Lecture 14

§1 Non-homogeneous Poisson process

标准 Poisson process (λ) 具有无记忆性，且 interarrival times 为 i.i.d. $Exp(\lambda)$.

由于这是一个强 assumption，我们需要一个不具有无记忆性的 more general / flexible model

1. **Definition:** Non-homogeneous Poisson process（非齐次泊松过程）

   一个 counting process $\{N_t : t \geq 0\}$ 被称为 a non-homogeneous Poisson process with a continuous intensity function $\lambda(t) : t \geq 0$，若其满足

   ① $N(0) = 0$

   ② $\{N_t, t \geq 0\}$ 有 independent increments

   ③ 对于足够小的 positive $h \geq 0$，有

   - $P\{N_{t+h} - N_t \geq 2\} = o(h)$
   - $P\{N_{t+h} - N_t = 1\} = \lambda(t) h + o(h)$
   - $(P\{N_{t+h} - N_t = 0\} = 1 - \lambda(t) h + o(h))$（若前两条满足，则自然满足）

   注：类比 Poisson process，有 $N_{t+h} - N_t \sim Ber(\lambda(t) \cdot h)$，唯一的区别在于用 $\lambda(t)$ 替换 λ

2. **Lemma:** Poisson approximation (non-homogeneous version)

   令 $\{X_i\}_{1 \leq i \leq n}$ 为 a sequence of independent $Ber(p_i)$ ($Ber(p)$) r.v.'s with respective success probability (p_i)，并假设

   $$p_1 + \cdots + p_n \to \lambda > 0 \quad (p_n \to \lambda > 0), \quad \text{when } n \to \infty$$

   则 $S_n = X_1 + \cdots + X_n$ converges to $Poi(\lambda)$, when $n \to \infty$

3. **Theorem:** increments 的 distribution

   定义 $m_t = \int_0^t \lambda(u) du$, for $t \geq 0$.

   对于一个 non-homogeneous Poisson process with intensity function $\lambda(t)$，对于 $0 \leq s < t$，有

   $$N_t - N_s \sim Poi(m(t) - m(s)) = Poi\left(\int_s^t \lambda(u) du\right)$$

   注：① The increment distribution 不仅取决于 length $t-s$，还取决于起始时间 s 和结束时间 t，因此 increment 不为 stationary

   ② interarrival time 不服从指数分布

   ③ 若 $s=0$，则有 $N_t \sim Poi(m(t))$，此时 $m(t)$ 被称为 the mean value function of the non-homogeneous Poisson process

   证明：

   $$N(t) - N(0) = \sum_{i=1}^{n} (N(t_i) - N(t_{i-1}))$$

   由于 $N(t_i) - N(t_{i-1}) \sim Ber(\lambda(t_{i-1}) \cdot (t_i - t_{i-1}))$

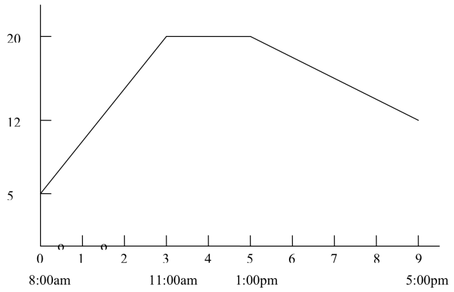
   $$\sum_{i=1}^{n} \lambda(t_{i-1}) \cdot (t_i - t_{i-1}) \to \int_0^t \lambda(u) du \quad \text{as } n \to \infty$$

   因此，$N(t) - N(0) \sim Poi\left(\int_0^t \lambda(u) du\right)$

   $$N(t) - N(s) \sim Poi\left(\int_s^t \lambda(u) du\right)$$

例1: **Siegbert runs a hot dog stand** that opens at 8 A.M. and

- From 8 until 11 A.M. customers seem to arrive, on the average, at a steadily increasing rate that starts with an initial rate of 5 customers per hour at 8 A.M. and reaches a maximum of 20 customers per hour at 11 A.M.
- From 11 A.M. until 1 P.M. the (average) rate seems to remain constant at 20 customers per hour.
- However, the (average) arrival rate then drops steadily from 1 P.M. until closing time at 5 P.M. at which time it has the value of 12 customers per hour.



$$\lambda(t)=\begin{cases}5+5t, & t\in[0,3],\\ 20, & t\in[3,5],\\ 20-2(t-5), & t\in[5,9].\end{cases}$$

$$\int_{0.5}^{1.5}\lambda(t)dt=\int_{0.5}^{1.5}(5+5t)dt=10.$$

- If we assume that the numbers of customers arriving at Siegbert's stand during disjoint time periods are independent, then what is a good probability model for the preceding?
- What is the probability that no customers arrive between 8:30 A.M. and 9:30 A.M. on Monday morning?
- What is the expected number of arrivals in this period?

① $\int_{0.5}^{1.5}\lambda(t)\,dt=\int_{0.5}^{1.5}(5+5t)\,dt=5t+\frac{5}{2}t^2\Big|_{0.5}^{1.5}=10$

$N_{1.5}-N_{0.5}\sim Poi(10)$

$P(N_{1.5}-N_{0.5}=0)=\dfrac{10^0\cdot e^{-10}}{0!}=e^{-10}$

② $E[N_{1.5}-N_{0.5}]=10$

例2: 6. The capital of a bank grows proportionally with time t: at time t, the bank holds ct units of capital, where c is a positive constant. The bank has to undergo stress tests, which occur according to a Poisson process with rate λ. The bank passes a stress test if it holds at least a certain amount of capital at the moment of the test. The required amounts of capitals are independently distributed as Pareto distribution, with probability density function given by:

$$f(y)=\frac{24}{(2+y)^4},\quad y>0.$$

(a) Suppose that a stress test occurs at time s. Determine the probability that the bank fails this stress test. **[4 marks]**

$8(2+cs)^{-3}$

(b) Let $N(t)$ denote the number of stress tests the bank fails before time t. Find the distribution of $N(t)$. **[7 marks]**

$N(t)\sim Poi\left(\frac{\lambda}{c}\left(1-\frac{4}{(2+ct)^2}\right)\right)$

(c) Determine the probability that the bank can pass all the stress tests which have ever occurred. **[4 marks]**

$e^{-\lambda/c}$

[Total: 15 marks]

(a) $p(s)=P(Y>cs)=\int_{cs}^{\infty}24(2+y)^{-4}dy=\dfrac{8}{(2+cs)^3}$

(b) ① $N(0)=0$

② independent increment 显然满足

③ 令 $\bar{N}(t)$ 表示 stress test 的出现数，则

$N(t+h)-N(t)\le\bar{N}(t+h)-\bar{N}(t)$

$\cdot\ P(N(t+h)-N(t)\ge 2)\le P(\bar{N}(t+h)-\bar{N}(t)\ge 2)=o(h)$

$$P(N(t+h)-N(t)=1) = P(\bar{N}(t+h)-\bar{N}(t)=1,\ fail)$$
$$= P(\bar{N}(t+h)-\bar{N}(t)=1)\cdot P(fail \mid \bar{N}(t+h)-\bar{N}(t)=1)$$
$$= (\lambda h + o(h))\cdot p(t)$$
$$= \lambda p(t) h + o(h)$$

因此 $\{N_t\}$ 为一个 non-homogeneous Poisson process with intensity function $\lambda(t) = \lambda p(t)$

因此 N_t 服从泊松分布，parameter 为

$$m_t = \int_0^t \lambda(s)\,ds = \int_0^t \lambda\cdot\frac{8}{(2+cs)^3}\,ds = -\frac{4\lambda}{c(2+cs)^2}\Big|_0^t = \frac{\lambda}{c}\left\{1-\frac{4}{(2+ct)^2}\right\}$$

即

$$N_t \sim Poi\left(\frac{\lambda}{c}\left\{1-\frac{4}{(2+ct)^2}\right\}\right)$$

(c) $P(N_t=0) = e^{-m(t)}$

$$\lim_{t\to\infty} P(N_t=0) = \lim_{t\to\infty} \exp\left\{-\frac{\lambda}{c}\left\{1-\frac{4}{(2+ct)^2}\right\}\right\} = e^{-\frac{\lambda}{c}}$$