1. Prove Pascal's Formula $\binom{\alpha}{k} = \binom{\alpha-1}{k-1} + \binom{\alpha-1}{k}$ for any $\alpha \in \mathbb{R}$ and k $\in \mathbb{Z}_{\geq 0}$. (Note: You will need to use the falling factorial definition.)

$$\begin{aligned}
\binom{\alpha}{k} &= \binom{\alpha-1}{k-1} + \binom{\alpha-1}{k} \\
&= \frac{(\alpha-1)!}{((\alpha-1)-(k-1))!(k-1)!} + \frac{(\alpha-1)!}{((\alpha-1)-k)!k!} \\
&= \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} + \frac{(\alpha-1)!}{(\alpha-1-k)!k!} \\
&= \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} + \frac{(\alpha-1)!(\alpha-k)\frac{1}{k}}{(\alpha-k)!(k-1)!} \\
&= \frac{(\alpha-1)! + (\alpha-1)!(\alpha-k)\frac{1}{k}}{(\alpha-k)!(k-1)!} \\
&= \frac{k(\alpha-1)! + (\alpha-1)!(\alpha-k)}{(\alpha-k)!k!} \\
&= \frac{\alpha(\alpha-1)!}{(\alpha-k)!k!} \\
&= \frac{\alpha!}{(\alpha-k)!k!}
\end{aligned}$$

2. Determine the generating function for each of the following sequences:

a. 1, r, r^2, r^3, ...
   1 + rx + r^2x^2 ... $\longrightarrow \frac{1}{1-rx}$

b. 1, − 1, 1, − 1, ...
   1 − x + x^2 − x^3 $\longrightarrow \frac{1}{1+x}$

c. $\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, -\binom{\alpha}{3}, \ldots$
   $\binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \binom{\alpha}{3}x^3$ ...
   $1 - \alpha\, x + \frac{\alpha(\alpha-1)}{2*1}x^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3*2*1}x^3$ ...
   $1 - \alpha\, x + \frac{[\alpha]_{(2)}}{[2]_{(2)}}x^2 - \frac{[\alpha]_{(3)}}{[3]_{(3)}}x^3$ ...
   $\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}x^k$
   $(1-x)^{\alpha}$

d. 1, $\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}$, ...
   $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$...
   e^x

e. 1, $\frac{-1}{1!}, \frac{1}{2!}, \frac{-1}{3!}, \frac{1}{4!}$, ...
   $1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3$...
   $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots - 2(\frac{1}{1!}x + \frac{1}{3!}x^3 \ldots)$
   e^x − sinh x

f. $\binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{3}{2}$, ...
   $\binom{0}{2} + \binom{1}{2}x + \binom{2}{2}x^2 + \binom{3}{2}x^3$ ...
   $-\frac{1}{2} + 0x + \frac{[2]_{(2)}}{[2]_{(2)}}x^2 + \frac{[3]_{(2)}}{[2]_{(2)}}x^3$ ...
   $-\frac{1}{2} + 0x + \frac{[2]_{(2)}}{2}x^2 + \frac{[3]_{(2)}}{2}x^3$ ...

Is this the right process? How do you know when to use EGF vs GF?

3. Given the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$,

   a. Solve the recursion by writing it as a linear homogenous recursion and finding the characteristic polynomial. Write your answer in the form $c_1 {q_1}^n + c_2 {q_2}^n$. (Note: we have already solved this up to finding the constants in class. Finish the problem.)

   $f_n = f_{n-1} + f_{n-2}$

   $0 = f_n$ - $f_{n-1} - f_{n-2}$

   $q^n - q^{n-1} - q^{n-2} = 0$

   $q^{n-2}(q^2 - q^1 - 1) = 0$

   Thus, the solution has the form $f_n = c_1(?)^n \ c_2(?)^n$.

   $q = \frac{1 \pm \sqrt{5}}{2}$

   $f_n = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}$

   $f_0 = c_1 + c_2$

   $f_1 = c_1(\frac{1+\sqrt{5}}{2})^1 + c_2(\frac{1-\sqrt{5}}{2})^1$

   Let $f_0 = 0$, $f_1 = 1$. Solving for c_1 and c_2 gives us $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = \frac{-1}{\sqrt{5}}$

   Thus, $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

   b. Solve the recursion by using generating functions. (Note: Use a partial fraction decomposition to finish the problem.)

   $f_n = f_{n-1} + f_{n-2}$

   $h_n = h_{n-1} + h_{n-2}$

   $0 = h_n - h_{n-1} - h_{n-2}$

   Let $g(x) = h_0 + h_1 x^1 + h_2 x^2$ ...

   Then,

$$
\begin{aligned}
g(x) =& h_0 + h_1 x^1 + h_2 x^2 ... \\
-xg(x) =& -h_0 x^1 - h_1 x^2 - h_2 x^3 ... \\
-x^2 g(x) =& -h_0 x^2 - h_1 x^3 - h_2 x^4 ...
\end{aligned}
$$

   Thus,

$$
(1 - x - x^2)g(x) = h_0 + (h_1 - h_0)x^1 + (h_2 - h_1 - h_0)x^2 + (h_3 - h_2 - h_1)x^3 + ...
$$

   But since $0 = h_n - h_{n-1} - h_{n-2}$,

$$
(1 - x - x^2)g(x) = h_0 + (h_1 - h_0)x^1
$$

$$
g(x) = \frac{h_0 + (h_1 - h_0)x}{(1 - x - x^2)}
$$

   Plugging in $h_0 = 0$ and $h_1 = 1$,

$$
g(x) = \frac{x}{(1 - x - x^2)}
$$

$$
g(x) = \frac{x}{(1 - x - x^2)}
$$

$$
g(x) = \frac{x}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}
$$

$$
g(x) = \frac{A}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})} + \frac{B}{(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}
$$

$$g(x) = \frac{1/2}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})} + \frac{1/2}{(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}$$
$$g(x) = \frac{1}{2(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})} + \frac{1}{2(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}$$
$$g(x) = \frac{1}{2}\frac{1}{((-\frac{1}{2} + \frac{\sqrt{5}}{2}) + x)} - \frac{1}{2}\frac{1}{((-\frac{1}{2} + \frac{\sqrt{5}}{2}) - x)}$$

At this point, I'm not sure how to convert to Power Series
$f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

4. Prove that the Fibonacci number f_n is even if, and only if, divisible by 3.

   Wait.. 2 is a fibonacci number that is even and not divisible by 3.. So is 8.

   $\longrightarrow$

   Assume: f_n is even (i.e. $\exists$ t $\in \mathbb{Z}$ such that f_n = 2t)

   $\longleftarrow$

   Assume: 3 divides f_n (i.e. $\exists$ t $\in \mathbb{Z}$ such that f_n = 3t)

5. Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the colors red, white, or blue. Let h_n be the number of colorings in which there is an even number of red squares (the example from class).

   a. Reproduce the exponential generating function solution from class.

   b. Solve this by using a standard generating function and partial fractions.

   c. Reproduce the associated recursion for h_n.

   d. Using your answer from part c, solve the recursion using the generating function method for non-homogeneous recursions.

6. Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the colors red or blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find a recurrence relation that h_n satisfies, then derive a formula for h_n.

7. Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which apples % 2 = 0, oranges $\leq$ 2, bananas % 3 = 0, and pears $\leq$ 1. Then find a formula for h_n from the generating function.

8. Determine the exponential generating function for the following sequence:

   a. 0!, 1!, 2!, ...

$$g^{(e)}(x) = \frac{0!}{0!} + \frac{1!}{1!}x + \frac{2!}{2!}x^2...$$
$$g^{(e)}(x) = 1 + x + x^2...$$

   b. $[\alpha]_{(\underline{0})}$, $[\alpha]_{(\underline{1})}$, $[\alpha]_{(\underline{2})}$, $[\alpha]_{(\underline{3})}$, ... (Note: $[\alpha]_{(\underline{n})}$ is the falling factorial.)

$$g^{(e)}(x) = \frac{\alpha}{0!} + \frac{\alpha(\alpha-1)}{1!}x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^2...$$
$$g^{(e)}(x) = \sum_{n=0}^{\infty} \frac{\alpha!}{(\alpha-n-1)!n!}$$

9. Let h_n denote the number of ways to color the square of a 1-by-n board with the colors red, white, blue, and green in such a way that the numbers of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence, then find a simple formula for h_n.

   Colors: RWBG. R is even, W is odd.

   EGF:

$$(1+\frac{x^2}{2!}+\frac{x^4}{4!}...)(x+\frac{x^3}{3!}+\frac{x^5}{5!}...)(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}...)^2$$

$$\frac{1}{2}(e^x+e^{-x})*\frac{1}{2}(e^x-e^{-x})*e^x*e^x$$

$$\frac{1}{4}(e^{2x}-e^{-2x})*e^{2x}$$

$$\frac{1}{4}(e^{4x}-1)$$

   $f(x) = \frac{1}{4}(e^{4x}-1) \longrightarrow f'(x) = \frac{1}{4}(4e^{4x}) \longrightarrow f''(x) = \frac{1}{4}16e^{4x} \longrightarrow f'''(x) = \frac{1}{4}64e^{4x} \longrightarrow f^n(x) = \frac{1}{4}(4^n e^{4x})$

   So,

   $h_n = \frac{4^n}{4n!}$ **Which doesn't seem right since n! grows faster than 4^n. Also, where goes - 1?**

10. Determine the number of ways to color the squares of a 1-by-n board using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

    Colors: RGBO. R, G are even.

    GF:

$$(1+x^2+x^4...)^2(1+x+x^2+x^3...)^2$$

$$\frac{1}{1-x^2}\frac{1}{1-x^2}\frac{1}{1-x}\frac{1}{1-x}$$

$$\frac{1}{(1-x)^4(1+x)^2}$$

$$\frac{A}{(1-x)}+\frac{B}{(1-x)^2}+\frac{C}{(1-x)^3}+\frac{D}{(1-x)^4}+\frac{E}{(1+x)}+\frac{F}{(1+x)^2}$$

$$A\sum x^n + E\sum(-1)^n x^n$$

11. Determine the number of n-digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.

12. Solve the recurrence relation:

    a. $h_n = 4h_{n-2}$, $h_0 = 0$, $h_1 = 1$, and $n \geq 2$.

       0, 1, 0, 4, 0, 16, 0, 64...

       $h_n - 4h_{n-2} = 0$

       $q^{n-2}(q^2 - 4) = 0$

       $h_n = a(2)^n + b(-2)^n$

       $0 = a + b$ and $1 = 2a - 2b$

       $b = -\frac{1}{4}$, $a = \frac{1}{4}$

       $h_n = \frac{1}{4}2^n - \frac{1}{4}(-2)^n$

b. $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$. $n \geq 3$.
$q^{n-3}(q^3 - q^2 - 9q^1 - 9) = 0$
$(q^2 - 9)(q^1 + 1) = 0$
$(q - 3)(q + 3)(q + 1) = 0$
$h_n = a(3)^n + b(-3)^n + c(-1)^n$
So, $0 = a + b + c$, $1 = 3a - 3b - c$, $2 = 9a + 9b + c$
Hence, $a = \frac{1}{4}$, $b = 0$, $c = -\frac{1}{4}$
$h_n = \frac{1}{4}(3)^n + -\frac{1}{4}(-1)^n$

c. $h_n = 4h_{n-1} + 4^n$, $h_0 = 3$ and $n \geq 1$.
3, 16, 80, 384...

13. Let h_n = the number of ternary strings of length n made up of 0's, 1's, and 2's, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$h_n = h_{n-1} + 2h_{n-2}$$

with $h_0 = 1$, $h_1 = 3$, and then find a formula for h_n.

14. Compute the Stirling numbers of the first and second kind up to n = 6 using their recursive formulas.
But stirling numbers take 2 parameters: s(p, k); where does n fit?

15. Prove the Stirling numbers of the second kind satisfy:
Recall: $S(p, k) = k\,S(p - 1, k) + S(p - 1, k - 1)$
   a. $S(n, 1) = 1$
   b. $S(n, 2) = 2^{n-1} - 1$
   c. $S(n, n - 1) = \binom{n}{2}$

16. Prove the Stirling numbers of the first kind satisfy:
   a. $s(n, 1) = (n - 1)!$
   b. $s(n, n - 1) = \binom{n}{2}$

17. Write $[n]_{(\underline{k})}$ as a polynomial in n for k = 5, 6, 7. (Do not use distribution!)
$[n]_{(\underline{k})} = n(n - 1)(n - 2)...(n - k)$
$[n]_{(\underline{k})} = \sum_{p=0}^{k}(-1)^{k-p}s(k,p)n^p$
$[n]_{(\underline{5})} = \sum_{p=0}^{5}(-1)^{5-p}s(5,p)n^p$
$[n]_{(\underline{5})} = -s(5,0) + s(5,1)n - s(5,2)n^2 + s(5,3)n^3 - s(5,4)n^4 + s(5,5)n^5$
$[n]_{(\underline{5})} = 4!n - s(5,2)n^2 + s(5,3)n^3 - \binom{5}{2}n^4 + n^5$
$s(5, 2) = 4s(4, 2) + 3!$ and $s(5, 3) = 4\binom{4}{2} + s(4, 2)$

18. Find a closed formula for the sequence: 1, 6, 15, 28, 45, 66, 91, ... (Use a difference table.)

	1	6	15	28	45	66	91
		5	9	13	17	21	25
			4	4	4	4	4
				0	0	0	0

$h_n = 1\binom{n}{0} + 5\binom{n}{1} + 4\binom{n}{2}$