

1. Prove Pascal's Formula $\binom{\alpha}{k} = \binom{\alpha-1}{k-1} + \binom{\alpha-1}{k}$ for any $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. (Note: You will need to use the falling factorial definition.)

$$\begin{aligned}
 \binom{\alpha}{k} &= \binom{\alpha-1}{k-1} + \binom{\alpha-1}{k} \\
 &= \frac{(\alpha-1)!}{((\alpha-1)-(k-1))!(k-1)!} + \frac{(\alpha-1)!}{((\alpha-1)-k)!k!} \\
 &= \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} + \frac{(\alpha-1)!}{(\alpha-1-k)!k!} \\
 &= \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} + \frac{(\alpha-1)!(\alpha-k)^{\frac{1}{k}}}{(\alpha-k)!(k-1)!} \\
 &= \frac{(\alpha-1)! + (\alpha-1)!(\alpha-k)^{\frac{1}{k}}}{(\alpha-k)!(k-1)!} \\
 &= \frac{k(\alpha-1)! + (\alpha-1)!(\alpha-k)}{(\alpha-k)!k!} \\
 &= \frac{\alpha(\alpha-1)!}{(\alpha-k)!k!} \\
 &= \frac{\alpha!}{(\alpha-k)!k!}
 \end{aligned}$$

2. Determine the generating function for each of the following sequences:

a. $1, r, r^2, r^3, \dots$

$$1 + rx + r^2x^2 + \dots \longrightarrow \frac{1}{1-rx}$$

b. $1, -1, 1, -1, \dots$

$$1 - x + x^2 - x^3 + \dots \longrightarrow \frac{1}{1+x}$$

c. $\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, -\binom{\alpha}{3}, \dots$

$$\binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \binom{\alpha}{3}x^3 + \dots$$

$$1 - \alpha x + \frac{\alpha(\alpha-1)}{2*1}x^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3*2*1}x^3 + \dots$$

$$1 - \alpha x + \frac{[\alpha]_{(2)}}{[2]_{(2)}}x^2 - \frac{[\alpha]_{(3)}}{[3]_{(3)}}x^3 + \dots$$

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} x^k$$

$$(1-x)^\alpha$$

d. $1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots$

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$e^x$$

e. $1, \frac{-1}{1!}, \frac{1}{2!}, \frac{-1}{3!}, \frac{1}{4!}, \dots$

$$1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \dots$$

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots - 2\left(\frac{1}{1!}x + \frac{1}{3!}x^3 + \dots\right)$$

$$e^x - \sinh x$$

f. $\binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{3}{2}, \dots$

$$\binom{0}{2} + \binom{1}{2}x + \binom{2}{2}x^2 + \binom{3}{2}x^3 + \dots$$

$$-\frac{1}{2} + 0x + \frac{[2]_{(2)}}{[2]_{(2)}}x^2 + \frac{[3]_{(2)}}{[2]_{(2)}}x^3 + \dots$$

$$-\frac{1}{2} + 0x + \frac{[2]_{(2)}}{2}x^2 + \frac{[3]_{(2)}}{2}x^3 + \dots$$

Is this the right process? How do you know when to use EGF vs GF?

3. Given the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$,
- a. Solve the recursion by writing it as a linear homogenous recursion and finding the characteristic polynomial. Write your answer in the form $c_1 q_1^n + c_2 q_2^n$. (Note: we have already solved this up to finding the constants in class. Finish the problem.)

$$f_n = f_{n-1} + f_{n-2}$$

$$0 = f_n - f_{n-1} - f_{n-2}$$

$$q^n - q^{n-1} - q^{n-2} = 0$$

$$q^{n-2}(q^2 - q^1 - 1) = 0$$

Thus, the solution has the form $f_n = c_1(?)^n + c_2(?)^n$.

$$q = \frac{1 \pm \sqrt{5}}{2}$$

$$f_n = c_1 \frac{1 + \sqrt{5}}{2}^n + c_2 \frac{1 - \sqrt{5}}{2}^n$$

$$f_0 = c_1 + c_2$$

$$f_1 = c_1 \left(\frac{1 + \sqrt{5}}{2}\right)^1 + c_2 \left(\frac{1 - \sqrt{5}}{2}\right)^1$$

Let $f_0 = 0$, $f_1 = 1$. Solving for c_1 and c_2 gives us $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = \frac{-1}{\sqrt{5}}$

$$\text{Thus, } f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

- b. Solve the recursion by using generating functions. (Note: Use a partial fraction decomposition to finish the problem.)

$$f_n = f_{n-1} + f_{n-2}$$

$$h_n = h_{n-1} + h_{n-2}$$

$$0 = h_n - h_{n-1} - h_{n-2}$$

$$\text{Let } g(x) = h_0 + h_1 x^1 + h_2 x^2 + \dots$$

Then,

$$\begin{aligned} g(x) &= h_0 + h_1 x^1 + h_2 x^2 + \dots \\ -xg(x) &= -h_0 x^1 - h_1 x^2 - h_2 x^3 + \dots \\ -x^2 g(x) &= -h_0 x^2 - h_1 x^3 - h_2 x^4 + \dots \end{aligned}$$

Thus,

$$(1 - x - x^2)g(x) = h_0 + (h_1 - h_0)x^1 + (h_2 - h_1 - h_0)x^2 + (h_3 - h_2 - h_1)x^3 + \dots$$

But since $0 = h_n - h_{n-1} - h_{n-2}$,

$$(1 - x - x^2)g(x) = h_0 + (h_1 - h_0)x^1$$

$$g(x) = \frac{h_0 + (h_1 - h_0)x}{(1 - x - x^2)}$$

Plugging in $h_0 = 0$ and $h_1 = 1$,

$$g(x) = \frac{x}{(1 - x - x^2)}$$

$$g(x) = \frac{x}{(1 - x - x^2)}$$

$$g(x) = \frac{x}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2}))(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2}))}$$

$$g(x) = \frac{A}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2}))} + \frac{B}{(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2}))}$$

$$g(x) = \frac{1/2}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2}))} + \frac{1/2}{(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2}))}$$

$$g(x) = \frac{1}{2(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2}))} + \frac{1}{2(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2}))}$$

$$g(x) = \frac{1}{2} \frac{1}{((-\frac{1}{2} + \frac{\sqrt{5}}{2}) + x)} - \frac{1}{2} \frac{1}{((-\frac{1}{2} + \frac{\sqrt{5}}{2}) - x)}$$

At this point, I'm not sure how to convert to Power Series

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

4. Prove that the Fibonacci number f_n is even if, and only if, divisible by 3.

Wait.. 2 is a fibonacci number that is even and not divisible by 3.. So is 8.

→

Assume: f_n is even (i.e. $\exists t \in \mathbb{Z}$ such that $f_n = 2t$)

←

Assume: 3 divides f_n (i.e. $\exists t \in \mathbb{Z}$ such that $f_n = 3t$)

5. Consider a 1-by- n chessboard. Suppose we color each square of the chessboard with one of the colors red, white, or blue. Let h_n be the number of colorings in which there is an even number of red squares (the example from class).

- a. Reproduce the exponential generating function solution from class.

Colors: R, W, B. R is even.

EGF:

$$\begin{aligned} & \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right) \left(1 + \frac{x^1}{1!} + \frac{x^2}{2!} \dots\right)^2 \\ & \frac{1}{2} (e^x + e^{-x}) e^x e^x \\ & \frac{1}{2} (e^{3x} + e^x) \\ & \frac{1}{2} \left(\sum \frac{3^n x^n}{n!} + \sum \frac{x^n}{n!} \right) \\ & \frac{1}{2} \left(\sum (3^n + 1) \frac{x^n}{n!} \right) \\ & \Rightarrow \sum \left(\frac{3^n + 1}{2} \right) \frac{x^n}{n!} \end{aligned}$$

- b. Solve this by using a standard generating function and partial fractions.

GF:

$$\begin{aligned} & (1 + x^2 + x^4 \dots)(1 + x + x^2 \dots)^2 \\ & \frac{1}{1-x^2} \frac{1}{1-x} \frac{1}{1-x} \\ & \frac{1}{(1+x)(1-x)^3} \\ & \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3} \end{aligned}$$

After partial fractions:

$$\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 1 \\
-3 & -1 & 0 & 1 & 0 \\
3 & -1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
A = \frac{1}{8}, B = \frac{1}{8}, C = \frac{1}{4}, D = \frac{1}{2}
\end{array}$$

$$\Rightarrow \frac{1}{8} \sum (-1)^n x^n + \frac{1}{8} \sum x^n + \frac{1}{4} \sum n x^{n-1} + \frac{1}{2} \sum n(n-1) x^{n-2}$$

How do I get from above to below?

$$\sum \left(\frac{3^n + 1}{2} \right) x^n$$

- c. Reproduce the associated recursion for h_n .
- d. Using your answer from part c, solve the recursion using the generating function method for non-homogeneous recursions.
6. Consider a 1-by- n chessboard. Suppose we color each square of the chessboard with one of the colors red or blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find a recurrence relation that h_n satisfies, then derive a formula for h_n .

Colors: R, B.

h_n is the number of colorings such that no two adjacent squares are red.

In other words, $B \Rightarrow R$ or B and $R \Rightarrow B$.

$$h_1 = 2, h_2 = 3, h_3 = 5, h_4 = 8, h_5 = 13 \dots$$

It looks like the number of ways that color the n th square blue is just h_{n-1} , and the number of ways to color the n th square red is the number of ways that color the $(n-1)$ th square blue, which is just h_{n-2} .

So it looks like a recurrence relation would be $h_n = h_{n-1} + h_{n-2}$

Since this is Fibonacci (and we already did this):

$$h_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

7. Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which apples $\% 2 = 0$, oranges ≤ 2 , bananas $\% 3 = 0$, and pears ≤ 1 . Then find a formula for h_n from the generating function.

GF:

$$\begin{aligned}
& (1 + x^2 + x^4 + \dots)(1 + x + x^2)(1 + x^3 + x^6 \dots)(1 + x) \\
& \frac{1}{(1-x)^2} \frac{1-x^3}{1-x} \frac{1}{(1-x)^3} \frac{1-x^2}{1-x} \\
& \frac{1}{(1+x)(1-x)} \frac{(1-x)(1+x+x^2)}{1-x} \frac{1}{(1-x)^3} \frac{(1-x)(1+x)}{1-x} \\
& \frac{(1+x+x^2)}{(1-x)^4} \\
& \frac{1}{(1-x)^2} + \frac{-3}{(1-x)^3} + \frac{3}{(1-x)^4}
\end{aligned}$$

$$\sum x^{2n} - 3 \sum x^{3n} + 3 \sum x^{4n} \longrightarrow \sum x^{2n} - 3x^{3n} + 3x^{4n}$$

8. Determine the exponential generating function for the following sequence:

a. $0!, 1!, 2!, \dots$

$$g^{(e)}(x) = \frac{0!}{0!} + \frac{1!}{1!}x + \frac{2!}{2!}x^2 \dots$$

$$g^{(e)}(x) = 1 + x + x^2 \dots$$

b. $[\alpha]_{(0)}, [\alpha]_{(1)}, [\alpha]_{(2)}, [\alpha]_{(3)}, \dots$ (Note: $[\alpha]_{(n)}$ is the falling factorial.)

$$g^{(e)}(x) = \frac{\alpha}{0!} + \frac{\alpha(\alpha-1)}{1!}x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^2 \dots$$

$$g^{(e)}(x) = \sum_{n=0}^{\infty} \frac{\alpha!}{(\alpha-n-1)!n!}$$

9. Let h_n denote the number of ways to color the square of a 1-by- n board with the colors red, white, blue, and green in such a way that the numbers of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence, then find a simple formula for h_n .

Colors: RWBG. R is even, W is odd.

EGF:

$$(1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots)(x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots)^2$$

$$\frac{1}{2}(e^x + e^{-x}) * \frac{1}{2}(e^x - e^{-x}) * e^x * e^x$$

$$\frac{1}{4}(e^{2x} - e^{-2x}) * e^{2x}$$

$$\frac{1}{4}(e^{4x} - 1)$$

$$f(x) = \frac{1}{4}(e^{4x} - 1) \longrightarrow f'(x) = \frac{1}{4}(4e^{4x}) \longrightarrow f''(x) = \frac{1}{4}16e^{4x} \longrightarrow f'''(x) = \frac{1}{4}64e^{4x} \longrightarrow f^n(x) = \frac{1}{4}(4^n e^{4x})$$

So,

$$h_n = \frac{4^n}{4n!} \text{ Which doesn't seem right since } n! \text{ grows faster than } 4^n. \text{ Also, where goes } -1?$$

10. Determine the number of ways to color the squares of a 1-by- n board using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

Colors:RGBO. R, G are even.

GF:

$$(1 + x^2 + x^4 \dots)^2(1 + x + x^2 + x^3 \dots)^2$$

$$\frac{1}{1-x^2} \frac{1}{1-x^2} \frac{1}{1-x} \frac{1}{1-x}$$

$$\frac{1}{(1-x)^4(1+x)^2}$$

$$\frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1-x)^4} + \frac{E}{(1+x)} + \frac{F}{(1+x)^2}$$

$$A \sum x^n + E \sum (-1)^n x^n$$

11. Determine the number of n -digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.

d_1, d_2, \dots, d_n where $d_i \% 2 = 1$

1 and 3 each occur a nonzero, even number of times. 5, 7, and 9 can occur any amount of times.

So:

$$(x^2 + x^4 + x^6 \dots)^2 (1 + x + x^2 \dots)^3$$

which is

$$\begin{aligned} & \left(\frac{1}{(1-x)^2} - 1 \right)^2 \left(\frac{1}{1-x} \right)^3 \\ & \left(\frac{1 - (1-x)^2}{(1-x)^2} \right)^2 \left(\frac{1}{1-x} \right)^3 \\ & \left(\frac{2x - x^2}{(1-x)^2} \right)^2 \left(\frac{1}{1-x} \right)^3 \\ & \frac{(2x - x^2)^2}{(1-x)^7} \\ & \frac{x^4 - 4x^3 + 4x^2}{(1-x)^7} \end{aligned}$$

which decomposes into:

$$\begin{aligned} & \frac{-1}{(x-1)^3} + \frac{2}{(x-1)^5} + \frac{-1}{(x-1)^7} \\ & - \sum x^{3n} + 2 \sum x^{5n} - \sum x^{7n} \rightarrow \sum -x^{3n} + 2x^{5n} - x^{7n} \end{aligned}$$

12. Solve the recurrence relation:

- a. $h_n = 4h_{n-2}$, $h_0 = 0$, $h_1 = 1$, and $n \geq 2$.

$$0, 1, 0, 4, 0, 16, 0, 64, \dots$$

$$h_n - 4h_{n-2} = 0$$

$$q^{n-2}(q^2 - 4) = 0$$

$$h_n = a(2)^n + b(-2)^n$$

$$0 = a + b \text{ and } 1 = 2a - 2b$$

$$b = -\frac{1}{4}, a = \frac{1}{4}$$

$$h_n = \frac{1}{4}2^n - \frac{1}{4}(-2)^n$$

- b. $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$. $n \geq 3$.

$$q^{n-3}(q^3 - q^2 - 9q^1 - 9) = 0$$

$$(q^2 - 9)(q^1 + 1) = 0$$

$$(q - 3)(q + 3)(q + 1) = 0$$

$$h_n = a(3)^n + b(-3)^n + c(-1)^n$$

$$\text{So, } 0 = a + b + c, 1 = 3a - 3b - c, 2 = 9a + 9b + c$$

$$\text{Hence, } a = \frac{1}{4}, b = 0, c = -\frac{1}{4}$$

$$h_n = \frac{1}{4}(3)^n - \frac{1}{4}(-1)^n$$

- c. $h_n = 4h_{n-1} + 4^n$, $h_0 = 3$ and $n \geq 1$.

$$3, 16, 80, 384, \dots$$

13. Let h_n = the number of ternary strings of length n made up of 0's, 1's, and 2's, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$h_n = h_{n-1} + 2h_{n-2}$$

with $h_0 = 1$, $h_1 = 3$, and then find a formula for h_n .

14. Compute the Stirling numbers of the first and second kind up to $n = 6$ using their recursive formulas.

$$S(6, 6) = 6 S(5, 6) + S(5, 5)$$

$$s(6, 6) = s(5, 5) - 5s(5, 6)$$

15. Prove the Stirling numbers of the second kind satisfy:

Recall: $S(p, k) = k S(p - 1, k) + S(p - 1, k - 1)$ and

$$S(p, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^p$$

a. $S(n, 1) = 1$

b. $S(n, 2) = 2^{n-1} - 1$

c. $S(n, n-1) = \binom{n}{2}$

16. Prove the Stirling numbers of the first kind satisfy:

Recall: $s(p, k) = s(p - 1, k - 1) - (p - 1)s(p - 1, k)$ and

$$s(p, k) = \sum_{i=k}^p p^{i-k} s(p+1, i+1)$$

a. $s(n, 1) = (n-1)!$

b. $s(n, n-1) = \binom{n}{2}$

17. Write $[n]_{(k)}$ as a polynomial in n for $k = 5, 6, 7$. (Do not use distribution!)

$$[n]_{(k)} = n(n-1)(n-2)\dots(n-k)$$

$$[n]_{(k)} = \sum_{p=0}^k (-1)^{k-p} s(k, p) n^p$$

$$[n]_{(5)} = \sum_{p=0}^5 (-1)^{5-p} s(5, p) n^p$$

$$[n]_{(5)} = -s(5, 0) + s(5, 1)n - s(5, 2)n^2 + s(5, 3)n^3 - s(5, 4)n^4 + s(5, 5)n^5$$

$$[n]_{(5)} = 4!n - s(5, 2)n^2 + s(5, 3)n^3 - \binom{5}{2}n^4 + n^5$$

$$s(5, 2) = 4s(4, 2) + 3! \text{ and } s(5, 3) = 4\binom{4}{2} + s(4, 2)$$

18. Find a closed formula for the sequence: 1, 6, 15, 28, 45, 66, 91, ... (Use a difference table.)

	1	6	15	28	45	66	91
		5	9	13	17	21	25
			4	4	4	4	4
				0	0	0	0

$$h_n = 1\binom{n}{0} + 5\binom{n}{1} + 4\binom{n}{2}$$