Theorem 4.3.8

a. If $\{s_n\}$ is an unbounded increasing sequence, then $\lim_{n\to\infty} s_n = \infty$

b. If $\{s_n\}$ is an unbounded decreasing sequence, then $\lim_{n\to\infty} s_n = -\infty$

Proof.

(a)
Since $s_1 \leq s_n \; \forall \; n \in \mathbb{N}$
Thus, if $\{s_n\}$ is unbounded, then it must be unbounded above.
Thus, for $m \in \mathbb{R}$, $\exists \; N \in \mathbb{N}$ st $s_N > m$
Because it's increasing,
$s_n \geq s_N > m$ for $n \geq N$
This is the definition of
Hence,
$\lim_{n\to\infty} s_n = \infty$

(b) is similar.

□

Cauchy Sequences

Definition 4.3.9

A sequence $\{s_n\}$ is **Cauchy** if $\forall \; \epsilon > 0$, $\exists \; N(\epsilon) \in \mathbb{N}$ st

$$|s_n - s_m| < \epsilon, \text{ for m, n } \geq N$$

Lemma 4.3.10

Every convergent sequence is Cauchy.

Proof.

Let: $\{s_n\}$ converge to s.
$\forall \; \epsilon > 0$, $\exists \; N(\epsilon) \in \mathbb{N}$ st

$$|s_n - s| < \frac{\epsilon}{2}, \text{ for n } \geq N$$

Then

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for n, m } \geq N$$

Hence, $\{s_n\}$ is Cauchy. □

Lemma 4.3.11

Every Cauchy sequence is bounded (similar to exam question: Every convergent sequence is bounded)

Proof.

This also appeared in a similar context in the HW: Example 13, page 186

□

Theorem 4.3.12 - Cauchy Convergence Criterion

A sequence of real numbers is convergent iff it is a Cauchy sequence.

Proof.

$\longrightarrow$
Assume that $\{s_n\}$ is convergent.
Then, by Lemma 4.3.10, $\{s_n\}$ is Cauchy.
$\longleftarrow$
Conversely, assume that $\{s_n\}$ is Cauchy.
Want to show: $\{s_n\}$ converges
Let: $S = \{s_n : n \in \mathbb{N}\}$ be the range of $\{s_n\}$

i) S is finite.

Thus, $\exists$ k $\in \mathbb{N}$ and $\{n_1, n_2, \dots n_k\} \subset \mathbb{N}$ st

$$S = \{s_{n_1}, s_{n_2}, ...s_{n_k}\}$$

Define m:

$$m = \{|s_{n_i} - s_{n_j}| : 1 \leq i \leq j \leq k\}$$
$$m = \{|s_{n_i} - s_{n_j}| : i, j, \in \{n, k\} \text{ and } i \neq j\}$$

Now, for $\epsilon = \frac{m}{2}$, $\exists$ $N(\epsilon) \in \mathbb{N}$ st

$$|s_n - s_m| < \frac{m}{2}, \text{ for n, m } \geq N$$

In particular,

$$|s_n - s_N| < \frac{m}{2} \text{ for } n \geq N \tag{1}$$

Now, $\exists$ $l \in \{1, 2, \dots k\}$ st $s_N = s_{n_l}$

Thus, **(1)** implies that

$$|s_n - s_{n_l}| < \frac{m}{2} \text{ for } n \geq N$$

Thus, $s_n = s_{n_l}$ $\forall$ $n \geq N$

Hence, $\lim\limits_{n\to\infty} s_n = s_{n_l}$

ii) S is infinite.

□

Side Note

Better ratio test:
$\{s_n\}$ is a sequence.
Test:
$\lim\limits_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = L < 1$
If $\lim\limits_{n\to\infty} |s_n| = 0$,
$\lim\limits_{n\to\infty} |s_n| = 0 = \lim\limits_{n\to\infty} |s_n - 0| = \lim\limits_{n\to\infty} s_n = 0$
The reason is because, if you're not careful, you can conclude something like, say, $\lim\limits_{n\to\infty} s_n = (-2)^n = 0$
$s_n = (-2)^n$
$\frac{s_{n+1}}{s_n} = \frac{(-2)^{n+1}}{(-2)^n} = -2 < 1$
which would tell you, in theory, that the limit is 0. Which is **not** true.