Assignment Set: 6, 7, 15, 17, 19, 21 from pages 141 - 142

6)

Find the closure of each set:

a. $\{ \frac{1}{n} : \text{n} \in \mathbb{N} \}$

Answer: $\emptyset$

b. $\mathbb{N}$

Answer: $\mathbb{N}$

c. $\mathbb{Q}$

Answer: $\mathbb{R}$

d. $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

Answer: $\emptyset$

e. $\{ \text{x} : |x - 5| \leq \frac{1}{2}\}$

$[4.5, 5.5]$

Answer: $[4.5, 5.5]$

f. $\{ \text{x} : x^2 > 0\}$

$(0, \infty)$

Answer: $[0, \infty)$

7)

Let S, T $\subset \mathbb{R}$. Find a counterexample of each of the following:

a. If P is the set of all isolated points of S, then P is a closed set.

Answer: Let S $= \mathbb{N}$

b. Every open set contains at least two points.

Answer: $\emptyset$

c. If S is closed, then cl(int S) = S.

Answer: Let S $= \mathbb{Q}$

d. If S is open, then int (cl S) = S.

Answer: Let S $= (-1, 0) \cup (0, 1)$

e. bd (cl S) = bd S

Answer: Let S $= (-1, 0) \cup (0, 1)$

f. bd (bd S) = bd S

Answer: Let S $= \mathbb{Q}$. Then bd S is $\mathbb{R}$, and bd (bd S) $= \emptyset \neq \mathbb{R}$.

g. bd (S $\cup$ T) = (bd S) $\cup$ (bd T)

Answer: Let S $= \mathbb{R}$, T $= (0, 1)$. bd (S $\cup$ T) $= \emptyset$, but bd S $\cup$ bd T $= \emptyset \cup \{0, 1\}$

h. bd (S $\cap$ T) = (bd S) $\cap$ (bd T)

Answer: Let S $= (0, 1)$, T $= (1, 2)$. bd (S $\cap$ T) $= \emptyset$, but bd S $\cap$ bd T = 1.

15)

Prove: If x is an accumulation point of the set S, then every neighborhood of x contains infinitely many points of S.

Proof.
Suppose that $\exists$ a deleted neighborhood of x, called N, that contains n points x_1, x_2, ... x_n of S where n is a finite amount and $x_1 \leq x_2, \leq ... x_n$
x is an accumulation point on S if $\forall \epsilon > 0$, N*(x, ϵ) $\cap$ S $\neq \emptyset$.
N is a deleted neighborhood of S if $\forall$ x $\in \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$, x $\in$ N.
Let $\hat{\epsilon} = \epsilon + \epsilon$, and $x_0 = x_1 - \hat{\epsilon}$.
By definition, $x_0 \in$ N, since N is a neighborhood $\forall \epsilon > 0$.
However, N only has n elements. A contradiction.
So, N can't be a deleted neighborhood since it has a finite number of elements, which means x can't be an accumulation point.

□

17)

Prove: S$'$ is a closed set.

Proof.

By definition, $\forall s \in S'$, $\forall \epsilon > 0$, N*(s, ϵ) $\cap$ S $\neq \emptyset$
Want to show: $\mathbb{R} \setminus$ S$'$ is open.

□

19)

Suppose S is a nonempty bounded set and let m = sup S. Prove or give a counter example: m is a boundary point of S.

Proof.

By definition,
s $\leq$ m, $\forall$ s $\in$ S, and,
$\forall$ $\epsilon > 0$, $\exists$ s' $\in$ S st m $- \epsilon <$ s'
By the second part of the definition of the supremum of S, N(m, ϵ) $\cap$ S $\neq \emptyset$.
Notice also that, by the first part of the definition of the supremum of S, (m + ϵ) $\notin$ S. This means that N(m, ϵ) $\cap$ $\mathbb{R} \setminus$ S $\neq \emptyset$.
By definition, m is a boundary point.

□

21)

Let A be a nonempty open subset of $\mathbb{R}$ and let Q $\subset \mathbb{Q}$. Prove: A $\cap$ Q $\neq \emptyset$.

Proof.
Notice that Q $\subset \mathbb{Q} \subset \mathbb{R}$.
Since A is nonempty, $\exists$ at least one element a $\in \mathbb{R}$.
Since A is nonempty and open, a + $\epsilon \in$ A.
If a $\in \mathbb{Q}$, then result.
If a $+\epsilon \in \mathbb{Q}$, then result.
If a $\notin \mathbb{Q}$ and (a + ϵ) $\notin \mathbb{Q}$, then:
Let x = a, y = a + ϵ, z = y − x.
By Archimedes' axiom, $\exists$ n st n > $\frac{1}{z}$
nz > 1
ny − nx > 1
Since the difference between ny and nx is bigger than 1,
$\exists m \in \mathbb{Z}$ st nx < m < ny.
See that since x < $\frac{m}{n} < y$, $\frac{m}{n}$ is a rational number, and $\frac{m}{n} \in$ A.
Hence, result. $\square$