Lecture 34

Friday, April 12, 2024 10:39 AM

Agenda

- Mote Carlo methods
- sampling

Last class we talked about vanilla monte-carlo and what's called change of measure

Sp we wish to calculate some integral $\int_A f\,dx$ but this is analytically intractable

Instead of not calculating the integral, we can do the following:

we will change $\int_A f\,dx = E\left(\frac{f(X)}{g_X(X)}\right)$ where $g_X(x)$ is the density of some r.v. X defined on A. Then, using the law of large numbers, $\frac{1}{n}\sum \frac{f(X_i)}{g_X(X_i)} \to E\left(\frac{f(X)}{g_X(X)}\right) = \int_A f\,dx$ when $X_1, X_2, \dots, X_n \sim P$ where P is dist'n of X

How is this different from just using $\frac{1}{n}\sum f(X_i)$ to approximate $\int_A f\,dx$

We may guess that $\frac{1}{n}\sum f(X_i) \to \int_A f\,dx$ by WLLN but this is <u>not</u> <u>true</u>

This is wrong because $\frac{1}{n}\sum f(X_i) \xrightarrow{P} E(f(X_i)) = \int f(x)\, g_X(x)\,dx \neq \int f(x)\,dx$

When we want to find $\int_0^1 f(x)\,dx = \int_0^1 \frac{f(x)}{1/1}\cdot\frac{1}{1}\,dx = \int_0^1 \frac{f(x)}{g_X(x)}\, g_X(x)\,dx$ where g_X is density of $X \sim \text{unif}(0,1)$

In MOM or bootstrap, we wanted to estimate $E(f(X))$ so we can directly use $\frac{1}{n}\sum f(X_i)$ by WLLN

EX) $\int_{-\infty}^{\infty} e^{-x^4}\,dx$ Calculate this

sol'n

$$\int_{-\infty}^{\infty} e^{-x^4}\,dx = \int_{-\infty}^{\infty} \frac{e^{-x^4}}{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}}\underbrace{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}}_{g_X(x)}\,dx = \int_{-\infty}^{\infty} \overbrace{\sqrt{2\pi}\,e^{-x^2(x^2-\frac{1}{2})}}^{h(x)}\underbrace{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}}_{g_X(x)}\,dx = E\left(\sqrt{2\pi}\,e^{-X^2(X^2-\frac{1}{2})}\right)$$

density choice

algebra

def'n of expectation

$X \sim N(0,1)$

WLLN

$$\approx \frac{1}{n}\sum \sqrt{2\pi}\,e^{-X_i^2(X_i^2-\frac{1}{2})}$$

See code

EX) Calculate $\int_0^5 e^{-x^4}\,dx$

sol'n

$$\int_0^5 e^{-x^4}\,dx = \int_0^5 \frac{e^{-x^4}}{1/5}\cdot\frac{1}{5}\,dx = E(5e^{-X^4}) \approx \frac{1}{n}\sum 5e^{-X_i^4}$$

See code

soln

$$\int_0^5 e^{-x^4}dx = \int_0^5 \frac{e^{-x^4}}{1/5}\cdot\frac{1}{5}dx = E(5e^{-x^4}) \approx \frac{1}{n}\sum 5e^{-x_i^4}$$ see code

$X \sim unif(0,5)$

what happens if we naively apply the WLLN?

$$\frac{1}{n}\sum f(x_i) \overset{???}{\rightarrow} \int_A f(x)dx$$ see code

How can we generate random samples?

Suppose we want a sample of $Y \sim P$ and we know the CDF of Y, $F_Y(y)$.

we will do a two step process

1) generate $U \sim unif(0,1)$
2) set $Y = F_Y^{-1}(u)$, then $Y \sim P$

EX) Find the transformation of $U \sim unif(0,1)$ that gives $Y \sim exp(\lambda)$

soln

Recall that if $Y \sim exp(\lambda)$, then $F_Y(y) = 1 - e^{-y/\lambda}$, thus

$$u = F_Y(y) = 1 - e^{-y/\lambda} \Rightarrow y = -\lambda \ln(1-u) \sim exp(\lambda)$$

Some distns we can sample from:

1) $Y = -2\sum_{i=1}^{n} \ln(U_i) \sim \chi^2(2n)$
2) $Y = -\beta\sum_{i=1}^{a} \ln(U_i) \sim gamma(a,\beta)$
3) $Y = \dfrac{\sum_{i=1}^{a} \ln(U_i)}{\sum_{i=1}^{a+b} \ln(U_i)} \sim beta(a,b)$

what about Discrete random variables?

Since if X is discrete, F_X has no inverse, we will use the quantile function

$$F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$$

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