Lecture 18

§1 Natural Logarithmic Function: Algebraic Properties

1. Theorem: Algebraic properties of ln

For any $b \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{>0}$

1. $\ln(bx) = \ln b + \ln x$
2. $\ln(b/x) = \ln b - \ln x$
3. $\ln(1/x) = -\ln x$
4. $\ln(x^r) = r\ln x$, for any $r \in \mathbb{Q}$

2. Proof of 1:

Let $f(x) = \ln(bx)$, defined on $(0,\infty)$. Then

$$f'(x) = \frac{b}{bx} = \frac{1}{x} = \ln' x$$

So $f(x) = \ln x + C$ for some constant C, $\forall x \in (0,+\infty)$

Substituting $x=1$ yields $\ln b = f(1) = \ln 1 + C = C$

So $f(x) = \ln x + \ln b$

3. Proof of 4

Let $f(x) = \ln(x^r)$, defined on $(0,\infty)$. Then

$$f'(x) = \frac{r x^{r-1}}{x^r} = r \cdot \frac{1}{x} = \frac{d}{dx}(r\ln x)$$

So $f(x) = r\ln x + C$ for some constant C, $\forall x \in (0,\infty)$

Substituting $x=1$ yields

$$f(1) = \ln 1^r = 0 = r\ln 1 + C = C$$

So $C=0$ and $\ln(x^r) = r\ln x$

4. Proof of 2

$$\ln\left(\frac{b}{x}\right) = \ln(bx^{-1})$$
$$= \ln b + \ln(x^{-1})$$

$= \ln b - \ln x$

5. Proof of 3

$$\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x$$
$$= -\ln x$$

§2 Natural Logarithmic Function: Graph and Range

1. Range

We already know that ln is differentiable and increasing on $(0, \infty)$

Since $\ln 2 = \int_1^2 \frac{1}{t}\,dt > (2-1)\left(\frac{1}{2}\right) = \frac{1}{2}$

It follows that for any $n \in \mathbb{Z}_+$,

$$\ln(2^n) = n \ln 2 > \frac{n}{2}$$

Since ln is increasing,

$$\ln(x) > \ln(2^n) > \frac{n}{2} \text{ for all } x \in [2^n, \infty)$$

As $n \to \infty$, $\ln(2^n) \to \infty$, and

$$\lim_{x \to \infty} \ln x = \infty$$

Also,

$$\lim_{x \to 0^+} \ln x = \lim_{u \to \infty} \ln\left(\frac{1}{u}\right) = \lim_{u \to \infty} (-\ln u) = -\infty$$

Now let y_0 be any fixed real number.

By the two limits above, there exists x_1 and x_2 in $(0, +\infty)$ such that

$$\ln(x_1) < y_0 \qquad \ln(x_2) > y_0$$

By IVT, there exists $c \in [x_1, x_2]$ such that $\ln c = y_0$

This shows that range$(\ln) = \mathbb{R}$

2. Concavity

Since $\ln''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} < 0$ for all $x \in (0, \infty)$, the curve $y=\ln x$ is concave down

3. Limit of derivatives

Note that $\lim_{x\to 0^+} \ln'(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$

Also $\lim_{x\to\infty} \ln'(x) = \lim_{x\to\infty} \frac{1}{x} = 0$

§3 Natural Logarithmic Function: Composite Function $\ln\circ g$

1. Composite function $\ln\circ g$

Let $g: D \to R > 0$ be differentiable on D. Then:

$$(\ln\circ g)'(x) = \ln'(g(x))\cdot g'(x) = \frac{g'(x)}{g(x)}$$

If we take $g(x) = |x|$ with $D = R\setminus\{0\}$, then:

$$g'(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

So $$g'(x) = \frac{|x|}{x}$$

By the formula above we have

$$\frac{d}{dx}\ln|x| = \frac{1}{|x|}\cdot\frac{|x|}{x} = \frac{1}{x}$$

This means that for $x\in R\setminus\{0\}$, $\ln|x|$ is an antiderivative of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

More generally, if $g(x) = |f(x)|$ where f is differentiable and never zero, then by the chain rule:

$$g'(x) = \frac{|f(x)|}{f(x)}\cdot f'(x)$$

So $$\frac{d}{dx}\ln|f(x)| = \frac{1}{|f(x)|}\cdot\frac{|f(x)|}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}$$

Then $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

is valid on any interval contained in the domain of f, $f(x)\neq 0$.

e.g. Find $\int \tan x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$
$$= -\int \frac{-\sin x}{\cos x}\,dx$$
$$= -\ln|\cos x| + C$$
$$= \ln|\sec x| + C$$

e.g. Find $\int \sec x\,dx$

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$
$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x}\,dx$$
$$= \ln|\sec x + \tan x|\,dx$$

2. Integrals of the tangent, cotangent, secant, cosecant functions.

1° $\int \tan u\,du = \ln|\sec u| + C$

2° $\int \cot u\,du = \ln|\sin u| + C$

3° $\int \sec u\,du = \ln|\sec u + \tan u| + C$

4° $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

e.g. Find $\int_0^{\frac{\pi}{6}} \tan 2x\,dx$

Let $u=2x$, then $dx = \frac{1}{2}du$, $u=0$ when $x=0$, $u=\frac{\pi}{3}$ when $x=\frac{\pi}{6}$

$$\int_0^{\frac{\pi}{6}} \tan 2x\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{2}\tan u\,du$$
$$= \frac{1}{2}\left[\ln|\sec u|\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\ln 2$$

§4 Natural Logarithmic Function: Logarithmic Differentiation

1. Logarithmic differentiation

Suppose $F(x)$ involves complicated products, quotients and powers, e.g.

$$F(x) = \frac{f_1(x)^{m_1} f_2(x)^{m_2}}{f_3(x)^{m_3}}$$

Taking ln on both sides we get:

$$\ln F(x) = m_1 \ln f_1(x) + m_2 \ln f_2(x) - m_3 \ln f_3(x)$$

Differentiating both sides yields:

$$\frac{F'(x)}{F(x)} = m_1 \frac{f_1'(x)}{f_1(x)} + m_2 \frac{f_2'(x)}{f_2(x)} - m_3 \frac{f_3'(x)}{f_3(x)}$$

Move $F(x)$ to the right to get $F'(x)$.

e.g. Find y' where $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{y'}{y} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

$$y' = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{5}{3x+2}\right)$$

§5 Natural Exponential Function: Definition and Derivative

1. Definition

Since $\ln : (0,\infty) \to R$ is bijective,

it has an inverse function $\exp : R \to (0,\infty)$

called the natural exponential function.

Since $\ln e = 1$ by definition, we have $e = \exp(1)$

Since $e > 0$, e^r is defined for any rational power r.

Since $\ln(e^r) = r \ln e \overset{\text{def}}{=} r$, we have $\exp(r) = e^r$, $\forall r \in Q$

For irrational power x we simply define e^x as follows: $e^x \overset{\text{def}}{=} \exp(x)$, $\forall x \in R \setminus Q$

Hence, $\exp(x) = e^x$ for all $x \in R$

Consequently

$e^{\ln x} = x, \quad \forall x \in (0, \infty)$

$\ln(e^x) = x, \quad \forall x \in R$

2. Derivative

Let $y = \exp(x)$, then $\ln y = x$

Apply dy/dx to both sides gives

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = \exp(x)$$

So $\exp'(x) = \exp(x)$ or $\frac{d}{dx} e^x = e^x$

e.g. Find $\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$

Let $u = \sin x$, then $du = \cos x \, dx$, $u=1$ when $x = \frac{\pi}{2}$, $u=0$ when $x=0$,

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx = \int_0^1 e^u \, du$$

$$= [e^u]_0^1$$

$$= e - 1$$

§6 Natural Exponential Function: Algebraic Properties

1. Theorem: Algebraic properties of exp

For any real number x_1, x_2 and x:

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^x)^r = e^{rx}$, for any $r \in Q$

2. Proof of 1

Let $y_1 = e^{x_1}$, $y_2 = e^{x_2}$. Then

$e^{x_1 + x_2} = e^{\ln y_1 + \ln y_2}$

$= e^{\ln(y_1 y_2)}$

$= y_1 y_2$

$= e^{x_1} \cdot e^{x_2}$

3. Proof of 4.

Let $y = e^x$. Then

$e^{rx} = e^{r \ln y}$

$= e^{\ln(y^r)}$

$= y^r$

$= (e^x)^r$

4. Proof of 3

$e^{x_1 - x_2} = e^{x_1} \cdot e^{-x_2}$

$= \frac{e^{x_1}}{e^{x_2}}$

5. Proof of 2

$e^{-x} = e^{0-x}$

$= \frac{1}{e^x}$