Lecture 10

§1 Remarks on metric spaces

1. **Remark 1**: 度量空间中的极限四则运算

对于 general metric space，四则运算可能未被定义，因此极限的运算可能不成立

e.g. ① 对于 $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$. In $X = R^N, C$, true

② 对于 $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$. In $X = C$, true

In $X = R^N$, true if $a_n b_n$ understood as $\vec{a_n}\cdot\vec{b_n}$

③ 对于 $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$. In $X = C$, true

2. **Remark 2**: 度量空间中的大小比较

对于除去 R 的 metric space X，通常未定义 ordering $a \le b$，因此 comparison of limits ($a_n \le b_n \Rightarrow \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$), monotone convergence theorem, upper & lower limits 通常不成立

3. **Remark 3**: 度量空间中的 Cauchy sequence

对于 general metric space，一个 Cauchy sequence ($d(q_m, q_n) < \varepsilon$) may not converge in X

e.g. 对于 $\{\frac{1}{n}\}_{n=1}^{\infty}$，它在 R 中收敛向 0，但在 $X = (0,1)$ 中不收敛.

Definition: Complete metric space (完备度量空间)

A metric space in which every Cauchy sequence converges is called Complete metric space

4. **Remark 4**: 度量空间中的级数

对于 general metric space X，通常未定义 "+"，因此不存在级数

但对于 $X = R^N$ 或 $X = C$，存在级数 $\sum_{n=n_0}^{\infty} a_n$

且存在 Absolute convergence test，即 $\sum_{n=1}^{\infty}|a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

5. **Remark 5**: 度量空间中的幂级数

对于 general metric space X，因此不存在幂级数

但对于 $X = C$，$a_n \in C$，$z \in C$，存在幂级数 $\sum_{n=n_0}^{\infty} a_n z^n$ (complex power series)

且存在收敛半径 $R = \frac{1}{\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}}}$ (在复平面内形成一个圆)

§2 Basic facts about metric space

1. **Fact 1**: 邻域一定为开集

Any neighborhood $N_r(p)$ is open

证明:

(只要证 p 点邻域中的任意点 q 均为内点，即 $\forall q \in N_r(p)$，$\exists \delta$, s.t. $N_\delta(q) \subset N_r(p)$)

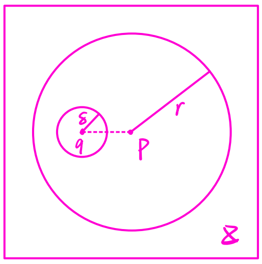
Just need to prove $\forall q \in N_r(p)$, q is an interior point of $N_r(p)$

Use $\delta = \frac{r - d(p,q)}{2}$ as radius to construct $N_\delta(q)$

(W.T.S. $N_\delta(q) \subset N_r(p)$)

$\forall s \in N_\delta(q)$,

$$
\begin{aligned}
d(s,p) &\le d(s,q) + d(p,q) \\
&\le \delta + d(p,q) \\
&= \frac{r-d(p,q)}{2} + d(p,q) \\
&= \frac{r+d(p,q)}{2} \\
&< \frac{r+r}{2} \\
&= r
\end{aligned}
$$



$\therefore\ s \in N_r(p)$

$\therefore\ N_\delta(q) \subset N_r(p)$

$\therefore\ N_r(p)$ is open

2. Fact 2: 极限点的等价条件

p is limit point of $E \subset X$ ①

$\Longleftrightarrow$ $\forall r>0$, $N_r(p)$ contains infinitely many points of E ②

$\Longleftrightarrow$ $\exists \{q_n\}_{n=1}^{\infty} \subset E$ s.t. $q_n \neq q_m$ for $n \neq m$ & $q_n \to p$ as $n \to \infty$

证明:

② $\Rightarrow$ ①:

Obvious (由 limit point 的定义可证得)

① $\Rightarrow$ ②:

Argue by contradiction.

Suppose $\exists$ bad $r_{bad} > 0$, s.t. $N_r(p)$ contains finitely many points $\{e_1, e_2, \cdots, e_m\}$ of E

Define $r_0 = \min\{d(p,e_1), d(p,e_2), \cdots, d(p,e_m)\}/2$

$\therefore$ $N_{r_0}(p)$ contains no points of E.

Contradicting with the definition of limit point.

Q.E.D.

③ $\Rightarrow$ ①:

$\because$ $d(q_n,p) \to 0$ as $n \to \infty$

$\therefore$ $\forall r>0$, $\exists$ integer $N>0$, s.t. if $n \ge N$, $d(q_n,p) < r$

$\therefore$ $q_N \in N_r(p)$

$q_{N+1} \in N_r(p)$

$\therefore$ one of them must $\neq p$

$\therefore$ p is limit point of $E \subset X$

① $\Rightarrow$ ③:

$\because$ p is a limit point of E

$\therefore$ $\forall r>0$, $N_r(p)$ contains points $\neq p$ of E

Take $r = r_1 = 1$, $N_{r_1}(p)$ contains point $q_1 \in E$, Take $r = r_2 = \frac{d(p,q_1)}{2} < \frac{1}{2}$, $N_{r_2}(p)$ contains point $q_2 \in E$, $q_2 \neq q_1$

Take $r = r_3 = \frac{d(p,q_2)}{2} < \frac{1}{2^2}$, $N_{r_3}(p)$ contains point $q_3 \in E$, $q_3 \neq q_2 \neq q_1$

$\cdots\cdots$

For $\{q_n\}_{n=1}^{\infty}$, $q_n \neq q_m$ if $n \neq m$,

$$0 \leq d(q_n, p) = 2r_{n+1} < \frac{2}{2^{n+1}} \ , \forall n > 1$$

$\because \frac{2}{2^{n+1}} \to 0$ as $n \to \infty$

$\therefore d(q_n, p) \to 0$ as $n \to \infty$

$\therefore q_n \to p$ as $n \to \infty$

3. Fact 3: 开集和闭集互补

Let $E \subset X$. Define complement of E by $E^c = X \setminus E$. Then

E is open $\iff$ E^c is closed

证明:

proof of " $\Rightarrow$ ":

(W.T.S. $(E^c)' \subset E^c$)

Argue by contradiction, suppose E^c is not closed.

Then $\exists \ p \in (E^c)'$, $p \in E$.

$\because$ E is open

$\therefore$ $\exists$ neighborhood $N_r(p) \subset E$

$\therefore$ $N_r(p)$ contains no points of E^c

$\therefore$ p is not a limit point (contradiction)

proof of " $\Leftarrow$ ":

(W.T.S $\forall p \in E$ is an interior point)

Argue by contradiction, suppose E is not open.

Then $\exists p \in E$, p is Not an interior point

$\therefore$ $\forall r > 0$, $N_r(p) \not\subset E$

$\therefore$ $N_r(p)$ contains point in E^c

$\therefore$ p is a limit point of E^c

$\because$ E^c is closed

$\therefore$ $p \in (E^c)' \subset E^c$ (contradiction)