

Exercise 1

Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3\}$. For each of the relations R from A to B listed below list all pairs $(a, b) \in \mathbb{R}$ and write the corresponding $\{0, 1\}$ -indicator-matrix.

a. $a = b : (0, 0), (1, 1), (2, 2), (3, 3)$

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	0	0	0

b. $a + b = 4 : (1, 3), (2, 2), (3, 1), (4, 0)$

0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	1
1	0	0	0

c. $a > b : (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)$

0	0	0	0
1	1	0	0
1	0	0	0
1	1	1	0
1	1	1	1

d. a divides $b : (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1, 2), (2, 2), (1, 3)$

0	0	0	0
1	1	1	1
1	0	1	0
1	0	0	0
1	0	0	0

Exercise 2

For each of these relations on the set $\{1, 2, 3, 4\}$ decide whether or not it is reflexive, symmetric, antisymmetric, and transitive.

- $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
- $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
- $\{(2, 4), (4, 2)\}$
- $\{(1, 2), (2, 3), (3, 4)\}$
- $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

<i>Relation</i>	<i>R</i>	<i>S</i>	<i>A</i>	<i>T</i>
<i>a</i>	0	0	0	1
<i>b</i>	1	1	0	1
<i>c</i>	0	1	0	1
<i>d</i>	0	0	1	0
<i>e</i>	1	1	1	1
<i>f</i>	0	0	0	1

Exercise 3

Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$ on the set $A = \{1, 2, 3, 4\}$

- Find $R \cup S$
 $\{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- Find $R \cap S$
 $\{(3, 1)\}$
- Find $R \circ S$
 $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Exercise 4

Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$ on the set $A = \{1, 2, 3, 4\}$.

- Find the reflexive closure of R .
 $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 1), (3, 3), (4, 4)\}$
- Find the symmetric closure of R .
 $\{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- Find the transitive closure of R .
 $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (1, 4)\}$

Exercise 5

Prove the following:

- a. A relation R is reflexive iff R^{-1} is reflexive (where R^{-1} is the inverse relation that just reverses the order).

→

Assume R is reflexive.

Let $(a, a) \in R$

Then $(a, a) \in R^{-1}$

Hence, R^{-1} is reflexive.

←

Assume R^{-1} is reflexive.

Let $(a, a) \in R^{-1}$

Then $(a, a) \in R$

Hence, R is reflexive.

- b. A relation R is symmetric iff $R = R^{-1}$.

→

Assume R is symmetric.

Let $(a, b) \in R$.

Want to show: $(a, b) \in R^{-1}$.

Notice: $(b, a) \in R$.

Thus, $(a, b) \in R^{-1}$.

Hence, $R = R^{-1}$.

←

Assume $R = R^{-1}$.

Let $(a, b) \in R$.

Then $(a, b) \in R^{-1}$.

$(a, b) \in R \Rightarrow (b, a) \in R^{-1}$.

But since $R^{-1} = R$, $(b, a) \in R$.

So, $(a, b) \in R \Rightarrow (b, a) \in R$.

Hence, R is symmetric..

- c. A relation R is anti-symmetric iff $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$

→

Assume R is anti-symmetric.

Then $(a, b), (b, a) \in R \Rightarrow a = b$.

So, $R \cap R^{-1}$ will only contain tuples such that $a = b$.

←

Assume $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$.

Let $(a, b) \in R$. If $a \neq b$, then $(a, b) \notin R \cap R^{-1}$. Thus, $(a, b) \notin R^{-1}$.

Hence, R is anti-symmetric.

Exercise 6

Let R be the relation represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the matrices for the relations:

a. R^2

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b. R^3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c. R^4

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 7

Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? If they are not, why?

a. $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Yes.

b. $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

No, $(1, 1)$ isn't in there.

c. $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

Yes.

d. $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

No, $(1, 2)$ isn't in there.

e. $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Yes.

Exercise 8

List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a. $\{0\}, \{1, 2\}, \{3, 4, 5\}$

$(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (3, 4), (4, 5), (3, 5), (5, 3), (4, 3)...$

b. $\{0, 1\}, \{2, 3\}, \{4, 5\}$

c. $\{0, 1, 2\}, \{3, 4, 5\}$

d. $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

Exercise 9

Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? If they are not, why?

- a. $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Yes.

- b. $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

No: $(0, 2)$ and $(2, 0)$ are both in there.

- c. $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

No: $(1, 2)$ and $(2, 1)$ are both in there.

- d. $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

No: $(1, 3)$ and $(3, 1)$ are both in there.

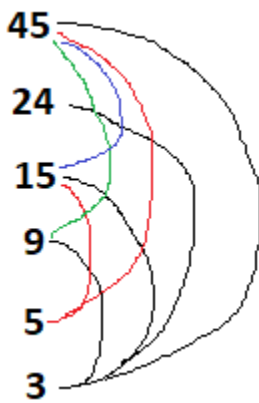
- e. $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

No: $(0, 1)$ and $(1, 0)$ are both in there.

Exercise 10

Answer these questions for the divides poset $(\{3, 5, 9, 15, 24, 45\}; |)$.

- a. Draw the Hasse diagram



- b. List the maximal and minimal elements.

Maximal: $\{45, 24\}$. Minimal: $\{3, 5\}$

- c. Is there a greatest element? A least element?

There is no element greater than nor less than all others.

- d. Find all upper bounds of $\{3, 5\}$. Find the least upper bound of $\{3, 5\}$, if it exists.

$UB(\{3, 5\})$: $\{15, 45\}$. $LUB(\{3, 5\})$: $\{15\}$

- e. Find all the lower bounds of $\{15, 45\}$. Find the greatest lower bound of $\{15, 45\}$, if it exists.

$LB(\{15, 45\})$: $\{3, 5, 15\}$. $GLB(\{15, 45\})$: $\{15\}$

Exercise 11

Prove the following:

- a. There is exactly one greatest element of a poset, if such an element exists.

Suppose $\exists a, b \in P$ such that a and b are the greatest elements of P .

Then $a \geq x$ and $b \geq x \forall x \in P$.

So $a \geq b$ and $b \geq a$.

Thus, $a = b$.

- b. There is exactly one maximal element in a poset with a greatest element.

Let P be a poset and let a be the greatest element in P .

Let $b \in P$ such that $b \neq a$.

Then, by definition, $a \leq b$.

Thus, a is the only maximal element in P .

- c. The least upper bound of a set in a poset is unique if it exists.

Let P be a poset and $a \in P$.

Suppose $\exists U_1$ and $U_2 \in P$ such that U_1 and U_2 are least upper bounds for a and $U_1 \neq U_2$.

Then, by definition, $U_1 \leq U_2$ and $U_2 \leq U_1$.

Hence, $U_1 = U_2$.

Exercise 12

Determine whether these posets are lattices.

- a. $(\{1, 3, 6, 9, 12\}; |)$

No, 9 join 6 doesn't have a LUB.

- b. $(\{1, 5, 25, 125\}; |)$

Yes.

- c. $(\mathbb{Z}; \geq)$

Yes, but it's not a complete lattice.

- d. $(\mathcal{P}(S), \subset)$, where $\mathcal{P}(S)$ is the power set of a set S .

Yes.

Exercise 13

Show that every totally ordered set is a lattice.

Let T be a totally ordered set, and let $a, b \in T$.

Since T is totally ordered, either $a \leq b$ or $b \leq a$.

Case:

- i) $a \leq b$

Then $a \text{ meet } b = a$, and $a \text{ join } b = b$.

- ii) $b \leq a$

Then $b \text{ meet } a = b$, and $b \text{ join } a = a$.

Hence, any two elements have a LUB and GLB.

Exercise 14

Show that every finite lattice has a least element and a greatest element.

Exercise 15

Give an example of an infinite lattice with

- a. neither a least nor a greatest element.
- b. a least but not a greatest element.
- c. a greatest but not a least element.
- d. both a least and a greatest element.

Exercise 16

Show that in any lattice $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. Note: $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$ was shown in class.)

Exercise 17

Show that in any lattice $x \vee (x \wedge y) = x$. Note: the dual absorption law was shown in class.

Exercise 18

Show that any lattice $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$. Note: the dual distributive inequality was shown in class.

Exercise 19

Show that the two distributive equalities are equivalent. That is, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ if, and only if, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Exercise 20

Show that the distributive law implies the modular law. That is, if a lattice satisfies one (hence both, from problem 19), then $(x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z)$.

Exercise 21

Check if the lattice N_5 is distributive.