

Homework Due 10/5/17: (7 problems) Section 4.1 pages 169 - 170; 1, 6(b), 7(f), 9(a), 11, 12, 15

Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18

## Theorem 4.1.13

Every convergent sequence is bounded.

*Proof.*

**Assume:**  $s_n \rightarrow s$  as  $n \rightarrow \infty$

Then for  $\epsilon = 1$ ,  $\exists N \in \mathbb{N}$  st,  $\forall n \geq N$ ,

$|s_n - s| \leq 1$  (page 121, Ex 61(a))

Side Note

$x \leq |x| \forall x \in \mathbb{R}$

So  $|s_n| < 1 + |s|$ ,  $\forall n \geq N$

$|s_n| \leq |s_n - s| + |s| \leq 1 + |s| \forall n \geq N$

Then,

**Let:**  $m = \max \{|s_1|, |s_2|, \dots, |s_{N-1}|, |s|\}$

Then  $|s_n| \leq m$ ,  $\forall n \in \mathbb{N}$

Hence,  $\{s_n\}$  is bounded.

□

## Theorem 4.1.14

If a sequence converges, then its limit is unique.

*Proof.*

**Assume:**  $\{s_n\}$  is a sequence and  $s_n \rightarrow s$  as  $n \rightarrow \infty$  and  $s_n \rightarrow t$  as  $n \rightarrow \infty$

Then  $\forall \epsilon > 0$ ,  $\exists N_1(\epsilon)$  st

$|s_n - s| < \frac{\epsilon}{2}$ ,  $\forall n \geq N_1$  **(1)**

Also,

$\exists N_2(\epsilon) \in \mathbb{N}$  st

$|s_n - t| < \frac{\epsilon}{2}$ ,  $\forall n \geq N_2$  **(2)**

Set  $N = \max \{N_1, N_2\}$

From **(1)**, **(2)**

$|s - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ ,  $\forall n \geq N$

Hence,

$s = t$

□

## 4.2 Limit Theorems

### Theorem 4.2.1

Suppose that  $\{s_n\}$  and  $\{t_n\}$  are convergent sequences with  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ .

Then,

- $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$
- $\lim_{n \rightarrow \infty} ks_n = ks$  and  $\lim_{n \rightarrow \infty} (k + s_n) = k + s$ , for any  $k \in \mathbb{R}$
- $\lim_{n \rightarrow \infty} (s_n t_n) = st$
- $\lim_{n \rightarrow \infty} \left(\frac{s_n}{t_n}\right) = \frac{s}{t}$ , provided that  $t_n \neq 0 \forall n \in \mathbb{N}$  and  $t \neq 0$

*Proof.*

**(a)**

$$\begin{aligned} |t + s - (s_n + t_n)| &= \\ |(t - t_n) + (s - s_n)| &\leq |t - t_n| + |s - s_n| \quad \textbf{(1)} \\ \forall \epsilon > 0, \exists N_1(\epsilon), N_2(\epsilon) \text{ st} \end{aligned}$$

$$|t - t_n| < \frac{\epsilon}{2} \quad \forall n \geq N_1 \quad (2)$$

and

$$|s - s_n| < \frac{\epsilon}{2}, \quad \forall n \geq N_2 \quad (3)$$

**Let:**  $N = \max \{N_1, N_2\}$

From **(1) - (3)**,

$$|s + t - (s_n + t_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq N$$

Hence, result.

**(c)**

$$|st - s_n t_n| = |(st - s_n t) + (s_n t - s_n t_n)| \leq |st - s_n t| + |s_n t - s_n t_n| = |s - s_n| |t| + |s_n| |t - t_n|$$

By theorem 4.1.3,  $\exists M > 0$  st  $|s_n| \leq M \quad \forall n \in \mathbb{N}$

So,

$$|st - s_n t_n| \leq |t| |s - s_n| + M |t - t_n|$$

**Let:**  $\epsilon > 0$

Then  $\exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$  st

$$\forall |s - s_n| < \frac{|t|\epsilon}{|t|+M}, \quad \forall n \geq N_1$$

and

$$M |t - t_n| < \frac{M\epsilon}{|t|+M}, \quad \forall n \geq N_2$$

Set  $N = \max \{N_1, N_2\}$

Then

$$|st - s_n t_n| < |t| \frac{\epsilon}{|t|+M} + \frac{M\epsilon}{|t|+M} = \epsilon \left( \frac{|t|+M}{|t|+M} \right) = \epsilon \quad \forall n \geq N$$

Hence, result.

(d)

Since  $\frac{s_n}{t_n} = (\frac{1}{t_n})(s_n)$ , the proof follows from (c) if we can prove that  $\lim_{n \rightarrow \infty} \frac{1}{t_n} = \frac{1}{t}$

Now:

$$|\frac{1}{t} - \frac{1}{t_n}| = |\frac{t_n - t}{t t_n}| = \frac{|t_n - t|}{|t| |t_n|} \quad (1)$$

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Side Note

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$$|t_n| > 1$$

$$|t_n| \geq M$$


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Recall that:

$$|t| - |t_n| \leq ||t| - |t_n|| \leq |t - t_n| \text{ from page 121, example 6(a)}$$

For  $\epsilon = |t| > 0$ ,  $\exists N_1(\epsilon) \in \mathbb{N}$  st

$$|t - t_n| < \frac{|t|}{2}, \forall n \geq N_1$$

$$\text{Now } |t| - |t_n| \leq ||t| - |t_n|| \leq |t - t_n| < \frac{|t|}{2} \quad \forall n \geq N_1$$

$$\text{So } |t_n| > \frac{|t|}{2}$$

$$\text{Equivalently, } \frac{|t - t_n|}{|t| |t_n|} < \frac{2|t - t_n|}{|t| |t|} \quad (2)$$

From (1) and (2)

$$|\frac{1}{n} - \frac{1}{t_n}| < \frac{2|t_n - t|}{|t|^2}, \forall n \geq N_1 \quad (3)$$

Also,

$$\exists N_2(\epsilon) \in \mathbb{N} \text{ st}$$

$$|t_n - t| < \frac{\epsilon |t|^2}{2}, \forall n \geq N_2 \quad (4)$$

$$\text{Let: } N = \max \{N_1, N_2\}$$

Then from (3) and (4),

$$|\frac{1}{t} - \frac{1}{t_n}| < \frac{2}{|t|^2} \frac{\epsilon |t|^2}{2} = \epsilon \quad \forall n \geq N$$

Hence, result.

□

## Example 4.2.2

Find

$$\lim_{n \rightarrow \infty} \frac{(4n^2 - 3)}{(5n^2 - 2n)}$$

=

$$\lim_{n \rightarrow \infty} \frac{n^2(4 - \frac{3}{n^2})}{n^2(5 - \frac{2}{n})}$$

Now,

$$\lim_{n \rightarrow \infty} \frac{3}{n^2} = 0 = \lim_{n \rightarrow \infty} \frac{2}{n}$$

By Theorem 4.2.1, (b)

$$\lim_{n \rightarrow \infty} (4 - \frac{3}{n^2}) = 4$$

and

$$\lim_{n \rightarrow \infty} (5 - \frac{2}{n}) = 5$$

By Theorem 4.2.11 (d),

$$\lim_{n \rightarrow \infty} \frac{(4n^2 - 3)}{(5n^2 - 2n)} = \frac{4}{5}$$

## Theorem 4.2.4

Assume that

$$\lim_{n \rightarrow \infty} s_n = s$$

and

$$\lim_{n \rightarrow \infty} t_n = t$$

If  $s_n \leq t_n \forall n \in \mathbb{N}$

then  $s \leq t$

*Proof.*

Assume  $s > t$

Then  $s - t = 0$

$\exists N_1(s - t), N_2(s - t) \in \mathbb{N}$  st

$$|s - s_n|$$

$$|s_n - s| < \frac{s-t}{2}, \forall n \geq N_1 \quad (1)$$

and

$$|t_n - t| < \frac{s-t}{2}, \forall n \geq N_2 \quad (2)$$

**Let:**  $N = \max \{N_1, N_2\}$

From (1)

$$\frac{-(s-t)}{2} < s_n - s < \frac{s-t}{2}, \forall n \geq N \quad (3)$$

From (2)

$$\frac{-(s-t)}{2} < t_n - s < \frac{s-t}{2}, \forall n \geq N \quad (4)$$

Setting  $n = N$ :

$$(3) \longrightarrow s_n > s - \frac{s-t}{2} = s - \frac{s}{2} + \frac{t}{2} = \frac{s+t}{2} \quad (5)$$

$$(4) \longrightarrow t_n < \frac{s-t}{2} + t = \frac{s+t}{2} \quad (6)$$

(5) and (6) yield to the contradiction that  $t_n < \frac{s+t}{2} < s_n$

Hence, result.

□