Lecture 8

§1 Surfaces of revolution

1. Arc length differentials

给定参数曲线 $(x,y)=(f(t),g(t))$，$t\in I$，a为I内一定值，定义

$$S(t):=\int_a^t\sqrt{f'(u)^2+g'(u)^2}\,du$$

为点$(f(a),g(a))$沿曲线至点$(f(t),g(t))$的 signed distance（可以为负）

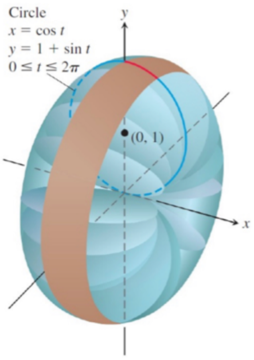
由 FTC，$\frac{ds}{dt}=\sqrt{f'(t)^2+g'(t)^2}=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}$

微分 $ds=\sqrt{f'(t)^2+g'(t)^2}\,dt$ 被称为 arc length differential

2. 旋转体表面积

将 $x=f(t)$，$y=g(t)\geq 0$ 绕 x轴旋转一周，形成的旋转体在 $a\leq t\leq b$上的表面积为 $S=\int_a^b 2\pi g(t)\underbrace{\sqrt{f'(t)^2+g'(t)^2}\,dt}_{ds}$

例：Using the formula, the surface area drawn in the following figure can be computed with a straightforward integral. (This is Example 9 in Chapter 11.2 of the book.)



$$S=\int_0^{2\pi}2\pi(1+\sin t)\cdot\sqrt{\sin^2 t+\cos^2 t}\,dt$$
$$=2\pi\int_0^{2\pi}(1+\sin t)\,dt$$
$$=4\pi^2$$

§2 Polar coordinates

1. 定义

Definition

For a point $(x,y)\in\mathbb{R}^2$ on the xy-plane in Cartesian coordinates:

- let r be the length of the line segment L joining the points $(0,0)$ and (x,y), and;
- let θ be the angle made by L and the positive x-axis, with the positive sign indicating the counterclockwise measurement.

Then the point (r,θ) is called a polar coordinate of the point (x,y).

注：1° 若一点的极坐标为(r,θ)，$r<0$，则该点的坐标也可表示为$(-r,\theta+\pi)$

2° 一点的极坐标不唯一，θ可随意$\pm 2k\pi$

2. Conversion between Cartesian coordinates (笛卡尔坐标) and polar coordinates

1° 给定 (r,θ): $x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$

给定 (x,y): $r^2 = x^2+y^2$, $\tan\theta = \frac{y}{x}$ (if $x\neq 0$)

若 $x=0$, 则 $\theta=\frac{\pi}{2}$ if $y>0$, $\theta=\frac{3\pi}{2}$ if $y<0$

2° 极坐标曲线: $F(r,\theta)=0$ 是 xy 平面内极坐标满足这一方程的点集

例: Express the curve $r=\frac{4}{2\cos\theta-\sin\theta}$ with a Cartesian equation

$$r=\frac{4}{2\cos\theta-\sin\theta}$$

$$\Rightarrow 2r\cos\theta - r\sin\theta = 4$$

$$\Rightarrow 2x-y=4$$

$$\Rightarrow y=2x-4$$

例: Express the curve $x^2+xy+y^2=1$ with a polar equation

$$x^2+xy+y^2=1$$

$$\Rightarrow r^2\cos^2\theta + r\cdot\cos\theta\cdot r\cdot\sin\theta + r^2\sin^2\theta = 1$$

$$\Rightarrow r^2(1+\sin\theta\cos\theta)=1$$

$$\Rightarrow r^2=\frac{1}{1+\cos\theta\sin\theta}$$

3. Sketch polar curves

1° 利用对称性

① 若 $(r,-\theta)$ 或 $(-r,\pi-\theta)$ 在曲线上

$\Rightarrow$ symmetry about x-axis

② 若 $(r,\pi-\theta)$ 或 $(-r,-\theta)$ 在曲线上

$\Rightarrow$ symmetry about y-axis

③ 若 $(r,\theta+\pi)$ 或 $(-r,\theta)$ 在曲线上

$\Rightarrow$ symmetry about origin by rotation of 180°

2° 作出 $r-\theta$ 图像, 据此描出 x-y 图像

例: Sketch $r=1-\cos\theta$ on the xy-plane

· since $\cos(-\theta)=\cos\theta$, $f(-\theta)=f(\theta)$, so the curve is symmetric about the x-axis.

Only need to investigate $\theta\in[0,\pi]$

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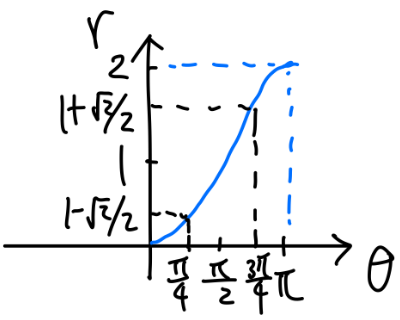

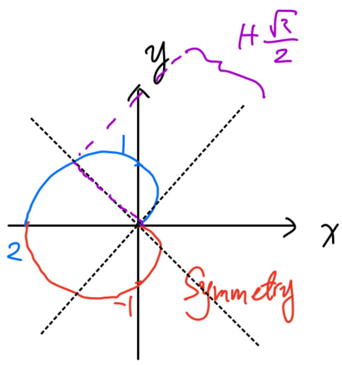

例: Sketch $r^2=\sin 2\theta$ on the xy-plane

- Since $(-r)^2=r^2$, curve is symmetric about the origin
- For $\theta\in[0,2\pi]$, $\sin 2\theta\geq 0$,
  so $\theta\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3\pi}{2}]$
- By symmetry, suffices to investigate $\theta\in[0,\frac{\pi}{2}]$
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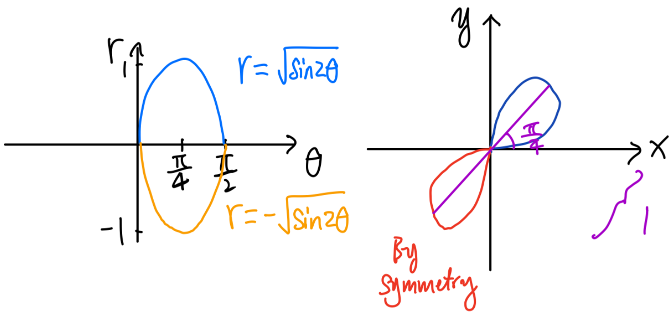


4. Slope

对于一个极坐标曲线 $r=f(\theta)$，曲线上的点满足

$x=f(\theta)\cdot\cos\theta$, $y=f(\theta)\cdot\sin\theta$

可利用此参数方程形式（θ为参数）进行计算.

极坐标曲线上 $\theta=\theta_0$ 处的切线斜率为

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0}=\left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\theta_0}=\left.\frac{f(\theta)\cos\theta+f'(\theta)\cdot\sin\theta}{-f(\theta)\sin\theta+f'(\theta)\cdot\cos\theta}\right|_{\theta=\theta_0}\quad(\text{分母不为}0)$$

5. Arc lengths

Consider a curve on the xy-plane given in polar coordinates by

$$r=f(\theta),\ a\leq\theta\leq b,$$

where f' is continuous. If the curve is traversed exactly once, then by the arc length formula ~~on Page 8~~ in 11.2, its length L is given by

$$L=\int_a^b\sqrt{(f(\theta))^2+f'(\theta))^2}\,d\theta=\int_a^b\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

证明:

Since $x=f(\theta)\cos\theta$, $y=f(\theta)\cdot\sin\theta$

$$\begin{aligned}L&=\int_a^b\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta\\&=\int_a^b\sqrt{(f'(\theta)\cdot\cos\theta-f(\theta)\sin\theta)^2+(f'(\theta)\cdot\sin\theta+f(\theta)\cdot\cos\theta)^2}\,d\theta\\&=\int_a^b\sqrt{f'(\theta)^2\cos^2\theta+f(\theta)^2\sin^2\theta+f'(\theta)^2\sin^2\theta+f^2(\theta)\cos^2\theta}\,d\theta\\&=\int_a^b\sqrt{f'(\theta)^2+f(\theta)^2}\,d\theta\end{aligned}$$

例: Find the arc length of the polar curve $r=1-\cos\theta$

Sol: Curve is traced out exactly once for $0\leq\theta\leq2\pi$

$$\begin{aligned}L&=\int_0^{2\pi}\sqrt{(1-\cos\theta)^2+\sin^2\theta}\,d\theta\\&=\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta\end{aligned}$$

$= \sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$

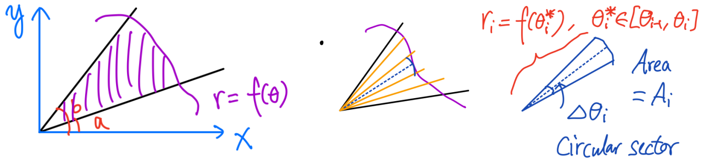
$= \sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2\frac{\theta}{2}}\,d\theta$

$= 2\cdot 2\cdot \cos\frac{\theta}{2}\Big|_0^{2\pi}$

$= 8$

6. Areas: Fan-shaped regions

1° xy平面内满足 $0\le r\le f(\theta)$, $a\le\theta\le b$ $(b-a\le 2\pi)$ 的区域

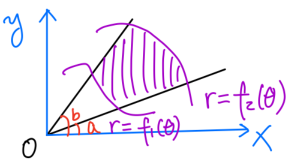


· Partition the θ-interval [a, b]

· $A = \lim_{n\to\infty}\sum_{i=1}^{n} A_i = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\Delta\theta_i}{2\pi}\pi\cdot f(\theta_i^*)^2 = \lim_{n\to\infty}\frac{1}{2}\sum_{i=1}^{\infty} f(\theta_i^*)^2\Delta\theta_i$

$A = \frac{1}{2}\int_a^b f(\theta)^2\,d\theta$

2° xy平面内满足 $0\le f_1(\theta)\le r\le f_2(\theta)$, $a\le\theta\le b$ $(b-a\le 2\pi)$ 的区域



$A = \frac{1}{2}\int_a^b \left(f_2(\theta)^2 - f_1(\theta)^2\right)d\theta$

例: Find the area of the region enclosed by the curve

$r = 2(1+\cos\theta)$, $0\le\theta\le 2\pi$

Sol: $A = \int_0^{2\pi}\frac{1}{2}\cdot 4(1+\cos\theta)^2\,d\theta$

$= 2\int_0^{2\pi} 1+2\cos\theta+\cos^2\theta\,d\theta$

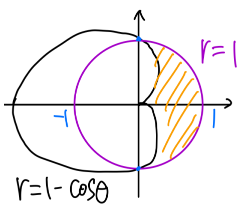
$= 2\cdot\left(2\pi + 4\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$

$= 6\pi$

例: Consider the region S enclosed by the curve

$r = 1-\cos\theta$, $0\le\theta\le 2\pi$

Find the area of the region that lies outside S and inside $r=1$



$1-\cos\theta\le r\le 1$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

Sol: $A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1^2-(1-\cos\theta)^2\right)d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2\cos\theta-\cos^2\theta\right)d\theta$

$= \frac{1}{2}\left(2\times 2 - 2\times\frac{1}{2}\times\frac{\pi}{2}\right)$

$= 2-\frac{\pi}{4}$