Part of HW 11: 1, 2, and 5 from section 5.3
Hint: 5^x and x^4 are polynomials, so you can assume g(x) is continuous.
Look at 5.3.8 if you have a chance.

Theorem 5.3.2

f : D $\longrightarrow \mathbb{R}$ is continuous $\Rightarrow$ f(D) is compact (i.e. D is compact $\Rightarrow$ f(D) is compact)

Proof.

A set is compact iff every open cover of a set has a finite subcover.
Assume: f(D) $\subset \bigcup_{\alpha \in I}$ G_α and G_α is open $\forall$ $\alpha \in$ I
Thus, D $\subset \bigcup_{\alpha \in I}$ $f^{-1}(G_\alpha)$

(i.e. $\bigcup_{\alpha \in I}$ $f^{-1}(G_\alpha) = \bigcup_{\alpha \in I}$ $(H_\alpha \cap D) \subset \bigcup_{\alpha \in I}$ H_α)

Side Note

Let: x $\in$ D
Then f(x) $\in$ f(D)
So, $\exists$ $\alpha_0 \in$ I st f(x) $\in G_{\alpha_0}$
and x $\in f^{-1}(G_{\alpha_0}) \subset \bigcup_{\alpha \in I}$ $f^{-1}(G_\alpha)$

From

$$f^{-1}(G_\alpha) = H_\alpha \cap D, \text{ where } H_\alpha \text{ is open (Theorem 5.2.14)}$$

Since D is compact,
D $\subset \bigcup_{i=1}^{n} H_{\alpha_i}$
Thus,
D $\subset \bigcup_{i=1}^{n}$ $(H_{\alpha_i} \cap D) = \bigcup_{i=1}^{n}$ $f^{-1}(G_{\alpha_i})$
f(D) $\subset \bigcup_{i=1}^{n} G_{\alpha_i}$
Hence, f(D) is compact.

Side Note

Let: x $\in$ D
Then x $\in f^{-1}(G_{\alpha_k})$ where k $\in$ {1, 2.. n}
So f(x) $\in G_{\alpha_k} \subset \bigcup_{i=1}^{n} G_{\alpha_i}$
and
f(D) $\subset \bigcup_{i=1}^{n} G_{\alpha_i}$

$\square$

Corollary 5.3.3

f : D $\longrightarrow \mathbb{R}$ is continuous and D is compact.
Then $\exists$ x_1, $x_2 \in$ D st

$$f(x_1) \leq f(x) \leq f(x_2) \qquad \forall x \in D$$

where f(x_2) is the max of f(D) and f(x_1) is the min.

Proof.

f(D) is compact, so by Heine-Borel, f(D) is closed and bounded.
Thus, sup f(D) exists.
Now, sup f(D) is an accumulation point of f(D).
Since f(D) is closed,
sup f(D) $\in$ f(D) and sup f(D) = f(x_2) for some $x_2 \in$ D
i.e. y $\leq$ sup f(D) $\forall$ y $\in$ f(D)
and
$\exists$ $y_n \in$ f(D) st sup f(D) $- \frac{1}{n} <$ $y_n \leq$ sup f(D) $<$ sup f(D) + $\frac{1}{n}$

□

Section 5.4: Uniform Continuity

Definition 5.4.1

Let: f : D $\longrightarrow \mathbb{R}$
We say that f is uniformly continuous on D if for $\epsilon > 0$, $\exists$ $\delta > 0$ st |f(x) $-$ f(y)| $< \epsilon$ whenever |x $-$ y| $< \delta$ and x, y $\in$ D

Recall the old definition of continuity:
f is continuous at c $\in$ D iff $\forall$ $\epsilon > 0$, $\exists$ $\delta > 0$ st |x $-$ c| and x $\in$ D $\Rightarrow$ |f(x) $-$ f(c)| $< \epsilon$

The old one depends on c and ϵ , where as the new definition only depends on ϵ
Notice that if f is uniformly continuous on D, then f is certainly continuous on D.

Example 5.4.2

Prove that f : $\mathbb{R} \longrightarrow \mathbb{R}$ where f(x) = 2x is uniformly continuous on $\mathbb{R}$

Proof.

Let: x, y $\in \mathbb{R}$ and $\epsilon > 0$
|f(x) $-$ f(y)| = |2x $-$ 2y| = 2|x $-$ y| $< \epsilon$
whenever |x $-$ y| $< \delta = \frac{\epsilon}{2}$

□

Practice 5.4.3

Negating the definition:
$\exists$ $\epsilon > 0$ st $\forall$ $\delta > 0$, $\exists$ x, y $\in$ D st |x $-$ y| $< \delta$ but |f(x) $-$ f(y)| $\geq \epsilon$
in logic, but is another word for and (where but tells you something different is coming)

Example 5.4.4

Let: f : $\mathbb{R} \longrightarrow \mathbb{R}$ where f(x) = x^2
Prove that f is not uniformly continuous on $\mathbb{R}$

Proof.

Let: $\epsilon = 1$, x $\in \mathbb{R}^+$ and choose $\delta > 0$
Let: y = x + $\frac{\delta}{2}$
Then

$$|x - y| = \frac{\delta}{2} < \delta$$

and

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| = |x + y|\frac{\delta}{2} \tag{1}$$

Let: x = $\frac{1}{\delta}$
Then

$$|x + y| = |\frac{1}{\delta} + \frac{1}{\delta} + \frac{\delta}{2} = \frac{2}{\delta} + \frac{\delta}{2} > \frac{2}{\delta} \tag{2}$$

From **(1)** and **(2)**,

$$|f(x) - f(y)| > \frac{2}{\delta}\frac{\delta}{2} = 1$$

Hence,
f is NOT uniformly continuous on $\mathbb{R}$

$\square$

Example 5.4.5

Let: f : $[-5, 5] \longrightarrow \mathbb{R}$ where f(x) = x^2
Show that f is uniformly continuous on $[-5, 5]$

Proof.

Let: $\epsilon > 0$ and x, y $\in$ D
Then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \leq |x - y|(|x| + |y|) \leq |x - y|10 < \epsilon$$

whenever $|x - y| < \delta = \frac{\epsilon}{10}$

$\square$

Theorem 5.4.6

Assume: f : D $\longrightarrow \mathbb{R}$ is continuous on a compact set D.
Then, f is uniformly continuous on D

Proof.

Let: $\epsilon > 0$ and c $\in$ D
Then $\exists$ δ (c) > 0 st

$$f(x) - f(c) < \frac{\epsilon}{2}$$

whenever |x $-$ c| $< \delta$ (c) and x $\in$ D
(this is just the definition of continuity)
Thus,
D $\subset \bigcup_{c \in D}$ N(c, $\frac{\delta(c)}{2}$)
Since D is compact and N(c, δ (c)) is an open set $\forall$ c $\in$ D (which we proved a long time ago),
$\exists$ n $\in \mathbb{N}$ st

$$D \subset \bigcup_{i=1}^{n} N(c_i, \frac{\delta(c)_i}{2})$$

Let: $\delta = \min \{\frac{\delta(c)_1}{2}, \frac{\delta(c)_2}{2}, ... \frac{\delta(c)_i}{2}\}$
Now let x, y $\in$ D with |x $-$ y| $< \delta$
Then x $\in$ N(c_k, $\frac{\delta(c)_k}{2}$ for some k $\in$ {1, 2, .. n}
Then |x $-$ c_k| $< \frac{\delta(c)_k}{2} < \delta$ $(c)_k$
From **(1)**,
|f(x) $-$ f(c_k)| $< \frac{\epsilon}{2}$ **(2)**
Also, |y $-$ c_k| = |(y $-$ x) + (x $-$ c_k)| $\leq$ |y $-$ x| + |x $-$ c_k| $< \delta + \frac{\delta(c)_k}{2} \leq \frac{\delta(c)_k}{2} + \frac{\delta(c)_k}{2} = \delta$ $(c)_k$
From **(1)**,
|f(y) $-$ f(c_k)| $< \frac{\epsilon}{2}$ **(3)**
Hence, from **(2)** and **(3)**,

$$|f(x) - f(y)| < |f(x) - f(c_k)| + |f(x) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, f is uniformly continuous.

□