Due 4/25 (Wednesday):
All (turn in): Chapter 10, page 206, 14-18, 20, 24

Chapter 10

Recall:
A homomorphism ϕ from a group G to a group G$'$ is a mapping from G into G$'$ that preserves the group operation; that is, ϕ(ab) = ϕ(a)ϕ(b) for a, b $\in$ G.
The kernel of a homomorphism ϕ from a group G to a group G$'$ with identity e$'$ is the set {x $\in$ G : ϕ(x) = e$'$ }.

Exercise 14

Explain why the correspondence x $\longrightarrow$ 3x from Z_{12} to Z_{10} is not a homomorphism.

Because ϕ is not OP:
$\phi(3 * 4) = \phi(12) = \phi(0) = (3 * (0 \bmod 12)) \bmod 10 = e$, and
$\phi(3)\phi(4) = (3 * (3 \bmod 12) * 3 * (4 \bmod 12)) \bmod 10 = (9 * 3 * 4) \bmod 10 = (108) \bmod 10 = 8$

Exercise 15

Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and Ker ϕ = {0, 10, 20}.
If $\phi(23) = 9$, determine all elements that map to 9.

$\phi(ab \bmod 30) = \phi(a \bmod 30)\phi(b \bmod 30)$
$\phi(23) = 9$.
$\phi(0) = \phi(10) = \phi(20) = 0$
It looks like it's $\phi(x) = 3x$:
$\phi(23) = 3 * 23 \bmod 30 = 69 \bmod 30 = 9$.
Thus,
$\phi(3)$, $\phi(13)$, and $\phi(23)$ all map to 9.

Exercise 16

Prove that there is no homomorphism from $Z_8 \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Suppose $\exists$ ϕ: $Z_8 \bigoplus Z_2 \longrightarrow Z_4 \bigoplus Z_4$, such that ϕ is a homomorphism.
Because Z_8 is of order 8, and $|Z_2|$ divides 8, there is an element of order 8 in $Z_8 \bigoplus Z_2$, let's call it z_8.
Thus, $z_8 \in Z_8 \bigoplus Z_2$.
Because ϕ is OP, $\exists$ z $\in Z_4 \bigoplus Z_4$ such that $\phi(z_8) = z$ and $|z| = 8$.
However, there is no element of order 8 in $Z_4 \bigoplus Z_4$. A contradiction.
Hence, no homomorphism exists.

Exercise 17

Prove that there is no homomorphism from $Z_{16} \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Suppose $\exists$ ϕ: $Z_{16} \bigoplus Z_2 \longrightarrow Z_4 \bigoplus Z_4$, such that ϕ is a homomorphism.
Since G / Ker ϕ is isomorphic to ϕ(G),
|G / Ker ϕ| = |ϕ(G)| = 16.
Since |G| = 32, |Ker ϕ| = 2.
Since Ker $\phi \leq$ G, e $\in$ Ker ϕ.
Since Ker ϕ is of order 2, the other element in Ker ϕ must have order 2, let's call it k.
The only possibilities for k are: (8, 0), (8, 1), (0, 1)
Since |G / Ker ϕ| has order 16, the possibilities for the order of each c $\in$ G / Ker ϕ are factors of 16: 1, 2, 4, 8, and 16.
Case:

i) Ker ϕ = {(0, 0), (8, 0)}

   Let's look at the coset c = Ker ϕ + (1, 1) $\in$ G / Ker ϕ

   Notice that the order of c is 8.

ii) Ker ϕ = {(0, 0), (8, 1)}

   Let's look at the coset c = Ker ϕ + (1, 1) $\in$ G / Ker ϕ

   Notice that the order of c is 16.

iii) Ker ϕ = {(0, 0), (0, 1)}

   Let's look at the coset c = Ker ϕ + (1, 1) $\in$ G / Ker ϕ

   Notice that the order of c is 16.

However, because the homomorphism is onto, there is an isomorphism from G / Ker ϕ to $Z_4 \bigoplus Z_4$.
Thus, for any element g $\in$ G / Ker ϕ of order p, $\exists$ ϕ(g) of order p.
However, the maximum order of $Z_4 \bigoplus Z_4$ is 4. A contradiction.
Hence, there is no homomorphism.

Exercise 18

Can there be a homomorphism from $Z_4 \bigoplus Z_4$ onto Z_8? Can there be a homomorphism from Z_{16} onto $Z_2 \bigoplus Z_2$? Explain your answers.

Exercise 20

How many homomorphisms are there from Z_{20} onto Z_8? How many are there to Z_8?

Let $\phi : Z_{20} \longrightarrow Z_8$
Since ϕ is onto, |ϕ(G)| = 8.
However, since G is finite, |ϕ(G)| divides |G|.
Therefore, 8 divides 20. A contradiction.
Therefore, there are no homomorphisms from Z_{20} to Z_8.

Let ϕ be a homomorphism from Z_8 onto Z_8.
Since ϕ is onto, and the groups are the same order, that means ϕ is an isomorphism.
As far as how many of those there are, I'm not sure. 8, I suppose?

Exercise 24

Suppose that ϕ: $Z_{50} \longrightarrow Z_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine ϕ(x).

   If $\phi(7) = 6$, and $\phi(0) = 0$, then I suppose $\phi(x) = 3x$

b. Determine the image of ϕ.

   {0, 3, 6, 9, 12}

c. Determine the kernel of ϕ.

   {0, 5, 10, 15, 20, 25, 30, 35, 40, 45}

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.

   {1, 6, 11, 16, 21, 26, 31, 36, 41, 46}