HW 8: pages 193, #1, 2, 3, 5, 9, 10, 17
For 2(c), see Theorem 1 and Example 9 from Lecture 15
Make sure when you do these problems, justify the answer by either writing down the theorem name or providing a counter example.

Exercise 1

Mark each statement True or False. Justify each answer.

a. A sequence (s_n) converges to s iff every subsequence of (s_n) converges to s.

   True. By Theorem 4.4.4.

b. Every bounded sequence is convergent.

   False.

   Counter example: (s_n) = $(-1)^n$

c. Let (s_n) be a bounded sequence. If (s_n) oscillates, then the set S of subsequential limits of (s_n) contains at least two points.

   True. If S oscillates, then lim inf S < lim sup S. This implies that these are two different points.

d. Let (s_n) be a bounded sequence and let m = lim sup s_n.

   Then, $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st N $\geq$ n implies $s_n > m - \epsilon$

   True.

   Proof.
   Let: $\epsilon > 0$

   Since s_n is bounded, let S be the set containing the range of s_n.

   By definition, $\exists$ some s_{n_k} st lim s_{n_k} = m where k $\in \mathbb{N}$

   Since lim s_{n_k} = m,

   $\exists$ N $\in \mathbb{N}$ st N $\geq n_k$ implies $|s_{n_k} - m| < \epsilon$

   $|s_{n_k} - m| < \epsilon$

   $-\epsilon < s_{n_k} - m < \epsilon$

   $m - \epsilon < s_{n_k} < m + \epsilon$ **(1)**

   So, by **(1)**,

   $\exists$ some N $\in \mathbb{N}$ st n $\geq$ N implies $s_n > m - \epsilon$

   □

e. If (s_n) is unbounded above, then (s_n) contains a subsequence that has ∞ as a limit.

   True. By Theorem 4.4.8.

Exercise 2

Mark each statement True or False. Justify each answer.

a. Every sequence has a convergent subsequence.

   False. Let $s_n = n$

b. The set of subsequential limits of a bounded sequence is always nonempty.

   True. By Theorem 4.4.8

c. (s_n) converges to s iff lim inf s_n = lim sup s_n = s

   True. By Definition 4.4.9 and exercise 9.

d. Let (s_n) be a bounded sequence and let m = lim sup s_n. Then, $\forall\ \epsilon > 0$, there are infinitely many terms in the sequence greater than $m - \epsilon$.

   True. By Theorem 4.4.7, s_n has a convergent subsequence.

   Let t_n be a subsequence of s_n st $\lim\limits_{n\to\infty} t_n = m$

   By definition,

   $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - m| < \epsilon$

   so,

   $-\epsilon < t_n - m < \epsilon$

   $m - \epsilon < t_n$

   Pick ϵ_2 to be $\frac{\epsilon}{2}$

   Then,

   $\exists\ N(\epsilon_2)$ st $m - \epsilon < t_{N(\epsilon_2)}$

   Inductively, we can let $\epsilon_3 = \frac{\epsilon_2}{2}$, and so on.

   Hence, since there are infinitely many terms in t_n greater than $m - \epsilon$, the same is true for s_n.

e. If (s_n) is unbounded above, then lim inf s_n = lim sup $s_n = \infty$

   True.

   Suppose: s_n has a subsequence t_n such that $\lim\limits_{n\to\infty} t_n = t$ where $t \neq \infty$ (but could be negative infinity)

   So,

   $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - t| < \epsilon$

   Notice also, that since s_n is unbounded above,

   $\forall\ m \in \mathbb{R}\ , \exists\ N_m \in \mathbb{N}$ st $s_{N_m} > m$

   That means that $\exists$ some N for t_n st $t_N > m$

   If we let m = t, then

   $\exists$ some N_1 for t_n st $t_{N_1} > t = m$

   If we let m = t + 1, then

   $\exists$ some N_2 for t_n st $t_{N_2} > m = t + 1$

   Inductively, t_n has an infinite amount of values above t, and is increasing: a contradiction.

   Thus, t_n is unbounded above.

Exercise 3

For each sequence, find the set S of subsequential limits, the limit inferior, and the limit inferior.

a. $s_n = 1 + (-1)^n$

b. $t_n = (0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7})$

c. $u_n = n^2(-1 + (-1)^n)$

d. $v_n = n \sin \frac{n\pi}{2}$

Exercise 5

Use exercise 4.3.14 to find the limit of each sequence:

a. $s_n = (1 + \frac{1}{2n})^{2n}$

b. $s_n = (1 + \frac{1}{n})^{2n}$

c. $s_n = (1 + \frac{1}{n})^{n-1}$

d. $s_n = (\frac{n}{n+1})^n$

e. $s_n = (1 + \frac{1}{2n})^n$

f. $s_n = (\frac{n+2}{n+1})^{n+3}$

Exercise 9

Let (s_n) be a bounded sequence.
Assume: $\liminf s_n = \limsup s_n = s$
Prove that (s_n) is convergent and that $\lim s_n = s$

Exercise 10

Assume: $x > 1$
Prove that $\lim x^{\frac{1}{n}} = 1$

Exercise 17

Prove that if $\limsup s_n = \infty$ and $k > 0$, then $\limsup (ks_n) = \infty$