Theorem 3.3: The Completeness Axiom

Recall the Fundamental Theorem of Arithmetic:
if n $\in \mathbb{N}$ with n $\geq$ 2, then n may be expressed as the product of prime numbers (the prime factorization (PF)).
The PF is unique with respect to (WRT) order.
Ex: $12 = 2 * 2 * 2 * 3$

Theorem 3.3.1

Let: p be a prime number
Then $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$

Proof.

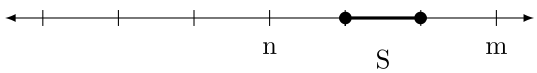
Assume: $\sqrt{p} \in \mathbb{Q}$
Then $\sqrt{p} = \frac{a}{b}$, where a, b $\in \mathbb{N}$ and gcd(a, b) = 1
So,
p = $\frac{a^2}{b^2}$
$a^2 = pb^2$
therefore,

$$p|a^2 \tag{1}$$

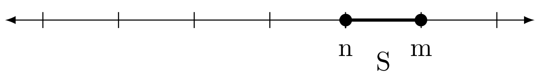
$p|a^2 \Rightarrow \exists k \in \mathbb{Z}$ st $a^2 = pk$
Since the PF of a^2 and a contain exactly the same distinct primes,
(i.e. a = $p_1\times$ $p_2\times$ ... p_n $\Rightarrow$ a^2 = $p_1^2\times$ $p_2^2\times$ ... p_n^2)
and since p is prime (i.e. p is a component of a^2 but can't be, say, p_2^2 because that would mean it has an integer square root and therefore isn't prime), it has to be one of the p_n's,
p | a.
Thus, $\exists$ k $\in \mathbb{Z}$ st. a = pk.
Then $a^2 = p^2k^2 = pb^2$ from **(1)**.
Thus, $b^2 = pk^2$, and we see that p | b^2.
However, we obtain the contradiction that p | b and p | a.
Hence, $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

Definition 3.3.7

Let S $\subset \mathbb{R}$.
If $\exists$ m $\in \mathbb{R}$ st s $\leq$ m $\forall$ s $\in S$,
then m is an upper bound of S and we say that S is **bounded above**.
Similarly, we can define **bounded below**.
If S is bounded above and below, then S is said to be **bounded**.
S can be open or closed. The example below is closed.



If an upper bound m of S is a member of S, then m is called the maximum (or largest element) of S, and we say that m = **max S**.
–n–[–S–m–
Similarly, we may decline **minimum of S (min S)**.



Theorem 1

If a set S $\subset \mathbb{R}$ possesses a max element, then it is unique. A similar result holds for a minimum element.

Proof.
Suppose: $\exists m_1, m_2 \in \mathbb{R}$ st m_1 = max S, m_2 = max S
Thus, m_1, $m_2 \in S$ and, $\forall s \in$ S

$$s \leq m_1 \tag{1}$$

$$s \leq m_2 \tag{2}$$

Let max = m_1 in **(1)** and max = m_2 in **(2)** to obtain that $m_2 \leq m_1$ and $m_1 \leq m_2$,
Hence, $m_1 = m_2$. ☐

Definition 3.3.5 (supremum defined)

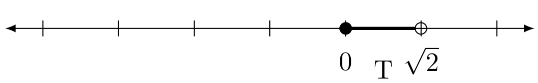
Let $\emptyset \neq$ S $\subset \mathbb{R}$ if S is bounded above,
then the **least upper bound** of S is called the **supremum** of S, denoted by sup S $\in \mathbb{R}$
iff:

a. s $\leq$ sup S $\forall$ s $\in$ S

b. $\exists$ s' $\in$ S st sup S $- \epsilon <$ s' $\forall\ \epsilon > 0$

Axiom of Completeness of the set of Real Numbers: $\mathbb{R}$

Every $\emptyset \neq$ S $\subset \mathbb{R}$ that is bounded above has a least upper bound (i.e. sup S $\in \mathbb{R}$ exists).
A similar statement can be made about inf S.
Remark: In practice 3.3.4, the set T = $\{$q $\in \mathbb{Q} : 0 \leq q \leq \sqrt{2}\}$ is bounded.



But $\sqrt{2}$ is not rational, so the set wouldn't have a least upper bound.
We need to fill in the gaps to make analysis work.

Examples (#3, page 132):

a. S = {1, 3}

   sup S = 3 $\longrightarrow$ since s $\leq$ 3 $\forall$ s $\in$ S and 3 $-\epsilon <$ 3

b. similar to a.

c. S = (0, 4]

   sup S = 4, max S = 4

d. S = (0, 4)

   sup S = 4 $\longrightarrow$ since s $\leq$ 4 $\forall$ s $\in$ S and 4 $-\epsilon < s' \in$ S

   max S = undefined. There is no max.

e. S = { $\frac{1}{2n}$: n $\in \mathbb{N}$ }

   sup S = $\frac{1}{2}$

   max S = $\frac{1}{2}$ (if the supremum is in the set, then max = sup)

f. S = { 1 $-$ $\frac{1}{2n}$: n $\in \mathbb{N}$ }

   sup S = 1

   max S = undefined (1 $\notin \mathbb{R}$)