Lecture 11

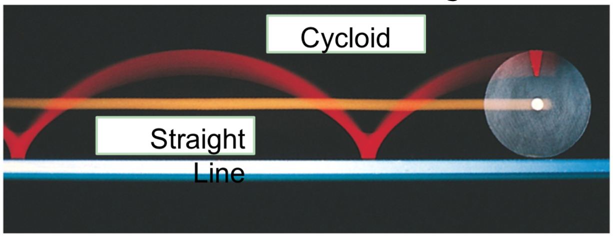
§1 Rolling

1. Rolling

1° 沿平面 rolling smoothly，无 slipping（滑动）/ bouncing（跳跃）

2° Pure translational motion in center

Cycloid trace at the edge



2. Translation in rolling

1° 滚轮的质心以速度 $\vec{v}_{com}$ 运动

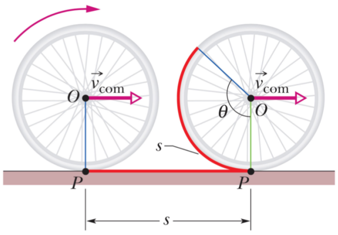
2° 对于 smooth rolling motion（无滑滚动/纯滚动）：

angular displacement：θ

translation displacement：s

$$s = \theta R$$

$$v_{com} = \frac{ds}{dt} = \frac{d\theta}{dt} \cdot R = \omega R$$



$$s = \theta R$$

$$v_{com} = \frac{s}{\Delta t} = \frac{ds}{dt} = \frac{d\theta}{dt} R = \omega R$$

Smooth rolling motion

- COM of a uniform wheel: has a translational speed

2

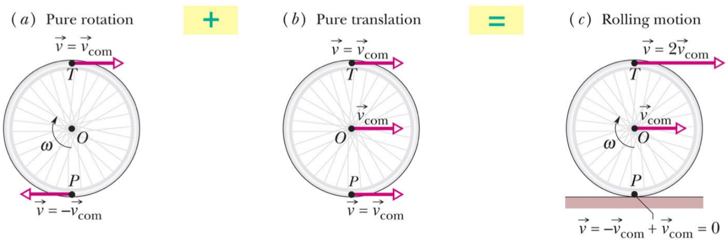
3. Rolling as translation and rotation combined（对 rolling 的第一种理解）

1° 滚轮的上部比下部运动更快

2° 对于 each point on the rim：

$$\vec{v} = \vec{v}_{com} + \vec{v}_t$$

$$v_t = \omega R$$



4. Rolling as pure rotation（对 rolling 的第二种理解）

1° 滚动运动可以视作关于点 P 的 pure rotation

P 为滚轮与地面的接触点，

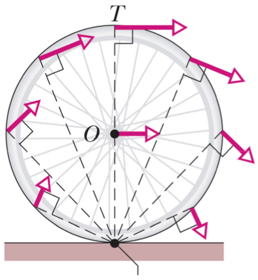
2° 角速度：

① 与绕O旋转时的角速度ω相同(对圆上任意点,均满足)

② 所有点,均可视作以相同的ω绕P点旋转

- Rolling motion of a wheel may be viewed as pure rotation about an axis P where the wheel contacts the ground as the wheel moves
- What is the angular speed?
- Same ω as rotating about O
- All points rotates at ω about P

$$v_{com} = \omega R$$
$$v_{top} = \omega \times 2R = 2v_{com}$$
$$v_P = 0$$



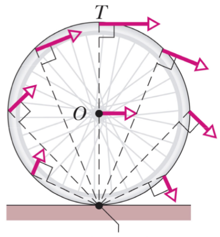
Rotation axis at P

- In combination, at the top

$$v_{top} = 2v_{com} = 2\omega_{com}R$$

- Rolling about P:

$$v_{top} = \omega_P \times 2R = 2\omega_P R$$
$$2\omega_{com}R = 2\omega_P R$$
$$\omega_{com} = \omega_P$$



Rotation axis at P

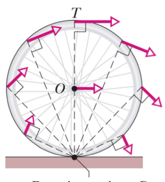
- All points, e.g., T and O rotates about P with same angular speed as the angular speed rotates about O.

* 若将P点,换成圆上任意一点,上述结论仍成立

3° 线速度方向

① 理解一: 轮上任一点: 永远垂直于该点相对于P点的位置向量.

② 理解二: $\vec{v} = \vec{v}_{com} + \vec{v}_t$

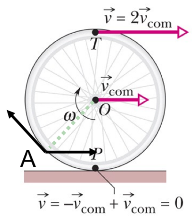


Rotation axis at P

- At any point on the wheel: always perpendicular to the position vector from P to that point.
- Or as the direction of the vector sum:

$$\vec{v} = \vec{v}_{com} + \vec{v}_t$$

- The point may or may not be located on the rim of the wheel.

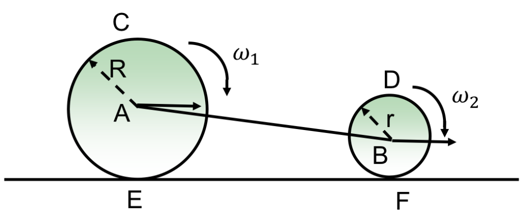


例:

Checkpoint 1

The rear wheel on a clown's bicycle has twice the radius of the front wheel. (a) When the bicycle is moving, is the linear speed at the very top of the rear wheel greater than, less than, or the same as that of the very top of the front wheel? (b) Is the angular speed of the rear wheel greater than, less than, or the same as that of the front wheel?

$$v_A = v_B$$
$$v_C = 2v_A$$
$$v_D = 2v_B$$
$$R = 2r$$
$$v_A = \omega_1 R$$
$$v_B = \omega_2 r$$
$$\omega_1 = 0.5\omega_2$$



Answer: (a) the same (b) less than

§2 Newton's 2nd Law for smooth rolling

1. Newton's 2nd Law for smooth rolling

1° 对于平动：$F_{net} = m a_{com} = m \alpha \cdot R$
　 对于转动：$\tau_{net} = I_{com}\alpha = I_{com}\frac{a_{com}}{R}$

2° 有两种方式来 initialize rolling
① 施加 torque 以创造 rotation
② 施加 force 以创造 translation

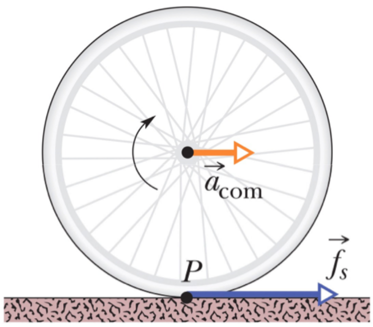
3° 若要发生纯滚动，必须满足 $a_{com} = \alpha \cdot R$

2. 施加 torque 以创造 rotation

1° 对于平动：F_{net} 仅受地面静摩擦力 f_s 影响
① 因为 F_{net} 与 f_s 相同，所以 f_s 的方向与实际加速度 $\vec{a}_{com}$ 相同
　匀速：无摩擦力
　加速：向右
　减速：向左
② $F_{net} = f_s = m \cdot a_{com}$

2° 对于转动：τ_{net} 同时受施加的力矩 $\tau_{applied}$ 与摩擦力的力矩 $f_s R$ 影响
$\tau_{net} = \tau_{applied} - f_s \cdot R = I \cdot \alpha$

3° 纯滚动条件：$a_{com} = \alpha \cdot R$



4° 得出方程：
$$\begin{cases} f_s = m \cdot a_{com} \\ \tau_{applied} - f_s \cdot R = I \cdot \alpha \\ a_{com} = \alpha \cdot R \end{cases}$$

5° 分析：
① 过程分析：
a. 施加力矩 $\tau_{applied}$，产生运动趋势
b. 地面产生一个合适的静摩擦力 f_s，确保 $a_{com} = \alpha \cdot R$
即 $\frac{f_s}{m} = \frac{\tau_{applied} - f_s \cdot R}{I} \cdot R$

得：$f_s = \tau_{applied} \cdot \frac{mR}{mR^2 + I}$
c. 同时产生对应的 a_{com} 与 α
② 静摩擦力分析：
a. 范围：
$f_s \le \mu_s N$
b. 若 a_{com} 超过 $\frac{\mu_s N}{M}$（f_s 超过 $\mu_s N$），则出现 slipping

3. 施加 force 以创造 translation

1° 对于平动：F_{net} 同时受施加的力 $F_{applied}$ 与摩擦力 f_s 影响

① $F_{applied}$ 的方向与实际加速度 $\vec{a}_{com}$ 相同

② $F_{net} = F_{applied} - f_s = m a_{com}$

2° 对于滚动：τ_{net} 仅受地面静摩擦力的力矩 $f_s R$ 影响

① f_s 的方向与实际运动趋势相反

② $\tau_{net} = f_s R = I \alpha$

3° 纯滚动条件：$a_{com} = \alpha \cdot R$

4° 得出方程：

$$\begin{cases} F_{applied} - f_s = m a_{com} \\ f_s R = I \alpha \\ a_{com} = \alpha \cdot R \end{cases}$$

4. 自行车分析

1° 后轮：施加 torque 以创造 rotation

Example: Frictional Force for Driving Wheel

$$F_{net} = m a_{com}$$
$$f_s = F_{net} = m a_{com}$$
$$\tau_{net} = \tau_{applied} - f_s R = I\alpha = I\frac{a_{com}}{R}$$
$$a_{com} = \frac{\tau_{applied}}{mR}\frac{1}{(1+\frac{I}{mR^2})}$$

11

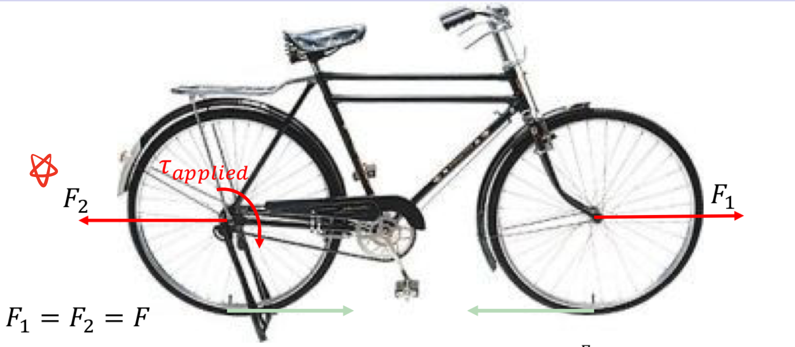
2° 前轮：施加 force 以创造 translation

Example: Frictional Force for Passive Wheel

$$F_{net} = m a_{com}$$
$$F_{applied} - f_s = F_{net} = m a_{com}$$
$$\tau_{net} = f_s R = I\alpha = I\frac{a_{com}}{R}$$
$$a_{com} = \frac{F_{applied}}{m}\frac{1}{(1+\frac{I}{mR^2})}$$

3° Connected wheels

Bicycle: Connected Wheels



$$F_{net} = ma_{com}$$
$$f_{s,back} - F = F_{net} = ma_{com} \quad (1)$$
$$\tau_{net} = \tau_{applied} - f_{s,back}R = I\alpha = I\frac{a_{com}}{R} \quad (2)$$

$$F_{net} = ma_{com}$$
$$F - f_{s,front} = F_{net} = ma_{com} \quad (3)$$
$$\tau_{net} = f_{s,front}R = I\alpha = I\frac{a_{com}}{R} \quad (4)$$

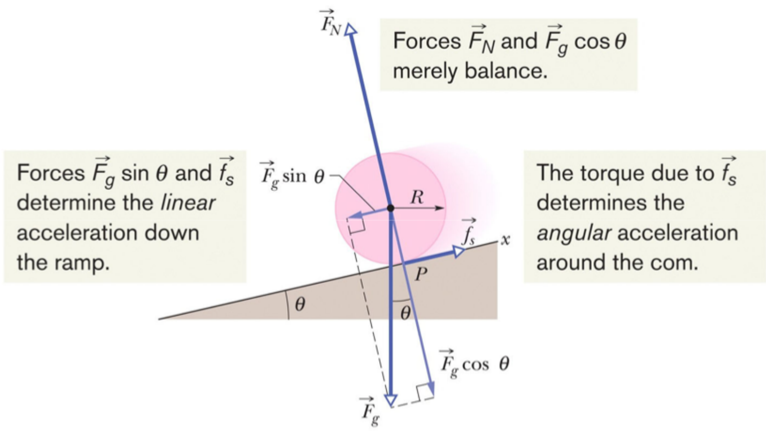
$$f_{s,back} - f_{s,front} = 2ma_{com} \quad (5)$$

$$a_{com} = \frac{\tau_{applied}}{2mR}\frac{1}{(1+\frac{I}{mR^2})} \quad (6)$$

5. Rolling down 分析 (Force applied case)

A round uniform body of mass M and radius R moving down a ramp at angle θ, along an x axis, Find the a_{com}?

- If no friction? How does it move?



1° 无摩擦时
- pure translational motion
- $a_{com} = g\sin\theta$, down along the surface

2° 无滑滚动

$$\begin{cases} Mg\sin\theta - f_s = Ma_{com} \\ f_s \cdot R = I_{com} \cdot \alpha \\ a_{com} = \alpha \cdot R \end{cases}$$

$$\begin{cases} f = \dfrac{MIg\sin\theta}{I_{com} + MR^2} \\ a_{com} = \dfrac{g\sin\theta}{1 + \frac{I_{com}}{MR^2}} \end{cases}$$

3° 若 $f_s \leq \mu_s F_N$, 则为非纯滚动

Summary

- Rolling Smoothly

$$v_{com} = \frac{s}{\Delta t} = \frac{ds}{dt} = \frac{d\theta}{dt}R = \omega R$$

$$a_{com} = \frac{dv_{com}}{dt} = \frac{d\omega}{dt}R = \alpha R$$

- Rolling on a ramp

$$a_{com,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2}$$