Lecture 11

§1 Basic facts about metric space (接上)

1. Fact 4: 开(闭)集的交(并)

Let A be an index set

(i) Let $\{G_\alpha\}_{\alpha\in A}$ be a family of open sets of X. Then $\bigcup_{\alpha\in A} G_\alpha$ is open (开集的并仍开)

(ii) Let $\{F_\alpha\}_{\alpha\in A}$ be a family of closed sets of X. Then $\bigcap_{\alpha\in A} F_\alpha$ is closed (闭集的交仍闭)

(iii) In (i), if A is finite, then $\bigcap_{\alpha\in A} G_\alpha$ is open (有限个开集的交仍开)

(iv) In (ii), if A is finite, then $\bigcup_{\alpha\in A} F_\alpha$ is closed (有限个闭集的并仍闭)

证明:

① proof of (i)

($\bigcup_{\alpha\in A} G_\alpha$ 中任意一点 p 都会属于某个 G_{α_p}，由 G_{α_p} 为开，可得 p 的邻域包含于 G_{α_p}，因此包含于 $\bigcup_{\alpha\in A} G_\alpha$)

$\forall p \in \bigcup_{\alpha\in A} G_\alpha \Rightarrow p \in$ some G_{α_p}

$\because$ G_{α_p} is open

$\therefore$ $\exists$ neighbourhood $N_r(p) \subset G_{\alpha_p} \subset \bigcup_{\alpha\in A} G_\alpha$

$\therefore$ p is an interior point of $\bigcup_{\alpha\in A} G_\alpha$

$\therefore$ $\bigcup_{\alpha\in A} G_\alpha$ is open

② proof of (ii)

Use Fact 3 (开集和闭集互补)

$(\bigcap_{\alpha\in A} F_\alpha)^c = \bigcup_{\alpha\in A} F_\alpha^c$

$\because$ F_α is closed

$\therefore$ F_α^c is open

By Fact 4 (i), $\bigcup_{\alpha\in A} F_\alpha^c$ is open

By Fact 3, $\bigcap_{\alpha\in A} F_\alpha = (\bigcup_{\alpha\in A} F_\alpha^c)^c$ is closed

③ proof of (iii)

(由于 p 属于 $\bigcap_{i=1}^n G_i$，因此 p 属于任意 G_i，对每个 G_i 都能找到一个被其包含的邻域 $N_{r_i}(p)$，选取 r_i 中最小的一个，以此为半径的邻域一定属于 $\bigcap_{i=1}^n G_i$.

$\because$ A is finite

$\therefore$ $A = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$

Denote G_{α_i} by G_i, $1\le i\le n$,

$\forall p \in \bigcap_{i=1}^n G_i \Rightarrow p \in G_i$, $1\le i\le n$

$\because$ G_i is open

$\therefore$ $\exists N_{r_i}(p) \subset G_i$

Now take $r = \min(r_1, \cdots, r_n)$, then $N_r(p) \subset N_{r_i}(p) \subset G_i$, $\forall i = 1, \cdots, n$

$\therefore\ N_r(p) \subset \bigcap_{i=1}^{n} G_i$

$\therefore$ p is an interior point of $\bigcap_{i=1}^{n} G_i$

$\therefore\ \bigcap_{\alpha\in A} G_\alpha$ is open

④ proof of (iv)

Use Fact 3（开集和闭集互补）

$(\bigcup_{\alpha\in A} F_\alpha)^c = \bigcap_{\alpha\in A} F_\alpha^c$

$\because\ F_\alpha$ is closed

$\therefore\ F_\alpha^c$ is open

By Fact 4 (i), $\bigcap_{\alpha\in A} F_\alpha^c$ is open

By Fact 3, $\bigcup_{\alpha\in A} F_\alpha = (\bigcap_{\alpha\in A} F_\alpha^c)^c$ is closed

注：若(iii)中A可为无限集，则 $\bigcap_{\alpha\in A} G_\alpha$ 不一定为开集

反例：令 $G_n = (-\frac{1}{n}, \frac{1}{n}),\ \forall n\geq 1$，$X = \mathbb{R}$．则 $\bigcap_{n=1}^{\infty} G_n = \{0\}$ Not open !

2. Fact 5: 关于 closure 的 facts

令 $E \subset X$，$\bar{E} = E \cup E'$，则

(i) closure $\bar{E}$ of E is closed

(ii) $E = \bar{E} \iff E$ is closed

(iii) If F is closed in X & $E \subset F \Rightarrow \bar{E} \subset F$（$\bar{E}$ 是包含E的最小的闭集）

证明：

① proof of (i)

(先证 $N_r(p) \subset E^c$，再证 $N_r(p) \subset (\bar{E})^c$，因此 $(\bar{E})^c$ 为开，即 $\bar{E}$ 为闭)

(使用 Fact 3，W.T.S. $(\bar{E})^c$ 为开)

$\forall p \in (\bar{E})^c \Rightarrow p \notin E,\ p \notin E'$

$\therefore\ \exists$ nbhd $N_r(p) \cap E = \varnothing$（极限点定义的反面：$\exists N_r(p)$，其中任意点均不属于E）

$\therefore\ N_r(p) \subset E^c$

(W.T.S. $N_r(p) \cap E' = \varnothing$，由此可得 $N_r(p) \subset (\bar{E})^c$)

Suppose $N_r(p) \cap E' \neq \varnothing$, then $\exists q \in N_r(p) \cap E'$

$\therefore$ q is limit point of E

$\therefore\ \forall r'$, $N_{r'}(p)$ contains a point of E

Take $r' = \frac{r - d(p,q)}{2}$, then $N_{r'}(p) \subset N_r(p)$

$\therefore\ N_r(p)$ contains a point of E (contradiction)

$\therefore\ N_r(p) \cap E' = \varnothing$

$\therefore\ N_r(p) \subset (\bar{E})^c$

$\therefore\ (\bar{E})^c$ is open

$\therefore\ \bar{E}$ is closed

② proof of (ii)

"$\Rightarrow$": obvious by (i)

"$\Leftarrow$": $\because$ E is closed

$\therefore E' \subset E$

$\therefore E \cup E' = \bar{E} \subset E$

$\because E \subset \bar{E}$

$\therefore E = \bar{E}$

③ proof of (iii)

$\because E \subset F$

$\therefore E' \subset F'$

$\therefore E \cup E' \subset F \cup F'$

$\bar{E} \subset \bar{F}$

$\because$ F is closed

$\therefore F = \bar{F}$

$\therefore \bar{E} \subset F$

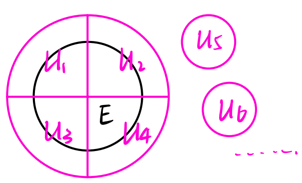
§2 Compact sets

1. Definition: Compact sets (紧集)

A set E of metric X is said to be compact, if $\forall$ open covering (开覆盖) $\bigcup_{\alpha \in A} G_\alpha$ of E (i.e. $E \subset \bigcup_{\alpha \in A} G_\alpha$, each G_α is open), $\exists$ finitely many $\alpha_1, \cdots, \alpha_n \in A$, s.t. $E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$

(若E的任意开覆盖,都存在有限子覆盖,则E为紧集)

如 $S = u_1 \cup u_2 \cup \cdots$, s.t. $E \subset S$　　如 $S' = u_1 \cup u_2 \cup u_3 \cup u_4$, s.t. $E \subset S'$



e.g. ① $E =$ empty set $\varnothing$ 为紧集

② $E =$ finite set $= \{x_1, \cdots, x_n\}$ 为紧集

③ $E = [-1, 1]$ 为紧集

④ $E = (-1, 1)$ 不是紧集, 取 $G_n = (-1+\frac{1}{n}, 1-\frac{1}{n})$, 需要无限个子覆盖才能盖住E

2. Fact 1: 紧集为闭集

Compact set E is closed

证明:

Just need to show E^c is open.

$\forall p \in E^c$

Observe $\cdot\ \forall q \in E,\ \because p \neq q \quad \therefore \exists N_{r_q}(q) \not\ni p$

$\cdot\ E \subset \bigcup_{q\in E} N_{r_q}(q)$ (open covering)

$\because$ E is compact

$\therefore \exists$ finitely many $q_1, \cdots, q_n$, s.t. $E \subset \bigcup_{q\in E} N_{r_q}(q)$

Take $r < \min\left(\frac{d(p,q_1)-r_1}{2}, \cdots, \frac{d(p,q_n)-r_n}{2}\right)$, then

$N_r(p) \cap$ any $N_{r_{q_i}}(q_i) = \varnothing$

$\therefore N_r(p) \cap E = \varnothing$

$\therefore N_r(p) \subset E^c$

$\therefore$ p is an interior point of E^c

$\therefore E^c$ is open

注：逆命题不成立，即 E is closed $\not\Rightarrow$ E is compact

反例：$E=[0,\infty)$, $X=\mathbb{R}$

实数域内的紧集必须要有界

3. Fact 2：若 X 中任意选取的有限个紧集不互斥，则 X 中的任意紧集不互斥

Let $\{K_\alpha\}_{\alpha\in A}$ be a family of compact sets of X.

Suppose $\forall$ finite subset A' of A, $\bigcap_{\alpha\in A'} K_\alpha \neq \varnothing$

Then $\bigcap_{\alpha\in A} K_\alpha \neq \varnothing$

证明：

Argue by contradiction. Suppose $\bigcap_{\alpha\in A} K_\alpha = \varnothing$

Then $\bigcup_{\alpha\in A} K_\alpha^c = X$

Fix $\alpha_1 \in A$, then $K_{\alpha_1} \subset X = \bigcup_{\alpha\in A} K_\alpha^c$

$\because K_\alpha$ is a compact set

$\therefore K_\alpha$ is closed

$\therefore K_\alpha^c$ is open

（因此 $\bigcup_{\alpha\in A} K_\alpha^c$ 是 K_{α_1} 的开覆盖，K_{α_1} 为紧集）

$\therefore \exists \alpha_2, \cdots, \alpha_n \in A$, s.t. $K_{\alpha_1} \subset \bigcup_{i=2}^{n} K_{\alpha_i}^c$

$\therefore K_{\alpha_1}^c \supset \bigcap_{i=2}^{n} K_{\alpha_i}$

$\therefore \bigcap_{i=1}^{n} K_{\alpha_i} = \varnothing$ (contradiction)

$\therefore \bigcap_{\alpha\in A} K_\alpha \neq \varnothing$

4. Fact 3：若 $\{K_n\}_{n=1}^{\infty}$ 为一个非空递减紧集序列，则 $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$ （Fact 2 的推论）

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of nonempty compact sets of X s.t. $K_{n+1} \subset K_n$, $\forall n \geq 1$. Then $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$

证明：

$\forall$ finite subset A' of A, $\bigcap_{\alpha\in A'} K_\alpha = K_{\max(A')} \neq \varnothing$

$\therefore$ By Fact 2, $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$