

HW 8: pages 193, #1, 2, 3, 5, 9, 10, 17

For 2(c), see Theorem 1 and Example 9 from Lecture 15

Make sure when you do these problems, justify the answer by either writing down the theorem name or providing a counter example.

## Exercise 1

Mark each statement True or False. Justify each answer.

- a. A sequence  $(s_n)$  converges to  $s$  iff every subsequence of  $(s_n)$  converges to  $s$ .

**True.** By Theorem 4.4.4.

- b. Every bounded sequence is convergent.

**False.**

Counter example:  $(s_n) = (-1)^n$

- c. Let  $(s_n)$  be a bounded sequence. If  $(s_n)$  oscillates, then the set  $S$  of subsequential limits of  $(s_n)$  contains at least two points.

**True.** If  $S$  oscillates, then  $\liminf S < \limsup S$ . This implies that these are two different points.

- d. Let  $(s_n)$  be a bounded sequence and let  $m = \limsup s_n$ .

Then,  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  st  $N \geq n$  implies  $s_n > m - \epsilon$

**True.**

*Proof.*

**Let:**  $\epsilon > 0$

Since  $s_n$  is bounded, let  $S$  be the set containing the range of  $s_n$ .

By definition,  $\exists$  some  $s_{n_k}$  st  $\lim s_{n_k} = m$  where  $k \in \mathbb{N}$

Since  $\lim s_{n_k} = m$ ,

$\exists N \in \mathbb{N}$  st  $N \geq n_k$  implies  $|s_{n_k} - m| < \epsilon$

$|s_{n_k} - m| < \epsilon$

$-\epsilon < s_{n_k} - m < \epsilon$

$m - \epsilon < s_{n_k} < m + \epsilon$  **(1)**

So, by **(1)**,

$\exists$  some  $N \in \mathbb{N}$  st  $n \geq N$  implies  $s_n > m - \epsilon$

□

- e. If  $(s_n)$  is unbounded above, then  $(s_n)$  contains a subsequence that has  $\infty$  as a limit.

**True.** By Theorem 4.4.8.

## Exercise 2

Mark each statement True or False. Justify each answer.

- a. Every sequence has a convergent subsequence.

**False.** Let  $s_n = n$

- b. The set of subsequential limits of a bounded sequence is always nonempty.

**True.** By Theorem 4.4.8

- c.  $(s_n)$  converges to  $s$  iff  $\liminf s_n = \limsup s_n = s$

**True.** By Definition 4.4.9 and exercise 9.

- d. Let  $(s_n)$  be a bounded sequence and let  $m = \limsup s_n$ . Then,  $\forall \epsilon > 0$ , there are infinitely many terms in the sequence greater than  $m - \epsilon$ .

**True.** By Theorem 4.4.7,  $s_n$  has a convergent subsequence.

Let  $t_n$  be a subsequence of  $s_n$  st  $\lim_{n \rightarrow \infty} t_n = m$

By definition,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  st  $n \geq N$  implies  $|t_n - m| < \epsilon$

so,

$-\epsilon < t_n - m < \epsilon$

$m - \epsilon < t_n$

Pick  $\epsilon_2$  to be  $\frac{\epsilon}{2}$

Then,

$\exists N(\epsilon_2)$  st  $m - \epsilon < t_{N(\epsilon_2)}$

Inductively, we can let  $\epsilon_3 = \frac{\epsilon_2}{2}$ , and so on.

Hence, since there are infinitely many terms in  $t_n$  greater than  $m - \epsilon$ , the same is true for  $s_n$ .

- e. If  $(s_n)$  is unbounded above, then  $\liminf s_n = \limsup s_n = \infty$

**True.**

**Suppose:**  $s_n$  has a subsequence  $t_n$  such that  $\lim_{n \rightarrow \infty} t_n = t$  where  $t \neq \infty$  (but could be negative infinity)

So,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  st  $n \geq N$  implies  $|t_n - t| < \epsilon$

Notice also, that since  $s_n$  is unbounded above,

$\forall m \in \mathbb{R}, \exists N_m \in \mathbb{N}$  st  $s_{N_m} > m$

That means that  $\exists$  some  $N$  for  $t_n$  st  $t_N > m$

If we let  $m = t$ , then

$\exists$  some  $N_1$  for  $t_n$  st  $t_{N_1} > t = m$

If we let  $m = t + 1$ , then

$\exists$  some  $N_2$  for  $t_n$  st  $t_{N_2} > m = t + 1$

Inductively,  $t_n$  has an infinite amount of values above  $t$ , and is increasing: a contradiction.

Thus,  $t_n$  is unbounded above.

### Exercise 3

For each sequence, find the set  $S$  of subsequential limits, the limit inferior, and the limit superior.

a.  $s_n = 1 + (-1)^n$

$$S = \{0, 2\}, s_* = 0, s^* = 2$$

b.  $t_n = (0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7})$

$$S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7}\}, s_* = 0, s^* = \frac{6}{7}$$

c.  $u_n = n^2(-1 + (-1)^n)$

$$S = \{0\}, s_* = -\infty, s^* = 0$$

d.  $v_n = n \sin \frac{n\pi}{2}$

$$S = \{0\}, s_* = -\infty, s^* = \infty$$

### Exercise 5

Use exercise 4.3.14 to find the limit of each sequence:

**Known:**  $t_n = (1 + \frac{1}{n})^n$  and  $\lim_{n \rightarrow \infty} t_n = e$

a.  $s_n = (1 + \frac{1}{2n})^{2n}$

We can just think of  $s_n$  as a subsequence of  $t_n$  (the original  $e$  sequence),  
so therefore it has the same limit:  $e$ .

b.  $s_n = (1 + \frac{1}{n})^{2n}$

$$= ((1 + \frac{1}{n})^n)^2$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = e^2$$

c.  $s_n = (1 + \frac{1}{n})^{n-1}$

$$= (1 + \frac{1}{n})^n (1 + \frac{1}{n})^{-1}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = e * 1 = e$$

d.  $s_n = (\frac{n}{n+1})^n$

$$= \frac{1}{(\frac{n+1}{n})^n}$$

$$= \frac{1}{(1 + \frac{1}{n})^n}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = \frac{1}{e}$$

e.  $s_n = (1 + \frac{1}{2n})^n$

$$= ((1 + \frac{1}{2n})^{2n})^{\frac{1}{2}}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = \sqrt{e}$$

f.  $s_n = (\frac{n+2}{n+1})^{n+3}$

$$= (\frac{n+2}{n+1})^n (\frac{n+2}{n+1})^3$$

$$= (\frac{n}{n+1} + \frac{2}{n+1})^n (\frac{n+2}{n+1})^3$$

$$\text{Now, } \lim_{n \rightarrow \infty} (\frac{n}{n+1} + \frac{2}{n+1})^n (\frac{n+2}{n+1})^3 = (e + 0) \times 1 \text{ by (d)}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = e$$

## Exercise 9

Let  $(s_n)$  be a bounded sequence.

**Assume:**  $\liminf s_n = \limsup s_n = s$

Prove that  $(s_n)$  is convergent and that  $\lim s_n = s$

Let  $S \subset \mathbb{R}$  be the range of limits for any subsequence of  $s_n$ .

Since  $\liminf s_n = s$ ,  $\inf S = s$ .

Since  $\limsup s_n = s$ ,  $\sup S = s$ .

By Corollary 4.4.12,  $S$  contains  $s$ .

Since  $\inf S = \sup S = s$ , the range of  $S$  is just  $\{s\}$ . (1)

Since  $s_n$  is bounded, it can't diverge to  $\infty$  or  $-\infty$ .

However, suppose  $s_n$  diverges in general.

Then,  $\exists \epsilon (s_n)$  st  $|s_n - s| > \epsilon (s_n)$  for all  $n \geq$  some  $N \in \mathbb{N}$

Since there are infinitely many  $n \geq N$ ,  $\exists s_{n_k}$  (a subsequence of  $s_n$ ) st

$|s_{n_k} - s| \geq \epsilon (s_n)$  where  $n_k = N + k$ ,  $k \in \mathbb{N}$

Since  $s_{n_k}$  is bounded (because  $s_n$  is bounded), it itself has a convergent subsequence (for notation reasons let's call it  $t_{n_k}$ )

Notice that  $t_{n_k}$  is a convergent subsequence of  $s_n$ , but its limit is not  $s$  (since  $\exists$  an  $\epsilon$  st  $|s_n - s| > \epsilon$ ), a contradiction.

Hence,  $s_n$  must converge to  $s$ .

## Exercise 10

**Assume:**  $x > 1$

Prove that  $\lim x^{\frac{1}{n}} = 1$

$\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1$  if

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  st  $N \geq n$  implies  $|x^{\frac{1}{n}} - 1| < \epsilon$

**Let:**  $\epsilon > 0$

$|x^{\frac{1}{n}} - 1| < \epsilon$

Since  $x > 1$  and  $n \in \mathbb{N}$ ,

$x^{\frac{1}{n}} - 1 < \epsilon$

$x^{\frac{1}{n}} < \epsilon + 1$

$(x^{\frac{1}{n}})^n < (\epsilon + 1)^n$

$x < (\epsilon + 1)^n$

$\ln x < n \ln (\epsilon + 1)$

$\frac{\ln x}{\ln (\epsilon + 1)} < n$

So, if  $\frac{\ln x}{\ln (\epsilon + 1)} < N$ ,

then  $\exists N$  st  $|x^{\frac{1}{n}} - 1| < \epsilon$

Hence, result.

## Exercise 17

Prove that if  $\limsup s_n = \infty$  and  $k > 0$ , then  $\limsup (ks_n) = \infty$

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Side Note

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**Question:** Is it a valid proof to say that since

$$t_n = \sum_{i=1}^n \frac{1}{n}$$

is the slowest possible diverging sequence (without constants of course),  
since

$$\lim_{n \rightarrow \infty} kt_n = k\infty = \infty$$

then  $\lim_{n \rightarrow \infty}$  of  $k \times$  any sequence diverging to  $\infty$  is also  $\infty$ ?

So, therefore  $\limsup (k * \text{any sequence diverging to } \infty)$  is also  $\infty$ ?

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**Let:**  $t_n$  be a subsequence of  $s_n$  st  $\lim_{n \rightarrow \infty} t_n = \infty$

Algebraically,

$$k \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} ks_n = \lim_{n \rightarrow \infty} ks_1, ks_2, ks_3 \dots ks_n = k\infty = \infty$$

Since the limit of any subsequence is the same as the limit of the sequence, and by Theorem 4.4.14,

$$k \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} kt_n = \lim_{n \rightarrow \infty} kt_1, kt_2, kt_3 \dots kt_n = k\infty = \infty$$

So, since  $kt_n$  is a subsequence of  $ks_n$ ,

$$\limsup (ks_n) = \infty$$