Exercise 1 (page 69)

Exercise 2

Let Q be the group of rational numbers under addition and let $\mathbf{Q}^*$ be the group of nonzero rational numbers under multiplication.
In Q, list the elements in in $\langle \frac{1}{2} \rangle$.
$\langle \frac{1}{2} \rangle = \{ \frac{n}{2} \ \mid n \in \mathbb{Z} \} = \{ ... \frac{-3}{2}, \frac{-2}{2}, \frac{-1}{2}, \frac{-0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, ... \}$
In $\mathbf{Q}^*$, list the elements in $\langle \frac{1}{2} \rangle$.
$\langle \frac{1}{2} \rangle = \{ (\frac{1}{2})^n \ \mid n \in \mathbb{Z} \} = \{ ... (\frac{1}{2})^{-3}, (\frac{1}{2})^{-2}, (\frac{1}{2})^{-1}, (\frac{1}{2})^{0}, (\frac{1}{2})^{1}, (\frac{1}{2})^{2}, (\frac{1}{2})^{3}, ... \}$

Exercise 4

Prove that in any group, an element and its inverse have the same order.

Proof.

Let g $\in$ G. By definition, $g^{-1} \in$ G exists.
Let n = |g| and m = $|g^{-1}|$
Want to show: n = m
Suppose not. Suppose that either n > m or m > n.
By definition,
$g^n = e$
$(g^{-1})^m = e$
So,
$g^n * (g^{-1})^m = e * e = e$
g * g * g ... (n times) * $g^{-1} * g^{-1} * g^{-1}$ ... (m times) = e
Without loss of generality, let's assume m is bigger.
Then $g^{-1} * g^{-1}$ ... (m − n times) = e.
However, since n is a positive integer, m − n < m
(a contradiction, since m is the smallest possible positive integer such that $(g^{-1})^m = e$)

□

Exercise 13

For any group elements a, x $\in$ G, prove that $|xax^{-1}| = |a|$. This exercise is referred to in Chapter 13.

Proof.

Let m be the order of xax^{-1}, and n be the order of a.
Want to show: m = n
By definition,
$(xax^{-1})^m = e$
$(xax^{-1}) * (xax^{-1}) * ... (xax^{-1})$ (m times) = e
$xa^m x^{-1} = e$
$x^{-1}\ xa^m x^{-1} = x^{-1}\ e$
$a^m x^{-1} = x^{-1}$
$a^m x^{-1} x = x^{-1} x$
$a^m = e$

Since both a^m = e and a^n = e, and both are defined to be the minimum positive integer that makes the equation true, they both have to be the same minimum positive integer.

□