Exercise 1

Let A = {0, 1, 2, 3, 4} and B = {0, 1, 2, 3}. For each of the relations R from A to B listed below list all pairs (a, b) $\in \mathbb{R}$ and write the corresponding {0, 1}-indicator-matrix.

a. a = b : (0, 0), (1, 1), (2, 2), (3, 3)

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

b. a + b = 4 : (1, 3), (2, 2), (3, 1), (4, 0)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

c. a > b : (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

d. a divides b : (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1, 2), (2, 2), (1, 3)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Exercise 2

For each of these relations on the set {1, 2, 3, 4} decide whether or not it is reflexive, symmetric, antisymmetric, and transitive.

a. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

b. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

c. {(2, 4), (4, 2)}

d. {(1, 2), (2, 3), (3, 4)}

e. {(1, 1), (2, 2), (3, 3), (4, 4)}

f. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Relation	*R*	*S*	*A*	*T*
a	0	0	0	1
b	1	1	0	1
c	0	1	0	1
d	0	0	1	0
e	1	1	1	1
f	0	0	0	1

Exercise 3

Let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)}, and let S be the relation {(2, 1), (3, 1), (3, 2), (4, 2)} on the set A = {1, 2, 3, 4}

a. Find R ∪ S

   {(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)}

b. Find R ∩ S

   {(3, 1)}

c. Find R o S

   {(1, 1), (1, 2), (2, 1), (2, 2)}

Exercise 4

Let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} on the set A = {1, 2, 3, 4}.

a. Find the reflexive closure of R.

   {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 1), (3, 3), (4, 4)}

b. Find the symmetric closure of R.

   {(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)}

c. Find the transitive closure of R.

   {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (1, 4)}

Exercise 5

Prove the following:

a. A relation R is reflexive iff R^{-1} is reflexive (where R^{-1} is the inverse relation that just reverses the order).

$\longrightarrow$

Assume R is reflexive.

Let $(a, a) \in R$

Then $(a, a) \in R^{-1}$

Hence, R^{-1} is reflexive.

$\longleftarrow$

Assume R^{-1} is reflexive.

Let $(a, a) \in R^{-1}$

Then $(a, a) \in R$

Hence, R is reflexive.

b. A relation R is symmetric iff $R = R^{-1}$.

$\longrightarrow$

Assume R is symmetric.

Let $(a, b) \in R$.

Want to show: $(a, b) \in R^{-1}$.

Notice: $(b, a) \in R$.

Thus, $(a, b) \in R^{-1}$.

Hence, $R = R^{-1}$.

$\longleftarrow$

Assume $R = R^{-1}$.

Let $(a, b) \in R$.

Then $(a, b) \in R^{-1}$.

$(a, b) \in R \ \Rightarrow (b, a) \in R^{-1}$.

But since $R^{-1} = R$, $(b, a) \in R$.

So, $(a, b) \in R \ \Rightarrow (b, a) \in R$.

Hence, R is symmetric..

c. A relation R is anti-symmetric iff $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$

$\longrightarrow$

Assume R is anti-symmetric.

Then $(a, b), (b, a) \in R \ \Rightarrow a = b$.

So, $R \cap R^{-1}$ will only contain tuples such that $a = b$.

$\longleftarrow$

Assume $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$.

Let $(a, b) \in R$. If $a \neq b$, then $(a, b) \notin R \cap R^{-1}$. Thus, $(a, b) \notin R^{-1}$.

Hence, R is anti-symmetric.

Exercise 6

Let R be the relation represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the matrices for the relations:

a. R^2

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b. R^3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c. R^4

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 7

Which of these relations on {0, 1, 2, 3} are equivalence relations? If they are not, why?

a. {(0, 0), (1, 1), (2, 2), (3, 3)}

   Yes.

b. {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

   No, (1, 1) isn't in there.

c. {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

   Yes.

d. {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

   No, (1, 2) isn't in there.

e. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

   Yes.

Exercise 8

List the ordered pairs in the equivalence relations produced by these partitions of {0, 1, 2, 3, 4, 5}.

a. {0}, {1, 2}, {3, 4, 5}

   (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (3, 4), (4, 5), (3, 5), (5, 3), (4, 3)...

b. {0, 1}, {2, 3}, {4, 5}

c. {0, 1, 2}, {3, 4, 5}

d. {0}, {1}, {2}, {3}, {4}, {5}

Exercise 9

Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? If they are not, why?

a. $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
   Yes.

b. $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
   No: $(0, 2)$ and $(2, 0)$ are both in there.

c. $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
   No: $(1, 2)$ and $(2, 1)$ are both in there.

d. $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   No: $(1, 3)$ and $(3, 1)$ are both in there.

e. $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$
   No: $(0, 1)$ and $(1, 0)$ are both in there.

Exercise 10

Answer these questions for the divides poset $(\{3, 5, 9, 15, 24, 45\}; |)$.

a. Draw the Hasse diagram

b. List the maximal and minimal elements

c. Is there a greatest element? A least element?

d. Find all upper bounds of $\{3, 5\}$. Find the least upper bound of $\{3, 5\}$, if it exists.

e. Find all the lower bounds of $\{15, 45\}$. Find the greatest lower bound of $\{15, 45\}$, if it exists.

Exercise 11

Prove the following:

a. There is exactly one greatest element of a poset, if such an element exists.

b. There is exactly one maximal element in a poset with a greatest element.

c. The least upper bound of a set in a poset is unique if it exists.

Exercise 12

Determine whether these posets are lattices.

a. $(\{1, 3, 6, 9, 12\}; |)$

b. $(\{1, 5, 25, 125\}; |)$

c. $(\mathbb{Z} ; \geq)$

d. $(\mathcal{P}(S), \subset)$, where $\mathcal{P}(S)$ is the power set of a set S.

Exercise 13

Show that every totally ordered set is a lattice.

Exercise 14

Show that every finite lattice has a least element and a greatest element.

Exercise 15

Give an example of an infinite lattice with

a. neither a least nor a greatest element.

b. a least but not a greatest element.

c. a greatest but not a least element.

d. both a least and a greatest element.

Exercise 16

Show that in any lattice $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. Note: $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$ was shown in class.)

Exercise 17

Show that in any lattice $x \vee (x \wedge y) = x$. Note: the dual absorption law was shown in class.

Exercise 18

Show that any lattice $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$. Note: the dual distributive inequality was shown in class.

Exercise 19

Show that the two distributive equalities are equivalent. That is, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ if, and only if, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Exercise 20

Show that the distributive law implies the modular law. That is, if a lattice satisfies one (hence both, from problem 19), then $(x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z)$.

Exercise 21

Check if the lattice N_5 is distributive.