

Lecture 10

§1 Remarks on metric spaces

1. Remark 1: 度量空间中的极限四则运算

对于 general metric space, 四则运算可能未被定义, 因此极限的运算可能不成立

e.g. ① 对于 $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$. In \mathbb{R}, \mathbb{C} , true

② 对于 $\lim (a_n b_n) = \lim a_n \cdot \lim b_n$. In $\mathbb{R} = \mathbb{C}$, true

In $\mathbb{R} = \mathbb{R}^N$, true if $a_n b_n$ understood as $\vec{a}_n \cdot \vec{b}_n$

③ 对于 $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$.

In $\mathbb{R} = \mathbb{C}$, true

2. Remark 2: 度量空间中的大小比较

对于除去 \mathbb{R} 的 metric space \mathbb{X} , 通常未定义 ordering $a \leq b$, 因此 comparison of limits ($a_n \leq b_n \Rightarrow \lim a_n \leq \lim b_n$), monotone convergence theorem, upper & lower limits 通常不成立

3. Remark 3: 度量空间中的 Cauchy sequence

对于 general metric space, 一个 Cauchy sequence ($d(q_m, q_n) < \epsilon$) may not converge in \mathbb{X}

e.g. 对于 $\{1/n\}_{n=1}^{\infty}$, 它在 \mathbb{R} 中收敛向 0, 但在 $\mathbb{X} = (0, 1)$ 中不收敛.

Definition: Complete metric space (完备度量空间)

A metric space in which every Cauchy sequence converges is called Complete metric space

4. Remark 4: 度量空间中的级数

对于 general metric space \mathbb{X} , 通常未定义 "+", 因此不存在级数

但对于 $\mathbb{X} = \mathbb{R}^N$ 或 $\mathbb{X} = \mathbb{C}$, 存在级数 $\sum_{n=0}^{\infty} a_n$

且存在 Absolute convergence test, 即 $\sum_{n=0}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=0}^{\infty} a_n$ converges

5. Remark 5: 度量空间中的幂级数

对于 general metric space \mathbb{X} , 因此不存在幂级数

但对于 $\mathbb{X} = \mathbb{C}$, $a_n \in \mathbb{C}$, $z \in \mathbb{C}$, 存在幂级数 $\sum_{n=0}^{\infty} a_n z^n$ (complex power series)

且存在收敛半径 $R = \frac{1}{\limsup |a_n|^{1/n}}$ (在复平面内形成一个圆)

§2 Basic facts about metric space

1. Fact 1: 邻域一定为开集

Any neighborhood $N_r(p)$ is open

证明:

(只要证 p 点邻域中的任意点 q 均为内点, 即 $\forall q \in N_r(p), \exists \delta, \text{ s.t. } N_\delta(q) \subset N_r(p)$)

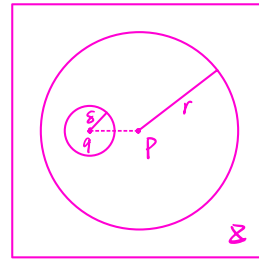
Just need to prove $\forall q \in N_r(p)$, q is an interior point of $N_r(p)$

Use $\delta = \frac{r - d(p, q)}{2}$ as radius to construct $N_\delta(q)$

(W.T.S. $N_\delta(q) \subset N_r(p)$)

$\forall s \in N_\delta(q)$,

$$\begin{aligned}
d(s, p) &\leq d(s, q) + d(p, q) \\
&\leq \delta + d(p, q) \\
&= \frac{r - d(p, q)}{2} + d(p, q) \\
&= \frac{r + d(p, q)}{2} \\
&< \frac{r + r}{2} \\
&= r
\end{aligned}$$



$$\therefore s \in N_r(p)$$

$$\therefore N_\delta(q) \subset N_r(p)$$

$$\therefore N_r(p) \text{ is open}$$

2. Fact 2: 极限点的等价条件

p is limit point of $E \subset \mathbb{R}$ ①

$\Leftrightarrow \forall r > 0, N_r(p)$ contains infinitely many points of E ②

$\Leftrightarrow \exists \{q_n\}_{n=1}^\infty \subset E$ s.t. $q_n \neq q_m$ for $n \neq m$ & $q_n \rightarrow p$ as $n \rightarrow \infty$

证明:

② \Rightarrow ①:

Obvious (由 limit point 的定义可证得)

① \Rightarrow ②:

Argue by contradiction.

Suppose \exists bad $r_{\text{bad}} > 0$, s.t. $N_{r_{\text{bad}}}(p)$ contains finitely many points $\{e_1, e_2, \dots, e_m\}$ of E

Define $r_0 = \min \{d(p, e_1), d(p, e_2), \dots, d(p, e_m)\} / 2$

$\therefore N_{r_0}(p)$ contains no points of E .

Contradicting with the definition of limit point.

Q.E.D.

③ \Rightarrow ①:

$\therefore d(q_n, p) \rightarrow 0$ as $n \rightarrow \infty$

$\therefore \forall r > 0, \exists$ integer $N > 0$, s.t. if $n \geq N, d(q_n, p) < r$

$\therefore q_N \in N_r(p)$

$q_{N+1} \in N_r(p)$

\therefore one of them must $\neq p$

$\therefore p$ is limit point of $E \subset \mathbb{R}$

① \Rightarrow ③:

$\therefore p$ is a limit point of E

$\therefore \forall r > 0, N_r(p)$ contains points $\neq p$ of E

Take $r = r_1 = 1, N_{r_1}(p)$ contains point $q_1 \in E$,

Take $r = r_2 = \frac{d(p, q_1)}{2} < \frac{1}{2}$, $N_{r_2}(p)$ contains point $q_2 \in E$, $q_2 \neq q_1$.

Take $r = r_3 = \frac{d(p, q_2)}{2} < \frac{1}{2}$, $N_{r_3}(p)$ contains point $q_3 \in E$, $q_3 \neq q_2 \neq q_1$.

For $\{q_n\}_{n=1}^{\infty}$, $q_n \neq q_m$ if $n \neq m$,

$$0 \leq d(q_n, p) = 2r_{n+1} < \frac{2}{2^{n+1}}, \forall n > 1$$

$$\therefore \frac{2}{2^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\therefore d(q_n, p) \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\therefore q_n \rightarrow p \text{ as } n \rightarrow \infty$$

3. Fact 3: 开集和闭集互补

Let $E \subset \mathbb{R}$. Define complement of E by $E^c = \mathbb{R} \setminus E$. Then

E is open $\iff E^c$ is closed

证明:

proof of " \Rightarrow ":

(W.T.S. $(E^c)' \subset E^c$)

Argue by contradiction, suppose E^c is not closed.

Then $\exists p \in (E^c)'$, $p \notin E^c$.

$\therefore E$ is open

$\therefore \exists$ neighborhood $N_r(p) \subset E$

$\therefore N_r(p)$ contains no points of E^c

$\therefore p$ is not a limit point (contradiction)

proof of " \Leftarrow ":

(W.T.S. $\forall p \in E$ is an interior point)

Argue by contradiction, suppose E is not open.

Then $\exists p \in E$, p is Not an interior point

$\therefore \forall r > 0$, $N_r(p) \not\subset E$

$\therefore N_r(p)$ contains point in E^c

$\therefore p$ is a limit point of E^c

$\therefore E^c$ is closed

$\therefore p \in (E^c)' \subset E^c$ (contradiction)