Lecture 10

§1 Torque

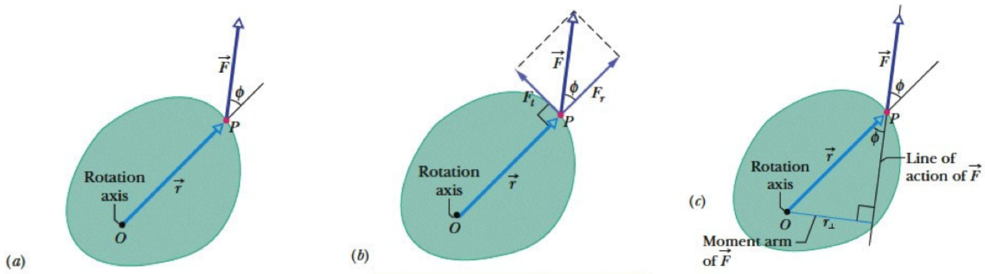
1. Torque (转矩)

1° 用于旋转的力的作用效用取决于
① 力的角度
② 力的作用位置

2° 我们定义 torque (τ)
① $\tau = r \cdot (F \sin\phi) = r F_\perp$
② $\tau = (r \sin\phi) \cdot F = r_\perp F$
$r_\perp$: line of action (作用线) of $\vec{F}$
moment arm (力臂) of $\vec{F}$

3° SI Unit: $N \cdot m$
* 虽然 $1J = 1N \cdot m$，但 torque 的单位不用 J，torque 不是 energy

4° 正方向：导致逆时针旋转
负方向：导致顺时针旋转



(a) The torque due to this force causes rotation around this axis (which extends out toward you).

(b) But actually only the *tangential* component of the force causes the rotation.

(c) You calculate the same torque by using this moment arm distance and the full force magnitude.

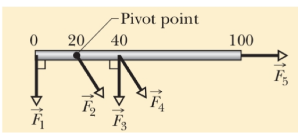
2. Torque 的矢量性

1° Torque 可以当作矢量
$\vec{\tau} = \vec{r} \times \vec{F}$

2° 绕单轴旋转时，方向服从右手定则
Sweep from r towards F!

3° Net torque
τ_{net} 为 individual torques 的和

例:
Checkpoint 6
The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.



Answer: F_1 & F_3, F_4, F_2 & F_5

3. Newton's Second Law for rotation

仅有 tangential component F_t 能沿路径方向加速质点，

$$a_t = \alpha \cdot r$$

$$F_t = m a_t = m \alpha r$$

$$\tau = F_t \cdot r$$

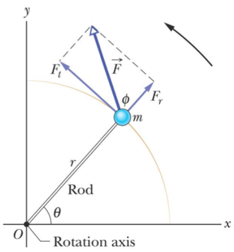
$$= m \alpha r^2$$

$$= I \alpha$$

由 $\tau = F \cdot r \cdot \sin\theta$ 得：

$$F r \sin\theta = I \alpha$$

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.



- Only the tangential component F_t of the applied force F can accelerate the particle along the path

$$a_t = \alpha r$$

$$F_t = m a_t = m \alpha r$$

$$\tau = F_t r = m \alpha r \times r = m \alpha r^2 = (m r^2)\alpha = I\alpha$$

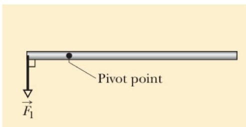
- For more than one force applied

Newton's Second Law for Rotation

例：

Checkpoint 7

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, are applied to the stick. Only $\vec{F}_1$ is shown. Force $\vec{F}_2$ is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of $\vec{F}_2$, and (b) should F_2 be greater than, less than, or equal to F_1?

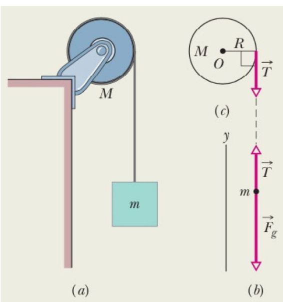


Answer: (a) F_2 should point downward, and

(b) should have a smaller magnitude than F_1

6

例：



A uniform disk, with mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed horizontal axle. A block with mass $m = 1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the **acceleration** of the falling block, the **angular acceleration** of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.

$$\begin{cases} T \cdot R = I \alpha \\ \alpha \cdot R = a \\ a = \dfrac{mg - T}{m} \end{cases}$$

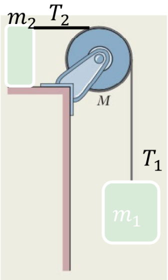
$$I = \frac{1}{2} M R^2$$

$$T = \frac{mMg}{2m + M} = 6N$$

$a = 4.8\ m/s^2$

$\alpha = \frac{TR}{\frac{1}{2}MR^2} = 24\ rad/s^2$

例:



- m_1 , m_2 and mass of pulley M
- What is the tensions (T_1 and T_2) and acceleration?
- What if M=0? What if $m_2 = 0$?

$$\begin{cases} m_1 g - T_1 = m_1 a \\ T_2 = m_2 a \\ (T_1 - T_2)\cdot R = I\frac{a}{R} = \frac{1}{2}MaR \end{cases}$$

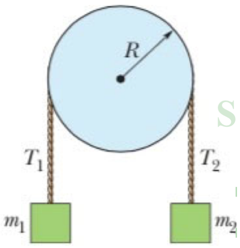
$$a = \frac{m_1 g}{m_1 + m_2 + \frac{1}{2}M}$$

$$T_1 = \frac{m_1 m_2 + \frac{1}{2}m_1 M}{m_1 + m_2 + \frac{1}{2}M}\cdot g$$

$$T_2 = \frac{m_1 m_2}{m_1 + m_2 + \frac{1}{2}M}\cdot g$$

If $M=0$, $a = \frac{m_1 g}{m_1 + m_2}$, $T_1 = T_2 = \frac{m_1 m_2}{m_1 + m_2} g$

例:



- m_1, m_2 ($m_2 \geq m_1$) and mass of pulley M
- What is the tensions (T_1 and T_2) and acceleration?
- What if M=0?

$$\begin{cases} m_2 g - T_2 = m_2 a \\ T_1 - m_1 g = m_1 a \\ (T_2 - T_1)R = I\frac{a}{R} \end{cases}$$

$$I = \frac{1}{2}MR^2$$

$$a = \frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M} g$$

$$T_1 = \frac{2m_1 m_2 + \frac{1}{2}m_1 M}{m_1 + m_2 + \frac{1}{2}M} g$$

$$T_2 = \frac{2m_1 m_2 + \frac{1}{2}m_2 M}{m_1 + m_2 + \frac{1}{2}M} g$$

If $M=0$, $a = \frac{m_2 - m_1}{m_1 + m_2} g$ $\quad T_1 = T_2 = \frac{2m_1 m_2}{m_1 + m_2} g$

4. Work and rotational kinetic energy

1° $\Delta K = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = W$

2° For rotational motion about fixed Axis

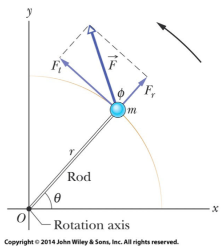
$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$

3° Constant torque

$W = \tau (\theta_f - \theta_i)$

Proof of Equations

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.



- Only the tangential force doing work

$$dW = F_t ds$$
$$s = \theta r$$
$$ds = r d\theta$$
$$dW = F_t r d\theta = \tau d\theta$$
$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$$

4° Power

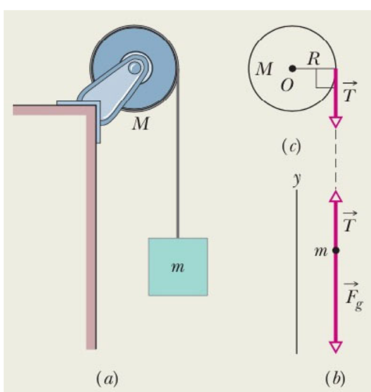
$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega$

5. 平动与转动的比较

Table 10-3 Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\, dx$	Work	$W = \int \tau\, d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

例: **Problem**



A uniform disk, with mass ***M***, mounted on a fixed horizontal axle. A block with mass ***m*** hangs from a massless cord that is wrapped around the rim of the disk. Find the kinetic energy of M and m, respectively, after traveling for a distance **S**.

Solution:

- Mechanical Energy Changed?

$$\Delta E_{mec} = 0$$
$$\Delta K + \Delta U = 0$$
$$(K_m - 0) + (K_M - 0) + (-mgS) = 0$$
$$mgS = K_m + K_M$$

- For m, the initial velocity is 0, the final velocity is v, the kinetic energy is

$$K_m = \frac{1}{2}mv^2$$

- For M, the angular velocity is ω, we have

$$\omega = v/R$$

$$K_M = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\frac{1}{2}MR^2\times\frac{v^2}{R^2} = \frac{1}{4}Mv^2$$

$$mgS = K_m + K_M = \frac{1}{2}\left(m+\frac{1}{2}M\right)v^2$$

$$v^2 = \frac{2mgS}{m+\frac{1}{2}M}$$

For m, the kinetic energy is

$$K_m = \frac{1}{2}mv^2 = \frac{m^2gS}{m+\frac{1}{2}M}$$

For M, the kinetic energy is

$$K_M = \frac{1}{4}Mv^2 = \frac{\frac{1}{2}mMgS}{m+\frac{1}{2}M}$$

If the pulley is massless, $M = 0$

$$K_m = mgS$$

$$K_M = 0$$

No matter M is 0 or not,

All the potential energy is transferred into the KE of the system.

Summary

Radian Measure

- Agular Position

$$\theta = \frac{s}{r}$$

$$1\,rev = 360° = \frac{2\pi r}{r} = 2\pi\,rad$$

- Angular Displacement

$$\Delta\theta = \theta_2 - \theta_1$$

- Angular Velocity and Speed

$$\omega_{avg} = \frac{\theta_2-\theta_1}{t_2-t_1} = \frac{\Delta\theta}{\Delta t} \text{ and } \omega = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- Angular Acceleration

$$\alpha_{avg} = \frac{\omega_2-\omega_1}{t_2-t_1} = \frac{\Delta\omega}{\Delta t} \text{ and } \alpha = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Summary

Radian Measure

- Comparison between Linear and Rotational Motion

Linear Equation	Angular Equation
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Summary

Radian Measure

- Linear and Angular Variables Related

$$v = \omega r$$

$$a_t = \alpha r$$

- For uniform circular motion

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} \text{ and } a_r = \frac{v^2}{r} = \omega^2 r$$

- Rotational Kinetic Energy

$$K = \frac{1}{2}I\omega^2$$

- Rotational Inertia

$$I = \sum m_i r_i^2 = \int r^2 dm$$

Summary

Radian Measure

- The parallel-Axis Theorem

$$I = I_{com} + Mh^2$$

- Torque

$$\tau = (r)(F sin\emptyset) = rF_{\perp} = (r sin\emptyset)(F) = r_{\perp}F$$

  – Line of action & Moment Arm of $\vec{F}$

- Newton's Second Law in Angular Form

$$\tau_{net} = I\alpha$$

- Work and the Rotation Kinetic Energy

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$$

Summary

Radian Measure

- Power for Rotational Motion

$$P = \frac{dW}{dt} = \tau\omega$$

- Work-Kinetic Energy Theorem for rotation bodies

$$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$$