Theorem 1 (infinum definition)

Let: $\emptyset \neq S \subset \mathbb{R}$, S is bounded below.
Then S possesses a greatest lower bound denoted by **inf S** (the **infinum** of S), where inf S $\in \mathbb{R}$, satisfying:

   i) inf S $\leq$ s $\forall$ s $\in$ S

   ii) $\forall \epsilon > 0, \exists s_1$ st inf S $+ \epsilon > s_1$

Proof.

Let: S be bounded below
Then

$$\exists m \in \mathbb{R} \text{ st } m \leq s \ \forall s \in S \tag{1}$$

Define the set -S to be $\{-s : s \in S\}$
So, $\emptyset \neq$ -S $\subset \mathbb{R}$
From **(1)**, -S $\leq -m \ \forall$ s $\in$ S.
Thus, $-m$ is an upper bound for -S.
By the Axiom of Completeness of $\mathbb{R}$ (AoC), sup(-S) $\in \mathbb{R}$ exists.
By definition,

$$-s \leq sup(-S), \ \forall s \ \in S \tag{2}$$

and $\forall \epsilon > 0, \exists$ -$s_1 \in$ S st

$$sup(-S) - \epsilon < -s_1 \text{ where } s_1 \in S \tag{3}$$

From **(2)**,

$$-sup(-S) \leq S \ \forall s \in S \tag{4}$$

Want to show: $-$sup(-S) = inf S
From **(3)**,

$$-sup(-S) + \epsilon > s_1 \text{ where } s_1 \in S \tag{5}$$

We see that from **(4)** and **(5)**,
inf S $= -$sup(-S).
Hence, result. $\qquad\qquad\square$

Theorem 3.3.7

Given nonempty subsets of A, B (A, B $\subset \mathbb{R}$),
Let: C = {x + y: x $\in$ A, y $\in$ B}
If A and B have suprema, then C has a supremum: sup C = sup A + sup B

Proof.

Let: c $\in$ C
Then c = x + y for some x $\in$ A, y $\in$ B.
It follows that:
x $\leq$ sup A, y $\leq$ sup B
x + y $\leq$ sup A + sup B
c $\leq$ sup A + sup B

$$c \leq sup \ A + sup \ B \ \forall \ c \in C \tag{1}$$

By the AoC, sup C $\in \mathbb{R}$ exists.
For $\epsilon > 0$, $\exists$ $x_0 \in$ A, $y_0 \in$ B st

$$sup \ A - \frac{\epsilon}{2} < x_0 \tag{2}$$

$$sup \ B - \frac{\epsilon}{2} < y_0 \tag{3}$$

From **(2)** and **(3)**,
sup A - $\frac{\epsilon}{2}$ + sup B - $\frac{\epsilon}{2}$ < x_0 + y_0 = $c_0 \in$ C
So,

$$sup \ A + sup \ B - \epsilon < c_0 \tag{4}$$

From **(1)** and **(4)**, sup C = sup A + sup B
Hence, result.

$\square$

Theorem 3.3.8

Suppose $\emptyset \neq D \subset \mathbb{R}$ and

f : D $\longrightarrow \mathbb{R}$

g : D $\longrightarrow \mathbb{R}$

$f(D) = \{f(x) : x \in D\}$

If $\forall$ x, y $\in$ D, $f(x) \leq g(y)$, then

f(D) is bounded above and g(D) is bounded below.

Furthermore, $\sup(f(D)) \leq \inf(g(D))$

Proof.

Let: $y_0 \in D$

Then $f(x) \leq g(y_0) \ \forall \ x \in D$

So, f(D) is bounded above by $g(y_0)$.

By the AoC, $\sup(f(D))$ exists and $\sup(f(D)) \leq g(y_0)$

Since $y_0 \in D$ was arbitrary, we see that

$\sup(f(D)) \leq g(y) \ \forall \ y \in D$

Thus, $\sup(f(D))$ is a lower bound for g(D)

$g(D) = \{g(y) : y \in D\}$

Hence, $\inf(g(D)) \in \mathbb{R}$ exists and

$\sup(f(D)) \leq \inf(g(D))$

Hence, result.

$\square$

Theorem 3.3.9: Archimedian Property / Principle of $\mathbb{R}$ (AP)

The set $\mathbb{N} = \{1, 2, 3...\}$ is unbounded above in $\mathbb{R}$

Proof.

Suppose: $\mathbb{N}$ is bounded above.

By the AoC, $\sup \mathbb{N} \in \mathbb{R}$ exists.

So,

   i) $n \leq \sup \mathbb{N} \ \forall \ n \in \mathbb{N}$ **(1)**

   ii) $\forall \ \epsilon > 0, \ \exists \ n \in \mathbb{N}$ st $\sup \mathbb{N} - \epsilon < n_0$ **(2)**

Using **(2)** with $\epsilon = 1$, $\exists \ n_0(1) \in \mathbb{N}$ st $\sup \mathbb{N} - \epsilon < n_0$

Then, $\sup \mathbb{N} < 1 + n_0$ **(3)**

See that **(3)** contradicts **(1)** with $n = 1 + n_0 \in \mathbb{N}$

By contradiction, $\mathbb{N}$ is unbounded.

$\square$