

HW 8: pages 193, #1, 2, 3, 5, 9, 10, 17

For 2(c), see Theorem 1 and Example 9 from Lecture 15

Make sure when you do these problems, justify the answer by either writing down the theorem name or providing a counter example.

Exercise 1

Mark each statement True or False. Justify each answer.

- a. A sequence (s_n) converges to s iff every subsequence of (s_n) converges to s .

True. By Theorem 4.4.4.

- b. Every bounded sequence is convergent.

False.

Counter example: $(s_n) = (-1)^n$

- c. Let (s_n) be a bounded sequence. If (s_n) oscillates, then the set S of subsequential limits of (s_n) contains at least two points.

True. If S oscillates, then $\liminf S < \limsup S$. This implies that these are two different points.

- d. Let (s_n) be a bounded sequence and let $m = \limsup s_n$.

Then, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $N \geq n$ implies $s_n > m - \epsilon$

True.

Proof.

Let: $\epsilon > 0$

Since s_n is bounded, let S be the set containing the range of s_n .

By definition, \exists some s_{n_k} st $\lim s_{n_k} = m$ where $k \in \mathbb{N}$

Since $\lim s_{n_k} = m$,

$\exists N \in \mathbb{N}$ st $N \geq n_k$ implies $|s_{n_k} - m| < \epsilon$

$|s_{n_k} - m| < \epsilon$

$-\epsilon < s_{n_k} - m < \epsilon$

$m - \epsilon < s_{n_k} < m + \epsilon$ **(1)**

So, by **(1)**,

\exists some $N \in \mathbb{N}$ st $n \geq N$ implies $s_n > m - \epsilon$

□

- e. If (s_n) is unbounded above, then (s_n) contains a subsequence that has ∞ as a limit.

True. By Theorem 4.4.8.

Exercise 2

Mark each statement True or False. Justify each answer.

- a. Every sequence has a convergent subsequence.

False. Let $s_n = n$

- b. The set of subsequential limits of a bounded sequence is always nonempty.

True. By Theorem 4.4.8

- c. (s_n) converges to s iff $\liminf s_n = \limsup s_n = s$

True. By Definition 4.4.9 and exercise 9.

- d. Let (s_n) be a bounded sequence and let $m = \limsup s_n$. Then, $\forall \epsilon > 0$, there are infinitely many terms in the sequence greater than $m - \epsilon$.

True. By Theorem 4.4.7, s_n has a convergent subsequence.

Let t_n be a subsequence of s_n st $\lim_{n \rightarrow \infty} t_n = m$

By definition,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - m| < \epsilon$

so,

$-\epsilon < t_n - m < \epsilon$

$m - \epsilon < t_n$

Pick ϵ_2 to be $\frac{\epsilon}{2}$

Then,

$\exists N(\epsilon_2)$ st $m - \epsilon < t_{N(\epsilon_2)}$

Inductively, we can let $\epsilon_3 = \frac{\epsilon_2}{2}$, and so on.

Hence, since there are infinitely many terms in t_n greater than $m - \epsilon$, the same is true for s_n .

- e. If (s_n) is unbounded above, then $\liminf s_n = \limsup s_n = \infty$

True.

Suppose: s_n has a subsequence t_n such that $\lim_{n \rightarrow \infty} t_n = t$ where $t \neq \infty$ (but could be negative infinity)

So,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - t| < \epsilon$

Notice also, that since s_n is unbounded above,

$\forall m \in \mathbb{R}, \exists N_m \in \mathbb{N}$ st $s_{N_m} > m$

That means that \exists some N for t_n st $t_N > m$

If we let $m = t$, then

\exists some N_1 for t_n st $t_{N_1} > t = m$

If we let $m = t + 1$, then

\exists some N_2 for t_n st $t_{N_2} > m = t + 1$

Inductively, t_n has an infinite amount of values above t , and is increasing: a contradiction.

Thus, t_n is unbounded above.

Exercise 3

For each sequence, find the set S of subsequential limits, the limit inferior, and the limit superior.

a. $s_n = 1 + (-1)^n$

$$S = \{0, 2\}, s_* = 0, s^* = 2$$

b. $t_n = (0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7})$

$$S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7}\}, s_* = 0, s^* = \frac{6}{7}$$

c. $u_n = n^2(-1 + (-1)^n)$

$$S = \{0\}, s_* = -\infty, s^* = 0$$

d. $v_n = n \sin \frac{n\pi}{2}$

$$S = \{0\}, s_* = -\infty, s^* = \infty$$

Exercise 5

Use exercise 4.3.14 to find the limit of each sequence:

Known: $t_n = (1 + \frac{1}{n})^n$ and $\lim_{n \rightarrow \infty} t_n = e$

a. $s_n = (1 + \frac{1}{2n})^{2n}$

We can just think of s_n as a subsequence of t_n (the original e sequence),
so therefore it has the same limit: e .

b. $s_n = (1 + \frac{1}{n})^{2n}$

$$= ((1 + \frac{1}{n})^n)^2$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = e^2$$

c. $s_n = (1 + \frac{1}{n})^{n-1}$

$$= (1 + \frac{1}{n})^n (1 + \frac{1}{n})^{-1}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = e * 1 = e$$

d. $s_n = (\frac{n}{n+1})^n$

$$= \frac{1}{(\frac{n+1}{n})^n}$$

$$= \frac{1}{(1 + \frac{1}{n})^n}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = \frac{1}{e}$$

e. $s_n = (1 + \frac{1}{2n})^n$

$$= ((1 + \frac{1}{2n})^{2n})^{\frac{1}{2}}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = \sqrt{e}$$

f. $s_n = (\frac{n+2}{n+1})^{n+3}$

$$= (\frac{n+2}{n+1})^n (\frac{n+2}{n+1})^3$$

$$= (\frac{n}{n+1} + \frac{2}{n+1})^n (\frac{n+2}{n+1})^3$$

$$\text{Now, } \lim_{n \rightarrow \infty} (\frac{n}{n+1} + \frac{2}{n+1})^n (\frac{n+2}{n+1})^3 = (e + 0) \times 1 \text{ by (d)}$$

$$\text{so, } \lim_{n \rightarrow \infty} s_n = e$$

Exercise 9

Let (s_n) be a bounded sequence.

Assume: $\liminf s_n = \limsup s_n = s$

Prove that (s_n) is convergent and that $\lim s_n = s$

Let $S \subset \mathbb{R}$ be the range of limits for any subsequence of s_n .

Since $\liminf s_n = s$, $\inf S = s$.

Since $\limsup s_n = s$, $\sup S = s$.

By Corollary 4.4.12, S contains s .

Since $\inf S = \sup S = s$, the range of S is just $\{s\}$. (1)

Since s_n is bounded, it can't diverge to ∞ or $-\infty$.

However, suppose s_n diverges in general.

Then, $\exists \epsilon (s_n)$ st $|s_n - s| > \epsilon (s_n)$ for all $n \geq$ some $N \in \mathbb{N}$

Since there are infinitely many $n \geq N$, $\exists s_{n_k}$ (a subsequence of s_n) st

$|s_{n_k} - s| \geq \epsilon (s_n)$ where $n_k = N + k$, $k \in \mathbb{N}$

Since s_{n_k} is bounded (because s_n is bounded), it itself has a convergent subsequence (for notation reasons let's call it t_{n_k})

Notice that t_{n_k} is a convergent subsequence of s_n , but its limit is not s (since \exists an ϵ st $|s_n - s| > \epsilon$), a contradiction.

Hence, s_n must converge to s .

Exercise 10

Assume: $x > 1$

Prove that $\lim x^{\frac{1}{n}} = 1$

$\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1$ if

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $N \geq n$ implies $|x^{\frac{1}{n}} - 1| < \epsilon$

Let: $\epsilon > 0$

$|x^{\frac{1}{n}} - 1| < \epsilon$

Since $x > 1$ and $n \in \mathbb{N}$,

$x^{\frac{1}{n}} - 1 < \epsilon$

$x^{\frac{1}{n}} < \epsilon + 1$

$(x^{\frac{1}{n}})^n < (\epsilon + 1)^n$

$x < (\epsilon + 1)^n$

$\ln x < n \ln (\epsilon + 1)$

$\frac{\ln x}{\ln (\epsilon + 1)} < n$

So, if $\frac{\ln x}{\ln (\epsilon + 1)} < N$,

then $\exists N$ st $|x^{\frac{1}{n}} - 1| < \epsilon$

Hence, result.

Exercise 17

Prove that if $\limsup s_n = \infty$ and $k > 0$, then $\limsup (ks_n) = \infty$

Side Note

Question: Is it a valid proof to say that since

$$t_n = \sum_{i=1}^n \frac{1}{n}$$

is the slowest possible diverging sequence (without constants of course),
since

$$\lim_{n \rightarrow \infty} kt_n = k\infty = \infty$$

then $\lim_{n \rightarrow \infty}$ of $k \times$ any sequence diverging to ∞ is also ∞ ?

So, therefore $\limsup (k * \text{any sequence diverging to } \infty)$ is also ∞ ?

Let: t_n be a subsequence of s_n st $\lim_{n \rightarrow \infty} t_n = \infty$

Algebraically, $k \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} kt_n = k\infty = \infty$

Hence, $\limsup (ks_n) = \infty$