Exercise 1

Let A = {0, 1, 2, 3, 4} and B = {0, 1, 2, 3}. For each of the relations R from A to B listed below list all pairs (a, b) $\in \mathbb{R}$ and write the corresponding {0, 1}-indicator-matrix.

a. a = b : (0, 0), (1, 1), (2, 2), (3, 3)

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

b. a + b = 4 : (1, 3), (2, 2), (3, 1), (4, 0)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

c. a > b : (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

d. a divides b : (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1, 2), (2, 2), (1, 3)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Exercise 2

For each of these relations on the set {1, 2, 3, 4} decide whether or not it is reflexive, symmetric, antisymmetric, and transitive.

a. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

b. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

c. {(2, 4), (4, 2)}

d. {(1, 2), (2, 3), (3, 4)}

e. {(1, 1), (2, 2), (3, 3), (4, 4)}

f. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Relation	*R*	*S*	*A*	*T*
a	0	0	0	1
b	1	1	0	1
c	0	1	0	1
d	0	0	1	0
e	1	1	1	1
f	0	0	0	1

Exercise 3

Let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)}, and let S be the relation {(2, 1), (3, 1), (3, 2), (4, 2)} on the set A = {1, 2, 3, 4}

a. Find R ∪ S

   {(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)}

b. Find R ∩ S

   {(3, 1)}

c. Find R o S

   {(1, 1), (1, 2), (2, 1), (2, 2)}

Exercise 4

Let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} on the set A = {1, 2, 3, 4}.

a. Find the reflexive closure of R.

   {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 1), (3, 3), (4, 4)}

b. Find the symmetric closure of R.

   {(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)}

c. Find the transitive closure of R.

   {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (1, 4)}

Exercise 5

Prove the following:

a. A relation R is reflexive iff R^{-1} is reflexive (where R^{-1} is the inverse relation that just reverses the order).

$\longrightarrow$

Assume R is reflexive.

Let $(a, a) \in R$

Then $(a, a) \in R^{-1}$

Hence, R^{-1} is reflexive.

$\longleftarrow$

Assume R^{-1} is reflexive.

Let $(a, a) \in R^{-1}$

Then $(a, a) \in R$

Hence, R is reflexive.

b. A relation R is symmetric iff $R = R^{-1}$.

$\longrightarrow$

Assume R is symmetric.

Let $(a, b) \in R$.

Want to show: $(a, b) \in R^{-1}$.

Notice: $(b, a) \in R$.

Thus, $(a, b) \in R^{-1}$.

Hence, $R = R^{-1}$.

$\longleftarrow$

Assume $R = R^{-1}$.

Let $(a, b) \in R$.

Then $(a, b) \in R^{-1}$.

$(a, b) \in R \Rightarrow (b, a) \in R^{-1}$.

But since $R^{-1} = R$, $(b, a) \in R$.

So, $(a, b) \in R \Rightarrow (b, a) \in R$.

Hence, R is symmetric..

c. A relation R is anti-symmetric iff $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$

$\longrightarrow$

Assume R is anti-symmetric.

Then $(a, b), (b, a) \in R \Rightarrow a = b$.

So, $R \cap R^{-1}$ will only contain tuples such that $a = b$.

$\longleftarrow$

Assume $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$.

Let $(a, b) \in R$. If $a \neq b$, then $(a, b) \notin R \cap R^{-1}$. Thus, $(a, b) \notin R^{-1}$.

Hence, R is anti-symmetric.

Exercise 6

Let R be the relation represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the matrices for the relations:

a. R^2

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b. R^3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c. R^4

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 7

Which of these relations on {0, 1, 2, 3} are equivalence relations? If they are not, why?

a. {(0, 0), (1, 1), (2, 2), (3, 3)}

Yes.

b. {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

No, (1, 1) isn't in there.

c. {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

Yes.

d. {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

No, (1, 2) isn't in there.

e. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Yes.

Exercise 8

List the ordered pairs in the equivalence relations produced by these partitions of {0, 1, 2, 3, 4, 5}.

a. {0}, {1, 2}, {3, 4, 5}

(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (3, 4), (4, 5), (3, 5), (5, 3), (4, 3)...

b. {0, 1}, {2, 3}, {4, 5}

c. {0, 1, 2}, {3, 4, 5}

d. {0}, {1}, {2}, {3}, {4}, {5}

Exercise 9

Which of these relations on {0, 1, 2, 3} are partial orderings? If they are not, why?

a. {(0, 0), (1, 1), (2, 2), (3, 3)}

   Yes.

b. {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

   No: (0, 2) and (2, 0) are both in there.

c. {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

   No: (1, 2) and (2, 1) are both in there.

d. {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

   No: (1, 3) and (3, 1) are both in there.

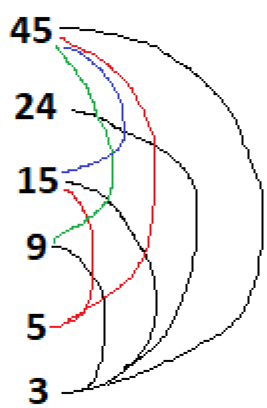
e. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

   No: (0, 1) and (1, 0) are both in there.

Exercise 10

Answer these questions for the divides poset ({3, 5, 9, 15, 24, 45}; |).

a. Draw the Hasse diagram



b. List the maximal and minimal elements.

   Maximal: {45, 24}. Minimal: {3, 5}

c. Is there a greatest element? A least element?

   There is no element greater than nor less than all others.

d. Find all upper bounds of {3, 5}. Find the least upper bound of {3, 5}, if it exists.

   UB({3, 5}): {15, 45}. LUB({3, 5}): {15}

e. Find all the lower bounds of {15, 45}. Find the greatest lower bound of {15, 45}, if it exists.

   LB({15, 45}): {3, 5, 15}. GLB({15, 45}): {15}

Exercise 11

Prove the following:

a. There is exactly one greatest element of a poset, if such an element exists.

   Suppose $\exists$ a, b $\in$ a poset P, such that a and b are the greatest elements of P.

   Then a $\geq$ x and b $\geq$ x $\forall$ x $\in$ P.

   So a $\geq$ b and b $\geq$ a.

   Thus, a = b.

b. There is exactly one maximal element in a poset with a greatest element.

   Let P be a poset and let a be the greatest element in P.

   Let b $\in$ P such that b $\neq$ a.

   Then, by definition, a $\leq$ b.

   Thus, a is the only maximal element in P.

c. The least upper bound of a set in a poset is unique if it exists.

   Let P be a poset and a $\in$ P.

   Suppose $\exists$ U_1 and U_2 $\in$ P such that U_1 and U_2 are least upper bounds for a and $U_1 \neq U_2$

   Then, by definition, $U_1 \leq U_2$ and $U_2 \leq U_1$.

   Hence, $U_1 = U_2$

Exercise 12

Determine whether these posets are lattices.

a. $(\{1, 3, 6, 9, 12\}; |)$

   No, 9 join 6 doesn't have a LUB.

b. $(\{1, 5, 25, 125\}; |)$

   Yes.

c. $(\mathbb{Z}\ ;\ \geq\ )$

   Yes, but it's not a complete lattice.

d. $(\mathcal{P}(S), \subset\ )$, where $\mathcal{P}(S)$ is the power set of a set S.

   Yes.

Exercise 13

Show that every totally ordered set is a lattice.
Let T be a totally ordered set, and let a, b $\in$ T.
Since T is totally ordered, either a $\leq$ b or b $\leq$ a.
Case:

i) a $\leq$ b

   Then a meet b = a, and a join b = b.

ii) b $\leq$ a

   Then b meet a = b, and b join a = a.

Hence, any two elements have a LUB and GLB.

Exercise 14

Show that every finite lattice has a least element and a greatest element.

Let L be a finite lattice.
Suppose there are two least elements in L: l_1, l_2 such that $l_1 \neq l_2$
Let $l = l_1$ meet l_2 (which exists because L is a lattice)
Case:

i) $l = l_1$: a contradiction, since l_2 is the least element.

ii) $l = l_2$: a contradiction, since l_1 is the least element.

iii) $l \neq l_1$ and $l \neq l_2$: a contradiction, since l_1 and l_2 are the least elements.

Thus, the least element in L is unique, if it exists.
Let A = a_1 meet a_2 meet ... a_n where n = |L| and $a_i \in$ L
Since A exists and is the least possible element, L has a least element.
WLOG, the same is true for a greatest element. (can I do this?)

Exercise 15

Give an example of an infinite lattice with

a. neither a least nor a greatest element.

   $(\mathbb{Z}, \leq)$

b. a least but not a greatest element.

   $(\mathbb{Z}^+, \leq)$

c. a greatest but not a least element.

   $(\mathbb{Z}^-, \leq)$

d. both a least and a greatest element.

   $(\mathbb{Q}^{[0,1]}, \leq)$

Exercise 16

Show that in any lattice $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. Note: $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$ was shown in class.)

Proof of $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$:
$Z \leq Z$
$(X \wedge Y) \wedge Z \leq Z$ **(1)**
We also know:
$(X \wedge Y) \wedge Z \leq X \wedge Y \leq X$ **(2)**
$(X \wedge Y) \wedge Z \leq X \wedge Y \leq Y$ **(3)**
And:
$(X \wedge Y) \wedge Z \leq X \wedge Z$ by **(1)** and **(2)**
And:
$(X \wedge Y) \wedge Z \leq X \wedge (Y \wedge Z)$ by **(1)**, **(2)** and **(3)**

Proof of $(x \wedge y) \wedge z \geq x \wedge (y \wedge z)$:
$X \geq X$
$X \geq X \wedge (Y \wedge Z)$

$Y \wedge Z \geq X \wedge (Y \wedge Z)$

$Y \geq Y \wedge Z \geq X \wedge (Y \wedge Z)$
$Z \geq Y \wedge Z \geq X \wedge (Y \wedge Z)$
Thus,
$(X \wedge Y) \wedge Z \geq X \wedge (Y \wedge Z)$

Exercise 17

Show that in any lattice $x \vee (x \wedge y) = x$. Note: the dual absorption law was shown in class.

$X \vee (X \wedge Y) \geq X$ **(1)**

$X \wedge Y \leq X$
$X \vee (X \wedge Y) \leq X \vee X = X$
$X \vee (X \wedge Y) \leq X$ **(2)**
By **(1)**, **(2)**, and antisymmetry,
$X \vee (X \wedge Y) = X$

Exercise 18

Show that any lattice $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$. Note: the dual distributive inequality was shown in class.

$X \vee Y \geq X$
$X \vee Y \geq Y \geq Y \wedge Z$
$X \vee Y \geq X \vee (Y \wedge Z)$
$X \vee Z \geq X \vee (Y \wedge Z)$
$(X \vee Y) \wedge (X \vee Z) \geq X \vee (Y \wedge Z)$

Exercise 19

Show that the two distributive equalities are equivalent. That is, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ if, and only if, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
$\longrightarrow$
WTS: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$(x \wedge y) \vee (x \wedge z) = (x \vee y) \wedge (x \vee z)$
$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$
$\longleftarrow$
WTS: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$
$(x \wedge y) \vee (x \wedge z) = (x \vee y) \wedge (x \vee z)$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Exercise 20

Show that the distributive law implies the modular law. That is, if a lattice satisfies one (hence both, from problem 19), then ($x \leq z \;\Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$).

Exercise 21

Check if the lattice N_5 is distributive.