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Let S be the set of real numbers. If a, b $\in$ S, define a $\sim$ b if a $-$ b is an integer.

a. **Show that $\sim$ is an equivalence relation on S.**

Properties of an equivalence relation:

Reflexive: $\forall$ a $\in$ S, a $\sim$ a

Symmetric: a $\sim$ b $\Rightarrow$ b $\sim$ a

Transitive: a $\sim$ b and b $\sim$ c $\Rightarrow$ a $\sim$ c

Proof.

Let a $\in$ S

a $\in \mathbb{R}$ $\Rightarrow$ a = a.

Therefore, a $-$ a $= 0 \in \mathbb{Z}$

Hence, (a, a) is a member of the relation $\forall$ a $\in$ S.

Thus, $\sim$ is a reflexive relation on S.

Let a, b $\in$ S such that a $-$ b = c where c $\in \mathbb{Z}$

a $-$ b = c

a = c + b

a $-$ c = b

$-$c = b $-$ a

Notice that c $\in \mathbb{Z}$ $\Rightarrow$ $-$c $\in \mathbb{Z}$

Thus, if a $-$ b yields an integer, then b $-$ a yields an integer.

Hence, $\sim$ is a symmetric relation on S.

Let a, b, c $\in$ S such that a $\sim$ b and b $\sim$ c.

Thus, $\exists$ d, e $\in \mathbb{Z}$ such that a $-$ b = d and b $-$ c = e.

Notice that d + e = a $-$ b + b $-$ c = a $-$ c

Since d, e $\in \mathbb{Z}$ $\Rightarrow$ (d + e) $\in \mathbb{Z}$, a $-$ c yields an integer.

Hence, $\sim$ is a transitive relation on S, and that completes the proof.

$\square$

b. **Describe the equivalence classes of S.**

Given a, b $\in$ S, a $\sim$ b if a $-$ b = c where c $\in \mathbb{Z}$

So each equivalence class is a set of real numbers each separated by some integer.