

Theorem 3.3: The Completeness Axiom

Recall the Fundamental Theorem of Arithmetic:

if $n \in \mathbb{N}$ with $n \geq 2$, then n may be expressed as the product of prime numbers (the prime factorization (PF)).

The PF is unique with respect to (WRT) order.

Ex: $12 = 2 * 2 * 2 * 3$

Theorem 3.3.1

Let: p be a prime number

Then $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$

Proof.

Assume: $\sqrt{p} \in \mathbb{Q}$

Then $\sqrt{p} = \frac{a}{b}$, where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$

So,

$$p = \frac{a^2}{b^2}$$

$$a^2 = pb^2$$

therefore,

$$p | a^2 \tag{1}$$

$$p | a^2 \Rightarrow \exists k \in \mathbb{Z} \text{ st } a^2 = pk$$

Since the PF of a^2 and a contain exactly the same distinct primes,

$$(\text{i.e. } a = p_1 \times p_2 \times \dots \times p_n \Rightarrow a^2 = p_1^2 \times p_2^2 \times \dots \times p_n^2)$$

and since p is prime (i.e. p is a component of a^2 but can't be, say, p_2^2 because that would mean it has an integer square root and therefore isn't prime), it has to be one of the p_n 's,

$$p | a.$$

Thus, $\exists k \in \mathbb{Z}$ st. $a = pk$.

Then $a^2 = p^2 k^2 = pb^2$ from (1).

Thus, $b^2 = pk^2$, and we see that $p | b^2$.

However, we obtain the contradiction that $p | b$ and $p | a$.

Hence, $\sqrt{p} \in \mathbb{R} \setminus \mathbb{Q}$. □

Definition 3.3.7

Let $S \subset \mathbb{R}$.

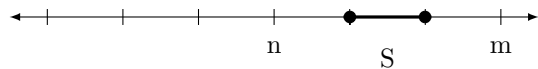
If $\exists m \in \mathbb{R}$ st $s \leq m \forall s \in S$,

then m is an upper bound of S and we say that S is **bounded above**.

Similarly, we can define **bounded below**.

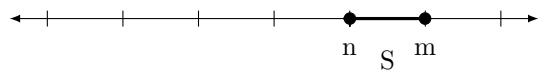
If S is bounded above and below, then S is said to be **bounded**.

S can be open or closed. The example below is closed.



If an upper bound m of S is a member of S , then m is called the maximum (or largest element) of S , and we say that $m = \mathbf{max} S$.

Similarly, we may define **minimum of S ($\min S$)**.



Theorem 1

If a set $S \subset \mathbb{R}$ possesses a max element, then it is unique. A similar result holds for a minimum element.

Proof.

Suppose: $\exists m_1, m_2 \in \mathbb{R}$ st $m_1 = \max S, m_2 = \max S$

Thus, $m_1, m_2 \in S$ and, $\forall s \in S$

$$s \leq m_1 \tag{1}$$

$$s \leq m_2 \tag{2}$$

Let $\max = m_1$ in (1) and $\max = m_2$ in (2) to obtain that $m_2 \leq m_1$ and $m_1 \leq m_2$,

Hence, $m_1 = m_2$. □

Definition 3.3.5 (supremum defined)

Let $\emptyset \neq S \subset \mathbb{R}$ if S is bounded above,

then the **least upper bound** of S is called the **supremum** of S , denoted by $\sup S \in \mathbb{R}$

iff:

a. $s \leq \sup S \forall s \in S$

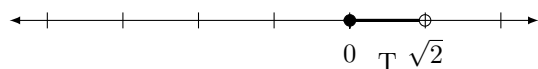
b. $\exists s' \in S$ st $\sup S - \epsilon < s' \forall \epsilon > 0$

Axiom of Completeness of the set of Real Numbers: \mathbb{R}

Every $\emptyset \neq S \subset \mathbb{R}$ that is bounded above has a least upper bound (i.e. $\sup S \in \mathbb{R}$ exists).

A similar statement can be made about $\inf S$.

Remark: In practice 3.3.4, the set $T = \{q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2}\}$ is bounded.



But $\sqrt{2}$ is not rational, so the set wouldn't have a least upper bound.
We need to fill in the gaps to make analysis work.

Examples (#3, page 132):

a. $S = \{1, 3\}$

$$\sup S = 3 \longrightarrow \text{since } s \leq 3 \ \forall s \in S \text{ and } 3 - \epsilon < 3$$

b. similar to a.

c. $S = (0, 4]$

$$\sup S = 4, \max S = 4$$

d. $S = (0, 4)$

$$\sup S = 4 \longrightarrow \text{since } s \leq 4 \ \forall s \in S \text{ and } 4 - \epsilon < s' \in S$$

$\max S = \text{undefined}$. There is no max.

e. $S = \{ \frac{1}{2^n} : n \in \mathbb{N} \}$

$$\sup S = \frac{1}{2}$$

$$\max S = \frac{1}{2} \text{ (if the supremum is in the set, then } \max = \sup \text{)}$$

f. $S = \{ 1 - \frac{1}{2^n} : n \in \mathbb{N} \}$

$$\sup S = 1$$

$$\max S = \text{undefined} \ (1 \notin \mathbb{R})$$