving:
not p $\Rightarrow$ not q

Want to show: $\exists$ a sequence $\{s_n\}$ in D st $s_n \neq c$, $\forall$ n $\in \mathbb{N}$ and $\lim_{n \to \infty} s_n = c$
but, $\{f(s_n)\}$ to fail to converge to L.
Let: $\delta_n = \frac{1}{n}$
Now, for each n $\in \mathbb{N}$, $\exists\ s_n \in$ D st

$$0 < |s_n - c| < \frac{1}{n} \text{ and } |f(s_n) - L| \geq \epsilon_0 \tag{3}$$

Notice that $s_n \neq c$, $\forall$ n $\in \mathbb{N}$ and $\lim_{n \to \infty} s_n = c$.

Side Note

Is $\lim_{n \to \infty} f(s_n) = L$?
For $\epsilon = \frac{\epsilon_0}{2} > 0$,
$\exists$ N $\in \mathbb{N}$ st

$$|f(s_n) - L| < \frac{\epsilon_0}{2}$$

for n $\geq$ N
So, no. $\lim_{n \to \infty} f(s_n) \neq L$

From **(3)**, $\lim_{n \to \infty} f(s_n) \neq L$

(page 166, Theorem 4.1.8 says
$|s_n - s| \leq k|a_n|$ for n $\geq$ N
and
if $\lim_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} s_n = s$)

$\square$