

Lecture 1

Theorem 3.2.8 - pg 118

Let $x, y \in \mathbb{R}$

- a. If $x \leq y + \epsilon \ \forall \ \epsilon > 0$, then $x \leq y$
- b. If $|x - y| \leq \epsilon \ \forall \ \epsilon > 0$, then $|x - y| = 0$ or, evidently, $x = y$

Definition 3.2.9

If $x \in \mathbb{R}$,

$$|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem 3.2.10

Let $x, y \in \mathbb{R}$ and $a \geq 0$

Then

- a. $|x| \geq 0$
- b. $|x| \leq a$ iff $-a \leq x \leq a$
- c. $|xy| = |x||y|$
- d. $|x + y| \leq |x| + |y|$ (equality holds only if signs are the same)

Lecture 2

Theorem 3.3: The Completeness Axiom

Recall the Fundamental Theorem of Arithmetic:

if $n \in \mathbb{N}$ with $n \geq 2$, then n may be expressed as the product of prime numbers (the prime factorization (PF)).

The PF is unique with respect to (WRT) order.

Ex: $12 = 2 * 2 * 2 * 3$

Theorem 3.3.1

Let: p be a prime number

Then $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$

Definition 3.3.7

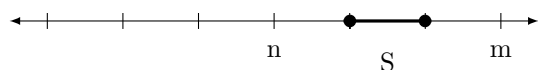
Let $S \subset \mathbb{R}$. If $\exists m \in \mathbb{R}$ st $s \leq m \forall s \in S$,

then m is an upper bound of S and we say that S is **bounded above**.

Similarly, we can define **bounded below**.

If S is bounded above and below, then S is said to be **bounded**.

S can be open or closed. The example below is closed.



If an upper bound m of S is a member of S , then m is called the maximum (or largest element) of S , and we say that $m = \mathbf{\max S}$. Similarly, we may define **minimum** of S ($\mathbf{n = \min S}$).

Theorem 1

If a set $S \subset \mathbb{R}$ possesses a max element, then it is unique. A similar result holds for a minimum element.

Definition 3.3.5 (supremum defined)

Let $\emptyset \neq S \subset \mathbb{R}$ if S is bounded above,

then the **least upper bound** of S is called the **supremum** of S , denoted by $\sup S \in \mathbb{R}$

iff:

a. $s \leq \sup S \forall s \in S$

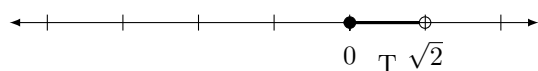
b. $\exists s' \in S$ st $\sup S - \epsilon < s' \forall \epsilon > 0$

Axiom of Completeness of the set of Real Numbers: \mathbb{R}

Every $\emptyset \neq S \subset \mathbb{R}$ that is bounded above has a least upper bound (i.e. $\sup S \in \mathbb{R}$ exists).

A similar statement can be made about $\inf S$.

Remark: In practice 3.3.4, the set $T = \{q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2}\}$ is bounded.



But $\sqrt{2}$ is not rational, so the set wouldn't have a least upper bound.

We need to fill in the gaps to make analysis work.

Lecture 3

Theorem 1 (infimum definition)

Let: $\emptyset \neq S \subset \mathbb{R}$, S is bounded below.

Then S possesses a greatest lower bound denoted by **inf** S (the **infimum** of S), where $\inf S \in \mathbb{R}$, satisfying:

- i) $\inf S \leq s \forall s \in S$
- ii) $\forall \epsilon > 0, \exists s_1$ st $\inf S + \epsilon > s_1$

Theorem 3.3.7

Given nonempty subsets of A, B ($A, B \subset \mathbb{R}$),

Let: $C = \{x + y : x \in A, y \in B\}$

If A and B have suprema, then C has a supremum: $\sup C = \sup A + \sup B$

Theorem 3.3.8

Suppose $\emptyset \neq D \subset \mathbb{R}$ and

$f : D \rightarrow \mathbb{R}$

$g : D \rightarrow \mathbb{R}$

$f(D) = \{f(x) : x \in D\}$

If $\forall x, y \in D, f(x) \leq g(y)$, then

$f(D)$ is bounded above and $g(D)$ is bounded below.

Furthermore, $\sup(f(D)) \leq \inf(g(D))$

Theorem 3.3.9: Archimedian Property / Principle of \mathbb{R} (AP)

The set $\mathbb{N} = \{1, 2, 3, \dots\}$ is unbounded above in \mathbb{R}

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Theorem 3.3.10

Each of the following is equivalent to the AP:

- a. $\forall z \in \mathbb{R}, \exists n \in \mathbb{N} \text{ st } n > z$
- b. $\forall x > 0, y \in \mathbb{R}, \exists n \in \mathbb{N} \text{ st } nx > y$
- c. $\forall x > 0, \exists n \in \mathbb{N} \text{ st } 0 < \frac{1}{n} < x$

Theorems 3.3.13 and 3.3.15

Let: $x, y \in \mathbb{R} \text{ st } x < y$

Then:

- a. $\exists r \in \mathbb{Q} \text{ st } x < r < y$
- b. $\exists z \in \mathbb{R} \setminus \mathbb{Q} \text{ st } x < z < y$

Section 3.4: Topology of \mathbb{R}

Definitions 3.4.1 and 3.4.2

Let $x \in \mathbb{R}$ and $\epsilon > 0$.

- (a) An ϵ -neighborhood of x is: $N(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon\}$
- (b) A deleted ϵ -neighborhood of x is: $N^*(x, \epsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$

Open Set Topology: Definition 3.4.3 (interior / boundary point)

Let: $S \subset \mathbb{R}$

A point $x \in \mathbb{R}$ is an **interior point** of S if $\exists \epsilon > 0 \text{ st } N(x, \epsilon) \subset S$.

If, $\forall \epsilon > 0, N(x, \epsilon) \cap S \neq \emptyset$ and $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$

Then x is a **boundary point** of S .

The set of all interior points is denoted by **int** S .

The set of all boundary points is denoted by **bd** S .

Nota Bene (N.B.):

$\text{int } S \subset S$ and $\text{bd } S = \text{bd } (\mathbb{R} \setminus S)$

Theorem 1

Let: $x \in S \subset \mathbb{R}$

Then either $x \in \text{int } S$, or $x \in \text{bd } S$.

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Definition 3.4.6 - Def of Open/Closed Set

Let: $S \subset \mathbb{R}$

if $\text{bd } S \subset S$, then S is closed.

if $\text{bd } S \subset (\mathbb{R} \setminus S)$, then S is open.

Theorem 3.4.7

a. A set S is open iff $S = \text{int } S$; i.e. iff $\forall s \in S$, s is an **interior point**.

b. A set S is closed iff its complement, $\mathbb{R} \setminus S$ is open.

Equivalently, a set s is open iff $\mathbb{R} \setminus S$ is closed.

Theorem 2 (not in book)

Let: $x \in \mathbb{R}$, $\epsilon > 0$

Then $N(x, \epsilon)$, $N^*(x, \epsilon)$ are open sets.

Theorem 3.4.10

Let: I be an index set. $I \subset \mathbb{N} \subset \mathbb{R}$

Suppose: $G_\alpha \subset \mathbb{R}$ is an open set $\forall \alpha \in I$

Then,

a. $\bigcup_{\alpha \in I} G_\alpha$ is an open set.

b. If $G_i \subset \mathbb{R}$ is open $\forall i = 1, 2, \dots, n \in \mathbb{N}$, then $\bigcap_{i=1}^n G_i$ is open.

Corollary 3.4.11

a. Let F_α be closed $\forall \alpha \in I$, I is an index set.

Then $\bigcap_{\alpha \in I} F_\alpha$ is closed.

b. Let F_i be closed $\forall i$ from 1 to n .

Then $(\bigcup_{i=1}^n F_i)$ is closed.

Accumulation (or Limit) Points; Definition 3.4.14

Let: $S \subset \mathbb{R}$

If $\forall \epsilon > 0$, $N^*(x, \epsilon) \cap S \neq \emptyset$,

Then $x \in \mathbb{R}$ is an **accumulation** or **limit** point. (The set of all accumulation points of S is denoted by S')

If $x \in S \setminus S'$,

then x is an **isolated point**,

in which case, $\exists \epsilon > 0$ st $N(x, \epsilon) \cap S = \{x\}$

Definition 3.4.16 - Closures

Let: $S \subset \mathbb{R}$

Then the **closure** of S , denoted by $\text{cl } S$, is defined to be:

$\text{cl } S = S \cup S'$

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Theorem 3.4.17 - pg 118

Let: $S \subset \mathbb{R}$

Then

- a. S is closed iff $S' \subset S$
- b. $\text{cl } S$ is a closed set
- c. S is closed iff $S = \text{cl } S$
- d. $\text{cl } S = S \cup S' = S \cup \text{bd } S$

Lecture 7

Section 3.5: Compact Sets

Three big areas of analysis: compactedness, continuity, and connectedness.

Definition: Open Cover / Subcover

Let: $S \subset \mathbb{R}$

An **open cover** of S is a collection C of open sets st $S \subset \bigcup C$. The collection C of open sets is said to **cover** the set S .

A subset of sets from the collection C that still covers S is called a **subcover** of S .

Definition 3.5.1

A set $S \subset \mathbb{R}$ is said to be **compact** if every **open cover** has a finite **subcover**

(i.e. if $S \subset \bigcup_{\alpha \in I} G_{\alpha}$,

where G_{α} is open $\forall \alpha \in I$; then $\exists n \in \mathbb{N}$ and $\exists \{n_1, n_2, \dots, n_k\} \subset I$

st $S \subset \bigcup_{i=1}^n G_{\alpha_i}$

Lemma 3.5.4

If $\emptyset \neq S \subset \mathbb{R}$ and S is **closed** and **bounded**, then S has a maximum and a minimum.

In fact, in this, $\max S = \sup S$, and $\min S = \inf S$.

Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq S \subset \mathbb{R}$ is compact iff S is closed and bounded.

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Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq S \subset \mathbb{R}$ is compact iff S is closed and bounded.

Theorem 3.5.6: Bolzano–Weierstrass Theorem

If a bounded set $S \subset \mathbb{R}$ contains an infinite number of points, then \exists at least one point in \mathbb{R} that is an accumulation point of S .

Theorem 3.5.7 (F.I.P.)

Let: $\{K_\alpha\}_{\alpha \in I}$ be a family of compact sets, where I is an index.

Suppose that the intersection of any finite subfamily of the K_α 's has a nonempty intersection.

Then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Corollary 3.5.8 Nested Intervals Theorem

Let: $\{A_n\}_{n=1}^\infty$ be a family of nonempty closed bounded intervals in \mathbb{R} st $A_{n+1} \subset A_n \forall n \in \mathbb{N}$

Then:

$\bigcap_{n=1}^\infty A_n \neq \emptyset$

Lecture 9

Definition 1: Sequence

A **sequence** is a function $S: \mathbb{N} \rightarrow \mathbb{R}$

We write $S(n) = S_n \forall n \in \mathbb{N}$ and refer to $\{S_n\}$ (the book uses (S_n)) as the **sequence**.

We refer to the set $\{S_n : n \in \mathbb{N}\}$ as the range of the sequence.

Side Note

$$S_n = (-1)^n \forall n \in \mathbb{N}$$

$$\{(-1)^n\}$$

$$\text{range}\{S_n\} = \{-1, 1\}$$

$$\text{Here } \{S_n\} = \{1, -1, 1, -1, \dots\}$$

An alternative to writing $\{S_n\}$ for a sequence is to list the elements: S_1, S_2, \dots, S_n

Sometimes the domain of the sequence is $\mathbb{N} \cup \{0\}$ or $\{n \in \mathbb{N} : n \geq m\}$ for some $m \in \mathbb{N}$.

In this case, we write $\{S_n\}_{n=0}^\infty$ or $\{S_n\}_{n=m}^\infty$

Note 1: A denumerable set (or a countably infinite set) S is a set for which there is a bijection $S: \mathbb{N} \rightarrow \mathbb{R}$

This bijection may be thought of as a sequence $\{S_n\}$, where $S_n = S(n) \forall n \in \mathbb{N}$ of distinct terms.

Definition 4.1.2

A sequence $\{S_n\}$ is said to **converge** to $s \in \mathbb{R}$ provided that $\forall \epsilon > 0$

$\exists N \in \mathbb{N} \leq n$ st

$$|S_n - s| < \epsilon$$

Theorem 4.1.8

Let: $\{S_n\}$ and $\{a_n\}$ be sequences, $s \in \mathbb{R}$

If some $k > 0$ and some $m \in \mathbb{N}$, we have:

$$|S_n - s| \leq k|a_n|, \forall n \geq m \quad (1)$$

and if $\lim_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} S_n = s$.

Lecture 10

Theorem 4.1.13

Every convergent sequence is bounded.

Theorem 4.1.14

If a sequence converges, then its limit is unique.

4.2 Limit Theorems

Theorem 4.2.1

Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences with $\lim_{n \rightarrow \infty} s_n = s$ and $\lim_{n \rightarrow \infty} t_n = t$.
Then,

- a. $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$
- b. $\lim_{n \rightarrow \infty} ks_n = ks$ and $\lim_{n \rightarrow \infty} (k + s_n) = k + s$, for any $k \in \mathbb{R}$
- c. $\lim_{n \rightarrow \infty} (s_n t_n) = st$
- d. $\lim_{n \rightarrow \infty} \left(\frac{s_n}{t_n}\right) = \frac{s}{t}$, provided that $t_n \neq 0 \forall n \in \mathbb{N}$ and $t \neq 0$

Theorem 4.2.4

Assume that

$$\lim_{n \rightarrow \infty} s_n = s$$

and

$$\lim_{n \rightarrow \infty} t_n = t$$

If $s_n \leq t_n \forall n \in \mathbb{N}$

then $s \leq t$