1 Elasticity & Equilibrium

§1 Stress (应力) and strain (应变)

1. Stress: 单位面积上的受力情况: $\frac{F}{A}$ (N/m^2)
2. Strain: 物体在外力作用下的相对形变, Stress = Modulus × strain

§2 Tension and compression

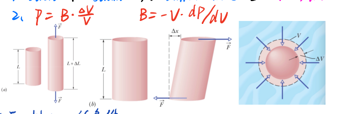
1. stress: $\frac{F}{A}$ strain: $\frac{\Delta L}{L}$ Young's modulus: E (N/m^2, Pa)
2. $F/A = E \cdot \Delta L/L$

§3 Shearing

1. stress: F/A strain: $\frac{\Delta x}{L}$ shear modulus: G (N/m^2, Pa)
2. $F/A = G \cdot \Delta x/L$

§4. Hydraulic stress

1. stress: P strain: $\Delta V/V$ Bulk modulus: B (N/m^2, Pa)
2. $P = B \cdot \frac{\Delta V}{V}$ $B = -V \cdot dP/dV$



§5 Equilibrium 的条件

① $\vec{F}_{net} = \frac{d\vec{P}}{dt} = 0$ ② $\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0$ (以任意点测量)

2. Gravitation

§1 Gravitation

1. Newton's Law: $F = G\frac{m_1 m_2}{r^2}$
gravitational constant $G = 6.67\times10^{-11} N\cdot m^2/kg^2$ ($m^3/kg\cdot s^2$)
2. Shell theorem 1
一个均匀的shell,吸引其外部的一个质点,等效于将该shell的质量集中于球心,该球心吸引另一质点.
3. Principle of superposition
$\vec{F}_{net} = \sum_{i=2}^{n} \vec{F}_{1i}$ / $\vec{F}_1 = \int d\vec{F}$
4. Gravitation near Earth's surface
$F = G\frac{Mm}{r^2} \Rightarrow a_g = \frac{GM}{r^2}$
$ma_g - F_N = m\omega^2 R$ 其中 F_N (=mg) 为 measured weight
$\Rightarrow g = a_g - \omega^2 R$
5. Shell theorem 2
一个均匀 shell 对其内部一点,施加的 gravitational force 为 0

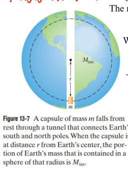


Figure 13-7 A capsule of mass m falls from rest through a tunnel that connects Earth's south and north poles. When the capsule is at distance r from Earth's center, the portion of Earth's mass that is contained in a sphere of that radius is M_{ins}.

The magnitude of the gravitation force is
$$F = \frac{GmM_{ins}}{r^2}$$
We assume that the density if uniform
$$\rho = \frac{M_{ins}}{\frac{4}{3}\pi r^3} = \frac{M}{\frac{4}{3}\pi R^3}$$
The inner mass is
$$M_{ins} = \frac{4}{3}\pi r^3 \rho = \frac{M}{R^3}r^3.$$
We have
$$F = \frac{GmM}{R^3}r,$$
In the vector form
$$\vec{F} = -K\vec{r},$$

6. Gravitational potential energy
1° 靠近地表: 重力 conservative, difference = mgh
2° 远离地表: reference 为无穷远处. At $r\to\infty$, U=0
对两点组成的系统, $U = -GMm/r$

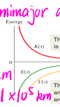
证: The work done by the gravitational force is
$$W = \int_R^\infty \vec{F}(r)\cdot d\vec{r}.$$
For the dot product,
$$\vec{F}(r)\cdot d\vec{r} = F(r)\,dr\cos\phi,$$
Since $\phi = 180^o$,
$$\vec{F}(r)\cdot d\vec{r} = -\frac{GMm}{r^2}dr,$$
We have
$$W = -GMm\int_R^\infty \frac{1}{r^2}dr = \left[\frac{GMm}{r}\right]_R^\infty = 0 - \frac{GMm}{R} = -\frac{GMm}{R}.$$
The mechanical energy is conservative,
$$U_\infty - U = -W.$$
Taking $U_\infty = 0$, we have
$$U = W = -\frac{GMm}{R}.$$

* 三质点系统: $U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_2m_3}{r_{23}} + \frac{Gm_1m_3}{r_{13}}\right)$
3° Path independence: $\Delta U = U_f - U_i = -W$
4° Escape speed (脱离速度): $K+U=0 \Rightarrow V = \sqrt{\frac{2GM}{R}}$

§2 Planets and satellites

1. Kepler's Laws: Law of orbits
所有行星绕太阳运动的轨道都是椭圆,太阳在椭圆的一个焦点上.
2. Kepler's Laws: Law of areas
对于一个行星,其与太阳连线在相等时间间隔内扫过面积相等
① $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega$ ② $L = rp_\perp = rmv_\perp = rm\omega r = m\omega r^2$
①+② ⇒ $dA/dt = L/2m$ (L: 角动量)
3. Kepler's Laws: Law of periods
所有行星绕太阳一周的恒星时间 (T_i) 的平方与它们轨道半长轴的立方成正比 $T_1^2/T_2^2 = a_1^3/a_2^3$
4. Satellites: orbits and energy (圆轨道)
· 势能: $U = -GMm/r$
· 动能: $GMm/r^2 = mv^2/r \Rightarrow K = GMm/2r = -U/2$
· 机械能: $E = K+U = -GMm/2r = -K$
椭圆轨道: $E = -GMm/2a$ (a: semimajor axis)

* 地球: 质量: 5.97×10^{24}kg 半径: 6400km
月球: 质量: 7.34×10^{22}kg 半径: 1737km
太阳: 质量: 1.99×10^{30}kg 半径: 6.96×10^5km
日地距: 1.496×10^8km 地月距: $3.6\sim4.1\times10^5$km



3. Fluid

§1 Fluid at rest

1. Pressure: $P = dF/dA$ (Pa, N/m^2) (1atm = 1.01×10^5 Pa)
2. hydrostatic pressure (由静止的流体产生)
1° Absolute pressure (深度为h): $P = P_0 + \rho gh$ (P_0: 大气压)
2° gauge pressure: ρgh (绝对压强与大气压的差值)
3° Atmospheric pressure: $P = P_0 - \rho_{air} gd$ (d: 到参考平面距离)
3. Measuring pressure
1° Mercury barometer (水银气压计): $P_0 = \rho gh$
2° Open-tube manometer: 测 gauge pressure $P_g = P - P_0 = \rho gh$
4. Pascal's principle
$P = P_{ext} + \rho gh$ / $\Delta P = \rho - \rho gh = P_{ext}$
应用: hydraulic lever (液压杆)
5. Archimedes' principle
buoyant force (浮力) $= F_b = m_f g$ (排水质量) $= F_g$ (若悬浮)
Apparent weight: $weight_{app}$ = weight − F_b

§2 Ideal fluids in motion

1. 关于 ideal fluid 的四个假设:
1° Steady flow: 流动液中任意一点的速度不随时间改变
2° Incompressible flow: ideal fluid is incompressible, 它的密度均匀不变
3° Nonviscous (无粘性的) flow: 无 viscous drag force
4° Irrotational flow: test body 不会绕着 com 旋转
2. The equation of continuity
Volume flow rate $R_V = Av$ = a constant (m^3/s)
Mass flow rate $R_m = \rho Av$ = a constant (kg/s)
3. Bernoulli's equation
$p + \frac{1}{2}\rho v^2 + \rho gh$ = constant

证:
- Work done on the system
$$W = W_g + W_p = \Delta K$$
- Work done by Gravitational Force
$$W_g = -\Delta mg(y_2 - y_1) = -\rho\Delta Vg(y_2 - y_1)$$
- Work done by the system
$$F\Delta x = pA\Delta x = p\Delta V$$
$$W_p = p_1\Delta V + (-p_2\Delta V) = -\Delta V(p_2 - p_1)$$
- K.E gained:
$$\Delta K = \frac{1}{2}\Delta mv_2^2 - \frac{1}{2}\Delta mv_1^2 = \frac{1}{2}\rho\Delta V(v_2^2 - v_1^2)$$
- Thus,
$$-\rho\Delta Vg(y_2 - y_1) - \Delta V(p_2 - p_1) = \frac{1}{2}\rho\Delta V(v_2^2 - v_1^2)$$
$$p_1 + \frac{1}{2}\rho v_1^2 + \rho gy_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho gy_2$$

4. Oscillation

§1 Simple harmonic motion

1. Displacement function: $x(t) = x_m\cos(\omega t + \phi)$
2. 物理量: ① frequency f (Hz, s^{-1}) ② period T (s): $T = \frac{1}{f}$
③ Angular frequency ω (rad/s): $\omega = \frac{2\pi}{T} = 2\pi f$
④ Amplitude x_m (m) ⑤ Phase $\omega t + \phi$ (rad, °)
⑥ Phase constant φ (rad)
3. 速度: $v(t) = -\omega x_m\sin(\omega t+\phi) = \omega x_m\cos(\omega t + \phi + \frac{\pi}{2})$
注: 速度的相位比位置方程的快 $\frac{\pi}{2}$ ($+\frac{\pi}{2}$)
velocity amplitude: $v_m = \omega\cdot x_m$, 在 x=0 时取到
4. 加速度: $a(t) = -\omega^2 x_m\cos(\omega t + \phi) = -\omega^2 x(t)$
注: ① 加速度方向始终与位移方向相反
② Acceleration amplitude: $a_m = \omega^2\cdot x_m$, 在 $x = \pm x_m$ 时取到
③ 加速度的相位比位置方程的快 π (+π)
5. 力 $\begin{cases} 牛二: F = -m\omega^2 x \\ 胡克: F = -kx \end{cases}$
⇒ ① $k = m\omega^2$ ② $\omega = \sqrt{k/m}$ ③ $T = 2\pi\sqrt{\frac{m}{k}}$

- Newton's 2nd Law & Hooke's Law
$$F = ma = -kx$$
$$ma + kx = 0$$
$$m\frac{d^2x}{dt^2} + kx = 0$$
- Solution: Let $x = Ae^{i(\omega t+\phi)}$
$$\frac{dx}{dt} = i\omega Ae^{i(\omega t+\phi)}$$
$$\frac{d^2x}{dt^2} = -\omega^2 Ae^{i(\omega t+\phi)}$$
- Put it back to Newton's and Hooke's Law
$$-\omega^2 Ae^{i(\omega t+\phi)}\times m + kAe^{i(\omega t+\phi)} = 0$$
$$\omega = \pm\sqrt{\frac{k}{m}} \to \omega = \sqrt{\frac{k}{m}}$$

6. 能量: ① 势能: $U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2(\omega t+\phi)$
② 动能: $K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2x_m^2\sin^2(\omega t+\phi)$
③ 机械能: $E = U(t) + K(t) = \frac{1}{2}kx_m^2 = \frac{1}{2}mv_m^2$

§2 Pendulum

1. Torsion pendulum (Angular simple harmonic motion)
$T = 2\pi\sqrt{\frac{I}{K}}$
通过测量不同物体的T,可以由已知物体的I求出未知物体的I
2. Simple pendulum
$\begin{cases} \alpha = -\frac{mgL}{I}\cdot\sin\theta \approx -\frac{mgL}{I}\cdot\theta \\ \alpha(t) = -\omega^2\theta(t) \end{cases} \Rightarrow \begin{cases} \omega = \sqrt{mgL/I} \\ T = 2\pi\sqrt{I/mgL} \end{cases}$
由 $I = mL^2$ 得: $T = 2\pi\sqrt{\frac{L}{g}}$, $\omega = \sqrt{\frac{g}{L}}$, $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$
3. Physical pendulum
$T = 2\pi\sqrt{I/mgh}$ (h: COG到 pivot point O 的距离)
(I: 绕O点的转动惯量)
用直尺测算g:
$\begin{cases} h = \frac{1}{2}L \\ I = I_c + md^2 = \frac{1}{3}mL^2 \end{cases} \Rightarrow \begin{cases} T = 2\pi\sqrt{2L/3g} \\ g = 8\pi^2L/3T^2 \end{cases}$

§3 Damped oscillation

1. Damping force: $F_d = -bv$ (b: damping constant (kg/s))
2. Damped oscillation:
牛二: $F_{net} = F_d + F_{spring} = ma \Rightarrow -bv - kx = ma$
$\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$
解得: $x(t) = x_m e^{-\frac{bt}{2m}}\cos(\omega' t + \phi)$
其中 Angular frequency 满足 $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
若 $b \ll \sqrt{km}$, $\omega' \approx \omega = \sqrt{\frac{k}{m}}$
Damped energy: 随时间递减
若 damping 比较小, 可以用 $x_m e^{-\frac{bt}{2m}}$ 代替 x_m
$E(t) \approx \frac{1}{2}kx_m^2 e^{-\frac{bt}{m}}$

§4 Forced oscillation and resonance

受到周期性外力而振动, 振子与driving force的角频率 ω_d 相同
$x(t) = x_m\cos(\omega_d t + \phi)$
当 $\omega_d = \omega$ 时, displacement amplitude x_m (approximately) 与 velocity amplitude v_m 到达最大值 (resonance)

Newton's 2nd Law and Hooke's Law
$$F_{net} = ma$$
When the oscillator is driven with a sinusoidal Force
$$F(t) = F_0\cos(\omega_d t)$$
Then,
$$F(t) + (-kx) = ma$$
$$m\frac{d^2x}{dt^2} + kx = F_0\cos(\omega_d t)$$
The general solution is
$$x_g = x_c + x_d$$
Where, x_c is the (complementary) solution to the
Let's assume
$$\text{equation } m\frac{d^2x}{dt^2} + kx = 0.$$
$$x_c = x_0\cos\omega_0 t$$
- Where ω_0 is the natural angular frequency
$$\omega_0 = \sqrt{\frac{k}{m}}$$
- Let's assume
$$x_d = A\cos\omega_d t$$
$$\frac{d^2x_d}{dt^2} = -A\omega_d^2\cos\omega_d t$$
- Put it back into the differential equation
$$m\frac{d^2x}{dt^2} + kx = F_0\cos(\omega_d t)$$
$$m\times -A\omega_d^2\cos\omega_d t + k\times A\cos\omega_d t = F_0\cos(\omega_d t)$$
$$(k - m\omega_d^2)A = F_0$$
$$A = \frac{F_0}{k - m\omega_d^2} = \frac{F_0}{m(\omega_0^2 - \omega_d^2)}$$
Thus,
$$x_g = x_c + x_d = x_0\cos\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega_d^2)}\cos\omega_d t$$

5. Transverse waves

§1 Transverse waves

1. Wave equation: $\frac{1}{v^2}\cdot\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 h}{\partial x^2}$
2. Sinusoidal function: $y(x,t) = y_m\sin(kx - \omega t)$
· Amplitude y_m: element 相对于平衡位置的最大位移的大小
· Phase $kx-\omega t$: 在一个特定的位置x, phase 随时间线性变化
· Wavelength λ: 沿着波传播方向, 相同两个波形间的距离.
· Angular wave number k: $k = 2\pi/\lambda = \omega/v$
3. 质点振动方程
在x=0处, 质点做SHM: $y = y_m\sin(-\omega t)$, 且 $\omega = \frac{2\pi}{T} = 2\pi f$
4. Phase constant $y(x,t) = y_m\sin(kx - \omega t + \phi)$
φ为正: 图像往x轴负方向移, φ为负: 图像往x轴正方向移
5. 波速 $v = \pm\frac{dx}{dt} = \lambda/T = \lambda f$ (向正方向传取+)
注意区分波速与 oscillating speed $v = \frac{dy}{dt}$, $v_m = \omega f$(振幅)
波速取决于介质的性质 $v = \sqrt{\tau/\mu}$
6. Find wave speed by wave equation

- We can find the wave speed through the wave equation, that governs the *travel of waves of any type.*
$$F_{net,y} = ma_y$$
$$F_{2y} - F_{1y} = dma_y$$
$$dm = \mu dx;\ a_y = \frac{d^2y}{dt^2}$$
- The force
$$\tau = \sqrt{F_{2x}^2 + F_{2y}^2}$$
- Assume the slope is $S = \frac{dy}{dx}$
$$F_{2y} = \tau S_2;\ F_{1y} = \tau S_1$$
$$\tau S_2 - \tau S_1 = \mu dx\times\frac{d^2y}{dt^2}$$
$$\tau(S_2 - S_1) = \mu dx\times\frac{d^2y}{dt^2}$$
$$\frac{S_2 - S_1}{dx} = \frac{\mu}{\tau}\times\frac{d^2y}{dt^2}$$
- The difference between S_2 and S_1 can be a differential amount dS
$$dS = d\left(\frac{dy}{dx}\right)$$
$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\mu}{\tau}\times\frac{d^2y}{dt^2}$$
- Since the wave equation for all traveling wave is
$$\frac{\partial^2y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2y}{\partial t^2}$$
- Therefore, the wave speed is
$$v = \sqrt{\frac{\tau}{\mu}}$$

§2 Energy and power

1. 对于 a region of string zero displacement: maximum energy
maximum displacement: no energy
2. The rate of energy transmitting (power)
$P_{avg} = \frac{1}{2}\mu v\omega^2 y_m^2$

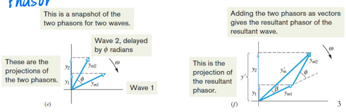
证: 考虑一个 element dm 的动能 dK
$dK = \frac{1}{2}dm\times u^2$
$u = \frac{\partial y}{\partial t} = -\omega y_m\cos(kx - \omega t)$
$dm = \mu dx = \mu\times v\times dt$
· 因此, $dK = \frac{1}{2}dm\,u^2$
$= \frac{1}{2}[-\omega y_m\cos(kx-\omega t)]^2\times\mu\times v\times dt$
$= \frac{1}{2}\mu\omega y_m^2\cos^2(kx-\omega t)\times v\times dt$

| 类: $= \frac{1}{2}\mu v\omega^2 y_m^2\cos^2(kx-\omega t)$
考虑势能 $dU = dK$
考虑平均值 $P_{avg} = dU_{avg} + dK_{avg}$
$= 2\times\frac{1}{2}\mu v\omega^2 y_m^2[\cos^2(kx-\omega t)]_{avg}$
$= \mu v\omega^2 y_m^2[\cos^2(kx-\omega t)]_{avg}$
cosine square 在一个波长上的平均值为 $\frac{1}{2}$
$P_{avg} = \frac{1}{2}\mu v\omega^2 y_m^2$

- For Transverse Sinusoidal Wave:
$$y(x,t) = y_m\sin(kx - \omega t)$$
$$u(x,t) = \frac{\partial y}{\partial t} = -\omega y_m\cos(kx - \omega t)\text{ and }\frac{\partial y}{\partial x} = ky_m\cos(kx - \omega t)$$
$$dK = \frac{1}{2}dmu^2 = \frac{1}{2}\mu dx\times[-\omega y_m\cos(kx - \omega t)]^2$$
$$= \frac{1}{2}\mu\omega^2 y_m^2\cos^2(kx - \omega t)dx$$
- For potential energy: The amount of stretch is
$$\sqrt{dx^2 + dy^2} - dx = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - dx$$
$$\approx dx\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \cdots\right] - dx \approx \frac{dx}{2}\left(\frac{\partial y}{\partial x}\right)^2$$
$$P.E. = dU = \text{tension}\times\text{amount of stretch} = \tau\times\frac{dx}{2}\times\left(\frac{\partial y}{\partial x}\right)^2$$
$$dU = \frac{1}{2}\mu v^2k^2y_m^2\cos^2(kx - \omega t)dx = \frac{1}{2}\mu\omega^2 y_m^2\cos^2(kx - \omega t)dx = dK$$

§3 Superposition and interference

1. Superposition: $y' = y_1(x,t) + y_2(x,t)$
2. Interference: 当两个正弦波沿着一个stretched string有相同的振幅、波长,传播方向时,会干涉产生一个沿相同方向传播的 resultant wave
Wave 1: $y_1(x,t) = y_m\sin(kx - \omega t)$
Wave 2: $y_2(x,t) = y_m\sin(kx - \omega t + \phi)$
由和差化积公式: $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
$y' = y_1(x,t) + y_2(x,t) = [2y_m\cos(\frac{1}{2}\phi)]\sin(kx - \omega t + \frac{1}{2}\phi)$
3. Phasor



$y_1 = y_{m1}\sin(kx - \omega t + \phi_1)$ $y_2 = y_{m2}\sin(kx - \omega t + \phi_2)$
$y_m' = \sqrt{y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos(\phi_2 - \phi_1)}$
$\tan\phi' = \frac{y_{m1}\sin\phi_1 + y_{m2}\sin\phi_2}{y_{m1}\cos\phi_1 + y_{m2}\cos\phi_2}$ $y = y_m'\sin(kx - \omega t + \phi')$

§4 Standing wave

1. Standing wave
若两个正弦波有相同的振幅、波长,且往相反方向传,则它们干涉会形成驻波. nodes never move, antinodes move the most.
2. Displacement function
$y(x,t) = y_m\sin(kx - \omega t) + y_m\sin(kx + \omega t)$
$y'(x,t) = 2y_m\sin(kx)\cos(\omega t)$
3. 波节与波腹
波节: $2y_m\sin(kx) = 0 \Rightarrow kx = \frac{2\pi}{\lambda}x = n\pi \Rightarrow x = n\cdot\frac{\lambda}{2}$ (n=……)
波腹: $2y_m\sin(kx) = \pm2y_m \Rightarrow kx = \frac{2\pi}{\lambda}x = (n+\frac{1}{2})\pi \Rightarrow x = (n+\frac{1}{2})\cdot\frac{\lambda}{2}$ (n=……)
4. 能量
maximum curvature: No K.E. nodes have maximum energy (P.E.)
straight: Minimum P.E. antinodes have maximum energy (K.E.)
- The mechanical energy flows back and forth from a node to an antinode as the string goes from maximum curvature to straight, and does not propagate along the string.
5. Reflecting and resonance
1° 两端均为 Hard boundaries. $L = n\cdot\frac{\lambda}{2}$ (n=1,2,3……)
resonance frequencies f: $f = \frac{v}{\lambda} = \frac{nv}{2L}$ (n=1,2,3……)
2° 一端 hard, 一端 free. $L = n\cdot\frac{\lambda}{4}$ (n=1,3,5……)
resonance frequencies f: $f = \frac{v}{\lambda} = \frac{nv}{4L}$ (n=1,3,5……)
3° 两端 free. $L = n\cdot\frac{\lambda}{2}$ (n=1,2,3……)
resonance frequencies f: $f = \frac{v}{\lambda} = \frac{nv}{2L}$ (n=1,2,3……)

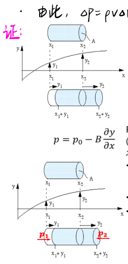
6. Longitudinal Waves

§1 Longitudinal Waves

1. Sinusoidal displacement function: $s(x,t) = s_m\cos(kx - \omega t)$
2. 加速度与速度
· ① 位移: compression / expansion
· $u(x,t) = \frac{\partial s}{\partial t} = \omega s_m\sin(kx - \omega t)$
· $a(x,t) = \frac{\partial u}{\partial t} = -\omega^2 s_m\cos(kx - \omega t)$
3. 声速: $v = \sqrt{B/\rho}$ (B: Bulk modulus, ρ: density)

证: 对于一个 moving air element: $\Delta m = \rho A\Delta x = \rho Av\Delta t$
· 其加速度为: $a = \frac{\Delta v}{\Delta t}$
· 作用于该物体上的力: $F = (P + \Delta P)A - PA = \Delta PA$
· 利用牛二: $F = \Delta ma \Rightarrow \Delta pA = \rho Av\Delta t\times\frac{\Delta v}{\Delta t} \Rightarrow \Delta p = \rho v\Delta v$
· 空气压缩 ΔV 后的压力 $\Delta p = B\frac{\Delta V}{V} = B\frac{\Delta v A\Delta t}{vA\Delta t} = B\frac{\Delta v}{v}$
· 由此, $\Delta p = \rho v\Delta v = B\frac{\Delta v}{v} \Rightarrow V = \sqrt{\frac{B}{\rho}}$

证:



- Originally undisturbed element column:
$p = p_0$, and $V = A(x_2 - x_1) = A\Delta x$
- Disturbed due to sound wave:
$p = p_0 + \Delta p$
$\Delta V = A(x_2 + y_2 - x_1 - y_1) - V = A\Delta y$
$$\Delta p = -B\frac{\Delta V}{V} = -B\frac{\Delta y}{\Delta x} \xrightarrow{\Delta x\to0} -B\frac{\partial y}{\partial x}$$
$$p = p_0 - B\frac{\partial y}{\partial x}$$
Pressure of fluid element (initially at equilibrium position x) with average displacement y.
- Net force on the fluid element:
$$F_{net} = p_1A - p_2A = AB\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = AB\Delta\left(\frac{\partial y}{\partial x}\right)$$
- Mass: $\Delta m = \rho A(x_2 - x_1) = \rho A\Delta x$
- Acceleration: $a = \frac{\partial^2y}{\partial t^2}$
Newton's 2nd law: $F_{net} = \Delta ma$
$$\lim_{\Delta x\to0}\frac{\Delta\left(\frac{\partial y}{\partial x}\right)}{\Delta x} = \frac{\partial^2y}{\partial x^2} = \frac{1}{(B/\rho)}\frac{\partial^2y}{\partial t^2}$$
Traveling wave equation: $\frac{\partial^2y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2y}{\partial t^2}$
$$v = \sqrt{\frac{B}{\rho}}$$

4. Pressure variation
$\Delta p(x,t) = \Delta p_m\sin(kx - \omega t)$, 其中 $\Delta p_m = (v\rho\omega)s_m$
· 对于弹性形变: $\Delta p = -B\frac{\Delta V}{V}$
· $V = A\Delta x$, $\Delta V = A\Delta s \Rightarrow \Delta p = -B\frac{A\Delta s}{A\Delta x} = -B\frac{\Delta s}{\Delta x}$
· $\frac{\Delta s}{\Delta x} = -ks_m\sin(kx - \omega t) \Rightarrow \Delta p = Bks_m\sin(kx - \omega t)$
· $\Delta p_m = Bks_m = v^2\rho ks_m$
$v = \frac{\omega}{k} \Rightarrow \Delta p_m = v^2\rho ks_m = v\rho\omega s_m$
5. Interference
· 两列波有相同的振幅、波长,向x轴正方向传播,相差为φ
Resultant wave: $s'(x,t) = 2s_m\cos(\frac{\phi}{2})\cos(kx - \omega t + \frac{\phi}{2})$
$s_m' = |2s_m\cos(\frac{\phi}{2})|$
相差φ取决于 path length differences, $\Delta L = |L_2 - L_1|$
· Fully constructive: $\Delta L = n\lambda$ (n=0,1,2……)
· Fully destructive: $\Delta L = (n+\frac{1}{2})\lambda$ (n=0,1,2……)
$\phi = 2\pi\frac{\Delta L}{\lambda}$
6. Standing waves in pipes
1° Two open ends. $L = n\cdot\frac{\lambda}{2}$ (n=1,2,3……)
resonance frequencies f: $f = \frac{v}{\lambda} = \frac{nv}{2L}$ (n=1,2,3……)
2° 一端 open, 一端 close. $L = n\cdot\frac{\lambda}{4}$ (n=1,3,5……)
resonance frequencies f: $f = \frac{v}{\lambda} = \frac{nv}{4L}$ (n=1,3,5……)
7. Net wave
The quality of a sound 取决于 the shape of its waveform.
The pitch of a sound 取决于 the fundamental frequency
Fundamental frequency 会被 overtones 加强, 且形成自己独特的音色.
Fundamental frequency 与 overtones 合成之后形成最后的波形.

§2 Intensity of sound

1. Intensity of sound
单位时间单位面积穿过该曲面的能量 $I = \frac{P}{A}$ (W/m^2)
P为 power (time rate of energy transferring)
对于一个 isotropic point source with power P_s: $I = \frac{P_s}{4\pi r^2}$
2. 声强与振幅 S_m
$I = \frac{1}{2}\rho v\omega^2 S_m^2$
证: a thin slice of air of thickness dx, area A, and mass dm
· 考虑 K.E. $dK = \frac{1}{2}dm\times u^2$, $u = \frac{\partial S}{\partial t} = \omega S_m\sin(kx - \omega t)$
$dK = \frac{1}{2}(\rho Adx)[\omega S_m\sin(kx - \omega t)]^2$
$= \frac{1}{2}\rho Avdt\,\omega^2 s_m^2\sin^2(kx - \omega t)$
$\frac{dK}{dt}_{avg} = \frac{1}{4}\rho Av\omega^2 s_m^2$
· 势能与动能相等, $P_{avg} = (\frac{dE}{dt})_{avg} = 2\times(\frac{dK}{dt})_{avg} = \frac{1}{2}\rho Av\omega^2 s_m^2$
· 因此, $I = \frac{P_{avg}}{A} = \frac{1}{2}\rho v\omega^2 s_m^2$
3. Sound level β
$\beta = (10dB)\lg I/I_0$ (dB, decibel)
Standard reference intensity: $I_0 = 10^{-12}\,W/m^2$

§3 Beat, Doppler effect, Supersonic speeds

1. Beat
两个有相同振幅,相似频率的波发生干涉时, resultant wave 会有 intensity fluctuating, 频率为两波的 frequency difference —— beat phenomena.
$f_{beat} = |f_1 - f_2|$

证:
- Wave (a)
$$s_1 = s_m\cos(kx + \omega_1 t)$$
- Wave (b)
$$s_2 = s_m\cos(kx + \omega_2 t)$$
- The net wave
$$s' = s_1 + s_2$$
- Since
$$\cos\alpha + \cos\beta = 2\cos[\tfrac{1}{2}(\alpha - \beta)]\cos[\tfrac{1}{2}(\alpha + \beta)]$$
- Thus,
$$s' = 2s_m\cos\omega' t\cos(kx + \omega t)$$

Where $\omega' = \frac{1}{2}(\omega_1 - \omega_2)$ and $\omega = \frac{1}{2}(\omega_1 + \omega_2)$
$$s' = 2s_m\cos\omega' t\cos(kx + \omega t)$$
- The frequencies are slightly different from each other
$$\omega \gg \omega'$$
- Regard the function as a cosine function with amplitude of s_m' fluctuating with time.
$$s_m' = |2s_m\cos\omega' t|$$
$$\omega_{beat} = 2\omega' = |\omega_1 - \omega_2|$$
$$f_{beat} = |f_1 - f_2|$$

2. Doppler effect: $f' = f\frac{V \pm V_D}{V \pm V_S}$
3. Supersonic speeds
when $V < V_S$, the Mach cone angle $\sin\theta = \frac{Vt}{V_st} = \frac{V}{V_s}$

7. Rigid Body

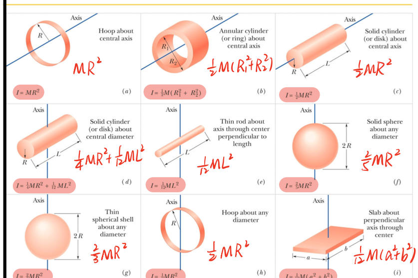
§1 Rotational motion

1. Angular variables
1° angular position: $\theta = \frac{s}{r}$ (rad)
2° angular displacement: $\Delta\theta = \theta_2 - \theta_1$ (+: 逆时针)
3° angular velocity: $\omega = d\theta/dt$ (rad/s), $\vec{v} = \vec{\omega}\times\vec{r}$
4° angular acceleration: $\alpha = d\omega/dt$ (rad/s^2)
5° 公式 ① $\omega = \omega_0 + \alpha t$
② $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
③ $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
④ $\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
⑤ $\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$
2. linear variables
1° $s = \theta\cdot r$
2° tangential velocity: $v = \omega\cdot r$, 方向与轨迹相切
3° radial acceleration a_r: $a_r = \omega^2 r$ (inward)
tangential acceleration a_t: $a_t = \frac{dv}{dt} = \alpha\cdot r$

§2 Rotational inertia

1. Kinetic energy of the motion: $K = \frac{1}{2}I\omega^2$
2. Rotational inertia: $I = \int r^2 dm$ (kg·m^2)

Table 10-2 Some Rotational Inertias



(b) $dM = \rho\cdot dv = 2\pi\rho lr\,dr$
$I = \int r^2 dM = \int_{R_1}^{R_2} 2\pi r^3\rho l\,dr = \frac{1}{2}M(R_2^2 + R_1^2)$
(c) $dM = \rho\cdot2\pi rl\,dr$ $I = \int r^2 dM = \int_0^R \rho\cdot2\pi r^3 l\,dr = \frac{1}{2}MR^2$
(h) $dm = \frac{M}{2\pi R}R\,d\theta$ $I = 4\int_0^{\frac{\pi}{2}} R^2\sin^2\theta\,\frac{M}{2\pi R}\cdot R\,d\theta = \frac{1}{2}MR^2$
(d) $dM = \frac{M}{L}\cdot dh$ $dI = \frac{1}{4}dMR^2 + dMh^2 = \frac{1}{4}\cdot\frac{M}{L}\cdot R^2dh + \frac{M}{L}h^2dh$
$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} dI = \int_{-\frac{L}{2}}^{\frac{L}{2}}\left(\frac{1}{4}\frac{M}{L}R^2 + \frac{M}{L}h^2\right)\cdot dh = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
(d') Solid disk with width Δh
$I_{Hoop} = \frac{1}{2}MR^2$ $dM = \frac{M}{\pi R^2}\cdot2\pi r\cdot dr$
$dI_{Hoop} = \frac{1}{2}r^2dM = \frac{1}{2}r^2\cdot\frac{M}{\pi R^2}\cdot2\pi r\cdot dr = \frac{M}{R^2}\cdot r^3dr$
$I = \int_0^R dI_{Hoop} = \int_0^R \frac{M}{R^2}\cdot r^3dr = \frac{1}{4}MR^2$
(e) 由 $I_{cyl} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ 取 R=0: $I_{thin\ rod}$: $\frac{1}{12}ML^2$
(g) $dM = \frac{M}{4\pi R^2}\cdot2\pi R\sin\theta\cdot R\,d\theta = \frac{1}{2}M\sin\theta\,d\theta$
$I = 2\cdot\int_0^{\frac{\pi}{2}} R^2\sin^2\theta\cdot\frac{1}{2}M\sin\theta\,d\theta = \frac{2}{3}MR^2$
(f) $dM = \frac{M}{\frac{4}{3}\pi R^3}\cdot4\pi r^2dr$ 由 $I_{thin\ shell} = \frac{2}{3}Mr^2$
$dI = \frac{2}{3}dM\cdot r^2 = 2\frac{M}{R^3}\cdot r^4dr$ $I = \int_0^R dI = \int_0^R 2\frac{M}{R^3}r^4dr = \frac{2}{5}MR^2$
(i) $\Delta M = \frac{M}{ab}\cdot dx\cdot dy$ $I = \int_{-\frac{b}{2}}^{\frac{b}{2}}dy\int_{-\frac{a}{2}}^{\frac{a}{2}}(x^2+y^2)\cdot\frac{M}{ab}dx = \frac{1}{12}M(a^2+b^2)$
3. Parallel axis theorem
① 两条轴平行
② 其中一条轴通过质心 ⇒ $I = I_{com} + Mh^2$
其中 h 为两轴间的 perpendicular distance

§3 Torque

1. Torque: $\vec{\tau} = \vec{r}\times\vec{F} = rF_\perp = r_\perp F$ (N·m)
2. 牛二: $F_t = ma_t = m\alpha\cdot r \Rightarrow \tau = m\alpha r^2 = I\alpha$
3. Work and Kinetic
$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$
$W = \int_{\theta_i}^{\theta_f}\tau\,d\theta = \tau(\theta_f - \theta_i)$ (若τ恒定)
4. Power $P = \tau\omega$

§4 Rolling

1. Translation in rolling: $S=\theta R$, $V=\omega R$
2. Rolling
   1° 对于 each point on the rim: $\vec{v}=\vec{v}_{com}+\vec{v}_t$, $v_t=\omega R$
   2° 可以视作关于点P的 pure rotation, P为与地面的接触点, 角速度: 与绕O旋转时的角速度 ω 相同 (圆上任意点)
3. 牛二:
   1° 对于平动: $F_{net}=m a_{com}=m\alpha R$
   对于转动: $\tau_{net}=I_{com}\alpha=I_{com}\frac{a_{com}}{R}$
   2° 有两种方式来 initialize rolling
   ① 施加 torque 以创造 rotation
   ② 施加 force 以创造 translation
   3° 若要发生纯滚动, 必须满足 $a_{com}=\alpha R$
4. 施加 torque 以创造 rotation
   1° 对于平动: F_{net} 仅受地面静摩擦力 f_s 影响
   因为 F_{net} 与 f_s 相同, 所以 f_s 的方向与实际加速度 a_{com} 相同
   $F_{net}=f_s=m\cdot a_{com}$
   2° 对于转动: τ_{net} 同时受施加的力矩 $\tau_{applied}$ 与摩擦力的力矩 f_sR 影响 $\tau_{net}=\tau_{applied}-f_sR=I\alpha$
   3° 纯滚动条件: $a_{com}=\alpha R$
   4° $\begin{cases} f_s=m\cdot a_{com}\\ \tau_{applied}-f_sR=I\alpha\\ a_{com}=\alpha R\end{cases} \Rightarrow a_{com}=\frac{\tau_{applied}}{mR}\frac{1}{(1+\frac{I}{mR^2})}$
5. 施加 force 以创造 translation
   1° 对于平动: F_{net} 同时受施加的力 $F_{applied}$ 与摩擦力 f_s 影响
   ① $F_{applied}$ 的方向与实际加速度 a_{com} 相同
   ② $F_{net}=F_{applied}-f_s=m\,a_{com}$
   2° 对于滚动: τ_{net} 仅受地面静摩擦力的力矩 f_sR 影响
   ① f_s 的方向与实际运动趋势相反
   ② $\tau_{net}=f_sR=I\alpha$
   3° 纯滚动条件: $a_{com}=\alpha R$
   4° $\begin{cases} F_{applied}-f_s=m\,a_{com}\\ f_sR=I\alpha\\ a_{com}=\alpha R\end{cases} \Rightarrow a_{com}=\frac{F_{applied}}{m}\frac{1}{(1+\frac{I}{mR^2})}$
6. Connected wheel

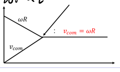
$F_1=F_2=F$

$F_{net}=ma_{com}$, $f_{s,back}-F=F_{net}=ma_{com}$ (1)
$\tau_{net}=\tau_{applied}-f_{s,back}R=I\alpha=I\frac{a_{com}}{R}$ (2)
$F_{net}=ma_{com}$, $F-f_{s,front}=F_{net}=ma_{com}$ (3)
$\tau_{net}=f_{s,front}R=I\alpha=I\frac{a_{com}}{R}$ (4)
$f_{s,back}-f_{s,front}=2ma_{com}$ (5)
$a_{com}=\frac{\tau_{applied}}{2mR}\frac{1}{(1+\frac{I}{mR^2})}$ (6)

7. Rolling down (Force applied)
   $\begin{cases} Mg\sin\theta-f_s=M\,a_{com}\\ f_sR=I_{com}\cdot\alpha\\ a_{com}=\alpha\cdot R\end{cases} \Rightarrow f=\frac{MI_{com}g\sin\theta}{I_{com}+MR^2}$, $a_{com}=\frac{g\sin\theta}{1+\frac{I_{com}}{MR^2}}$

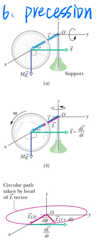
§5. Kinetic energy of rolling

1. Kinetic energy of rolling: $K=\frac12 I_{com}\omega^2+\frac12 Mv_{com}^2$
2. 纯滚动无机械能损耗
3. 沿斜面下滚: $mgh=\frac12 mv_{com}^2+\frac12 I_{com}\omega^2$, $v_{com}=\omega\cdot R$
4. The yo-yo
   1° 情况一: 无滑滚动
   $\begin{cases} mg-T=m\,a_{com}\\ T\cdot R_0=I\cdot\alpha\\ a_{com}=\alpha\cdot R_0\end{cases} \Rightarrow \begin{cases} T=\frac{Img}{I+mR_0^2}\\ \alpha=\frac{R_0mg}{I+mR_0^2}\\ a_{com}=\frac{R_0^2mg}{I+mR_0^2}\end{cases}$ downward
   2° 情况二: 有滑滚动
   半径为R的 yo-yo, 有角速度 ω_0, 在 t=0s 时, 它与平面接触.
   ① 接触时: 有滑滚动, 摩擦力为 f_k
   平动 $\begin{cases} f_k=\mu_k mg=m\,a_{com}\\ V_{com}=a_{com}\cdot t\ (a_{com}\neq\alpha R)\end{cases}$ 滚动 $\begin{cases} f_kR=\mu_k mgR=I_{com}\alpha\\ \omega=\omega_0-\alpha t\end{cases}$
   ② 当 $V_{com}=\omega R$ 时, 开始做无滑滚动
   $t=\frac{\omega_0 R}{\mu_k g(1+\frac{mR^2}{I_{com}})}$



§6 Angular momentum

1. Angular momentum: $\vec{l}=\vec{r}\times\vec{p}=m\cdot(\vec{r}\times\vec{v})$ ($kg\,m^2/s$, $J\cdot s$)
2. 牛二: $\tau_{net}=dl/dt$
3. Angular momentum of a system of particles
   $\vec{L}=\sum\vec{l}_i$ $\vec{\tau}_{net}=d\vec{L}/dt$
4. 绕定轴转动的刚体的角动量: $L=I\omega$
5. Conservation of angular momentum
   1° 若作用于一个系统的合外力矩为0, 则系统角动量保持不变
   $\vec{L}_i=\vec{L}_f$ (closed, isolate system) $I_i\omega_i=I_f\omega_f$
   2° 若作用于系统沿某定轴上的合外力矩分量为0, 则系统沿该轴方向角动量保持不变
   $\tau_{z,net}=\frac{dL_z}{dt}=0$ L_z = constant
6. precession of gyroscope



- No spinning: The torque created by the gravitational force
  $$\vec{\tau}=\frac{d\vec{L}}{dt}$$
  $$\tau=Mgr\sin 90°=Mgr$$
- Spinning: Precession, rotate horizontally about O
  $$L=I\omega$$
  $$dL=L\,d\phi$$
  $$\tau=\frac{dL}{dt}=Mgr$$
  $$dL=Mgrdt$$
  $$d\phi=\frac{dL}{L}=\frac{Mgrdt}{I\omega}$$
  Procession rate: $\Omega=\frac{d\phi}{dt}=\frac{Mgr}{I\omega}$

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8. 一些数学公式

§1 和差化积

$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ $\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ $\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\sin\alpha\cos\beta=\frac12[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\cos\alpha\sin\beta=\frac12[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\cos\alpha\cos\beta=\frac12[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\sin\alpha\sin\beta=\frac12[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

§2 Taylor series

$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$ $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$

§3 其他公式

$e^{ix}=\cos x+i\sin x$

Assignment Answers

§1 Chapter 13 Gravitation

• As defined earlier, gravitational potential energy is $U=mgy$ and is positive for a body of mass m above the earth's surface (which is at $y=0$). But in this chapter, gravitational potential energy is $U=-Gm_Em/r$, which is negative for a body of mass m above the earth's surface (which is at $r=R_E$). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?

Solution: According to the definition of gravitational acceleration on the surface of the earth, one finds $mg=\frac{Gm_Em}{R_E^2}$, which indicates $g=\frac{Gm_E}{R_E^2}$.

Second, write $U(r)=-Gm_Em/r+C$ and set $U(R_E)=0$ as the zero point potential energy, thus one finds $U(r)=-\frac{Gm_Em}{r}+\frac{Gm_Em}{R_E}$. Noe let $r=y+R_E$ and expand the above potential energy assuming $y\ll R_E$ as follows

$$U=-\frac{Gm_Em}{y+R_E}+\frac{Gm_Em}{R_E}\approx-\frac{Gm_Em}{R_E}\left(1-\frac{y}{R_E}\right)+\frac{Gm_Em}{R_E}=\frac{Gm_Em}{R_E^2}y=mgy.$$

Therefore, the gravitational potential energy $U=mgy$ is simply an approximate expression of the gravitational potential energy $U=-Gm_Em/r$ near the surface of the earth after setting the zero point at $r=R_E$.

•• Binary Star with Equal Masses
Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.
(c) How much energy would be required to separate the two stars to infinity?
Answers: $W=\frac{GM^2}{4R}$. $\Delta E=W=E_2-(K_1+U_1)=2\cdot\frac12Mv^2+\frac{GM^2}{2R}=\frac{GM^2}{4R}$.

•• Using Kepler's third law. We found that for two objects with masses m_1 and m_2 orbiting in a circular orbit at distance R from each other under gravity (about their center of mass), we have

$$\omega^2R^3=G(m_1+m_2),$$

where ω is the angular velocity of the orbital motion. This formula is very useful and let us see some applications.

(a) For communication purposes it is very useful to have satellites that orbit the Earth once a day exactly. This means an antenna on Earth aimed at the satellite remains aimed at it as the Earth rotates. At what radius do such geostationary satellites orbit?
Solution: Geostationary satellites: noting that $T=1day=24\times3600$ s and $\omega=\frac{2\pi}{T}$, therefore, after neglecting the mass of the satellite, one gets

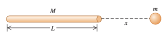
$$R=\left(\frac{Gm_ET^2}{(2\pi)^2}\right)^{1/3}=\left(\frac{R_E^2gT^2}{(2\pi)^2}\right)^{1/3}=4.22\times10^7\,m.$$

(b) The star S2 orbits near the center of the Milky Way. We observe that its orbit has a radius of about 930 AU (an AU, or Astronomical Unit, is the radius of the Earth's orbit about the Sun and equal to 1.5×10^{11} m. S2 has an orbital period of about 15.6 years. Find the mass of the object S2 is orbiting, in terms of the mass of the Sun $M_\odot=2.0\times10^{30}$ kg. Can you guess what the object is?
Solution: Star S2: Define the mass at the center of the orbit as M and use the Kepler's third law

$$M\approx m_{S2}=\frac{4\pi^2R^3}{T^2G}=6.59\times10^{36}\,kg\approx3.31\times10^6M_\odot.$$

For a normal star, the mass is roughly around the same order of the solar mass $M_\odot=1.989\times10^{30}$kg. Therefore, we can neglect the mass m_{S2} and find $M=3.31\times10^6M_\odot$, which is the mass of a super massive black hole.

•• A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod.



(a) Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L.
Answers: $U=-\frac{GMm}{L}\ln\left(1+\frac{L}{x}\right)$, which reduces to $-\frac{GMm}{x}$ as expected in the $x\gg L$ limit.
(b) Use $F_x=-dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod. Show that your answer reduces to the expected result when x is much larger than L.
Answers: $F_x=-\frac{\partial U}{\partial x}=-\frac{GMm}{x^2+Lx}$, which goes to $-\frac{GMm}{x^2}$ when $x\gg L$. When x is much larger than L the rod car be treated as a point mass.

••• A comet orbits the sun (mass m_S) in an elliptical orbit of semi-major axis a and eccentricity e. Find expression for the speeds of the comet at perihelion and aphelion.
Solution: Let v_p and v_a be the velocities of the comet at perihelion and aphelion, respectively. According to the energy conservation, angular momentum conservation, and the geometric relations shown in following figure, on can write down the following equations

Mechanical energy conservation: $\frac12mv_p^2-\frac{GM_Sm}{r_p}=\frac12mv_a^2-\frac{GM_Sm}{r_a}$,
Angular momentum conservation: $mv_pr_p=mv_ar_a$,
Geometric relations: $r_p=(1-e)a$, and $r_a=(1+e)a$.

Solving the above equations gives that the speeds at perihelion and aphelion are

$$v_p=\sqrt{\frac{Gm_S}{a}\frac{1+e}{1-e}},\quad\text{and}\quad v_a=\sqrt{\frac{Gm_S}{a}\frac{1-e}{1+e}}.$$

§2 Chapter 14 Fluid

•• (Textbook C14-P3)
In the figure below, water stands at depth $D=30.0$ m behind the vertical upstream face of a dam of width $W=250$ m.
(a) Find the net horizontal force on the dam from the gauge pressure of the water (gauge pressure is the pressure difference w.r.t. the external air).
Answers: $F=\frac12\rho gWD^2=1.10\times10^9$ N.
(b) Find the net torque due to that force about a horizontal line through O parallel to the width of the dam. This torque tends to rotate the dam around that line, which would cause the dam to fail.
Answers: $\tau=\frac16\rho gWD^3=1.10\times10^{10}$ N·m.
(c) Find the effective moment arm of the torque $r_{eff}=\tau/F$.
Answers: The effective arm $r_{eff}=H/3=10$ m.

(a) $F=\int_0^D\rho gyW\,dy=\frac12\rho gWD^2=\frac12(1.00\times10^3\,kg/m^3)(9.80\,m/s^2)(250\,m)(30.0\,m)^2=1.10\times10^9$N.

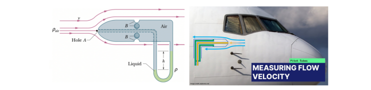
(b) $d\tau=dF(D-y)=\rho gyW(D-y)dy$

and the total torque of the water is

$$\tau=\int_0^D\rho gW(D-y)y\,dy=\rho gW\left(\frac12D^3-\frac13D^3\right)=\frac16\rho gWD^3=\frac16(1.00\times10^3\,kg/m^3)(9.80\,m/s^2)(250\,m)(30.0\,m)^3=1.10\times10^{10}\,N\cdot m.$$

(c) $r=\frac{\tau}{F}=\frac{\frac16\rho gWD^3}{\frac12\rho gWD^2}=\frac{D}{3}=\frac{30.0\,m}{3}=10.0$ m.

••• (Textbook C14-P48)
A pitot tube (Figures below) is used to determine the air-speed of an airplane (relative to the air). It consists of an outer tube with a number of small holes B (four are shown) that allow air into the tube; that tube is connected to one arm of a U-tube. The other arm of the U-tube is connected to hole A at the front end of the device, which points in the direction the plane is headed. At A the air becomes stagnant so that $v_A=0$. At B, however, the speed of the air presumably equals the airspeed v of the plane.



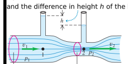
(a) Use Bernoulli's equation to show that

$$v=\sqrt{\frac{2\rho gh}{\rho_{air}}},$$

where ρ is the density of the liquid in the U-tube and h is the difference in liquid levels in that tube.
Solution:

48. (a) Bernoulli's equation gives $p_A=p_B+\frac12\rho_{air}v^2$. However, $\Delta p=p_A-p_B=\rho gh$ in order to balance the pressure in the two arms of the U-tube. Thus $\rho gh=\frac12\rho_{air}v^2$, or

••• As shown below, the Venturi meter can be used to measure flow speed in a pipe. The narrow pipe part of the pipe is called the throat. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2 and the difference in height h of the liquid levels in the two vertical tubes.



$$v_1=\sqrt{\frac{2gh}{(A_1/A_2)^2-1}}.$$

••• A large bucket of height H and cross-sectional area A_1 is filled with water. The top is open to the atmosphere. There is an opening facet of area A_2, which is much smaller than A_1, at the bottom of the bucket.
(a) Show that when the height of the water is h, the speed of the water leaving the faucet is approximately $\sqrt{2gh}$.
(b) Show that if $A_2\ll A_1$, the rate of change of the height h of the water is given by

$$\frac{dh}{dt}=-\frac{A_2}{A_1}\sqrt{2gh}.$$

(c) Find h as a function of time if $h=H$ at $t=0$.
Answers:

$$h=\left(-\frac{A_2}{A_1}\sqrt{\frac{g}{2}}t+\sqrt{H}\right)^2$$

(d) Find the total time needed to drain the bucket if $H=2.00$ m, $A_1=0.800m^2$, and $A_2=1.00\times10^{-4}A_1$.
Answers: 6.39×10^3s

Solution:
(a) From Bernoulli's equation between the surface and the bottom of the bucket, one gets

$$p_0+\rho gh+\frac12\rho\cdot0^2=p_0+\rho g\cdot0+\frac12\rho v^2\ \Rightarrow\ v=\sqrt{2gh}\quad(5)$$

with p_0 the atmosphere pressure.
(b) During short time dt, according to the conservation of fluid matter volume, one gets the volume that goes out of the bucket from the bottome equals the volume decrease from the top surface. Thus

$$vdtA_2=-dhA_1\ \Rightarrow\ \frac{dh}{dt}=-\frac{A_2}{A_1}v=-\frac{A_2}{A_1}\sqrt{2gh},\quad(6)$$

where $dh<0$ means the height of the water surface is decreasing.
(c) Solve ordinary differential equation by separating the variables h from t, the write

$$\frac{dh}{\sqrt{h}}=-\frac{A_2}{A_1}\sqrt{2g}dt\ \Rightarrow\quad(7)$$
$$\int\frac{dh}{\sqrt{h}}=-\frac{A_2}{A_1}\sqrt{2g}\int dt+C\ \Rightarrow\quad(8)$$
$$2\sqrt{h}=-\frac{A_2}{A_1}\sqrt{2g}t+C\quad(9)$$

From initial condition $h(0)=H\Rightarrow C=2\sqrt{H}$, one finds $\sqrt{h}=-\frac{A_2}{A_1}\sqrt{\frac{g}{2}}t+\sqrt{H}$, and eventually writes $h=\left(-\frac{A_2}{A_1}\sqrt{\frac{g}{2}}t+\sqrt{H}\right)^2$.
(d) From the above results, one can solve for time (t) as a function of h and obtain

$$t=(\sqrt{H}-\sqrt{h})\frac{A_1}{A_2}\sqrt{\frac{2}{g}}=(\sqrt{2.00}-0)\frac{A_1}{1.00\times10^{-4}A_1}\sqrt{\frac{2}{9.81}}=6.39\times10^3s.\quad(10)$$

§3 Chapter 15 Oscillation

* Figure below (left) shows a pendulum of length L with a bob of mass M. The bob is attached to a spring that has a force constant k, as shown. When the bob is directly below the pendulum support, the spring is unstressed. Derive an expression for the period of this oscillating system for small-amplitude vibrations.

Answers: The period reads

$$T=\frac{2\pi}{\sqrt{\frac{g}{L}+\frac{k}{M}}}.$$

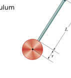
Solution:
For the pendulum coupled to a spring, we use Newton's second law and project to the tangential direction

$$-Mg\sin\theta-k(L\theta)\cos\theta=ML\frac{d^2\theta}{dt^2}\ \Rightarrow\quad(6)$$
$$-Mg\theta-k(L\theta)=ML\frac{d^2\theta}{dt^2}\quad(7)$$

in the limit of small θ where $\sin\theta\simeq\theta$ and $\cos\theta\simeq1$. Therefore, one can reach the equation of motion for the spring pendulum system

$$\frac{d^2\theta}{dt^2}=-\left(\frac{g}{L}+\frac{k}{M}\right)\theta\ \Rightarrow$$
$$\omega=\sqrt{\frac{g}{L}+\frac{k}{M}}\quad\text{and}\quad T=\frac{2\pi}{\omega}=\frac{2\pi}{\sqrt{\frac{g}{L}+\frac{k}{M}}}$$

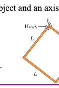
4. ** (Textbook C15-P7) In Figure above (right), the pendulum consists of a uniform disk with radius $r=10.0$ cm and mass $M=500$ g attached to a uniform rod with length $L=500$ mm and mass $m=250$ g.



(a) Calculate the rotational inertia of the pendulum about the pivot point.
Answers: $I=\frac12Mr^2+M(r+L)^2+\frac13mL^2=0.203$ kg m²
(b) What is the distance between the pivot point and the center of mass of the pendulum
Answers: $d=0.483$ m.
(c) Calculate the period of oscillation.
Answers: $T=1.50$ s.

$$d=\frac{Ml_d+ml_r}{M+m}=0.483\,m.\qquad T=2\pi\sqrt{\frac{I}{(M+m)gd}}=1.50\,s.$$

** As shown below (left), a square object of total mass M is constructed of four identical uniform thin sticks, each of length L, attached together. This object is hung on a hook at its upper corner. If it is rotated slightly to the left and then released, at what frequency will it swing back and forth? **Answers:** $\omega=0.921\sqrt{g/L}$ and $f=0.921\sqrt{g/L}/(2\pi)$.



EXECUTE: The parallel-axis theorem gives I for each stick for an axis at the center of the square to be $\frac{1}{12}mL^2+m(L/2)^2=\frac13mL^2$ and the total I for this axis is $\frac43mL^2$. For the entire object and an axis at the hook, applying the parallel-axis theorem again to the object of mass $4m$ gives $I=\frac43mL^2+4m(L/\sqrt2)^2=\frac{10}{3}mL^2$.

$$f=\frac{1}{2\pi}\sqrt{\frac{m_{object}gd}{I}}=\frac{1}{2\pi}\sqrt{\frac{4m_{object}gL/\sqrt2}{\frac{10}{3}m_{object}L^2}}=\sqrt{\frac{6}{5\sqrt2}}\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right)=0.921\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right).$$

** Reduced Mass.
(a) If we attach two blocks that have masses m_1 and m_2 to either end of a spring that has a force constant k and set them into oscillation by releasing them from rest with the spring stretched, show that the oscillation frequency is given by $\omega=\sqrt{k/\mu}$, where $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass of the system.
Hint: First consider the separate motions of these two blocks and write their equation of motions as follows

$$m_1\frac{d^2x_1}{dt^2}=-k(x_1-x_2)\quad\text{and}\quad m_2\frac{d^2x_2}{dt^2}=-k(x_2-x_1),$$

from which you can obtain the two equations which describe their relative displacement x_1-x_2 (SHM) and their center of mass x_{com} (free motion), respectively.
Solution:
Use Newton's second law and apply it to both blocks separately and obtain

$$m_1\frac{d^2x_1}{dt^2}=-k(x_1-x_2),\quad(11)$$
$$m_2\frac{d^2x_2}{dt^2}=-k(x_2-x_1),\quad(12)$$

where x_1 and x_2 are displacements of m_1 and m_2 from their equilibrium positions, respectively. It is interesting to first notice that Eq. (11) + Eq (12) tells us that the center of mass of these two blocks is at rest or moving with constant velocity since the external force is zero.
Let us now compute Eq. (11) / m_1 − Eq (12)/ m_2,

$$\frac{d^2x_1}{dt^2}-\frac{d^2x_2}{dt^2}=-k(1/m_1+1/m_2)(x_1-x_2)\ \Rightarrow$$
$$\frac{d^2}{dt^2}(x_1-x_2)=-k\frac{m_1+m_2}{m_1m_2}(x_1-x_2)\ \Rightarrow$$
$$\mu\frac{d^2x}{dt^2}=-kx\quad\text{with}\quad\mu=\frac{m_1m_2}{m_1+m_2},$$

where $x=x_1-x_2$. By using analogy and comparing with the EOM of SHM, we can obtain $\omega=\sqrt{\frac{k}{\mu}}$. This implies that the relative motion $x=x_1-x_2$ between these two blocks is SHM with the oscillation frequency $\omega=\sqrt{\frac{k}{\mu}}$.

(b) In one of your chemistry labs, you determine that one of the vibrational modes of the HCl molecule has a frequency of $f=\omega/(2\pi)=8.97\times10^{13}Hz$. Using the result of Part (a), find the "effective spring constant" between the H atom and the Cl atom in the HCl molecule.
Answers: $k=514$ N/m

Solution:
From $f=8.969\times10^{13}$ Hz derive ⇒ $\omega=2\pi f=5.635\times10^{14}s^{-1}$
Note that the masses are $m_H=1.67\times10^{-27}$ kg and $m_{Cl}=35.45m_H=5.92\times10^{-26}$ kg, thus one finds the reduced mass is

$$\mu=\frac{m_Hm_{Cl}}{m_H+m_{Cl}}=1.62\times10^{-27}\,kg.$$

and the spring constant $k=\mu\omega^2=1.62\times10^{-27}\times(5.635\times10^{14})^2=514$ N/m.

*** Damped oscillation
Show that the solution to the following EOM of the damped oscillation

$$m\frac{d^2x}{dt^2}=-b\frac{dx}{dt}-kx$$

is given by

$$x=Ae^{-\frac{b}{2m}t}\cos(\omega t+\delta),\quad\text{with}\quad\omega=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}.$$

Solution: Let us compute $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ with the above assumed form of $x(t)$

$$\frac{dx(t)}{dt}=-A\frac{b}{2m}e^{-\frac{b}{2m}t}\cos(\omega t+\delta)-\omega Ae^{-\frac{b}{2m}t}\sin(\omega t+\delta),$$
$$\frac{d^2x(t)}{dt^2}=A\left(\frac{b}{2m}\right)^2e^{-\frac{b}{2m}t}\cos(\omega t+\delta)+A\omega\frac{b}{m}e^{-\frac{b}{2m}t}\sin(\omega t+\delta)-\omega^2Ae^{-\frac{b}{2m}t}\cos(\omega t+\delta).$$

Comparing the coefficients of the sin and cos terms in the EOM, one finds

$$mA\omega\frac{b}{m}e^{-\frac{b}{2m}t}\sin(\omega t+\delta)=-b(-\omega Ae^{-\frac{b}{2m}t}\sin(\omega t+\delta)),$$
$$Ae^{-\frac{b}{2m}t}\cos(\omega t+\delta)\left[m\left(\frac{b}{2m}\right)^2-m\omega^2\right]=Ae^{-\frac{b}{2m}t}\cos(\omega t+\delta)\left[\left(\frac{b}{2m}\right)b-k\right],$$

where the first equation is automatically satisfied while the second equation is satisfied when $\omega=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$

*** Resonance effects.
Consider the case with damped oscillation with a periodic driving force $F(t)=F_d\cos\omega_dt$. Thus, based on what we have learnt in damped oscillations and Newton's second law, the corresponding EoM can be cast into

$$m\frac{d^2x}{dt^2}=-b\frac{dx}{dt}-kx+F_d\cos(\omega_dt).$$

Show that the solution to the above equation consists of two parts, the transient solution ($x_{transient}$) and the steady-state solution (x_{steady}), which read

$$x(t)=\underbrace{A_0e^{-\frac{b}{2m}t}\cos(\omega t+\delta)}_{x_{transient}}+\underbrace{A_d\cos(\omega_dt-\delta_d)}_{x_{steady}},$$

where $x_{transient}$ is the general solution to damped oscillation discussed in the previous problem and it decays to zero after long enough time.
In the steady-state solution, the amplitude A_d and the phase constant δ_d are given by

$$A_d=\frac{F_d}{\sqrt{m^2(\omega_0^2-\omega_d^2)^2+b^2\omega_d^2}},$$
$$\tan\delta_d=\frac{b\omega_d}{k-m\omega_d^2}=\frac{b\omega_d}{m(\omega_0^2-\omega_d^2)},$$
$$\text{with the natural frequency:}\quad\omega_0\equiv\frac{k}{m}.$$

The resonance effect occurs when the frequency of the driving force ω_d coincides with the natural frequency ω_0.
Solution:
Following the hint, one can show that the sum of these two solutions give

$$m\frac{d^2x_{transient}}{dt^2}=-b\frac{dx_{transient}}{dt}-kx_{transient},$$
$$+\quad m\frac{d^2x_{steady}}{dt^2}=-b\frac{dx_{steady}}{dt}-kx_{steady}+F_d\cos(\omega_dt),$$
$$\Rightarrow\quad m\frac{d^2x(t)}{dt^2}=-b\frac{dx(t)}{dt}-kx(t)+F_d\cos(\omega_dt),$$

where $x(t)=x_{transient}+x_{steady}$. This indicates we only need to check $A_d\cos(\omega_dt-\delta_d)$ is a special solution to the inhomogeneous equation.
Let us write $x_s=A_d\cos(\omega_dt-\delta_d)$ and compute $\frac{d^2x_s}{dt^2}$ and $\frac{dx_s}{dt}$ as follows

$$\frac{dx_s}{dt}=-A_d\omega_d\sin(\omega t-\delta_d),$$
$$\frac{d^2x_s}{dt^2}=-A_d\omega_d^2\cos(\omega t-\delta_d).$$

Plug in the above expressions and write

$$F_d\cos(\omega_dt)=F_d\cos\delta_d\cos(\omega t-\delta_d)-F_d\sin\delta_d\sin(\omega t-\delta_d),$$

then compare the coefficients of $\sin(\omega t-\delta_d)$ and $\cos(\omega t-\delta_d)$ in the following EOM

$$\left(-mA_d\omega_d^2+kA_d\right)\cos(\omega t-\delta_d)-bA_d\omega_d\sin(\omega t-\delta_d)=F_d\cos\delta_d\cos(\omega t-\delta_d)-F_d\sin\delta_d\sin(\omega t-\delta_d).$$

This leads to two equations

$$F_d\cos\delta_d=-mA_d\omega_d^2+kA_d,$$
$$F_d\sin\delta_d=bA_d\omega_d.$$

By taking the ratio of these two equations, one gets the result for $\tan\delta_d$. By squaring these two equations and summing them together, one gets the expression for A_d.

§4 Chapter 16 Transverse waves

8. ** On a real string with the linear mass density μ, some of the energy of a wave dissipates as it travels down the string. Such a situation can be described by a wave function whose amplitude depends on x: $y=A(x)\sin(kx-\omega t)$, where $A(x)=A_0e^{-bx}$. What is the power transported by the wave as a function of x, where $x>0$?

Solution:
The power is defined as

$$P(x,t)=-\frac{\partial y}{\partial x}F_T\frac{\partial y}{\partial t}\quad\text{with}\quad y=A_0e^{-bx}\sin(kx-\omega t).$$

Therefore, by differentiating $y(x,t)$ with x and t

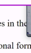
$$P(x,t)=F_TA^2(x)k\omega\cos^2(kx-\omega t)+\frac{dA(x)}{dx}F_TA(x)\sin(kx-\omega t)\cos(kx-\omega t).$$

Now consider the time average of the power P by noting that

$$\frac1T\int_0^Tdt\cos^2(kx-\omega t)=\frac12$$
$$\frac1T\int_0^Tdt\sin(kx-\omega t)\cos(kx-\omega t)=0,$$

then the time-average power at fixed position is
$P=\frac12\mu v\omega^2A^2(x)$
where μ is the linear density, $\omega=kv$ and $F_T=\mu v^2$.
Then, plug in $A=A_0e^{-bx}$,
$P=\frac12\mu v\omega^2(A_0e^{-bx})^2=\frac12\mu v\omega^2A_0^2e^{-2bx}$.

10. *** Waves of Arbitrary Shape.
(a) Explain why any wave described by a function of the form $y(x,t)=f(x-vt)$ moves in the +x-direction with speed v.
(b) Show that $y(x,t)=f(x-vt)$ satisfies the wave equation, no matter what the functional form of f.
(c) A wave pulse is described by the function $y(x,t)=De^{-(Bx-Ct)^2}$, where B, C, and D are all positive constants. What is the speed of this wave?



Solution:
SET UP: If $u=x-vt$, then $\frac{\partial u}{\partial t}=-v$ and $\frac{\partial u}{\partial x}=1$.
EXECUTE: (a) As time goes on, someone moving with the wave would need to move in such a way that the wave appears to have the same shape. If this motion can be described by $x=vt+b$, with b a constant, then $y(x,t)=f(b)$, and the waveform is the same to such an observer.
(b) $\frac{\partial^2y}{\partial x^2}=\frac{d^2f}{du^2}$ and $\frac{\partial^2y}{\partial t^2}=v^2\frac{d^2f}{du^2}$, so $y(x,t)=f(x-vt)$ is a solution to the wave equation with wave speed v. $\frac{1}{v^2}\frac{\partial^2y}{\partial t^2}=$
(c) This is of the form $y(x,t)=f(u)$, with $u=x-vt$ and $f(u)=De^{-B^2(x-Ct/B)^2}$. The result of part (b) may be used to determine the speed $v=C/B$.
EVALUATE: The wave in part (c) moves in the +x-direction. The speed of the wave is independent of the constant D.

§5 Chapter 17 Longitudinal Waves

4. * For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6.0×10^{-5} Pa. (This is a very faint sound and the displacement and pressure amplitudes are very small.) At 20 °C for this sound wave (At 20 °C, note that the bulk modulus for air is 1.42×10^5 Pa and $v=344$ m/s.), calculate
(a) the intensity;
(b) the sound intensity level; (defined as ten times the logarithm in base 10 with $I_0=1\times10^{12}\,W/m^2$) $\beta=(10\,dB)\log_{10}\frac{I}{I_0}$
(c) the displacement amplitude.

$v=\sqrt{\frac{B}{\rho}}$

Solution:
(a) Note that the pressure amplitude p_m and displacement amplitude A_m are related by

$$p_m=\rho v^2kA_m=BkA_m=\left(\frac{2\pi f}{v}\right)BA_m=\omega A_mB/v$$
$$I=\frac12\rho v\omega^2A_m^2=\frac12\rho v\left(\frac{p_mv}{B}\right)^2=\frac12\rho v\frac{p_m^2}{B^2}\frac{B}{\rho}=\frac{vp_m^2}{2B}=4.4\times10^{12}\,W/m^2$$

$I=\frac{P_{avg}}{A}=\frac12\rho v\omega^2s_m^2$

$\beta=(10\,dB)\log\frac{I}{I_0}$

(b) $\beta=(10\,dB)\log_{10}\frac{I}{I_0}=(10\,dB)\log_{10}4.4=6.4\,dB$
(c) From above relation, one gets $A_m=\frac{vp_m}{2\pi fB}=5.8\times10^{-11}\,m$, which is very small size in length.

§6 Chapter 11 Rolling, torque, angular momentum

** Billiard Physics.
A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball (Figure below). The force of the hit F is much greater than the friction force f that the table surface exerts on the ball. The hit lasts for a very short time Δt.
(a) For what value of h will the ball roll without slipping? **Answers:** $h=\frac{2R}{5}$.
Solution: For rolling without slipping, we can have two ways to do the calculation. First way: choose the instantaneous axis on the ground and let $J=F\Delta t$ be the impulse exerted on the cue ball, thus

$$\text{Angular Momentum}:\ -J(h+R)=L=I\omega=\frac75mR^2\omega,\quad\text{(clockwise AM is negative)}\quad(30)$$
$$\text{Linear Momentum}:\ J=mv_{cm},\quad(31)$$
$$\text{No slipping condition}:\ v_{cm}+\omega R=0.\quad(32)$$

Solving the above equations yields $h=\frac25R$.
Second way, choose COM as the axis, then the equation for the angular momentum becomes $-Jh=I_{cm}\omega=\frac25mR^2\omega$ while the rest remains the same. You will the same answer.

(b) If you hit the ball dead center $h=0$ and give it an initial speed v_0, the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then? **Answers:** $\frac57v_0$.
Solution: In this case, it is not obvious where the instantaneous axis is right away. So let choose COM as the axis. Then immediately we find $J*(0)=I_{cm}\omega=0$, thus no rotation. Simply from the impulse-momentum theorem you find $J=mv_0$.
Now we know that the cue ball starts to slide at first and there must be kinetic friction which is going against the direction of motion. The equations of motion are then

$$-f=ma\ \Rightarrow\ a=-f/m\ \Rightarrow\ v_{cm}=v_0-\frac{ft}{m}.\quad(33)$$
$$-fR=I_{cm}\alpha\ \Rightarrow\ \alpha=-\frac52\frac{f}{mR}\ \Rightarrow\ \omega=-\frac52\frac{ft}{mR}.\quad(34)$$

We see that v_{cm} is decreasing while the magnitude of ω is increasing. When $v_{cm}+\omega R=0$, the point that touches the ground will become instantaneously stationary and the ball is going to start rolling without slipping. Let us find that time which is given by

$$v_0-\frac{ft}{m}+\left(-\frac52\frac{ft}{mR}\right)R=0,$$

namely, $t=\frac{2mv_0}{7f}$. At this moment and the time after, $v_{cm}=\frac57v_0$.

** A ladybug of mass m lies on the rim of a uniform disk of mass $4m$ and radius R that can rotate freely about its center like a merry-go-round. Initially the ladybug and disk rotate together with an angular velocity of ω_0. Then the ladybug walks halfway to the center of the disk ($R/2$).
(a) What then is the angular velocity ω of the ladybug–disk system? **Answers:** $\omega=4\omega_0/3$.
Solution: The angular momentum is conserved about the center axis in this process, simply because of vanishing torque. The magnitude of the angular momentum of the ladybug-disk system is

$$L=mr^2\omega+I\omega,\quad(20)$$

where $I=\frac124mR^2=2mR^2$ for the uniform disk with mass $4m$ and ω is the angular velocity, and r is the distance between the ladybug and the disk center.
Initially, the ladybug is at the rim, thus $r=R$, $\omega=\omega_0$, and the angular momentum is

$$L_i=mR^2\omega_0+2mR^2\omega_0.\quad(21)$$

After the ladybug has completed its walk, it reaches to the final location $r=R/2$. The final angular momentum is then

$$L_f=m\frac{R^2}{4}\omega_f+2mR^2\omega_f.\quad(22)$$

From $L_i=L_f$, we obtain

$$\omega_f=\frac{3mR^2\omega_0}{m\frac{R^2}{4}+2mR^2}=\frac43\omega_0.\quad(23)$$

(b) What is the ratio K_f/K_0 of the new kinetic energy of the system to its initial kinetic energy? **Answers:** $K_f/K_0=4/3$.
Solution: The total kinetic energy of the system is

$$K=\frac12m(\omega r)^2+\frac12I\omega^2.\quad(24)$$

By comparing with the expression of the angular momentum, we can write the total kinetic energy as $K=\frac12L\omega$. Therefore, one can find the ratio of kinetic energy as follows

$$\frac{K_f}{K_0}=\frac{L_f\omega_f}{L_i\omega_0}=\frac43.\quad(25)$$

Of course, one can directly compute the ratio $\frac{K_f}{K_0}$ by plugging in the corresponding value of r and ω as in part (a) and obtain

$$\frac{K_f}{K_0}=\frac{\frac12(mR^2/4+I)\omega_f^2}{\frac12(mR^2+I)\omega_0^2}=\frac{(mR^2/4+2mR^2)\omega_f^2}{(mR^2+2mR^2)\omega_0^2}=\frac34\left(\frac43\right)^2=\frac43.\quad(26)$$

(c) What accounts for the change in the kinetic energy?
Answers: The ladybug (the centripetal force) does positive work while walking toward the center of the disk, increasing the total kinetic energy of the system.
Solution: For inertial frame observer, the ladybug only experiences the centripetal force as it walks towards the center of the disk. The force is inward and the displacement is also pointing to the center, therefore it does positive work. (To be more specific, the ladybug gives a force to the disk , the reaction force from the disk provides the centripetal force for the ladybug. Therefore, it is the ladybug who does the work and increases the total energy of the ladybug-disk system.)

** A spherical ball with mass M and rotational inertia $I_{cm}=(2/5)MR^2$ is given an initial clockwise angular velocity ω_0 and zero linear velocity $v_{cm}=0$ before it is placed upon a horizontal surface with kinetic friction coefficient μ_k.
(a) Which direction is the ball going to move towards? **Answers:** Right.
Solution: Consider the relative motion between the contact point of the ball and the ground. Since the ball is rotating clockwise, the bottom of the ball is moving to the left w.r.t. the ground, therefore the friction is pointing to the right against the direction of relative motion. The friction is the only horizontal force exerted on the ball, thus the ball is going to move to the right.
(b) How long does it take for the ball to roll without slipping? **Answers:** $t=\frac{2\omega_0R}{7\mu_kg}$.
Solution: Choose the COM as the axis and the counterclockwise as the positive direction for the rotation, and write down the equations of motion as follows

$$fR=\frac25MR^2\alpha,\ \Rightarrow\ \mu_kMgR=\frac25MR^2\alpha\ \Rightarrow\ \alpha=\frac{5\mu_kg}{2R},\quad(12)$$
$$f=Ma_{cm},\ \Rightarrow\ a_{cm}=\mu_kg.\quad(13)$$

Since both α and a_{cm} are constants, one can immediately find the angular velocity and linear velocity of the ball after taking into account the initial condition

$$\omega=-\omega_0+\frac{5\mu_kg}{2R}t,\quad(14)$$
$$v_{cm}=\mu_kgt,\quad(15)$$

where the minus sign in $-\omega_0$ comes from the fact that initial angular velocity is clockwise.
It is important to compute the velocity of the bottom contact point of the ball with the ground as follows

$$v_B=v_{cm}+\omega R=\mu_kgt-\omega_0R+\frac{5\mu_kg}{2}t.\quad(16)$$

When v_B becomes zero, the ball stops sliding and it starts to roll without slipping, thus the kinetic friction disappears starting from that moment. Setting $v_B=0$ gives $t=\frac{2\omega_0R}{7\mu_kg}$.
(c) How much energy does it lose during this process? **Solution:** $\Delta K=-MR^2\omega_0^2/7$.
Solution: At $t=\frac{2\omega_0R}{7\mu_kg}$, one finds

$$\omega_f=-\frac27\omega_0,\quad\text{and}\quad v_f=\frac27\omega_0R.\quad(17)$$

Thus the final kinetic energy is

$$K_f=\frac12Mv_f^2+\frac12\frac25MR^2\omega_f^2=\frac{2}{35}MR^2\omega_0^2.\quad(18)$$

Noting that initially there is only rotational energy, thus we find the kinetic energy change is

$$\Delta K=K_f-K_0=\frac{2}{35}MR^2\omega_0^2-\frac12\frac25MR^2\omega_0^2=-\frac17MR^2\omega_0^2,\quad(19)$$

where the minus sign indicate energy is lost due to friction in this process.

** A yo-yo is made from two uniform disks, each with mass $m/2$ and radius R, connected by a light axle of radius b. This yo-yo's total moment of inertia about an axis passing through the center can be approximated by $mR^2/2$. A light, thin string is wound several times around the axis, and then the yo-yo is placed upright on a flat table. As shown in the right figure above, the string is pulled with a horizontal force F to the right.
(a) Suppose the yo-yo is rolling without slipping, find the acceleration of the yo-yo.
Solution: Suppose the friction is applied to the yo-yo as shown in the above right figure. Choose counterclockwise as the positive direction for rotation and write down the torque equation and force equation as follows

$$\tau=I\alpha=-F(R-b)\quad\text{instantaneous axis with}\quad I=\frac32mR^2,\quad(6)$$
$$F-f=ma.\quad(7)$$

Together with the relation $a=R\alpha$ for rolling without slipping, one finds the following solutions

$$f=\frac{2b+R}{3R}F,\quad\text{and}\quad a=\frac{2F}{m}\left(1-\frac{b}{R}\right).\quad(8)$$

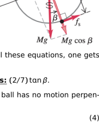
Alternatively, you can write down the torque equation and force equation as follows

$$\tau=I_{cm}\alpha=bF-fR\quad\text{center of mass axis with}\quad I_{cm}=\frac12mR^2,\quad(9)$$
$$F-f=ma,\quad(10)$$

which leads to the same answer.
(b) Given the coefficient of static friction between the yo-yo and the table is μ_s, what is the maximum magnitude of the pulling force F for which the Yo-Yo rolls without slipping?
From the above result, plus the static friction condition $f\le\mu_smg$, one finds the maximum magnitude the pulling force is given by

$$F\le\frac{3R}{2b+R}\mu_smg.\quad(11)$$

** A Ball Rolling Uphill.
A ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal. Treat the ball as a uniform solid sphere.
(a) Draw the free-body diagram for the ball. Explain why the friction force must be directed uphill.
Solution: For an uphill rolling (without slipping) ball, its angular velocity and lin simultaneously $v=\omega R$. Choose the COM as the axis and then find the only non-ze friction. It has to be uphill in order to decelerate the angular velocity and keep the b with its linear motion. ($\omega R=v$)



(b) What is the acceleration of the center of mass of the ball? **Answers:** $a_{cm}=(5/7$
Solution: From Newton's second law and its rotational analog, one gets

$$ma_{cm}=mg\sin\beta-f_s,$$
$$I\alpha=\frac25mR^2\alpha=f_sR.$$

The acceleration and angular acceleration are related by $a_{cm}=R\alpha$. Combining all these equations, one gets $a_{cm}=(5/7)g\sin\beta$.
(c) What minimum coefficient of static friction is needed to prevent slipping? **Answers:** $(2/7)\tan\beta$.
Solution: First, one can also solve for f_s and find $f_s=\frac27mg\sin\beta$. In addition, the ball has no motion perpendicular to the ramp, thus

$$n=mg\cos\beta,\ \Rightarrow\ \mu_s\ge f_s/n=(2/7)\tan\beta.\quad(4)$$

§7 Chapter 12 Elasticity & Equilibrium

** Knocking Over a Post.
One end of a post weighing 400 N and with height h rests on a rough horizontal surface with $\mu_s=0.30$. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post (Fig. below). A horizontal force $\vec{F}$ is exerted on the post as shown. If the force F is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip? **Answers:** $F=400$ N.

EXECUTE: (a) Taking torques about the point where the rope is fastened to the ground, the lever arm of the applied force is $h/2$ and the lever arm of both the weight and the normal force is $h\tan\theta$, and so $F\frac{h}{2}=(n-w)h\tan\theta$.

Taking torques about the upper point (where the rope is attached to the post), $fh=F\frac{h}{2}$. Using $f\le\mu_sn$ and solving for F, $F\le2w\left(\frac{1}{\mu_s}-\frac{1}{\tan\theta}\right)^{-1}=2(400\text{ N})\left(\frac{1}{0.30}-\frac{1}{\tan36.9°}\right)^{-1}=400$ N.

** Bulk Modulus of an Ideal Gas.
The equation of state (the equation relating pressure, volume, and temperature) for an ideal gas is $pV=nRT$, where n and R are constants.
(a) Show that if the gas is compressed while the temperature T is held constant, the bulk modulus B is equal to the pressure p.
Solution: The bulk modulus B is defined as

$$B=-\frac{\Delta p}{\Delta V/V}=-V\frac{dp}{dV}.\quad(9)$$

From the equation of motion $pV=nRT$ together with constant T, one gets $d(pV)=0=Vdp+pdV$, which is $dp/dV=-p/V$. Therefore, $B=p$ for constant T.
(b) When an ideal gas is compressed without the transfer of any heat into or out of it, the pressure and volume are related by pV^γ = constant, where γ is a constant having different values for different gases. Show that, in this case, the bulk modulus is given by $B=\gamma p$.
Solution: For the adiabatic process (no heat transfer), similarly one gets $d(pV^\gamma)=0$, which implies $V^\gamma dp+\gamma pV^{\gamma-1}dV=0$ and $dp/dV=-\gamma p/V$. Plugging it into the definition of the bulk modulus B

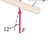
$$B=-\frac{\Delta p}{\Delta V/V}=-V\frac{dp}{dV}=\gamma p.\quad(10)$$

Usually $\gamma>1$ and it means that ideal gas is harder to compress without heat transfer than the gas with constant temperature.

** Downhill Hiking.
During vigorous downhill hiking, the force on the knee cartilage (the medial and lateral meniscus) can be up to eight times body weight. Depending on the angle of descent, this force can cause a large shear force on the cartilage and deform it. The cartilage has an area of about 10 cm² and a shear modulus of 12 MPa. If the hiker plus his pack have a combined mass of 110 kg (not unreasonable), and if the maximum force at impact is 8 times his body weight (which, of course, includes the weight of his pack) at an angle of 12° with the cartilage (Fig. below), through what angle (in degrees) will his knee cartilage be deformed? (Recall that the bone below the cartilage pushes upward with the same force as the downward force.)
Answers: The target answer is the angle of deformation of the cartilage $\phi\equiv\frac{x}{h}=0.15$rad, which is also known as the shear strain.

Solution: Vigorous downhill hiking produces a shear force on the knee cartilage which could deform the cartilage. The shear strain, or the angle of deformation, can be obtained from the definition as follows

$$\text{Shear Strain}:\quad\phi\equiv\frac{x}{h}=\frac{\text{Shear Stress}}{\text{Shear Modulus}}=\frac{F_{\parallel}/A}{S}=\frac{8*110*9.8*\sin(12°)/(10\times10^{-4})}{12\times10^6}=0.15\text{r}$$



The angle of 0.15rad is 8.6° in degrees.