Last Lecture.
Final is over 5.1 through 5.4 and 6.1

Theorem 6.1.10: The Chain Rule

Recall:
f(x) = x^n implies f′ (x) = nx^{n-1} $\forall$ n $\in \mathbb{Z} \setminus \{0\}$
Let: I, J be intervals in $\mathbb{R}$ and f : I $\longrightarrow \mathbb{R}$ and g : J $\longrightarrow \mathbb{R}$, where f(I) $\subset$ J
Also,
Let: c $\in$ I
If f′ (c) exists and g′ (f(c)) exists, then (g o f)′ (c) = g′ (f(c)) * f′ (c)

Proof.

Since g′ (f(c)) exists,

$$\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c))$$

Define the function h : J $\longrightarrow \mathbb{R}$ where

$$h(y) = \frac{g(y) - g(f(c))}{y - f(c)}$$

if y $\neq$ f(c), and

$$h(y) = g'(f(c))$$

if y = f(c).
Notice that $\lim_{y \to f(c)}$ h(y) = g′ (f(c)) = h(f(c)) **(1)**
Since f(c) $\in$ J, which is an interval, f(c) is a limit/accumulation point of J.
Recall: I is an interval if where ever x_1, $x_2 \in$ I, $x_1 < x_2$, and $x_1 < x < x_2$ then x $\in$ I
Then, by Theorem 5.2.2(a)(d),
(a) h is continuous at f(c)
(d) $\lim_{y \to f(c)}$ h(y) = h(f(c))
h is continuous at f(c).
Also, since f is differentiable at c, it follows by Theorem 6.1.6 that f is continuous at c.
Also, h is continuous at f(c).
By Theorem 5.2.12, h o f is continuous at c.
By Theorem 5.2.2 (a)(d), we see that

$$\lim_{x \to c}(h \text{ o } f)(x) = \lim_{x \to c} h(f(x)) = h(f(c)) = g'(f(c)) \text{ from (1)}$$

Now, let x $\in$ I with x $\neq$ c.
Then,

$$(g \text{ o } f)(c) = \lim_{x \to c} \frac{(g \text{ o } f)(x) - (g \text{ o } f)(c)}{x - c} = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \tag{2}$$

Notice that h(y)(y − f(c)) = g(y) − g(f(c)), $\forall$ y $\in$ J.
Thus,

$$h(f(x))[f(x) - f(c)] = g(f(x)) - g(f(c)) \tag{3}$$

Substituting **(3)** into **(2)** yields that

$$(g \text{ o } f)'(c) = \lim_{x \to c} \frac{h(f(x))[f(x) - f(c)]}{x - c} = \lim_{x \to c}[h(f(x))] \lim_{x \to c}[\frac{f(x) - f(c)}{x - c}] = g'(f(c)) * f'(c)$$

Hence, result.
You might be wondering, though, why can't you do this?

$$(g \text{ o } f)'(c) = \lim_{x \to c}[\frac{g(f(x)) - g(f(c))}{x - c} * \frac{f(x) - f(c)}{f(x) - f(c)}] = \lim_{x \to c}[\frac{g(f(x)) - g(f(c))}{f(x) - f(c)}] * \lim_{x \to c}[\frac{f(x) - f(c)}{x - c}]$$

You can't let f(x) − f(c) = 0. It might be 0.

□