Do 20 if you finish 19 on the HW

Theorem 5.1.13 (as seen in Lec 20)

Let f : D $\longrightarrow \mathbb{R}$, g : D $\longrightarrow \mathbb{R}$ and let c $\in$ D$'$
If $\lim_{x\to c}$ f(x) = L and $\lim_{x\to c}$ g(x) = M, then

a. $\lim_{x\to c}$ (f + g) = L + M

b. Let k $\in \mathbb{R}$, $\lim_{x\to c}$ kf = kL

c. $\lim_{x\to c}$ (fg) = LM

d. $\lim_{x\to c}$ ($\frac{f}{g}$) = $\frac{L}{M}$, provided that M $\neq$ 0

Proof.
(a) through **(c)** are similar to **(d)**.
(d): Let {s$_n$} be a sequence in D st s$_n \neq$ c $\forall$ n $\in \mathbb{N}$ and $\lim_{n\to\infty}$ s$_n$ = c.
Then, by Theorem 5.1.8, $\lim_{n\to\infty}$ f(s$_n$) = L.
Now, $\lim_{n\to\infty}$ g(x) = M $\neq$ 0.
So $\exists$ N $\in \mathbb{N}$ st

$$g(s_n) \neq 0 \text{ for } n \geq N$$

(ask why? next time)
Then, $\lim_{n\to\infty}$ ($\frac{f}{g}$)(s$_n$) = $\lim_{n\to\infty} \frac{f(s_n)}{g(s_n)} = \frac{\lim_{n\to\infty} f(s_n)}{\lim_{n\to\infty} g(s_n)}$ (by Theorem 4.2.11d) = $\frac{L}{M}$
Recall:
|x| − |y| $\leq$ ||x| − |y|| $\leq$ |x − y|
|y| $\geq$ |x| − |x − y|
So,
|g(s$_n$)| $\geq$ |M| − |M − g(s$_n$)|
and since,
$\lim_{n\to\infty}$ g(s$_n$) = M $\neq$ 0
|g(s$_n$) − M| < $\frac{|M|}{2}$
−|g(s$_n$) − M| > $\frac{-|M|}{2}$
for n $\geq$ N
So,
|g(s$_n$)| > |M| − $\frac{|M|}{2}$ = $\frac{|M|}{2}$ > 0 for n $\geq$ N ∎

Also, for the homework:
$\lim_{x\to c}$ P(x) = P(c) where P is a polynomial.

Example 5.1.14

Since $\lim_{x\to c}$ x^1 = c (Example 5.1.3),
then it follows by induction that $\lim_{x\to c}$ x^n = c^n $\forall$ n $\in \mathbb{N}$
Assume: for k $\in \mathbb{N}$, $\lim_{x\to c}$ x^k = c^k
Want to show: $\lim_{x\to c}$ x^{k+1} = c^{k+1}
Define: f(x) = x^k, g(x) = x

Then $\lim_{x \to c} f(x) = c^k$, $\lim_{x \to c} g(x) = c$
So,

$$\lim_{x \to c} x^{k+1} = \lim_{x \to c}(x^k x^1) = \lim_{x \to c}(f(x)g(x)) = \lim_{x \to c} f(x) \lim_{x \to c} g(x) = c^k c = c^{k+1}$$

Hence, $\lim_{x \to c} x^n = c^n$, $\forall$ n $\in \mathbb{N}$ by induction.
Combining the result with Theorem 5.1.13, we see that if P is a polynomial and $c \in \mathbb{R}$, then $\lim_{x \to c} P(x) = P(c)$.
To see this,
Let: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ where $a_i \in \mathbb{R}$ for i = 0, 1, 2, ... n

Example 5.1.5

Find $\lim_{x \to 1} \frac{2x^2-3x+1}{x-1}$
We need the 0 < part of the limit definition so that the limit can exist even if the function is undefined at that point.
Notice that for $x \neq 1$,

$$\frac{2x^2 - 3x + 1}{x - 1} = \frac{(x-1)(2x-1)}{(x-1)}$$

So,

$$\lim_{x \to 1} \frac{2x^2 - 3x + 1}{x - 1} = \lim_{x \to 1} 2x - 1 = 2 - 1 = 1$$

If we let $q = f(x) = \frac{2x^2-3x+1}{x-1}$, then $f(x) = 2x - 1$ with a hole at x = 1.

One Sided Limits

Let: $f : D \longrightarrow \mathbb{R}$ and let $c \in D'$
Then,

i) We write $\lim_{x \to c-} f(x) = L$ iff, for $\epsilon > 0$, $\exists\ \delta > 0$ st $|f(x) - L| < \epsilon$ whenever $c - \delta < x < c$ and $x \in D$

ii) We write $\lim_{x \to c+} f(x) = L$ iff, for $\epsilon > 0$, $\exists\ \delta > 0$ st $|f(x) - L| < \epsilon$ whenever $c < x < c + \delta$ and $x \in D$

Of course, if $\lim_{x \to c} f(x) = L$ iff both $\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) = L$

5.2 Continuous Functions

Definition 5.2.1

Let: $f : D \longrightarrow \mathbb{R}$ and $c \in D$ (we don't know that c is an accumulation point)
We say that f is **continuous at c** if
for any $\epsilon > 0$, $\exists\ \delta > 0$ st

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta \text{ and } x \in D$$

N.B.: f(c) must be defined in order for f(x) to be continuous at x = c

Theorem 5.2.2

Let: f : D $\longrightarrow \mathbb{R}$ and c $\in$ D
Then the following are equivalent:

a. f is continuous at c
b. If $\{x_n\}$ is any sequence in D st $x_n \longrightarrow c$ as n $\longrightarrow \infty$ (x_n can actually be c), then $\lim_{n\to\infty} f(x_n) = f(c)$
c. For every neighborhood V of f(c), $\exists$ a neighborhood U of c st $f(U \cap D) \subset V$
   Furthermore, if c $\in D'$, then the above are all equivalent to d)
d. f has a limit at c and $\lim_{x\to c} f(x) = f(c)$

Proof.

Case:

i) $c \in D \setminus D'$ (i.e. c is an isolated point)
Thus, $\exists$ a neighborhood $U \subset \mathbb{R}$ of c st

$$U \cap D = \{c\}$$

(i.e. $U = (c - \delta , c + \delta) = \{c\}$)

(a)

Want to show: f is continuous at x = c

For $\epsilon > 0$, $\exists\ \delta > 0$ st $(c - \delta , c + \delta) \subset U$.

This follows since a neighborhood is open. Thus,

$$|f(x) - f(c)| = 0 < \epsilon \text{ whenever} |x - c| < \delta \text{ and } x \in D$$

This means by definition that f(x) is continuous at x = c.

(b)

Let: $\{x_n\} \subset D$ st $x_n \longrightarrow c$ as n $\longrightarrow \infty$

and

For $\epsilon > 0$, $\exists\ \delta > 0$ st $(c - \delta , c + \delta) \subset U$

Want to show: $\lim_{n\to\infty} f(x_n) = f(c)$.

Since U is open, $\exists$ N $\in \mathbb{N}$ st

$$|x_n - c| < \delta \text{ for } n \geq N$$

Thus, $x_n \in U$ for $n \geq N$

We see that

$$|f(x_n) - f(c)| = 0 < \epsilon \text{ for } n \geq N$$

Hence, $\lim_{n\to\infty} f(x_n) = f(c)$

(c)

Now,

Let: V be a neighborhood of f(c)

Then, using U as defined prior to (a):

$$f(U \cap D) \subset V$$

Hence, a, b, and c, are equivalent if $c \in D \setminus D'$

ii) $c \in D \cap D'$ (i.e. c is an accumulation point)

$\square$