

Section 3.5: Compact Sets

Three big areas of analysis: compactness, continuity, and connectedness.

Definition 3.5.1

A set $S \subset \mathbb{R}$ is said to be compact if every **open cover** has a finite **subcover**

(i.e. if $S \subset \bigcup_{\alpha \in I} G_\alpha$,

where G_α is open $\forall \alpha \in I$; then $\exists n \in \mathbb{N}$ and $\exists \{n_1, n_2, \dots, n_k\} \subset I$

st $S \subset \bigcup_{i=1}^n G_{\alpha_i}$

Example 3.5.2

a. Show that $S = (0, 2)$ is not compact.

b. Show that $S = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}$ is compact.

(a)

Notice that:

$$(0, 2) \subset \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 3\right) \quad (1)$$

If $(0, 2)$ were compact, then from (1) there would exist a **finite** subcover.

Assume: $(0, 2)$ is compact.

So $\exists k \in \mathbb{N}$ and $\{n_1, n_2, \dots, n_k\} \subset \mathbb{N}$ st

$$(0, 2) \subset \bigcup_{i=1}^k \left(\frac{1}{n_i}, 3\right) \quad (2)$$

Choose $m = \max \{n_1, n_2, \dots, n_k\}$

Then, notice that $\left(\frac{1}{n_i}, 3\right) \subset \left(\frac{1}{m}, 3\right) \forall i = 1, 2, \dots, k$

From (1), $(0, 2) \subset \left(\frac{1}{m}, 3\right)$.

Notice that $0 < \frac{1}{m+1} < \frac{1}{m}$

and $\frac{1}{m+1} \in (0, 2)$.

However, $\frac{1}{m+1} \notin \left(\frac{1}{m}, 3\right)$.

(b)

Suppose that $S \subset \bigcup_{\alpha \in I} G_\alpha$ ($\alpha \in I$)

where I is an index set and G_α is open $\forall \alpha \in I$.

$\forall i = 1, 2, \dots, n, \exists \alpha_i \in I$ st $x_i \in G_{\alpha_i}$

Then, $S \subset \bigcup_i G_{\alpha_i}$

We see that a **finite** subset of \mathbb{R} is compact.

Lemma 3.5.4

If $\emptyset \neq S \subset \mathbb{R}$ and S is **closed** and **bounded**, then S has a maximum and a minimum. In fact, in this, $\max S = \sup S$, and $\min S = \inf S$.

Proof.

Since S is bounded, $\inf S, \sup S \in \mathbb{R}$ both exist.

Want to show: $\max S = \sup S$

For $\epsilon > 0$, $\exists s_1(\epsilon) \in S$ st
 $\sup S - \epsilon < s_1 \leq \sup S < \sup S + \epsilon$.
 So,
 $-\epsilon < s_1 - \sup S \leq \epsilon$
 Thus, $s_1 \in N(\sup S, \epsilon)$.
 So,

$$N(\sup S, \epsilon) \cap S \neq \emptyset \quad (1)$$

Also, $\sup S + \frac{\epsilon}{2} \in N(\sup S, \epsilon)$ and $\sup S + \frac{\epsilon}{2} \in \mathbb{R} \setminus S$.
 ($s \leq \sup S \forall s \in S$, and $\sup S \in S$)

$$N(\sup S, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset \quad (2)$$

From (1) and (2), $\sup S \in \text{bd } S \subset S$, since S is closed. Hence, $\sup S = \max S$.

□

Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq S \subset \mathbb{R}$ is compact iff S is closed and bounded.

Proof.

→

Suppose: S is compact

Want to show: S_∞ is bounded

Notice that $S \subset \bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}$, ???

where $(-n, n) = N(0, n)$ is open $\forall n \in \mathbb{N}$.

$G_n \subset G_{n+1} \forall n \in \mathbb{N}$.

Since S is compact, $\exists k \in \mathbb{N}$ and $\{n_1, n_2, \dots, n_k\} \subset \mathbb{N}$ st

$S \subset \bigcup_{i=1}^k (-n_i, n_i)$,

Let: $m = \max \{n_1, n_2, \dots, n_k\}$.

Then, $(-n_i, n_i) \subset (-m, m) \forall i = 1, 2, \dots, k$.

Thus, $S \subset (-m, m)$.

So, $|S| < m, \forall s \in S$.

Or, equivalently,

$-m < s < m, \forall s \in S$.

Hence, S is bounded.

Want to show: S is closed

Suppose: S is not closed

Thus, $\exists p \in \text{cl } S \setminus S$, i.e. $p \in S'$.

Side Note

S is closed iff $\text{cl } S = S \cup S' = S$

$S \subset S \cup S'$

If $\text{cl } S \neq S$, then $S \subset S \cup S'$

Notice that:

$\bigcap_{n=1}^{\infty} [p - \frac{1}{n}, p + \frac{1}{n}] = \{p\}$

So, is equal to $\mathbb{R} \setminus \{p\}$.

$$\begin{aligned}
S &\subset \mathbb{R} \setminus \bigcap_{n=1}^{\infty} [p - \frac{1}{n}, p + \frac{1}{n}] \\
S &\subset \bigcup_{n=1}^{\infty} \mathbb{R} \setminus [p - \frac{1}{n}, p + \frac{1}{n}] \\
S &\subset \bigcup_{n=1}^{\infty} [(-\infty, p - \frac{1}{n}) \cup (p + \frac{1}{n}, \infty)]
\end{aligned}$$

Since S is compact, $\exists k \in \mathbb{N}$ and $\{n_1, n_2, \dots, n_k\} \subset \mathbb{N}$ st

$$S \subset \bigcup_{i=1}^k [(-\infty, p - \frac{1}{n_i}) \cup (p + \frac{1}{n_i}, \infty)]$$

Let: $m = \max \{n_1, n_2, \dots, n_k\}$

Then $(-\infty, p - \frac{1}{n_i}) \cup (p + \frac{1}{n_i}, \infty) \subset (-\infty, p - \frac{1}{m}) \cup (p + \frac{1}{m}, \infty)$.

Thus, $S \subset [(-\infty, p - \frac{1}{m}) \cup (p + \frac{1}{m}, \infty)]$

←

Conversely,

Suppose: S is closed and bounded

Want to show: S is compact

Let: $S \subset \bigcup_{\alpha \in I} G_{\alpha}$, where G_{α} is open $\forall \alpha \in I$ (some index)

$\forall x \in \mathbb{R}$, define:

$$S_x = S \cap (-\infty, x]$$

Also define the set:

$$\beta = \{x \in \mathbb{R} : S_x \text{ is covered by a finite collection of the } G_{\alpha} \text{'s}\}$$

Notice that S is closed and bounded, so $\sup S = \max S$, $\inf S = \min S$,

and $S_{\inf S} = S \cap (-\infty, \inf S] = \{\inf S\} = \{\min S\}$ (since by Lemma 3.5.4, $\inf S = \min S$)

Now, since $\min S = \inf S \in S$, then $\exists \alpha_0 \in I$ st $\inf S \in G_{\alpha_0}$.

This proves that

$$S_{\inf S} = \{\inf S\} \subset G_{\alpha_0}$$

Hence, $\inf S \in \beta \neq \emptyset$.

□