Structural Optimization for Large-Scale Problems

Lecture 7: Huge-scale optimization

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Outline

Problems sizes

Random coordinate search

Confidence level of solutions

Sparse Optimization problems

Sparse updates for linear operators

Fast updates in computational trees

Simple subgradient methods

Application examples

Nonlinear Optimization: problems sizes

§ 1 Huge-scale problem

Class	Operations	Dimension	Iter.Cost	Memory	
Small-size	All	$10^0 - 10^2$	$n^4 \to n^3$	Kilobyte:	10^3
Medium-size	A^{-1}	$10^3 - 10^4$	$n^3 \to n^2$	Megabyte:	10^6
Large-scale	Ax	$10^5 - 10^7$	$n^2 \to n$	Gigabyte:	10^9
Huge-scale	$x + y$	$10^8 - 10^{12}$	$n \to \log n$	Terabyte:	10^{12}

Sources of Huge-Scale problems

- Internet (New)
- Telecommunications (New)
- Finite-element schemes (Old)
- Partial differential equations (Old)

Very old optimization idea: Coordinate Search

Problem: $\min_{x \in \mathbb{R}^n} f(x)$ (f is convex and differentiable).

Coordinate relaxation algorithm

For $k \geq 0$ iterate

1. Choose active coordinate i_k.
2. Update $x_{k+1} = x_k - h_k \nabla_{i_k} f(x_k) e_{i_k}$ ensuring $f(x_{k+1}) \leq f(x_k)$.

   (e_i is ith coordinate vector in $\mathbb{R}^n$.)

Main advantage: Very simple implementation.

Possible strategies

1. Cyclic moves. (Difficult to analyze.)
2. Random choice of coordinate (Why?)
3. Choose coordinate with the maximal directional derivative.

Complexity estimate: assume 对 complexity 的分析

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^n.$$

Let us choose $h_k = \frac{1}{L}$. Then

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \frac{1}{2L}|\nabla_{i_k} f(x_k)|^2 \geq \frac{1}{2nL}\|\nabla f(x_k)\|^2 \\ &\geq \frac{1}{2nLR^2}(f(x_k) - f^*)^2. \end{aligned}$$

Hence, $f(x_k) - f^* \leq \frac{2nLR^2}{k}$, $k \geq 1$. (For Grad.Method, drop n.)

This is the only known theoretical result known for CDM before 2012!

Criticism

Theoretical justification:

- Complexity bounds are not known for the most of the schemes.
- The only justified scheme needs computation of the whole gradient.

  (Why don't use GM?) (为什么不用 full-dimensional gradient method?)

Computational complexity:

- Fast differentiation: if function is defined by a sequence of operations, then $T(\nabla f) \leq 4T(f)$.
- Can we do anything without computing the function's values?

Result: CDM were almost out of the computational practice for decades.

Google problem

Let $E \in \mathbb{R}^{n \times n}$ be an incidence matrix of a graph. Denote $e = (1, \ldots, 1)^T$ and

$$\bar{E} = E \cdot \operatorname{diag}(E^T e)^{-1}.$$

Thus, $\bar{E}^T e = e$. Our problem is as follows:

($\bar{E}$ 为 column-stochastic matrix)

$$\text{Find } x^* \geq 0 : \quad \bar{E}x^* = x^*.$$

(x^* 为 dominating eigenvector)

Optimization formulation:

$$f(x) \stackrel{\text{def}}{=} \frac{1}{2}\|\bar{E}x - x\|^2 + \frac{\gamma}{2}[\langle e, x\rangle - 1]^2 \quad \to \quad \min_{x \in \mathbb{R}^n}$$

Huge-scale problems

Main features

- The size is very big ($n \geq 10^7$).
- The data is distributed in space.
- The requested parts of data are not always available.
- The data may be changing in time.

Consequences

The simplest operations are expensive or infeasible:

- Update of the full vector of variables.
- Matrix-vector multiplication.
- Computation of the objective function's value, etc.

Structure of the Google Problem

Let ua look at the gradient of the objective:

$$\begin{aligned} \nabla_i f(x) &= \langle a_i, g(x)\rangle + \gamma[\langle e, x\rangle - 1],\ i = 1, \ldots, n, \\ g(x) &= \bar{E}x - x \in \mathbb{R}^n, \quad (\bar{E} = (a_1, \ldots, a_n)). \end{aligned}$$

Main observations:

- The coordinate move $x_+ = x - h_i \nabla_i f(x) e_i$ needs $O(p_i)$ a.o.

  (p_i is the number of nonzero elements in a_i.)
- $d_i \stackrel{\text{def}}{=} \operatorname{diag}\left(\nabla^2 f \stackrel{\text{def}}{=} \bar{E}^T \bar{E} + \gamma e e^T\right)_i = \gamma + \frac{1}{p_i}$ are available.

  We can use them for choosing the step sizes ($h_i = \frac{1}{d_i}$).

Reasonable coordinate choice strategy? Random!

Random coordinate descent methods (RCDM)

§2 RCDM

$$\min_{x \in \mathbb{R}^N} f(x), \quad (f \text{ is convex and differentiable})$$

Main Assumption:

$$|f_i'(x + h_i e_i) - f_i'(x)| \leq L_i |h_i|, \quad h_i \in R, \ i = 1, \ldots, N,$$

where e_i is a coordinate vector. Then 关于第i个 coordinate 的 descent lemma

$$f(x + h_i e_i) \leq f(x) + f_i'(x) h_i + \frac{L_i}{2} h_i^2. \quad x \in \mathbb{R}^N, \ h_i \in \mathbb{R}.$$

Define the coordinate steps: $T_i(x) \stackrel{\text{def}}{=} x - \frac{1}{L_i} f_i'(x) e_i.$ Then,

$$f(x) - f(T_i(x)) \geq \frac{1}{2L_i} [f_i'(x)]^2, \quad i = 1, \ldots, N.$$

Random choice for coordinates

We need a special random counter $\mathcal{R}_\alpha$, $\alpha \in \mathbb{R}$:

$$Prob\,[i] \;=\; p_\alpha^{(i)} \;=\; L_i^\alpha \cdot \left[\sum_{j=1}^{N} L_j^\alpha\right]^{-1}, \qquad i = 1, \ldots, N.$$

Note: $\mathcal{R}_0$ generates uniform distribution.

Method $RCDM(\alpha, x_0)$

For $k \geq 0$ iterate:

1) Choose $i_k = \mathcal{R}_\alpha$.

2) Update $x_{k+1} = T_{i_k}(x_k)$.

Complexity bounds for RCDM

We need to introduce the following norms for $x, g \in \mathbb{R}^N$:

$$\|x\|_\alpha = \left[\sum_{i=1}^N L_i^\alpha [x^{(i)}]^2\right]^{1/2}, \quad \|g\|_\alpha^* = \left[\sum_{i=1}^N \frac{1}{L_i^\alpha} [g^{(i)}]^2\right]^{1/2}.$$

After k iterations, $RCDM(\alpha, x_0)$ generates random output x_k, which depends on $\xi_k = \{i_0, \ldots, i_k\}$. Denote $\phi_k = E_{\xi_{k-1}} f(x_k)$.

i的选取历史

Theorem. For any $k \geq 1$ we have

第k次迭代的函数值期望

$$\boxed{\phi_k - f^* \leq \frac{2}{k} \cdot \left[\sum_{j=1}^N L_j^\alpha\right] \cdot R_{1-\alpha}^2(x_0),}$$

where $R_\beta(x_0) = \max_x \left\{ \max_{x_* \in X^*} \|x - x_*\|_\beta : \ f(x) \leq f(x_0) \right\}$.

Interpretation

Denote $S_\alpha = \sum_{i=1}^{N} L_i^\alpha$.

1. $\alpha = 0$. Then $S_0 = N$, and we get

$$\phi_k - f^* \quad \leq \quad \frac{2N}{k} \cdot R_1^2(x_0).$$

Note

- We use the metric $\|x\|_1^2 = \sum_{i=1}^{N} L_i [x^{(i)}]^2$.
- Matrix with diagonal $\{L_i\}_{i=1}^N$ can have its norm equal to n.
- Hence, for GM we can guarantee the same bound. $\frac{2nLR^2}{k}$

  But its cost of iteration is much higher!

Interpretation

2. $\alpha = \frac{1}{2}$. Denote

$$D_\infty(x_0) = \max_x \left\{ \max_{y \in X^*} \max_{1 \le i \le N} |x^{(i)} - y^{(i)}| : \; f(x) \le f(x_0) \right\}.$$

Then, $R^2_{1/2}(x_0) \le S_{1/2} D^2_\infty(x_0)$, and we obtain

$$\boxed{\phi_k - f^* \;\; \le \;\; \frac{2}{k} \cdot \left[\sum_{i=1}^N L_i^{1/2} \right]^2 \cdot D^2_\infty(x_0).}$$

Note:

- For the first order methods, the worst-case complexity of minimizing over a box does depend on N.
- Since $S_{1/2}$ can be bounded, RCDM can be applied in situations when the usual GM fails.

Interpretation

3. $\alpha = 1$. Then $R_0(x_0)$ is the size of the initial level set in the standard Euclidean norm. Hence,

$$\phi_k - f^* \quad \leq \quad \frac{2}{k} \cdot \left[\sum_{i=1}^{N} L_i \right] \cdot R_0^2(x_0) \quad \equiv \quad \frac{2N}{k} \cdot \left[\frac{1}{N} \sum_{i=1}^{N} L_i \right] \cdot R_0^2(x_0).$$

Rate of convergence of GM can be estimated as

$$f(x_k) - f^* \leq \frac{\gamma}{k} R_0^2(x_0),$$

where γ satisfies condition $f''(x) \preceq \gamma \cdot I$, $x \in \mathbb{R}^N$.

Note: maximal eigenvalue of symmetric matrix can reach its trace.

In the worst case, the rate of convergence of GM is the same as that of *RCDM*.

§3 RCDM 的 confidence level

Minimizing the strongly convex functions

Theorem. Let $f(\cdot)$ be strongly convex with respect to $\|\cdot\|_{1-\alpha}$ with convexity parameter $\sigma_{1-\alpha} > 0$.

Then, for $\{x_k\}$ generated by $RCDM(\alpha, x_0)$ we have

$$\boxed{\phi_k - \phi^* \quad \leq \quad \left(1 - \frac{\sigma_{1-\alpha}}{S_\alpha}\right)^k (f(x_0) - f^*).}$$

Proof: Let x_k be generated by *RCDM* after k iterations.

Let us estimate the expected result of the next iteration.

$$f(x_k) - E_{i_k}(f(x_{k+1})) = \sum_{i=1}^N p_\alpha^{(i)} \cdot [f(x_k) - f(T_i(x_k))]$$

$$\geq \sum_{i=1}^N \frac{p_\alpha^{(i)}}{2L_i} [f_i'(x_k)]^2 \;=\; \frac{1}{2S_\alpha}(\|f'(x_k)\|_{1-\alpha}^*)^2$$

$$\geq \frac{\sigma_{1-\alpha}}{S_\alpha}(f(x_k) - f^*).$$

It remains to compute expectation in ξ_{k-1}. □

Confidence level of the answers

Note: We have proved that the expected values of random $f(x_k)$ are good.

Can we guarantee anything after a single run?

Confidence level: Probability $\beta \in (0,1)$, that some statement about random output is correct.

Main tool: Markov inequality $(\xi \geq 0)$:

$$Prob\,[\xi \geq T] \quad \leq \quad \frac{E(\xi)}{T}.$$

Our situation:

$$Prob\,[f(x_k) - f^* \geq \epsilon] \quad \leq \quad \frac{1}{\epsilon}[\phi_k - f^*] \;\leq\; 1 - \beta.$$

We need $\phi_k - f^* \leq \epsilon \cdot (1-\beta)$. Too expensive for $\beta \to 1$?

Regularization technique

Consider $f_\mu(x) = f(x) + \frac{\mu}{2}\|x - x_0\|^2_{1-\alpha}$. It is strongly convex.

Therefore, we can obtain $\phi_k - f^*_\mu \leq \epsilon \cdot (1-\beta)$ in

$$O\left(\frac{1}{\mu} S_\alpha \ln \frac{1}{\epsilon \cdot (1-\beta)}\right) \text{ iterations.}$$

Theorem. Define $\alpha = 1$, $\mu = \frac{\epsilon}{4R_0^2(x_0)}$, and choose

$$k \quad \geq \quad 1 + \frac{8S_1 R_0^2(x_0)}{\epsilon}\left[\ln \frac{2S_1 R_0^2(x_0)}{\epsilon} + \ln \frac{1}{1-\beta}\right].$$

Let x_k be generated by $RCDM(1, x_0)$ as applied to f_μ. Then

$$Prob\left(f(x_k) - f^* \leq \epsilon\right) \quad \geq \quad \beta.$$

Note: $\beta = 1 - 10^{-p} \quad \Rightarrow \quad \ln 10^p = 2.3p.$

Implementation details: Random Counter

Given the values L_i, $i = 1, \ldots, N$, generate efficiently random $i \in \{1, \ldots, N\}$ with probabilities $Prob\,[i = k] = L_k / \sum_{j=1}^{N} L_j$.

Solution: a) Trivial $\Rightarrow$ $O(N)$ operations.

b). Assume $N = 2^p$. Define $p + 1$ vectors $S_k \in \mathbb{R}^{2^{p-k}}$, $k = 0, \ldots, p$:

$$S_0^{(i)} = L_i,\ i = 1, \ldots, N.$$

$$S_k^{(i)} = S_{k-1}^{(2i)} + S_{k-1}^{(2i-1)},\ i = 1, \ldots, 2^{p-k}, \quad k = 1, \ldots, p.$$

Algorithm: Make the choice in p steps, from top to bottom.

- If the element i of S_k is chosen, then choose in S_{k-1} either $2i$ or $2i - 1$ in accordance to probabilities $\frac{S_{k-1}^{(2i)}}{S_k^{(i)}}$ or $\frac{S_{k-1}^{(2i-1)}}{S_k^{(i)}}$.

Difference: for $n = 2^{20} > 10^6$ we have $p = \log_2 N = 20$.

Sparse problems

Problem: $\min_{x \in Q} f(x)$, where Q is closed and convex in $\mathbb{R}^N$, and

- $f(x) = \Psi(Ax)$, where Ψ is a simple *convex function*:

$$\Psi(y_1) \geq \Psi(y_2) + \langle \Psi'(y_2), y_1 - y_2 \rangle, \quad y_1, y_2 \in \mathbb{R}^M,$$

- $A : \mathbb{R}^N \to \mathbb{R}^M$ is a *sparse* matrix.

Let $p(x) \stackrel{\text{def}}{=} \#$ of nonzeros in x. Sparsity coefficient: $\gamma(A) \stackrel{\text{def}}{=} \frac{p(A)}{MN}$.

Example 1: Matrix-vector multiplication

- Computation of vector Ax needs $p(A)$ operations.
- Initial complexity MN is reduced in $\gamma(A)$ times.

Gradient Method

$$x_0 \in Q, \quad x_{k+1} = \pi_Q(x_k - hf'(x_k)), \quad k \geq 0.$$

Main computational expenses

- Projection onto a simple set Q needs $O(N)$ operations.
- Displacement $x_k \to x_k - hf'(x_k)$ needs $O(N)$ operations.
- $f'(x) = A^T \Psi'(Ax)$. If Ψ is simple, then the main efforts are spent for two matrix-vector multiplications: $2p(A)$.

Conclusion: As compared with *full* matrices, we accelerate in $\gamma(A)$ times.

Note: For Large- and Huge-scale problems, we often have $\gamma(A) \approx 10^{-4} \ldots 10^{-6}$. **Can we get more?**

Sparse updating strategy

Main idea

- After update $x_+ = x + d$ we have $y_+ \stackrel{\text{def}}{=} Ax_+ = \underbrace{Ax}_{y} + Ad$.
- What happens if d is *sparse*?

Denote $\sigma(d) = \{j : d^{(j)} \neq 0\}$. Then $\boxed{y_+ = y + \sum\limits_{j \in \sigma(d)} d^{(j)} \cdot Ae_j.}$

Its complexity, $\kappa_A(d) \stackrel{\text{def}}{=} \sum\limits_{j \in \sigma(d)} p(Ae_j)$, can be VERY small!

$$\begin{aligned} \kappa_A(d) &= M \sum_{j \in \sigma(d)} \gamma(Ae_j) = \gamma(d) \cdot \tfrac{1}{p(d)} \sum_{j \in \sigma(d)} \gamma(Ae_j) \cdot MN \\ &\leq \gamma(d) \max_{1 \leq j \leq m} \gamma(Ae_j) \cdot MN. \end{aligned}$$

If $\gamma(d) \leq c\gamma(A)$, $\gamma(A_j) \leq c\gamma(A)$, then $\boxed{\kappa_A(d) \leq c^2 \cdot \gamma^2(A) \cdot MN}$.

Expected acceleration: $(10^{-6})^2 = 10^{-12}$ $\Rightarrow$ 1 sec $\approx$ 32 000 years

When it can work?

- Simple methods: No full-vector operations! (Is it possible?)
- Simple problems: Functions with *sparse* gradients.

Examples

1. Quadratic function $f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$. The gradient
$$f'(x) = Ax - b, \quad x \in \mathbb{R}^N,$$
is *not* sparse even if A is sparse.
2. Piece-wise linear function $g(x) = \max\limits_{1\le i\le m}[\langle a_i, x\rangle - b^{(i)}]$. Its *subgradient* $f'(x) = a_{i(x)}$, $i(x):\ f(x) = \langle a_{i(x)}, x\rangle - b^{(i(x))}$, can be sparse if a_i is sparse!

But: We need a fast procedure for updating *max-operations*.

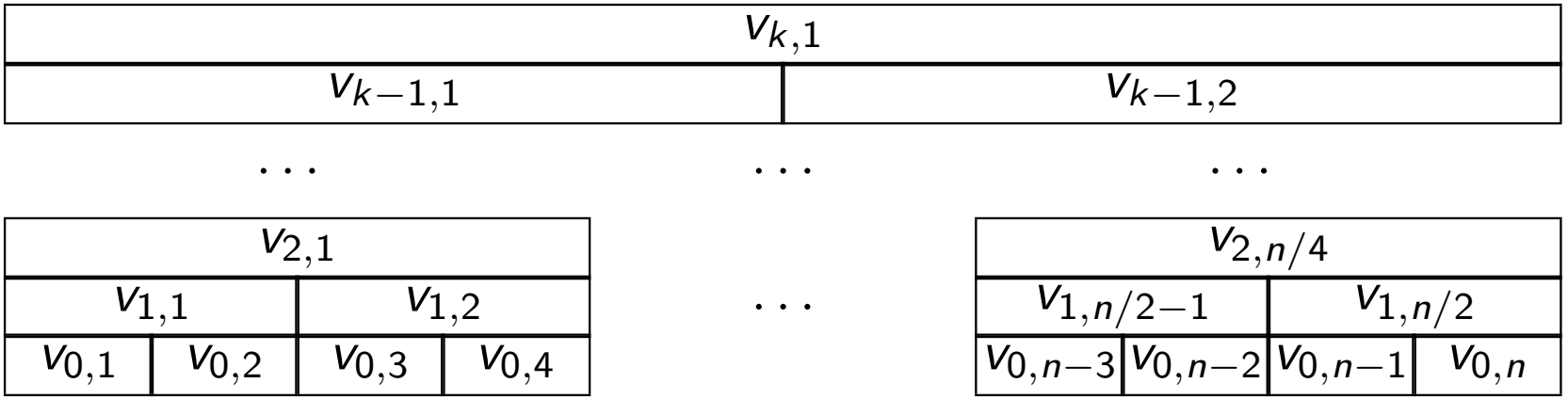
Fast updates in short computational trees

Def: Function $f(x)$, $x \in \mathbb{R}^n$, is *short-tree representable*, if it can be computed by a short binary tree with the height $\approx \ln n$.

Let $n = 2^k$ and the tree has $k+1$ levels: $v_{0,i} = x^{(i)}$, $i = 1, \ldots, n$.
Size of the next level halves the size of the previous one:

$$v_{i+1,j} \quad = \quad \psi_{i+1,j}(v_{i,2j-1}, v_{i,2j}), \quad j = 1, \ldots, 2^{k-i-1}, \; i = 0, \ldots, k-1,$$

where $\psi_{i,j}$ are some bivariate functions.



Main advantages

- Important examples (symmetric functions)

$$
\begin{aligned}
f(x) &= \|x\|_p, \quad p \geq 1, & \psi_{i,j}(t_1, t_2) &\equiv [\, |t_1|^p + |t_2|^p \,]^{1/p}, \\
f(x) &= \ln \left(\sum_{i=1}^{n} e^{x^{(i)}} \right), & \psi_{i,j}(t_1, t_2) &\equiv \ln \left(e^{t_1} + e^{t_2} \right), \\
f(x) &= \max_{1 \leq i \leq n} x^{(i)}, & \psi_{i,j}(t_1, t_2) &\equiv \max \{ t_1, t_2 \}.
\end{aligned}
$$

- The binary tree requires only $n - 1$ auxiliary cells.
- Its value needs $n - 1$ applications of $\psi_{i,j}(\cdot, \cdot)$ ($\equiv$ *operations*).
- If x_+ differs from x in one entry only, then for re-computing $f(x_+)$ we need only $k \equiv \log_2 n$ operations.

Thus, we can have pure subgradient minimization schemes with

Sublinear Iteration Cost

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§5 Pure subgradient scheme with sublinear iteration cost 的方法

Simple subgradient methods

I. Problem: $f^* \stackrel{\text{def}}{=} \min\limits_{x \in Q} f(x)$, where

- Q is a closed and convex and $\|f'(x)\| \leq L(f)$, $x \in Q$,
- the optimal value f^* is known.

Consider the following optimization scheme (B.Polyak, 1967): (f^* 已知)

$$x_0 \in Q, \quad x_{k+1} = \pi_Q \left(x_k - \frac{f(x_k) - f^*}{\|f'(x_k)\|^2} f'(x_k) \right), \quad k \geq 0.$$

Denote $f_k^* = \min\limits_{0 \leq i \leq k} f(x_i)$. Then for any $k \geq 0$ we have:

$$f_k^* - f^* \leq \frac{L(f)\|x_0 - \pi_{X_*}(x_0)\|}{(k+1)^{1/2}},$$

$$\|x_k - x^*\| \leq \|x_0 - x^*\|, \quad \forall x^* \in X_*.$$

Proof:

Let us fix $x^* \in X_*$. Denote $r_k(x^*) = \|x_k - x^*\|$. Then

$$\begin{aligned} r_{k+1}^2(x^*) &\leq \left\| x_k - \tfrac{f(x_k)-f^*}{\|f'(x_k)\|^2} f'(x_k) - x^* \right\|^2 \\ &= r_k^2(x^*) - 2\tfrac{f(x_k)-f^*}{\|f'(x_k)\|^2} \langle f'(x_k), x_k - x^* \rangle + \tfrac{(f(x_k)-f^*)^2}{\|f'(x_k)\|^2} \\ &\leq r_k^2(x^*) - \tfrac{(f(x_k)-f^*)^2}{\|f'(x_k)\|^2} \leq r_k^2(x^*) - \tfrac{(f_k^*-f^*)^2}{L^2(f)}. \end{aligned}$$

From this reasoning, $\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2$, $\forall x^* \in X^*$. □

Corollary: Assume X_* has recession direction d_*. Then

$$\|x_k - \pi_{X_*}(x_0)\| \leq \|x_0 - \pi_{X_*}(x_0)\|, \quad \langle d_*, x_k \rangle \geq \langle d_*, x_0 \rangle.$$

(Proof: consider $x^* = \pi_{X_*}(x_0) + \alpha d_*$, $\alpha \geq 0$.) □

Constrained minimization

II. Problem: $\min_{x \in Q} \{ f(x) : \; g(x) \leq 0 \}$, where

- Q is closed and convex,
- f, g have uniformly bounded subgradients.

Consider the following method. It has step-size parameter $h > 0$.

If $g(x_k) > h \, \|g'(x_k)\|$, then (A): $x_{k+1} = \pi_Q \left(x_k - \frac{g(x_k)}{\|g'(x_k)\|^2} \, g'(x_k) \right)$,

else (B): $x_{k+1} = \pi_Q \left(x_k - \frac{h}{\|f'(x_k)\|} \, f'(x_k) \right)$.

Let $\mathcal{F}_k \subseteq \{0, \ldots, k\}$ be the set (B)-iterations, and $f_k^* = \min_{i \in \mathcal{F}_k} f(x_i)$.

Theorem: If $k > \|x_0 - x^*\|^2 / h^2$, then $\mathcal{F}_k \neq \emptyset$ and

$$f_k^* - f(x) \leq hL(f), \quad \max_{i \in \mathcal{F}_k} g(x_i) \; \leq \; hL(g).$$

Computational strategies

1. Constants $L(f)$, $L(g)$ are known (e.g. Linear Programming)

We can take $h = \frac{\epsilon}{\max\{L(f),L(g)\}}$. Then we need to decide on the number of steps N (easy!).

Note: The standard advice is $h = \frac{R}{\sqrt{N+1}}$ (much more difficult!)

2. Constants $L(f)$, $L(g)$ are not known

- Start from a guess.
- Restart from scratch each time we see the guess is wrong.
- The guess is doubled after restart.

3. Tracking the record value f_k^*

Double run. Other ideas are welcome!

Application examples

Observations:

1. Very often, Large- and Huge- scale problems have repetitive sparsity patterns and/or limited connectivity.
   - Social networks.
   - Mobile phone networks.
   - Truss topology design (local bars).
   - Finite elements models (2D: four neighbors, 3D: six neighbors).
2. For p-diagonal matrices $\kappa(A) \leq p^2$.

Nonsmooth formulation of Google Problem

Main property of spectral radius ($A \geq 0$)

If $A \in \mathbb{R}^{n \times n}_+$, then $\rho(A) = \min\limits_{x \geq 0} \max\limits_{1 \leq i \leq n} \frac{1}{x^{(i)}} \langle e_i, Ax \rangle$.

The minimum is attained at the corresponding eigenvector.

Since $\rho(\bar{E}) = 1$, our problem is as follows:

$$f(x) \stackrel{\text{def}}{=} \max_{1 \leq i \leq N} [\langle e_i, \bar{E}x \rangle - x^{(i)}] \quad \to \quad \min_{x \geq 0}.$$

Interpretation: Maximizing the self-esteem!

Since $f^* = 0$, we can apply Polyak's method with sparse updates.

Additional feature: the optimal set X^* is a *convex cone*.

If $x_0 = e$, then the whole sequence is separated from zero:

$$\langle x^*, e \rangle \quad \leq \quad \langle x^*, x_k \rangle \ \leq \ \|x^*\|_1 \cdot \|x_k\|_\infty \ = \ \langle x^*, e \rangle \cdot \|x_k\|_\infty.$$

Goal: Find $\bar{x} \geq 0$ such that $\|\bar{x}\|_\infty \geq 1$ and $f(\bar{x}) \leq \epsilon$.

(First condition is satisfied automatically.)

Computational experiments: Iteration Cost

We compare Polyak's GM with sparse update (GM_s) with the standard one (GM).

Setup: Each agent has exactly p random friends. Thus, $\kappa(A) \approx p^2$.

Iteration Cost: $GM_s \approx p^2 \log_2 N, \quad GM \approx pN.$

NB: $\log_2 10^3 \approx 10, \quad \log_2 10^6 \approx 20, \quad \log_2 10^9 \approx 30.$

Time for 10^4 iterations ($p = 32$)

N	$\kappa(A)$	GM_s	GM
1024	1632	3.00	2.98
2048	1792	3.36	6.41
4096	1888	3.75	15.11
8192	1920	4.20	139.92
16384	1824	4.69	408.38

Time for 10^3 iterations ($p = 16$)

N	$\kappa(A)$	GM_s	GM
131072	576	0.19	213.9
262144	592	0.25	477.8
524288	592	0.32	1095.5
1048576	608	0.40	2590.8

1 sec $\approx$ 100 min!

Convergence of GM_s: Medium Size

Let $N = 131072$, $p = 16$, $\kappa(A) = 576$, and $L(f) = 0.21$.

Iterations	$f - f^*$	Time (sec)
$1.0 \cdot 10^5$	0.1100	16.44
$3.0 \cdot 10^5$	0.0429	49.32
$6.0 \cdot 10^5$	0.0221	98.65
$1.1 \cdot 10^6$	0.0119	180.85
$2.2 \cdot 10^6$	0.0057	361.71
$4.1 \cdot 10^6$	0.0028	674.09
$7.6 \cdot 10^6$	0.0014	1249.54
$1.0 \cdot 10^7$	0.0010	1644.13

Dimension and accuracy are sufficiently high, but the time is still reasonable.

Convergence of GM_s: Large Scale

Let $N = 1048576$, $p = 8$, $\kappa(A) = 192$, and $L(f) = 0.21$.

Iterations	$f - f^*$	Time (sec)
0	2.000000	0.00
$1.0 \cdot 10^5$	0.546662	7.69
$4.0 \cdot 10^5$	0.276866	30.74
$1.0 \cdot 10^6$	0.137822	76.86
$2.5 \cdot 10^6$	0.063099	192.14
$5.1 \cdot 10^6$	0.032092	391.97
$9.9 \cdot 10^6$	0.016162	760.88
$1.5 \cdot 10^7$	0.010009	1183.59

Final point $\bar{x}_*$: $\|\bar{x}_*\|_\infty = 2.941497$, $R_0^2 \overset{\text{def}}{=} \|\bar{x}_* - e\|_2^2 = 1.2 \cdot 10^5$.

Theoretical bound: $\frac{L^2(f)R_0^2}{\epsilon^2} = 5.3 \cdot 10^7$. **Time for GM:** $\approx$ 1 year!

Conclusion

1. Sparse GM is an efficient and reliable method for solving Large- and Huge- Scale problems with uniform sparsity.
2. We can treat also dense rows. Assume that inequality $\langle a, x \rangle \leq b$ is dense. It is equivalent to the following *system*:

$$\begin{array}{rcl} y^{(1)} & = & a^{(1)} x^{(1)}, \quad y^{(j)} = y^{(j-1)} + a^{(j)} x^{(j)}, \quad j = 2, \ldots, n, \\ y^{(n)} & \leq & b. \end{array}$$

   We need *new variables* $y^{(j)}$ for all nonzero coefficients of a.
   - Introduce $p(a)$ additional variables and $p(A)$ additional equality constraints. (No problem!)
   - Hidden drawback: the above equalities are satisfied with *errors*.
   - May be it is not too bad?
3. Similar technique can be applied to dense columns.

Theoretical consequences

Assume that $\kappa(A) \approx \gamma^2(A)n^2$. Compare three methods:

- Sparse updates (SU). Complexity $\gamma^2(A)n^2 \frac{L^2R^2}{\epsilon^2} \log n$ operations.
- Smoothing technique (ST). Complexity $\gamma(A)n^2 \frac{LR}{\epsilon}$ operations.
- Polynomial-time methods (PT). Complexity $(\gamma(A)n + n^3)n \ln \frac{LR}{\epsilon}$ operations.

There are three possibilities.

- Low accuracy: $\gamma(A)\frac{LR}{\epsilon} < 1$. Then we choose SU.
- Moderate accuracy: $1 < \gamma(A)\frac{LR}{\epsilon} < n^2$. We choose ST.
- High accuracy: $\gamma(A)\frac{LR}{\epsilon} > n^2$. We choose PT.

NB: For Huge-Scale problems usually $\gamma(A) \approx \frac{1}{n} \Rightarrow \boxed{\frac{LR}{\epsilon} \bigvee n}$