Exercise 1

Let f : D $\longrightarrow \mathbb{R}$ and let c $\in$ D. Mark each statement True or False. Justify each answer.

a. f is continuous at c iff $\forall\ \epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in D$

   True. By definition of continuous.

b. if f(D) is a bounded set, then f is continuous on D

   False.

   Let: f : D $\longrightarrow \mathbb{R}$ be defined by D = $\{\mathbb{R}\}$ and f = {1 if x $\neq$ 0, 0 otherwise.}

   Pick $\epsilon = 0.5$. Notice that there is no δ such that $|f(x) - f(0)| < 0.5$

c. if c is an isolated point of D, then f is continuous at c

   True. If you just pick a δ st only x = c fits in $|x - c| < \delta$ (which is possible since it's an isolated point), then that works for any $\epsilon > 0$ since $|f(x) - f(c)|$ will always be 0.

d. if f is continuous at c and (x_n) is a sequence in D, then $x_n \longrightarrow c$ whenever $f(x_n) \longrightarrow f(c)$

   True.

   So, what we are asking is:

   (1) f is continuous at c, **(2)** (x_n) is a sequence in D, and **(3)** $f(x_n) \longrightarrow f(c)$ implies $x_n \longrightarrow c$

   True or false?

   Want to show: $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|x_n - c| < \epsilon$

   $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ st x $\in$ D and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$

   So, if we let x = x_n (since $x_n \in$ D),

   For $\epsilon > 0$, $\exists\ \delta > 0$ st $x_n \in$ D and $|x_n - c| < \delta$ implies $|f(x_n) - f(c)| < \epsilon$ **(4)**

   We also know that, by **(3)**,

   $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|f(x_n) - f(c)| < \epsilon$

   So, by **(1)** and **(3)**,

   For $\epsilon > 0$, $\exists\ \delta > 0$ st $x_n \in$ D and $|x_n - c| < \delta$ implies $|f(x_n) - f(c)| < \epsilon$

   and for this same ϵ , there is an N $\in \mathbb{N}$ st n $\geq$ N implies the same.

   I'm not sure at this point, but I think that $\exists$ N $\in$ n st n $\geq$ N implies $|f(x_n) - f(c)| < \epsilon$ implies that $|x_n - c|$ goes to 0 as well.

e. if f is continuous at c, then for every neighborhood V of f(c), there exists a neighborhood U of c such that f(U $\cap$ D) = V

   True. By Theorem 5.2.2 (c).

Exercise 2 (omit d)

Let f : D $\longrightarrow$ R and let c $\in$ D. Mark each statement True or False. Justify each answer.

a. if f is continuous at c and c is an accumulation point of D, then $\lim_{x \to c}$ f(x) = f(c)

   True, by Theorem 5.2.2 (d).

b. Every polynomial is continuous at each point in $\mathbb{R}$

   True, by Theorem 5.1.13. Since every polynomial can be obtained by Theorem 5.1.13's operations on continuous functions, every polynomial is continuous as well.

c. if $\{(x_n)\}$ is a Cauchy sequence in D, then $\{f(x_n)\}$ is convergent.

   False.

   It is true that all Cauchy sequences are convergent, but just because $\{x_n\}$ is a sequence in D doesn't mean that x_n converges to L st L $\in$ D.

   If L $\notin$ D, then f(L) won't be defined.

   In order for $\{f(x_n)\}$ to be convergent, that has to be the case. It can't converge to an undefined number.

d. if f : $\mathbb{R}$ $\longrightarrow$ $\mathbb{R}$ and g : $\mathbb{R}$ $\longrightarrow$ $\mathbb{R}$ are both continuous on $\mathbb{R}$, then f o g and g o f are both continuous on $\mathbb{R}$

   True. Since f and g are defined for all real numbers, f o g and g o f are also defined for all real numbers. Since both functions are continuous everywhere, there are no discontinuities no matter what input either function takes (even if the input is the output of another function, i.e. a composition).

   Also true by Theorem 5.2.12.

Exercise 3

Let: f(x) = (x^2 + 4x − 21)/(x − 3) for x $\neq$ 3.
How should f(3) be defined so that f will be continuous at 3?

$$f(x) = \frac{x^2 + 4x - 21}{x - 3} = \frac{(x + 7)(x - 3)}{(x - 3)}$$

f(3) should be defined as 10.

Exercise 5 (prove the result)

Find an example of a function f : $\mathbb{R} \longrightarrow \mathbb{R}$ that is continuous at exactly one point.
Let: f(x) = 1 and D = {0}
In order for f(x) to be continuous, it must satisfy this definition:
$\forall\ \epsilon > 0, \exists\ \delta > 0$ st $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$
If we let c = 0, notice that f(0) = 1, 0 is in the domain, and $|f(x) - f(c)| < \epsilon$ for any epsilon (since f(x) and f(c) will always be the same number under our definitions).
Notice also that f(x) is only defined at x = 0.
Since f(x) is continuous, and f(x) is only defined at one point, f(x) is only continuous at one point.

Exercise 10

a. Let f : D $\longrightarrow \mathbb{R}$ and define |f| : D $\longrightarrow \mathbb{R}$ by $|f|(x) = |f(x)|$. Suppose that f(x) is continuous at $c \in D$. Prove that |f| is continuous at c.

   Want to show: $\forall\ \epsilon > 0, \exists\ \delta > 0$ st $x \in D$ and $|x - c| < \delta$ implies $||f|(x) - |f|(c)| < \epsilon$

   Since f(x) is continuous at $c \in D$,

   $\forall\ \epsilon > 0, \exists\ \delta > 0$ st $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$

   By the triangle inequality,

   $||f(x)| - |f(c)|| \leq |f(x) - f(c)|$

   Since $|f|(x) = |f(x)|$,

   $||f(x)| - |f(c)|| \leq |f(x) - f(c)| < \epsilon$

   $||f(x)| - |f(c)|| < \epsilon$

   $||f|(x) - |f|(c)| < \epsilon$

   Hence, |f| is continuous at c.

b. if |f| is continuous at c, does it follow that f is continuous at c? Justify your answer.

   No.

   For example, if $f(x) = \{1 \text{ for } x \neq 0, -1 \text{ otherwise}\}$, |f|(x) is defined as 1 for all $x \in \mathbb{R}$, making it continuous everywhere. However, f(x) is not continuous at x = 0.

Exercise 11 (just prove the "max" result)

Define max(f, g) and min(f, g) as in Example 2.11.
Example 2.11
max(f, g)(x) = max {f(x), g(x)}

Show that:
max(f, g) = $\frac{1}{2}$(f + g) + $\frac{1}{2}$|f − g|

Let: h = $\frac{1}{2}$(f + g) + $\frac{1}{2}$|f − g|
h − $\frac{1}{2}$(f + g) = $\frac{1}{2}$|f − g|
2h − (f + g) = |f − g|
So,
Case

i) 2h − (f + g) = f − g (i.e. f − g is non-negative and f ≥ g)

   2h = f − g + f + g

   2h = 2f

   h = f

ii) −(2h − (f + g)) = f − g (i.e. f − g is negative and g > f)

   2h − (f + g) = g − f

   2h = g − f + f + g

   2h = 2g

   h = g

Hence, result.

Exercise 13

Let: f : D $\longrightarrow \mathbb{R}$ be continuous at c ∈ D
Assume: f(c) > 0
Prove that ∃ α > 0 and a neighborhood U of c st f(x) > α , ∀ x ∈ U ∩ D
Want to show: ∃ a neighborhood V of f(c) st ∀ v ∈ V, v > 0
Since f(c) > 0, if we let $\epsilon = \frac{f(c)-0}{4}$ and V = N(f(c), ϵ), then we can see that ∀ v ∈ V, v > 0
(which we can do since f is continuous at c)
By Theorem 5.2.2 (c), for this neighborhood V, ∃ a neighborhood U of c st f(U ∩ D) ⊂ V
Let: $\alpha = \frac{f(c)}{2} > 0$
Notice that ∀ v ∈ V, α < v
Notice also that f(U ∩ D) ⊂ V
Thus,
f(x) > α ∀ x ∈ U ∩ D

Exercise 16

I got super lost on this problem..
(First prove that for any $\mathrm{H} \subset \mathbb{R}$, $\mathrm{f}^{-1}(\mathbb{R} \setminus \mathrm{H}) = \mathbb{R} \setminus \mathrm{f}^{-1}(\mathrm{H})$, use this in conjunction with Theorem 5.2.14)
Let: $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}$
Prove that f is continuous on $\mathbb{R}$ iff $\mathrm{f}^{-1}(\mathrm{H})$ is a closed set whenever H is a closed set.
(vs H is a closed set whenever f(H) is a closed set?)

Proof.
$\longrightarrow$
Let: $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$
Want to show: H is a closed set $\Rightarrow \mathrm{f}^{-1}(\mathrm{H})$ is a closed set
Theorem 5.2.14 (since $\mathrm{D} = \mathbb{R}$):
for every open set $\mathrm{H} \subset \mathbb{R}$, $\exists$ an open set $\mathrm{G} \subset \mathbb{R}$ st $\mathrm{f}^{-1}(\mathrm{H}) = \mathrm{G}$
Corollary 5.2.15:
A function $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous iff $\mathrm{f}^{-1}(\mathrm{G})$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$

So at this point we know that:
G is open in $\mathbb{R} \Rightarrow \mathrm{f}^{-1}(\mathrm{G})$ is open in $\mathbb{R}$

Let: $\mathrm{H} \subset \mathbb{R}$ be open.
***Somehow, from here, we show:

$$\text{for } H \subset \mathbb{R}, \qquad f^{-1}(\mathbb{R} \setminus \mathrm{H}) = \mathbb{R} \setminus \mathrm{f}^{-1}(\mathrm{H}) \tag{1}$$

Suppose: $\mathrm{f}^{-1}(\mathrm{H})$ is closed
Then, $\mathbb{R} \setminus \mathrm{f}^{-1}(\mathrm{H})$ must be open.
Which means that $\mathrm{f}^{-1}(\mathbb{R} \setminus \mathrm{H})$ is open.
We also know that $\mathbb{R} \setminus \mathrm{H}$ is closed since H is open.
However, by Corollary 5.2.15,
if $\mathbb{R} \setminus \mathrm{H}$ is open then $\mathrm{f}^{-1}(\mathbb{R} \setminus \mathrm{H})$ must be open,
a contradiction.
So, $\mathrm{f}^{-1}(\mathrm{H})$ must be open.

$\longleftarrow$
Let: $\mathrm{f}^{-1}(\mathrm{H})$ be a closed set whenever H is a closed set
So,
H is an open set whenever $\mathrm{f}^{-1}(\mathrm{H})$ is an open set.
Want to show: for any $\mathrm{H} \subset \mathbb{R}$, $\mathrm{f}^{-1}(\mathbb{R} \setminus \mathrm{H}) = \mathbb{R} \setminus \mathrm{f}^{-1}(\mathrm{H})$
Then,
Want to show: for every open set $\mathrm{G} \subset \mathbb{R}$, $\exists$ an open set $\mathrm{H} \subset \mathbb{R}$ st $\mathrm{f}^{-1}(\mathrm{G}) = \mathrm{H} \cap \mathrm{D}$ (which implies continuity on $\mathbb{R}$)

$\square$