

**Exercise 1**

Let  $A = \{0, 1, 2, 3, 4\}$  and  $B = \{0, 1, 2, 3\}$ . For each of the relations  $R$  from  $A$  to  $B$  listed below list all pairs  $(a, b) \in \mathbb{R}$  and write the corresponding  $\{0, 1\}$ -indicator-matrix.

a.  $a = b : (0, 0), (1, 1), (2, 2), (3, 3)$

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	0	0	0

b.  $a + b = 4 : (1, 3), (2, 2), (3, 1), (4, 0)$

0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	1
1	0	0	0

c.  $a > b : (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)$

0	0	0	0
1	1	0	0
1	0	0	0
1	1	1	0
1	1	1	1

d.  $a$  divides  $b : (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1, 2), (2, 2), (1, 3)$

0	0	0	0
1	1	1	1
1	0	1	0
1	0	0	0
1	0	0	0

**Exercise 2**

For each of these relations on the set  $\{1, 2, 3, 4\}$  decide whether or not it is reflexive, symmetric, antisymmetric, and transitive.

- $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
- $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
- $\{(2, 4), (4, 2)\}$
- $\{(1, 2), (2, 3), (3, 4)\}$
- $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

<i>Relation</i>	<i>R</i>	<i>S</i>	<i>A</i>	<i>T</i>
<i>a</i>	0	0	0	1
<i>b</i>	1	1	0	1
<i>c</i>	0	1	0	1
<i>d</i>	0	0	1	0
<i>e</i>	1	1	1	1
<i>f</i>	0	0	0	1

**Exercise 3**

Let  $R$  be the relation  $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$ , and let  $S$  be the relation  $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$  on the set  $A = \{1, 2, 3, 4\}$

- Find  $R \cup S$   
 $\{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- Find  $R \cap S$   
 $\{(3, 1)\}$
- Find  $R \circ S$   
 $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

**Exercise 4**

Let  $R$  be the relation  $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$  on the set  $A = \{1, 2, 3, 4\}$ .

- Find the reflexive closure of  $R$ .  
 $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 1), (3, 3), (4, 4)\}$
- Find the symmetric closure of  $R$ .  
 $\{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- Find the transitive closure of  $R$ .  
 $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (1, 4)\}$

**Exercise 5**

Prove the following:

- a. A relation  $R$  is reflexive iff  $R^{-1}$  is reflexive (where  $R^{-1}$  is the inverse relation that just reverses the order).

→

Assume  $R$  is reflexive.

Let  $(a, a) \in R$

Then  $(a, a) \in R^{-1}$

Hence,  $R^{-1}$  is reflexive.

←

Assume  $R^{-1}$  is reflexive.

Let  $(a, a) \in R^{-1}$

Then  $(a, a) \in R$

Hence,  $R$  is reflexive.

- b. A relation  $R$  is symmetric iff  $R = R^{-1}$ .

→

Assume  $R$  is symmetric.

Let  $(a, b) \in R$ .

Want to show:  $(a, b) \in R^{-1}$ .

Notice:  $(b, a) \in R$ .

Thus,  $(a, b) \in R^{-1}$ .

Hence,  $R = R^{-1}$ .

←

Assume  $R = R^{-1}$ .

Let  $(a, b) \in R$ .

Then  $(a, b) \in R^{-1}$ .

$(a, b) \in R \Rightarrow (b, a) \in R^{-1}$ .

But since  $R^{-1} = R$ ,  $(b, a) \in R$ .

So,  $(a, b) \in R \Rightarrow (b, a) \in R$ .

Hence,  $R$  is symmetric..

- c. A relation  $R$  is anti-symmetric iff  $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$

→

Assume  $R$  is anti-symmetric.

Then  $(a, b), (b, a) \in R \Rightarrow a = b$ .

So,  $R \cap R^{-1}$  will only contain tuples such that  $a = b$ .

←

Assume  $R \cap R^{-1} \subset \Delta : \Delta = \{(a, a) : a \in A\}$ .

Let  $(a, b) \in R$ . If  $a \neq b$ , then  $(a, b) \notin R \cap R^{-1}$ . Thus,  $(a, b) \notin R^{-1}$ .

Hence,  $R$  is anti-symmetric.

**Exercise 6**

Let  $R$  be the relation represented by the matrix  $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ . Find the matrices for the relations:

- a.  $R^2$
- b.  $R^3$
- c.  $R^4$

**Exercise 7**

Which of these relations on  $\{0, 1, 2, 3\}$  are equivalence relations? If they are not, why?

- a.  $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
- b.  $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
- c.  $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
- d.  $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
- e.  $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

**Exercise 8**

List the ordered pairs in the equivalence relations produced by these partitions of  $\{0, 1, 2, 3, 4, 5\}$ .

- a.  $\{0\}, \{1, 2\}, \{3, 4, 5\}$
- b.  $\{0, 1\}, \{2, 3\}, \{4, 5\}$
- c.  $\{0, 1, 2\}, \{3, 4, 5\}$
- d.  $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

**Exercise 9**

Which of these relations on  $\{0, 1, 2, 3\}$  are partial orderings? If they are not, why?

- a.  $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
- b.  $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
- c.  $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
- d.  $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
- e.  $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

**Exercise 10**

Answer these questions for the divides poset  $(\{3, 5, 9, 15, 24, 45\}; |)$ .

- Draw the Hasse diagram
- List the maximal and minimal elements
- Is there a greatest element? A least element?
- Find all upper bounds of  $\{3, 5\}$ . Find the least upper bound of  $\{3, 5\}$ , if it exists.
- Find all the lower bounds of  $\{15, 45\}$ . Find the greatest lower bound of  $\{15, 45\}$ , if it exists.

**Exercise 11**

Prove the following:

- There is exactly one greatest element of a poset, if such an element exists.
- There is exactly one maximal element in a poset with a greatest element.
- The least upper bound of a set in a poset is unique if it exists.

**Exercise 12**

Determine whether these posets are lattices.

- $(\{1, 3, 6, 9, 12\}; |)$
- $(\{1, 5, 25, 125\}; |)$
- $(\mathbb{Z}; \geq)$
- $(\mathcal{P}(S), \subset)$ , where  $\mathcal{P}(S)$  is the power set of a set  $S$ .

**Exercise 13**

Show that every totally ordered set is a lattice.

**Exercise 14**

Show that every finite lattice has a least element and a greatest element.

**Exercise 15**

Give an example of an infinite lattice with

- neither a least nor a greatest element.
- a least but not a greatest element.
- a greatest but not a least element.
- both a least and a greatest element.

**Exercise 16**

Show that in any lattice  $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ . Note:  $(x \wedge y) \wedge z \leq x \wedge (y \wedge z)$  was shown in class.)

**Exercise 17**

Show that in any lattice  $x \vee (x \wedge y) = x$ . Note: the dual absorption law was shown in class.

**Exercise 18**

Show that any lattice  $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ . Note: the dual distributive inequality was shown in class.

**Exercise 19**

Show that the two distributive equalities are equivalent. That is,  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$  if, and only if,  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ .

**Exercise 20**

Show that the distributive law implies the modular law. That is, if a lattice satisfies one (hence both, from problem 19), then  $(x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z)$ .

**Exercise 21**

Check if the lattice  $N_5$  is distributive.