Let A be a nonempty set, and let

$$S_A = \{f : A \longrightarrow A : \text{f is both 1 to 1 and onto}\}$$

Show that S$_A$ is a group under composition. Is S$_A$ an Abelian group?

a. **Closure:** Want to show that, $\forall$ f, g $\in$ S$_A$, f o g $\in$ S$_A$

Let f, g $\in$ S$_A$, and let a $\in$ A.

Since both f and g are well defined, f(a) and g(a) exist.

Since both f and g map to A, f(a) $\in$ A and g(a) $\in$ A. **(1)**

Since both f and g are one to one, f(a) and g(a) are unique. **(2)**

By **(1)** and **(2)**, f(g(a)) and g(f(a)) both exist and are unique.

Therefore, both f o g and g o f are one-to-one.

Now, we want to show that they're onto.

Suppose $\exists$ a$_0$ $\in$ A such that f(g(a)) $\neq$ a$_0$ (or that g(f(a)) $\neq$ a$_0$), $\forall$ a $\in$ A.

However, if a$_0$ $\in$ A, then it gets mapped onto by both f and g.

So that means there exists some a$_f$ and a$_g$ in A such that f(a$_g$) = a$_0$ (or g(a$_f$) = a$_0$).

And since a$_f$ and a$_g$ are in A, they get mapped to by f and g, respectively.

Thus, a contradiction.

b. **Associativity:** Want to show that, $\forall$ f, g, h $\in$ S$_A$, (f o g) o h = f o (g o h).

Let f, g, h $\in$ S$_A$, and let a $\in$ A.

Let h(a) = a$_h$, g(h(a)) = a$_{gh}$, f(a) = a$_f$, f(g(a)) = a$_{fg}$, which are all defined since f, g, and h are all well defined and onto.

Notice that ((f o g) o h)(a) = f(g(a$_h$)) and (f o (g o h))(a) = f(a$_{gh}$).

Want to show: g(a$_h$) = a$_{gh}$.

Well, g(a$_h$) = g(a(h)) by definition, and a$_{gh}$ = g(a(h)) by definition.

Hence, result.

c. **Identity:** Want to show that $\exists$ I $\in$ S$_A$ such that I o f = f o I = f, $\forall$ f $\in$ S$_A$.

d. **Inverse:** Want to show that, $\forall$ f $\in$ S$_A$, $\exists$ f^{-1} such that f(f^{-1}(a)) = f^{-1}(f(a)) = a, $\forall$ a $\in$ A.

S$_A$ is **NOT** an Abelian group (since function composition is not commutative).