Homework 7: pages 184 - 185 numbers 1, 2(a)(b), 3(e), 4, 10, 13, 14 $\longleftarrow$ 14 is difficult, but not impossible!
(want to show that lim $(1 + \frac{1}{n})^n$ exists)
Hint:
$(1 + \text{b})^n = 1 + \text{nb} + \frac{n(n-1}{2!}\text{b}^n + ... + \frac{n(n-1)...(n-(r-1))}{r!}\text{b}^r + ... + \text{b}^n$
In our problem, b = $\frac{1}{n}$
Look at it as $1 + \sum_{r=1}^{n} \frac{n(n-1)...(n-(r-1))}{r!}\frac{1}{n^r}$
$(1 + \frac{1}{n})^n$ goes in there somewhere somehow.

Problem 1

Mark each statement True or False. Justify each answer.

a. If a monotone sequence is bounded, then it is convergent.

   True

   by Theorem 4.3.3

b. If a bounded sequence is monotone, then it is convergent.

   True

   by Theorem 4.3.3

c. If a convergent sequence is monotone, then it is bounded.

   True

   by Theorem 4.3.3

Problem 2(a)(b)

Mark each statement True or False. Justify each answer.

a. If a convergent sequence is bounded, then it is monotone.

   False.

   Counterexample: s_n =

$$(-1)^n \frac{1}{n}$$

b. If (s_n) is an unbounded increasing sequence, then lim $s_n = +\infty$

   True.

   By Theorem 4.3.8

   Assume: (s_n) is an unbounded, increasing sequence.

   Then, $\forall$ n $\in \mathbb{N}$, $s_n \leq s_{n+1}$

   and

   $\forall$ m $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st. n $\geq$ N implies $s_n > m$

   By Definition 4.2.9:

   We say a sequence diverges to ∞ if $\forall$ m $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n > m$

   Hence, result.

Problem 3(e)

Prove that each sequence is monotone and bounded. Then, find the limit.
(e) $s_1 = 5$ and $s_{n+1} = \sqrt{4s_n + 1}$ for $n \geq 1$
s_n is monotone if it's either increasing or decreasing.
$s_1 = 5$, $s_2 = \sqrt{21} = 4.58257569496$, $s_3 = \sqrt{4\sqrt{21}+1} = \sqrt{\sqrt{336}+1} = \sqrt{19.3303028} = 4.39662402304$
Hmm, limit's probably 4. Let's see.

Conjecture
$\{s_n\}$ is decreasing and $4 \leq s_n \leq 5$, $\forall$ n $\in \mathbb{N}$
P(n) (Proposition as a function of n):
$s_n \geq s_{n+1}$, $\forall$ n $\in \mathbb{N}$
$s_1 = 5 > \sqrt{21} = s_2$
Suppose that, $\forall$ k $\in \mathbb{N}$,

$$\sqrt{4s_k + 1} \geq \sqrt{4s_{k+1} + 1}$$

Now,

$$s_{k+1} = \sqrt{4s_{k+1} + 1} \geq \sqrt{4s_{k+2} + 1} = s_{k+2}$$

So,

$$s_k \geq s_{k+1}$$

Hence, by induction, P(n): $s_n \geq s_{n+1}$ is true $\forall$ n $\in \mathbb{N}$
Q(n): $s_n \geq 4$ $\forall$ n $\in \mathbb{N}$
$s_1 = 5 > 4$
Assume for k $\in \mathbb{N}$ that $s_k > 4$

$$s_{k+1} = \sqrt{4s_k + 1} > \sqrt{4(3.75) + 1} = 4$$

Hence, by induction, Q(n): $s_n > 4$ is true $\forall$ n $\in \mathbb{N}$
By the Montone Convergence Theorem,
$\exists$ s $\in \mathbb{R}$ st
$\lim_{n\to\infty} s_n = s$
By HW problem 11, page 170.
Thus,
$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} s_n = s$
So, we claim that $\lim_{n\to\infty} s_{n+1} = s = \lim_{n\to\infty} \sqrt{4s_n + 1} = \sqrt{4s+1}$
From Example 4.2.6,
$\lim_{n\to\infty} \sqrt{t_n} = \sqrt{t}$ if $\lim_{n\to\infty} t_n = t$
Also, by Theorem 4.2.1 (b), $\lim_{n\to\infty} \sqrt{1+s_n} = \sqrt{1+s}$
(which is like saying $\lim_{n\to\infty} t_n = t$)
Hence,

$$s = \sqrt{4s+1}$$
$$s^2 = 4s + 1$$
$$s^2 - 4s - 1 = 0$$
$$s = \frac{4 \pm \sqrt{20}}{2}$$

But one of those limits can't be true since limits are unique.
Since $s_n \geq 0$, $\forall$ n $\in \mathbb{N}$,
then $\lim\limits_{n\to\infty} s_n = s \geq 0$, $\forall$ n $\in \mathbb{N}$
(By Corollary 4.2.5)
Hence,
$s = \frac{4+\sqrt{20}}{2} = 2 + \sqrt{5}$

Problem 4

Find an example of a sequence of real numbers satisfying each set of properties.

a. Cauchy, but not monotone.

   $s_n = (-1)^n \frac{1}{n}$

b. Monotone, but not cauchy.

   $s_n = n$

c. Bounded, but not cauchy.

   $s_n = (-1)^n$

Problem 10

a. Suppose that $|r| < 1$. Recall from Exercise 3.1.7 that

$$1 + r + r^2 + ...r^n = \frac{1 - r^{n-1}}{1 - r}$$

   Find $\lim\limits_{n\to\infty}$ (1 + r + r^2 + ... r^n).

   In other words, find $\lim\limits_{n\to\infty} \frac{1-r^{n-1}}{1-r}$

   P(n): $s_n = \frac{1-r^{n-1}}{1-r}$ is increasing and bounded.

   $s_1 = 0$, $s_2 = 1$

   Assume: $s_k \leq s_{k+1}$

$$s_{k+1} =$$

b. If we let the infinite repeating decimal 0.9999... stand for the limit:

$$\lim_{n\to\infty} (\frac{9}{10} + \frac{9}{10^2} + ... + \frac{9}{10^n}),$$

   Show that 0.99999.... = 1.

Problem 13

Prove Lemma 4.3.11:
Every Cauchy sequence is bounded. (Similar to the proof of Theorem 4.1.13)

Problem 14

Let (s_n) be the sequence defined by $s_n = (1 + \frac{1}{n})^n$.
Use the binomial theorem (Exercise 3.1.30) to show that (s_n) is an increasing sequence with $s_n < 3 \; \forall$ n.
Conclude that (s_n) is convergent. The limit of (s_n) is referred to as e and is used as the base for natural logarithms. The approximate value of e is 2.71828.