Lecture 14

§1 More about FTC

1. About $\int_a^x f(t)\,dt$

1° $\int_a^x f(t)\,dt = \int_a^x f(s)\,ds$: the inner variable is a "dummy variable", so the choice of letter is not important.

2° However, $\int_a^x f(x)\,dx$ makes no sense!

3° FTC 1 can be used to prove FTC 2

4° Some elementary functions have antiderivatives not expressible in terms of an elementary function, e.g.

$$f(x) = \frac{\sin x}{x}$$

But we know an antiderivative:

$$F(x) = \int_a^x \frac{\sin t}{t}\,dt$$

where $a \neq 0$ is a constant, and x has the same sign as a.

2. "Real life" meanings

1° If $C(x)$ is the total cost for producing x units of goods, then by FTC 2,

$$\int_a^b C'(x)\,dx = C(b) - C(a)$$

$\int_a^b C'(x)\,dx$ is the extra cost for increasing production from a to b units.

3. Mathematical consequence

The average slope of all the tangent lines to the curve $y=f(x)$ over the interval $[a,b]$ is:

$$\frac{\int_a^b f'(x)\,dx}{b-a} = \frac{f(b)-f(a)}{b-a}$$

Slope of secant = average of the slopes of the tangents to the curve between a and b.

4. Differentiation and integration

By FTC 1: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

By FTC 2: $\int_a^x f'(t)\,dt = f(x) - f(a)$

Hence, apply integration ($\int_a^x f(t)\,dt$) and then differentiation to a continuous function f, or vice-versa (反之亦然), gives you f back. (Subject to a difference by a constant)

§2 Areas Between Two Curves

1. Definition

Let f and g be functions that are integrable on $[a,b]$. Then the area A between the graph of $y=f(x)$ and the graph of $y=g(x)$, from $x=a$ to $x=b$, is defined by

$$A = \int_a^b |f(x)-g(x)|\,dx$$

2. Area between $y=f(x)$ and the x-axis

Take $g(x)=0$, then the area becomes

$$A = \int_a^b |f(x)|\,dx$$

If f is non-negative, then

$$A = \int_a^b f(x)\,dx$$

e.g. Find the area A between the graph of $y=f(x)$ and the x-axis, from $x=a$ and $x=b$.

$f(x)=\sin x$; $a=0$ and $b=2\pi$

$$\begin{aligned} A &= \int_0^{2\pi} |\sin x|\,dx \\ &= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx \\ &= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx \end{aligned}$$

$$= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi}$$

$$= 4$$

e.g. Find the area A between the graph of $y=f(x)$ and the graph of $y=g(x)$, from $x=a$ to $x=b$.

$f(x)=(x-2)^2$ and $g(x)=2x-1$; $a=0$ and $b=8$

$$f(x)-g(x) = x^2-4x+4-2x+1$$

$$= x^2-6x+5$$

$$= (x-5)(x-1)$$

$$\text{Area} = \int_0^8 |f(x)-g(x)|\,dx$$

$$= \int_0^8 |x^2-6x+5|\,dx$$

$$= \int_0^1 (x^2-6x+5)\,dx - \int_1^5 (x^2-6x+5)\,dx + \int_5^8 (x^2-6x+5)\,dx$$

$$= \left[\tfrac{1}{3}x^3-3x^2+5x\right]_0^1 - \left[\tfrac{1}{3}x^3-3x^2+5x\right]_1^5 + \left[\tfrac{1}{3}x^3-3x^2+5x\right]_5^8$$

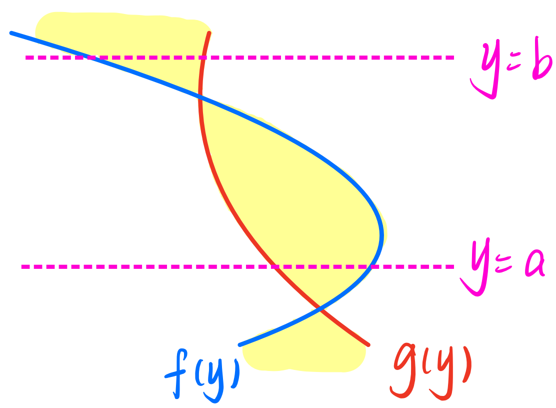
$$= \left(\tfrac{1}{3}-3+5\right) - \left(\tfrac{125}{3}-75+25-\left(\tfrac{1}{3}-3+5\right)\right) + \left(\tfrac{8^3}{3}-3\times8^2+40 - \left(\tfrac{125}{3}-75+25\right)\right)$$

$$= 40$$

3. Remark

Area A between curves $x=f(y)$ and $x=g(y)$, from $y=a$ to $y=b$, can be defined similarly:

$$A = \int_a^b |f(x)-g(x)|\,dy$$



§3 Substitution Method (Change of Variable)

1. Theorem 5.5.6 — The Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

e.g. Find an antiderivative of $f(x) = \frac{1}{\sqrt{x}} \cos\sqrt{x}$

$f(x) = \frac{1}{\sqrt{x}} \cos\sqrt{x}$

$f(x) = 2\frac{d}{dx}(\sin\sqrt{x}) = \frac{d}{dx}(2\sin\sqrt{x})$

$F(x) = 2\sin\sqrt{x}$

e.g. Find $\int \sin^3 x\, dx$

$\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx$

Let's try the substitution $u = -\cos x$

Then $\frac{du}{dx} = \sin x$, or $du = \sin x\, dx$

$$\begin{aligned}\int \sin^3 x\, dx &= \int (1-u)^2 du \\ &= u - \frac{1}{3}u^3 + C \\ &= -\cos x + \frac{1}{3}\cos^3 x + C\end{aligned}$$

2. Proof of substitution rule

Since f is continuous, by FTC I it has an antiderivative F.

i.e. $f(x) = F'(x)$

$$\begin{aligned}\int f(g(x))\, g'(x)\, dx &= \int F'(g(x))\, g'(x)\, dx \\ &= \int (F\circ g)'(x)\, dx \\ &= (F\circ g)(x) + C \\ &= F(g(x)) + C \\ &= F(u) + C \\ &= \int F'(u)\, du + C\end{aligned}$$

$$= \int f(u)\,du + C$$

e.g. Find $\int x\sqrt{2x+1}\,dx$

Try $u = 2x+1 \Rightarrow x = \frac{u-1}{2}$

Then $\frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2}du$

$$\int x\sqrt{2x+1}\,dx = \int \frac{u-1}{2}\sqrt{u}\cdot\frac{1}{2}du$$
$$= \int \frac{u-1}{4}\sqrt{u}\,du$$
$$= \frac{1}{4}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$$
$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C$$
$$= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{\frac{5}{2}} - \frac{2}{3}(2x+1)^{\frac{3}{2}}\right) + C$$

3. General conclusion

If f is continuous on an interval I and $F' = f$ on I, then

$$\int f(Ax+B)\,dx = \int f(u)\frac{1}{A}du$$
$$= \frac{1}{A}F(u) + C$$
$$= \frac{1}{A}F(Ax+B) + C$$

e.g. $\int \sec^2(5x+1)\,dx = \frac{1}{5}\tan(5x+1) + C$

4. Use $d(g(x))$

We may write $d(g(x))$ instead of du if $u = g(x)$

e.g. $\int \sin^3 x\,dx$

$$= \int (1-\cos^2 x)\sin x\,dx$$
$$= \int (\cos^2 x - 1)(-\sin x\,dx)$$
$$= \int (\cos^2 x - 1)\,d(\cos x)$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

e.g. $I = \int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$. Find I.

Method 1: Let $u = x^3+1$, then $du = 3x^2\,dx$

Also, $u=0$ when $x=-1$, $u=2$ when $x=1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du$$

Method 2: Find the antiderivative first:

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1}\,d(x^3+1)$$
$$= \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C$$

Apply FTC 2:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \frac{4}{3}\sqrt{2}$$

5. Theorem 5.6.7 —— Substitution in definite integrals

If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x)=u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof:

Let F be an antiderivative of f on range (g). Then

$$\int_{g(a)}^{g(b)} f(u)\,du = F(g(b)) - F(g(a))$$
$$= (F\circ g)(b) - (F\circ g)(a)$$
$$= \int_a^b (F\circ g)'(x)\,dx$$
$$= \int_a^b f(g(x))\cdot g'(x)\,dx$$