Lecture 25

§1 Surfaces and areas

1. Surfaces

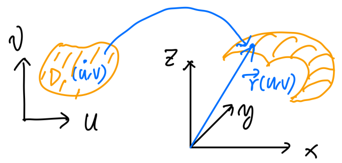
Def: A surface (in $\mathbb{R}^3$) is the range of a continuous function $\vec{r}: D \to \mathbb{R}^3$ where D is a 2-dimensional connected subset of $\mathbb{R}^2$:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D.$$

注: 1° $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ 为平面的 parametric equations

2° D 被称为 parameter domain.

3° $\vec{r}$ 在 D 内部通常是 one-to-one 的



例: Example

(a) Find two different parametric representations for the cone given by

$$z = \sqrt{x^2 + y^2}.$$

(b) Find a parametric representation for the sphere given by

$$x^2 + y^2 + z^2 = a^2,$$

where $a > 0$.

(c) Do the same for the cylinder $x^2 + (y-3)^2 = 9$, $0 \le z \le 5$.

(a) ① $x = u$, $y = v$, $z = \sqrt{u^2+v^2}$

② $x = r\cos\theta$, $y = r\sin\theta$, $z = r$, $r \ge 0$, $0 \le \theta \le 2\pi$

(b) $$\begin{cases} x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{cases}, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

(c) $x^2 + (y-3)^2 = 9$

$\Rightarrow x^2 + y^2 - 6y = 0$

$\Rightarrow r^2 - 6r\sin\theta = 0$

$\Rightarrow r = 6\sin\theta$, $0 \le \theta \le \pi$

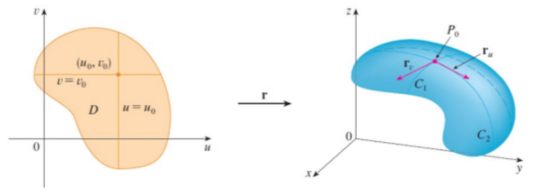
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} x = 6\sin\theta\cos\theta \\ y = 6\sin\theta\sin\theta \\ z = z \end{cases}$$

2. Tangent plane to parametric surface

Let S be a parametric surface given by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle,$$

~~where x, y and z have continuous partial derivatives.~~ Let $P_0 = \mathbf{r}(u_0, v_0)$ be a point in S. If we hold u constant by setting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a parametrization (with parameter v) of a curve C_1 lying on S, as shown below.



A tangent vector to C_1 at P_0 is given by

$$\mathbf{r}_v := \left\langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right\rangle.$$

Similarly, by holding v constant, $\mathbf{r}(u, v_0)$ gives a curve C_2 lying on S, and its tangent vector at P_0 is

$$\mathbf{r}_u := \left\langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right\rangle.$$

Def: • A parametric surface S is called smooth if $\vec{r}_u$ and $\vec{r}_v$ are cts, and $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ for every point in the interior of the parameter domain. The tangent plane to a smooth parametric surface at a point is the plane through the point with normal vector $\vec{r}_u \times \vec{r}_v$.

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v), \frac{\partial z}{\partial v}(u,v) \right\rangle$$

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v) \right\rangle$$

$$\vec{n} = \vec{r}_v \times \vec{r}_u$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v) & \frac{\partial z}{\partial v}(u,v) \\ \frac{\partial x}{\partial u}(u,v), & \frac{\partial y}{\partial u}(u,v), & \frac{\partial z}{\partial u}(u,v) \end{vmatrix}$$

$$= \left(\frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\cdot\frac{\partial y}{\partial u}\right)\vec{i} + \left(\frac{\partial z}{\partial v}\cdot\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}\cdot\frac{\partial z}{\partial u}\right)\vec{j} + \left(\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\cdot\frac{\partial x}{\partial u}\right)\vec{k}$$

3. Surface areas: parametric forms

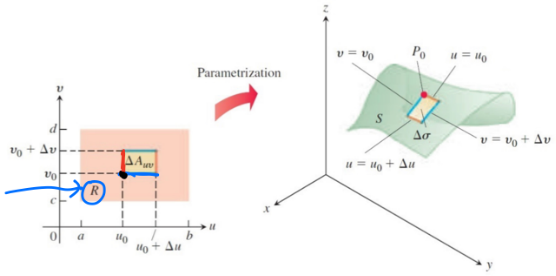
Surface Areas: Parametric Forms

Suppose $\vec{r}$ is one-to-one in the interior of the parameter domain D.

Let S be a smooth parametric surface. The surface area of S is equal to the sum of the areas of many small subregions, as shown below. The area $\Delta\sigma$ of each subregion is approximately the area of a parallelogram, which is

$$|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.$$

Parameter domain; may not be a rectangle in general



$\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0)$

$\approx \vec{r}_v \Delta v$

$\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$

$\approx \frac{\partial \vec{r}}{\partial u}(u_0, v_0)\Delta u$

$= \vec{r}_u \Delta u$

By adding terms of the form $|\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$ and taking limit as $\|P\| \to 0$, we define the surface area of S to be $\iint_S d\sigma$, where

Area of a small region of S

$$\iint_S d\sigma := \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv,$$

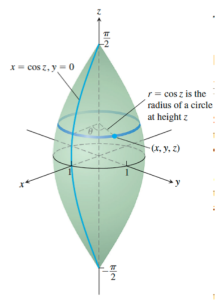
and R is the parameter domain.

注：曲面的面积元素可表示为

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$
$$= \sqrt{EG - F^2}\, du\, dv$$

其中
$$\begin{cases} E = x_u^2 + y_u^2 + z_u^2 \\ F = x_u x_v + y_u y_v + z_u z_v \\ G = x_v^2 + y_v^2 + z_v^2 \end{cases}$$

例：e.g. Parametrize the following surface "American football", and find its area.



- Parametrization is
$$\begin{cases} x = \cos u \cos v \\ y = \cos u \sin v \\ z = u \end{cases} \qquad 0 \le v \le 2\pi, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$$

- Surface area $= \iint_S d\sigma = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv$

- $\vec{r}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle = \langle -\sin u \cos v, -\sin u \sin v, 1 \rangle$

  $\vec{r}_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle = \langle -\cos u \sin v, \cos u \cos v, 0 \rangle$

  $|\vec{r}_u \times \vec{r}_v| = \cos u \sqrt{1 + \sin^2 u}$

- Area $= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos u \sqrt{1 + \sin^2 u}\, du\, dv$

  $= 2\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$

4. Surface areas: Explicit form: $z=f(x,y)$

对于一个由 $z=f(x,y)$, $(x,y)\in R$ 给出的 smooth surface S，有

$$x=u,\quad y=v,\quad z=f(u,v),\quad (u,v)\in R$$

此时 $\vec{n}=\vec{r_x}\times\vec{r_y}=\langle -f_x,-f_y,1\rangle$，surface area of S 为

$$A=\iint_R\sqrt{f_x^2+f_y^2+1}\,dx\,dy$$

R 为 f 的 domain (i.e. S 在 xy-plane 上投影)

例：

Example

Let S be the surface given by $z=\sqrt{x^2+y^2}$, $z\le 2$.

(a) Find the tangent plane to S at $(\sqrt{2}/2,\sqrt{2}/2,1)$.

(b) Find the surface area of S.

(a) $\vec{r_x}=\langle 1,0,\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2x\rangle=\langle 1,0,\frac{x}{\sqrt{x^2+y^2}}\rangle$

$\vec{r_y}=\langle 0,1,\frac{y}{\sqrt{x^2+y^2}}\rangle$

At P_0, $\vec{r_x}=\langle 1,0,\frac{\sqrt{2}}{2}\rangle$, $\vec{r_y}=\langle 0,1,\frac{\sqrt{2}}{2}\rangle$

$$\vec{r_x}\times\vec{r_y}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&\frac{\sqrt{2}}{2}\\0&1&\frac{\sqrt{2}}{2}\end{vmatrix}=\langle -\tfrac{\sqrt{2}}{2},-\tfrac{\sqrt{2}}{2},1\rangle$$

Since $(0,0,0)\in S$, (切平面恰好过 $(0,0,0)$)

tangent plane: $-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y+z=0$

(b) $A=\iint_R|\vec{r_x}\times\vec{r_y}|\,dx\,dy$

$=\iint_R\sqrt{2}\,dx\,dy$

$=\int_0^{2\pi}\int_0^2\sqrt{2}\,r\,dr\,d\theta$

$=4\sqrt{2}\pi$

5. Surface areas: Implicit form

利用隐函数求导公式 $\frac{dz}{dx}=-\frac{F_x}{F_z}$, $\frac{dz}{dy}=-\frac{F_y}{F_z}$

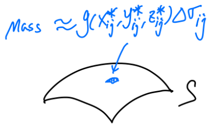
§2 Surface integrals (第一类曲面积分)

1. Motivation: mass

Surface Integrals

Let S be a smooth surface. Suppose that the planar density of of S at (x,y,z) is given by $g(x,y,z)$, say kg/m^2. Then the mass of S is given by the surface integral of g,

$$\iint_S g(x,y,z)\,d\sigma,$$



where $\iint_S g(x,y,z)\,d\sigma$ is the limit of $\sum_k g(x_k^*,y_k^*,z_k^*)\Delta\sigma_k$ as the areas of the small regions $(\Delta\sigma_k)$ goes to 0.

2. 第一类曲面积分的计算：parametric form

$$\iint_S g(x,y,z)\,d\sigma=\iint_R g(\vec{r}(u,v))\,|r_u\times r_v|\,du\,dv$$

R 为 parameter domain

If S is in parametric form given by $(x, y, z) = \mathbf{r}(u, v)$, then

$$\iint_S g(x,y,z)d\sigma = \iint_R g(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv,$$

where R is the parameter domain.

3、第一类曲面积分的计算：explicit form

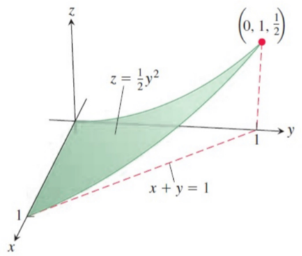
$$\iint_S g(x,y,z)d\sigma = \iint_R g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dxdy$$

If S is in explicit form $z = f(x, y)$, where f is defined on R, then

$$\iint_S g(x,y,z)d\sigma = \iint_R g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dx\,dy.$$

例：Example

Evaluate $\iint_S \sqrt{x(1+2z)}\,d\sigma$ on the portion of the surface $z = y^2/2$ over the triangular region R: $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the xy-plane.



$$\begin{aligned}
I &= \iint_R \sqrt{x(1+y^2)}\sqrt{0+y^2+1}\,dxdy \\
&= \iint_R \sqrt{x}\,(1+y^2)\,dxdy \\
&= \int_0^1\int_0^{1-x} \sqrt{x}(1+y^2)\,dydx \\
&= \int_0^1 \sqrt{x}\cdot\left(y+\tfrac{1}{3}y^3\right)\Big|_0^{1-x}dx \\
&= \int_0^1 \sqrt{x}\cdot\left(1-x+\tfrac{1}{3}(1-x)^3\right)dx \\
&= \int_0^1 -\tfrac{1}{3}x^{\frac{7}{2}}+x^{\frac{5}{2}}-2x^{\frac{3}{2}}+x^{\frac{1}{2}}\,dx \\
&= \frac{284}{945}
\end{aligned}$$

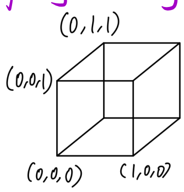
4. Piecewise - smooth surfaces（分段光滑曲面）

令 $S = S_1 \cup S_2 \cup \cdots \cup S_n$，其中

- 每个 S_i 均 smooth
- $S_i \cap S_j$ 为 $\varnothing$，一个点，或一条曲线

则 $\iint_S f(x,y,z)\,d\sigma = \sum_{i=1}^{n}\iint_{S_i} f(x,y,z)\,d\sigma$

例：令 $f(x,y,z) = xyz$ 且 S 为 cube surface．求 $I := \iint_S f(x,y,z)\,d\sigma$



- There are six faces, corresponding to $S_1: x=0$, $S_2: x=1$, $S_3: y=0$, $S_4: y=1$, $S_5: z=0$, $S_6: z=1$.
- $\iint_{S_1} f(x,y,z)\,d\sigma = \iint_{S_3} f(x,y,z)\,d\sigma = \iint_{S_5} f(x,y,z)\,d\sigma = 0$

  Since $f(x,y,z)=0$ on these faces.
- For S_6, by explicit form, we have

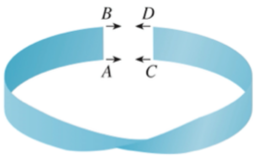
  $$\iint_{S_6} f(x,y,z)\,d\sigma = \int_0^1\int_0^1 xy\sqrt{0^2+0^2+1}\,dx\,dy = \frac{1}{4}$$
- By symmetry, $\iint_{S_2} = \iint_{S_4} = \frac{1}{4}$.

  Hence $I = 0\cdot 3 + \frac{1}{4}\cdot 3 = \frac{3}{4}$

§2 Surface integrals (第二类曲面积分)

1. Orientable surfaces (双侧曲面)

- Roughly speaking, these are surfaces with two sides.
- 不论P沿怎样的路径在S上连续移动(不跨越S的边缘), 当P返回其起始点时, $\vec{n}(P)$的指向没有改变.
- 并非所有 surface 都 orientable. e.g. the Möbius strip.



2. Motivation: flux

Suppose that some fluid with constant density ρ passes through some planar region D with area A, at a constant velocity v perpendicular to D. The mass flow rate of the fluid across D is given by

$$\rho v A, \qquad \frac{kg}{m^3}\cdot\frac{m}{sec}\cdot m^2 = \frac{kg}{sec}$$

and it measures the mass of fluid passing through D per unit time.

If the fluid passes through D with a velocity $\mathbf{v}$ which is not perpendicular to D, then we need to consider the component of $\mathbf{v}$ in the direction of $\mathbf{n}$, where $\mathbf{n}$ is a unit normal vectors of D. In this case, the mass flow rate is equal to

$$\rho(\mathbf{v}\cdot\mathbf{n})A.$$

If $\mathbf{v}$ is a velocity field of some fluid, which has density function $\rho(x,y,z)$, then the mass flow rate of the fluid across an orientable surface S (which is a surface that has "two sides") is given by

$$\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,d\sigma := \iint_S \rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\,d\sigma.$$

Here $\vec{n}$ has a fixed orientation (i.e., always points to the same side of S).

3. Definition: flux of $\vec{F}$ across S (第二类曲面积分)

Definition
Let $\mathbf{F}$ be a continuous vector field in $\mathbb{R}^3$, defined on an orientable surface S, with a unit normal vector $\mathbf{n}$ specifying the orientation of S. The surface integral of $\mathbf{F}$ over S, or the flux of $\mathbf{F}$ across S, is defined to be

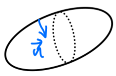
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

i.e., $\vec{n}$ corresponds to a "side" of the surface.

- $\vec{n} = \vec{n}(x,y,z)$
- $\vec{n}$ specifies 曲面的一边与 flux 的 measuring direction.

e.g.

"upward flux"



"inward flux"

注: $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S \vec{F}\cdot d\vec{\sigma} = \iint_S M\,dydz + N\,dxdz + P\,dxdy$

4. Flux: computation

If S is a surface with parametrization $\mathbf{r}(u,v)$, then the unit normal vectors to S at a point are

$$\mathbf{n} = \pm\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}.$$

Here, the sign is determined by the context or question.

Suppose that the flux direction is given by the one with "+". Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\,|\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv$$
$$= \iint_R \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv. \quad (1)$$

若 S 由 $z=f(x,y)$, $(x,y)\in R$ 给出, 则

$\vec{r}_x = \langle 1, 0, f_x\rangle$, $\vec{r}_y = \langle 0, 1, f_y\rangle$

$\vec{r}_x\times\vec{r}_y = \langle -f_x, -f_y, 1\rangle$

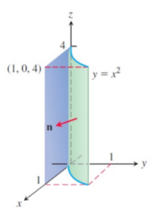
第二类曲面积分的计算变为

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R \vec{F}\cdot(\vec{r}_x\times\vec{r}_y)\,dxdy$$
$$= \iint_R (-Mf_x - Nf_y + P)\,dxdy$$

例: **Example**
Find the flux of $\mathbf{F} := \langle yz, x, -z^2\rangle$ across the surface

$$S := \{(x,y,z) : y = x^2,\ 0\le x\le 1,\ 0\le z\le 4\},$$

in the direction of $\mathbf{n}$ indicated in the figure.



- $\vec{r}_x = \langle 1, 2x, 0\rangle$
  $\vec{r}_z = \langle 0, 0, 1\rangle$

$\vec{r}_x \times \vec{r}_z = \langle 2x, -1, 0 \rangle$

- $\vec{r}_x \times \vec{r}_z \big|_{(0,0,0)} = \langle 0, -1, 0 \rangle$

same side as picture, take "+"

- $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$

$= \iint_D \langle x^2 z, x, -z^2 \rangle \cdot \langle 2x, -1, 0 \rangle \, dx\,dz$

$= \int_0^4 \int_0^1 2x^3 z - x \, dx\,dz$

$= 2$

例: Find the inward flux of $\vec{F} = \langle z, y, x \rangle$ across the sphere $x^2+y^2+z^2=1$.

- $\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases}$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$\vec{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle$

$\vec{r}_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle$

- look at $(1,0,0)$, $\theta = 0$ & $\phi = \frac{\pi}{2}$

$\vec{r}_\phi \times \vec{r}_\theta = \langle 1, 0, 0 \rangle \Rightarrow$ take "$-$"

- $\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_D - \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta \rangle \cdot \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle \, d\phi\, d\theta$

$= -\int_0^\pi \int_0^{2\pi} 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta \, d\theta\, d\phi$

$= -\int_0^\pi \pi \sin^3\phi \, d\phi$

$= -\frac{4}{3}\pi$