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Exercise 16

Determine 7^{1000} mod 6.
Let a, b, k, n $\in \mathbb{Z}$ such that k > 0 and n > 0.
If a $\equiv$ b mod n, then $a^k \equiv b^k$ mod n
Base Case:
k = 1
$a^1 \equiv b^1$ holds true
Inductive Step:
Suppose: $a^k \equiv b^k$ mod n
$a^{k+1} \equiv a^1a^k$
$a^1a^k \equiv a^1b^k$ mod n by inductive hypothesis
$a^{k+1} \equiv b^kb$ mod n by base case
$a^{k+1} \equiv b^{k+1}$ mod n
Thus, $a^k = b^k$ mod n
Since 7 $\equiv$ 1 mod 6
$7^{1000} \equiv 1^{1000} \equiv 1$ mod 6

Determine 6^{1001} mod 7.
6^{1001} mod 7 $\equiv 6 * 6^{1000}$ mod 7

$$\begin{aligned} 6^{1001} \mod 7 &\equiv 6 * 6^{1000} \mod 7 \\ &\equiv 6 * (6^2)^{500} \mod 7 \\ &\equiv 6 * (36)^{500} \mod 7 \\ &\equiv 6 * (1)^{500} \mod 7 \\ &\equiv 6 \mod 7 \end{aligned}$$

Exercise 20

Let p_1, p_2, ... p_n, be prime numbers. Show that $p_1 * p_2 * ... p_n * p_{n+1}$ is not divisible by any of the n + 1 primes.
We will prove this by contradiction.
Suppose there are finitely many primes which are the ones listed.
Then, consider $p_1 * p_2 * ... p_n * p_{n+1}$.
This number is either composite or prime.
If it's prime, we just created a new prime, a contradiction.
If it's composite, that means it must be divisible by some prime.
By the Fundamental Theorem of Arithmetic, $\exists$ t $\in \mathbb{Z}$ such that
$p_1t = q = p_1 * p_2 * ... p_n * p_{n+1}$, which implies that $p_i \mid 1$ for i $\in \{1, 2, ... n\}$
This holds if and only if $p_i = 1$, a contradiction of the definition of a prime number.
Hence, there are infinitely many primes.