

Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18 (for 5(i) define t_n to be 1 over s_n , and then show that 1 over s_n goes to 0)

Problem 1

Mark each statement True or False. Justify each answer.

- a. If (s_n) and (t_n) are convergent sequences with $s_n \rightarrow s$ and $t_n \rightarrow t$, then $\lim (s_n + t_n) = s + t$ and $\lim (s_n t_n) = st$.

True. By Theorem 4.2.1 (a) and (c).

- b. If (s_n) converges to s and $s_n > 0 \forall n \in \mathbb{N}$, then $s > 0$.

False. Counter example: $(s_n) = \frac{1}{n}$ ($s = 0$, but the moment you define n , $s_n > 0$)

- c. The sequence (s_n) converges to s iff $\lim s_n = s$.

False. The sequence converges to s iff s exists **as a real number**. If $s = +\infty$ then it can't converge.

- d. $\lim s_n = +\infty$ iff $\lim (\frac{1}{s_n}) = 0$.

False. If $\lim (\frac{1}{s_n}) = 0$ but $(s_n) = -1, -2, -3, \dots$ then s_n does not diverge to $+\infty$

Problem 2

Mark each statement True or False. Justify each answer.

- a. If $s_n = s$ and $\lim t_n = t$, then $\lim (s_n t_n) = st$.

False. We don't know s_n 's limit (which could be, for example, $(s_n) = n$, which diverges)

- b. If $\lim s_n = +\infty$, then (s_n) is said to converge to $+\infty$.

False. You can only converge to a finite number.

- c. Given sequences (s_n) and (t_n) with $s_n \leq t_n \forall n \in \mathbb{N}$, if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

True.

Suppose \exists sequences (s_n) and (t_n) st $s_n \leq t_n \forall n \in \mathbb{N}$ where $\lim s_n = +\infty$ and $\lim t_n$ is NOT $+\infty$.
 t_n diverges to $+\infty$ if $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

Let: $M \in \mathbb{R}$

We know that

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

Since $s_n \leq t_n \forall n \in \mathbb{N}$

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $s_n \geq t_n > M$

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $s_n > M$

This is the definition of diverging to $+\infty$, a contradiction.

Hence, result.

- d. Suppose (s_n) is a sequence st the sequence of ratios $(\frac{s_{n+1}}{s_n})$ converges to L . If $L < 1$, then $\lim s_n = 0$.

False.

Let: $s_n = n(1)^{-n} \rightarrow (\frac{s_{n+1}}{s_n}) = \frac{(n+1)(1)^{-(n+1)}}{n(1)^{-n}}$

which converges to -1 which is less than 1 but does not have a limit of 0 .

Problem 4

a. Prove Theorem 4.2.1(b):

Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. Then

(b) $\lim (ks_n) = ks$ and $\lim (k + s_n) = k + s$, for any $k \in \mathbb{R}$

We know that since s_n and t_n are convergent sequences with limits s and t , respectively.

So,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |s_n - s| < \epsilon$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |t_n - t| < \epsilon$$

Want to show: $\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |ks_n - ks| < \epsilon$

$$|ks_n - ks| = |k(s_n - s)| = |k||s_n - s|$$

So,

$$|ks_n - ks| = |k||s_n - s| < \epsilon$$

$$|s_n - s| < |k|\epsilon = \epsilon_1(\epsilon)$$

Since

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |s_n - s| < \epsilon$$

thus,

$$\forall \epsilon_1 > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |s_n - s| < \epsilon_1$$

and

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |ks_n - ks| < \epsilon$$

Hence, $\lim (ks_n) = ks$

Want to show: $\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |k + s_n - (k + s)| < \epsilon$

We know:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |s_n - s| < \epsilon$$

So,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |s_n + k - s - k| < \epsilon$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |s_n + k - (s + k)| < \epsilon$$

Since this is true,

$$\lim (s_n + k) = k + s$$

b. Prove Corollary 4.2.5:

If (t_n) converges to t and $t_n \geq 0 \forall n \in \mathbb{N}$, then $t \geq 0$.

We know that

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |t_n - t| < \epsilon$$

Suppose: $t < 0$

Let: $\epsilon = |t|$

$$\forall n \in \mathbb{N}, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |t_n - t| < |t|$$

Since t is negative,

$$\forall n \in \mathbb{N}, \exists N \in \mathbb{N} \text{ st } n \geq N \text{ implies } |t_n + |t|| < |t|$$

Since $t_n \geq 0$,

$\forall n \in \mathbb{N}, \exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n + |t| < |t|$

So,

$\forall n \in \mathbb{N}, \exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n < 0$

but $t_n \geq 0$, a contradiction.

Hence, result.

Problem 5

For s_n given by the following formulas, determine the convergence or divergence of the sequence (s_n) . Find any limits that exist.

a. $s_n = \frac{3-2n}{1+n} \rightarrow \frac{1}{2}$

b. $s_n = \frac{(-1)^n}{n+3} \rightarrow 0$

c. $s_n = \frac{(-1)^n}{2n-1} \rightarrow 0$

d. $s_n = \frac{2^{3n}}{3^{2n}} = \frac{8^n}{9^n} \rightarrow 0$

e. $s_n = \frac{n^2-2}{n+1} \rightarrow \infty$

f. $s_n = \frac{3+n-n^2}{1+2n} \rightarrow -\infty$

g. $s_n = \frac{1-n}{2^n} \rightarrow 0$

h. $s_n = \frac{3^n}{n^3+5} \rightarrow \infty$

i. $s_n = \frac{n!}{2^n} \rightarrow \infty$

j. $s_n = \frac{n!}{n^n} = \frac{1*2*3*4*5}{5*5*5*5*5}$ where $n = 5 \rightarrow 0$

k. $s_n = \frac{n^2}{2^n} \rightarrow 0$

l. $s_n = \frac{n^2}{n!} \rightarrow 0$

Problem 9

Prove Theorem 4.2.12:

Suppose that (s_n) and (t_n) are sequences st $s_n \leq t_n \forall n \in \mathbb{N}$

a. If $\lim s_n = +\infty$ then $\lim t_n = +\infty$

Suppose \exists sequences (s_n) and (t_n) st $s_n \leq t_n \forall n \in \mathbb{N}$ where $\lim s_n = +\infty$ and $\lim t_n$ is NOT $+\infty$.

t_n diverges to $+\infty$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

Let: $M \in \mathbb{R}$

We know that

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n > M$

Since $s_n \leq t_n \forall n \in \mathbb{N}$,

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $s_n \geq t_n > M$

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $s_n > M$

This is the definition of diverging to $+\infty$, a contradiction.

Hence, s_n diverges to $+\infty$.

b. If $\lim t_n = -\infty$ then $\lim s_n = -\infty$

Suppose \exists sequences (s_n) and (t_n) st $s_n \leq t_n \forall n \in \mathbb{N}$ where $\lim s_n = -\infty$ and $\lim t_n$ is NOT $-\infty$.

t_n diverges to $-\infty$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n < M$

Let: $M \in \mathbb{R}$

We know that

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $t_n < M$

Since $s_n \leq t_n \forall n \in \mathbb{N}$,

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $s_n \leq t_n < M$

$\exists N \in \mathbb{N}$ st $n \geq N$ implies $s_n < M$

This is the definition of diverging to $-\infty$, a contradiction.

Hence, s_n diverges to $-\infty$.

Problem 10

Prove the converse part of Theorem 4.2.13:

Let (s_n) be a sequence of positive numbers. Then, $\lim s_n = +\infty$ iff $\lim (\frac{1}{s_n}) = 0$.

—→

Assume: $\lim s_n = +\infty$

Given any $\epsilon > 0$, let $M = \frac{1}{\epsilon}$. Then there exists a natural number N st $n \geq N$ implies that $s_n > M = \frac{1}{\epsilon}$.

Since each s_n is positive, we have:

$|\frac{1}{s_n} - 0| < \epsilon$, whenever $n \geq N$

Thus, $\lim (\frac{1}{s_n}) = 0$.

←

Problem 17

a. Show that $\lim_{n \rightarrow \infty} \frac{k^n}{n!} = 0 \forall k \in \mathbb{R}$

b. What can be said about $\lim_{n \rightarrow \infty} \frac{n!}{k^n}$?

Problem 18

Assume that (s_n) is a convergent sequence with $a \geq s_n \geq b \forall n \in \mathbb{N}$.

Prove that $a \leq \lim s_n \leq b$.