Lecture 35
Monday, April 15, 2024 10:39 AM

Agenda:

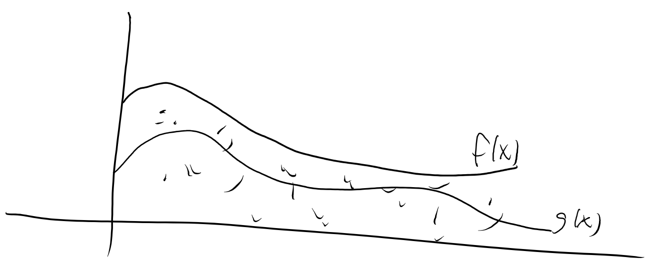
- Sampling
- Gibbs Sampler

Rejection Sampling:

Sp we wish to sample from a dist'n P where we know the density $g(x)$. If we can't sample from P but we can sample from another dist'n Q with density $f(x)$.

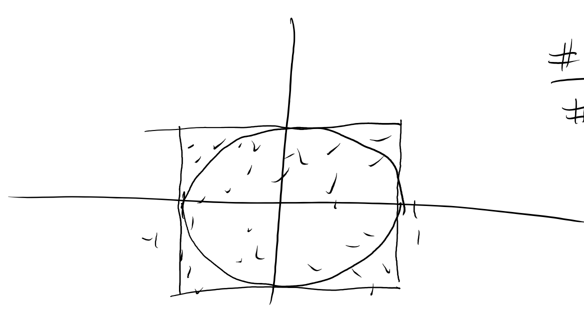
If we find some $M \in \mathbb{R}$ s.t. $f(x) \leq M g(x)$ then we can can perform rejection sampling

pictorially



If a dot is below both curves, accept the X value as a sample from P

EX) Sp we want to estimate π, we can draw a unit circle and the circumscribed square as follows:



$$\frac{\#\text{ points in circle}}{\#\text{ total points}} = \frac{\pi}{4}$$

Formally:

Pick some Q that we can sample from

Generate $U \sim \text{unif}(0,1)$

Generate $X \sim Q$

if $\left(\frac{f(x)}{M S(X)} \leq u \right)$

keep X as a sample from p

repeat until sample # is sufficient

Gibbs Sampler:

Sp we have some multivariate Dist'n, P, that we want to sample from, we will take P to be defined on $\mathbb{R}^2$ but we can't. If we have the property that the conditional densities fully specify the joint, and we can sample from the two conditional dist'ns, then we can obtain a sample from P

If $(X, Y) \sim P$ and we can sample $X|y$ and $Y|X$, then we can create a sample from P

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Start at (X_0, Y_0)
for (i in 1,2, ..., n){
    X_i ~ X|Y_{i-1}
    Y_i ~ Y|X_i
    store (X_i, Y_i)
}
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EX) Sp that we observe data from a mixture of poissons

$$Y_i \sim p \underbrace{pois(\lambda_1)}_{Z_i = 1} + (1-p) \underbrace{pois(\lambda_2)}_{Z_i = 0}$$

We will define a latent variable Z_i s.t. $Z_i | p \sim bern(p)$ and $Y_i | Z_i = 1 \sim pois(\lambda_1)$ and $Y_i | Z_i = 0 \sim pois(\lambda_2)$

Lets allow $\lambda_1 \sim gamma(\alpha, \beta)$ $\lambda_2 \sim gamma(\alpha, \beta)$ and $p \sim beta(a, b)$

$$Y_i | Z_i, \lambda_i, p = Y_i | Z_i, \lambda_i \sim pois(\lambda_i)$$

$$Z_i | Y_i, \lambda_i, p = Z_i | p \sim bern(p)$$

We want to find the dist'ns of $\lambda_i | Z_i, Y_i, p$ and $p | Z_i, \lambda_i, Y_i$

$\lambda_i \mid z_i, y_i, p = \lambda_i \mid z_i, y_i$

$p(y_i \mid z_i, \lambda_i) \sim pois(\lambda_i)$ $\qquad$ $p(y_i, z_i \mid \lambda_i) = p(y_i \mid z_i, \lambda_i) \cdot p(z_i)$

$$p(\lambda_i \mid z_i, y_i) = \frac{p(y_i, z_i \mid \lambda_i) \cdot p(\lambda_i)}{p(z_i, y_i)} = \frac{p(z_i)}{p(z_i, y_i)} \, p(y_i \mid z_i \lambda_i)\, p(\lambda_i) = c \cdot \underline{f(y_i \mid z_i, \lambda_i)}\, \underline{g(\lambda_i)}$$

$$f(\lambda_i \mid z_i, y_i) = c \cdot \left[\prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!} \right] \frac{1}{\Gamma(\alpha) \beta^{\alpha}} \lambda_i^{\alpha - 1} e^{-\lambda_i / \beta}$$

$$= c_2 \, \lambda_i^{\alpha + \Sigma y_i I(z_i = j)} e^{-\lambda_i (\Sigma I(z_i = j) + 1/\beta)}$$

$$\lambda_1 \mid z_i, y_i \sim gamma\left(\alpha + \Sigma y_i z_i, \frac{1}{\Sigma z_i + 1/\beta}\right)$$

$$\lambda_2 \mid z_i, y_i \sim gamma\left(\alpha + \Sigma y_i (1 - z_i), \frac{1}{\Sigma (1 - z_i) + 1/\beta}\right)$$

Lastly

$p(y_i, z_i, \lambda_i = p(y_i, z_i$ $\qquad$ $z_i \mid p \sim bern(p)$

$$p(p \mid z_i, y_i) = \frac{p(z_i, y_i \mid p) \cdot p(p)}{p(z_i, y_i)} = c \cdot p(z_i \mid p) \cdot p(p)$$

$$f(p \mid z_i, y_i, \lambda_i) = c \left[\prod_{i=1}^{n} p^{z_i} (1-p)^{1 - z_i} \right] \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1} (1-p)^{b-1}$$

$$= c \, p^{a + \Sigma z_i - 1} (1-p)^{b + n - \Sigma z_i - 1}$$

$$p \mid z_i, y_i, \lambda_i \sim beta(a + \Sigma z_i, b + n - \Sigma z_i)$$