Lecture 26

§1 Curl

1. Curl (旋度)

Definition
The curl of a vector field $\mathbf{F} = \langle M, N, P\rangle$ is defined to be

$$\operatorname{curl}\mathbf{F} := \underbrace{\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)}\mathbf{i} + \underbrace{\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)}\mathbf{j} + \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}\mathbf{k}.$$

Circulation density projected on yz-plane　　" zx-plane　　" xy-plane

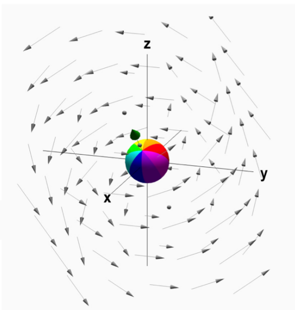
Remark
If we think of ∇ as the symbolic vector $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$, then the curl can be remembered as the symbolic expression

$$\boxed{\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}.}$$

注: ① $\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$

② $\operatorname{curl}\vec{F}$ 描述由 $\vec{F}$ 造成的 rotation behaviours. 在一点 P_0 处:

- $\operatorname{curl}\vec{F}$ 的方向与以 P_0 为圆心的非常小的球体的旋转的 positive axis 方向相同 (遵从右手定则)
- $\operatorname{curl}\vec{F}$ 的长度描述球体旋转的速度.

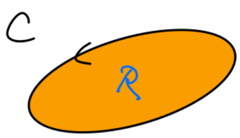


§2 Stokes' Theorem

1. 引例

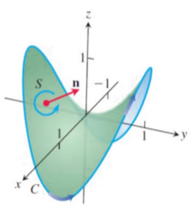
Stokes' Theorem

Recall the circulation version of Green's theorem:

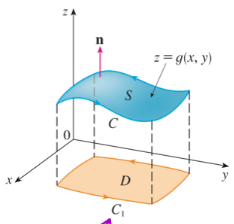


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Q: What if C is a space curve? $\oint_C \vec{F}\cdot d\vec{r} = ?$



Let us investigate a special case below first.



- Assume $\vec{F} = \langle M, N, P \rangle$, M, N, P have cts partials.
- Suppose g has cts second partials.
- $\oint_C \vec{F} \cdot d\vec{r} = ?$

$C_1: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad a \le t \le b \qquad C: \begin{cases} x = x(t) \\ y = y(t) \\ z = g(x(t), y(t)) \end{cases}, \quad a \le t \le b$

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \oint_C M\,dx + N\,dy + P\,dz \\
&= \int_a^b M x'(t)\,dt + N y'(t)\,dt + P z'(t)\,dt \\
&= \int_a^b M x'(t)\,dt + N y'(t)\,dt + P(g_x x'(t) + g_y y'(t))\,dt \\
&= \int_a^b (M + P g_x) x'(t)\,dt + (N + g_y) y'(t)\,dt \\
&= \oint_{C_1} (\bar{M} + \bar{P} g_x)\,dx + (\bar{N} + \bar{P} g_y)\,dy
\end{aligned}$$

(Here $\bar{M}(x,y) := M(x, y, g(x,y))$; same for $\bar{N}$ & $\bar{P}$)

$$\begin{aligned}
&\overset{\text{Green's}}{=} \iint_D \left(\frac{\partial(\bar{N} + \bar{P} g_y)}{\partial x} - \frac{\partial(\bar{M} + \bar{P} g_x)}{\partial y} \right) dA \\
&\overset{\text{Product rule}}{=} \iint_D \left(\frac{\partial \bar{N}}{\partial x} + \bar{P} g_{yx} + \frac{\partial \bar{P}}{\partial x} g_y - \frac{\partial \bar{M}}{\partial y} - \bar{P} g_{xy} - \frac{\partial \bar{P}}{\partial y} g_x \right) dA \\
&= \iint_D \left(\frac{\partial \bar{N}}{\partial x} + \frac{\partial \bar{P}}{\partial x} g_y - \frac{\partial \bar{M}}{\partial y} - \frac{\partial \bar{P}}{\partial y} g_x \right) dA \\
&\overset{\text{Chain rule}}{=} \iint_D \left(\frac{\partial N}{\partial x} + \frac{\partial N}{\partial z} g_x + \left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} g_x\right) g_y - \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} g_z\right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \cdot g_y\right) g_x \right) dA \\
&= \iint_D \left[-\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) g_x - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) g_y + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \right] dA \\
&= \iint_D \operatorname{curl} \vec{F} \cdot \langle -g_x, -g_y, 1 \rangle\, dA \\
&= \iint_S (\operatorname{curl} \vec{F} \cdot \vec{n})\, d\sigma
\end{aligned}$$

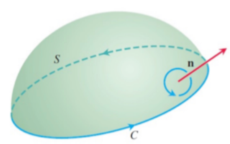
2. Stokes' Theorem (斯托克斯公式)

Theorem (Stokes' Theorem) orientable, with a side given by $\vec{n}$

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C, and let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. If C is positively oriented (counterclockwise) with respect to the surface's unit normal $\mathbf{n}$, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

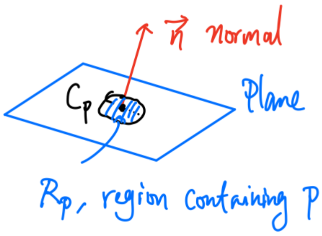
If $\vec{n}$ is "up", then C is counterclockwise when viewed from above.



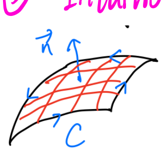
注: ① 格林公式为斯托克斯公式的特例: C 与 S 均全部位于 xy 平面上, $\vec{n} = \vec{k} = \langle 0, 0, 1 \rangle$.

② $(\operatorname{curl} \vec{F} \cdot \vec{n})(P)$ gives the circulation density on a tiny plane containing P with unit normal $\vec{n}$:

$$(\operatorname{Curl} \vec{F} \cdot \vec{n})(P) = \lim_{A(R_P) \to 0} \frac{\oint_{C_P} \vec{F} \cdot d\vec{r}}{A(R_P)}$$



③ Intuition of Stokes' theorem:



Total circulation around C

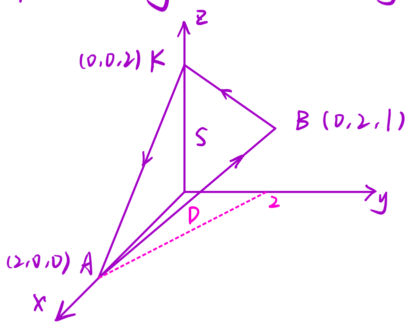
$= \sum$ circulation around small closed curve

(due to cancellation as in Green's thm)

$\approx \sum$ circulation around small closed plane curve

$= \sum (\text{curl}\vec{F}\cdot\vec{n})\,\Delta\sigma$

例: $\vec{F} = \langle x+2y,\ x+3z,\ 2x+y\rangle$. find $\oint_C \vec{F}\cdot\vec{T}\,ds$.



- C is counterclockwise when $\vec{n}$ is upward (z-component > 0)
- $\vec{AB}\times\vec{AK} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ -2 & 0 & 2 \end{vmatrix} = \langle 4, 2, 4\rangle$

  upward normal
- S is given by $4(x-2)+2y+4z=0$

  $\Rightarrow f(x,y) := z = -x-\frac{1}{2}y+2$

  $\Rightarrow D: \{(x,y) \mid 0\le x\le 2,\ 0\le y\le 2-x\}$
- $\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & x+3z & 2x+y \end{vmatrix} = \langle -2,-2,-1\rangle$
- $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\vec{n}\,d\sigma$

  $= \iint_D \text{curl}\vec{F}\cdot\langle -f_x, -f_y, 1\rangle\,dA$ $(dxdy)$

  $= \iint_D (2f_x+2f_y-1)\,dA$

  $= \iint_D -4\,dA$

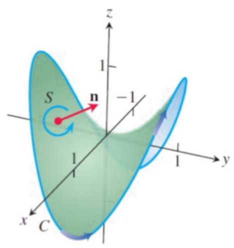
  $= -8$

例: Let S be the surface $z = y^2-x^2$ inside the cylinder $x^2+y^2\le 1$.
Find upward flux of curl $\vec{F}$ across S, where $\vec{F} = \langle y, -x, x^2\rangle$

- Flux $= \iint_S (\text{curl}\vec{F})\cdot\vec{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}$

  where $C = \{(x,y,y^2-x^2) : x^2+y^2=1\}$

  $= \{(\cos t, \sin t, \sin^2 t-\cos^2 t) : 0\le t\le 2\pi\}$



- $\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx+N\,dy+P\,dz$

  $= \int_0^{2\pi} -1+4\cos^3 t\sin t\ dt$

  $= -2\pi$

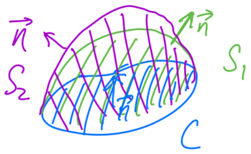
3. Surface independence (旋度场的性质一)

← having $\vec{n}$ pointing the same side

If S_1 and S_2 are two oriented surfaces having the same boundary curve, which is counterclockwise, then Stokes' theorem implies that

that is a simple closed curve

$$\iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_2} \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,d\sigma.$$



注: 换言之, 任意旋度场均 "surface independent", 即
一个旋度场穿过有界曲面的通量仅取决于 boundary 而非 surface.

例: Find circulation of $\vec{F} = \langle x^2+z,\ y^2+2x,\ z^2-y\rangle$ along C, where C is intersection of S_1: $x^2+y^2+z^2=1$ and S_2: $z=\sqrt{x^2+y^2}$, counterclockwise when viewed from above.

- $\begin{cases} x^2+y^2+z^2=1 \\ z=\sqrt{x^2+y^2} \end{cases}$

  $\Rightarrow x^2+y^2+(x^2+y^2)=1$

  $\Rightarrow C: x^2+y^2=\frac{1}{2}$, in plane $z=\frac{1}{\sqrt{2}}$

- Can take surface S lying on the plane i.e.

  $S=\{(x,y,\frac{1}{\sqrt{2}}) : x^2+y^2\le \frac{1}{2}\}$

- Take upward unit normal $\vec{n}=\vec{k}$: by Stokes' theorem:

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \iint_S \text{curl}\,\vec{F}\cdot\vec{k}\,d\sigma \\ &= \iint_D \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA \\ &= 2\iint_D dA \\ &= \pi\end{aligned}$$

4. 3D Component Test

若 $\vec{F}=\langle M,N,P\rangle$, M,N,P 在 $D\in R^3$ (D open, 单连通) 上有连续偏导, 则

$\vec{F}$ is conservative on D ⓵ $\Leftrightarrow$ $P_y=N_z,\ M_z=P_x,\ N_x=M_y$ on D ⓶

注: ⓶ 也可以写作 $\text{curl}\,\vec{F}\equiv\vec{0}$ on D

证明:

⓵ $\Rightarrow$ ⓶:

- 若 $\vec{F}$ is conservative on D,

  则 $\vec{F}=\langle f_x, f_y, f_z\rangle$ for some f on D.

- 则 $P_y=f_{zy}=f_{yz}=N_z$

  $M_z=f_{xz}=f_{zx}=P_x$

  $N_x=f_{yx}=f_{xy}=M_y$

② ⇒ ①:

- Let C be any closed curve in D. May assume C is simple. Take any piecewise smooth surface with C being its boundary curve, say S. By Stoke's theorem,
  $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,d\sigma$
- Since $\text{curl}\,\vec{F} \equiv \vec{0}$ on D by ②, so RHS $= 0$, i.e. $\oint_C \vec{F}\cdot d\vec{r} = 0$
  D satisfies the loop property, so it is conservative on D.

注: 若对于D上的每一个点, $\text{curl}\,\vec{F} \equiv 0$, 则 $\vec{F}$ 被称作 irrotational on D.
The 3D component test 表明 irrotational field (无旋场) 等同于保守场
(on an open simply connected domain)