Homework Due 10/5/17 (7 problems):
Section 4.1 pages 169 - 170; 1, 6(b), 7(f), 9(a), 11, 12, 15

#1

Mark each statement True or False. Justify each answer.

a. If (s_n) is a sequence and $s_i = s_j$ then i = j.

   False.

   Let: $(s_n) = \{1^n\}$

b. If $s_n \longrightarrow s$, then, for every $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|s_n - s| < \epsilon$.

   True.

   A sequence $\{s_n\}$ is said to **converge** to s $\in \mathbb{R}$ provided that $\forall\ \epsilon > 0$

   $\exists$ N $\in \mathbb{N} \leq$ n st

   $|s_n - s| < \epsilon$

   This is the definition of convergence, so this implies that $s_n \longrightarrow s$

c. If $s_n \longrightarrow k$ and $t_n \longrightarrow k$, then $s_n = t_n$ $\forall$ n $\in \mathbb{N}$.

   False.

   Let: $s_n = \sum_{i=0}^{\infty} \frac{1}{2^i}$, $t_n = 2 - \sum_{i=0}^{\infty} \frac{1}{2^i}$

d. Every convergent sequence is bounded.

   By Theorem 4.1.13, this is true.

6(b)

Definition 4.1.2
A sequence $\{s_n\}$ is said to **converge** to s $\in \mathbb{R}$ provided that $\forall\ \epsilon > 0$
$\exists$ N $\in \mathbb{N}$ st
n $\geq$ N $\longrightarrow |s_n - s| < \epsilon$

Using only definition 4.1.2, prove the following:
For k > 0, k $\in \mathbb{R}$, $\lim\limits_{n\to\infty}(\frac{1}{n^k}) = 0$

Proof.

Let: $\{s_n\} = \frac{1}{n^k}$, s = 0
$|s_n - s| = |\frac{1}{n^k} - 0| = |\frac{1}{n^k}|$
Want to show: $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N $\longrightarrow |\frac{1}{n^k}| < \epsilon$
Let: $\epsilon > 0$, N $\in \mathbb{N}$, k $\in \mathbb{R} > 0$
Want to show: $\exists$ N $\in \mathbb{N}$ st $|\frac{1}{N^k}| < \epsilon$
Let: $|\frac{1}{N^k}| < \epsilon$
$\frac{1}{|N^k|} < \epsilon$
$\frac{1}{\epsilon} < |N^k|$
$|N^k| = N^k$ since N $\in \mathbb{N}$ and k > 0 **(1)**
$\frac{1}{\epsilon} < N^k$
$(\frac{1}{\epsilon})^{\frac{1}{k}} < N$
If N is the ceiling of $(\frac{1}{\epsilon})^{\frac{1}{k}} + 1$, then N exists.

Want to show: $|\frac{1}{(N+1)^k}| < \epsilon$
If we know that $|\frac{1}{N^k}| < \epsilon$,
then showing
$|\frac{1}{(N+1)^k}| < |\frac{1}{N^k}|$

shows

$|\frac{1}{(N+1)^k}| < \epsilon$

$|\frac{1}{(N+1)^k}| < |\frac{1}{N^k}|$
$\frac{1}{|(N+1)^k|} < \frac{1}{|N^k|}$
$|N^k| < |(N+1)^k|$
From **(1)**,
$|N^k| = N^k < |(N+1)^k| = (N+1)^k$
$N^k < (N+1)^k$
This is true since N $\in \mathbb{N}$ and k > 0
So, $|\frac{1}{N^k}|$ decreases as N grows.
Since $\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|\frac{1}{n^k}| < \epsilon$,
$\lim\limits_{n\to\infty}\frac{1}{n^k} = 0$

□

7(f)

Using any of the results in this section (4.1), prove the following:
If |x| < 1, then $\lim_{n\to\infty} x^n = 0$

Proof.

$|x| < 1$ implies $0 \le |x| < 1$ **(1)**
Let: $s_n = x^n$, $s = 0$
Want to show: $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \ge N$ implies $|s_n - s| < \epsilon$
Let: $\epsilon > 0$
$|s_n - s| < \epsilon = |x^n| < \epsilon$
Want to show: $\exists\ N \in \mathbb{N}$ st $|x^N| < \epsilon$
$|x^N| < \epsilon$
$||x^N|| < |\ \epsilon\ |$
We know that because of **(1)** and because $N \in \mathbb{N}$,
$|x^{N+1}| < |x^N|$
We also know that $\epsilon > 0$
So, $0 < |x^{N+\ k}| < ... < |x^{N+1}| < |x^N|$ where $k \in \mathbb{N}$

□

9(a)

For each of the following, prove or give a counter example:
If (s_n) converges to s, then $(|s_n|)$ converges to $|s|$.

Proof.

If s_n converges to s, then by definition,
$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $N \le n$ implies $|s_n - s| < \epsilon$
Want to show: $||s_n| - |s|| < \epsilon$
Case 1: s_n and s are the same sign.
$||s_n| - |s|| = |s_n - s|$
Therefore, $N \le n$ implies $||s_n| - |s|| < \epsilon$
If we let $s_n = |s_n|$ and $|s| = s$, then $|s_n|$ converges to $|s|$.
Case 2: s_n and s are different signs.
$||s_n| - |s|| \le |s_n - s| < \epsilon$
$||s_n| - |s|| < \epsilon$
Therefore, $N \le n$ implies $||s_n| - |s|| < \epsilon$
If we let $s_n = |s_n|$ and $|s| = s$, then $|s_n|$ converges to $|s|$.
Hence, result.

□

11

Given the sequence (s_n), k $\in \mathbb{N}$, let (t_n) be the sequence defined by $t_n = s_{n+k}$. That is, the terms in (t_n) are the same as that of the terms in (s_n) after the first k terms have been skipped. Prove that (t_n) converges iff (s_n) converges, and if they converge, show that lim t_n = lim s_n. Thus, the convergence of a sequence is not affected by omitting (or changing) a finite number of terms.

Proof.

$\longrightarrow$
(t_n) converges $\longrightarrow$ (s_n) converges
If t_n converges, then by definition,
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|t_n - L| < \epsilon$
(or)
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st $|t_n - L| < \epsilon\ \forall$ n $\geq$ N
Since $t_n = s_{n+k}$,
we know that s_{n+k} converges.
Let: n_1 = n + k
So,
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st $n_1 \geq$ N implies $|s_{n_1} - L| < \epsilon$
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n + k $\geq$ N implies $|s_{n_1} - L| < \epsilon$
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n $\geq$ N $-$ k implies $|s_{n_1} - L| < \epsilon$
Notice that N $-$ k $\in \mathbb{N}$. Let's call it N_1
$\forall\ \epsilon > 0, \exists\ N_1 \in \mathbb{N}$ st n $\geq N_1$ implies $|s_{n_1} - L| < \epsilon$
Since there is still a natural number N_1 st n $\geq N_1$ implies $|s_{n_1} - L| < \epsilon$,
If t_n converges, then s_n converges.
$\longleftarrow$
(s_n) converges $\longrightarrow$ (t_n) converges
If s_n converges, then by definition,
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $|s_n - L| < \epsilon$
Since $t_n = s_{n+k}$, $t_{n-k} = s_n$
So since s_n converges, t_{n-k} converges.
If we let n_1 = n $-$ k,
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st $n_1 \geq$ N implies $|t_{n_1} - L| < \epsilon$
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n $-$ k $\geq$ N implies $|t_{n_1} - L| < \epsilon$
$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n $\geq$ (N + k) implies $|t_{n_1} - L| < \epsilon$
Notice that N + k $\in \mathbb{N}$. Let's call it N_1
$\forall\ \epsilon > 0, \exists\ N_1 \in \mathbb{N}$ st n $\geq N_1$ implies $|t_{n_1} - L| < \epsilon$
Since there is still a natural number N_1 st n $\geq N_1$ implies $|t_{n_1} - L| < \epsilon$,
If s_n converges, then t_n converges.

□

12

a. Assume that lim $s_n = 0$. If (t_n) is a bounded sequence, prove that lim($s_n t_n$) = 0.

   If lim $s_n = 0$, then by definition,

   $\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st if n $\geq$ N, then $|s_n$ - $0| < \epsilon$

   If t_n is a bounded sequence, then $\forall$ n $\in \mathbb{N}$, a $\leq t_n \leq$ b, where a, b $\in \mathbb{R}$

We know that t_n will always be between two constants a and b, so lets let $t_n = c$, where $a \leq c \leq b$.

Since s_n converges,

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n - 0| < \epsilon$

can be simplified to

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n| < \epsilon$

Want to show: $\lim(s_n t_n) = 0$

$\lim(s_n t_n) = 0$ if

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n t_n| < \epsilon$

Since we let $t_n = c$, some bounded real number, this is equivalent to

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|cs_n| < \epsilon$

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n| < |c|\epsilon$

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n| < \epsilon_1$

which is equivalent to

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n| < \epsilon$

Hence, result.

b. Show by example that the boundedness of (t_n) is a necessary condition in part (a).

If $\lim s_n = 0$, then by definition,

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n - 0| < \epsilon$

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n| < \epsilon$

However, if we let t_n be unbounded (i.e. let $t_n = e^n$), this doesn't work. See below:

$s_n t_n$ is bounded if

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st if $n \geq N$, then $|s_n t_n| < \epsilon$

Suppose: $s_n = \frac{1}{n}$

Then $s_n t_n = \frac{e^n}{n}$

Since e^n grows faster than $\frac{1}{n}$, $s_n t_n$ grows overall as n approaches infinity.

Hence, the boundedness of t_n is necessary.

15

a. Prove that x is an accumulation point of a set S iff $\exists$ a sequence (s_n) of points in S $\setminus$ {x} st (s_n) converges to x.

$\longrightarrow$

Let: $x \in S'$

This means that $N^*(x, \epsilon) \cap S \neq \emptyset, \forall \epsilon > 0$. **(1)**

$N^*(x, \epsilon)$ means $\{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$

If $\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - x| < \epsilon$,

Then (s_n) converges to x.

Let: $s_n \in N^*(x, \frac{1}{n}) \cap S \neq \emptyset$

Then

$|s_n - x| < \frac{1}{n}$ and $s_n \in S \setminus \{x\}$

Let: $\epsilon > 0$

$\exists N(\epsilon) \in \mathbb{N}$ st $\frac{1}{N} < \epsilon$ (By AP)

Thus, from **(1)**,

$|s_n - x| < \epsilon, \forall n \geq N$.

Hence, $\lim s_n = x$ and $s_n \in S \setminus \{x\} \ \forall n \in \mathbb{N}$

$\longleftarrow$

Conversely,

Assume: $\{s_n\}$ is a sequence in $S \setminus \{x\}$ st $\lim_{n\to\infty} s_n = x$

Want to show: $x \in S'$

$\forall \epsilon > 0, \exists N(\epsilon)$ (as in N is chosen based on ϵ) $\in \mathbb{N}$ st

$|s_n - x| < \epsilon \ \forall n \geq N$ and $s_n \in S \setminus \{X\}$

$s_n \in (x - \epsilon, x + \epsilon), s_n \neq x$

(theorem 4.2.1 is a possibility on test)

4.2.4, 4.2.7 not on exam

Thus, $N^*(x, \epsilon) \cap S \neq \emptyset$

So, $x \in S'$

Hence result.

b. Prove that a set S is closed iff, whenever (s_n) is a convergent sequence of points in S, it follows that $\lim s_n$ is in S.

$\longrightarrow$

Let: S be a closed set.

Want to show: (s_n) is a sequence in S st $\lim_{n\to\infty} s_n = s$ implies $s \in S$

If S is closed, then $S = \text{cl } S = S \cup S'$, $\text{bd } S \subset S$

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ st $|s_n - s| < \epsilon \ \forall n \geq N$

Want to show: $s \in S$

Case

i) $s \in S$.
   In this case, we are done.

ii) $s \notin S$
   Hence, (s_n) is a sequence in $S \setminus \{s\}$ st $\lim_{n\to\infty} s_n = s$. By **(a)**, $s \in S'$.
   Since S is closed, $s \in S$.

Note: The above is what we did in class. Below (until $\longleftarrow$) is my original answer. Can you tell me if the next 11 or so lines are valid?

Suppose: $\lim s_n \notin S$

This implies that $s \notin S$ where

$\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$

Since S is closed,

Let: u be the closest boundary point to s

Now, let $\epsilon = |\frac{u-s}{2}|$

We know that $|s_n - s| < \epsilon$ for this epsilon.

$|s_n - s| < |\frac{u-L}{2}|$

Which implies that the distance between s_n and s is less than the distance between s_n and the nearest boundary point of S.

This means there is an s_n st $s_n \notin S$, a contradiction.

So, s_n is not a convergent sequence of points in S if $\lim s_n$ is not in S.

$\longleftarrow$

Conversely,

Assume: whenever (s_n) is a sequence in S st $\lim_{n\to\infty} s_n = s$, then $s \in S$

Want to show: S is closed

We will use Theorem 3.4.17 (a): S is closed iff $S' \subset S$

Let: $s \in S'$

$s \in S'$ means $\forall\ \epsilon > 0$, $N^*(s, \epsilon)\ \cap\ S \neq \emptyset$

Let: $s_n \in N^*(s, \frac{1}{n})\ \cap\ S \neq \emptyset$, $n \in \mathbb{N}$

So,

$|s_n - s| < \frac{1}{n}$, $s_n \in S\ \forall\ n \in \mathbb{N}$

Hence $\lim_{n\to\infty} s_n = s$

We know $\exists\ N \in \mathbb{N}$ st $\frac{1}{N} < \epsilon$ by AP.

From **(1)**, $|s_n - s| < \frac{1}{n} < \epsilon\ \forall\ n \geq N$

Hence, $\lim_{n\to\infty} s_n = s$, $s_n \in S$, $\forall\ n \in \mathbb{N}$

By assuming $s \in S$, S is closed.