Lecture 21

§1 Wave speed on a stretched string

1. Wave speed

1° 取决于介质的性质

2° 与波长、频率有关，但不由其决定

3° 若一个波经过一介质，波会导致介质中的质点振动，这需要 both mass (K.E.) and elasity (P.E.)

4° mass 与 elasity 决定了波能传多快

2. Find wave speed by wave equation

可以由波动方程求得波速（适用于任一种波）

- 由牛顿第二定律：

$$F_{net,y} = ma_y$$

$$F_{2y} - F_{1y} = dm \cdot a_y$$

$$dm = \mu dx \; ; \; a_y = \frac{d^2y}{dt^2}$$

- F_1 与 F_2 大小相等，为

$$\tau = \sqrt{F_{2x}^2 + F_{2y}^2}$$

- 令斜率为 $S = \frac{dy}{dx}$，

$$F_{2y} = \tau S_2$$

$$F_{1y} = \tau S_1$$

结合上式，得：

$$\tau S_2 - \tau S_1 = \mu dx \cdot \frac{d^2y}{dt^2}$$

$$\frac{S_2 - S_1}{dx} = \frac{\mu}{\tau} \cdot \frac{d^2y}{dt^2}$$

- 将 $S_2 - S_1$ 视作 dS，

$$dS = d\left(\frac{dy}{dx}\right)$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\mu}{\tau} \cdot \frac{d^2y}{dt^2}$$

- 由波动方程：

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 y}{\partial t^2}$$

比较得：

$$v = \sqrt{\frac{\tau}{\mu}}$$

3. 正弦波的波动方程

Wave Equation for Sinusoidal Function

- For a transverse wave:

$$y(x,t) = y_m sin(kx - \omega t)$$

$$\frac{\partial y}{\partial t} = -\omega y_m cos(kx - \omega t) \rightarrow \frac{\partial^2 y}{\partial t^2} = -\omega^2 y_m sin(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = k y_m cos(kx - \omega t) \rightarrow \frac{\partial^2 y}{\partial x^2} = -k^2 y_m sin(kx - \omega t)$$

- Since

$$v = \frac{\omega}{k} \rightarrow \omega^2 = v^2 k^2$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y_m sin(kx - \omega t) = -v^2 k^2 y_m sin(kx - \omega t) = v^2 \frac{\partial^2 y}{\partial x^2}$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

4. Wave speed on a stretched string

1° 沿着 stretched ideal string 的波速仅取决于 tension 与 linear density.

2° 频率由波源决定

3° 波长为 $\lambda = \frac{v}{f}$

例:

Problem

Checkpoint 3

You send a traveling wave along a particular string by oscillating one end. If you increase the frequency of the oscillations, do (a) the speed of the wave and (b) the wavelength of the wave increase, decrease, or remain the same? If, instead, you increase the tension in the string, do (c) the speed of the wave and (d) the wavelength of the wave increase, decrease, or remain the same?

Answer: (a) remain the same
(b) decrease
(c) increase
(d) increase

§2 Energy and power of a wave traveling along a string

1. Energy

1° 波的移动会同时传递动能与势能

2° 对于一个 element of dm

动能: 平衡位置 max, 波峰/谷 min.

势能: 由 wave 拉伸 string 产生

平衡位置 max, 波峰/谷 min.

3° 对于一个 region of string

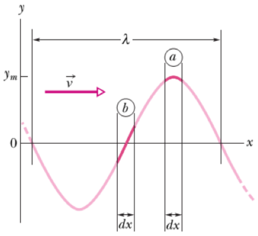
maximum displacement: no energy

zero displacement: maximum energy

- As the wave moves, it transports energy as both Kinetic Energy and Potential Energy.

For an element of Δm

- Kinetic Energy: Maximum at b; zero at a.
- Potential Energy: the wave must necessarily stretch the string. → Maximum at b; zero at a.



- The region of the string at its maximum displacement has no energy, and the regions at zero displacement has maximum energy.
- The wave *transports* the energy along the string.

2. The rate of energy transmitting (power)

· 考虑一个 element dm 的动能 dK

$$dK = \frac{1}{2} dm \times u^2$$

$$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$$

$$dm = \mu dx = \mu \times v \times dt$$

· 因此，

$$dK = \frac{1}{2} dm\, u^2$$

$$= \frac{1}{2} [-\omega y_m \cos(kx - \omega t)]^2 \times \mu \times v \times dt$$

$$= \frac{1}{2} \mu \omega^2 y_m^2 \cos^2(kx - \omega t) \times v \times dt$$

$$\frac{dK}{dt} = \frac{1}{2} \mu v \omega^2 y_m^2 \cos^2(kx - \omega t)$$

· 考虑势能 dU

$$dU = dK$$

· 考虑平均值

$$P_{avg} = dU_{avg} + dK_{avg}$$

$$= 2 \times \frac{1}{2} \mu v \omega^2 y_m^2 [\cos^2(kx - \omega t)]_{avg}$$

$$= \mu v \omega^2 y_m^2 [\cos^2(kx - \omega t)]_{avg}$$

cosine square 在一个波长上的平均值为 $\frac{1}{2}$

$$P_{avg} = \frac{1}{2} \mu v \omega^2 y_m^2$$

Additional Information

- For **Transverse Sinusoidal Wave**:

$$y(x,t) = y_m \sin(kx - \omega t)$$

$$u(x,t) = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t) \text{ and } \frac{\partial y}{\partial x} = k y_m \cos(kx - \omega t)$$

$$\boldsymbol{dK} = \frac{1}{2} dm u^2 = \frac{1}{2} \times \mu dx \times [-\omega y_m \cos(kx - \omega t)]^2$$

$$= \frac{1}{2} \mu \omega^2 y_m^2 \cos^2(kx - \omega t) dx$$

- **For potential energy**: The amount of stretch is

$$\sqrt{dx^2 + dy^2} - dx = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - dx$$

$$\approx dx\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \cdots\right] - dx \approx \frac{dx}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

$$P.E. = dU = tension \times amount\ of\ stretch = \tau \times \frac{dx}{2} \times \left(\frac{\partial y}{\partial x}\right)^2$$

$$\boldsymbol{dU} = \frac{1}{2} \mu v^2 k^2 y_m^2 \cos^2(kx - \omega t) dx = \frac{1}{2} \mu \omega^2 y_m^2 \cos^2(kx - \omega t) dx = \boldsymbol{dK}$$

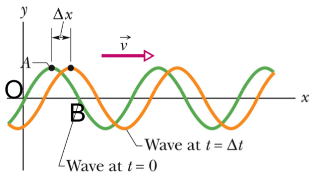
Additional Information

- At t=0 s,

$$y(x,0) = y_m sin(kx)$$

$$u(x,0) = \frac{\partial y}{\partial t} = -\omega y_m cos(kx)$$

- The angular wavenumber is $k = \frac{2\pi}{\lambda}$



- Therefore,

$$y(A,0) = y_m sin\left(\frac{2\pi}{\lambda} \times \frac{\lambda}{4}\right) = y_m$$

$$u(A,0) = -\omega y_m cos\left(\frac{2\pi}{\lambda} \times \frac{\lambda}{4}\right) = 0 \rightarrow dK = 0 = dU$$

$$u(B,0) = -\omega y_m cos\left(\frac{2\pi}{\lambda} \times \frac{\lambda}{2}\right) = \omega y_m$$

$$dK = \frac{1}{2} dm u^2 = \frac{1}{2} \mu dx \times (\omega y_m)^2 = \frac{1}{2} \mu \omega^2 y_m^2 dx = dU$$

- At point B, the element has both its maximum Kinetic energy and the potential energy.

例:

Problem

A string has linear density μ= 525 g/m and is under tension τ = 45 N. We send a sinusoidal wave with frequency f =120 Hz and amplitude y_m =8.5 mm along the string. At what average rate does the wave transport energy?

Solution:

$$P_{avg} = \frac{1}{2} \mu v \omega^2 y_m^2$$

$$\omega = 2\pi f = 2\pi \times 120\ Hz = 754\ rad/s$$

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{45\ N}{0.525\ kg/m}} = 9.26\ m/s$$

$$P_{avg} = \frac{1}{2} \mu v \omega^2 y_m^2 = 0.5 \times 0.525 \times 9.26 \times 754^2 \times 0.0085^2$$
$$= 100\ W$$

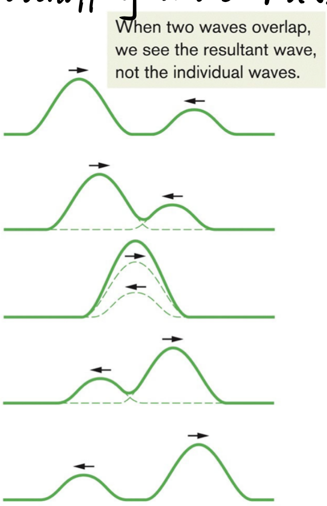
§3 Superposition and interference

1. Principle of superposition (叠加原理)

- Resultant wave (or net wave) 为 Overlapping waves 的 algebraically sum

$$y' = y_1(x,t) + y_2(x,t)$$

- Overlapping waves 不会影响各自的传播



- Two or more waves pass simultaneously through the same region
- As shown in the left, two waves travel simultaneously along the same stretched string.

$$y' = y_1(x,t) + y_2(x,t)$$

- **Principle of Superposition**: When several effects occur simultaneously, the net effect is the sum of the individual effects.
- Overlapping waves algebraically add to produce a resultant wave (or net wave).
- Overlapping waves do not in any way alter the travel of each other.

2. Interference (干涉) of waves

当两个正弦波沿着一个 stretched string 有相同的振幅、波长、传播方向时，会干涉产生一个沿相同方向传播的 resultant sinusoidal wave

Wave 1: $y_1(x,t) = y_m \sin(kx - \omega t)$

Wave 2: $y_2(x,t) = y_m \sin(kx - \omega t + \phi)$

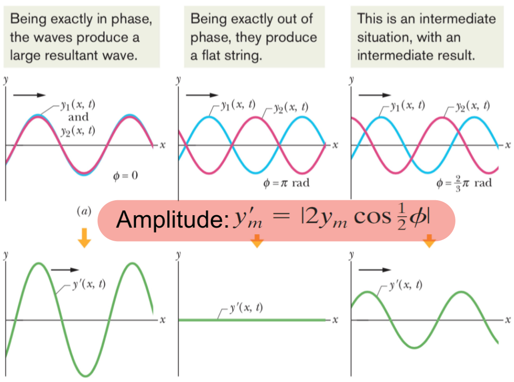
$y' = y_1(x,t) + y_2(x,t)$

$= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$

由和差化积公式: $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$y' = [2y_m \cos(\frac{1}{2}\phi)] \cdot \sin(kx - \omega t + \frac{1}{2}\phi)$

Interference of Waves



Fully Constructive | Fully destructive | Intermediate

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Interference of Waves

- The wave repeats itself every 2π rad
- A phase difference $\phi = 2\pi$, a shift of the wave relative to the other by a distance of λ.

Table 16-1 Phase Difference and Resulting Interference Types[a]

Phase Difference, in: Degrees	Radians	Wavelengths	Amplitude of Resultant Wave	Type of Interference
0	0	0	$2y_m$	Fully constructive
120	$\frac{2}{3}\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$\frac{4}{3}\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

$$y' = \left[2y_m \cos\left(\frac{1}{2}\phi\right)\right] \sin(kx - \omega t + \frac{1}{2}\phi)$$

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香港中文大學(深圳) The Chinese University of Hong Kong, Shenzhen

例: **Problem**

Checkpoint 4

Here are four possible phase differences between two identical waves, expressed in wavelengths: 0.20, 0.45, 0.60, and 0.80. Rank them according to the amplitude of the resultant wave, greatest first.

$$y'_m = |2y_m \cos(\frac{1}{2}\phi)|$$

$$\phi = 2\pi \times wavelength\ difference$$

Answer: 0.2 and 0.8, 0.60, and 0.45,

Summary

- Wave Speed v on a stretched String

$$v = \sqrt{\frac{\tau}{\mu}}$$

- Average Power:

$$P_{avg} = \frac{1}{2}\mu v \omega^2 y_m^2$$

- Superposition of Waves: Sum of the displacement
- Interference of Waves with same k and ω:
  - Same direction with same amplitude: **Traveling Sinusoidal Wave**

$$y' = \left[2y_m \cos\left(\frac{1}{2}\emptyset\right)\right] \sin(kx - \omega t + \frac{1}{2}\emptyset)$$