Lecture 23

§1 Integration by Partial Fractions : General Approach

1. Definition of degree

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_1 x + a_0$ with $a_n \neq 0$

then $\deg(P(x)) = n$

Here $\deg(P(x))$ is called the degree of $P(x)$

2. Long division

If $\deg(P(x)) \geq \deg(Q(x))$, then by long division, we can find polynomials $S(x)$ and $R(x)$ such that

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

with $\deg(R(x)) < \deg(Q(x))$.

e.g. $\dfrac{x^3-4x^2+2x-3}{x+2}$

$$\begin{array}{r|l}
 & \quad x^2 - 6x + 14 \\ \hline
x+2 & x^3 - 4x^2 + 2x - 3 \\
 & x^2 + 2x^2 \\ \hline
 & \quad -6x^2 + 2x \\
 & \quad -6x^2 - 12x \\ \hline
 & \qquad\quad 14x - 3 \\
 & \qquad\quad 14x + 28 \\ \hline
 & \qquad\qquad -31
\end{array}$$

So, $\dfrac{x^3-4x^2+2x-3}{x+2} = x^2 - 6x + 14 - \dfrac{31}{x+2}$

3. Definition of irreducible quadratic polynomial

Quadratic polynomial ax^2+bx+c is called irreducible if it has no real root, i.e. if $b^2-4ac<0$.

e.g. x^2+1 is irreducible in R

4. A fact

Every non-zero polynomial $Q(x)$ can be written as:

$$Q_1(x)\cdot Q_2(x)\cdots\cdots Q_k(x)$$

where each $Q_i(x)$ has one of the following forms:

1° $a_i x+b_i$, $a_i\neq 0$

2° $a_i x^2+b_i x+c_i$, $a_i\neq 0$, $b_i^2-4a_ic_i<0$

e.g. $$x^4+1 = x^4+2x^2+1-2x^2$$
$$=(x^2+1)^2-2x^2$$
$$=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

5. General approach

Now we look at $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x))<\deg(Q(x))$.

1° Case 1: $Q(x)$ is a product of distinct linear factors.

$$Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots\cdots(a_kx+b_k).$$

In this case, there exist constants $A_1, A_2, \cdots\cdots, A_k$ such that

$$\frac{P(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots\cdots+\frac{A_k}{a_kx+b_k}$$

e.g. Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

Step 1: Factorize the denominator

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(x+2)(2x-1)$$

Step 2: Split the fraction up, keeping A, B, C as "undetermined coefficients"

$$\frac{x^2+2x-1}{2x^3+3x^2-2x}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{2x-1}$$

Step 3: Find the value of A, B, C

$$x^2+2x-1 = A(x+2)(2x-1)+Bx(2x-1)+Cx(x+2)$$
$$= (2A+2B+C)x^2+(3A-B+2C)x-2A$$

$$\begin{cases} 2A+2B+C=1 \\ 3A-B+2C=2 \\ -2A=-1 \end{cases} \Longrightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{10} \\ C=\frac{1}{5} \end{cases}$$

Step 4: Integrate the fraction

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} = \int \frac{1}{2x}\,dx - \int \frac{1}{10(x+2)}\,dx + \int \frac{1}{5(2x-1)}\,dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{10}\ln|2x-1| + C$$

2° Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose that a linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ appears in the factorization of $Q(x)$.

In this case, instead of the single term $A_1/(a_1x+b_1)$, we should use

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots\cdots + \frac{A_r}{(a_1x+b_1)^r}$$

e.g. Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Step 1: Applying long division

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

Step 2: To split up $\frac{4x}{x^3-x^2-x+1}$, first factorize the dominator

$$x^3-x^2-x+1 = x^2(x-1)-(x-1)$$
$$=(x+1)(x-1)^2$$

Step 3: Split up the fraction

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Step 4: Find the coefficients A, B, C

$$4x = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$\begin{cases} A+C=0 \\ B-2C=4 \\ -A+B+C=0 \end{cases} \Longrightarrow \begin{cases} A=1 \\ B=2 \\ C=-1 \end{cases}$$

Step 5: Solve the integral of the fraction

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = \int \frac{dx}{x-1} + \int \frac{2dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$= \ln|x-1| + (-2)(x-1)^{-1} - \ln|x+1| + C$$

$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

Step 6: Solve the integral of the whole expression

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{4x}{x^3-x^2-x+1}\,dx$$

$$= \frac{1}{2}(x+1)^2 + \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

3° Case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has a factor ax^2+bx+c, where $b^2-4ac<0$.

In this case, the expression of $P(x)/Q(x)$ will have a term of

$$\frac{Ax+B}{ax^2+bx+c}$$

e.g. Find $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

Step 1: Split up the fraction

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Step 2: Find the coefficients

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

Solving given $A=2$, $B=1$, $C=-2$, $D=1$

Step 3 Solve the integral

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x\cdot dx}{x^2+1} + \int \frac{dx}{x^2+1} - \int \frac{2dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

$$= \ln|x^2+1| + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C$$

4° Case 4: Q(x) contains a repeated irreducible quadratic factor.

If $Q(x)$ has a factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of a single term $Ax+B/(ax^2+bx+c)$, we should use:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots\cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

e.g. Find $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx = \int \frac{(x^2-2x+2)+(2x-1)}{(x^2-2x+2)^2}\,dx$$

$$= \int \frac{1}{x^2-2x+2}\,dx + \int \frac{2x-1}{(x^2-2x+2)^2}\,dx$$

$$= \underbrace{\int \frac{1}{x^2-2x+2}\,dx}_{J} + \underbrace{\int \frac{2x-2}{(x^2-2x+2)^2}\,dx}_{K} + \underbrace{\int \frac{1}{(x^2-2x+2)^2}\,dx}_{L}$$

$$J = \int \frac{1}{(x-1)^2+1}\,dx = \arctan(x-1) + C_1$$

$$K = \int \frac{1}{(x^2-2x+2)^2}\,d(x^2-2x+2) = -(x^2-2x+2)^{-1} + C_2$$

$$L = \int \frac{1}{((x-1)^2+1)^2}\,dx$$

Let $x-1 = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$$\begin{aligned} L &= \int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta \\ &= \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta \\ &= \int \cos^2\theta\, d\theta \\ &= \int \frac{1+\cos 2\theta}{2}\,d\theta \\ &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C_3 \\ &= \frac{1}{2}(\theta + \tan\theta\cdot\cos^2\theta) + C_3 \\ &= \frac{1}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) + C_3 \\ &= \frac{1}{2}\left(\arctan(x-1) + \frac{x-1}{x^2-2x+2}\right) + C_3 \end{aligned}$$

$$\begin{aligned} \int \frac{x^2+1}{(x^2-2x+2)^2}\,dx &= J + K + L \\ &= \arctan(x-1) - \frac{1}{x^2-2x+2} + \frac{1}{2}\left(\frac{x-1}{x^2-2x+2} + \arctan(x-1)\right) + C \end{aligned}$$

§2 Integration by Partial Fractions : Finding Undetermined Coefficients

1、Heaviside "cover-up" method

In case 1 where

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)\cdots\cdots(x-r_n)} = \frac{A_1}{x-r_1} + \cdots\cdots + \frac{A_n}{x-r_n}$$

Multiplying both sides by $(x-r_1)$

$$\frac{f(x)}{(x-r_2)\cdots\cdots(x-r_n)} = A_1 + (x-r_1)\left(\frac{A_2}{x-r_2} + \cdots\cdots + \frac{A_n}{x-r_n}\right)$$

Substituting $x = r_1$ yields:

$$A_1 = \frac{f(r_1)}{(r_1-r_2)+\cdots+(r_1-r_n)}$$

(cover up $(x-r_1)$, then substitute $x = r_1$)

In general, to find A_i we remove $x-r_i$ from the bottom of the fraction, then substitute in $x = r_i$.

e.g. $$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$A = \frac{0^2+2\times 0-1}{(2\times 0-1)(0+2)} = \frac{1}{2}$$

$$B = \frac{(\frac{1}{2})^2+1-1}{1/2\,(\frac{1}{2}+2)} = \frac{1}{5}$$

$$C = \frac{(-2)^2+2\times(-2)-1}{-2\times(-2\times 2-1)} = -\frac{1}{10}$$

2. Differentiating method

e.g. $$\frac{f(x)}{(x-r)^3} = \frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3}$$ where $\deg(f(x)) \leq 2$

Multiple by $(x-r)^3$:

$$f(x) = A(x-r)^2 + B(x-r) + C \quad ①$$

Set $x = r$:

$$f(r) = C$$

Differentiate ①:

$$f'(x) = 2A(x-r) + B \quad ②$$

Set $x = r$:

$$f'(r) = B$$

Differentiate ②:

$$f''(x) = 2A$$

Since $\deg(f(x)) \leq 2$, $f''(x)$ is a constant K, so

$$A = \frac{f''(x)}{2} = \frac{K}{2}$$