Lecture 6

§1 Conditional probability

若部分 outcome is fixed, i.e. 我们事先得到了一些信息, 则计算它的 occurrence 的概率时, 需要限制在一个基于已知信息的 smaller sample space 内

e.g. **Example 3.6.**

Consider 10 objects with the following classifications regarding shape and color.

	Red	Blue	Total
Ball	5	1	6
Cube	1	3	4
Total	6	4	10

Suppose we draw an object at random from these 10 objects. We can evaluate the probabilities of many events that we are interested:

$$\Pr(\text{ball}) = \frac{6}{10}, \quad \Pr(\text{blue}) = \frac{4}{10}, \quad \Pr(\text{blue ball}) = \frac{1}{10}, \quad \Pr(\text{red cube}) = \frac{1}{10}, \quad \ldots$$

Suppose when the object was drawn we found that it was blue by a glimpse. Then this information will alter our uncertainty about its shape. Denote the probability that it is a ball given that it is blue by Pr(ball|blue). Then,

$$\Pr(\text{ball}|\text{blue}) = \frac{\text{no. of blue balls}}{\text{no. of blue objects}} = \frac{1}{4}.$$

Similarly,

$$\Pr(\text{ball}|\text{red}) = \frac{5}{6}, \qquad \Pr(\text{blue}|\text{ball}) = \frac{1}{6}, \qquad \ldots$$

These are called *conditional probabilities.*

It can be easily observed that

$$\begin{aligned}\Pr(\text{ball}|\text{blue}) &= \frac{\text{no. of blue balls/total no. of objects}}{\text{no. of blue objects/total no. of objects}} \\ &= \frac{\Pr(\text{ball} \cap \text{blue})}{\Pr(\text{blue})}.\end{aligned}$$

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1. Definition: Conditional probability (条件概率)

Definition 3.2.
For any two events A and B, the conditional probability of A given the occurrence of B is written as $\Pr(A|B)$ and is defined as

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)},$$

provided that $\Pr(B) > 0$.

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e.g. **Example 3.7.** (Boy or Girl paradox)

A mother has two kids. You ask, "is anyone of them a boy?" The mother says "Yes". What is the probability that they are both boys?

Solution:

Take $\Omega = \{BB, BG, GB, GG\}$ as the sample space of equally likely outcomes.

Let $A = \{\text{at least one of them is a boy}\} = \{BB, BG, GB\}$,

$B = \{\text{both kids are boys}\} = \{BB\}$.

$$\Pr(B|A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

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e.g. **Example 3.8.** (Boy or Girl paradox)

A mother has two kids. You ask, "is your elder kid a boy?" The mother says "Yes". What is the probability that they are both boys?

Solution:

Take $\Omega = \{BB, BG, GB, GG\}$ as the sample space of equally likely outcomes.

Let $A = \{\text{the elder kid is a boy}\} = \{BB, BG\}$,

$B = \{\text{both kids are boys}\} = \{BB\}$.

$$\Pr(B|A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

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e.g. **Example 3.9.** (Simpson's paradox)

There are 2 treatments for a disease, A and B. Applying A to some patients and B to others results in the following observations:

	Men (M)		Women (W)	
	Recovered (R)	Dead (D)	Recovered (R)	Dead (D)
Treatment A	20	80	40	20
Treatment B	50	160	15	5

For men, the probabilities of recovery given the two treatments are

$$\begin{aligned}\Pr(R|M\cap A) &= \frac{\Pr(R\cap M\cap A)}{\Pr(M\cap A)} = \frac{20}{100} = \frac{1}{5},\\ \Pr(R|M\cap B) &= \frac{\Pr(R\cap M\cap B)}{\Pr(M\cap B)} = \frac{50}{210} = \frac{5}{21} > \Pr(R|M\cap A).\end{aligned}$$

For women, the probabilities of recovery given the two treatments are

$$\begin{aligned}\Pr(R|W\cap A) &= \frac{\Pr(R\cap W\cap A)}{\Pr(W\cap A)} = \frac{40}{60} = \frac{2}{3},\\ \Pr(R|W\cap B) &= \frac{\Pr(R\cap W\cap B)}{\Pr(W\cap B)} = \frac{15}{20} = \frac{3}{4} > \Pr(R|W\cap A).\end{aligned}$$

Therefore for both men and women, B is a better treatment than A.

However, if we combine the two tables for the two genders, i.e., for all patients, we have

$$\begin{aligned}\Pr(R|A) &= \frac{\Pr(R\cap A)}{\Pr(A)} = \frac{20+40}{100+60} = \frac{3}{8},\\ \Pr(R|B) &= \frac{\Pr(R\cap B)}{\Pr(B)} = \frac{50+15}{210+20} = \frac{13}{46} < \frac{3}{8} = \Pr(R|A),\end{aligned}$$

which implies that A is better!

This example illustrates what has come to be known as the *Simpson's paradox*, a reversal of the direction of a comparison or an association when data from several groups are combined to form a single group.

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2. Theorem: Multiplication Theorem (概率乘法定理)

对于任意两个事件 A与B, Pr(B)>0, 有

$$\Pr(A\cap B) = \Pr(B)\cdot\Pr(A|B)$$

对于任意三个事件 A,B,C, Pr(B∩C)>0, 有

$$\Pr(A\cap B\cap C) = \Pr(C)\cdot P(B|C)\cdot P(A|B\cap C)$$

e.g. **Example 3.10.**

What is the probability of drawing three aces in a row from a poker deck?

Solution:

Denote A_i, $i=1,2,3$, as the event that the i-th drawn card is an ace. Then,

$$\begin{aligned}\Pr(A_1\cap A_2\cap A_3) &= \Pr(A_1)\times\Pr(A_2|A_1)\times\Pr(A_3|A_1\cap A_2)\\ &= \frac{4}{52}\times\frac{3}{51}\times\frac{2}{50} = \frac{1}{5525}.\end{aligned}$$

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3. Definition: Independence (独立性)

两个 events A与B 被称为 independent 当且仅当

$$\Pr(A\cap B) = \Pr(A)\cdot\Pr(B)$$

若 Pr(A)>0, 则 A与B 被称为 independent 当且仅当

$$\Pr(B|A) = \Pr(B)$$

e.g. **Example 3.11.**

Opinion poll on building an incinerator in Hong Kong:

	Support (S)	Opposed (S^c)
CUHK students (H)	0.459	0.441
Non-CUHK students (H^c)	0.051	0.049

If a person is drawn at random, then

$$\begin{aligned}\Pr(H) &= 0.459+0.441 = 0.9,\\ \Pr(S) &= 0.459+0.051 = 0.51,\\ \Pr(H\cap S) &= 0.459,\\ \Pr(H)\Pr(S) &= 0.9\times 0.51 = 0.459 = \Pr(H\cap S).\end{aligned}$$

Hence being a CUHK student or not and opinion are independent in this poll.

注：关于互斥与独立

1° 证明：若事件A与事件B发生的概率都不为0，那么独立和互斥有这样一层关系：

互斥不独立，独立不互斥

① 若A、B互斥，则$A\cap B=\varnothing$

那么$P(A\cap B)=0$

而$P(A)\cdot P(B)\neq 0$，

因此$P(A\cap B)\neq P(A)\cdot P(B)$，即A、B不独立

② 若A、B独立，则$P(A\cap B)=P(A)\cdot P(B)$

假设A、B互斥，则$A\cap B=\varnothing$

那么$P(A\cap B)=0$

而$P(A)\cdot P(B)\neq 0$

则$P(A\cap B)\neq P(A)\cdot P(B)$

矛盾

因此A、B独立，则A、B不互斥

2° 零概率事件与不可能事件

零概率事件：$P(A)=0$

不可能事件：$A=\varnothing$

在离散情况下，两者等价.

在连续情况下，零概率事件可能发生

e.g. X服从在$(0,1)$上均匀分布

则事件$\{X=0.5\}$发生的概率为0

但事件$\{X=0.5\}$有可能发生

$P(A)=0 \;\not\Leftrightarrow\; A=\varnothing$

3° 证明：零概率事件与任何事件独立

设$P(A)=0$

对于任意事件B，有$A\cap B\subseteq A$

那么$P(A\cap B)\leq P(A)$

则$P(A\cap B)=0$

于是有$P(A\cap B)=P(A)\cdot P(B)=0$

即A、B独立

4° 证明：不可能事件与任意事件互斥

设A为不可能事件

则$A=\varnothing$

对任何事件B，有$A\cap B=\varnothing$

则 A、B 互斥

5° 综上：不可能事件与任何事件既独立又互斥

6° 不独立可能不互斥，不互斥可能不独立

4. Definition: mutually (joint) independence (相互独立)

事件 $A_1, A_2, \cdots, A_k$ 被称为 (mutually) independent 当且仅当这些事件任意组合后的交集的概率等于这几个事件概率的乘积

例如，A_1，A_2，A_3 独立当且仅当

$$\Pr(A_1 \cap A_2) = \Pr(A_1) \cdot \Pr(A_2)$$

$$\Pr(A_1 \cap A_3) = \Pr(A_1) \cdot \Pr(A_3)$$

$$\Pr(A_2 \cap A_3) = \Pr(A_2) \cdot \Pr(A_3)$$

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3)$$

e.g. **Example 3.12.**

Two fair coins are tossed. Denote A as the event that the first coin lands on head, B as the event that the second coin lands on tail, C as the event that both coin land on the same face.

Are these events independent?

Solution:

$\Omega = \{HH, HT, TH, TT\}$, assuming equally likely outcomes.

$A = \{HH, HT\}$, $B = \{HT, TT\}$, $C = \{HH, TT\}$.

Obviously, $\Pr(A) = \Pr(B) = \Pr(C) = \dfrac{1}{2}$.

$\Pr(A \cap B) = \Pr(\{HT\}) = \dfrac{1}{4} = \Pr(A)\Pr(B) \quad \Longrightarrow A$ and B are independent.

$\Pr(A \cap C) = \Pr(\{HH\}) = \dfrac{1}{4} = \Pr(A)\Pr(C) \quad \Longrightarrow A$ and C are independent.

$\Pr(B \cap C) = \Pr(\{TT\}) = \dfrac{1}{4} = \Pr(B)\Pr(C) \quad \Longrightarrow B$ and C are independent.

However, $\Pr(A \cap B \cap C) = \Pr(\varnothing) = 0 \neq \Pr(A)\Pr(B)\Pr(C)$.

Thus, the events A, B, and C are not mutually independent, they are just *pairwisely independent.*

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