Lecture 9

§1 The 3D space

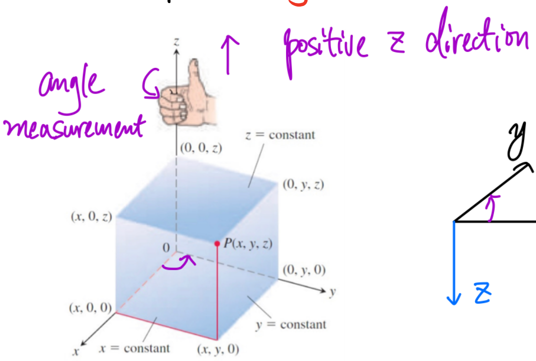
1、Coordinate axes and their placement

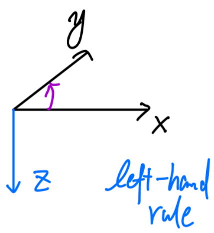
1° three-dimensional space 的表示：$\mathbb{R}^3$

$$\mathbb{R}^3 := \mathbb{R}\times\mathbb{R}\times\mathbb{R} := \{(x,y,z) : x\in\mathbb{R},\ y\in\mathbb{R},\ z\in\mathbb{R}\}$$

x,y,z 表示点 $P(x,y,z)$ 的位置，关于 coordinate axes (x-axis, y-axis, z-axis)

2° 坐标轴通常遵循 right-hand rule





2、基本的 plane (平面), octant (卦限), set of points

1° plane

① $x=0$: yz-plane

② $y=0$: xz-plane

③ $z=0$: xy-plane

2° octant

First octant: $\{(x,y,z) : x>0, y>0, z>0\}$

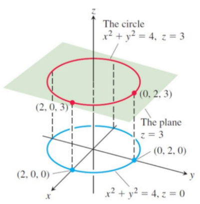
3° set of points

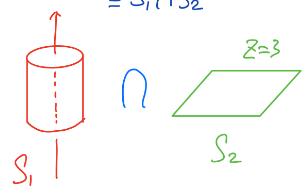
The set of all points (x, y, z) which satisfy

$$\underbrace{x^2+y^2=4}_{S_1},\quad \underbrace{z=3}_{S_2}$$

$\{(x,y,z): x^2+y^2=4\ \&\ z=3\}$
$= \{(x,y,z): x^2+y^2=4\}$
$\cap \{(x,y,z): z=3\}$
$= S_1\cap S_2$

can be represented geometrically by the pink disk in the following figure.





3、distance & spheres (球面)

1° 定义

Definition

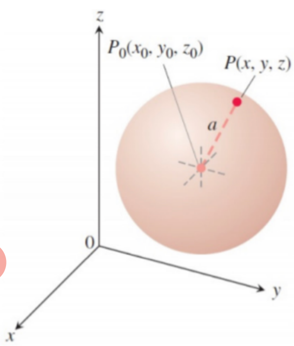
The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined by

$$|P_1P_2| := \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

The sphere with radius a centered at the point (x_0, y_0, z_0) is the set of all points (x, y, z) that satisfy

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

when $a=0$, it is a "degenerate sphere" which is a point.



2° R^n中两点的距离

The distance between $P_1 := (x_1, x_2, \cdots, x_n)$ and $P_2 := (y_1, y_2, \cdots, y_n)$ in R^n is defined by

$$|P_1P_2| := \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\cdots+(y_n-x_n)^2}$$
$$= \left(\sum_{i=1}^{n}(y_i-x_i)^2\right)^{\frac{1}{2}}$$

3° 球面

Consider the equation

$$x^2+y^2+z^2+\alpha x+\beta y+\gamma z+\delta=0 \quad (*)$$

Since

$$(*) \Leftrightarrow \left(x+\frac{\alpha}{2}\right)^2+\left(y+\frac{\beta}{2}\right)^2+\left(z+\frac{\gamma}{2}\right)^2=\left(\frac{\alpha}{2}\right)^2+\left(\frac{\beta}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2-\delta$$

若 $K := \left(\frac{\alpha}{2}\right)^2+\left(\frac{\beta}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2-\delta \geq 0$, then

$(*)$ describes a sphere of radius $\sqrt{K}$ centered at $\left(-\frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}\right)$

注: With center (x_0, y_0, z_0) and radius a:

sphere: $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$

closed ball: $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$

open ball: $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$

$(0,0,0)$ 可被看作 sphere 或 closed ball

open ball 也被表示为 $B_a(x_0, y_0, z_0)$

例:

Example

Show that $x^2+y^2+z^2+4x-6y+2z+6\overset{(*)}{=}0$ is the equation of a sphere, and find its center and radius.

Sol: $(*) \Leftrightarrow (x+2)^2+(y-3)^2+(z+1)^2=8$

center: $(-2, 3, -1)$, radius: $2\sqrt{2}$

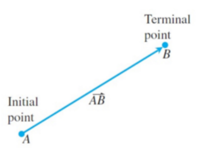
§2 Vectors

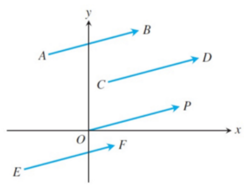
1. vectors

Roughly speaking, a vector is a directed line segment that goes from a point A (the initial point) to a point B (the terminal point). Such a vector is denoted by $\overrightarrow{AB}$.

A&B in the same space R^n.

- The two points A and B can be the same. $\overrightarrow{AA}$: zero vector.
- There are two defining properties of a vector: length and direction.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are considered exactly the same if they have the same length and direction, even if $A \neq C$.





$\overrightarrow{AB}$ is also written as $\vec{AB}$.

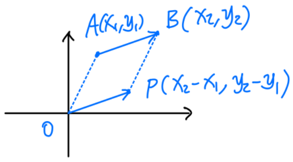
2、向量的表示

Let $\overrightarrow{AB}$ be a vector in the xy-plane, where $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Consider translating A to the origin O. It is not hard to see that $\overrightarrow{OP} = \overrightarrow{AB}$ if and only if

$$P = (x_2 - x_1, y_2 - y_1).$$

Therefore $\overrightarrow{AB}$ can be recorded by the position of P alone, and we write

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle \quad \text{or} \quad \overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}.$$



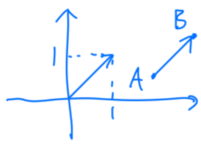
注：若 $A=(x_1,y_1,z_1)$，$B=(x_2,y_2,z_2)$ 是3D空间内的点，且 $\overrightarrow{AB}=\overrightarrow{OP}$
$P=(x_2-x_1, y_2-y_1, z_2-z_1)$，则
$\overrightarrow{AB} = \langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle$ 或 $\overrightarrow{AB} = \begin{bmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{bmatrix}$

3、Component forms and length

- In general, every vector in the xy-plane can be represented as a vector $\vec{v} = \langle v_1, v_2 \rangle$. The numbers v_1 and v_2 are called the components of $\vec{v}$.
- A common alternative notation for $\vec{v}$ is $\mathbf{v}$. or $\vec{v}$ ← Usually used for printing.
- The length of the vector $\vec{v} = \langle v_1, v_2 \rangle$ is defined to be (or norm or magnitude)

$$\sqrt{v_1^2 + v_2^2}.$$

  The length of $\vec{v}$ is commonly denoted by $|\vec{v}|$ or $\|\vec{v}\|$.



$\overrightarrow{AB} = \langle 1, 1 \rangle$.

- Similarly, every vector in the space can be written as $\vec{v} = \langle v_1, v_2, v_3 \rangle$, with

$$|\vec{v}| := \|\vec{v}\| := \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

注：在n维空间中，向量 $\vec{v}$ 可以被表示为
$\langle v_1, v_2, \cdots\cdots, v_n \rangle$
$|\vec{v}| := \|\vec{v}\| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

4、Algebraic operations（代数运算）of vectors

Definition

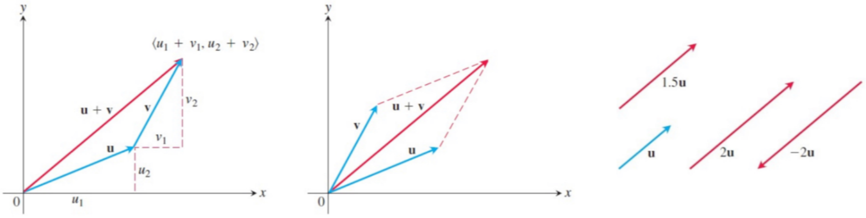
Given two vectors $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$, $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and a scalar $k \in \mathbb{R}$, define additions and scalar multiplications as follows:

- $\vec{u} + \vec{v} := \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle$.
- $k\vec{u} := \langle ku_1, ku_2, \ldots, ku_n \rangle$.

Similar to real numbers, we write $\vec{u} - \vec{v}$ to mean $\vec{u} + (-1)\vec{v}$.

These two operations are componentwise

The ~~geometry~~ geometric effect of these operations on vectors in $\mathbb{R}^2$ is displayed in the following figure.



5. Properties of algebraic operations

Definition

The zero vector, denoted by $\vec{0}$ or $\mathbf{0}$, is the vector whose components are all zeros.

Properties of Vector Operations

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars. in the same space $\mathbb{R}^n$

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $0\mathbf{u}=\mathbf{0}$
6. $1\mathbf{u}=\mathbf{u}$
7. $a(b\mathbf{u})=(ab)\mathbf{u}$
8. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$
9. $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$

6. Unit vectors and standard unit vectors

Definition

- A unit vector is a vector whose length is 1.
- The standard unit vectors in $\mathbb{R}^3$ are the unit vectors

$$\vec{i}:=\mathbf{i}:=\langle 1,0,0\rangle,\quad \vec{j}:=\mathbf{j}:=\langle 0,1,0\rangle,\quad \vec{k}:=\mathbf{k}:=\langle 0,0,1\rangle.$$

std. unit vectors in $\mathbb{R}^2$: $\vec{i}=<1,0>$ & $\vec{j}=<0,1>$.

Remark

- Any vector $\vec{v}=\langle v_1,v_2,v_3\rangle$ in $\mathbb{R}^3$ can be written as a linear combination of $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$$\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}.$$

- Whenever $|\vec{v}|\neq 0$, the vector $\vec{v}/|\vec{v}|$ is a unit vector, which is called the direction of $\vec{v}$.

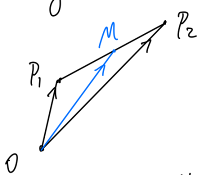
$$\left|\frac{1}{|\vec{v}|}\vec{v}\right|=\frac{1}{|\vec{v}|}|\vec{v}|=1$$

For any nonzero vector $\vec{v}$, $\vec{v}=\underbrace{|\vec{v}|}_{\text{length}}\underbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)}_{\text{direction}}$

例：若 $\vec{v}=<3,-4>$ 为 velocity vector，则 $|\vec{v}|=5$ 为 speed，$<\frac{3}{5},\frac{-4}{5}>$ 为 direction

7. Midpoint formula

The midpoint of the line segment P_1P_2 joining $P_1(x_1,y_1,z_1)$ & $P_2(x_2,y_2,z_2)$ is $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$. This can be derived using vectors:

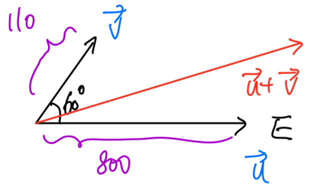


$$\begin{aligned}\vec{OM}&=\vec{OP_1}+\frac{1}{2}\vec{P_1P_2}\\&=<x_1+\frac{x_2-x_1}{2},\ y_1+\frac{y_2-y_1}{2},\ z_1+\frac{z_2-z_1}{2}>\\&=<\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}>.\end{aligned}$$

Hence $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$.

8. Applications of vectors

例: **EXAMPLE 8** A jet airliner, flying due east at 800 km/h in still air, encounters a 110 km/h tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?



Sol: $\vec{u} = <800, 0>$

$V_1 = 110 \cdot \cos\frac{\pi}{3} = 55$

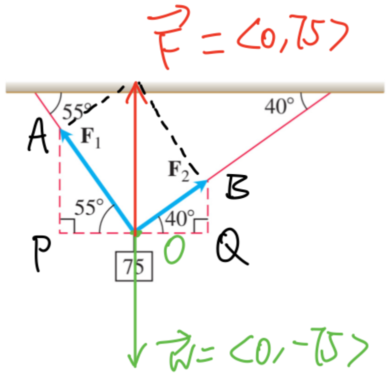
$V_2 = 110 \cdot \sin\frac{\pi}{3} = 55\sqrt{3}$

$\vec{u}+\vec{V} = <855, 55\sqrt{3}>$

speed $= S = |\vec{u}+\vec{v}| = \sqrt{855^2+55^2\cdot 3} \approx 860.3$ km/h

direction $= \frac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

例: **EXAMPLE 9** A 75-N weight is suspended by two wires, as shown in Figure 12.18a. Find the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in both wires.



Sol: $\vec{F_1}+\vec{F_2} = \vec{F} = <0, 75>$

$\vec{F_1} = <-|\vec{F_1}|\cdot\cos 55°, |\vec{F_1}|\cdot\sin 55°>$

$\vec{F_2} = <|\vec{F_2}|\cdot\cos 40°, |\vec{F_2}|\cdot\sin 40°>$

$$\begin{cases} 0 = |\vec{F_2}|\cdot\cos 40° - |\vec{F_1}|\cos 55° \\ 75 = |\vec{F_2}|\sin 40° + |\vec{F_1}|\sin 55° \end{cases}$$

$$\begin{cases} |\vec{F_1}| \approx 57.67\ N \\ |\vec{F_2}| \approx 43.18\ N \end{cases}$$

§3 Dot products

1. Dot products

> **Definition**
> Given two vectors $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$, the dot product of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}\cdot\vec{v}$, is defined by
>
> $$\vec{u}\cdot\vec{v} := u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Example

(a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$
$= -6 - 4 + 3 = -7$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right)\cdot(4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$

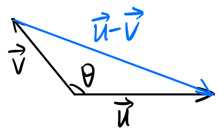
2. Properties of dot products

> **Properties of the Dot Product**
> If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then
>
> 1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
> 2. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$
> 3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
> 4. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
> 5. $\mathbf{0}\cdot\mathbf{u} = 0$.

3. dot products & angles in R^2 & R^3

Consider $\vec{u}$ and $\vec{v}$ in R^2 (or R^3), with angle θ made between them $(0 \le \theta \le \pi)$.

Then by the law of cosine,



$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\cos\theta\,|\vec{u}||\vec{v}|$$

Since $|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$,

$$\vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\cos\theta\,|\vec{u}||\vec{v}|$$

$\Rightarrow$ $\vec{u}\cdot\vec{v} = \cos\theta\,|\vec{u}|\cdot|\vec{v}|$

注: · 若 $\vec{u}$ 与 $\vec{v}$ 不为 $\vec{0}$，则

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} \Rightarrow \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}\right)$$

· 若 $\vec{u}$ 与 $\vec{v}$ 为 unit vectors，则

$$\cos\theta = \vec{u}\cdot\vec{v} \Rightarrow \theta = \arccos(\vec{u}\cdot\vec{v})$$

· 若 $\vec{u}\cdot\vec{v} = 0$，则

$\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$

一般的，

> **Def**: For $\vec{u}, \vec{v}$ in $\mathbb{R}^n$:
>
> · If $\vec{u} \ne \vec{0}$ and $\vec{v} \ne \vec{0}$, then the angle made by $\vec{u}$ and $\vec{v}$ is $\theta := \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$.
>
> · $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if $\vec{u}\cdot\vec{v} = \vec{0}$ (note that $\vec{u}$ and $\vec{v}$ can be zero).