

HW 11: page 220 - 221, #1, 2, 5 and page 226-227, # 1 - 3, 4(a)(b), 5, 11

Exercise 1 (pages 220 - 221)

Mark each statement True or False. Justify each answer.

- a. Let D be a compact subset of \mathbb{R} and suppose that $f : D \rightarrow \mathbb{R}$ is continuous. Then $f(D)$ is compact.

True, by Theorem 5.3.2.

- b. Suppose that $f : D \rightarrow \mathbb{R}$ is continuous. Then, there exists a point x_1 in D st $f(x_1) \geq f(x) \forall x \in D$

False.

Let: $f(x) = x$ and $D = \mathbb{R}$

Suppose: $\exists x_1 \in D$ st $f(x_1) \geq f(x) \forall x \in D$

Notice that $(f(x_1) + 1) \in \mathbb{R}$, and if $x_2 = (f(x_1) + 1)$, then $f(x_2) = (f(x_1) + 1) > f(x_1)$. A contradiction.

- c. Let D be a bounded subset of \mathbb{R} and assume that $f : D \rightarrow \mathbb{R}$ is continuous. Then $f(D)$ is bounded.

False.

Let: $f : (0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$

Suppose: $\exists f(x_1)$ st $f(x_1) \geq f(x) \forall x \in (0, \infty)$

Notice that $(f(x_1) + 1) \in \mathbb{R}$, and if $x_2 = \frac{1}{f(x_1)+1}$, then $f(x_2) = (f(x_1) + 1) > f(x_1)$. A contradiction.

Exercise 2 (pages 220 - 221)

Mark each statement True or False. Justify each answer.

- a. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and assume $f(a) < 0 < f(b)$. Then there exists a point $c \in (a, b)$ st $f(c) = 0$.

True, by Theorem 5.3.6 (IVT).

- b. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and assume $f(a) \leq k \leq f(b)$. Then there exists a point $c \in [a, b]$ st $f(c) = k$.

True, by Theorem 5.3.6 (IVT). Also because this statement is just (a) above with $k = 0$, except weaker.

- c. If $f : D \rightarrow \mathbb{R}$ is continuous and bounded on D , then f assumes maximum and minimum values on D .

False.

Let: $f : (0, 1) \rightarrow \mathbb{R}$ be defined by $f(x) = x$

Suppose: f has $x \in D$, a maximum value on D

Notice that $0 < x < 1$, and that $x < x + \frac{1-x}{2}$.

However, notice also that $x + \frac{1-x}{2} < 1$

But x is a maximum value on D . A contradiction.

WLOG, a minimum value on D is similar.

Exercise 5 (pages 220 - 221)

Show that the equation $5^x = x^4$ has at least one real solution.

Let: $f : [-1, 0] \rightarrow \mathbb{R}$ be defined by $f(x) = 5^x - x^4$

Notice that $f(-1) = -0.8$ and $f(0) = 1$

Since $5^x - x^4 = 0$ means $5^x = x^4$, and $-0.8 < 0 < 1$,

by Theorem 5.3.6, since $f(x)$ is continuous on \mathbb{R} ,

$\exists c \in [-1, 0]$ st $f(c) = 0$.

Exercise 1 (pages 226 - 227)

Let $f : D \rightarrow \mathbb{R}$. Mark each statement True or False. Justify each answer.

- a. f is uniformly continuous on D iff for every $\epsilon > 0$ there exists a $\delta > 0$ st $|f(x) - f(y)| < \delta$ whenever $|x - y| < \epsilon$ and $x, y \in D$.

This isn't the definition, but I can't find a counter example for it...

- b. If $D = \{x\}$, then f is uniformly continuous at x .

True. Since x is the only element in the domain, and since f is a function, $f(x)$ is the only element in the range of f which makes $|f(x) - f(y)|$ always less than any $\epsilon > 0$ since there is only one object in the range, making them the same object in any possible case.

- c. If f is continuous and D is compact, then f is uniformly continuous on D .

True, by Theorem 5.4.6.

Exercise 2 (pages 226 - 227)

Let $f : D \rightarrow \mathbb{R}$. Mark each statement True or False. Justify each answer.

- a. In the definition of uniform continuity, the positive δ depends only on the function f and the given $\epsilon > 0$.

False. The positive δ depends on the given $x, y \in D$ as well.

- b. If f is continuous and (x_n) is a Cauchy sequence in D , then $(f(x_n))$ is a Cauchy sequence.

False.

Let: $x_n = \frac{1}{n}$, $n \in \mathbb{N}$ and $f : (0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$

Notice that $f(x_n) = 1, 2, 3, \dots$

This is not a Cauchy sequence.

- c. If $f : (a, b) \rightarrow \mathbb{R}$ can be extended to a function that is continuous on $[a, b]$, then f is uniformly continuous on (a, b) .

True, by Theorem 5.4.9.

Exercise 3 (pages 226 - 227)

Determine which of the following continuous functions are uniformly continuous on the given set. Justify your answers.

- a. $f(x) = x$ on $[2, 5]$ since f is continuous and D is compact, f is uniformly continuous (by Theorem 5.4.6)
- b. $f(x) = x$ on $(0, 2)$ since $\tilde{f} : [0, 2] \rightarrow \mathbb{R}$ is continuous, f is uniformly continuous (by Theorem 5.4.9)
- c. $f(x) = x^2 + 2x - 7$ on $[0, 5]$ since f is continuous and D is compact, f is uniformly continuous (by Theorem 5.4.6)
- d. $f(x) = x^2 + 2x - 7$ on $(1, 4)$ since $\tilde{f} : [1, 4] \rightarrow \mathbb{R}$ is continuous, f is uniformly continuous (by Theorem 5.4.9)
- e. $f(x) = \frac{1}{x^2}$ on $(0, 1)$ Since $\lim_{x \rightarrow 0} f(x)$ does not exist, $f(x)$ cannot be extended to a continuous function. Therefore, f is not uniformly continuous.
- f. $f(x) = \frac{1}{x^2}$ on $(0, \infty)$ Since $\lim_{x \rightarrow 0} f(x)$ does not exist, $f(x)$ cannot be extended to a continuous function. Therefore, f is not uniformly continuous.
- g. $f(x) = \frac{x^2 - 4}{x - 2}$ on $(2, 4)$ Since $\lim_{x \rightarrow 2} f(x)$ and $\lim_{x \rightarrow 4} f(x)$ exist, $f(x)$ can be extended to a continuous function. Therefore, f is uniformly continuous.
- h. $f(x) = x \sin(\frac{1}{x})$ on $(0, 1)$ Since $\lim_{x \rightarrow 0} f(x) = 0$ and $\lim_{x \rightarrow 1} f(x) = \sin(1)$, $f(x)$ can be extended to a continuous function. Therefore, f is uniformly continuous.

Exercise 4(a)(b) (pages 226 - 227)

Prove that each function is uniformly continuous on the given set by directly verifying the ϵ - δ property in Definition 4.1.

Definition 5.4.1:

$f : D \rightarrow \mathbb{R}$ is uniformly continuous on D if

$\forall \epsilon > 0, \exists \delta > 0$ st $0 < |x - y| < \delta$ and $x, y \in D$ implies $|f(x) - f(y)| < \epsilon$

a. $f(x) = x^3$ on $[0, 2]$

$\forall \epsilon > 0, \exists \delta > 0$ st $0 < |x - y| < \delta$ and $x, y \in D$ implies $|x^3 - y^3| < \epsilon$

$$|x^3 - y^3|$$

$$|(x - y)(x^2 + xy + y^2)|$$

$$|(x - y)(x^2 + xy + y^2)| \leq |(x - y)|(|x^2| + |xy| + |y^2|) \leq 12|x - y| < \epsilon$$

so, whenever $|x - y| < \delta = \frac{\epsilon}{12}$, $|x^3 - y^3| < \epsilon$

b. $f(x) = \frac{1}{x}$ on $[2, \infty)$

$\forall \epsilon > 0, \exists \delta > 0$ st $0 < |x - y| < \delta$ and $x, y \in D$ implies $|\frac{1}{x} - \frac{1}{y}| < \epsilon$

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{y - x}{xy}|$$

Since all elements in the domain are positive,

$$|\frac{y - x}{xy}| = |y - x| \frac{1}{xy} = |x - y| \frac{1}{xy} < \epsilon$$

So, since $\frac{1}{x}$ is maximum at $x = 2$ and $\frac{1}{y}$ is maximum at $y = 2$,

$$|x - y| < xy\epsilon$$

$$|x - y| < (2)(2)\epsilon$$

$$|x - y| < \delta = 4\epsilon$$

so, whenever $|x - y| < \delta = 4\epsilon$, $|\frac{1}{x} - \frac{1}{y}| < \epsilon$

Exercise 5 (pages 226 - 227)

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Let: $\epsilon > 0$

Choose $\delta = \text{SOMETHING}$ to make $|x - y| < \delta$. We know that:

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{|\sqrt{x} + \sqrt{y}|} < \frac{\delta}{|\sqrt{x} + \sqrt{y}|}$$

So, let

$$\delta \frac{1}{|\sqrt{x} + \sqrt{y}|} = \epsilon \Rightarrow \delta \frac{1}{|1 + 1|} = \epsilon \Rightarrow \delta = 2\epsilon$$

then $|\sqrt{x} - \sqrt{y}| < \epsilon$ if $|x - y| < \delta$ and $x, y \in [0, \infty)$

Exercise 11 (pages 226 - 227)

Let $f : D \rightarrow \mathbb{R}$ be uniformly continuous on the bounded set D . Prove that f is bounded on D .
Use Theorem 5.4.8. The hint is that it's bounded.

Theorem 5.4.8

Let: $f : D \rightarrow \mathbb{R}$ be uniformly continuous on D

Assume: $\{x_n\}$ is a Cauchy sequence in D

Then,

$\{f(x_n)\}$ is a Cauchy sequence.

Lemma 4.3.11

Every Cauchy sequence is bounded.

Proof.

Any Cauchy sequence x_n in D means that $\{f(x_n)\}$ is a Cauchy sequence, and if $\{f(x_n)\}$ is a Cauchy sequence then it's bounded.

So, our strategy will be to somehow make a Cauchy sequence x_n that has a limit at c such that $f(c) = \max(f(D))$ and, WLOG, d such that $f(d) = \min(f(D))$.

Either that, or figure out a way to make a list of Cauchy sequences that hit all values in the domain.

□