Lecture 32

Friday, April 5, 2024 11:12 AM

exp disp fam

$$f(y_i; \varphi_i; \theta_i) = e^{\frac{y_i \theta_i - b(\theta_i)}{\varphi_i} + c(y_i; \varphi_i)}$$

$$g(E(Y_i)) = X_i^T \beta$$

Agenda

- excel
- logistic regression

Logistic Regression

our random component is that $Y_i \sim bin(m, p_i)$

$$f(y_i) = \binom{m}{y_i} p_i^{y_i} (1-p_i)^{m-y_i} = \exp\left(\ln\left(\binom{m}{y_i}\right) + \ln(p_i^{y_i}) + \ln((1-p_i)^{m-y_i}) \right)$$

$$= \exp\left(y_i \ln\left(\frac{p_i}{1-p_i}\right) + m \ln(1-p_i) + \ln\left(\binom{m}{y_i}\right) \right)$$

$$= \exp\left(\left[\frac{y_i}{m} \ln\left(\frac{p_i}{1-p_i}\right) - (-\ln(1-p_i)) \right] m + \ln\left(\binom{m}{y_i}\right) \right)$$

In this form

$$\theta_i = \ln\left(\frac{p_i}{1-p_i}\right) \qquad b(\theta_i) = -\ln(1-p_i) \qquad c(y_i; \varphi_i) = \ln\left(\binom{m}{y_i}\right) \qquad \varphi_i = \frac{1}{m}$$

solve for p_i

$$\theta_i = \ln\left(\frac{p_i}{1-p_i}\right) \qquad e^{\theta_i} = \frac{p_i}{1-p_i} \qquad (1-p_i)e^{\theta_i} = p_i \qquad p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}} = \text{expit}(\theta_i)$$

$$\ln\left(\frac{p_i}{1-p_i}\right) = \text{logit}(p_i)$$

$$W = \text{diag}\left\{ \frac{1}{\varphi_i v(p_i) [g'(p_i)]^2} : i = 1, 2, \dots, n \right\}$$

we need $g'(p_i)$ and $v(p_i)$

For $v(p_i)$:

recall that for $Y_i \sim bin(m, p_i)$ $\quad Var(Y_i) = m p_i (1-p_i) = \frac{1}{\varphi_i} v(p_i) \quad v(p_i) = p_i(1-p_i)$

For $g'(p_i)$:

The canonical link is when $g = (b')^{-1}$. So we need the function $b(\theta_i)$

$$b(\theta_i) = -\ln(1-p_i) = -\ln\left(1 - \frac{e^{\theta_i}}{1+e^{\theta_i}}\right) = -\ln\left(\frac{1}{1+e^{\theta_i}}\right) = \ln(1+e^{\theta_i})$$

$$b(\theta_i) = -\ln(1-p_i) = -\ln\left(1-\frac{e^{\theta_i}}{1+e^{\theta_i}}\right) = -\ln\left(\frac{1}{1+e^{\theta_i}}\right) = \ln(1+e^{\theta_i})$$

$$b'(\theta_i) = \frac{e^{\theta_i}}{1+e^{\theta_i}} \qquad g(p_i) = (b')^{-1}(p_i) = \ln\left(\frac{p_i}{1-p_i}\right)$$

Thus

$$g'(p_i) = \frac{1}{p_i} + \frac{1}{1-p_i} = \frac{1-p_i+p_i}{p_i(1-p_i)} = \frac{1}{p_i(1-p_i)}$$

So

$$W = \operatorname{diag}\left\{\frac{m}{p_i(1-p_i)\left[\frac{1}{p_i(1-p_i)}\right]^2}, i=1,2,\dots,n\right\} = \operatorname{diag}\{m\,p_i(1-p_i) : i=1,2,\dots,n\}$$

Lastly,

$$u_i = (y_i - p_i)\,g'(p_i) = \frac{(y_i-p_i)}{p_i(1-p_i)}$$