Lecture 10

§1 Graph sketching

1. Key components

1° Domain D and symmetry (even or odd function)

2° Critical points

3° Intervals of monotonicity

4° Points of inflection and intervals of concavity

5° Asymptotes

6° x- and y- intercepts (截距)

e.g. Sketch $y=f(x)$ where $f(x)=\frac{x^2+4}{2x}$

① Domain D and symmetry (even or odd function)

$D=(-\infty,0)\cup(0,+\infty)$

$f(-x)=-f(x)$, so f is an odd function

$\Rightarrow$ symmetric about origin

For analysis below we will only consider $x>0$

② Critical points

$$f'(x)=\frac{2x(2x)-2x^2-8}{4x^2}=\frac{x^2-4}{2x^2}$$

$f'(x)=0 \iff x=2$

Critical point on graph at $(2,2)$

③ Intervals of monotonicity

	0		2	
$f'(x)$		<0		>0
f		$\searrow$		$\nearrow$

④ Points of inflection and intervals of concavity

$$f'(x)=\frac{x^2-4}{2x^2}=\frac{1}{2}-2x^{-2}$$

$f''(x) = 4x^{-3}$

$\Rightarrow f''(x) > 0 \quad \forall x > 0$

f concave up on $(0, +\infty)$

No point of inflection

⑤ Asymptotes

Oblique asymptote: $y = Ax + B$, where

$A = \lim_{x\to\infty} \frac{f(x)}{x}$ $\qquad$ $B = \lim_{x\to\infty} (f(x) - Ax)$

$$A = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2+4}{2x^2} = \frac{1}{2}$$

$$B = \lim_{x\to\infty} (f(x) - Ax) = \lim_{x\to\infty} \left(\frac{x^2+4}{2x^2} - \frac{1}{2}x \right) = \lim_{x\to\infty} \left(\frac{4}{2x} \right) = 0$$

$\Rightarrow y = \frac{1}{2}x$ is the oblique asymptote of $y = f(x)$ as $x \to \infty$

Vertical asymptote: f is only undefined at 0

$$\lim_{x\to 0^+} \frac{x^2+4}{2x} = \infty$$

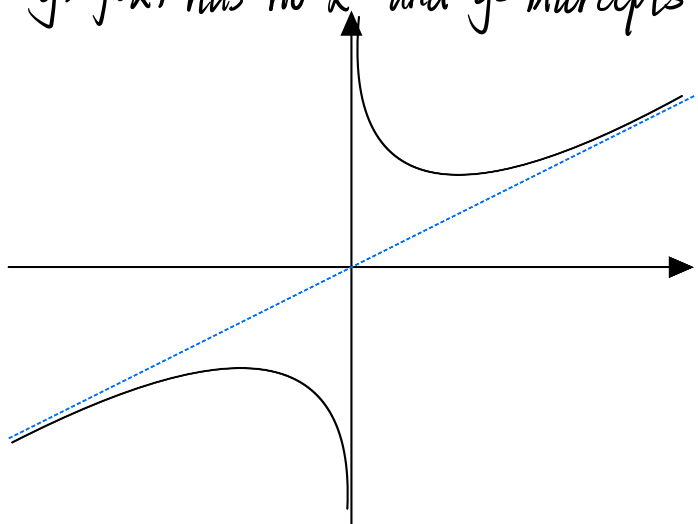
$\Rightarrow x = 0$ is the vertical asymptote

⑥ x- and y- intercepts

$f(0)$ is undefined

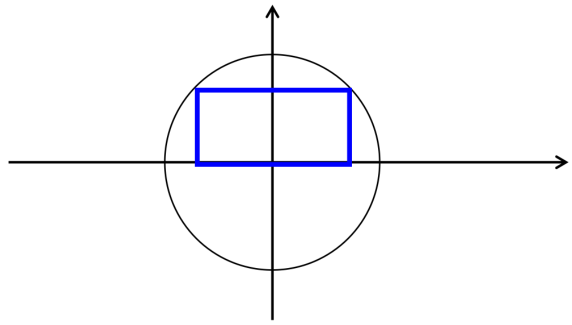
$y = f(x) = 0$ has no real solution

$y = f(x)$ has no x- and y- intercepts



§2 Applied optimization

e.g. A rectangle is to be inscribed (内切) in a semicircle of radius 2. What is the largest area the rectangle can have and what are its dimensions?



Let the rectangle width $=2x$ and height $=y$

Then $x^2+y^2=4$, $y=\sqrt{4-x^2}$

Area $A(x)=2xy=2x\sqrt{4-x^2}$, continuous on $[0,2]$

$$A'(x)=2\sqrt{4-x^2}+2x\left(\tfrac{1}{2}\right)(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$=\frac{2(4-x^2)-2x^2}{\sqrt{4-x^2}}$$

$$=\frac{-4x^2+8}{\sqrt{4-x^2}}$$

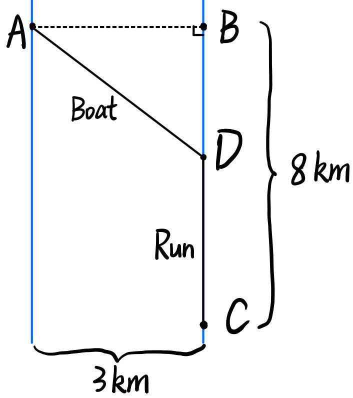
For $A'(x)=0$, $x=\sqrt{2}$ $(x\in(0,2))$, $y=\sqrt{2}$

x	0		$\sqrt{2}$		2
$A'(x)$		>0		<0	
$A(x)$	0		4		0

Dimensions are: Width $=2\sqrt{2}$, Height $=\sqrt{2}$

Largest area $=4$

e.g. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see figure). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible.)



Let $x = |CD|$, then $t(x) = \frac{|AD|}{6} + \frac{|DB|}{8}$

$$t(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}, \quad x \in [0, 8]$$

$$t'(x) = \frac{1}{6} \cdot \frac{1}{2} \cdot (x^2+9)^{-\frac{1}{2}} \cdot 2x - \frac{1}{8} = \frac{4x - 3\sqrt{x^2+9}}{24\sqrt{x^2+9}}$$

For $t'(x) = 0$, $x = \frac{9\sqrt{7}}{7}$

x	0	$\frac{9\sqrt{7}}{7}$	8
$A'(x)$	<0	>0	

He should land to D where $BD = \frac{9\sqrt{7}}{7}$ km

e.g. x = number of video game consoles, million units.

Cost: $C(x) = x^3 - 6x^2 + 15x$

Revenue: $R(x) = 9x$

Profit: $P(x) = R(x) - C(x)$

Question: Find x that maximizes profit, if any.

$$P(x) = -x^3 + 6x^2 - 6x$$

$$P'(x) = -3x^2 + 12x - 6 = -3(x^2 - 4x + 2)$$

For $P'(x) = 0$, $x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$

x	0		$2-\sqrt{2}$		$2+\sqrt{2}$	
$P'(x)$		<0		>0		<0
$P(x)$	0	$\searrow$		$\nearrow$	$4+4\sqrt{2}$	$\searrow$

Abs. max at $x = 2 + \sqrt{2} \approx 3.414$ million units

Remark:

If the profit function P is differentiable, then when P is maximized at x_0, $P'(x_0)=0$, in which case marginal revenue = marginal cost, i.e. $R'(x_0) = C'(x_0)$

§3 Concavity (extended discussion)

1. Theorem: Concavity and secant lines

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then the graph of f lies above the secant line joining $(a,f(a))$ and $(b,f(b))$ on (a,b).

(ii) If f is concave up on (a,b), then the graph of f lies below the secant line joining $(a,f(a))$ and $(b,f(b))$ on (a,b).

Proof of (i):

The secant line has a graph

$$y = g(x) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a) \quad ①$$

We will show that $f(x) > g(x)$ for all $x \in (a,b)$, given that $f(x)$ is concave down on (a,b).

Let $x \in (a,b)$. By MVT,

$$f(x) = f(a) + f'(c_1)(x-a) \text{ for some } c_1 \in (a,x). \quad ②$$

and $f(b) = f(x) + f'(c_2)(b-x)$ for some $c_2 \in (x,b)$.

Since $f(x)$ is concave down on (a,b), $c_1 < c_2$

$$f'(c_2) < f'(c_1)$$

Now $f(b) = f(a) + f'(c_1)(x-a) + f'(c_2)(b-x)$

$$< f(a) + f'(c_1)(x-a) + f'(c_1)(b-x)$$

$$= f(a) + f'(c_1)(b-a)$$

$$f(b) < f(a) + f'(c_1)(b-a)$$

$$\frac{f(b)-f(a)}{b-a} < f'(c_1) \quad ③$$

From ①②③, we have $f(x) > g(x)$

QED

2. Theorem: Concavity and tangent lines

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then for any $c \in (a,b)$ the tangent line to $y=f(x)$ at c lies above the graph of $y=f(x)$

(ii) If f is concave up on (a,b), then for any $c \in (a,b)$ the tangent line to $y=f(x)$ at c lies below the graph of $y=f(x)$

Proof of (i):

The tangent line at c has a graph

$$y = g(x) = f(c) + f'(c)(x-c)$$

We will show that $g(x) > f(x)$ for all $x \in [a,b] \setminus \{c\}$, given that $f(x)$ is concave down on (a,b).

Let $x \in [a,b] \setminus \{c\}$. Note that

$$g(x) > f(x)$$

$$\Leftrightarrow f(c) + f'(c)(x-c) > f(x)$$

$$\Leftrightarrow f'(c)(x-c) > f(x) - f(c)$$

$$\Leftrightarrow f'(x) \begin{cases} > \frac{f(x)-f(c)}{x-c} & \text{if } x > c \\ < \frac{f(x)-f(c)}{x-c} & \text{if } x < c \end{cases}$$

So it suffices to show that

$$f'(x) \begin{cases} > \frac{f(x)-f(c)}{x-c} & \text{if } x \in (c,b] \quad ① \\ < \frac{f(x)-f(c)}{x-c} & \text{if } x \in [a,c) \quad ② \end{cases}$$

Now assume that $x \in (c,b]$. By MVT

$$f'(c_1) = \frac{f(x)-f(c)}{x-c} \text{ for some } c_1 \in (c,x)$$

Since f is concave down on (a,b), $f'(c_1) < f'(c)$

Then $\frac{f(x)-f(c)}{x-c} < f'(c)$

which is identical to ①, i.e. we have proven ① is true.

② can be proved similarly, then statement (i) is proved.

§4 First derivative test (extended discussion)

The first derivative test doesn't always apply.

e.g. Consider $f(x) = \begin{cases} x^2(1+\sin\frac{\pi}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h(1+\sin\frac{\pi}{h}) = 0$

Hence, 0 is a critical point of f. But

$$f'(x) = 2x(1+\sin\frac{\pi}{x}) - \pi\cos\frac{\pi}{x} \text{ for } x \neq 0$$

No matter what value of a you pick $(a>0)$, in the interval $(0,a)$ there exist x_1 and x_2 such that $f'(x_1) > 0$ and $f'(x_2) < 0$.

So f is NEVER always increasing or always decreasing near 0, on the right. Hence none of the conditions in the first derivative test is satisfied.

Since $f(0)=0$, $f(x)\geq 0 \quad \forall x$

So $f(0)$ is an absolute (and hence local) minimum.