

Lecture 24

§1 Riemann integrable on a set

1. Definition: Riemann integrable on a set

$S = \text{subset of } \mathbb{R}^n, \text{ bdd.}$

Let $f(x)$ be defined on S and bdd on S . Take a large closed rectangle $Q \supset S$.

$$\text{Let } f_S(x) = \begin{cases} f(x) & x \in S \\ 0 & x \notin S \end{cases}$$

若 $f_S(x)$ is Riemann integrable on Q , $\Rightarrow f(x)$ is Riemann integrable on S .

$$\text{令 } \int_S f(x) dx = \int_Q f_S(x) dx$$

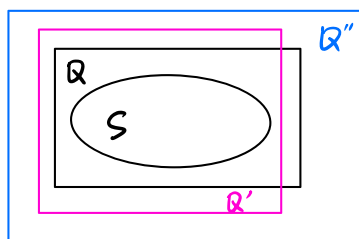
2. Question 1

Issue: 令 Q' 为 \mathbb{R}^n 中的另一个 closed rectangle, $Q' \supset S$

Q : 若 $\int_Q f_S(x) dx$ 存在, 则 $\int_{Q'} f_S(x) dx$ 是否存在?

$$\int_Q f_S(x) dx \stackrel{?}{=} \int_{Q'} f_S(x) dx$$

A: Yes; Yes.



证明:

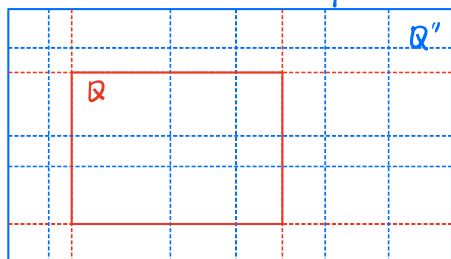
Take large closed rectangle $Q'' \supset Q', Q$

W.T.S. $\int_Q f_S(x) dx$ exists, then $\int_{Q'} f_S(x) dx$ exists

$$\text{and } \int_Q f_S(x) dx = \int_{Q'} f_S(x) dx$$

Observe: \forall partition P' of Q'' , it induces naturally a partition P of Q ;

$$\text{Moreover, } \sum_{\substack{R \in P' \cap Q \\ \forall x_k \in R}} f_S(x_k) |R| = \sum_{\substack{R \in P \\ \forall x_k \in R}} f_S(x_k) |R| \quad (\#)$$



Observe: as $\|P'\| \rightarrow 0$, we have $\|P\| \rightarrow 0$ & $\|P' \cup P\| \rightarrow 0$

Then RHS of $(\#) \rightarrow \int_Q f_S(x) dx$ as $\|P'\| \rightarrow 0$

LHS of $(\#) \rightarrow \int_{Q'} f_S(x) dx$

$$\Rightarrow \int_{Q'} f_S(x) dx \text{ exists \& } = \int_Q f_S(x) dx$$

3. Question 2

Q: If $f(x)$ continuous on S , is it R -integrable of S ? (It depends)

* Special Q: Is constant function $\mathbb{1}$ always integrable on S ?

A: define

$$\mathbb{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

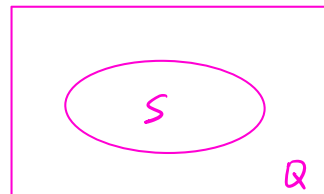
Take $\mathbb{Q} \supset S$, $\int_{\mathbb{Q}} \mathbb{1}_S(x) dx$ exists?

Let $D =$ set of discontinuous point of $\mathbb{1}_S(x)$

- $\mathbb{1}_S(x)$ continuous in $\overset{\circ}{S}$
- $\mathbb{1}_S(x)$ continuous in $\mathbb{Q} \setminus S$
- $\mathbb{1}_S(x)$ discontinuous at every point $\in \partial S$

$$\Rightarrow D = \partial S$$

$$\Rightarrow \int_{\mathbb{Q}} \mathbb{1}_S(x) dx \text{ exists} \iff |\partial S| = 0$$



Definition:

若 $|\partial S| = 0$, 则称 S 为 rectifiable

Definition:

若 S 为 bdd & rectifiable, 则 $\int_S \mathbb{1} dx$ exists 且被称为 the volume / measure of S

To answer Q,

Theorem:

If S is bdd & rectifiable & f is continuous on S , then $f(x)$ is R -integrable in S

证明:

$$\text{令 } f_S(x) = \begin{cases} f(x), & x \in S \\ 0, & x \notin S \end{cases}, \quad \mathbb{Q} \supset S$$

Observe: ① $f_S(x) = f(x)$ in $\overset{\circ}{S}$ & continuous at every point $\in \overset{\circ}{S}$

② $f_S(x) \equiv 0$ in $D \setminus \bar{S}$ & continuous at every point $\in D \setminus \bar{S}$

③ set D of discontinuous points $\subset \partial S$

$$\Rightarrow |D| = 0$$

$$\Rightarrow \int_{\mathbb{Q}} f_S(x) dx \text{ exists}$$

$$\Rightarrow f(x) \text{ is } R\text{-integrable in } S$$

4. Question 3

Q: User-friendly criterion for S to be rectifiable?

Definition:

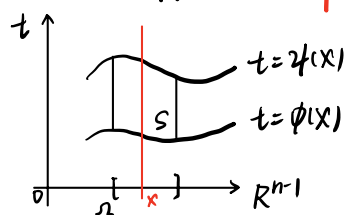
令 Ω 为 compact, rectifiable subset in \mathbb{R}^{n-1} ($n \geq 1$)

- γ & ϕ are continuous on Ω

• $\psi \geq \phi$ on Ω

Let $S = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \phi(x) \leq t \leq \psi(x), x \in \Omega\}$

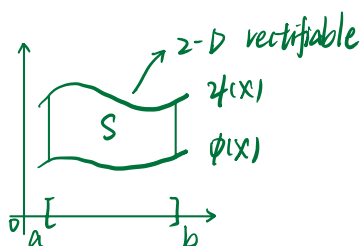
则 S 被称为 a simple region in \mathbb{R}^n



Theorem:

S is simple $\Rightarrow S$ compact & rectifiable

e.g. 1-D rectifiable



证明:

W.T.S. $|\partial S| = 0$

$\partial S = \text{graph of } \phi(x) \text{ \& } \psi(x) \text{ and } L = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x \in \partial\Omega, \phi(x) \leq t \leq \psi(x)\}$

(想利用 $|E_i| = 0, \forall i \geq 1 \Rightarrow |\bigcup_{i=1}^{\infty} E_i| = 0$)

Claim 1: $|L| = 0$

since Ω is rectifiable, $|\partial\Omega| = 0$ ($\partial\Omega$ as a subset of \mathbb{R}^{n-1})

$\Rightarrow \exists$ closed rectangles $\{Q_i\}_{i=1}^{\infty}$ in \mathbb{R}^{n-1} s.t.

• $\bigcup_{i=1}^{\infty} Q_i \supset \partial\Omega$

• $\sum_{i=1}^{\infty} |Q_i| < \varepsilon$

$\therefore \phi$ & ψ continuous on Ω , which is compact.

$\therefore \exists$ big $M > 0$ s.t. $-M \leq \phi(x) \leq \psi(x) \leq M, \forall x \in \Omega$

Observe

• $\bigcup_{i=1}^{\infty} \text{interior of } Q_i \times [-M-1, M+1] \supset L$

• $\sum_{i=1}^{\infty} |Q_i \times [-M-1, M+1]| = \sum_{i=1}^{\infty} |Q_i| \cdot (2M+2) < (2M+2)\varepsilon \Rightarrow |L| = 0$

Claim 2: $|\text{Graph of } t = \phi(x) \text{ over } \Omega| = 0$

$\therefore \phi$ is continuous on Ω & Ω is compact

$\therefore \phi$ is uniformly continuous on Ω , i.e.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x, y \in \Omega, |x - y| < \delta$, we have $|\phi(x) - \phi(y)| < \varepsilon$ (#)

Take large closed rectangle Q in \mathbb{R}^{n-1} s.t. $Q \supset \Omega$

Take partition P of Q with $\|P\|$ so small that diameters of subrectangles determined by $P < \delta$.

$\forall R \in P$ s.t. $R \cap \Omega \neq \emptyset \Rightarrow$ can take $x_R \in R \cap \Omega$

Then in $(\#)$, take $x = x_R$, y arbitrary point in $R \cap \Omega$

$$\Rightarrow |\phi(x) - \phi(y)| < \varepsilon, \forall y \in R \cap \Omega$$

$$\phi(x) - \varepsilon < \phi(y) < \phi(x) + \varepsilon$$

$$\Rightarrow \text{graph of } \phi \text{ over } \Omega \subset \bigcup_{\substack{R \in P \\ R \cap \Omega \neq \emptyset}} R \times (\phi(x_R) - \varepsilon, \phi(x_R) + \varepsilon)$$

$$\subset \bigcup_{\substack{R_{fat} \in P \\ R \cap \Omega \neq \emptyset}} R_{fat} \times (\phi(x_R) - \varepsilon, \phi(x_R) + \varepsilon) \quad (\text{interior of } R_{fat} = 2|R|)$$

$$\sum |R_{fat} \times [\phi(x_R) - \varepsilon, \phi(x_R) + \varepsilon]| = \sum 2^n |R| \cdot 2\varepsilon \leq 2^n |Q| \cdot 2\varepsilon$$

\Rightarrow Claim 2

R.E.D.

§2 Fubini's theorem on simple region

1. Theorem: Fubini's theorem on simple region

Suppose S is as given before (hence rectifiable in R^n) & $f: S \rightarrow R$ continuous, then

$$\int_S f(x,t) dx dt = \int_\Omega \left(\int_{\phi(x)}^{\psi(x)} f(x,t) dt \right) dx$$

证明:

Take a large closed rectangle $Q \subset R^n$ s.t. $Q \supset S, Q = Q_1 \times [a,b], Q_1 \subset R^{n-1}, \Omega \subset Q_1$

$\therefore S$ is rectifiable

$\therefore \int_S f(x,t) dx dt$ exists

i.e. $\int_Q f_S(x,t) dx dt$ exists

Fix $x_0 \in Q_1$.

If $x_0 \notin \Omega$, then $f_S(x_0,t) \equiv 0, \forall t \in [a,b] \Rightarrow \int_{[a,b]} f_S(x_0,t) dt = 0$

If $x_0 \in \Omega$, then $f_S(x_0,t)$ is continuous on $[a,b]$, possibly except at $t = \phi(x_0), t = \psi(x_0)$

$$\Rightarrow \int_{[a,b]} f_S(x_0,t) dt \text{ exists} = \int_{\phi(x_0)}^{\psi(x_0)} f(x_0,t) dt$$

By ole Fubini,

$$\int_{Q_1} \left(\int_{[a,b]} f_S(x,t) dt \right) dx \text{ exists \& } = \int_Q f_S(x,t) dx dt = \int_S f(x,t) dx dt$$

$$\int_\Omega \left(\int_{\phi(x_0)}^{\psi(x_0)} f(x,t) dt \right) dx$$