Lecture 33

Wednesday, April 10, 2024 10:57 AM

Agenda

- Poisson Regression
- Monte Carlo Methods

1st, is the poisson an exponential dispersion family?

$$f(y) = \frac{e^{-\lambda}\lambda^y}{y!} = \exp(-\lambda + y\ln\lambda - \ln(y!)) = \exp((y\ln\lambda - \lambda) + \ln(y!))$$

$$\theta_i = \ln(\lambda_i) \qquad b(\theta_i) = \lambda = e^{\theta_i} \qquad \varphi_i = 1 \qquad c(y_i; \varphi_i) = \ln(y_i!)$$

We want to find $\hat{W}_k = \text{diag}\left\{\frac{1}{v(\lambda_i)(g'(\lambda_i))^2} : i = 1, 2, \cdots, n\right\}$

$$= \text{diag}\{v(\lambda_i) : i = 1, 2, \cdots, n\}$$

$$= \text{diag}\{\hat{\lambda}_i : i = 1, 2, \cdots, n\}$$

$$\hat{u}_k = \begin{bmatrix} (Y_1 - \hat{\lambda}_1) g'(\hat{\lambda}_1) \\ \vdots \\ (Y_n - \hat{\lambda}_n) g'(\hat{\lambda}_n) \end{bmatrix} = \begin{bmatrix} \frac{Y_1 - \hat{\lambda}_1}{\hat{\lambda}_1} \\ \vdots \\ \frac{Y_n - \hat{\lambda}_n}{\hat{\lambda}_n} \end{bmatrix}$$

$$v(\lambda_i) = \frac{1}{g'(\lambda_i)} \qquad g'(\lambda_i) = \frac{1}{v(\lambda_i)}$$

$$\frac{1}{\hat{\lambda}_i} = g'(\hat{\lambda}_i)$$

Initialize $\hat{\lambda}_0$

while (not convergence)

$$\hat{W}_k = \text{diag}\{\hat{\lambda}_i : i = 1, 2, \cdots, n\}$$

$$\hat{u}_k = \begin{bmatrix} \frac{Y_1 - \hat{\lambda}_1}{\hat{\lambda}_1} \\ \vdots \\ \frac{Y_n - \hat{\lambda}_n}{\hat{\lambda}_n} \end{bmatrix}$$

$$\hat{Z}_k = X\hat{\beta}_k + \hat{u}_k$$

$$\hat{\beta}_{k+1} = [X^T \hat{W}_k X]^{-1} X^T \hat{W}_k \hat{Z}_k$$

$$\hat{\lambda}_i^{k+1} = e^{X_i^T \hat{\beta}_{k+1}}$$

Monte Carlo Methods:

Recall the WLLN that says

$$\bar{X}_n \xrightarrow{P} \mu \qquad \text{for } X_1, X_2, \dots \text{ iid with } E(X_n) = \mu \quad V(X_n) = \sigma^2 < \infty$$

Cts mapping thm)

If g is a cts function, and $X_n \xrightarrow{P} c$, then $g(X_n) \xrightarrow{P} g(c)$

We want to estimate some integral

$$\int f(x)\,dx = \sum_{i=1}^{n} f(x_i)\, g(x_i) \qquad \text{When we use the average, } g(x_i) = \frac{1}{n}$$

We also want to only use points that will be important in the calculation of the integral. (if $f(x)$ is arbitrarily small at $\pm\infty$, we don't want sample points that are too extreme). We do something called change of measure:

Sp that the density function, $g(x)$, places mass where $f(x)$ is non-zero, then we can write

$$\int f(x)\,dx = \int \frac{f(x)}{g(x)}\, g(x)\,dx = E\left(\frac{f(x)}{g(x)}\right) \approx \sum_{i=1}^{n} \frac{f(x_i)}{g(x_i)} \cdot \frac{1}{n}$$

This is called importance sampling

$$f(x) = e^{-x^4}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} e^{-x^4}\,dx = \int_{-\infty}^{\infty} \frac{e^{-x^4}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$\frac{e^{-x^4}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} = \sqrt{2\pi}\, e^{-x^2(x^2 + 1/2)}$$

$$E\left(\sqrt{2\pi}\, e^{-x^2(x^2+1/2)}\right) \approx \frac{\sqrt{2\pi}}{n} \sum_{i=1}^{n} e^{-x_i^2(x_i^2 + 1/2)}$$

where $X \sim N(0,1)$

EX) Find $\int_0^5 e^{-x^4}\,dx$

$$\int_0^5 e^{-x^4}\,dx = \int_0^5 \frac{e^{-x^4}}{1/5} \cdot \frac{1}{5}\,dx = E\left(5e^{-x^4}\right) \quad \text{where } X \sim \text{unif}(0,1)$$

$$\sum^{n} \quad {}^{-x^4}$$

$$\int_0 e^{-x^4}\,dx = \int_0 \frac{\quad}{1/5} \ldots = E(5e \ldots) \quad \text{where } X \sim \text{Unif}(0,1)$$

$$\approx \frac{1}{n}\sum_{i=1}^{n} 5e^{-x^4}$$