

## Definition 3.4.6 - Def of Open/Closed Set

**Let:**  $S \subset \mathbb{R}$

if  $\text{bd } S \subset S$ , then  $S$  is closed.

if  $\text{bd } S \subset (\mathbb{R} \setminus S)$ , then  $S$  is open.

## Theorem 3.4.7

a. A set  $S$  is open iff  $S = \text{int } S$ ; i.e. iff  $\forall s \in S$ ,  $s$  is an **interior point**.

b. A set  $S$  is closed iff its complement,  $\mathbb{R} \setminus S$  is open.

Equivalently, a set  $S$  is open iff  $\mathbb{R} \setminus S$  is closed.

*Proof.*

(a):

→

**Assume:**  $S$  is open

**Want to show:**  $S = \text{int } S$

By definition,  $\text{int } S \subset S$ .

**Want to show:**  $S \subset \text{int } S$

**Let:**  $x \in S$  (1)

**Want to show:**  $x \in \text{int } S$

Since  $S$  is open,  $\text{bd } S \subset \mathbb{R} \setminus S$

So,  $x \notin \text{bd } S$ .

Thus,  $\exists \epsilon > 0$  st  $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$

$\forall \epsilon > 0$ ,  $N(x, \epsilon) \cap S \neq \emptyset$

$x \in \text{bd } S$  if  $\forall \epsilon > 0$ ,

$N(x, \epsilon) \cap S \neq \emptyset$  and  $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$

Thus,  $N(x, \epsilon) \subset S$ .

So,  $x \in \text{int } S$ .

This proves that  $S \subset \text{int } S$

←

**Assume:**  $S = \text{int } S$

**Want to show:**  $S$  is open

**Let:**  $x \in \text{bd } S$

**Want to show:**  $x \in \mathbb{R} \setminus S$

Since  $x \in \text{bd } S$ , we conclude that  $x \notin \text{int } S$ .

Side Note

$x \in \text{bd } S$  if,  $\forall \epsilon > 0$ ,

$N(x, \epsilon) \cap S \neq \emptyset$  and  $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$

Thus,  $x \in \mathbb{R} \setminus S$ .

So,  $\text{bd } S \subset \mathbb{R} \setminus S$ .

So, by definition,

$S$  is open.

(b):  $S$  is closed iff  $\mathbb{R} \setminus S$  is open.

So,  $x \notin \text{bd } S$ .

Thus,  $\exists \epsilon > 0$  st  $N(x, \epsilon) \cap S \neq \emptyset$

Hence,  $N(x, \epsilon) \subset \mathbb{R} \setminus S$

So,  $\mathbb{R} \setminus S$  is open from (a).

←

**Assume:**  $\mathbb{R} \setminus S$  is open

**Want to show:**  $S$  is closed

**Let:**  $x \in \text{bd } S$

**Want to show:**  $x \in S$

Since  $x \in \text{bd } S$ ,  $\forall \epsilon > 0$ ,

$$N(x, \epsilon) \cap S \neq \emptyset \quad (1)$$

and

$$N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset \quad (2)$$

Since  $\mathbb{R} \setminus S$  is open,  $\forall s \in \mathbb{R} \setminus S$ ,  $s$  is an **interior point** of  $\mathbb{R} \setminus S$ .

Thus,  $x \in S$ .

We have shown that  $\text{bd } S \subset S$ .

By definition,  $S$  is closed.

□

### Example 3.4.8

a.  $[0, 5]$  is a closed set. ( $\mathbb{R} \setminus [0, 5] = (-\infty, 0) \cup (5, \infty)$ )

b.  $(0, 5)$  is an open set.

c.  $[0, 5)$  is neither open nor closed.

d.  $[2, \infty)$  is a closed set.

e.  $\mathbb{R}$  is both open and closed.

$$\text{bd } \mathbb{R} = \emptyset \subset \mathbb{R}$$

$$\text{Also, int } \mathbb{R} = \mathbb{R}$$

Also,  $\emptyset$  is both open and closed.

### Theorem 2 (not in book)

**Let:**  $x \in \mathbb{R}$ ,  $\epsilon > 0$

Then:

a.  $N(x, \epsilon)$  is an open set

b.  $N^*(x, \epsilon)$  is an open set

(a)

*Proof.*

$$N(x, \epsilon) = \{y : |y - x| < \epsilon\} \text{ i.e. } -\epsilon < y - x < \epsilon$$

So,  $y \in N(x, \epsilon)$  iff  $x - \epsilon < y < x + \epsilon$

**Let:**  $y \in N(x, \epsilon)$

We shall find  $\hat{\epsilon} > 0$  st

$N(y, \hat{\epsilon}) \subset N(x, \epsilon)$ , which will show that

$N(x, \epsilon)$  is open.

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Side Note

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x-ep, y-ephat, y, yplusEphat, x, xplusEp

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**Let:**  $\hat{\epsilon} = \min \{y - (x - \epsilon), x + \epsilon - y\}$  **(1)**

**Want to show:**  $N(y, \hat{\epsilon}) \subset N(x, \epsilon)$

**Let:**  $z \in N(y, \hat{\epsilon})$

Then,  $y - \hat{\epsilon} < z < y + \hat{\epsilon}$  **(2)**

From **(1)**,  $\hat{\epsilon} \leq y - (x - \epsilon)$  **(3)**

and

$\hat{\epsilon} \leq x + \epsilon - y$  **(4)**

So from **(4)**,

$$y + \hat{\epsilon} \leq y + x + \epsilon - y$$

$$y + \hat{\epsilon} \leq x + \epsilon$$

From **(3)**,  $(x - \epsilon) - y \leq -\hat{\epsilon}$  **(5)**

Then,

$$y + (x - \epsilon) - y \leq y - \hat{\epsilon}$$

$$x - \epsilon \leq y - \hat{\epsilon}$$
 **(6)**

From **(2)**, **(5)**, **(6)**,

$$x - \epsilon \leq y - \hat{\epsilon} < z < y + \hat{\epsilon} \leq x + \epsilon$$

Therefore,

$$x - \epsilon < z < x + \epsilon$$

Thus,  $z \in N(x, \epsilon)$ .

Hence,

$$N(y, \hat{\epsilon}) \subset N(x, \epsilon)$$

Which proves that

$N(x, \epsilon)$  is open.

**(b):**  $N^*(x, \epsilon)$  is an open set. Similar to **(a)**.

□

## Theorem 3.4.10

**Let:**  $I$  be an index set.  $I \subset \mathbb{N} \subset \mathbb{R}$

**Suppose:**  $G_\alpha \subset \mathbb{R}$  is an open set  $\forall \alpha \in I$

Then,

- a.  $\bigcup_{\alpha \in I} G_\alpha$  is an open set.

b. If  $G_i \subset \mathbb{R}$  is open  $\forall i = 1, 2, \dots, n$  where  $n \in \mathbb{N}$

Then  $\bigcap_{i=1}^n G_i$  is open.

*Proof.*

(a):

**Let:**  $x \in \bigcup_{\alpha \in I} G_\alpha$

Thus,  $\exists \alpha_0 \in I$  st  $x \in G_{\alpha_0}$ .

Since  $G_{\alpha_0}$  is open,  $\exists \epsilon_0 > 0$  st  $N(x, \epsilon_0) \subset G_{\alpha_0}$

Thus,  $N(x, \epsilon_0) \subset \bigcup_{\alpha \in I} G_\alpha$

This proves that  $x \in \text{int}(\bigcup_{\alpha \in I} G_\alpha)$

By Theorem 3.4.7 a),

$\bigcup_{\alpha \in I} G_\alpha$  is open.

(b):

**Let:**  $x \in \bigcap_{i=1}^n G_i$

Thus,  $x \in G_i \forall i = 1, 2, \dots, n$

Since  $G_i$  is open  $\forall i = 1, 2, \dots, n$

$\exists \epsilon_i > 0$  st  $N(x, \epsilon_i) \subset G_i \forall i$  from 1 to  $n$ .

Choose  $\epsilon = \min \{\epsilon_1, \epsilon_2, \dots, \epsilon_n\} > 0$

Then  $N(x, \epsilon) \subset N(x, \epsilon_i) \forall i$  from 1 to  $n$ .

Hence,  $N(x, \epsilon) \subset \bigcap_{i=1}^n G_i$

Hence,  $\bigcap_{i=1}^n G_i$  is open. □

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Side Note

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—(—(——)—)—

x-epi, x-ep, x, xplusEp, xplusEpi

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## Corollary 3.4.11

a. Let  $F_\alpha$  be closed  $\forall \alpha \in I$ ,  $I$  is an index set.

Then  $\bigcap_{\alpha \in I} F_\alpha$  is closed.

b. Let  $F_i$  be closed  $\forall i$  from 1 to  $n$ .

Then  $(\bigcup_{i=1}^n F_i)$  is closed.