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User Subscription Dynamics and Revenue Maximization in Communications Markets[†]

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***Abstract*—In order to understand the complex interactions between different technologies in a communications market, it is of fundamental importance to understand how technologies affect the demand of users and competition between network service providers (NSPs). To this end, we analyze user subscription dynamics and revenue maximization in monopoly and duopoly communications markets. First, by considering a monopoly market with only one NSP, we investigate the impact of technologies on the users' dynamic subscription. It is shown that, for any price charged by the NSP, there exists a unique equilibrium point of the considered user subscription dynamics. We also provide a sufficient condition under which the user subscription dynamics converges to the equilibrium point starting from any initial point. We then derive upper and lower bounds on the optimal price and market share that maximize the NSP's revenue. Next, we turn to the analysis of a duopoly market and show that, for any charged prices, the equilibrium point of the considered user subscription dynamics exists and is unique. As in a monopoly market, we derive a sufficient condition on the technologies of the NSPs that ensures the user subscription dynamics to reach the equilibrium point. Then, we model the NSP competition using a non-cooperative game, in which the two NSPs choose their market shares independently, and provide a sufficient condition that guarantees the existence of at least one pure Nash equilibrium in the market competition game.**

I. INTRODUCTION

Tremendous efforts have been dedicated in the past decade to enhancing the quality-of-service (QoS) of communications networks and expanding their network capacities. Nevertheless, it is the joint consideration of prices and QoS that determines the demand of users and the revenue of network service providers (NSPs). Moreover, investment[1] in technologies by NSPs affects QoS and in turn the pricing schemes they use. Hence, technologies, NSPs and users are closely coupled and interact in a complex way (as illustrated in Fig. 1).

In this paper, we are interested in the problem of revenue maximization in a communications market, and for this we consider the interaction between technologies, the subscription decisions of users, and the pricing strategies of NSPs. First, we focus on a monopoly market with only one resource-constrained NSP, which provides each user with a QoS that depends on the number of subscribers. In particular, to take into account the QoS degradation when more users join the network, the QoS is modeled as a non-increasing function in terms of the number of subscribers. By jointly considering the provided QoS and charged price, users can dynamically decide whether or not to subscribe to the NSP. Under the assumption that users make their subscription decisions based on the most recent QoS and the current price, we show that, for any QoS function and price, there exists a unique equilibrium point of the user subscription dynamics at which the number of subscribers does not change. Nevertheless, if the QoS degrades too fast when more users subscribe to the NSP, the user subscription dynamics may not converge to the equilibrium point. Hence, we find a sufficient condition which needs to be fulfilled by the QoS function to ensure the global convergence of the user subscription dynamics. We also derive upper and lower bounds on the optimal price and market share that maximize the NSP's revenue.

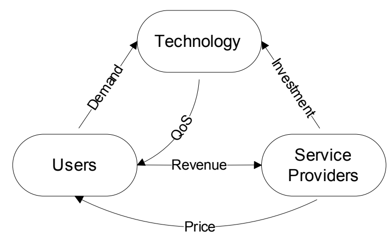

Fig. 1. Interaction between technology, NSPs and users.

Next, we analyze a duopoly market by adding another NSP providing a constant QoS to its subscribers. Given the provided QoS and charged prices, users dynamically select the NSP that yields a higher (positive) utility. We first show that, for any prices, the considered user subscription dynamics always admits a unique equilibrium point, at which no user wishes to change its subscription decision. We next obtain a sufficient condition that the QoS functions need to fulfill to guarantee the convergence of the user subscription dynamics. Then, we analyze the competition between the two NSPs. Specifically, modeled as a strategic player in a non-cooperative game, each NSP aims to maximize its own revenue by selecting its own market share while regarding the market share of its competitor as fixed. This is in sharp contrast with the existing related literature which typically studies price competition among NSPs. For the formulated market share competition game, we derive a sufficient condition on the QoS function that

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[1]Although investment decisions regarding technologies are not explicitly considered in this paper, our analysis shows how technologies affect the NSPs' revenues and hence, can provide a basis for investment decisions.

guarantees the existence of at least one pure Nash equilibrium (NE).

Now, we briefly review the existing literature related to our work. Communication markets have been attracting an unprecedented amount of attention from various research communities. For instance, [1] studied the technology adoption and competition between incumbent and emerging network technologies. Nevertheless, the model characterizing users' valuation of QoS is mainly restricted to uniform distribution and no NSP competition was addressed. In [2], the authors showed that non-cooperative communication markets suffer from unfair revenue distribution among NSPs and proposed a revenue-sharing mechanism that calls for cooperation among the NSPs. The behavior of users and its impact on the revenue distribution, however, were not explicitly considered. Without considering the interplay between different NSPs, the authors in [3] generalized the utility maximization problem (i.e., rate allocation in their work) by incorporating the participation of content providers into the model, and derived equilibrium prices and data rates. Another paper related to our work is [5] in which the authors examined the evolution of market segmentation in wireless social community networks. A key assumption, based on which the equilibrium was derived, is that the social community network provides a higher QoS to each user as the number of subscribers increases. While this assumption may hold if network coverage is a dominant factor that determines the QoS, it does not model the QoS degradation due to, for instance, user traffic congestion incurred at the NSP. In fact, more users using the network service will degrade the QoS if the NSP is resource-constrained [8][9].

In summary, the main contributions of this paper are as follows: (i) we study how different QoS functions (hence, different technologies) affect the user subscription, revenue maximization and market competition by assuming a general QoS model for one NSP and a general distribution of the users' valuations of the QoS. This is unlike most of the existing literature where linear QoS functions and a uniform distribution are assumed; (ii) we find a sufficient condition which the QoS provided by a resource-constrained NSP needs to fulfill in order to guarantee the convergence of the user subscription dynamics; and (iii) we analyze the competition between two NSPs choosing their market shares strategically and characterize QoS functions that ensure the existence of a NE of the market share competition game.

The rest of this paper is organized as follows. Section II describes the model and assumptions. In Sections III and IV, we consider monopoly and duopoly markets, respectively, to study the user subscription dynamics and revenue maximization. An illustrative example is shown in Section V. Finally, concluding remarks are offered in Section VI.

II. Model

We consider a communications market in which two NSPs, denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$, operate. There is a continuum of users, as in [1], that can potentially subscribe to one of the NSPs for communication services. The continuum model approximates well the real user population if there are a sufficiently large number of users in the market so that each individual user is negligible [9]. As in [5][8], we assume throughout this paper that each user can subscribe to at most one NSP at any time instant. We also assume that NSP $\mathcal{S}_1$ has sufficient resources to provide a guaranteed level of QoS to all of its subscribers [5][6], whereas NSP $\mathcal{S}_2$ is resource-constrained and thus is prone to congestion among subscribers.[2] In other words, the QoS provided by NSP $\mathcal{S}_1$ is the same regardless of the number of its subscribers, whereas the QoS provided by NSP $\mathcal{S}_2$ degrades with the number of its subscribers [9]. Let λ_i be the fraction of users subscribing to NSP $\mathcal{S}_i$ for $i = 1, 2$. Then λ_1 and λ_2 satisfy $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \leq 1$. Also, let q_i be the QoS provided by NSP $\mathcal{S}_i$ for $i = 1, 2$. Note that q_1 is independent of λ_1 while q_2 is non-increasing in λ_2. We use a function $g(\cdot)$ defined on $[0, 1]$ to express the QoS provided by NSP $\mathcal{S}_2$ as $q_2 = g(\lambda_2)$. For simplicity, we assume as in [3] that the cost of serving subscribers is fixed and smaller than the level that drives the NSPs out of the market so that we can use revenue maximization as the objective of the NSPs.

Users are heterogeneous in the sense that they may value the same level of QoS differently. Each user k is characterized by a non-negative real number α_k, which represents its valuation of QoS. Specifically, when user k subscribes to NSP $\mathcal{S}_i$, its utility is given by

$$u_{k,i} = \alpha_k q_i - p_i, \tag{1}$$

where p_i is the subscription price charged by NSP $\mathcal{S}_i$, for $i = 1, 2$.[3] Users that do not subscribe to either of the two NSPs obtain zero utility. Note that in our model the NSPs are allowed to engage in neither QoS discrimination nor price discrimination. That is, all users subscribing to the same NSP receive the same QoS and pay the same subscription price [5][6].

Now, we impose assumptions on the QoS function of NSP $\mathcal{S}_2$, user subscription decisions, and the users' valuation of QoS as follows.

Assumption 1: $g(\cdot)$ is a non-increasing and continuously differentiable[4] function, and $0 < g(\lambda_2) < q_1$ for all $\lambda_2 \in [0, 1]$.

Assumption 2: Each user k subscribes to NSP $\mathcal{S}_i$ if $u_{k,i} > u_{k,j}$ and $u_{k,i} \geq 0$ for $i, j \in \{1, 2\}$ and $i \neq j$. If $u_{k,1} = u_{k,2} \geq 0$, user k subscribes to NSP $\mathcal{S}_1$.[5]

Assumption 3: The users' valuation of QoS follows a probability distribution whose probability density function (PDF) $f(\cdot)$ is strictly positive and continuous on $[0, \beta]$ for some

[2] An example that fits into our assumptions on NSPs is a cognitive radio network in which NSP $\mathcal{S}_1$ is a licensed operator serving primary users while NSP $\mathcal{S}_2$ is a spectrum broker serving secondary users. Another example is a market in which NSP $\mathcal{S}_1$ serves each user using a dedicated channel while NSP $\mathcal{S}_2$ has its users share its limited resources or capacity [9].

[3] A similar quasilinear utility model has been used in [5][9][10].

[4] Since $g(\cdot)$ is defined on $[0, 1]$, we use a one-sided limit to define the derivative of $g(\cdot)$ at 0 and 1, i.e., $g'(0) = \lim_{\lambda_2 \to 0^+} [g(\lambda_2) - g(0)]/(\lambda_2 - 0)$ and $g'(1) = \lim_{\lambda_2 \to 1^-} [g(\lambda_2) - g(1)]/(\lambda_2 - 1)$.

[5] Specifying an alternative tie-breaking rule (e.g., random selection between the two NSPs) in case of $u_{k,1} = u_{k,2} \geq 0$ will not affect the analysis of this paper, since the fraction of indifferent users is zero under Assumption 3 and thus the revenue of the NSPs is independent of the tie-breaking rule. A similar remark holds for the tie-breaking rule between subscribing and not subscribing in the case of $u_{k,i} = 0 \geq u_{k,j}$ for $i, j = \{1, 2\}$ such that $i \neq j$.

$\beta > 0$. For completeness of definition, we have $f(\alpha) = 0$ for all $\alpha \notin [0, \beta]$. The cumulative density function (CDF) is given by $F(\alpha) = \int_{-\infty}^{\alpha} f(x)dx$ for $\alpha \in \mathbb{R}$.

We briefly discuss the above three assumptions. Assumption 1 captures the congestion effects that users experience when subscribing to resource-constrained NSP $\mathcal{S}_2$ (e.g., traffic congestion in [8][9]). The shape of the QoS function $g(\cdot)$ of NSP $\mathcal{S}_2$ is determined by various factors including the resource allocation scheme and the scheduling algorithm of NSP $\mathcal{S}_2$. Assumption 2 can be interpreted as a rational subscription decision. A rational user will subscribe to the NSP that provides a higher utility if at least one NSP provides a non-negative utility, and to neither NSP otherwise. Assumption 3 can be considered as an expression of user diversity in terms of the valuation of QoS. When the users' valuation of QoS is sufficiently diverse, its distribution can be described by a continuous positive PDF on a certain interval as in [9]. Note that the lower bound on the interval is set as zero to simplify the analysis, and this will be the case when there is enough diversity in the users' valuation of QoS so that there are non-subscribers for any positive price [6].

III. Monopoly Communication Market

In this section, we study user subscription dynamics and revenue maximization in a monopoly communications market where only one NSP operates. For convenience of presentation, we first analyze the monopoly market of NSP $\mathcal{S}_2$ and then apply the analysis to the monopoly market of NSP $\mathcal{S}_1$.

A. User Subscription Dynamics in the Monopoly Market of NSP $\mathcal{S}_2$

When only NSP $\mathcal{S}_2$ operates in the communications market, each user has a choice of whether to subscribe to NSP $\mathcal{S}_2$ or not at each time instant. Since the QoS provided NSP $\mathcal{S}_2$ is varying with the fraction of its subscribers[6], each user will form a belief, or expectation, on the QoS of NSP $\mathcal{S}_2$ when it makes a subscription decision. To describe the dynamics of user subscription, we construct and analyze a dynamic model which specifies how users form their beliefs and make decisions based on their beliefs. We consider a discrete-time model with time periods indexed $t = 1, 2, \ldots$. At each period t, user k holds a belief or expectation on the QoS of NSP $\mathcal{S}_2$, denoted by $\tilde{g}_k(\lambda_2^t)$ where λ_2^t is the fraction of subscribers at period t, and makes a subscription decision in a myopic way to maximize its expected utility in the current period.[7] Then, user k subscribes to NSP $\mathcal{S}_2$ at period t if and only if $\alpha_k \tilde{g}_k(\lambda_2^t) \geq p_2$. We specify that every user expects that the QoS in the current period is equal to that in the previous period. That is, we have $\tilde{g}_k(\lambda_2^t) = g(\lambda_2^{t-1})$ for $t = 1, 2, \ldots$, where λ_2^0 is the initial fraction of subscribers.[8]

[6]"Fraction of subscribers" of an NSP is used throughout this paper to mean the proportion of users in the market that subscribe to this NSP.

[7]An example consistent with our subscription timing is a "Pay-As-You-Go" plan in which a subscribing user pays a fixed service charge for a unit of time (day, week, or month) and is free to quit its subscription at any time period, effective from the next time unit.

[8]This model of belief formation is called naive or static expectations in [11]. A similar dynamic model of belief formation and decision making has been extensively adopted in the existing literature, e.g., [1][5][8].

Our model implies that the fraction of subscribers of NSP $\mathcal{S}_2$ evolves following a sequence $\{\lambda_2^t\}_{t=0}^{\infty}$ in $[0, 1]$ generated by

$$\lambda_2^t = h_m(\lambda_2^{t-1}) \triangleq 1 - F\left(\frac{p_2}{g(\lambda_2^{t-1})}\right), \tag{2}$$

for $t = 1, 2, \ldots$, starting from a given initial point $\lambda_2^0 \in [0, 1]$. Note that the price p_2 of NSP $\mathcal{S}_2$ is held fixed over time. Given the user subscription dynamics (2), we are interested in whether the fraction of subscribers will stabilize in the long run and, if so, to what value. As a first step, we define an equilibrium point of the user subscription dynamics.

Definition 1: λ_2^* is an *equilibrium* point of the user subscription dynamics in the monopoly market of NSP $\mathcal{S}_2$ if it satisfies

$$h_m(\lambda_2^*) = \lambda_2^*. \tag{3}$$

Definition 3 implies that once an equilibrium point is reached, the fraction of subscribers remains the same from that point on. Thus, equilibrium points are natural candidates for the fixed points in the long run. The following Proposition, whose proof is deferred to [13], establishes the existence and uniqueness of an equilibrium point.

Proposition 1. *For any non-negative price p_2, there exists a unique equilibrium point of the user subscription dynamics in the monopoly market of NSP $\mathcal{S}_2$.* □

Although Proposition 1 guarantees the existence of a unique equilibrium point, it does not provide us with an explicit expression of the equilibrium point as a function of the monopoly price. In order to obtain a closed-form expression of the equilibrium point, we consider a class of simple QoS functions defined below.

Definition 2: The QoS function $g(\cdot)$ is *linearly-degrading* if $g(\lambda_2) = \bar{q}_2 - c\lambda_2$ for all $\lambda_2 \in [0, 1]$, for some $\bar{q}_2 > 0$ and $c \in [0, \bar{q}_2)$. In particular, a linearly-degrading QoS function with $c = 0$, i.e., $g(\lambda_2) = \bar{q}_2$ for all $\lambda_2 \in [0, 1]$, is referred to as a constant QoS function.

Linearly-degrading QoS functions model a variety of applications including flow control and capacity sharing in [9]. It can also be viewed as the first-order Taylor approximation (around the zero point, i.e., Maclaurin series) of a complicated QoS function. With a linearly-degrading QoS function and uniformly distributed valuations of QoS [5][6][9], we can obtain a simple closed-form expression of the equilibrium point. Specifically, with $g(\lambda_2) = \bar{q}_2 - c\lambda_2$ for $\lambda_2 \in [0, 1]$ and $f(\alpha) = 1/\beta$ for $\alpha \in [0, \beta]$, the equilibrium point of the user subscription dynamics in the monopoly market of NSP $\mathcal{S}_2$ can be expressed as a function of p_2 as follows:

$$\lambda_2^*(p_2) = \begin{cases} \frac{\bar{q}_2 + c - \sqrt{(\bar{q}_2 - c)^2 + \frac{4cp_2}{\beta}}}{2c}, & \text{for } p_2 \in [0, \beta\bar{q}_2], \\ 0, & \text{for } p_2 \in (\beta\bar{q}_2, \infty), \end{cases} \tag{4}$$

if $c \in (0, \bar{q}_2)$, and $\lambda_2^*(p_2) = \max\{0, 1 - p_2/(\beta\bar{q}_2)\}$ if $c = 0$.

Our equilibrium analysis so far guarantees the existence of a unique stable point of the user subscription dynamics in the monopoly market. However, it does not discuss whether the unique stable point will be eventually reached. To answer this question, we turn to the analysis of the convergence properties of the user subscription dynamics. The convergence of the user subscription dynamics is not always guaranteed, especially when the QoS provided by the monopolist degrades rapidly with respect to the fraction of subscribers. As a hypothetical example, suppose that only a small fraction of users subscribe to NSP $\mathcal{S}_2$ at period t and each subscriber obtains a high QoS. In our model of belief formation, users expect that the QoS will remain high at period $t+1$, and thus a large fraction of users subscribe at period $t+1$, which will result in a low QoS at period $t+1$. This in turn will induce a small fraction of subscribers at period $t+2$. When the QoS is very sensitive to the fraction of subscribers, the user subscription dynamics may oscillate around or diverge away from the equilibrium point and thus convergence may not be obtained. The following theorem, the proof of which can be found in [13] provides a sufficient condition under which the user subscription dynamics always converges.

Theorem 1. *For any non-negative price p_2, the user subscription dynamics specified by (2) converges to the unique equilibrium point starting from any initial point $\lambda_2^0 \in [0,1]$ if*

$$\max_{\lambda_2 \in [0,1]} \left\{ -\frac{g'(\lambda_2)}{g(\lambda_2)} \right\} < \frac{1}{K}, \tag{5}$$

where $K = \max_{\alpha \in [0,\beta]} f(\alpha)\alpha$. □

By applying Theorem 1 to linearly-degrading QoS functions, we obtain the following result.

Corollary 1. *If the QoS function $g(\cdot)$ is linearly-degrading, i.e., $g(\lambda_2) = \bar{q}_2 - c\lambda_2$ for $\lambda_2 \in [0,1]$, and*

$$\frac{c}{\bar{q}_2} < \frac{1}{1+K}, \tag{6}$$

then the user subscription dynamics converges to the unique equilibrium point starting from any initial point $\lambda_2^0 \in [0,1]$. □

The condition (5) in Theorem 1 is sufficient but not necessary for the convergence of the user subscription dynamics. In particular, we observe through numerical simulations that in some cases (e.g., $g(\lambda_2) = 1 - 0.9\lambda_2$ for $\lambda_2 \in [0,1]$ and $f(\alpha) = 1$ for $\alpha \in [0,1]$) the user subscription dynamics converges for a wide range of prices although the condition (5) is violated. Nevertheless, the sufficient condition provides us with the insight that if QoS degradation is too fast (i.e., $-Kg'(\lambda_2)$ is larger than $g(\lambda_2)$ for some $\lambda_2 \in [0,1]$), the dynamics may oscillate or diverge.

It should be noted that we can generalize the user subscription dynamics by assuming that only ϵ fraction of users, where $\epsilon \in (0,1]$, can change their subscription decisions in each period while the users form their beliefs as before. Then the user subscription dynamics is generated by

$$\lambda_2^t = (1-\epsilon)\lambda_2^{t-1} + \epsilon h_m(\lambda_2^{t-1}) \tag{7}$$

for $t = 1, 2, \ldots$, starting from an initial point $\lambda_2^0 \in [0,1]$. Note that (7) is more general than (2) since (7) reduces to (2) when $\epsilon = 1$. Definition 1 still gives the definition of an equilibrium point of the user subscription dynamics (7), and thus the equilibrium (Proposition 1) and convergence analysis (Theorem 1) are still valid.

B. Revenue Maximization in the Monopoly Market of NSP $\mathcal{S}_2$

Building on the equilibrium analysis of the user subscription dynamics, we are now interested in finding an optimal price of NSP $\mathcal{S}_2$ that maximizes its *steady-state* or equilibrium revenue in the monopoly market.[9] The revenue of NSP $\mathcal{S}_2$ at price p_2 can be expressed as

$$R_2(p_2) = p_2 \lambda_2^*(p_2), \tag{8}$$

where $\lambda_2^*(p_2)$ is the equilibrium point of the user subscription dynamics at price p_2. It can be shown that $\lambda_2^*(0) = 1$, $\lambda_2^*(\cdot)$ is strictly decreasing on $[0, \beta g(0)]$, and $\lambda_2^*(p_2) = 0$ for all $p_2 \geq \beta g(0)$. As a result, NSP $\mathcal{S}_2$ will gain a positive revenue only if it sets a price p_2 in $(0, \beta g(0))$, and thus a revenue-maximizing price lies in $(0, \beta g(0))$. However, a direct method to find an expression of $p_2 \in (0, \beta g(0))$ that maximizes $R_2(p_2)$ is mathematically intractable even when the QoS function is linearly-degrading and the users' valuation of QoS is uniformly distributed, since $\lambda_2^*(p_2)$ is an involved function of p_2 as can be seen in (4). In the following analysis, we reformulate the revenue maximization problem by applying the marginal user principle[10] [10]. Specifically, we change the choice variable in the revenue maximization problem.

Suppose that a marginal user exists, whose valuation of QoS is denoted by α. Then from the utility function in (1), we can see that all the users with a valuation of QoS greater than α receive a positive utility and thus subscribe to NSP $\mathcal{S}_2$ [6][10]. Hence, when a marginal user has valuation of QoS $\alpha \in [0, \beta]$, the fraction of subscribers is given by $\lambda_2 = 1 - F(\alpha)$. Also, for a given price $p_2 \in [0, \beta g(0)]$, there exists a unique valuation of QoS of a marginal user $\alpha \in [0, \beta]$, and the relationship between p_2 and α is given by

$$p_2 = \alpha g(1 - F(\alpha)). \tag{9}$$

Based on the above relationships between p_2, α, and λ_2, we can formulate the revenue maximization problem using different choice variables as follows:

$$\begin{aligned} \max_{p_2 \in [0, \beta g(0)]} p_2 \lambda_2^*(p_2) &= \max_{\alpha \in [0,\beta]} \alpha g(1 - F(\alpha)) \left[1 - F(\alpha)\right] \\ &= \max_{\lambda_2 \in [0,1]} F^{-1}(1 - \lambda_2) g(\lambda_2) \lambda_2, \end{aligned} \tag{10}$$

where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$ defined on $[0,1]$.[11] It is clear that a solution to each of the above three problems exists, since the constraint set is compact and the objective function is continuous. Let p^*, α^*, and λ_2^{**} be a solution to each respective problem in (10). By imposing an assumption on the distribution of the users' valuation of QoS, we obtain upper and lower bounds on p^*, α^*, and λ_2^{**} in Proposition 2, whose proof is given in [13].

Proposition 2. *Suppose that $f(\cdot)$ is non-increasing on $[0, \beta]$. Then optimal variables solving the revenue maximization*

[9] By focusing on equilibrium revenue, we implicitly assume that the unique equilibrium point is reached within a finite number of time periods.

[10] In the monopoly market of NSP $\mathcal{S}_2$, marginal users are users that are indifferent between subscribing and not subscribing to NSP $\mathcal{S}_2$ given the received QoS and the charged price. In our model, a marginal user receives zero utility.

[11] We define $F^{-1}(0) = 0$ and $F^{-1}(1) = \beta$.

problem in (10) satisfy $F^{-1}(1/2)g(1/2) \le p_2^* < \beta g(0)$, $F^{-1}(1/2) \le \alpha^* < \beta$, *and* $0 < \lambda_2^{**} \le 1/2$. □

The non-increasing property of $f(\cdot)$ can be considered as representing a class of emerging markets where there are fewer users with higher valuations of QoS provided by the NSP [16]. Proposition 2 shows that when the monopolist maximizes its revenue in an emerging market, no more than a half of the users, only those whose valuation is sufficiently high, are served. Since a uniform distribution satisfies the non-increasing property, applying Proposition 2 to the case of a uniform distribution of the users' valuation of QoS (i.e., $f(\alpha) = 1/\beta$ and $F(\alpha) = \alpha/\beta$ for $\alpha \in [0, \beta]$) yields $(\beta/2)g(1/2) \le p_2^* < \beta g(0)$ and $\beta/2 \le \alpha^* < \beta$. If, in addition, the QoS function satisfies the sufficient condition (5) for convergence, we obtain tighter bounds on optimal variables.

Corollary 2. *Suppose that* $f(\alpha) = 1/\beta$ *for* $\alpha \in [0, \beta]$ *and* $-g'(\lambda_2)/g(\lambda_2) < 1$ *for all* $\lambda_2 \in [0, 1]$. *Then optimal variables solving the revenue maximization problem in (10) satisfy* $(\beta/2)g(1/2) \le p_2^* < [(\sqrt{5}-1)/2]\beta g((3-\sqrt{5})/2)$, $\beta/2 \le \alpha^* < [(\sqrt{5}-1)/2]\beta$, *and* $(3-\sqrt{5})/2 < \lambda_2^{**} \le 1/2$. □

With a uniform distribution of the users' valuation of QoS and a linearly-degrading QoS function, we can obtain an explicit expression of optimal variables of the revenue maximization problem as follows:

$$\alpha^* = \frac{2c - \bar{q}_2 + \sqrt{\bar{q}_2^2 + c^2 - c\bar{q}_2}}{3c}\beta, \tag{11}$$

$$\lambda_2^{**} = \frac{c + \bar{q}_2 - \sqrt{\bar{q}_2^2 + c^2 - c\bar{q}_2}}{3c}, \tag{12}$$

and $p_2^* = \alpha^*(\bar{q}_2 - c\lambda_2^{**})$. It can be shown that both p_2^* and λ_2^{**} decrease in c, which implies that the maximum revenue is smaller as c is larger (i.e., QoS degradation is faster).

Finally, we consider a monopoly market of NSP $\mathcal{S}_1$. Since NSP $\mathcal{S}_1$ can be considered as having a constant QoS function, we obtain the following results by specializing the monopoly analysis so far to a constant QoS function.

Proposition 3. *In the monopoly market of NSP* $\mathcal{S}_1$, *the following statements hold:*
1. For any non-negative price p_1, *there exists a unique equilibrium point* $\lambda_1^* = 1 - F(p_1/q_1)$, *and the user subscription dynamics generated by (2) always converges to the unique equilibrium point starting from any initial point* $\lambda_1^0 \in [0, 1]$.
2. If $f(\cdot)$ *is non-increasing on* $[0, \beta]$, *then there exists an optimal price* $p_1^* \in [F^{-1}(1/2)q_1, \beta q_1)$ *that maximizes the revenue of NSP* $\mathcal{S}_1$.
3. If $f(\alpha) = 1/\beta$ *for* $\alpha \in [0, \beta]$, *then the optimal values of the revenue maximization problem are* $p_1^* = (\beta q_1)/2$, $\alpha^* = \beta/2$, *and* $\lambda_1^{**} = 1/2$. □

IV. Duopoly Communication Market

In this section, we analyze user subscription dynamics and revenue maximization in a duopoly communications market where two competing NSPs $\mathcal{S}_1$ and $\mathcal{S}_2$ operate. We first analyze the equilibrium and convergence of user subscription dynamics, and then study revenue maximization in two different scenarios where the NSPs compete by choosing prices and market shares.

A. User Subscription Dynamics in the Duopoly Market

With the two NSPs operating in the market, each user has three possible choices at each time instant: subscribe to NSP $\mathcal{S}_1$, subscribe to NSP $\mathcal{S}_2$, and subscribe to neither. As in the monopoly market, we consider a dynamic model in which the users update their beliefs and make subscription decisions at discrete time period $t = 1, 2, \ldots$. The users expect that the QoS provided by NSP $\mathcal{S}_2$ in the current period is equal to that in the previous period and make their subscription decisions to myopically maximize their expected utility in the current period [1][5]. We assume that, other than the subscription price, there is no cost involved in subscription decisions (e.g., initiation fees, termination fees, device prices) when users switch between NSP $\mathcal{S}_1$ and NSP $\mathcal{S}_2$ [1]. By Assumption 2, at period $t = 1, 2 \cdots$, user k subscribes to NSP $\mathcal{S}_1$ if and only if

$$\alpha_k q_1 - p_1 \ge \alpha_k g(\lambda_2^{t-1}) - p_2 \text{ and } \alpha_k q_1 - p_1 \ge 0, \tag{13}$$

to NSP $\mathcal{S}_2$ if and only if

$$\alpha_k g(\lambda_2^{t-1}) - p_2 > \alpha_k q_1 - p_1 \text{ and } \alpha_k g(\lambda_2^{t-1}) - p_2 \ge 0, \tag{14}$$

and to neither NSP if and only if

$$\alpha_k q_1 - p_1 < 0 \text{ and } \alpha_k g(\lambda_2^{t-1}) - p_2 < 0. \tag{15}$$

Given the prices (p_1, p_2), the user subscription dynamics in the duopoly market is described by a sequence $\{(\lambda_1^t, \lambda_2^t)\}_{t=0}^{\infty}$ in $\Lambda = \{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \mid \lambda_1 + \lambda_2 \le 1\}$ generated by

$$\lambda_1^t = h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq 1 - F\left(\frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})}\right), \tag{16}$$

$$\lambda_2^t = h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq F\left(\frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})}\right) - F\left(\frac{p_2}{g(\lambda_2^{t-1})}\right) \tag{17}$$

if $p_1/q_1 > p_2/g(\lambda_2^{t-1})$, and by

$$\lambda_1^t = h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq 1 - F\left(\frac{p_1}{q_1}\right), \tag{18}$$

$$\lambda_2^t = h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq 0 \tag{19}$$

if $p_1/q_1 \le p_2/g(\lambda_2^{t-1})$, for $t = 1, 2, \ldots$, starting from a given initial point $(\lambda_1^0, \lambda_2^0) \in \Lambda$. Note that there are two regimes of the user subscription dynamics in the duopoly market, and which regime governs the dynamics depends on the relative values of the *prices per QoS*, i.e., p_1/q_1 and $p_2/g(\lambda_2^{t-1})$.

We give the definition of an equilibrium point, which is similar to Definition 1.

Definition 3: $(\lambda_1^*, \lambda_2^*)$ is an *equilibrium* point of the user subscription dynamics in the duopoly market if it satisfies

$$h_{d,1}(\lambda_1^*, \lambda_2^*) = \lambda_1^* \text{ and } h_{d,2}(\lambda_1^*, \lambda_2^*) = \lambda_2^*. \tag{20}$$

We establish the existence and uniqueness of an equilibrium point and provide equations characterizing it in Proposition 4, whose proof is deferred to [13].

Proposition 4. *For any non-negative price pair (p_1, p_2), there exists a unique equilibrium point $(\lambda_1^*, \lambda_2^*)$ of the user subscription dynamics in the duopoly market. Moreover, $(\lambda_1^*, \lambda_2^*)$ satisfies*

$$\begin{cases} \lambda_1^* = 1 - F\left(\frac{p_1}{q_1}\right), \ \lambda_2^* = 0, & \text{if } \frac{p_1}{q_1} \leq \frac{p_2}{g(0)}, \\ \lambda_1^* = 1 - F\left(\theta_1^*\right), \ \lambda_2^* = F\left(\theta_1^*\right) - F\left(\theta_2^*\right), & \text{if } \frac{p_1}{q_1} > \frac{p_2}{g(0)}, \end{cases} \tag{21}$$

where $\theta_1^* = (p_1 - p_2)/(q_1 - g(\lambda_2^*))$ and $\theta_2^* = p_2/g(\lambda_2^*)$. □

Proposition 4 indicates that, given any prices (p_1, p_2), the market shares of the two NSPs are uniquely determined when the fraction of users subscribing to each NSP no longer changes. It also shows that the structure of the equilibrium point depends on the relative values of p_1/q_1 and $p_2/g(0)$. Specifically, if the price per QoS of NSP $\mathcal{S}_1$ is always smaller than or equal to that of NSP $\mathcal{S}_2$, i.e., $p_1/q_1 \leq p_2/g(0)$, then no users subscribe to NSP $\mathcal{S}_2$ at the equilibrium point. On the other hand, if NSP $\mathcal{S}_2$ offers a smaller price per QoS to its first subscriber than NSP $\mathcal{S}_1$ does, i.e., $p_1/q_1 > p_2/g(0)$, then both NSP $\mathcal{S}_1$ and NSP $\mathcal{S}_2$ may attract a positive fraction of subscribers.

We now investigate whether the user subscription dynamics specified by (16)–(19) stabilizes as time passes. As in the case of monopoly, the user subscription dynamics is guaranteed to converge to the unique equilibrium in the duopoly market when the QoS degradation of NSP $\mathcal{S}_2$ is not too fast. In the following theorem, we provide a sufficient condition for convergence and the proof details can be found in [13].

Theorem 2. *For any non-negative price pair (p_1, p_2), the user subscription dynamics specified by (16)–(19) converges to the unique equilibrium point starting from any initial point $(\lambda_1^0, \lambda_2^0) \in \Lambda$ if*

$$\max_{\lambda_2 \in [0,1]} \left\{ -\frac{g'(\lambda_2)}{g(\lambda_2)} \cdot \frac{q_1}{q_1 - g(\lambda_2)} \right\} < \frac{1}{K}, \tag{22}$$

where $K = \max_{\alpha \in [0,\beta]} f(\alpha)\alpha$. □

Note that the condition (22) imposes a more stringent requirement on the QoS function $g(\cdot)$ than the condition (5) does, since $q_1/(q_1 - g(\lambda_2)) > 1$ for all $\lambda_2 \in [0, 1]$. However, the condition (22) provides us with a similar insight that, if QoS degradation is severe, the user subscription dynamics may exhibit oscillation or divergence.

B. Revenue Maximization in the Duopoly Market

We now study revenue maximization in the duopoly market. In the economics literature, competition among a small number of firms has been analyzed using game theory, following largely two distinct approaches: Bertrand competition and Cournot competition [14]. In Bertrand competition, firms choose prices independently while supplying quantities demanded at the chosen prices. On the other hand, in Cournot competition, firms choose quantities independently while prices are determined in the markets to equate demand with the chosen quantities. In the case of monopoly, whether the monopolist chooses the price or the quantity does not affect the outcome since there is a one-to-one relationship between the price and the quantity given a downward-sloping demand function. This point was illustrated with our model in Section III-B. On the contrary, in the presence of strategic interaction, whether firms choose prices or quantities can affect the outcome significantly. For example, it is well-known that identical firms producing a homogeneous good obtain zero profit in the equilibrium of Bertrand competition while they obtain a positive profit in the equilibrium of Cournot competition, if they have a constant marginal cost of production and face a linear demand function.

We first consider Bertrand competition between the two NSPs. Let $\lambda_i^*(p_1, p_2)$ be the market share of NSP $\mathcal{S}_i$, for $i = 1, 2$, at the unique equilibrium point of the considered user subscription dynamics given a price pair (p_1, p_2). $\lambda_i^*(\cdot)$ can be interpreted as a demand function of NSP $\mathcal{S}_i$, and the revenue of NSP $\mathcal{S}_i$ at the equilibrium point can be expressed as[12] $R_i(p_1, p_2) = p_i \lambda_i^*(p_1, p_2)$, for $i = 1, 2$. Bertrand competition in the duopoly market can be formulated as a non-cooperative game specified by

$$\mathcal{G}_B = \{\mathcal{S}_i, R_i(p_1, p_2), p_i \in \mathbb{R}_+ \mid i = 1, 2\}. \tag{23}$$

A price pair (p_1^*, p_2^*) is said to be a (pure) NE of $\mathcal{G}_B$ (or a Bertrand equilibrium) if it satisfies

$$R_i(p_i^*, p_{-i}^*) \geq R_i(p_i, p_{-i}^*), \ \forall\, p_i \in [0, +\infty), \forall\, i = 1, 2. \tag{24}$$

It can be shown that, if a Bertrand equilibrium (p_1^*, p_2^*) exists, it must satisfy

$$0 < \frac{p_2^*}{g(0)} < \frac{p_1^*}{q_1} < \beta \tag{25}$$

and $\lambda_i^*(p_1^*, p_2^*) \in (0, 1)$ so that $R_i(p_1^*, p_2^*) > 0$, for $i = 1, 2$. However, since the functions $\lambda_i^*(p_1, p_2)$, $i = 1, 2$, are defined implicitly by (21), it is difficult to provide a primitive condition on $g(\cdot)$ that guarantees the existence of a Bertrand equilibrium.

We now consider Cournot competition between the two NSPs. Let $\lambda_i \in [0, 1]$ be the market share chosen by NSP $\mathcal{S}_i$, for $i = 1, 2$. Suppose that $\lambda_1 + \lambda_2 \leq 1$ so that the chosen market shares are feasible. Let $p_i(\lambda_1, \lambda_2)$, $i = 1, 2$, be the prices that clear the market, i.e., the prices that satisfy $\lambda_i = \lambda_i^*(p_1(\lambda_1, \lambda_2), p_2(\lambda_1, \lambda_2))$ for $i = 1, 2$. Note first that, given a price pair (p_1, p_2), if a user k subscribes to NSP $\mathcal{S}_1$, i.e., $\alpha_k q_1 - p_1 \geq \alpha_k g(\lambda_2) - p_2$ and $\alpha_k q_1 - p_1 \geq 0$, then all the users whose valuation of QoS is larger than α_k also subscribe to NSP $\mathcal{S}_1$. Also, if a user k subscribes to one of the NSPs, i.e., $\max\{\alpha_k q_1 - p_1, \alpha_k g(\lambda_2) - p_2\} \geq 0$, then all the users whose valuation of QoS is larger than α_k also subscribe to one of the NSPs. Therefore, realizing positive market shares $\lambda_1, \lambda_2 > 0$ requires two types of marginal users whose valuations of QoS are specified by $\alpha_{m,1}$ and $\alpha_{m,2}$ with $\alpha_{m,1} > \alpha_{m,2}$. $\alpha_{m,1}$ is the valuation of QoS of a marginal user that is indifferent between subscribing to NSP $\mathcal{S}_1$ and NSP $\mathcal{S}_2$, while $\alpha_{m,2}$ is the valuation of QoS of a marginal user that is indifferent between subscribing to NSP $\mathcal{S}_2$ and neither. The expressions

[12] Without causing ambiguity, in the following analysis, we also express the revenue of an NSP as a function of the fraction of subscribers.

for $\alpha_{m,1}$ and $\alpha_{m,2}$ that realizes (λ_1, λ_2) such that $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 \leq 1$ are given by

$$\alpha_{m,1}(\lambda_1, \lambda_2) = z_1(\lambda_1) \triangleq F^{-1}(1 - \lambda_1), \tag{26}$$

$$\alpha_{m,2}(\lambda_1, \lambda_2) = z_2(\lambda_1, \lambda_2) \triangleq F^{-1}(1 - \lambda_1 - \lambda_2). \tag{27}$$

Also, by solving the indifference conditions, $\alpha_{m,1} q_1 - p_1 = \alpha_{m,1} g(\lambda_2) - p_2$ and $\alpha_{m,2} g(\lambda_2) - p_2 = 0$, we obtain a unique price pair that realizes (λ_1, λ_2) such that $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 \leq 1$,

$$\begin{aligned} p_1(\lambda_1, \lambda_2) = & F^{-1}(1 - \lambda_1)\left[q_1 - g(\lambda_2)\right] \\ & + F^{-1}(1 - \lambda_1 - \lambda_2) g(\lambda_2), \end{aligned} \tag{28}$$

$$p_2(\lambda_1, \lambda_2) = F^{-1}(1 - \lambda_1 - \lambda_2) g(\lambda_2). \tag{29}$$

Note that the expressions (26)–(29) are still valid even when $\lambda_i = 0$ for some $i = 1, 2$, although uniqueness is no longer obtained. Hence, we can interpret $p_i(\cdot)$, $i = 1, 2$, as a function defined on Λ (an inverse demand function in economics terminology). Then the revenue of $\mathcal{S}_i$ when the NSPs choose $(\lambda_1, \lambda_2) \in \Lambda$ is given by $R_i(\lambda_1, \lambda_2) = \lambda_i p_i(\lambda_1, \lambda_2)$, for $i = 1, 2$. We define $R_i(\lambda_1, \lambda_2) = 0$, $i = 1, 2$, if $\lambda_1 + \lambda_2 > 1$, i.e., if the market shares chosen by the NSPs are infeasible. Cournot competition in the duopoly market can be formulated as a non-cooperative game specified by

$$\mathcal{G}_C = \left\{\mathcal{S}_i, R_i(\lambda_1, \lambda_2), \lambda_i \in [0, 1] \mid i = 1, 2\right\}. \tag{30}$$

A market share pair $(\lambda_1^{**}, \lambda_2^{**})$ is said to be a (pure) NE of $\mathcal{G}_C$ (or a Cournot equilibrium) if it satisfies

$$R_i(\lambda_i^{**}, \lambda_{-i}^{**}) \geq R_i(\lambda_i, \lambda_{-i}^{**}), \ \forall \, \lambda_i \in [0, 1], \forall \, i = 1, 2. \tag{31}$$

Note that $(1, 1)$ is a NE of $\mathcal{G}_C$, which yields zero profit to both NSPs. To eliminate this inefficient and counterintuitive equilibrium, we can restrict the strategy space of each NSP to $[0, 1)$. Deleting 1 from the strategy space can also be justified by noting that $\lambda_i = 1$ is a weakly dominated strategy for NSP $\mathcal{S}_i$, for $i = 1, 2$, since $R_i(1, \lambda_{-i}) = 0 \leq R_i(\lambda_i, \lambda_{-i})$ for all $(\lambda_i, \lambda_{-i}) \in [0, 1]^2$.[13] We use $\tilde{\mathcal{G}}_C$ to represent the Cournot competition game with the restricted strategy space $[0, 1)$. The following lemma, the proof of which is available in [13], bounds the market shares that solve the revenue maximization problem of each NSP, when the PDF of the users' valuation of QoS satisfies the non-increasing property as in Proposition 2.

Lemma 1. *Suppose that $f(\cdot)$ is non-increasing on $[0, \beta]$. Let $\tilde{\lambda}_i(\lambda_{-i})$ be a market share that maximizes the revenue of NSP $\mathcal{S}_i$ provided that NSP $\mathcal{S}_{-i}$ chooses $\lambda_{-i} \in [0, 1)$, i.e., $\tilde{\lambda}_i(\lambda_{-i}) \in \arg\max_{\lambda_i \in [0,1)} R_i(\lambda_i, \lambda_{-i})$. Then $\tilde{\lambda}_i(\lambda_{-i}) \in (0, 1/2]$ for all $\lambda_{-i} \in [0, 1)$, for all $i = 1, 2$. Moreover, $\tilde{\lambda}_i(\lambda_{-i}) \neq 1/2$ if $\lambda_{-i} > 0$, for $i = 1, 2$.* □

Lemma 1 implies that, when the strategy space is specified as $[0, 1)$ and $f(\cdot)$ satisfies the non-increasing property, strategies $\lambda_i \in \{0\} \cup (1/2, 1)$ is strictly dominated for $i = 1, 2$.[14] Hence, if a NE $(\lambda_1^{**}, \lambda_2^{**})$ of $\tilde{\mathcal{G}}_C$ exists, then it must satisfy $(\lambda_1^{**}, \lambda_2^{**}) \in (0, 1/2)^2$, which yields positive revenues for both NSPs. Furthermore, since a revenue-maximizing NSP never uses a strictly dominated strategy, the set of NE of $\tilde{\mathcal{G}}_C$ is not affected by restricting the strategy space to $[0, 1/2]$. Based on the discussion so far, we can provide a sufficient condition on $f(\cdot)$ and $g(\cdot)$ that guarantees the existence of a NE of $\tilde{\mathcal{G}}_C$ in Theorem 3. The proof can be found in [13].

Theorem 3. *Suppose that $f(\cdot)$ is non-increasing and continuously differentiable on $[0, \beta]$.*[15] *If $f(\cdot)$ and $g(\cdot)$ satisfy (32) and (33) (shown on the top of the next page), for all $(\lambda_1, \lambda_2) \in [0, 1/2]^2$, then the game $\tilde{\mathcal{G}}_C$ has at least one NE.* □

Note that the condition (5) in Theorem 1 can be rewritten as $g(\lambda_2) + K g'(\lambda_2) > 0$ for all $\lambda_2 \in [0, 1]$, where $K = \max_{\alpha \in [0, \beta]} f(\alpha)\alpha$. Similarly, we can interpret the conditions (32) and (33) as providing upper bounds for $-g'(\lambda_2)/g(\lambda_2)$ that are determined by $f(\cdot)$.[16] When the users' valuation of QoS is uniformly distributed, the conditions (32) and (33) coincide and reduce to $g(\lambda_2) + \lambda_2 g'(\lambda_2) \geq 0$, and thus we obtain the following corollary.

Corollary 3. *Suppose that the users' valuation of QoS is uniformly distributed, i.e., $f(\alpha) = 1/\beta$ for $\alpha \in [0, \beta]$. If $g(\lambda_2) + \lambda_2 g'(\lambda_2) \geq 0$ for all $\lambda_2 \in [0, 1/2]$, then the game $\tilde{\mathcal{G}}_C$ has at least one NE.* □

Corollary 3 states that if the elasticity of the QoS provided by NSP $\mathcal{S}_2$ with respect to the fraction of its subscribers is no larger than 1 (i.e., $-[g'(\lambda_2)\lambda_2/g(\lambda_2)] \leq 1$), the Cournot competition game with the strategy space $[0, 1)$ has at least one NE. As mentioned before, this condition is analogous to the sufficient conditions for convergence in that it requires that the QoS provided by NSP $\mathcal{S}_2$ cannot degrade too fast with respect to the fraction of subscribers.

We briefly discuss an iterative process to reach a NE of the Cournot competition game. Theorem 3 is based on the fact that the Cournot competition game with the strategy space $[0, 1/2]$ can be transformed to a supermodular game [15] when (32) and (33) are satisfied. It is known that the largest and the smallest NE of a supermodular game can be obtained by iterated strict dominance, which uses the best response. However, a detailed analysis of this process requires an explicit expression of the best response correspondence of each NSP, which is not readily available without specific assumptions on $f(\cdot)$ and $g(\cdot)$. Finally, we mention that the existence result of NE, although important in its own right, is only the first step toward understanding competition between the two NSPs. Subsequent issues, including the uniqueness of NE, the effects of $f(\cdot)$ and $g(\cdot)$ on NE, and comparison between the monopoly outcome and the duopoly outcome, are left for our future work.

[13] $\lambda_i \in [0, 1]$ is a weakly dominated strategy for NSP $\mathcal{S}_i$ in $\mathcal{G}_C$ if there exists another strategy $\lambda_i' \in [0, 1]$ such that $R_i(\lambda_i, \lambda_{-i}) \leq R_i(\lambda_i', \lambda_{-i})$ for all $\lambda_{-i} \in [0, 1]$.

[14] $\lambda_i \in [0, 1)$ is a strictly dominated strategy for NSP $\mathcal{S}_i$ in $\tilde{\mathcal{G}}_C$ if there exists another strategy $\lambda_i' \in [0, 1)$ such that $R_i(\lambda_i, \lambda_{-i}) < R_i(\lambda_i', \lambda_{-i})$ for all $\lambda_{-i} \in [0, 1)$.

[15] We define the derivative of $f(\cdot)$ at 0 and β using a one-sided limit as in footnote 4.

[16] In fact, the result of Theorem 3 also holds when the directions of the inequalities in (32) and (33) are opposite. However, we choose the directions as in (32) and (33) in order to provide an analogous interpretation to that of the sufficient conditions (5) and (22) for convergence.

$$\left\{\frac{1}{f(F^{-1}(1-\lambda_1-\lambda_2))}+\frac{\lambda_1 f'(F^{-1}(1-\lambda_1-\lambda_2))}{[f(F^{-1}(1-\lambda_1-\lambda_2))]^3}\right\}g(\lambda_2)$$

$$+\left\{F^{-1}(1-\lambda_1)-\frac{\lambda_1}{f(F^{-1}(1-\lambda_1))}-F^{-1}(1-\lambda_1-\lambda_2)+\frac{\lambda_1}{f(F^{-1}(1-\lambda_1-\lambda_2))}\right\}g'(\lambda_2)\geq 0 \tag{32}$$

$$\left\{\frac{1}{f(F^{-1}(1-\lambda_1-\lambda_2))}+\frac{\lambda_2 f'(F^{-1}(1-\lambda_1-\lambda_2))}{[f(F^{-1}(1-\lambda_1-\lambda_2))]^3}\right\}g(\lambda_2)+\frac{\lambda_2}{f(F^{-1}(1-\lambda_1-\lambda_2))}g'(\lambda_2)\geq 0 \tag{33}$$

V. Illustrative Example

In this section, we apply the analysis to an illustrative communications market with two NSPs. To facilitate the illustration, for NSP $\mathcal{S}_2$, we consider linearly-degrading QoS functions for NSP $\mathcal{S}_2$ [9]. In the considered example, we have $q_1 = 2$, $g(\lambda_2) = 1 - c\lambda_2$ for $\lambda_2 \in [0,1]$, where $c \in [0,1)$ is constant, and $f(\alpha) = 1$ for $\alpha \in [0,1]$. Note that, for $g(\lambda_2) = 1 - c\lambda_2$, a larger value of the QoS degradation rate c means a worse technology in terms of QoS provisioning [9].

A. Monopoly Market

We first consider the monopoly market of NSP $\mathcal{S}_2$. Fig. 2(a) illustrates the convergence of the user subscription dynamics for a particular price $p_2 = 0.5$, when $c = 1/8$ and $c = 1/2$. Note that given any price $p_2 \geq 0$, convergence will always be obtained with $c = 1/8$, since the QoS function $g(\lambda_2) = 1-\lambda_2/8$ satisfies the sufficient condition for convergence given in Theorem 1. Although the sufficient condition is violated when $c = 1/2$, convergence is also observed for $p_2 = 0.5$. In Figs. 2(b) and 2(c), we show the graph of the revenue of NSP $\mathcal{S}_2$ when the choice variable is taken as the price and the market share, respectively. Fig. 2(c) verifies Proposition 2 that the optimal market share that maximizes the revenue of NSP $\mathcal{S}_2$ is upper bounded by $1/2$. From Figs. 2(b) and 2(c), we notice that a better technology (i.e., a smaller QoS degradation rate) leads NSP $\mathcal{S}_2$ to set a higher price, serve a larger fraction of subscribers, and thus earn a larger revenue. This result implies that a monopolistic NSP has an incentive to invest in advanced technologies to provide a higher QoS as long as the benefit from the investment exceeds the fixed cost.

B. Duopoly Market

Next, we show some numerical results regarding convergence and market share competition in a duopoly market. The convergence of user subscription dynamics is shown in Fig. 3(a) for a particular price pair. Starting from different initial points, Figs. 3(b) and 3(c) show the (best-response) iterations of revenues and market shares, respectively, when both $\mathcal{S}_1$ and $\mathcal{S}_2$ choose the optimal[17] market shares that maximize their own revenues while regarding the market share of its competitor as fixed. Since the considered QoS function satisfies Corollary 3, the game can be transformed into a supermodular game and hence the best-response dynamics is known to converge to a pure NE [15], as verified in Fig. 3(c). It is interesting to see that, regardless of the initial points[18], all the iterations converge to the same point, suggesting that there exists a unique NE in the market share competition game $\tilde{\mathcal{G}}$. Moreover, Lemma 1 is verified by Fig. 3(c) which shows that the desired market shares of both $\mathcal{S}_1$ and $\mathcal{S}_2$ are less than $\frac{1}{2}$, given any initial points. It can also be observed from Figs. 3(b) and 3(c) that if NSP $\mathcal{S}_2$ has a technology that provides a lower QoS (shown in dashed lines), it obtains a lower revenue, while NSP $\mathcal{S}_1$ obtains a higher revenue, even though the changes in market shares are not significant. This is because NSP $\mathcal{S}_2$ has to decrease its price in order to maintain the market share if it can only provide a lower QoS, while NSP $\mathcal{S}_1$ can charge a higher price without loosing the market share.

By comparing Figs. 2(c) and 3(c), we notice that competition reduces NSP $\mathcal{S}_2$'s market share, since some users prefer to subscribe to NSP $\mathcal{S}_1$ that can provide a higher constant QoS. Similarly, Figs. 2(c) and 3(b) indicate that competition reduces NSP $\mathcal{S}_2$'s optimal revenue, as NSP $\mathcal{S}_2$ charges a lower price and fewer users subscribe to NSP $\mathcal{S}_2$ when NSP $\mathcal{S}_1$ also operates in the market. In Table I, we compare the user welfare, which is defined as the sum utility of all the users that subscribe to either of the NSPs, in monopoly and duopoly markets. It shows that the users benefit significantly from NSP competition. This can be explained by the fact that the charged price is reduced and that a new NSP with sufficient resources is introduced in the duopoly market. Therefore, the users have more freedom to choose the NSPs and gain extra benefits when there is competition in the market.

TABLE I
Comparison of User Welfare

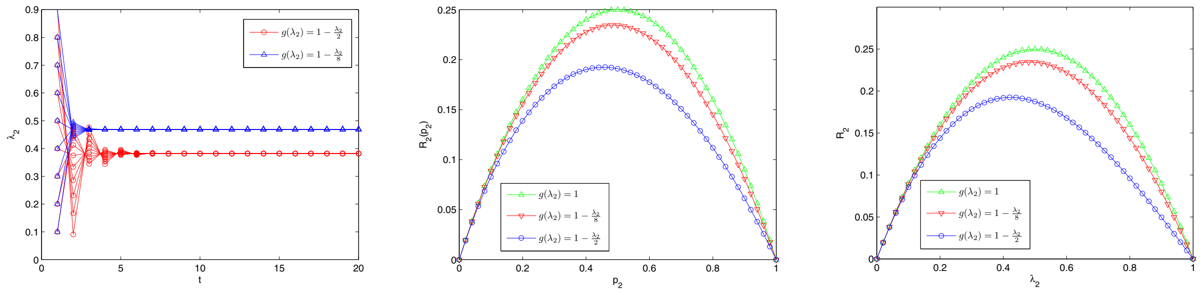
	$g(\lambda_2)=1-\frac{\lambda_2}{2}$	$g(\lambda_2)=1-\frac{\lambda_2}{8}$
Monopoly	0.0704	0.1098
Duopoly	0.3333	0.3389

VI. Conclusion

In this paper, we investigated the interaction between technologies, user subscription dynamics and pricing strategies. First, we focused on a monopoly market where only one resource-constrained NSP operates, providing each user with a QoS that depends on the number of subscribers. We showed that, for any price, there exists a unique equilibrium point of the user subscription dynamics at which the number of subscribers does not change. We also provided a sufficient

[17]The best response is computed numerically.

[18]In the illustrative example, we only show results given initial points that satisfy $\lambda_1 + \lambda_2 = 0.9$.

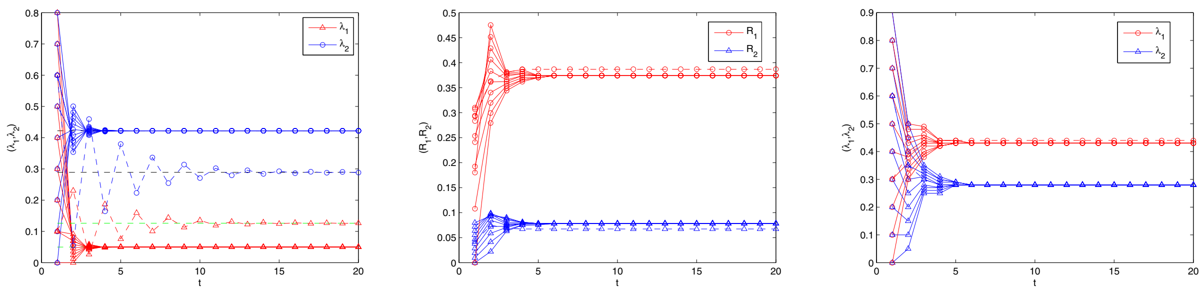


(a) Convergence of the user subscription dynamics $p_2 = 0.5$.

(b) Revenue v.s. price.

(c) Revenue v.s. λ_2.

Fig. 2. Monopoly market with NSP $\mathcal{S}_2$ only.



(a) Convergence of the user subscription dynamics. $p_1 = 1.5$ and $p_2 = 0.5$. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2}{2}$.

(b) Iteration of revenues under the best-response dynamics. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2}{2}$.

(c) Iteration of market shares under the best-response dynamics. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2}{2}$.

Fig. 3. Duopoly market with NSP $\mathcal{S}_1$ and NSP $\mathcal{S}_2$.

condition on the QoS function that ensures the global convergence of the user subscription dynamics. Under a non-increasing PDF of the users' valuations of QoS, we then derived upper and lower bounds on the optimal price and the resulting market share that maximize the NSP's revenue. Next, we analyzed a duopoly market by adding another NSP with sufficient resources to provide a constant QoS to its subscribers. It was shown that, for any prices, the considered user subscription dynamics always admits a unique equilibrium point. We further obtained a sufficient condition on the QoS function to guarantee the convergence of the user subscription dynamics. Then, we studied competition between the two revenue-maximizing NSPs, primarily focusing on market share competition. We modeled the NSPs as strategic players in a non-cooperative game where each NSP aims to maximize its own revenue by choosing its market share. We obtained a sufficient condition that ensures the existence of at least one NE of the game. Finally, the illustrative example showed that the QoS function significantly influences the subscription decisions of the users and competition between the NSPs in such a way that both the users and the resource-constrained NSP benefit from a higher QoS (i.e., better technology) of the NSP.

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