Assignment Set: 6, 7, 15, 17, 19, 21 from pages 141 - 142

6)

Find the closure of each set:

a. { $\frac{1}{n}$: n $\in \mathbb{N}$ }
   Answer: $\emptyset$

b. $\mathbb{N}$
   Answer: $\mathbb{N}$

c. $\mathbb{Q}$
   Answer: $\mathbb{R}$

d. $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$
   Answer: $\emptyset$

e. { x : $|x - 5| \leq \frac{1}{2}$}
   $[4.5, 5.5]$
   Answer: $[4.5, 5.5]$

f. { x : $x^2 > 0$}
   $(0, \infty)$
   Answer: $[0, \infty)$

7)

Let S, T $\subset \mathbb{R}$. Find a counterexample of each of the following:

a. If P is the set of all isolated points of S, then P is a closed set.
   Answer: Let S = $\mathbb{N}$

b. Every open set contains at least two points.
   Answer: $\emptyset$

c. If S is closed, then cl(int S) = S.
   Answer: Let S = $\mathbb{Q}$

d. If S is open, then int (cl S) = S.
   Answer: Let S = $(-1, 0) \cup (0, 1)$

e. bd (cl S) = bd S
   Answer: Let S = $(-1, 0) \cup (0, 1)$

f. bd (bd S) = bd S
   Answer: Let S = $\mathbb{Q}$. Then bd S is $\mathbb{R}$, and bd (bd S) = $\emptyset \neq \mathbb{R}$.

g. bd (S $\cup$ T) = (bd S) $\cup$ (bd T)
   Answer: Let S = $\mathbb{R}$, T = $(0, 1)$. bd (S $\cup$ T) = $\emptyset$, but bd S $\cup$ bd T = $\emptyset \cup \{0, 1\}$

h. bd (S $\cap$ T) = (bd S) $\cap$ (bd T)
   Answer: Let S = $(0, 1)$, T = $(1, 2)$. bd (S $\cap$ T) = $\emptyset$, but bd S $\cap$ bd T = 1.

15)

Prove: If x is an accumulation point of the set S, then every neighborhood of x contains infinitely many points of S.

17)

Prove: S' is a closed set.

19)

Suppose S is a nonempty bounded set and let m = sup S. Prove or give a counter example: m is a boundary point of S.

21)

Let A be a nonempty open subset of $\mathbb{R}$ and let $Q \subset \mathbb{Q}$. Prove: $A \cap Q \neq \emptyset$.

Let: $S \subset \mathbb{R}$
Then

a. S is closed iff $S' \subset S$

b. cl S is a closed set

c. S is closed iff S = cl S

d. cl S = S U S' = S $\cup$ bd S

Proof.

a)

S is closed iff $S' \subset S$
$\longrightarrow$
Suppose: S is closed.
Want to show: $S' \subset S$
Let: x $\in$ S'
Thus, $\forall \epsilon > 0$

$$\mathrm{N}(x, \epsilon) \cap \mathrm{S} = \emptyset \tag{1}$$

Want to show: x $\in S$
Assume: x $\notin$ S
Then, from **(1)**,

$$\mathrm{N}(x, \epsilon) \cap \mathrm{S} \neq \emptyset \tag{2}$$

and

$$\mathrm{N}(x, \epsilon) \cap \neg\mathrm{S} \neq \emptyset \tag{3}$$

From **(2)** and **(3)**,
x $\in$ bd S $\subset$ S by definition of a closed set. This is a contradiction.
Hence, x $\in$ S.
This proves:

$$S' \subset S$$

$\longleftarrow$
Conversely,
Suppose: $S' \subset$S
Want to show: $\mathbb{R} \setminus$ S is open $\Rightarrow$ S is closed.
Let: x $\in \mathbb{R} \setminus$ S
Want to show: $\exists \epsilon > 0$ st N(x, ϵ) $\subset \mathbb{R} \setminus$ S
Since x $\notin$ S, we see that x not $\notin$ S'.
Thus, $\exists \epsilon > 0$ st $\mathrm{N}(x, \epsilon) \cap \mathrm{S} = \emptyset$
Since x $\notin$ S, we have:

$$\mathrm{N}(x, \epsilon) \cap \mathrm{S} = \emptyset \tag{1}$$

Hence, N(x, ϵ) $\subset \mathbb{R} \setminus$ S , which proves that $\mathbb{R} \setminus$ S is open, or, equivalently, that S is closed.
This completes the proof of a).

b)

cl S is a closed set
Recall that cl S = S $\cup$ S$'$.
Want to show: $\mathbb{R} \setminus$ cl S is open $\Rightarrow$ cl S is closed
Let: x $\in$ cl ($\mathbb{R} \setminus$ S) (aka (S $\cup$ S$'$) Compliment)
We must find an $\epsilon > 0$ st N(x, ϵ) $\subset$ cl ($\mathbb{R} \setminus$ S)
Now x $\notin$ S and x $\notin$ S$'$.
$\exists\epsilon > 0$ st N*(x, ϵ) $\cap$ S = $\emptyset$
However, x $\notin$ S, so

$$\text{N}(x, \epsilon) \ \cap \ \text{S} = \emptyset \tag{1}$$

We claim that N(x, ϵ) $\cap$ S' = $\emptyset$
Since:

$$\neg[x \in S \cup S']$$
$$\neg[x \in S \text{ or } x \in S']$$
$$x \notin S \text{ and } x \notin S'$$

which is equivalent to N(x, ϵ) $\subset \mathbb{R} \setminus$ S$'$
Let: y $\in$ N(x, ϵ)
By Theorem 2(a), the set N(x, ϵ) is open.
So $\exists\hat{\epsilon} > 0$ st N(y, $\hat{\epsilon}$) $\subset$ N(x,ϵ).
In particular, y $\notin$ N(x, ϵ).
From **(1)**
N*(y, $\hat{\epsilon}$) $\cap$ S = $\emptyset$.
So, y $\notin$ S$'$ or, equivalently, y $\in \mathbb{R} \setminus$ S$'$.
This proves that N(x, ϵ) $\subset \mathbb{R} \setminus$ S$'$ or, equivalently,

$$\text{N}(x, \epsilon) \ \cap \ \text{S}' = \emptyset \tag{2}$$

From **(1)** and **(2)**, N(x, ϵ) $\cap$ (S $\cup$ S$'$) = $\emptyset$.
Hence,

$$N(x, \epsilon) \subset (S \cup S')^C = \text{ cl } S^C \tag{3}$$

Thus, **(3)** and * prove that cl S^C is open.
Hence, by Theorem 3.4.7, cl S is closed.

c)

S is closed iff S = cl S (= S $\cup$ S')
$\longrightarrow$
Suppose: S is closed.
Want to show: S = S $\cup$ S$'$.
By definition, S $\subset$ S $\cup$ S$'$.
Want to show: S $\cup$ S$'$ $\subset$ S
Let x $\in$ S $\cup$ S'.
If x $\in$ S, then we are finished.
If x $\in$ S' $\setminus$ S Venn Diagram: (S ( )xxS')
Then by a), S' $\subset$ S, since S is closed.

Hence, x ∈ S, and we are finished.
$\longleftarrow$
Conversely,
Suppose: S = $S \cup S'$
Want to show: S is closed.
By (b), cl S is closed.
Since, S = $S \cup S'$ = cl S, S is also closed.

d)

cl S = S ∪ S' = S ∪ bd S
Let: x ∈ S ∪ S'
If x ∈ S, then x ∈ S ∪ bd S.
So, S ∪ S ⊂ S ∪ bd S in this case.
If x ∈ $S' \setminus$ S, then $\forall \epsilon > 0$, N(x, ϵ) ∩ S ≠ ∅, which implies x ∈ $\mathbb{R} \setminus$ S and N(x, ϵ) ∩ $\mathbb{R} \setminus$ S ≠ ∅
Thus, x ∈ bd S ⊂ S ∪ bd S.
Hence, S ∪ S' ⊂ S ∪ bd S.
For the reverse conclusion, let x ∈ S ∪ bd S.
If x ∈ S, then x ∈ S ∪ S′. So, in this case, S ∪ bd S ⊂ S ∪ S′ = cl S.
if x ∈ bd $S \setminus$ S, then, in particular,
$\forall \epsilon > 0$,

$$N * (x, \epsilon) \cap S \neq \emptyset$$

which implies that x ∈ S' ⊂ S ∪ S'.
Hence, S ∪ bd S ⊂ S ∪ S′.
Hence, result.

□