Theorem 3.3: The Completeness Axiom

Recall the Fundamental Theorem of Arithmetic:
if n $\in \mathbb{N}$ with n $\geq$ 2, then n may be expressed as the product of prime numbers (the prime factorization (PF)).
The PF is unique with respect to (WRT) order.
Ex: $12 = 2 * 2 * 2 * 3$

Theorem 3.3.1

Let: p be a prime number
Then $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$

Proof.

Assume: $\sqrt{p} \in \mathbb{Q}$
Then $\sqrt{p} = \frac{a}{b}$, where a, b $\in \mathbb{N}$ and gcd(a, b) = 1
So,
p = $\frac{a^2}{b^2}$
$a^2 = pb^2$
therefore,

$$p|a^2 \tag{1}$$

$p|a^2 \Rightarrow \exists k \in \mathbb{Z}$ st $a^2 = pk$
Since the PF of a^2 and a contain exactly the same distinct primes,
(i.e. a = $p_1\times$ $p_2\times$ ... p_n $\Rightarrow$ a^2 = $p_1^2\times$ $p_2^2\times$ ... p_n^2)
and since p is prime (i.e. p is a component of a^2 but can't be, say, p_2^2 because that would mean it has an integer square root and therefore isn't prime), it has to be one of the p_n's,
p | a.
Thus, $\exists$ k $\in \mathbb{Z}$ st. a = pk.
Then a^2 = p^2k^2 = pb^2 from **(1)**.
Thus, b^2 = pk^2, and we see that p | b^2.
However, we obtain the contradiction that p | b and p | a.
Hence, $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$. □

Definition 3.3.7

Let S $\subset \mathbb{R}$.
If $\exists$ m $\in \mathbb{R}$ st s $\leq$ m $\forall$ s $\in S$,
then m is an upper bound of S and we say that S is **bounded above**.
Similarly, we can define **bounded below**.
If S is bounded above and below, then S is said to be **bounded**.
–n–[–S–]–m
If an upper bound m of S is a member of S, then m is called the maximum (or largest element) of S, and we say that m = **max S**.
–n–[–S–m–
Similarly, we may decline **minimum of S (min S)**.

Theorem 1

If a set $S \subset \mathbb{R}$ possesses a max element, then it is unique. A similar result holds for a minimum element.

Proof.
Suppose: $\exists m_1, m_2 \in \mathbb{R}$ st $m_1 =$ max S, $m_2 =$ max S
Thus, m_1, $m_2 \in S$ and, $\forall s \in$ S

$$s \leq m_1 \tag{1}$$

$$s \leq m_2 \tag{2}$$

Let $\mathrm{m} = \mathrm{m}_2$ in **(1)** and m =

□