Final is **not** cumulative! Covers this material and onward.

Chapter 5 - Limits and Continuity

5.1.1: Definition (Limits of Functions)

Let: $D \subset \mathbb{R}$, $f : D \longrightarrow \mathbb{R}$, and $c \in D'$ (i.e. c is an accumulation point)
We say that $L \in \mathbb{R}$ is a **limit** of f at c if, $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ st when $x \in D$ and $0 < |x - c| < \delta$, then

$$|f(x) - L| < \epsilon$$

(i.e. the limit as x goes to c of f(x) = L)

$$x \pm c$$
$$-\delta < x - c < \delta$$
$$c - \delta < x < c + \delta$$

Recall the definition of a limit:

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

where x is fixed.

Theorem 5.1.2

Let: $f : D \longrightarrow \mathbb{R}$, $c \in D'$
Then:
The $\lim_{x \to c}$ f(x) = L exists iff for each neighborhood V of L, $\exists$ a deleted neighborhood U of c st
f(U ∩ D) ⊂ V.

Proof.
$\longrightarrow$
Suppose $\lim_{x \to c}$ f(x) = L.
Then,
for each neighborhood V of L (i.e. for each $\epsilon > 0$, V = N(L, ϵ)), $\exists$ a deleted neighborhood U of c
(i.e. $\exists\ \delta(\epsilon) > 0$ st N*(c, δ) = U)
st f(U ∩ D) ⊂ V
$\longleftarrow$
The converse is similar.
Remember: definitions are iff

<div style="text-align: right;">□</div>

Example 5.1.3

Let: $k \in \mathbb{R}$
Define f : $\mathbb{R} \longrightarrow \mathbb{R}$ by f(x) = k, $\forall$ x $\in \mathbb{R}$
Let c $\in \mathbb{R}$
Show that $\lim_{x \to c}$ f(x) = k
Solution:
For each $\epsilon > 0$,

$$|f(x) - k| = |k - k| = 0 < \epsilon$$

whenever $0 < |\text{x - c}| < \epsilon$

Example 5.1.4

Confirm that $\lim_{x \to c} f(x) = c$ for the function f(x) = x, where $c \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$
Solution:
For each $\epsilon > 0$,

$$|f(x) - c| = |x - c| < \epsilon$$

whenever $0 < |\text{x - c}| < \delta = \epsilon$

Theorem 5.1.8

Let: $f : D \longrightarrow \mathbb{R}$, $c \in D'$
Then,
$\lim_{x \to c} f(x) = L$ iff for **every** sequence $\{s_n\}$ in D st $s_n \neq c$, $\forall n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = c$,
it follows that the sequence $\{f(s_n)\}$ converges to L.
(i.e. the values of s_n eventually get within a δ neighborhood of c)

Proof.
$\longrightarrow$
Assume: $\lim_{x \to c} f(x) = L$
Let: $\{s_n\}$ be a sequence in D st $s_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = c$
Want to show: $\lim_{n \to \infty} f(s_n) = L$
Now, $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ st

$$|f(x) - L| < \epsilon \tag{1}$$

whenever $0 < |\text{x} - \text{c}| < \delta$ **and x $\in$ D** (we need this part so that $|f(x) - L|$ makes sense).
I'd like to know that $|f(s_n) - L|$ gets close to 0, so:
Since $\lim_{n \to \infty} s_n = c$, $\exists N \in \mathbb{N}$ st

$$0 < |s_n - c| < \delta \tag{2}$$

for n $\geq$ N
From **(1)** and **(2)**,

$$|f(s_n) - L| < \epsilon$$

for n $\geq$ N
(if we think of $f(s_n)$ as our t_n, where $t_n \longrightarrow L$ as n $\longrightarrow \infty$)
By definition,
$\lim_{n \to \infty} f(s_n) = L$

$\longleftarrow$

Conversely, using the contrapositive,

Assume: $\lim\limits_{x \to c}$ f(x) does not exist.

───Side Note───

Negating that:

$\exists$ L $\in \mathbb{R}$ st

$\lim\limits_{x \to c}$ f(x) = L

$\forall \epsilon > 0, \exists \delta > 0$ st

$\forall$ x st $0 < |x - c| < \delta,$

$|f(x) - L| < \epsilon$

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Thus,

for each L $\in \mathbb{R}$, $\exists \epsilon_0 > 0$ st

$\forall \delta > 0, \exists$ x st $0 < |x - c| < \delta$ st

$$|f(x) - L| \geq \epsilon_0$$

───Side Note───

First we proved p $\Rightarrow$ q.

Now we're going to prove q $\Rightarrow$ p by proving:

not p $\Rightarrow$ not q

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Want to show: $\exists$ a sequence $\{s_n\}$ in D st $s_n \neq c, \forall$ n $\in \mathbb{N}$ and $\lim\limits_{n \to \infty} s_n = c$

but, $\{f(s_n)\}$ to fail to converge to L.

Let: $\delta_n = \frac{1}{n}$

Now, for each n $\in \mathbb{N}$, $\exists s_n \in$ D st

$$0 < |s_n - c| < \frac{1}{n} \text{ and } |f(s_n) - L| \geq \epsilon_0 \tag{3}$$

Notice that $s_n \neq c, \forall$ n $\in \mathbb{N}$ and $\lim\limits_{n \to \infty} s_n = c$.

───Side Note───

Is $\lim\limits_{n \to \infty}$ f(s_n) = L?

For $\epsilon = \frac{\epsilon_0}{2} > 0,$

$\exists$ N $\in \mathbb{N}$ st

$$|f(s_n) - L| < \frac{\epsilon_0}{2}$$

for n $\geq$ N

So, no. $\lim\limits_{n \to \infty}$ f(s_n) $\neq$ L

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From **(3)**, $\lim\limits_{n \to \infty}$ f(s_n) $\neq$ L

(page 166, Theorem 4.1.8 says

$|s_n - s| \leq k|a_n|$ for n $\geq$ N

and

if $\lim\limits_{n \to \infty} a_n = 0$, then $\lim\limits_{n \to \infty} s_n = s$)

$\square$

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