Homework: page 148-149, #1-4, 6, 8

Heine-Borel Theorem

$\emptyset \neq$ S $\subset \mathbb{R}$ is compact iff S is closed and bounded.

Proof.
$\longrightarrow$ Done. $\longleftarrow$ **Suppose:** S is closed and bounded.
Let: S $\subset \bigcup_{\alpha \in I}$ G_α where G_α is open $\forall\ \alpha \in$ I
Since is is bounded, sup S, inf S $\in \mathbb{R}$ both exist.
Define, for x $\in \mathbb{R}$,
S_x = S $\cap$ $(-\infty$, x].
S $\subset \bigcup_{x \in S}$ N(x, ϵ)
β = { x $\in \mathbb{R}$: S_x has a finite subcover from the G_α's}
$\beta \neq \emptyset$, inf S $\in \beta$
$S_{inf S}$ = S $\cap$ $(-\infty$, inf S]
We need to prove that S has a finite subcover of the G_α's.
If β is unbounded above, then $\exists$ z $\in \beta$ st z > sup S.
Then S_z = S $\cap$ $(-\infty$, z] = S
Since S_z = S has a finite subcover of the G_α's, we see that, in this case, S is compact.
We prove that β is unbounded above using contradiction.
Suppose: β is bounded above.
Thus, sup $\beta \in \mathbb{R}$ exists.
Case i: sup $\beta \in$ S.
In this case, $\exists\ \epsilon \in$ I st sup $\beta \in G_{\alpha_0}$
Since G_{α_0} is open, $\exists\ \epsilon\ _0 > 0$ st
N(sup β , $\epsilon\ _0$) = (sup $\beta - \epsilon\ _0$, sup $\beta + \epsilon\ _0$) $\subset G_{\alpha_0}$
By the definition of the supremum,
$\exists$ $x_0 \in \beta$ st
sup $\beta - \epsilon\ _0 < y_0 \leq$ sup B < sup B + $\frac{\epsilon_0}{2}$ < sup $\beta + \epsilon\ _0$
Since $x_0 \in \beta\ _1$, $\exists$ k $\in \mathbb{N}$ and $\{\alpha\ _1, \alpha\ _2, ...\ \alpha\ _n\} \subset$ I
st $S_{x_0} \subset \bigcup_{i=1}^{k} G_{\alpha_i}$

——————————————Side Note——————————————-

S_{x_0} = S $\cap$ $(-\infty$, x_0]
$S_{sup\beta}$ + $\frac{\epsilon_0}{2}$
= S $\cap$ $(-\infty$, sup β + $\frac{\epsilon_0}{2}$]

This produces the contradiction that sup $\beta + \frac{\epsilon_0}{2} \in \beta$
Case ii):
sup $\beta \in \mathbb{R}\ \setminus$ S, which is open since S is closed.
Thus, $\exists\ \epsilon\ _1 > 0$ st N(sup β , $\epsilon\ _1$) $\subset \mathbb{R}\ \setminus$ S

——————————————Side Note——————————————-

—-(——————-)—- sup - ep1, sup B, sup B + ep1/2, supB + ep1

As in case i), $\exists$ $x_1 \in \beta$ st
sup $\beta - \epsilon\ _1 < x_1 \leq$ sup β < sup $\beta + \frac{\epsilon_1}{2}$ < sup $\beta + \epsilon\ _1$
From **(1)**, N(sup β , $\epsilon\ _1$) = (sup $\beta - \epsilon\ _1$, sup $\beta + \epsilon\ _1 \cap$ S = $\emptyset$
—-(——]———-]—-)— supB-ep0, x0inB, supB, supBplusEpOver2, supBplusEp0

Notice that:
S_{x_1} = S $\cap$ ($-\infty$, x_1] = S $\cap$ ($-\infty$, sup $\beta + \frac{\epsilon_1}{2}$]
Again we obtain the contradiction that sup $\beta + \frac{\epsilon_1}{2} \in \beta$
Hence, result by contradiction.

□

Theorem 3.5.6: Bolzond-Weierstrass Theorem

If a bounded set S $\subset \mathbb{R}$ contains an infinite number of points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.

Proof.

Suppose: $\exists$ S $\subset \mathbb{R}$ where S has an infinite number of points and S is bounded but S$'$ = $\emptyset$
Since cl S = S $\cup$ S$'$ = S $\cup$ $\emptyset$ = S, we can see by Theorem 3.4.17 a) that S is closed.
Since S is also bounded, it follows by the Heire-Borel theorem that S is compact.
Let: x $\in$ S
Then x $\notin$ S$'$, so $\exists$ $\epsilon_{\ x} > 0$ st
N(x, $\epsilon_{\ x}$) $\cap$ S = {x}
——————————————Side Note——————————————-
—(—-———)— x-ep(x?), x, yMemS, xplusep(x?)
If x $\in$ S$'$, then:
$\neg$[$\forall$ $\epsilon > 0$, N*(x, ϵ) $\cap$ S $\neq \emptyset$]
$\exists$ $\epsilon > 0$ st N(x, ϵ) $\cap$ S = {x}

Then:
S $\subset \bigcup_{x \in S}$ N(x, $\epsilon_{\ x}$)
Since S is compact,
$\exists$ k $\in \mathbb{N}$ and {x_1, x_2, ... x_k} $\subset$ S
S $\subset \bigcup_{i=1}^{k}$ N(x_{i_1}, ϵ_{i_1})
However, S $\cap$ ($\bigcup_{i=1}^{k}$ N(x_{i_1}, ϵ_{i_1})) = {x_1, x_2, ... x_k}
This produces the contradiction that S contains a **finite** number of points.
Hence, result.

□

Theorem 3.5.7 (F.I.P.)

Let: $\{K_\alpha\}_{\alpha \in I}$ be a family of compact sets, where I is an index.
Suppose that the intersection of any finite subfamily of the K_α's has a nonempty intersection.
Then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Proof.

Assume that $\bigcap_{\alpha \in I} K_\alpha = \emptyset$
Then $\mathbb{R} \setminus (\bigcap_{\alpha \in I} K_\alpha) = \bigcup_{\alpha \in I} (\mathbb{R} \setminus K_\alpha) = \mathbb{R}$
Notice, by the Heine-Borel Theorem that $\mathbb{R} \setminus K_\alpha$ is open $\forall$ $\alpha \in$ I.
Let: $\alpha_{\ 0} \in$ I

Since K_{α_0} is compact,
$\exists$ k $\in \mathbb{N}$ and $\{\alpha_{\ 1}, \alpha_{\ 2}, \ldots \alpha_{\ n}\} \subset$ I st.
$K_{\alpha_0} \subset \bigcup_{\alpha \in I}$ ($\mathbb{R} \setminus$ K$_\alpha$)
$\subset \bigcup_{i=1}^{k}$ ($\mathbb{R} \setminus K_{\alpha_0}$)

Side Note

If A $\subset$ B, then $\mathbb{R} \setminus$ B $\subset \mathbb{R} \setminus$ A
Let x $\in \mathbb{R} \setminus$ B.
Then x $\notin$ B.
So, x $\notin$ A.
Thus, x $\in \mathbb{R} \setminus$ A

$\mathbb{R} \setminus$ ($\bigcup_{i=1}^{k}$($\mathbb{R} \setminus$ K$_\alpha$)) $\subset \mathbb{R} \setminus K_{\alpha_0}$
$\bigcap_{i=1}^{k} K_{\alpha_i} \subset \mathbb{R} \setminus K_{\alpha_0}$
We obtain the contradiction that:
$\bigcap_{i=0}^{k} K_{\alpha_i} = \emptyset$
Hence, result.

□

Corollary 3.5.8 Nested Intervals Theorem

Let: {A$_n$}$_{n=1}^{\infty}$ be a family of nonempty closed bounded intervals in $\mathbb{R}$ st A$_{n+1} \subset$ A$_n$ $\forall$ n $\in \mathbb{N}$
Then:
$\bigcap_{n=1}^{\infty}$ A$_n \neq \emptyset$

Proof.

We use Theorem 3.5.7.
Will this be contradiction?
Suppose: $\forall$ k $\in \mathbb{N}$, that {n$_1$, n$_2$, ... n$_k$} $\subset \mathbb{N}$
Then,
$\bigcap_{i=1}^{k}$ A$_{ni}$ = A$_m \neq \emptyset$
where
m = max {n$_1$, n$_2$, ... n$_k$}

Side Note

—-[—–[—[—]—]—]—- not imp, not imp, not imp, A3, A2, A1

□

Assignment Set: 6, 7, 15, 17, 19, 21 from pages 141 - 142

6)

Find the closure of each set:

a. { $\frac{1}{n}$: n $\in \mathbb{N}$ }
   Answer: $\emptyset$

b. $\mathbb{N}$
   Answer: $\mathbb{N}$

c. $\mathbb{Q}$
   Answer: $\mathbb{R}$

d. $\bigcap_{n=1}^{\infty}$ $(0, \frac{1}{n})$
   Answer: $\emptyset$

e. { x : $|x-5| \leq \frac{1}{2}$}
   $[4.5, 5.5]$
   Answer: $[4.5, 5.5]$

f. { x : $x^2 > 0$}
   $(0, \infty)$
   Answer: $[0, \infty)$

7)

Let S, T $\subset \mathbb{R}$. Find a counterexample of each of the following:

a. If P is the set of all isolated points of S, then P is a closed set.
   Answer: Let S = $\mathbb{N}$

b. Every open set contains at least two points.
   Answer: $\emptyset$

c. If S is closed, then cl(int S) = S.
   Answer: Let S = $\mathbb{Q}$

d. If S is open, then int (cl S) = S.
   Answer: Let S = $(-1, 0) \cup (0, 1)$

e. bd (cl S) = bd S
   Answer: Let S = $(-1, 0) \cup (0, 1)$

f. bd (bd S) = bd S
   Answer: Let S = $\mathbb{Q}$. Then bd S is $\mathbb{R}$, and bd (bd S) = $\emptyset \neq \mathbb{R}$.

g. bd (S $\cup$ T) = (bd S) $\cup$ (bd T)
   Answer: Let S = $\mathbb{R}$, T = $(0, 1)$. bd (S $\cup$ T) = $\emptyset$, but bd S $\cup$ bd T = $\emptyset \cup \{0, 1\}$

h. bd (S $\cap$ T) = (bd S) $\cap$ (bd T)
   Answer: Let S = $(0, 1)$, T = $(1, 2)$. bd (S $\cap$ T) = $\emptyset$, but bd S $\cap$ bd T = 1.

15)

Prove: If x is an accumulation point of the set S, then every neighborhood of x contains infinitely many points of S.

Proof.

Suppose that $\exists$ a deleted neighborhood of x, called N, that contains n points x_1, x_2, ... x_n of S where n is a finite amount and $x_1 \leq x_2, \leq ... x_n$
x is an accumulation point on S if $\forall \epsilon > 0$, N*(x, ϵ) $\cap$ S $\neq \emptyset$.
N is a deleted neighborhood of S if $\forall$ x $\in \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$, x $\in$ N.
Let $\hat{\epsilon} = \epsilon + \epsilon$, and $x_0 = x_1 - \hat{\epsilon}$.
By definition, $x_0 \in$ N, since N is a neighborhood $\forall \epsilon > 0$.
However, N only has n elements. A contradiction.
So, N can't be a deleted neighborhood since it has a finite number of elements, which means x can't be an accumulation point.

□

17)

Prove: S$'$ is a closed set.

Proof.

By definition, $\forall s \in S'$, $\epsilon > 0$, N*(s, ϵ) $\cap$ S $\neq \emptyset$
Notice that if S$'$ is empty or S$'$ is $\mathbb{R}$, then S$'$ is a closed set and we are done.
If S$'$ is not empty, $\exists$ at least one element.
Let: $\mathbb{R}$ $\setminus$ S' $\subset \mathbb{R}$, x $\in \mathbb{R}$ $\setminus$ S'
Want to show: $\mathbb{R}$ $\setminus$ S$'$ is open.
$\mathbb{R}$ $\setminus$ S$'$ is open iff $\mathbb{R}$ $\setminus$ S$'$ = int ($\mathbb{R}$ $\setminus$ S$'$)
int $\mathbb{R}$ $\setminus$ S$'$ = {s: N(s, ϵ) $\subset \mathbb{R}$ $\setminus$ S$'$ }

□

19)

Suppose S is a nonempty bounded set and let m = sup S. Prove or give a counter example: m is a boundary point of S.

Proof.

By definition,
s $\leq$ m, $\forall$ s $\in$ S, and,
$\forall$ $\epsilon > 0$, $\exists$ s' $\in$ S st m $- \epsilon <$ s'
By the second part of the definition of the supremum of S, N(m, ϵ) $\cap$ S $\neq \emptyset$.
Notice also that, by the first part of the definition of the supremum of S, (m + ϵ) $\notin$ S. This means that N(m, ϵ) $\cap$ $\mathbb{R}$ $\setminus$ S $\neq \emptyset$.
By definition, m is a boundary point.

□

21)

Let A be a nonempty open subset of $\mathbb{R}$ and let $Q \subset \mathbb{Q}$. Prove: $A \cap Q \neq \emptyset$.

Proof.
Notice that $Q \subset \mathbb{Q} \subset \mathbb{R}$.
Since A is nonempty, $\exists$ at least one element $a \in \mathbb{R}$.
Since A is nonempty and open, $a + \epsilon \in A$.
If $a \in \mathbb{Q}$, then result.
If $a + \epsilon \in \mathbb{Q}$, then result.
If $a \notin \mathbb{Q}$ and $(a + \epsilon) \notin \mathbb{Q}$, then:
Let $x = a$, $y = a + \epsilon$, $z = y - x$.
By Archimedes' axiom, $\exists$ n st $n > \frac{1}{z}$
$nz > 1$
$ny - nx > 1$
Since the difference between ny and nx is bigger than 1,
$\exists m \in \mathbb{Z}$ st $nx < m < ny$.
See that since $x < \frac{m}{n} < y$, $\frac{m}{n}$ is a rational number, and $\frac{m}{n} \in A$.
Hence, result. $\square$