Assignment Set: 1, 2, 3, 4, 6, 8 from pages 148 - 149

1)

Mark each statement as true or false. Justify each answer.

a. A set S is compact iff every open cover of S contains a finite subcover.

   True.

   The original definition is:

   A set S $\subset \mathbb{R}$ is said to be compact if every open cover has a finite subcover.

   $\longrightarrow$ (every open cover $\Rightarrow$ compact)

   Let: S $\subset \mathbb{R}$ be a set st every open cover has a finite subcover.

   Then, by definition, S is compact.

   $\longleftarrow$ (every open cover $\Leftarrow$ compact)

   Let: S $\subset \mathbb{R}$ be compact

   Then, by definition, S is a set st every open cover has a finite subcover.

   Hence, result.

b. Every finite set is compact.

   True.

   Let: S be a finite set, $0 < |S| < \infty$, S = $\{s_1, s_2, ... s_n\}$

   Let: l = min $\{s_1, s_2, ... s_n\}$, u = max $\{s_1, s_2, ... s_n\}$

   Since S has lower bound l and upper bound u, S is bounded.

   Since l, u $\in$ bd S by definition of maximum and minimum, S is closed.

   Hence, by Heine-Borel, S is compact.

c. No infinite set is compact.

   Not true.

   If S = [0, 1], there are infinite values between 0 and 1.

   However, [0, 1] is compact.

d. If a set is compact, then it has a maximum and a minimum.

   True.

   Let: S $\neq \emptyset$ be a compact set

   Since S is compact, S is also closed and bounded.

   Since S $\neq \emptyset$, there is at least one element.

   Since S is closed, bd S $\subset$ S.

   Since there is at least one element, and all boundary points of S are in S, there is a maximum and minimum element.

e. If a set has a maximum and a minimum, then it is compact.

   Let: S $\neq \emptyset$, l = min S, u = max S

   Since l, u $\in$ bd S by definition of minimum and maximum, S is bounded.

   Notice that l, u $\notin$ int S.

   Since $\exists$ s $\in$ S st s $\notin$ int S, S is not open.

   Therefore, S is closed.

   By the Heine-Borel theorem, since S is closed and bounded, S is compact.

2)

Mark each statement as true or false. Justify each answer.

a. Some unbounded sets are compact.

   False.

   By Heine-Borel, S is compact only if closed and bounded.

b. If S $\subset \mathbb{R}$ is compact, then $\exists$ x $\in \mathbb{R}$ st s $\in$ S$'$

   False.

   The empty set is compact and contains no elements.

c. If S is compact and s $\in$ S$'$, then s $\in$ S.

   True.

   By Heine-Borel, if S is compact, then S is closed and bounded.

   If S is closed, then S$'$ $\subset$ S by Theorem 3.4.17

   So, if s $\in$ S$'$, then s $\in$ S

d. If S is unbounded, then S has at least one accumulation point.

   False.

   $\mathbb{N}$ is a counter example.

e. **Let:** F = $\{$A$_i$, i $\in \mathbb{N}$ $\}$. Suppose that the intersection of any finite subfamily of F is nonempty. If $\cap$ F = $\emptyset$, then, for some k $\in \mathbb{N}$, A$_k$ is not compact.

   True.

   Suppose: $\forall$ A$_i$ $\in$ F, A$_i$ is compact

   Then, by Heine-Borel, A$_i$ is closed and bounded.

   If any finite subfamily of F is nonempty, but $\cap$ F = $\emptyset$, then $\lim_{i \to \infty}$ F = $\emptyset$, so F gets smaller as i grows.

   However, as F gets smaller, since A$_i$ is closed, bd A$_i$ $\subset$ A$_i$, which means eventually F should converge to a set with a single element in it.

   But this is not true, so $\exists$ A$_i$ $\in$ F st A$_i$ is not compact.

3)

Show that each subset of $\mathbb{R}$ is not compact by describing an open cover for it that has no finite subcover.

a. '[1, 3)

   $\{n \in \mathbb{N} : \bigcup_{i=1}^{n} (0, 2 + \sum_{k=1}^{i} \frac{1}{2^k})\}$

b. '[1,2)

   $\{n \in \mathbb{N} : \bigcup_{i=1}^{n} ((2 - \sum_{i=1}^{n} \frac{1}{2^i}), (3 + \sum_{i=1}^{n} \frac{1}{2^i}))\}$

c. $\mathbb{N}$

   $\{\bigcup_{n=1}^{\infty} (0, n)\}$

d. $\{\frac{1}{n} : n \in \mathbb{N} \}$

   $\{ n \in \mathbb{N} : \bigcup_{i=1}^{n} (0, \sum_{k=1}^{i} \frac{1}{2^k})\}$

e. $\{x \in \mathbb{Q} : 0 \leq x \leq 2\}$

   but wait, isn't this a closed and bounded set? It's obviously bounded, so why is that not closed?

   In any case, here's a cover:

   $\bigcup_{i=0}^{\infty} (-1, \sum_{k=0}^{i} \binom{2k}{k} \frac{1}{8^k}) \cup (\sqrt{2} - (\sqrt{2} - \sum_{k=0}^{i} \binom{2k}{k} \frac{1}{8^k}), 3)$

4)

Prove that the intersection of any collection of compact sets is compact.
Let: S be $\bigcap_{\alpha \in I} G_\alpha$ where I is an index set and all G_α's are compact
Let: S be nonempty (since if it's empty, then it's compact anyhow)
By Heine-Borel, G_α is both closed and bounded $\forall\ \alpha$.
Let: U be the set of all least upper bounds from each G_α, and L be the set of all greatest lower bounds from each G_α's
Since G_α is closed $\forall\ G_\alpha$, each element in U is a max, and each element in L is a min.
Since S is an intersection, its minimum will be max L, and its maximum will be min U, which we know exists because $S \neq \emptyset$
Since S has a min and a max, the min and max are members of its boundary set.
Since the min and max are members of the boundary set, they can't be interior points.
Since $\exists\ s \in S$ that isn't an interior point, S is not open and therefore closed.
Since S has a minimum and a maximum, it is also bounded.
Since S is closed and bounded, it is compact by Heine-Borel.

6)

Show that compactedness is necessary in Corollary 3.5.8. That is, find a family of intervals $\{A_n : n \in \mathbb{N} \}$ with $A_{n+1} \subset A_n\ \forall\ n$, $\bigcup_{n=1}^{\infty} A_n = \emptyset$, and such that:

a. The sets A_n are all closed.

   $\{n \in \mathbb{N} : \emptyset \}$

b. The sets A_n are all bounded.

   $\{n \in \mathbb{N} : (0, 0)\}$

8)

If S $\subset \mathbb{R}$ is compact and T $\subset$ S is closed, then T is compact.

a. Prove this using the definition of compactness.

   If S is compact, then $\forall \bigcup_{\alpha \in I}$ G_α that cover S (G_α is open), $\exists \bigcup_{i=1}^{n} G_{\alpha_i}$ - a finite subcover of S.

   Let: CV be an arbitrary open cover for S, SCV be CV's finite subcover for S

   By definition, S $\subset$ CV and S $\subset$ SCV

   Since T $\subset$ S, T $\subset$ CV and T $\subset$ SCV

   So, $\forall$ CV of S and $\forall$ SCV of S, CV and SCV are covers and finite subcovers for T.

   So, T is compact.

b. Prove using the Heine-Borel theorem: If S $\subset \mathbb{R}$ is compact and T $\subset$ S is closed, then T is compact.

   Since S is compact, by Heine-Borel, S is closed and bounded.

   Let: T $\neq \emptyset$ (since if T $= \emptyset$ then it is compact anyhow)

   Since T is closed, bd T $\subset$ T.

   Since T $\neq \emptyset$ and bd T $\subset$ T, T contains a maximum and a minimum.

   Therefore, T is bounded.

   Since T is both closed and bounded, T is compact.