Lecture 23

§1 可积的等价条件

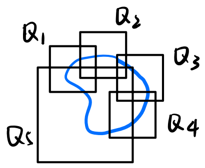
1、 **Definition:** measure 0

一个 subset $A\subset R^n$ 被称为 have "measure" ("volume") 0，若

$\forall \varepsilon>0$，$\exists$ seq $\{Q_i\}_{i=1}^{\infty}$ of closed rectangles in R^n, s.t.

(i) $\bigcup_{i=1}^{\infty} \mathring{Q}_i \supset A$ (countable)

(ii) $\sum_{i=1}^{\infty} |Q_i| < \varepsilon$ (countable)



此时记作 $|A|=0$

2、 **Facts**：关于 measure 0 的 facts

(i) $B\subset A$ & $|A|=0 \Rightarrow |B|=0$

(ii) If $|A_i|=0$, $\forall i\geq 0$, then $|\bigcup_{i=1}^{\infty} A_i|=0$

Corollary: If $A_i=\{a_i\}$, then $|A_i|=0 \Rightarrow |\bigcup_{i=1}^{\infty}\{a_i\}|=0$

$*$ In particular, $|\mathbb{Q}|=0$

证明: (proof of (ii))

Fix $i\geq 1$

$\because |A_i|=0$

$\therefore \forall \varepsilon>0$, $\exists \{Q_j^i\}_{j=1}^{\infty}$ s.t. (Q_j^i : closed rectangle)

- $\bigcup_{j=1}^{\infty} \mathring{Q}_j^i \supset A_i$
- $\sum_{j=1}^{\infty} |Q_j^i| < \frac{\varepsilon}{2^i}$

Now $\{Q_j^i\}_{i=1, j=1}^{\infty}$ satisfies

- $\bigcup_{i=1, j=1}^{\infty} \mathring{Q}_j^i \supset \bigcup_{i=1}^{\infty} A_i$
- $\sum_{i=1,j=1}^{\infty} |Q_j^i| = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|Q_j^i|\right) < \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i} = \varepsilon$

$\Rightarrow |\bigcup_{i=1}^{\infty} A_i|=0$

3、 **Jargon**: continuous almost everywhere

我们称 $f(x)$ is continuous almost everywhere on Q，若

$\exists D\subset Q$ s.t. f is continuous at every $x\in Q\setminus D$ and $|D|=0$

e.g. Recall: If f is monotone on $[a,b]$, then the set D of discontinuous points of f is at most countable

$\Rightarrow f(x)$ is continuous almost everywhere on $[a,b]$

4、 **Fact:** 可积的等价条件

Let f be bdd on Q. Then

f is R-integrable on $Q \iff f$ is continuous almost everywhere on Q.

证明: 仅证明 "$\Leftarrow$"

(Use Fact 2 (user-friendly version))

W.T.S.: $\forall \varepsilon > 0, \exists P$ s.t. $U(f;P) - L(f;P) < \varepsilon$

Let D be the set of discontinuous points of f in Q

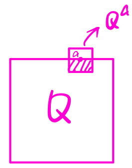
$\Rightarrow |D| = 0$

$\Rightarrow \forall \varepsilon > 0, \exists \{Q_i\}_{i=1}^{\infty}$ s.t. $\bigcup_{i=1}^{\infty} \mathring{Q}_i \supset D$ & $\sum_{i=1}^{\infty} |Q_i| < \varepsilon$ $(*)$

On the other hand, $\forall a \in Q \setminus D$, f continuous at a

$\Rightarrow \exists$ closed (small) rectangle Q^a s.t.

$a \in \mathring{Q}^a$ and $|f(x) - f(a)| < \frac{\varepsilon}{2}$, $\forall x \in Q^a \cap Q$



$\Rightarrow |f(x) - f(y)| < \varepsilon$, $\forall x, y \in Q^a \cap Q$ $(\#)$

Now observe $\{\mathring{Q}_i\}_{i=1}^{\infty}$ & $\{\mathring{Q}^a\}_{a \in Q \setminus D}$ cover Q, which is compact.

$\Rightarrow \exists$ finitely many $\mathring{Q}_i$'s & $\mathring{Q}^a$'s s.t. they cover Q

$\mathring{Q}_{i_1} \cup \mathring{Q}_{i_2} \cup \cdots \cup \mathring{Q}_{i_k} \cup \mathring{Q}^{a_1} \cup \mathring{Q}^{a_2} \cup \cdots \cup \mathring{Q}^{a_j} \supset Q$

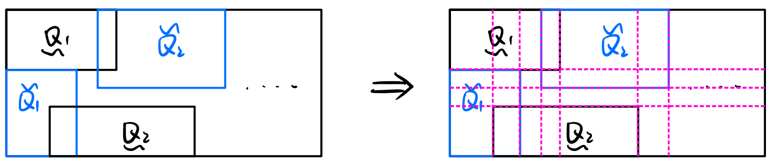
Let $\underset{\sim}{Q}_1 = Q_{i_1} \cap Q, \cdots, \underset{\sim}{Q}_k = Q_{i_k} \cap Q$

$\check{Q}_1 = Q^{a_1} \cap Q, \cdots, \check{Q}_j = Q^{a_j} \cap Q$

$\Rightarrow$ · $\underset{\sim}{Q}_1 \cup \cdots \cup \underset{\sim}{Q}_k \cup \check{Q}_1 \cup \cdots \cup \check{Q}_j = Q$

· $|\underset{\sim}{Q}_1| + |\underset{\sim}{Q}_2| + \cdots + |\underset{\sim}{Q}_k| < \varepsilon$ (由 $(*)$)

· $\forall x, y \in \check{Q}_r$ $(r = 1, 2, \cdots, j)$, $|f(x) - f(y)| < \varepsilon$ (由 $(\#)$) $(\clubsuit)$



Now extends all the sides of rectangles $\underset{\sim}{Q}$'s and $\check{Q}$'s to form a partition P of Q

$\Rightarrow \forall R \in P$, R contained in some $\underset{\sim}{Q}$ or $\check{Q}$

On the other hand, each $\underset{\sim}{Q}$ or $\check{Q}$ in union of several R's $\in P$

$$
\begin{aligned}
\Rightarrow U(f;P) - L(f;P) &= \sum_{R \in P} (M_R(f) - m_R(f)) |R| \\
&= \sum_{R \subset \text{some } \underset{\sim}{Q}} (M_R(f) - m_R(f)) |R| + \sum_{R \subset \text{some } \check{Q}} (M_R(f) - m_R(f)) |R| \\
&\le \sum_{R \subset \text{some } \underset{\sim}{Q}} 2M |R| + \sum_{R \subset \text{some } \check{Q}} \varepsilon |R| \quad (\text{由 } f(x) \le M \text{ on } Q \ \& \ (\clubsuit)) \\
&\le 2M \sum_{i=1}^{k} |\underset{\sim}{Q}_i| + \varepsilon |Q| \\
&< 2M\varepsilon + \varepsilon |Q| \\
&= (2M + |Q|) \varepsilon
\end{aligned}
$$

Q.E.D.

§2 Fubini's theorem on Q

1. Theorem: Fubini's theorem on Q

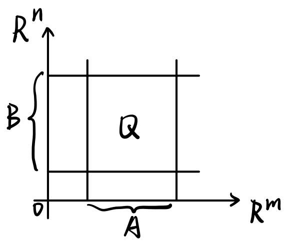
Let A be closed rectangle in R^m & B closed rectangle on R^n

Let $Q = A \times B \subset R^{m+n}$, write generic point in Q as (x,y) where $x \in A$, $y \in B$

Suppose $\int_Q f(x,y)\,d(x,y)$ $(dxdy)$ exists, and $\forall x \in A$, $\int_B f(x,y)\,dy$ exists.

Then $\int_A (\int_B (x,y)\,dy)\,dx$ exists

& $\int_A (\int_B (x,y)\,dy)\,dx = \int_{\underbrace{A\times B}_{Q}} f(x,y)\,dxdy$



证明:

$\because$ $\int_Q f(x,y)\,dxdy$ exists & $= I$

$\therefore$ $\forall \varepsilon, \exists \delta$ s.t. if $\|P\| < \delta$, then

$I - \varepsilon < \sum_{R \in P} f(x_R, y_R)\,|R| < I + \varepsilon$, $\forall (x_R, y_R) \in R$ $(*)$

W.T.S. if $\|P_A\| < \delta$, then $I - c\varepsilon < \sum_{R_A \in P_A} \int_B f(x_{R_A}, y)\,dy\,|R_A| < I + c\varepsilon$, $\forall x_{R_A} \in R_A$

Fix P_A & x_{R_A}'s

By def of $\int_B f(x_{R_A}, y)\,dy$, $\exists \delta_1 > 0$, s.t. if $\|P_B\| < \delta_1$, then

$\int_B f(R_A, y)\,dy - \varepsilon < \sum_{R_B \in P_B} f(x_{R_A}, y_{R_B})\,|R_B| < \int_B f(x_{R_A}, y)\,dy + \varepsilon$ $(\#)$

WLOG, assume $\delta_1 < \delta$.

Now let $P = (P_A, P_B)$ $\Rightarrow$ $\|P\| < \delta$

(Can use $(*)$ with $R = R_A \times R_B$, $x_R = x_{R_A}$, $y_R = y_{R_B}$)

$\therefore$ $I - \varepsilon < \sum_{\substack{R_A \in P_A \\ R_B \in P_B}} f(x_{R_A}, y_{R_B})\,|R_A||R_B| < I + \varepsilon$

$I - \varepsilon < \sum_{R_A \in P_B} (\sum_{R_B \in P_B} f(x_{R_A}, y_{R_B})\,|R_A||R_B|) < I + \varepsilon$

$\therefore$ $I - \varepsilon < \sum_{R_A \in P_A} |R_A| (\int_B f(x_{R_A}, y)\,dy + \varepsilon)$ (由 $(\#)$)

$\sum_{R_A \in P_A} |R_A| (\int_B f(x_{R_A}, y)\,dy - \varepsilon) < I + \varepsilon$

注意到 $\sum_{R_A \in P_A} |R_A| \varepsilon = |Q| \varepsilon$

$\Rightarrow$ $I - \varepsilon - |Q|\varepsilon < \sum_{R_A \in P_A} |R_A| \int_B f(x_{R_A}, y)\,dy < I - \varepsilon + |Q|\varepsilon$