

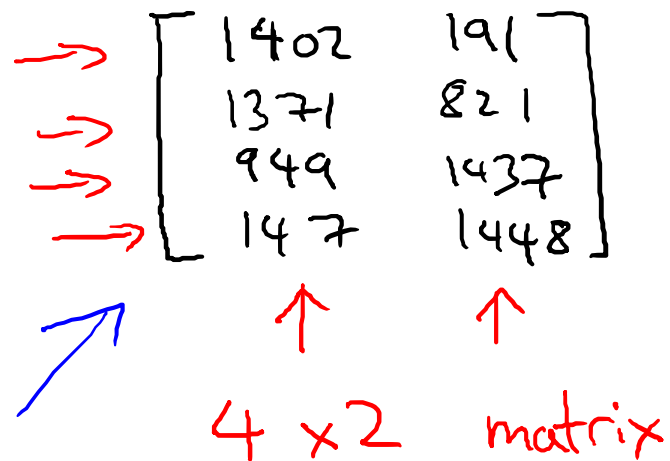
Machine Learning

Linear Algebra  
review (optional)

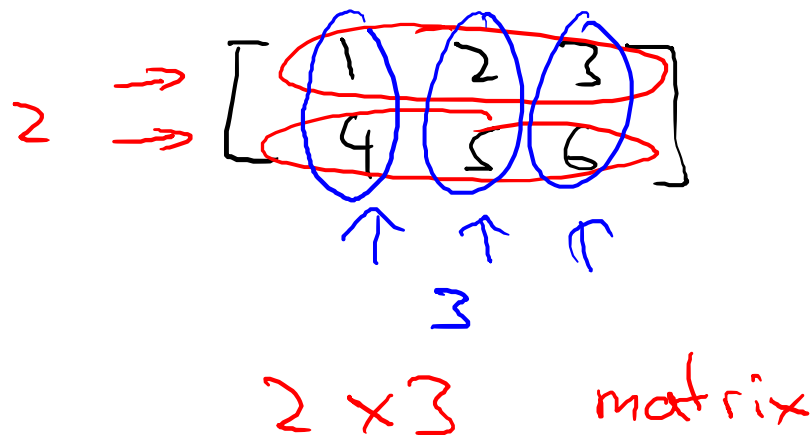
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Matrices and  
vectors

**Matrix:** Rectangular array of numbers:



→  $\mathbb{R}^{4 \times 2}$



$\mathbb{R}^{2 \times 3}$

Dimension of matrix: number of rows x number of columns

# Matrix Elements (entries of matrix)

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

$A_{ij}$  = " $i, j$  entry" in the  $i^{th}$  row,  $j^{th}$  column.

$$A_{11} = 1402$$

$$A_{12} = 191$$

$$A_{32} = 1437$$

$$A_{41} = 147$$

$$\cancel{A_{43}} = \text{Undefined (error)}$$

Vector: An  $n \times 1$  matrix.

$$\underline{y} = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

$n = 4$

← 4-dimensional vector.

~~$\mathbb{R}^{3 \times 2}$~~

$\mathbb{R}^4$

$y_i = i^{th}$  element

$$y_1 = 460$$

$$y_2 = 232$$

$$y_3 = 315$$

→ A, B, C, X

a, b, x, y

1-indexed vs 0-indexed:

$y[1]$

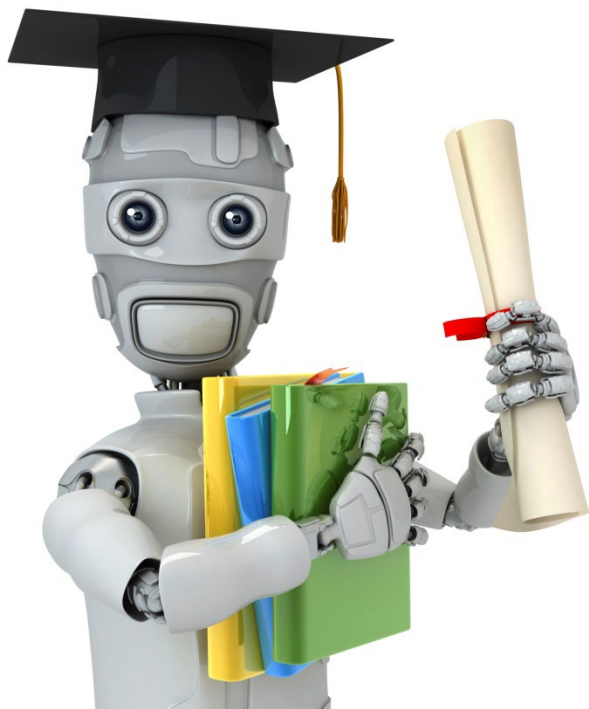
$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

1-indexed

$y[0]$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

0-indexed



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# Linear Algebra review (optional)

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## Addition and scalar multiplication

# Matrix Addition

$$\begin{array}{c} \downarrow \downarrow \\ \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0.5 \\ 4 & 10 \\ 3 & 2 \end{bmatrix} \\ \text{3x2 matrix} \quad \text{3x2} \quad \text{3x2} \end{array}$$

$$\begin{array}{c} \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \end{bmatrix} = \\ \text{3x2} \quad \text{2x2} \end{array} \quad \text{error}$$

# Scalar Multiplication

← real number

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 15 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \times 3$$

3x2                      3x2

$$\begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} / 4 =$$

$$\frac{1}{4} \begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} =$$

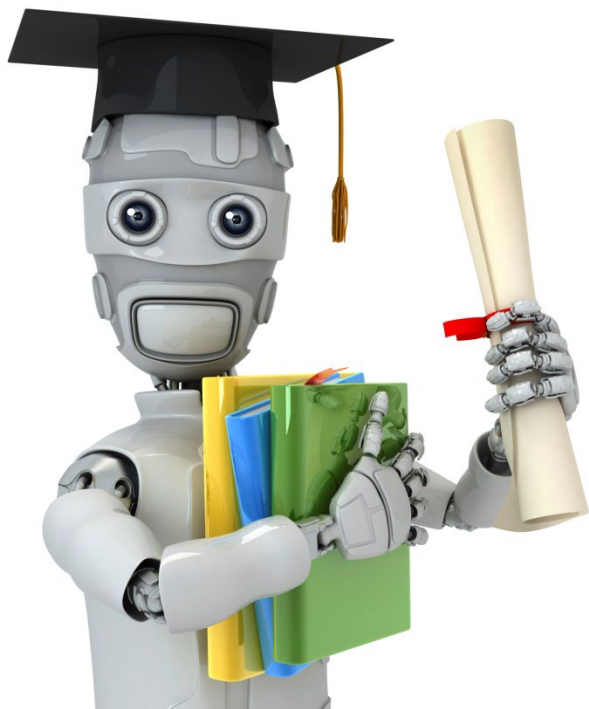
$$\begin{bmatrix} 1 & 0 \\ \frac{3}{2} & \frac{3}{4} \end{bmatrix}$$

# Combination of Operands

$$\begin{aligned}
 & \text{Scalar multiplication} \rightarrow 3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \text{ / 3} \quad \text{Scalar division} \\
 & = \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix} \quad \begin{array}{l} \text{matrix subtraction /} \\ \text{vector subtraction} \end{array} \\
 & = \begin{bmatrix} 2 \\ 12 \\ 10\frac{1}{3} \end{bmatrix} \quad \begin{array}{l} \text{matrix addition /} \\ \text{vector addition} \end{array}
 \end{aligned}$$

$3 \times 1$  matrix  
 $3$ -dimensional vector





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# Linear Algebra review (optional)

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## Matrix-vector multiplication

# Example

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}$$

$3 \times 2$        $2 \times 1$        $3 \times 1$  matrix

$$1 \times 1 + 3 \times 5 = 16$$

$$4 \times 1 + 0 \times 5 = 4$$

$$2 \times 1 + 1 \times 5 = 7$$

## Details:

$$\underline{A} \times \underline{x} = \underline{y}$$

The diagram illustrates the matrix multiplication  $\underline{A} \times \underline{x} = \underline{y}$ . Matrix  $\underline{A}$  is an  $m \times n$  matrix (m rows, n columns). Vector  $\underline{x}$  is an  $n \times 1$  matrix (n-dimensional vector). Vector  $\underline{y}$  is an  $m$ -dimensional vector. The diagram shows the dot product of rows of  $\underline{A}$  with elements of  $\underline{x}$  to produce elements of  $\underline{y}$ .

$m \times n$  matrix  
(m rows, n columns)

$n \times 1$  matrix  
(n-dimensional vector)

$m$ -dimensional vector

→ To get  $\underline{y}_i$ , multiply  $\underline{A}$ 's  $i^{th}$  row with elements of vector  $\underline{x}$ , and add them up.

# Example

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 4 \\ -1 & -2 & 0 & 0 \end{bmatrix}_{3 \times 4} \begin{matrix} \downarrow \\ 1 \\ 3 \\ 2 \\ 1 \end{matrix}_{4 \times 1} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}$$

$$\left. \begin{array}{l} 1 \times 1 + 2 \times 3 + 1 \times 2 + 5 \times 1 = 14 \\ 0 \times 1 + 3 \times 3 + 0 \times 2 + 4 \times 1 = 13 \\ -1 \times 1 + (-2) \times 3 + 0 \times 2 + 0 \times 1 = -7 \end{array} \right\}$$

House sizes:

- 2104
- 1416
- 1534
- 852

Matrix

$$\begin{bmatrix} 1 & 2104 \\ 1 & 1416 \\ 1 & 1534 \\ 1 & 852 \end{bmatrix}$$

4x2

$$h_{\theta}(x) = -40 + 0.25x$$

$h_{\theta}(x)$

2x1

Vector

$$\begin{bmatrix} -40 \\ 0.25 \end{bmatrix}$$

x

4x1 matrix

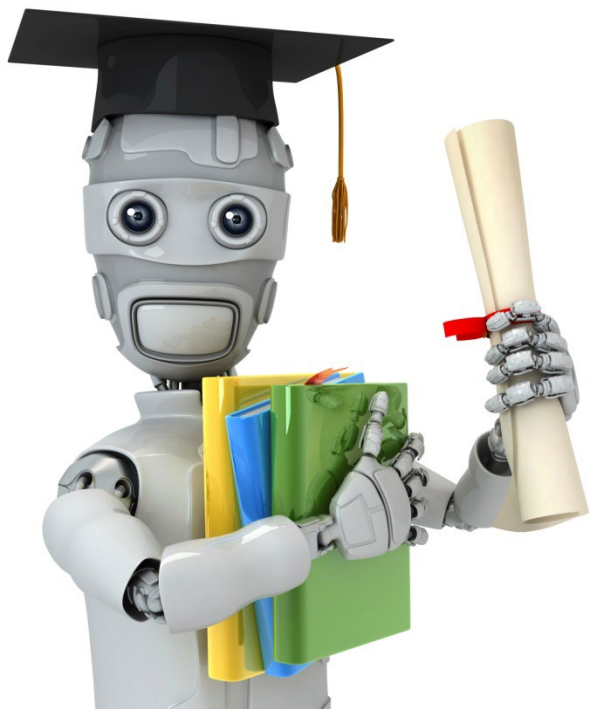
$$\begin{bmatrix} -40 \times 1 + 0.25 \times 2104 \\ -40 \times 1 + 0.25 \times 1416 \\ \phantom{-40 \times 1 + 0.25 \times 1416} \\ \phantom{-40 \times 1 + 0.25 \times 1416} \end{bmatrix}$$

$h_{\theta}(1416)$

$$\text{prediction} = \text{Data Matrix} \times \text{Parameters}$$

4x1

for  $i = 1, \dots, 1000$ ,  
prediction (i) = ...



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# Linear Algebra review (optional)

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## Matrix-matrix multiplication

# Example

$$\begin{array}{c}
 \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 0 & 1 \\ \hline 5 & 2 \\ \hline \end{array} = \begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix} \\
 \text{2} \times 3 \quad \text{3} \times 2 \\
 \hline
 \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 5 \\ \hline \end{array} = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \\
 \hline
 \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}
 \end{array}$$

Diagram illustrating matrix multiplication with dimensions and intermediate results:

- Top row:  $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 0 & 1 \\ \hline 5 & 2 \\ \hline \end{array} = \begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix}$ . Dimensions  $2 \times 3$  and  $3 \times 2$  are indicated. The result matrix is  $\begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix}$ .
- Middle row:  $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 5 \\ \hline \end{array} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$ . The dimension  $3 \times 2$  is indicated. The result is a column vector  $\begin{bmatrix} 11 \\ 9 \end{bmatrix}$ .
- Bottom row:  $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$ . The dimension  $3 \times 2$  is indicated. The result is a column vector  $\begin{bmatrix} 10 \\ 14 \end{bmatrix}$ .

Arrows indicate the correspondence between the dimensions and the elements in the result matrices.

## Details:

$$\begin{array}{c}
 \underline{A} \quad \times \quad \underline{B} \quad = \quad \underline{C} \\
 \left[ \begin{array}{c} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{array} \right] \times \left[ \begin{array}{c|c|c|c} \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \end{array} \right] = \left[ \begin{array}{c|c|c|c} \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \end{array} \right]
 \end{array}$$

$m \times n$  matrix  
 (m rows, n columns)

$n \times o$  matrix  
 (n rows, o columns)

$m \times o$  matrix

~~$n \times 1$~~

The  $i^{th}$  column of the matrix  $C$  is obtained by multiplying  $A$  with the  $i^{th}$  column of  $B$ . (for  $i = 1, 2, \dots, o$ )



# Example

$$\overset{2 \times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}} \overset{2 \times 2}{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}$$

$$= \overset{2 \times 2}{\begin{bmatrix} 9 & 7 \\ 15 & 12 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 0 + 3 \times 3 \\ 2 \times 0 + 5 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 15 \end{bmatrix} \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 3 \times 2 \\ 2 \times 1 + 5 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

House sizes:

$$\begin{cases} 2104 \\ 1416 \\ 1534 \\ 852 \end{cases}$$

Matrix

$$\begin{bmatrix} 1 & 2104 \\ 1 & 1416 \\ 1 & 1534 \\ 1 & 852 \end{bmatrix} \times$$

Matrix

$$\begin{bmatrix} -40 & 200 & -150 \\ 0.25 & 0.1 & 0.4 \end{bmatrix} =$$

Have 3 competing hypotheses:

1.  $h_{\theta}(x) = -40 + 0.25x$

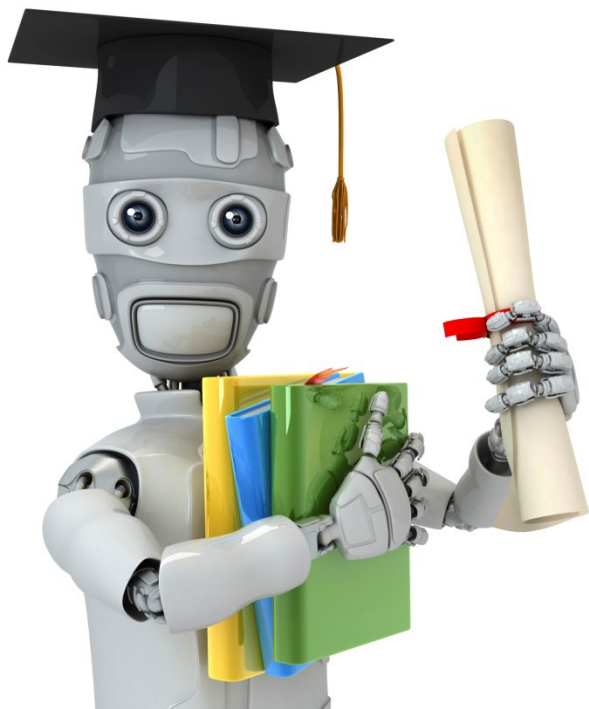
2.  $h_{\theta}(x) = 200 + 0.1x$

3.  $h_{\theta}(x) = -150 + 0.4x$

$$\begin{bmatrix} 486 & 410 & 692 \\ 314 & 342 & 416 \\ 344 & 353 & 464 \\ 173 & 285 & 191 \end{bmatrix}$$

Prediction  
of first  
 $h_{\theta}$

Predictions  
of 2<sup>nd</sup>  
 $h_{\theta}$



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# Linear Algebra review (optional)

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## Matrix multiplication properties


$$3 \times 5 = 5 \times 3$$


"Commutative"

Let  $A$  and  $B$  be matrices. Then in general,  
 $A \times B \neq B \times A$ . (not commutative.)

E.g.


$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$


$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$


$A \times B$   
 $m \times n$        $n \times m$

$A \times B$       is       $m \times m$

$B \times A$       is       $n \times n$



$$\underline{3 \times 5 \times 2}$$

$$3 \times 10 = 30 = 15 \times 2$$

$$3 \times (5 \times 2) = (3 \times 5) \times 2$$

"Associative"

$$A \times (B \times C)$$

$$(\underline{A \times B}) \times C$$



$$A \times B \times C.$$

Let  $D = B \times C$ . Compute  $A \times D$ .

Let  $E = A \times B$ . Compute  $E \times C$ .

$$A \times (B \times C)$$

$$(A \times B) \times C$$

Some  
answer.

# Identity Matrix

1 is identity.

$$1 \times z = z \times 1 = z$$

for any  $z$

Denoted  $I$  (or  $I_{n \times n}$ ).

Examples of identity matrices:

$$\begin{bmatrix} 1 \end{bmatrix}$$

$1 \times 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$3 \times 3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$4 \times 4$

Informally:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

For any matrix  $A$ ,

$$A \cdot \boxed{I} = \boxed{I} \cdot A = A$$

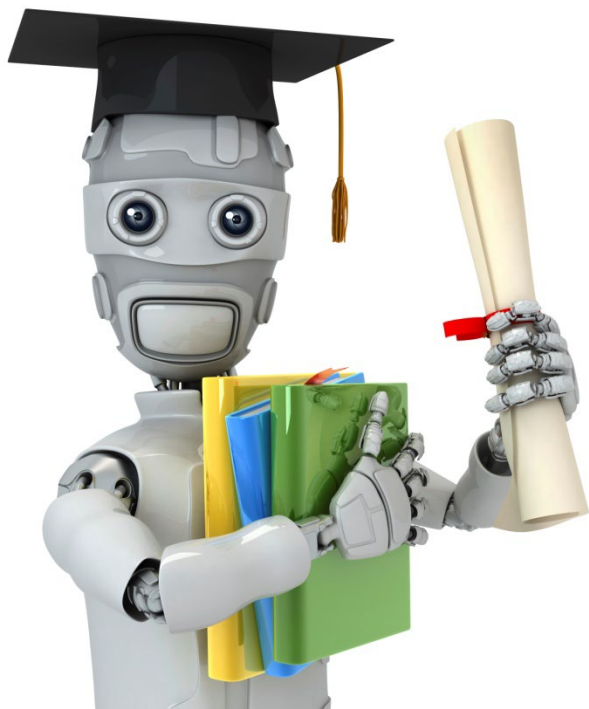
$m \times n$   $n \times n$   $m \times m$   $m \times n$

$$I_{n \times n}$$

Note:

$$AB \neq BA \text{ in general}$$

$$AI = \cancel{IA} IA \checkmark$$



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Linear Algebra  
review (optional)

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Inverse and  
transpose

$I$  = "identity."

$$3 \underbrace{(3^{-1})}_{\frac{1}{3}} = 1$$

$$12 \times \underbrace{(12^{-1})}_{\frac{1}{12}} = 1$$

$$0 \underbrace{(0^{-1})}_{\text{undefined}}$$

Not all numbers have an inverse.

**Matrix inverse:** ↖ square matrix  
(#rows = #columns)  $A^{-1}$

If  $A$  is an  $m \times m$  matrix, and if it has an inverse,

$$\rightarrow \underline{A(A^{-1})} = \underline{A^{-1}A} = \underline{I}.$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

↑

E.g.

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}}_{A \text{ (2x2)}} \underbrace{\begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A} = I_{2 \times 2}$$

Matrices that don't have an inverse are "singular" or "degenerate"



# Matrix Transpose

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix}$$

*Handwritten annotations: A red arrow points to the underlined A. A blue arrow points from the first row to the first column of B. A green oval encloses the first row [1, 2, 0]. A blue oval encloses the second row [3, 5, 9]. A red '2 x 3' is written below the matrix.*

$$\underline{B} = A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}$$

*Handwritten annotations: A blue arrow points to the underlined B. A green oval encloses the first column [1, 2, 0]. A blue oval encloses the second column [3, 5, 9]. A red '3 x 2' is written below the matrix.*

Let  $A$  be an  $m \times n$  matrix, and let  $B = A^T$ .

Then  $B$  is an  $n \times m$  matrix, and

$$\underline{B}_{ij} = \underline{A}_{ji}.$$

$$B_{12} = A_{21} = 2$$

$$B_{32} = 9$$

$$A_{23} = 9.$$