

Due 4/25 (Wednesday):

All (turn in): Chapter 10, page 206, 14-18, 20, 24

Chapter 10

Recall:

A homomorphism ϕ from a group G to a group G' is a mapping from G into G' that preserves the group operation; that is, $\phi(ab) = \phi(a)\phi(b)$ for $a, b \in G$.

The kernel of a homomorphism ϕ from a group G to a group G' with identity e' is the set $\{x \in G : \phi(x) = e'\}$.

Exercise 14

Explain why the correspondence $x \rightarrow 3x$ from Z_{12} to Z_{10} is not a homomorphism.

Because ϕ is not OP:

$$\phi(3 * 4) = \phi(12) = \phi(0) = (3 * (0 \bmod 12)) \bmod 10 = e, \text{ and}$$

$$\phi(3)\phi(4) = (3 * (3 \bmod 12) * 3 * (4 \bmod 12)) \bmod 10 = (9 * 3 * 4) \bmod 10 = (108) \bmod 10 = 8$$

Exercise 15

Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and $\text{Ker } \phi = \{0, 10, 20\}$.

If $\phi(23) = 9$, determine all elements that map to 9.

$$\phi(ab \bmod 30) = \phi(a \bmod 30)\phi(b \bmod 30)$$

$$\phi(23) = 9.$$

$$\phi(0) = \phi(10) = \phi(20) = 0$$

It looks like it's $\phi(x) = 3x$:

$$\phi(23) = 3 * 23 \bmod 30 = 69 \bmod 30 = 9.$$

Thus,

$\phi(3)$, $\phi(13)$, and $\phi(23)$ all map to 9.

Exercise 16

Prove that there is no homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$.

Suppose $\exists \phi: Z_8 \oplus Z_2 \rightarrow Z_4 \oplus Z_4$, such that ϕ is a homomorphism.

Because Z_8 is of order 8, and $|Z_2|$ divides 8, there is an element of order 8 in $Z_8 \oplus Z_2$, let's call it z_8 .

Thus, $z_8 \in Z_8 \oplus Z_2$.

Because ϕ is OP, $\exists z \in Z_4 \oplus Z_4$ such that $\phi(z_8) = z$ and $|z| = 8$.

However, there is no element of order 8 in $Z_4 \oplus Z_4$. A contradiction.

Hence, no homomorphism exists.

Exercise 17

Prove that there is no homomorphism from $Z_{16} \oplus Z_2$ onto $Z_4 \oplus Z_4$.

Suppose $\exists \phi: Z_{16} \oplus Z_2 \rightarrow Z_4 \oplus Z_4$, such that ϕ is a homomorphism.

Since $G / \text{Ker } \phi$ is isomorphic to $\phi(G)$,

$$|G / \text{Ker } \phi| = |\phi(G)| = 16.$$

Since $|G| = 32$, $|\text{Ker } \phi| = 2$.

Since $\text{Ker } \phi \leq G$, $e \in \text{Ker } \phi$.

Since $\text{Ker } \phi$ is of order 2, the other element in $\text{Ker } \phi$ must have order 2, let's call it k .

The only possibilities for k are: $(8, 0)$, $(8, 1)$, $(0, 1)$

Since $|G / \text{Ker } \phi|$ has order 16, the possibilities for the order of each $c \in G / \text{Ker } \phi$ are factors of 16: 1, 2, 4, 8, and 16.

Case:

i) $\text{Ker } \phi = \{(0, 0), (8, 0)\}$

Let's look at the coset $c = \text{Ker } \phi + (1, 1) \in G / \text{Ker } \phi$

Notice that the order of c is 8.

ii) $\text{Ker } \phi = \{(0, 0), (8, 1)\}$

Let's look at the coset $c = \text{Ker } \phi + (1, 1) \in G / \text{Ker } \phi$

Notice that the order of c is 16.

iii) $\text{Ker } \phi = \{(0, 0), (0, 1)\}$

Let's look at the coset $c = \text{Ker } \phi + (1, 1) \in G / \text{Ker } \phi$

Notice that the order of c is 16.

Exercise 18

Can there be a homomorphism from $Z_4 \oplus Z_4$ onto Z_8 ? Can there be a homomorphism from Z_{16} onto $Z_2 \oplus Z_2$? Explain your answers.

Exercise 20

How many homomorphisms are there from Z_{20} onto Z_8 ? How many are there to Z_8 ?

Let $\phi: Z_{20} \rightarrow Z_8$

Since ϕ is onto, $|\phi(G)| = 8$.

However, since G is finite, $|\phi(G)|$ divides $|G|$.

Therefore, 8 divides 20. A contradiction.

Therefore, there are no homomorphisms from Z_{20} to Z_8 .

Let ϕ be a homomorphism from Z_8 onto Z_8 .

Since ϕ is onto, and the groups are the same order, that means ϕ is an isomorphism.

As far as how many of those there are, I'm not sure. 8, I suppose?

Exercise 24

Suppose that $\phi: \mathbb{Z}_{50} \rightarrow \mathbb{Z}_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine $\phi(x)$.

If $\phi(7) = 6$, and $\phi(0) = 0$, then I suppose $\phi(x) = 3x$

b. Determine the image of ϕ .

$\{0, 3, 6, 9, 12\}$

c. Determine the kernel of ϕ .

$\{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.

$\{1, 6, 11, 16, 21, 26, 31, 36, 41, 46\}$