Section 3.5: Compact Sets

Three big areas of analysis: compactedness, continuity, and connectedness.

Definition 3.5.1

A set s $\subset \mathbb{R}$ is said to be compact if every **open cover** has a finite **subcover**
(i.e. if S $\subset \bigcup_{\alpha \in I}$ G_α) ,
where G_α is open $\forall$ $\alpha \in$ I; then $\exists$ n $\in \mathbb{N}$ and $\exists$ {n_1, n_2, ... n_k} $\subset$ I
st S $\subset \bigcup_{i=1}^{n} G_{\alpha_i}$

Example 3.5.2

a. Show that S = (0, 2) is not compact.

b. Show that S = {x_1, x_2, ... x_n} $\subset \mathbb{R}$ is compact.

(a)
Notice that:

$$(0,2) \subset \bigcup_{n=1}^{\infty} (\frac{1}{n}, 3) \tag{1}$$

If (0,2) were compact, then from **(1)** there would exist a **finite** subcover.
Assume: (0, 2) is compact.
So $\exists$ k $\in \mathbb{N}$ and {n_1, n_2, ... n_k} $\subset \mathbb{N}$ $_k$ st

$$(0,2) \subset \bigcup_{i=1}^{k} (\frac{1}{n_i}, 3) \tag{2}$$

Choose m = max {n_1, n_2, ... n_k}
Then, notice that ($\frac{1}{n_i}$, 3) $\subset$ ($\frac{1}{m}$, 3) $\forall$ i = 1, 2, ... k
From **(1)**, (0, 2) $\subset$ ($\frac{1}{m}$, 3).
Notice that $0 < \frac{1}{m+1} < \frac{1}{m}$
and $\frac{1}{m+1} \in$ (0, 2).
However, $\frac{1}{m+1} \notin$ ($\frac{1}{m}$, 3).
(b)
Suppose that S $\subset \bigcup_\alpha G_\alpha$ $(\alpha \in I)$
where I is an index set and G_α is open $\forall\alpha \in$ I.
$\forall i = 1, 2, ...$ n, $\exists \alpha_i \in$ I st $x_i \in G_\alpha$ (α_i).
Then, S $\subset \bigcup_i^n G_\alpha$ (α_i).
We see that a **finite** subset of $\mathbb{R}$ is compact.

Lemma 3.5.4

If $\emptyset \neq S \subset \mathbb{R}$ and S is **closed** and **bounded**, then S has a maximum and a minimum. In fact, in this, max S = sup S, and min S = inf S.

Proof.

Since S is bounded, inf S, sup S $\in \mathbb{R}$ both exist. **Want to show:** max S = sup S
For $\epsilon > 0$, $\exists s_1(\epsilon) \in$ S st
sup S $-\epsilon < S_1 \leq$ sup S $<$ sup S $+\epsilon$.

So,
$-\epsilon < s_1 - supS \leq \epsilon$
Thus, $s_1 \in$ N(sup S, ϵ).
So,

$$\text{N}(\text{sup S}, \epsilon) \ \cap \ \text{S} \neq \emptyset \tag{1}$$

Also, sup S $+\frac{\epsilon}{2} \in$ N(sup S, ϵ) and sup S $+\frac{\epsilon}{2} \in \mathbb{R} \setminus S$.
(s $\leq$ sup S $\forall$ s $\in$ S, and sup S $\in$ S)
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From **(1)** and **(2)**, sup S $\in$ bd S $\subset$ S, since S is closed. Hence, sup S = max S.

□

Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq$ S $\subset \mathbb{R}$ is compact iff S is closed and bounded.

Proof.
$\longrightarrow$
Suppose: S is compact
Want to show: S_∞ is bounded
Notice that S $\subset$ From n=1 to ∞, $\bigcup$ ($-$n, n) = $\mathbb{R}$,
where ($-$n, n) = N(0, n) is open $\forall n \in \mathbb{N}$.
$\text{G}_n \subset$ G sub n + 1 $\forall n \in \mathbb{N}$.
Since S is compact, $\exists k \in \mathbb{N}$ and {n_1, n_2, ... n_k} $\subset \mathbb{N}$ st
S $\subset$ from i=1 to k $\bigcup$ ($-\text{n}_i$, n_i)
Let: m = max {n_1, n_2, ... n_k}.
Then, ($-\text{n}_i$, n_i) $\subset$ ($-$m, m) $\forall i = 1, 2, ...k$.
Thus, S $\subset$ ($-$m, m).
So, $|S| <$ m, $\forall s \in$ S. Or, equivalently,
$-$m < s < m, $\forall$ s $\in$ S.
Hence, S is bounded.
Want to show: S is closed
Suppose: S is not closed
Thus, $\exists$ p $\in$ cl S $\setminus$ S, i.e. p $\in$ s$'$.
———-
S is closed iff cl S = S $\cup$ S$'$ = S
S $\subset$ S $\cup$ S$'$
If cl S $\neq$ S, then S $\subset$ S $\cup$ S$'$
———-
Notice that:
From n = 1 to ∞, $\bigcap$ [p $-\frac{1}{n}$, p $+\frac{1}{n}$] = {p}
So, $\mathbb{R}$ but not From n = 1 to ∞, $\bigcap$ [p $-\frac{1}{n}$, p $+\frac{1}{n}$] is equal to $\mathbb{R}$ but not {p}.
S $\subset \mathbb{R}$ but not From n = 1 to ∞, $\bigcap$ [p $-\frac{1}{n}$, p $+\frac{1}{n}$]
S $\subset$ From n = 1 to ∞, $\bigcup$ $\mathbb{R}$ but not [p $-\frac{1}{n}$, p $+\frac{1}{n}$]
S $\subset$ From n = 1 to ∞, $\bigcup$ [($-\infty$, p $-\frac{1}{n}$) $\bigcup$ (p $+\frac{1}{n}$, ∞)]
Since S is compact, $\exists$ k $\in \mathbb{N}$ and {n_1, n_2, ... n_k} $\subset \mathbb{N}$ st
S $\subset$ From i = 1 to k, $\bigcup$ [($-\infty$, p $-\frac{1}{n_i}$) $\bigcup$ (p $+\frac{1}{n_i}$, ∞)]
Let: m = max {n_1, n_2, ... n_k}
Then ($-\infty$, p $-\frac{1}{n_i}$) $\bigcup$ (p $+\frac{1}{n_i}$, ∞) $\subset$ ($-\infty$, p $-\frac{1}{m}$) $\bigcup$ (p $+\frac{1}{m}$, ∞).

Thus, S $\subset$ [($-\infty$, p $-\frac{1}{m}$) $\bigcup$ (p $+\frac{1}{m}$, ∞)]
$\longleftarrow$
Conversely,
Suppose: S is closed and bounded
Want to show: S is compact
Let us suppose that S $\subset$ when $\alpha \in I, \bigcup$ G$_\alpha$,
where G$_\alpha$ is open $\forall \alpha \in I$, where I is some index.
$\forall x \in \mathbb{R}$, define:
S$_x$ = S $\cap$ ($-\infty$, x]
Also define the set:
Beta = {x $\in \mathbb{R}$: S$_x$ is covered by a finite collection of the G$_\alpha$'s}
Notice that S is bounded, so inf S $\in \mathbb{R}$.
and S$_i nfS$ = S $\cap$ ($-\infty$, inf S] = {inf S}
Since by Lemma 3.5.4, inf S = min S.
Now, since min S = inf S $\in$ S, then $\exists$ $\alpha_0 \in$ I st inf S $\in$ G$_\alpha$ (α_0).
This proves that
S$_i nfS$ = {inf S} $\subset$ G$_\alpha$ (α_0)
Hence, inf S $\in$ Beta $\neq \emptyset$.

$\square$