Definition of a king:
For any x, either King $\longrightarrow$ x, or King $\longrightarrow$ y $\longrightarrow$ x for some path y.
Prop 1.4.30 - Every tournament has a king
——
A graph is **acyclic** if it has no cycle.
A graph is a **forest** if it is acyclic.
A graph is a **tree** if it is a connected acyclic graph.
¡pictures of trees¿
A **leaf** is a pendant vertex (i.e. a vertex with degree 1)
A **star** is ***
¡picture of a star¿
The **distance**, d(u, v), is the length of the shortest path between two vertices u and v.

Lemma 2.1.3:

Every tree G st $|V(G)| \geq 2$ has ≥ 2 leaves.
Deleting a leaf results in a smaller tree on n - 1 vertices.

Proof.

¡picture of maximal path. i.e. dot-dot-dot-dot¿
No leaf is an internal vertex of a path.
We would use an induction method to prove this:
B *** A(n) $\Rightarrow$ B(n)
A(n): T is a tree on n vertices
B(n): T has n - 1 edges
Want to show: num edges = num vertices - 1
¡picture from top right of Method of Induction page¿
Induction on n:
Step 1:
T′ = T - {a leaf}
T′is a tree on n - 1 vertex
Step 2:
T′has n - 2 edges (induction hypothesis)
Step 3:
T = T′+ {an edge}
T has n - 2 + 1 = n - 1 edges.

□

Theorem 2.1.A (or 4?)

a. connected, no cycle. n vertices (do I have n - 1 edges?)

b. connected, n - 1 edges

c. n - 1 edges, no cycle (not sure if connected)

d. For any u, v $\in$ V, $\exists$ exactly one u, v - path. No loops.

Proof.

We're going to say these three things are equivalent.
We did A $\Rightarrow$ B in previous slides. (induction on n)
For B $\Rightarrow$ C:
Want to show: G has no cycles
Suppose G has cycles (contradiction):
¡picture from Theorem 2.1.A (or 4)
G′ = G - {e_1, e_2, ...} is acyclic
acyclic, connected, n - 1 vertices = tree
G′is connected, (using any tree that has n vertices has n - 1 edges), G′has n - 1 edges
C $\Rightarrow$ A (if you have 3 and 2, then prove you have 1):
Suppose c(G) (number of components) = k (by contradiction).
¡pictures of n_1 vertices, n_2 vertices.. n_k vertices¿ ¡has n_1 - 1 edges, n_2 - 1 edges, etc...¿

$$n - 1 = e(G) = \sum_{i=1}^{k}(n_1 - 1) = \sum_{i=1}^{k}(n_i - k) = n - k$$

The only solution is that k = 1.

□

Corollary 2.1.5

a. Every edge of a tree is a cut-edge.

b. Adding one edge to a tree forms exactly one cycle.

c. Every connected graph contains a spanning tree.

A spanning subgraph of G is a subgraph of G that contains all the vertices of G.
A spanning tree is a spanning subgraph that is a tree.

Proposition 2.1.8 (or B)

Tree T has k edges, simple graph G has min(G) $\geq$ k (minimum degree bigger than or equal to k) $\longrightarrow$ T is a subgraph of G.
T′ = T - {a leaf} has k - 1 edges.
¡picture of G¿
To prove this, we would use induction on k.
min vertex(G) $\geq$ k $\geq$ k - 1
T′has k vertices.
Base: k = 1
If T has only 2 vertices, then T has 1 edge. This is a trivial case.
¡Missing some other stuff¿

Definition 2.1.9

eccentricity (for any connected graph) ϵ (u) = max{d(u, v) : v $\in$ V(G)}
¡picture below eccentricity¿ (where 4 is the radius, 7 is the diameter)
The radius, rad(G), is the minimum *** = min of ϵ (u) where u $\in$ V
The diameter, diam(G), is the maximum *** = max of ϵ (u) where u $\in$ V
ϵ (u) = d(u, v) for some leaf v

Theorem 2.1.13 (Jordan, 1869)

The center of a tree is always one edge or one vertex.

Proof.

We do induction on n.
Let: T′ = T - {all leaves}
$\epsilon_{T'}(\mathrm{u}) = \epsilon_{T}(\mathrm{u})$ - 1
If G ≠ a line segment with a vertex at each end, then no leaf can be a center vertex.

□

Theorem 2.1.10

Not on test

Theorem 2.1.11

G is simple, Diam(G) ≥ 3 $\longrightarrow$ Diam($\overline{G}$) ≤ 3

Proof.

Claim: x cannot be adjacent to both u and v, otherwise distance will be smaller than 3. So, at least one of them is not true.
Dotted lines signify non-adjacency.
In the case of neither x nor y being adjacent to u, then in $\overline{G}$, u is adjacent to both x and y.
In the case

□