Example 1: Page 195 #16(a)

Prove that $S^* = \lim\limits_{n\to\infty}$ (sup $\{s_{n+1}, s_{n+2}, s_{n+3}...\}$) = lim sup s_n, $s \in \mathbb{R}$
Want to show that, although we know $s^* = \lim\limits_{n\to\infty}$ (sup $\{s_{n+1}, s_{n+2}, s_{n+3}...\}$), s^* is in fact lim sup s_n.
Let t_n = sup $\{s_{n+1}, s_{n+2}, s_{n+3}...\}$

Side Note

$|s_n| \leq M \; \forall \; n \in \mathbb{N}$
$-M \leq s_n \leq M \; \forall \; n \in \mathbb{N}$

$\{t_n\}$ is a bounded, decreasing sequence and

$$t_{n+1} = sup\{s_{n+2}, s_{n+3}...\} \leq t_n = sup\{s_{n+1}, s_{n+3}, ...\} \; \forall \; n \in \mathbb{N}$$

If U is a bounded set in $\mathbb{R}$ and $V \subset U$, then sup V $\leq$ sup U.
So, sup U $\in \mathbb{R}$ exists.

i) $u \leq$ sup U $\forall \; u \in U$

ii) $\forall \; \epsilon > 0$, exs $u_1 \in U$ st sup U $- \epsilon < u_1$

Notice that for $v \in V$, $v \leq$ sup U.
So, sup V $\leq$ sup U.
Is t_n bounded?
It's bounded below since $-M \leq s_{n+1} \leq t_n \; \forall \; n \in \mathbb{N}$
By the monotonic convergence theorem,
$\lim\limits_{n\to\infty} t_n = s^*$ exists.
From Theorem 4.4.11, conditions (a) and (b):

a. $\forall \; \epsilon > 0, \exists \; N \in \mathbb{N}$ st $s_n < s^* + \epsilon$, $\forall \; n \geq N$

b. $\forall \; \epsilon > 0$, $i \in \mathbb{N}$, $\exists \; j > i$ st $s_j > s^* - \epsilon$

Notice that $s^* \leq t_n \; \forall \; n \in \mathbb{N}$
(since t_n is decreasing and it has a limit)
Let $n \in \mathbb{N}$ and notice that $t_{n+k} \leq t_n \; \forall \; k \in \mathbb{N}$
So,
$\lim\limits_{k\to\infty} t_{n+k} \leq t_n \; \forall \; n \in \mathbb{N}$
So,
$s^* = \lim\limits_{k\to\infty} t_{n+k}$, therefore $s^* \leq t_n \; \forall \; n \in \mathbb{N}$
Thus,
$\forall \; \epsilon > 0, \exists \; N_1 \in \mathbb{N}$ st

$$|t_n - s^*| < \epsilon \text{ for } n \geq N_1$$

$$t_n - s^* < \epsilon \text{ for } n \geq N_1$$

$$t_n < s^* + \epsilon \text{ for } n \geq N_1$$

$$s_{n+1} \leq t_n < s^* + \epsilon \text{ for } n + 1 \geq N_1 + 1$$

So,

$$s_M < s^* + \epsilon \text{ for } M \geq N_1 + 1$$

So, s^* satisfies (a).

Define t_n = sup $\{s_{n+1}, s_{n+2}, s_{n+3}...\}$
Now for any $\epsilon > 0$,

$$t_n \geq s^* > s^* - \frac{\epsilon}{2} \ \forall \text{ n} \in \mathbb{N} \tag{1}$$

Also, for any i = n $\in \mathbb{N}$, $\exists$ s_j where j > i st

$$t_n - \frac{\epsilon}{2} < s_j \tag{2}$$

Notice that $t_n - \frac{\epsilon}{2}$ is no longer a least upper bound for the set.
From **(1)** and **(2)**, $s_j > t_n - \frac{\epsilon}{2} > s^* - \epsilon$
So, since s^* satisfies **(a)** and **(b)**, s^* is the lim sup s_n.

Unbounded Sequences

S = {all subsequential limits of s_n}
We know that S is not empty if s_n is bounded since every bounded sequence has a convergent subsequence.
But what if s_n is bounded? page 192
Case:

i) $\{s_n\}$ is unbounded above.

From the proof of Theorem 4.4.8, $\exists$ a monotonic subsequence $\{s_{n_k}\}$ of $\{s_n\}$ st $\lim\limits_{k\to\infty} s_{n_k} = \infty$

Although ∞ is not a real number, we say that, if s_n is unbounded above, then lim sup $s_n = \infty$

ii) $\{s_n\}$ is bounded above but unbounded below.

Subcase i: $\exists$ a subsequence $\{s_{n_k}\}$ st $\lim\limits_{k\to\infty} s_{n_k} = \text{s} \in \mathbb{R}$. Then, set lim sup s_n = sup S

Subcase ii: There is no subsequence $\{s_{n_k}\}$ st $\lim\limits_{k\to\infty} s_{n_k} = \text{s} \in \mathbb{R}$ (a finite number).

Then, $\lim\limits_{n\to\infty} s_n = -\infty$

and, lim sup $s_n = -\infty$, so sup S $= -\infty$

which means, since lim inf $s_n \leq$ lim sup s_n, lim inf $= -\infty$