

Assignment Set: 6, 7, 15, 17, 19, 21 from pages 141 - 142

6)

Find the closure of each set:

a. $\{ \frac{1}{n} : n \in \mathbb{N} \}$

Answer: \emptyset

b. \mathbb{N}

Answer: \mathbb{N}

c. \mathbb{Q}

Answer: \mathbb{R}

d. $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

Answer: \emptyset

e. $\{ x : |x - 5| \leq \frac{1}{2} \}$

$[4.5, 5.5]$

Answer: $[4.5, 5.5]$

f. $\{ x : x^2 > 0 \}$

$(0, \infty)$

Answer: $[0, \infty)$

7)

Let $S, T \subset \mathbb{R}$. Find a counterexample of each of the following:

a. If P is the set of all isolated points of S , then P is a closed set.

Answer: Let $S = \mathbb{N}$

b. Every open set contains at least two points.

Answer: \emptyset

c. If S is closed, then $\text{cl}(\text{int } S) = S$.

Answer: Let $S = \mathbb{Q}$

d. If S is open, then $\text{int}(\text{cl } S) = S$.

Answer: Let $S = (-1, 0) \cup (0, 1)$

e. $\text{bd}(\text{cl } S) = \text{bd } S$

Answer: Let $S = (-1, 0) \cup (0, 1)$

f. $\text{bd}(\text{bd } S) = \text{bd } S$

Answer: Let $S = \mathbb{Q}$. Then $\text{bd } S$ is \mathbb{R} , and $\text{bd}(\text{bd } S) = \emptyset \neq \mathbb{R}$.

g. $\text{bd}(S \cup T) = (\text{bd } S) \cup (\text{bd } T)$

Answer: Let $S = \mathbb{R}$, $T = (0, 1)$. $\text{bd}(S \cup T) = \emptyset$, but $\text{bd } S \cup \text{bd } T = \emptyset \cup \{0, 1\}$

h. $\text{bd}(S \cap T) = (\text{bd } S) \cap (\text{bd } T)$

Answer: Let $S = (0, 1)$, $T = (1, 2)$. $\text{bd}(S \cap T) = \emptyset$, but $\text{bd } S \cap \text{bd } T = 1$.

15)

Prove: If x is an accumulation point of the set S , then every neighborhood of x contains infinitely many points of S .

Proof.

Suppose that \exists a deleted neighborhood of x , called N , that contains n points x_1, x_2, \dots, x_n of S where n is a finite amount and $x_1 \leq x_2 \leq \dots \leq x_n$.

x is an accumulation point on S if $\forall \epsilon > 0, N^*(x, \epsilon) \cap S \neq \emptyset$.

N is a deleted neighborhood of S if $\forall x \in \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}, x \in N$.

Let $\hat{\epsilon} = \epsilon + \epsilon$, and $x_0 = x_1 - \hat{\epsilon}$.

By definition, $x_0 \in N$, since N is a neighborhood $\forall \epsilon > 0$.

However, N only has n elements. A contradiction.

So, N can't be a deleted neighborhood since it has a finite number of elements, which means x can't be an accumulation point.

□

17)

Prove: S' is a closed set.

Proof.

By definition, $\forall s \in S', \forall \epsilon > 0, N^*(s, \epsilon) \cap S \neq \emptyset$

Want to show: $\mathbb{R} \setminus S'$ is open.

□

19)

Suppose S is a nonempty bounded set and let $m = \sup S$. Prove or give a counter example: m is a boundary point of S .

Proof.

By definition,

$s \leq m, \forall s \in S$, and,

$\forall \epsilon > 0, \exists s' \in S$ st $m - \epsilon < s'$

By the second part of the definition of the supremum of S , $N(m, \epsilon) \cap S \neq \emptyset$.

Notice also that, by the first part of the definition of the supremum of S , $(m + \epsilon) \notin S$. This means that $N(m, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$.

By definition, m is a boundary point.

□

21)

Let A be a nonempty open subset of \mathbb{R} and let $Q \subset \mathbb{Q}$. Prove: $A \cap Q \neq \emptyset$.

Proof.

Notice that $Q \subset \mathbb{Q} \subset \mathbb{R}$.

Since A is nonempty, \exists at least one element $a \in \mathbb{R}$.

Since A is nonempty and open, $a + \epsilon \in A$.

If $a \in Q$, then result.

If $a + \epsilon \in Q$, then result.

If $a \notin Q$ and $(a + \epsilon) \notin Q$, then:

Let $x = a$, $y = a + \epsilon$, $z = y - x$.

By Archimedes' axiom, $\exists n$ st $n > \frac{1}{z}$

$nz > 1$

$ny - nx > 1$

Since the difference between ny and nx is bigger than 1,

$\exists m \in \mathbb{Z}$ st $nx < m < ny$.

See that $x < \frac{m}{n} < y$, $\frac{m}{n}$ is a rational number, and $\frac{m}{n} \in A$.

Hence, result. □

Let: $S \subset \mathbb{R}$

Then

- a. S is closed iff $S' \subset S$
- b. $\text{cl } S$ is a closed set
- c. S is closed iff $S = \text{cl } S$
- d. $\text{cl } S = S \cup S' = S \cup \text{bd } S$

Proof.

a)

S is closed iff $S' \subset S$

\longrightarrow

Suppose: S is closed.

Want to show: $S' \subset S$

Let: $x \in S'$

Thus, $\forall \epsilon > 0$

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

Want to show: $x \in S$

Assume: $x \notin S$

Then, from (1),

$$N(x, \epsilon) \cap S \neq \emptyset \quad (2)$$

and

$$N(x, \epsilon) \cap \neg S \neq \emptyset \quad (3)$$

From (2) and (3),

$x \in \text{bd } S \subset S$ by definition of a closed set. This is a contradiction.

Hence, $x \in S$.

This proves:

$$S' \subset S$$

\longleftarrow

Conversely,

Suppose: $S' \subset S$

Want to show: $\mathbb{R} \setminus S$ is open $\Rightarrow S$ is closed.

Let: $x \in \mathbb{R} \setminus S$

Want to show: $\exists \epsilon > 0$ st $N(x, \epsilon) \subset \mathbb{R} \setminus S$

Since $x \notin S$, we see that $x \notin S'$.

Thus, $\exists \epsilon > 0$ st $N(x, \epsilon) \cap S = \emptyset$

Since $x \notin S$, we have:

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

Hence, $N(x, \epsilon) \subset \mathbb{R} \setminus S$, which proves that $\mathbb{R} \setminus S$ is open, or, equivalently, that S is closed.

This completes the proof of a).

b)

$\text{cl } S$ is a closed set

Recall that $\text{cl } S = S \cup S'$.

Want to show: $\mathbb{R} \setminus \text{cl } S$ is open $\Rightarrow \text{cl } S$ is closed

Let: $x \in \text{cl } (\mathbb{R} \setminus S)$ (aka $(S \cup S')$ Compliment)

We must find an $\epsilon > 0$ st $N(x, \epsilon) \subset \text{cl } (\mathbb{R} \setminus S)$

Now $x \notin S$ and $x \notin S'$.

$\exists \epsilon > 0$ st $N^*(x, \epsilon) \cap S = \emptyset$

However, $x \notin S$, so

$$N(x, \epsilon) \cap S = \emptyset \quad (1)$$

We claim that $N(x, \epsilon) \cap S' = \emptyset$

Since:

$$\begin{aligned} &\neg[x \in S \cup S'] \\ &\neg[x \in S \text{ or } x \in S'] \\ &x \notin S \text{ and } x \notin S' \end{aligned}$$

which is equivalent to $N(x, \epsilon) \subset \mathbb{R} \setminus S'$

Let: $y \in N(x, \epsilon)$

By Theorem 2(a), the set $N(x, \epsilon)$ is open.

So $\exists \hat{\epsilon} > 0$ st $N(y, \hat{\epsilon}) \subset N(x, \epsilon)$.

In particular, $y \notin N(x, \epsilon)$.

From **(1)**

$N^*(y, \hat{\epsilon}) \cap S = \emptyset$.

So, $y \notin S'$ or, equivalently, $y \in \mathbb{R} \setminus S'$.

This proves that $N(x, \epsilon) \subset \mathbb{R} \setminus S'$ or, equivalently,

$$N(x, \epsilon) \cap S' = \emptyset \quad (2)$$

From **(1)** and **(2)**, $N(x, \epsilon) \cap (S \cup S') = \emptyset$.

Hence,

$$N(x, \epsilon) \subset (S \cup S')^C = \text{cl } S^C \quad (3)$$

Thus, **(3)** and * prove that $\text{cl } S^C$ is open.

Hence, by Theorem 3.4.7, $\text{cl } S$ is closed.

c)

S is closed iff $S = \text{cl } S (= S \cup S')$

\longrightarrow

Suppose: S is closed.

Want to show: $S = S \cup S'$.

By definition, $S \subset S \cup S'$.

Want to show: $S \cup S' \subset S$

Let $x \in S \cup S'$.

If $x \in S$, then we are finished.

If $x \in S' \setminus S$ Venn Diagram: $(S \cup S') \setminus S$

Then by a), $S' \subset S$, since S is closed.

Hence, $x \in S$, and we are finished.

←

Conversely,

Suppose: $S = S \cup S'$

Want to show: S is closed.

By (b), $\text{cl } S$ is closed.

Since, $S = S \cup S' = \text{cl } S$, S is also closed.

d)

$\text{cl } S = S \cup S' = S \cup \text{bd } S$

Let: $x \in S \cup S'$

If $x \in S$, then $x \in S \cup \text{bd } S$.

So, $S \cup S \subset S \cup \text{bd } S$ in this case.

If $x \in S' \setminus S$, then $\forall \epsilon > 0$, $N(x, \epsilon) \cap S \neq \emptyset$, which implies $x \in \mathbb{R} \setminus S$ and $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$

Thus, $x \in \text{bd } S \subset S \cup \text{bd } S$.

Hence, $S \cup S' \subset S \cup \text{bd } S$.

For the reverse conclusion, let $x \in S \cup \text{bd } S$.

If $x \in S$, then $x \in S \cup S'$. So, in this case, $S \cup \text{bd } S \subset S \cup S' = \text{cl } S$.

if $x \in \text{bd } S \setminus S$, then, in particular,

$\forall \epsilon > 0$,

$$N(x, \epsilon) \cap S \neq \emptyset$$

which implies that $x \in S' \subset S \cup S'$.

Hence, $S \cup \text{bd } S \subset S \cup S'$.

Hence, result.

□