Due 4/23:
All (turn in): page 187, 2, 6, 14
Present (me): 10

Page 187

Exercise 2

Prove that A_n is normal in S_n.
WTS: $sA_ns^{-1} \subset A_n$ $\forall$ $s \in S_n$.
Let $s \in S_n$, $a \in A_n$.
Case:

i) $s \notin A_n$

Then s can be expressed as a product of k transpositions, where k is an odd integer.

Since s is odd, this means that s^{-1} is also odd.

Since sas^{-1} is a product of 2k plus an even number of transpositions, sas^{-1} is even.

Therefore, $sas^{-1} \in A_n$ $\forall$ $a \in A_n$.

Hence, $sA_ns^{-1} \subset A_n$ $\forall$ $s \notin A_n$ but still in S_n.

ii) $s \in A_n$

Then sas^{-1} is a product of 3k transpositions (where k is an even integer this time).

Thus, sas^{-1} is even.

Hence, $sA_ns^{-1} \subset A_n$ $\forall$ $s \in A_n$.

Hence, A_n is normal in S_n.

Exercise 6

Let H = { $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$: a, b, d $\in \mathbb{R}$ and ad $\neq 0$ }.
Is H a normal subgroup of GL(2, $\mathbb{R}$)?

WTS: $gHg^{-1} \subset H$ $\forall$ $g \in GL(2, \mathbb{R})$

Let $g \in GL(2, \mathbb{R})$, $h \in H$.
$ghg^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} h & -f \\ -g & e \end{bmatrix} \frac{1}{eh-fg}$: a, b, ... h $\in \mathbb{R}$, eh $-$ fg $\neq 0$ and ad $\neq 0$.
We want to check if the product of the first and fourth entries in our new matrix is 0 or not, and if the third entry is 0.
New matrix:
$\begin{bmatrix} e(ah-bg)-fdg & g(ah-bg)-hdg \\ e(be-fa)+fde & g(be-fa)+hde \end{bmatrix}$
This matrix is a member of H if

$$e(be-fa)+fde = ebe-efa+fde = 0$$

and

$$(e(ah-bg)-fdg)(g(be-fa)+hde) = (eah-ebg-fdg)(gbe-gfa+hde) \neq 0$$

Exercise 10

Let H = {(1), (12)(34)} in A_4.

a. Show that H is not normal in A_4.

   Well, recall that a subgroup H of G is normal iff $gH = Hg \ \forall\ g \in G$.

   So all we need to do is find a $g \in A_4$ such that $gH \neq Hg$.

   Notice: $(23) \in A_4$.

   (23)H = {(23)(1), (23)(12)(34)} = {(23), (1342)}

   H(23) = {(1)(23), (12)(34)(23)} = {(23), (1243)}

   {(23), (1342)} $\neq$ {(23), (1243)}

   Thus, H is not normal in A_4

b. Referring to the multiplication table for A_4 in Table 5.1 on page 105, show that, although $\alpha_6 H = \alpha_7 H$ and $\alpha_9 H = \alpha_{11} H$, it is not true that $\alpha_6 \alpha_9 H = \alpha_7 \alpha_{11} H$.

   α_6 = (243), α_7 = (142), α_9 = (132), and α_{11} = (234)

   So, let's look at both:

   $\alpha_6 \alpha_9 H \longleftarrow ? \longrightarrow \alpha_7 \alpha_{11} H$

   (243)(132)H $\longleftarrow ? \longrightarrow$ (142)(234)H

   {(243)(132)(1), (243)(132)(12)(34)} $\longleftarrow ? \longrightarrow$ {(142)(234)(1), (142)(234)(12)(34)}

   {(12)(34), (1)} $\longleftarrow ? \longrightarrow$ {(14)(23),(13)(24)}

   Nope! Those sets are not equal, so it's not true that $\alpha_6 \alpha_9 H = \alpha_7 \alpha_{11} H$.

c. Explain why this proves that the left cosets of H do not form a group under coset multiplication.

   Because the order of the permutations results in different output permutation, which means that coset multiplication isn't associative. Therefore, it can't form a group.

Exercise 14

What is the order of the element $14 + \langle 8 \rangle$ in the factor group $Z_{24}/\langle 8 \rangle$?