Lecture 17

§1 Global / Absolute extrema

1. Global / Absolute extrema

Definition

Let $(a_1, \dots, a_n)$ be a point in the domain D of a function f.

- We say that f has a global maximum (or an absolute maximum) at $(a_1, \dots, a_n)$ if
$$f(a_1, \dots, a_n) \geq f(x_1, \dots, x_n)$$
for every $(x_1, \dots, x_n) \in D$.
- We say that f has a global minimum (or an absolute minimum) at $(a_1, \dots, a_n)$ if
$$f(a_1, \dots, a_n) \leq f(x_1, \dots, x_n)$$
for every $(x_1, \dots, x_n) \in D$.

2. Theorem

Theorem

Let $f : D \to \mathbb{R}$ be a continuous function, where D is a closed and bounded set in $\mathbb{R}^2$. Then f has a global maximum $f(x_1, y_1)$ and a global minimum $f(x_2, y_2)$ at some (x_1, y_1) and (x_2, y_2) in D.

注: ✡1° In short，定义在有界闭区间的连续函数有最值（对于定义在R^n中的函数均成立）

2° 若定义域不为闭区间，即便有界，也可能无最值

e.g. $f(x,y) = x$, $D = \{(x,y) : x^2+y^2<1\}$

3. 策略: Finding global extrema for cts functions on closed & bounded D

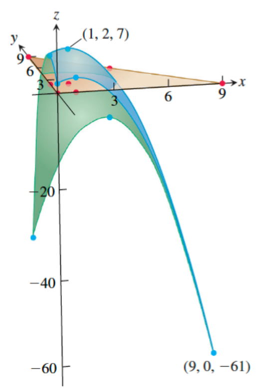
1° 找出D内所有的 critical points （无需判断是否为极值）

2° 找出D的 boundary 上的所有极值

3° 计算它们的大小并比较

例: **Example** (e.g. 14.7.6)

Find the global maximum and global minimum of the function
$$f(x,y) := 2 + 2x + 4y - x^2 - y^2 \quad \leftarrow \text{cts}$$
defined on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



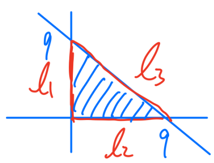
Sol: · Interior : critical points

$$\begin{cases} f_x = 2 - 2x = 0 \\ f_y = 4 - 2y = 0 \end{cases}$$

$\Rightarrow (x,y) = (1,2)$: in interior of D

· Boundary

Boundary is $l_1 \cup l_2 \cup l_3$, where



$l_1 : \{(0,y) : 0 \leq y \leq 9\}$

$l_2 : \{(x,0) : 0 \leq x \leq 9\}$

$l_3: \{(x, 9-x): 0 \le x \le 9\}$

- on l_3: $f(x,y) = f(x, 9-x) = -43 + 16x - 2x^2 = p(x)$

  $p'(x) = 0 \Rightarrow x = 4 \Rightarrow y = 9$, so

  $(x,y) = (4,5)$ is a candidate.

  At end points $x=0$ and $x=9$, we have

  $(x,y) = (0,9)$ and $(x,y) = (9,0)$ are candidates.

- By doing similar arithmetic for l_1 and l_2,

  we found the following candidates:

  l_1: $(0,0)$, $(0,2)$, $(0,9)$

  l_2: $(0,0)$, $(1,0)$, $(9,0)$

- Compare all candidates

  $f(1,2) = 7$, $f(0,0) = 2$, $f(0,2) = 6$, $f(0,9) = -43$,

  $f(1,0) = 3$, $f(9,0) = -61$, $f(4,5) = -11$

  Max: $f(1,2) = 7$; Min: $f(9,0) = -61$

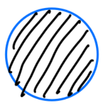
§2 Optimization with equality constraints (条件极值)

1. Optimization with one equality constraint

> **Optimization with One Equality Constraint**
>
> Maximize (or minimize) $f(x_1, \ldots, x_n)$
>
> Subject to (s.t.) $g(x_1, \ldots, x_n) = 0$

e.g. Q: Given a mountain altitude function (cts) $f(x,y)$, what is the highest altitude given by points in the unit disk $x^2+y^2 \le 1$?

- Find crit. pts. in the interior of D, and find the maximum in the boundary, then compare.



D: $x^2+y^2 \le 1$

Notation

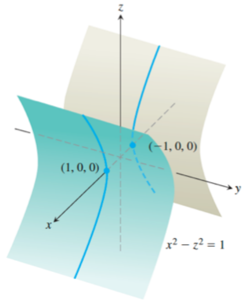
Maximize $f(x,y)$

Subject to (s.t.) $x^2+y^2 = 1$. ← Boundary

This means we are trying to find $\max\limits_{(x,y)\in C} f(x,y)$, where

$$C = \{(x,y) : x^2+y^2 = 1\}.$$

e.g. <u>Another example</u> Find all points on the surface $x^2-z^2-1=0$ that are closest to the origin.



<u>Notation</u> Minimize $\sqrt{x^2+y^2+z^2}$

Subject to $x^2-z^2-1=0$.

注：对于此类问题，由于可行集为平面内的一条线／空间内的一个面，定义域内不存在内点，因此无法直接利用偏导求解极值

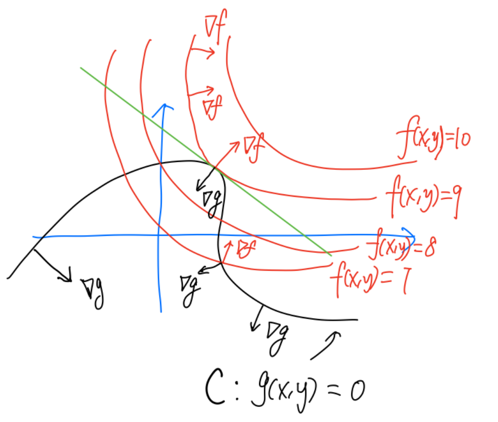
2. Theorem（拉格朗日数乘法的原理）

> **Theorem (14.8.12, 2D Version)**
>
> Let $f(x,y)$ be a differentiable function on D, and let C given by
>
> $$\vec{r}(t)=\langle x(t), y(t)\rangle,\quad t\in I$$
>
> be a smooth curve contained in the interior of D. If P_0 is an interior point of C that gives a local extremum of f relative to points in C, then $\nabla f \perp \vec{r}\,'$ at P_0.
>
> i.e., $P_0=(x(t_0), y(t_0))$, t_0 int. pt of I.
>
> This thm is also true for $f(x,y,z)$ and C: $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$

Consider a 2-variable case:

Max $f(x,y)$

s.t. $g(x,y)=0$.



证明：

- 考虑f的值，only on C
- 令 $g(t):=f(x(t),y(t))$，$(x(t),y(t))\in C$，令 $P_0=(x(t_0),y(t_0))$
- $f(x(t_0),y(t_0))$ 为f在C上的 local maximum，
  因此 $g(t_0)$ 为 $(t_0-\delta, t_0+\delta)$ 上的 maximum for some $\delta>0$.
  因此 $g'(t_0)=0$
- 因为 $g'(t_0)=f_x(P_0)\,x'(t_0)+f_y(P_0)\,y'(t_0)=\nabla f(P_0)\cdot\vec{r}\,'(t_0)$，
  因此
  $$\nabla f(P_0)\cdot\vec{r}\,'(t_0)=0$$

- 事实上，当f在点P_0处有相对于C的极值时，若C为$g(x,y)=0$，则∇f与∇g均在P_0点垂直于C，因此
  - $\nabla g = \langle 0,0\rangle$ 或
  - $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$ ($\nabla f \parallel \nabla g$)

3. The method of Lagrange multipliers (拉格朗日数乘法)

Suppose that the minimum of $f(x,y)$ subject to the constraint $g(x,y)=0$ exists, where f and g are differentiable. Assume that $\nabla g(x,y) \neq \vec{0}$ whenever $g(x,y)=0$. The following is a procedure for finding the constrained minimum:

1. Find all values of x, y and λ that simultaneously satisfy

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y)=0.$$

2. Evaluate f at all the points (x,y) obtained in step 1. The smallest value is the desired minimum.

If the constrained maximum exists, then the largest value in step 2 is the maximum.

例: Example (e.g. 14.8.3)

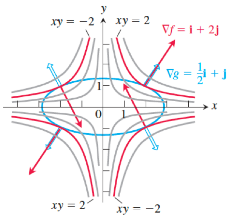
Find the maximum and minimum of the function

$$f(xy) := xy$$

defined on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Specify where the extrema occur on the ellipse.



Sol:
- $\nabla g = \langle g_x, g_y\rangle = \langle \frac{x}{4}, y\rangle = \langle 0,0\rangle \Rightarrow (x,y)=(0,0)$

  But $(0,0) \notin C$

- $\nabla f = \langle y, x\rangle$

  Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases}$

  $\Rightarrow \begin{cases} y = \lambda \cdot \frac{1}{4}x \\ x = \lambda y \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 \end{cases}$

  $\Rightarrow y = \frac{1}{4}\lambda^2 y$

  $\Rightarrow y = 0$ or $\lambda = \pm 2$

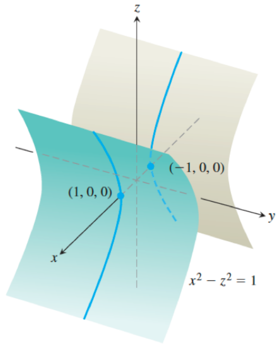
  - If $y=0 \Rightarrow x=0$, $(0,0)$ not on C
  - If $\lambda = 2 \Rightarrow y = \pm 1 \Rightarrow (x,y) = (2,1)$ or $(-2,-1)$
  - If $\lambda = -2 \Rightarrow y = \pm 1 \Rightarrow (x,y) = (2,-1)$ or $(-2,1)$

- $f(2,1) = f(-2,-1) = 2$, $f(2,-1) = f(-2,1) = -2$

  So max at $(2,1)$ and $(-2,-1)$; min at $(2,-1)$ and $(-2,1)$

＊ 拉格朗日数乘法对多元变量同样适用

例: Find all points on the surface $S: x^2-z^2-1=0$ that are closest to origin, and find the minimum distance.



Sol: $\min \sqrt{x^2+y^2+z^2}$ s.t. $x^2-z^2-1=0$ $\Rightarrow$ $\min x^2+y^2+z^2$ s.t. $x^2-z^2-1=0$

- $\nabla g = \langle 2x, 0, -2z \rangle = \langle 0,0,0 \rangle$

  $\Rightarrow x=z=0$, but $(0,y,0)$ not on S

- Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z)=0 \end{cases}$

  $\Rightarrow \begin{cases} 2x = \lambda 2x \\ 2y = \lambda \cdot 0 \\ 2z = \lambda \cdot (-2z) \\ x^2-z^2-1=0 \end{cases}$

  $\Rightarrow \lambda = 1,\ x=\pm 1,\ y=0,\ z=0$

＊ 因为定义域不是有界闭区间，因此最值可能不存在，因此仅仅比较 candidates 处的函数值无法得出最值，需要证明 candidates 处取到的就是最值

- $\forall (x_0,y_0,z_0)$ s.t. $x^2-z^2-1=0$

  $\Rightarrow x_0^2 = z_0^2+1$

  $\Rightarrow \sqrt{x_0^2+y_0^2+z_0^2} = \sqrt{2z_0^2+y_0^2+1} \geq 1$

  And $(\pm 1, 0, 0)$ are both 1 unit away from $(0,0,0)$

- so both $(\pm 1,0,0)$ give global min distance 1.

4. Two equality constraints

考虑 $\max/\min f(x,y,z)$
s.t. $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$

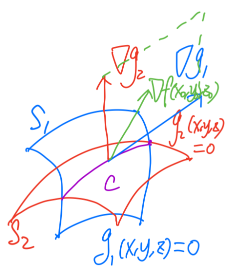
也就是，在 $S_1 \cap S_2$ 上找 f 的最值. $S_i = \{(x,y,z): g_i(x,y,z)=0\}$, $i \in \{1,2\}$

- 若 f 在 $P_0 \in S_1 \cap S_2$ 处有相对于其他 $S_1 \cap S_2$ 上的点的极值，则 $\nabla f(P_0)$ 一定垂直于 $C = S_1 \cap S_2$

- 因为 ∇g_1 与 ∇g_2 也垂直于 C，

  有 ∇f 在 $\nabla g_1(P_0)$ 与 $\nabla g_2(P_0)$ 形成的平面内.

  (假定 $\nabla g_1(P_0)$ 与 $\nabla g_2(P_0)$ 均 $\neq 0$，且不平行.)

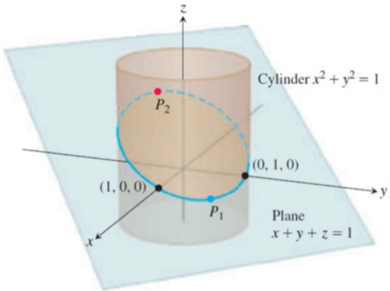


This means that the method of Lagrange multipliers can still be used, but the system to be solved is the following instead:

$$\begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}.$$

例: **Example** (e.g. 14.8.5)

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



Sol: max/min $x^2+y^2+z^2$

s.t. g_1: $x+y+z-1=0$

g_2: $x^2+y^2-1=0$

- $\nabla\vec{g}_1 = \langle 1,1,1\rangle \neq \vec{0}$

  $\nabla\vec{g}_2 = \langle 2x, 2y, 0\rangle \neq \vec{0}$ on ellipse

  $\nabla\vec{g}_1$ and $\nabla\vec{g}_2$ not parallel

- Solve $\begin{cases} \nabla f = \lambda_1 \nabla\vec{g}_1 + \lambda_2 \nabla\vec{g}_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$

$$\Rightarrow \begin{cases} 2x = \lambda_1 + \lambda_2 \cdot 2x \\ 2y = \lambda_1 + \lambda_2 \cdot 2y \\ 2z = \lambda_1 \\ x+y+z-1=0 \\ x^2+y^2-1=0 \end{cases}$$

$\Rightarrow$ All solutions $(x,y,z) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right), (0,1,0), (1,0,0)$

......

- Farthest: $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$

  Closest: $(0,1,0), (1,0,0)$