

## Lecture 1

### Theorem 3.2.8 - pg 118

Let  $x, y \in \mathbb{R}$

- a. If  $x \leq y + \epsilon \forall \epsilon > 0$ , then  $x \leq y$
- b. If  $|x - y| \leq \epsilon \forall \epsilon > 0$ , then  $|x - y| = 0$  or, evidently,  $x = y$

### Definition 3.2.9

If  $x \in \mathbb{R}$ ,

$$|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } x < 0. \end{cases}$$

### Theorem 3.2.10

Let  $x, y \in \mathbb{R}$  and  $a \geq 0$

Then

- a.  $|x| \geq 0$
- b.  $|x| \leq a$  iff  $-a \leq x \leq a$
- c.  $|xy| = |x||y|$
- d.  $|x + y| \leq |x| + |y|$  (equality holds only if signs are the same)

## Lecture 2

### Theorem 3.3: The Completeness Axiom

Recall the Fundamental Theorem of Arithmetic:

if  $n \in \mathbb{N}$  with  $n \geq 2$ , then  $n$  may be expressed as the product of prime numbers (the prime factorization (PF)).

The PF is unique with respect to (WRT) order.

Ex:  $12 = 2 * 2 * 2 * 3$

### Theorem 3.3.1

**Let:**  $p$  be a prime number

Then  $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$

### Definition 3.3.7

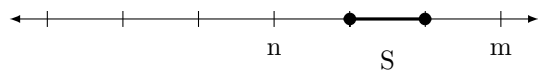
Let  $S \subset \mathbb{R}$ . If  $\exists m \in \mathbb{R}$  st  $s \leq m \forall s \in S$ ,

then  $m$  is an upper bound of  $S$  and we say that  $S$  is **bounded above**.

Similarly, we can define **bounded below**.

If  $S$  is bounded above and below, then  $S$  is said to be **bounded**.

$S$  can be open or closed. The example below is closed.



If an upper bound  $m$  of  $S$  is a member of  $S$ , then  $m$  is called the maximum (or largest element) of  $S$ , and we say that  $m = \mathbf{max} S$ . Similarly, we may define **minimum** of  $S$  ( $\mathbf{n} = \mathbf{min} S$ ).

### Theorem 1

If a set  $S \subset \mathbb{R}$  possesses a max element, then it is unique. A similar result holds for a minimum element.

### Definition 3.3.5 (supremum defined)

Let  $\emptyset \neq S \subset \mathbb{R}$  if  $S$  is bounded above,

then the **least upper bound** of  $S$  is called the **supremum** of  $S$ , denoted by  $\sup S \in \mathbb{R}$

iff:

a.  $s \leq \sup S \forall s \in S$

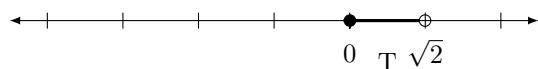
b.  $\exists s' \in S$  st  $\sup S - \epsilon < s' \forall \epsilon > 0$

### Axiom of Completeness of the set of Real Numbers: $\mathbb{R}$

Every  $\emptyset \neq S \subset \mathbb{R}$  that is bounded above has a least upper bound (i.e.  $\sup S \in \mathbb{R}$  exists).

A similar statement can be made about  $\inf S$ .

Remark: In practice 3.3.4, the set  $T = \{q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2}\}$  is bounded.



But  $\sqrt{2}$  is not rational, so the set wouldn't have a least upper bound.

We need to fill in the gaps to make analysis work.

## Lecture 3

### Theorem 1 (infimum definition)

**Let:**  $\emptyset \neq S \subset \mathbb{R}$ ,  $S$  is bounded below.

Then  $S$  possesses a greatest lower bound denoted by **inf**  $S$  (the **infimum** of  $S$ ), where  $\inf S \in \mathbb{R}$ , satisfying:

- i)  $\inf S \leq s \forall s \in S$
- ii)  $\forall \epsilon > 0, \exists s_1$  st  $\inf S + \epsilon > s_1$

### Theorem 3.3.7

Given nonempty subsets of  $A, B$  ( $A, B \subset \mathbb{R}$ ),

**Let:**  $C = \{x + y : x \in A, y \in B\}$

If  $A$  and  $B$  have suprema, then  $C$  has a supremum:  $\sup C = \sup A + \sup B$

### Theorem 3.3.8

Suppose  $\emptyset \neq D \subset \mathbb{R}$  and

$f : D \rightarrow \mathbb{R}$

$g : D \rightarrow \mathbb{R}$

$f(D) = \{f(x) : x \in D\}$

If  $\forall x, y \in D, f(x) \leq g(y)$ , then

$f(D)$  is bounded above and  $g(D)$  is bounded below.

Furthermore,  $\sup(f(D)) \leq \inf(g(D))$

### Theorem 3.3.9: Archimedian Property / Principle of $\mathbb{R}$ (AP)

The set  $\mathbb{N} = \{1, 2, 3, \dots\}$  is unbounded above in  $\mathbb{R}$

## Lecture 4

### Theorem 3.3.10

Each of the following is equivalent to the AP:

- a.  $\forall z \in \mathbb{R}, \exists n \in \mathbb{N} \text{ st } n > z$
- b.  $\forall x > 0, y \in \mathbb{R}, \exists n \in \mathbb{N} \text{ st } nx > y$
- c.  $\forall x > 0, \exists n \in \mathbb{N} \text{ st } 0 < \frac{1}{n} < x$

### Theorems 3.3.13 and 3.3.15

**Let:**  $x, y \in \mathbb{R} \text{ st } x < y$

Then:

- a.  $\exists r \in \mathbb{Q} \text{ st } x < r < y$
- b.  $\exists z \in \mathbb{R} \setminus \mathbb{Q} \text{ st } x < z < y$

## Section 3.4: Topology of $\mathbb{R}$

### Definitions 3.4.1 and 3.4.2

Let  $x \in \mathbb{R}$  and  $\epsilon > 0$ .

- (a) An  $\epsilon$ -neighborhood of  $x$  is:  $N(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon\}$
- (b) A deleted  $\epsilon$ -neighborhood of  $x$  is:  $N^*(x, \epsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$

### Open Set Topology: Definition 3.4.3 (interior / boundary point)

**Let:**  $S \subset \mathbb{R}$

A point  $x \in \mathbb{R}$  is an **interior point** of  $S$  if  $\exists \epsilon > 0 \text{ st } N(x, \epsilon) \subset S$ .

If,  $\forall \epsilon > 0, N(x, \epsilon) \cap S \neq \emptyset$  and  $N(x, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$

Then  $x$  is a **boundary point** of  $S$ .

The set of all interior points is denoted by **int**  $S$ .

The set of all boundary points is denoted by **bd**  $S$ .

**Nota Bene (N.B.):**

$\text{int } S \subset S$  and  $\text{bd } S = \text{bd } (\mathbb{R} \setminus S)$

### Theorem 1

**Let:**  $x \in S \subset \mathbb{R}$

Then either  $x \in \text{int } S$ , or  $x \in \text{bd } S$ .

## Lecture 5

### Definition 3.4.6 - Def of Open/Closed Set

**Let:**  $S \subset \mathbb{R}$

if  $\text{bd } S \subset S$ , then  $S$  is closed.

if  $\text{bd } S \subset (\mathbb{R} \setminus S)$ , then  $S$  is open.

### Theorem 3.4.7

a. A set  $S$  is open iff  $S = \text{int } S$ ; i.e. iff  $\forall s \in S$ ,  $s$  is an **interior point**.

b. A set  $S$  is closed iff its complement,  $\mathbb{R} \setminus S$  is open.

Equivalently, a set  $s$  is open iff  $\mathbb{R} \setminus S$  is closed.

### Theorem 2 (not in book)

**Let:**  $x \in \mathbb{R}$ ,  $\epsilon > 0$

Then  $N(x, \epsilon)$ ,  $N^*(x, \epsilon)$  are open sets.

### Theorem 3.4.10

**Let:**  $I$  be an index set.  $I \subset \mathbb{N} \subset \mathbb{R}$

**Suppose:**  $G_\alpha \subset \mathbb{R}$  is an open set  $\forall \alpha \in I$

Then,

a.  $\bigcup_{\alpha \in I} G_\alpha$  is an open set.

b. If  $G_i \subset \mathbb{R}$  is open  $\forall i = 1, 2, \dots, n \in \mathbb{N}$ , then  $\bigcap_{i=1}^n G_i$  is open.

### Corollary 3.4.11

a. Let  $F_\alpha$  be closed  $\forall \alpha \in I$ ,  $I$  is an index set.

Then  $\bigcap_{\alpha \in I} F_\alpha$  is closed.

b. Let  $F_i$  be closed  $\forall i$  from 1 to  $n$ .

Then  $(\bigcup_{i=1}^n F_i)$  is closed.

### Accumulation (or Limit) Points; Definition 3.4.14

**Let:**  $S \subset \mathbb{R}$

If  $\forall \epsilon > 0$ ,  $N^*(x, \epsilon) \cap S \neq \emptyset$ ,

Then  $x \in \mathbb{R}$  is an **accumulation** or **limit** point. (The set of all accumulation points of  $S$  is denoted by  $S'$ )

If  $x \in S \setminus S'$ ,

then  $x$  is an **isolated point**,

in which case,  $\exists \epsilon > 0$  st  $N(x, \epsilon) \cap S = \{x\}$

### Definition 3.4.16 - Closures

**Let:**  $S \subset \mathbb{R}$

Then the **closure** of  $S$ , denoted by  $\text{cl } S$ , is defined to be:

$\text{cl } S = S \cup S'$

## Lecture 6

### Theorem 3.4.17 - pg 118

**Let:**  $S \subset \mathbb{R}$

Then

- a.  $S$  is closed iff  $S' \subset S$
- b.  $\text{cl } S$  is a closed set
- c.  $S$  is closed iff  $S = \text{cl } S$
- d.  $\text{cl } S = S \cup S' = S \cup \text{bd } S$

## Lecture 7

### Section 3.5: Compact Sets

Three big areas of analysis: compactedness, continuity, and connectedness.

#### Definition: Open Cover / Subcover

**Let:**  $S \subset \mathbb{R}$

An **open cover** of  $S$  is a collection  $C$  of open sets st  $S \subset \bigcup C$ . The collection  $C$  of open sets is said to **cover** the set  $S$ .

A subset of sets from the collection  $C$  that still covers  $S$  is called a **subcover** of  $S$ .

#### Definition 3.5.1

A set  $S \subset \mathbb{R}$  is said to be **compact** if every **open cover** has a finite **subcover**

(i.e. if  $S \subset \bigcup_{\alpha \in I} G_{\alpha}$  ,

where  $G_{\alpha}$  is open  $\forall \alpha \in I$ ; then  $\exists n \in \mathbb{N}$  and  $\exists \{n_1, n_2, \dots, n_k\} \subset I$

st  $S \subset \bigcup_{i=1}^n G_{\alpha_i}$

#### Lemma 3.5.4

If  $\emptyset \neq S \subset \mathbb{R}$  and  $S$  is **closed** and **bounded**, then  $S$  has a maximum and a minimum.

In fact, in this,  $\max S = \sup S$ , and  $\min S = \inf S$ .

#### Theorem 3.5.5 (Heine-Borel)

A subset  $\emptyset \neq S \subset \mathbb{R}$  is compact iff  $S$  is closed and bounded.

## Lecture 8

### Theorem 3.5.5 (Heine-Borel)

A subset  $\emptyset \neq S \subset \mathbb{R}$  is compact iff  $S$  is closed and bounded.

### Theorem 3.5.6: Bolzano–Weierstrass Theorem

If a bounded set  $S \subset \mathbb{R}$  contains an infinite number of points, then  $\exists$  at least one point in  $\mathbb{R}$  that is an accumulation point of  $S$ .

### Theorem 3.5.7 (F.I.P.)

**Let:**  $\{K_\alpha\}_{\alpha \in I}$  be a family of compact sets, where  $I$  is an index.

Suppose that the intersection of any finite subfamily of the  $K_\alpha$ 's has a nonempty intersection.

Then  $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

### Corollary 3.5.8 Nested Intervals Theorem

**Let:**  $\{A_n\}_{n=1}^\infty$  be a family of nonempty closed bounded intervals in  $\mathbb{R}$  st  $A_{n+1} \subset A_n \forall n \in \mathbb{N}$

Then:

$\bigcap_{n=1}^\infty A_n \neq \emptyset$



## Lecture 9

### Definition 1: Sequence

A **sequence** is a function  $S: \mathbb{N} \rightarrow \mathbb{R}$

We write  $S(n) = S_n \forall n \in \mathbb{N}$  and refer to  $\{S_n\}$  (the book uses  $(S_n)$ ) as the **sequence**.

We refer to the set  $\{S_n : n \in \mathbb{N}\}$  as the range of the sequence.

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Side Note

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$$S_n = (-1)^n \forall n \in \mathbb{N}$$

$$\{(-1)^n\}$$

$$\text{range}\{S_n\} = \{-1, 1\}$$

$$\text{Here } \{S_n\} = \{1, -1, 1, -1, \dots\}$$


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An alternative to writing  $\{S_n\}$  for a sequence is to list the elements:  $S_1, S_2, \dots, S_n$

Sometimes the domain of the sequence is  $\mathbb{N} \cup \{0\}$  or  $\{n \in \mathbb{N} : n \geq m\}$  for some  $m \in \mathbb{N}$ .

In this case, we write  $\{S_n\}_{n=0}^\infty$  or  $\{S_n\}_{n=m}^\infty$

**Note 1:** A denumerable set (or a countably infinite set)  $S$  is a set for which there is a bijection  $S: \mathbb{N} \rightarrow \mathbb{R}$

This bijection may be thought of as a sequence  $\{S_n\}$ , where  $S_n = S(n) \forall n \in \mathbb{N}$  of distinct terms.

### Definition 4.1.2

A sequence  $\{S_n\}$  is said to **converge** to  $s \in \mathbb{R}$  provided that  $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$  st

$$|S_n - s| < \epsilon \quad \forall n \geq N$$

### Theorem 4.1.8

**Let:**  $\{S_n\}$  and  $\{a_n\}$  be sequences,  $s \in \mathbb{R}$

If some  $k > 0$  and some  $m \in \mathbb{N}$ , we have:

$$|S_n - s| \leq k|a_n|, \forall n \geq m \quad (1)$$

and if  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} S_n = s$ .

## Lecture 10

### Theorem 4.1.13

Every convergent sequence is bounded.

### Theorem 4.1.14

If a sequence converges, then its limit is unique.

## 4.2 Limit Theorems

### Theorem 4.2.1

Suppose that  $\{s_n\}$  and  $\{t_n\}$  are convergent sequences with  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ .  
Then,

- a.  $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$
- b.  $\lim_{n \rightarrow \infty} ks_n = ks$  and  $\lim_{n \rightarrow \infty} (k + s_n) = k + s$ , for any  $k \in \mathbb{R}$
- c.  $\lim_{n \rightarrow \infty} (s_n t_n) = st$
- d.  $\lim_{n \rightarrow \infty} \left(\frac{s_n}{t_n}\right) = \frac{s}{t}$ , provided that  $t_n \neq 0 \forall n \in \mathbb{N}$  and  $t \neq 0$

### Theorem 4.2.4

Assume that

$$\lim_{n \rightarrow \infty} s_n = s$$

and

$$\lim_{n \rightarrow \infty} t_n = t$$

If  $s_n \leq t_n \forall n \in \mathbb{N}$

then  $s \leq t$