Lecture 1

Theorem 3.2.8 - pg 118

Let x, y $\in \mathbb{R}$

a. If x $\leq$ y + ϵ $\forall$ $\epsilon > 0$, then x $\leq$ y

b. If $|x - y| \leq \epsilon$ $\forall$ $\epsilon > 0$, then $|x - y| = 0$ or, evidently, $x = y$

Definition 3.2.9

If x $\in \mathbb{R}$,

$$|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem 3.2.10

Let x, y $\in \mathbb{R}$ and a ≥ 0

Then

a. $|x| \geq 0$

b. $|x| \leq a$ iff $-a \leq x \leq a$

c. $|xy| = |x||y|$

d. $|x + y| \leq |x| + |y|$ (equality holds only if signs are the same)

Lecture 2

Theorem 3.3: The Completeness Axiom

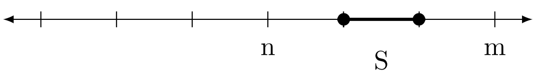
Recall the Fundamental Theorem of Arithmetic:
if n $\in \mathbb{N}$ with n ≥ 2, then n may be expressed as the product of prime numbers (the prime factorization (PF)).
The PF is unique with respect to (WRT) order.
Ex: $12 = 2 * 2 * 2 * 3$

Theorem 3.3.1

Let: p be a prime number
Then $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$

Definition 3.3.7

Let S $\subset \mathbb{R}$. If $\exists$ m $\in \mathbb{R}$ st s $\leq$ m $\forall$ s $\in S$,
then m is an upper bound of S and we say that S is **bounded above**.
Similarly, we can define **bounded below**.
If S is bounded above and below, then S is said to be **bounded**.
S can be open or closed. The example below is closed.



If an upper bound m of S is a member of S, then m is called the maximum (or largest element) of S, and we say that m = **max S**. Similarly, we may decline **minimum** of S **(n = min S)**.

Theorem 1

If a set S $\subset \mathbb{R}$ possesses a max element, then it is unique. A similar result holds for a minimum element.

Definition 3.3.5 (supremum defined)

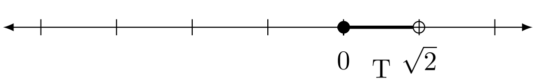
Let $\emptyset \neq$ S $\subset \mathbb{R}$ if S is bounded above,
then the **least upper bound** of S is called the **supremum** of S, denoted by sup S $\in \mathbb{R}$
iff:

a. s $\leq$ sup S $\forall$ s $\in$ S

b. $\exists$ s' $\in$ S st sup S $- \epsilon <$ s' $\forall \epsilon > 0$

Axiom of Completeness of the set of Real Numbers: $\mathbb{R}$

Every $\emptyset \neq$ S $\subset \mathbb{R}$ that is bounded above has a least upper bound (i.e. sup S $\in \mathbb{R}$ exists).
A similar statement can be made about inf S.
Remark: In practice 3.3.4, the set T = {q $\in \mathbb{Q}$: $0 \leq q \leq \sqrt{2}$} is bounded.



But $\sqrt{2}$ is not rational, so the set wouldn't have a least upper bound.
We need to fill in the gaps to make analysis work.

Lecture 3

Theorem 1 (infinum definition)

Let: $\emptyset \neq S \subset \mathbb{R}$, S is bounded below.
Then S possesses a greatest lower bound denoted by **inf S** (the **infinum** of S), where inf S $\in \mathbb{R}$, satisfying:

i) inf S $\leq$ s $\forall$ s $\in$ S

ii) $\forall\ \epsilon > 0$, $\exists\ s_1$ st inf S + $\epsilon > s_1$

Theorem 3.3.7

Given nonempty subsets of A, B (A, B $\subset \mathbb{R}$),
Let: C = {x + y: x $\in$ A, y $\in$ B}
If A and B have suprema, then C has a supremum: sup C = sup A + sup B

Theorem 3.3.8

Suppose $\emptyset \neq D \subset \mathbb{R}$ and
$f : D \longrightarrow \mathbb{R}$
$g : D \longrightarrow \mathbb{R}$
f(D) = {f(x) : x $\in$ D}
If $\forall$ x, y $\in$ D, f(x) $\leq$ g(y), then
f(D) is bounded above and g(D) is bounded below.
Furthermore, sup(f(D)) $\leq$ inf(g(D))

Theorem 3.3.9: Archimedian Property / Principle of $\mathbb{R}$ (AP)

The set $\mathbb{N}$ = {1, 2, 3...} is unbounded above in $\mathbb{R}$

Lecture 4

Theorem 3.3.10

Each of the following is equivalent to the AP:

a. $\forall$ z $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st n $>$ z

b. $\forall$ x $>$ 0, y $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st nx $>$ y

c. $\forall$ x $>$ 0, $\exists$ n $\in \mathbb{N}$ st 0 $< \frac{1}{n} <$ x

Theorems 3.3.13 and 3.3.15

Let: x, y $\in \mathbb{R}$ st x $<$ y
Then:

a. $\exists$ r $\in \mathbb{Q}$ st x $<$ r $<$ y

b. $\exists$ z $\in \mathbb{R} \setminus \mathbb{Q}$ st x $<$ z $<$ y

Section 3.4: Topology of $\mathbb{R}$

Definitions 3.4.1 and 3.4.2

Let x $\in \mathbb{R}$ and $\epsilon > 0$.

(a) An ϵ -neighborhood of x is: N(x, ϵ) = {y $\in \mathbb{R}$: $|y - x| < \epsilon$ }

(b) A deleted ϵ -neighborhood of x is: N*(x, ϵ) = {y $\in \mathbb{R}$: 0 $< |y - x| < \epsilon$ }

Open Set Topology: Definition 3.4.3 (interior / boundary point)

Let: S $\subset \mathbb{R}$
A point x $\in \mathbb{R}$ is an **interior point** of S if $\exists$ $\epsilon > 0$ st N(x, ϵ) $\subset$ S.

If, $\forall$ $\epsilon > 0$, $N(x, \epsilon) \cap$ S $\neq \emptyset$ and $N(x, \epsilon) \cap \mathbb{R} \setminus$ S $\neq \emptyset$
Then x is a **boundary point** of S.
The set of all interior points is denoted by **int S**.
The set of all boundary points is denoted by **bd S**.
Nota Bene (N.B.):
int S $\subset$ S and bd S = bd ($\mathbb{R} \setminus$ S)

Theorem 1

Let: x $\in$ S $\subset \mathbb{R}$
Then either x $\in$ int S, or x $\in$ bd S.

Lecture 5

Definition 3.4.6 - Def of Open/Closed Set

Let: $S \subset \mathbb{R}$

if bd $S \subset S$, then S is closed.
if bd $S \subset (\mathbb{R} \setminus S)$, then S is open.

Theorem 3.4.7

a. A set S is open iff S = int S; i.e. iff $\forall$ s $\in$ S, s is an **interior point**.

b. A set S is closed iff its compliment, $\mathbb{R} \setminus S$ is open.
   Equivalently, a set s is open iff $\mathbb{R} \setminus S$ is closed.

Theorem 2 (not in book)

Let: $x \in \mathbb{R}$, $\epsilon > 0$
Then N(x, ϵ), N*(x, ϵ) are open sets.

Theorem 3.4.10

Let: I be an index set. $I \subset \mathbb{N}$
Suppose: $G_\alpha \subset \mathbb{R}$ is an open set $\forall$ $\alpha \in I$
Then,

a. $\bigcup_{\alpha \in I} G_\alpha$ is an open set.

b. If $G_i \subset \mathbb{R}$ is open $\forall$ i = 1, 2, ... n $\in \mathbb{N}$, then $\bigcap_{i=1}^{n} G_i$ is open.

Corollary 3.4.11

a. Let F_α be closed $\forall$ $\alpha \in$ I, I is an index set.
   Then $\bigcap_{\alpha \in I} F_\alpha$ is closed.

b. Let F_i be closed $\forall$ i from 1 to n.
   Then ($\bigcup_{i=1}^{n} F_i$) is closed.

Accumulation (or Limit) Points; Definition 3.4.14

Let: $S \subset \mathbb{R}$
If $\forall$ $\epsilon > 0$, N*$(x, \epsilon) \cap S \neq \emptyset$,
Then $x \in \mathbb{R}$ is an **accumulation** or **limit** point. (The set of all accumulation points of S is denoted by $\mathbf{S}'$)
If $x \in S \setminus S'$,
then x is an **isolated point**,
in which case, $\exists$ $\epsilon > 0$ st N(x, ϵ) $\cap$ S = {x}

Definition 3.4.16 - Closures

Let: $S \subset \mathbb{R}$
Then the **closure** of S, denoted by **cl S**, is defined to be:
cl S = $S \cup S'$

Lecture 6

Theorem 3.4.17 - pg 118

Let: $S \subset \mathbb{R}$
Then

a. S is closed iff $S' \subset S$

b. cl S is a closed set

c. S is closed iff S = cl S

d. cl S = S U S' = S $\cup$ bd S

Lecture 7

Section 3.5: Compact Sets

Three big areas of analysis: compactedness, continuity, and connectedness.

Definition: Open Cover / Subcover

Let: S $\subset \mathbb{R}$
An **open cover** of S is a collection C of open sets st S $\subset \cup$ C. The collection C of open sets is said to **cover** the set S.
A subset of sets from the collection C that still covers S is called a **subcover** of S.

Definition 3.5.1

A set S $\subset \mathbb{R}$ is said to be **compact** if every **open cover** has a finite **subcover**
(i.e. if S $\subset \bigcup_{\alpha \in I}$ G_α) ,
where G_α is open $\forall$ $\alpha \in$ I; then $\exists$ n $\in \mathbb{N}$ and $\exists$ {n_1, n_2, ... n_k} $\subset$ I
st S $\subset \bigcup_{i=1}^{n} G_{\alpha_i}$

Lemma 3.5.4

If $\emptyset \neq S \subset \mathbb{R}$ and S is **closed** and **bounded**, then S has a maximum and a minimum.
In fact, in this, max S = sup S, and min S = inf S.

Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq$ S $\subset \mathbb{R}$ is compact iff S is closed and bounded.

Lecture 8

Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq$ S $\subset \mathbb{R}$ is compact iff S is closed and bounded.

Theorem 3.5.6: Bolzano−Weierstrass Theorem

If a bounded set S $\subset \mathbb{R}$ contains an infinite number of points, then $\exists$ at least one point in $\mathbb{R}$ that is an accumulation point of S.

Theorem 3.5.7 (F.I.P.)

Let: $\{K_\alpha\}_{\alpha \in I}$ be a family of compact sets, where I is an index set.
Suppose that the intersection of any finite subfamily of the K_α's has a nonempty intersection.
Then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Corollary 3.5.8 Nested Intervals Theorem

Let: $\{A_n\}_{n=1}^{\infty}$ be a family of nonempty closed bounded intervals in $\mathbb{R}$ st $A_{n+1} \subset A_n$ $\forall$ n $\in \mathbb{N}$
Then:
$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$

Lecture 9

Definition 1: Sequence

A **sequence** is a function S: $\mathbb{N} \longrightarrow \mathbb{R}$
We write S(n) = S_n $\forall$ n $\in \mathbb{N}$ and refer to $\{S_n\}$ (the book uses (S_n)) as the **sequence**.
We refer to the set $\{ S_n : n \in \mathbb{N} \}$ as the range of the sequence.

Side Note

$S_n = (-1)^n$ $\forall$ n $\in \mathbb{N}$
$\{(-1)^n\}$
range$\{S_n\} = \{-1, 1\}$
Here $\{S_n\} = \{1, -1, 1, -1...\}$

An alternative to writing $\{S_n\}$ for a sequence is to list the elements: S_1, S_2, ... S_n
Sometimes the domain of the sequence is $\mathbb{N} \cup \{0\}$ or $\{n \in \mathbb{N} : n \geq m\}$ for some m $\in \mathbb{N}$.
In this case, we write $\{S_n\}_{n=0}^{\infty}$ or $\{S_n\}_{n=m}^{\infty}$
Note 1: A denumerable set (or a countably infinite set) S is a set for which there is a bijection S: $\mathbb{N} \longrightarrow \mathbb{R}$
This bijection may be thought of as a sequence $\{S_n\}$, where S_n = S(n) $\forall$ n $\in \mathbb{N}$ of distinct terms.

Definition 4.1.2

A sequence $\{s_n\}$ is said to **converge** to s $\in \mathbb{R}$ provided that $\forall\ \epsilon > 0$
$\exists$ N $\in \mathbb{N}$ st N $\leq$ n implies $|s_n - s| < \epsilon$

Theorem 4.1.8

Let: $\{s_n\}$ and $\{a_n\}$ be sequences, s $\in \mathbb{R}$
If some k > 0 and some m $\in \mathbb{N}$, we have:
$|s_n - s| \leq k|a_n|$, $\forall$ n $\geq$ m **(1)**
and if $\lim_{n\longrightarrow\infty} a_n = 0$, then $\lim_{n\longrightarrow\infty} s_n = s$.

Lecture 10

Theorem 4.1.13

Every convergent sequence is bounded.

Theorem 4.1.14

If a sequence converges, then its limit is unique.

4.2 Limit Theorems

Theorem 4.2.1

Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$.
Then,

a. $\lim_{n\to\infty} (s_n + t_n) = s + t$

b. $\lim_{n\to\infty} ks_n = ks$ and $\lim_{n\to\infty} (k + s_n) = k + s$, for any $k \in \mathbb{R}$

c. $\lim_{n\to\infty} (s_n t_n) = st$

d. $\lim_{n\to\infty} (\frac{s_n}{t_n}) = \frac{s}{t}$, provided that $t_n \neq 0 \ \forall \ n \in \mathbb{N}$ and $t \neq 0$

Theorem 4.2.4

Assume that
$\lim_{n\to\infty} s_n = s$
and
$\lim_{n\to\infty} t_n = t$
If $s_n \leq t_n \ \forall \ n \in \mathbb{N}$
then $s \leq t$

Lecture 11

Spent doing Homework 5 Review

Lecture "12"

Spent taking Test 1

Lecture 13

Definition 4.3.1

A sequence (s_n) is **increasing** (or **decreasing**) if $s_n \leq s_{n+1}$ (or $s_{n+1} \leq s_n$) $\forall$ n $\in \mathbb{N}$. A sequence is **monotonic** if it is increasing or decreasing.

Theorem 4.3.3 (Monotone Convergence Theorem)

A **monotonic sequence** is convergent iff it is bounded.

Lecture 14

Theorem 4.3.8

a. If $\{s_n\}$ is an unbounded increasing sequence, then $\lim_{n\to\infty} s_n = \infty$

b. If $\{s_n\}$ is an unbounded decreasing sequence, then $\lim_{n\to\infty} s_n = -\infty$

Definition 4.3.9

A sequence $\{s_n\}$ is **Cauchy** if $\forall\ \epsilon > 0$, $\exists\ N(\epsilon\ ) \in \mathbb{N}$ st

$$|s_n - s_m| < \epsilon, \text{ for m, n } \geq N$$

Lemma 4.3.10

Every convergent sequence is Cauchy.

Lemma 4.3.11

Every Cauchy sequence is bounded (similar to exam question: Every convergent sequence is bounded)

Theorem 4.3.12 - Cauchy Convergence Criterion

A sequence of real numbers is convergent iff it is a Cauchy sequence.

4.4.1 Subsequences

Definition 4.4.1

Let: $\{s_n\}_{n=1}^{\infty}$ be a sequence
Also, let $\{n_k\}$ be a sequence $\in \mathbb{N}$ st

$$n_1 < n_2 < n_3 ...$$

The sequence $\{s_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence** of $\{s_n\}$.
Notice that, in this case, $n_k \geq k$ (i.e. $k \leq t_k$) $\forall\ k \in \mathbb{N}$
Thus, $\lim_{n\to\infty} n_k = \infty$