Lecture 21

§1 Change of variable formula (triple integrals)

There is a similar change of variables formula for triple integrals. Let T be a transformation given by

$$x = g(u, v, w), \quad y = h(u, v, w) \quad \text{and} \quad z = k(u, v, w),$$

which maps a region R in the uvw-space onto a region E in the xyz-space. The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Under assumptions similar to those in the theorem of change of variables for double integrals, the following formula holds for triple integrals:

$$\iiint_E f(x, y, z)\, dV$$
$$= \iiint_R f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

例: Evaluate $\int_0^3 \int_0^4 \int_{x=y/2}^{y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\,dy\,dz$

Sol: · Let $u = \frac{2x-y}{2}$, $v = \frac{z}{3}$, $w = \frac{y}{2}$

$x = u+w$, $y = 2w$, $z = 3v$

· $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} = -6$

$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = 6$

· $0 \le z \le 3 \Rightarrow 0 \le v \le 1$

$0 \le y \le 4 \Rightarrow 0 \le w \le 2$

$\frac{y}{2} \le x \le \frac{y}{2} + 1 \Rightarrow w \le u+w \le w+1 \Rightarrow 0 \le u \le 1$

· $I = \int_0^1 \int_0^1 \int_0^2 (u+v) \cdot 6\, dw\,dv\,du$

$= 12 \int_0^1 \int_0^1 (u+v)\, dv\,du$

$= 12 \int_0^1 u + \frac{1}{2}\, du$

$= 12$

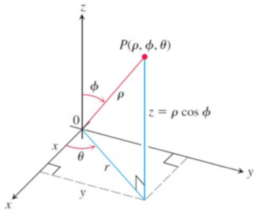
§2 Spherical coordinates

1. Spherical coordinates (球面坐标)

Given any point P in the xyz-space, we can represent the point by (ρ, ϕ, θ), where

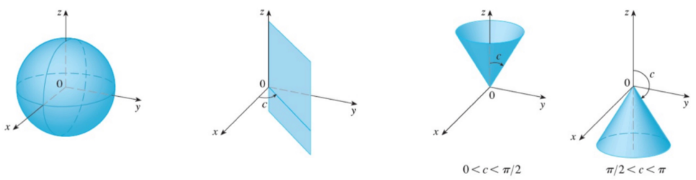
- ρ is the distance from P to the origin ($\rho \geq 0$);
- ϕ is the angle made from the positive z-axis to $\overrightarrow{OP}$ ($0 \leq \phi \leq \pi$), and;
- θ is the same as in cylindrical coordinates.

The coordinate system above in (ρ, ϕ, θ) is called the spherical coordinate system.



The figures below, from left to right, show the surfaces for:

1. $\rho = c$.
2. $\theta = c$.
3. $\phi = c$.

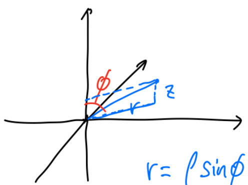


2. Conversion between coordinate systems

$$\begin{cases} x = \rho \cdot \sin\phi \cdot \cos\theta \\ y = \rho \cdot \sin\phi \cdot \sin\theta \\ z = \rho \cdot \cos\phi \end{cases}$$

$\rho = \sqrt{x^2 + y^2 + z^2}$, $r = \sqrt{x^2+y^2} = \rho \cdot \sin\phi$

Conversion Between Coordinate Systems



$r = \rho\sin\phi$
$z = \rho\cos\phi$

Let r be as in cylindrical coordinates. Then

$$z = \rho\cos\phi \quad \text{and} \quad r = \rho\sin\phi,$$

from which we can deduce

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.}$$

By definition, we also have (given x, y & z)

$$\boxed{\rho = \sqrt{x^2 + y^2 + z^2}.}$$

Then we can find ϕ and then θ as well.

例: e.g. (a) ~~(a)~~ Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z-1)^2 = 1.$$

(b) ~~(b)~~ Find a spherical coordinate equation for the cone

$$z = \sqrt{x^2 + y^2}.$$

(a) $x^2+y^2+(z-1)^2=1$

$\Rightarrow x^2+y^2+z^2-2z=0$

$\Rightarrow \rho^2-2\rho\cdot\cos\phi=0$

$\Rightarrow \rho=0$ or $\rho=2\cos\phi$ ($\rho=0$ 被包含于 $\rho=2\cos\phi$ 中)

$\Rightarrow \rho=2\cos\phi,\quad 0\le\phi\le\frac{\pi}{2}$

(b) $z=\sqrt{x^2+y^2}$

$\Rightarrow \rho\cos\phi=\rho\sin\phi$

$\Rightarrow \rho=0$ or $\phi=\frac{\pi}{4}$ $(0\le\phi\le\pi)$ ($\rho=0$ 被包含于 $\phi=\frac{\pi}{4}$ 中)

$\Rightarrow \phi=\frac{\pi}{4}$

3. Integration with spherical coordinates

When evaluating $\iiint_E f(x,y,z)\,dV$ with spherical substitutions

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi,$$

the absolute value of the Jacobian is

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right|=\rho^2\sin\phi,$$

so the change of differentials becomes

$$\boxed{dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.}$$

* check:

$x=\rho\sin\phi\cos\theta,\quad y=\rho\cdot\sin\phi\cdot\sin\theta,\quad z=\rho\cdot\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}$$

$$=\cos\phi\begin{vmatrix}\rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\end{vmatrix}-(-\rho\sin\phi)\begin{vmatrix}\sin\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta\end{vmatrix}$$

$$=\cos\phi\left(\rho^2(\sin\phi\cos\phi\cos^2\theta+\sin\phi\cos\phi\sin^2\theta)\right)+\rho\sin\phi\left(\rho(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta)\right)$$

$$=\cos\phi(\rho^2\cos\phi\sin\phi)+\rho\sin\phi(\rho\sin^2\phi)$$

$$=\rho^2\sin\phi(\cos^2\phi+\sin^2\phi)$$

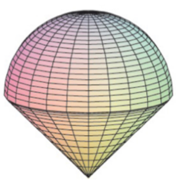
$$=\rho^2\sin\phi$$

Since $0\le\phi\le\pi$, $\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right|=\rho^2\sin\phi$

例: Example

(a) Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the closed ball centered at the origin with radius 1.

(b) Find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.



☺ An ice cream ☺

(a) $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$= 2\pi \cdot \left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^1 e^{\rho^3}\rho^2\, d\rho\right)$

$= \frac{4\pi}{3}(e-1)$

(b)
- $z = \sqrt{x^2+y^2}$

  $\Rightarrow \phi = \frac{\pi}{4}$

- $x^2+y^2+z^2 = z$

  $\Rightarrow \rho^2 = \rho\cos\phi$

  $\Rightarrow \rho = 0$ or $\rho = \cos\phi$, $0 \le \phi \le \frac{\pi}{2}$

- E: $0 \le \rho \le \cos\phi$

  $0 \le \phi \le \frac{\pi}{4}$

  $0 \le \theta \le 2\pi$

- $\mathrm{Vol}(E) = \iiint_E dV$

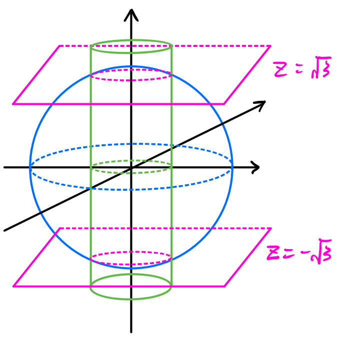
  $= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

  $= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{3}\cos^3\phi \sin\phi \, d\phi$

  $= 2\pi \cdot \left(-\frac{1}{12}\cos^4\phi \Big|_0^{\frac{\pi}{4}}\right)$

  $= \frac{\pi}{8}$

例: Find the volume of the solid E, where E is intersection of solid $x^2+y^2 \le 1$ and solid $x^2+y^2+z^2 \le 4$



- E_1: E_2:   , $V(E) = V(E_1) + 2V(E_2)$
- Solve $\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow z = \pm\sqrt{3}$
- $V(E_1) = (\pi 1^2)\cdot 2\sqrt{3} = 2\sqrt{3}\pi$
- $V(E_2) = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_{\sqrt{3}/\cos\phi}^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

  $= 2\pi \int_0^{\frac{\pi}{6}} \frac{1}{3}(8 - 3\sqrt{3}/\cos^3\phi)\cdot \sin\phi \, d\phi$

  $= \frac{2}{3}\pi \cdot \left(-8\cos\phi - \frac{3\sqrt{3}}{2}\cos^{-2}\phi \Big|_0^{\frac{\pi}{6}}\right)$

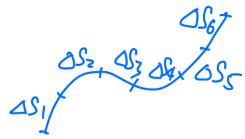
  $= \frac{\pi}{3}(16 - 9\sqrt{3})$

- $V(E) = V(E_1) + 2V(E_2) = \frac{32}{3}\pi - 4\sqrt{3}\pi$

§3 Line integrals (曲线积分) of real-valued functions (第一类曲线积分)

1. Definition

Suppose we would like to compute the mass of a wire C in the space with linear density $f(x, y, z)$. By breaking C into smaller pieces and picking points (x_k, y_k, z_k) from the ~~subintervals~~ Subarcs, the mass is approximately

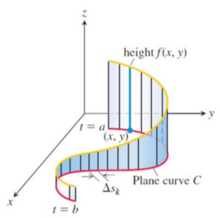


$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k.$$

This motivates the definition of line integrals.

> **Definition**
> If f is a real-valued defined on a curve C, then the **line integral of f along C** ~~(with respect to arc length)~~ is
>
> S is arc length parameter
>
> $$\int_C f(x, y, z)\, ds := \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k,$$
>
> provided that the limit exists. P partition C into pieces

- The line integral of a nonnegative function $f(x, y)$ along a plane curve can be interpreted as the area of the "fence" between the graph and the xy-plane, as indicated below.



(At least) Two meanings of $\int_C f(x,y)\,ds$ for nonnegative f:

- Mass of a wire.
- Area of a fence/wall.

注: · 对于 plane curve C, $f(x,y)$ 沿着 C 的 line integral 为 $\int_C f(x,y)\,ds$, 被定义为 $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta S_k$

· R^n 空间内的 n 元 line integral 也可以被 defined similarly

2. Computation

- A function $\mathbf{r} : I \to \mathbb{R}^n$ is called **smooth** if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for every t. A curve having a smooth parametrization is called a **smooth curve**. } Recall
- If C is given by a smooth parametrization

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b,$$

and f is continuous on C, then $\int_C f\,ds$ exists, and

$$\begin{aligned}\int_C f(x, y, z)\,ds &= \int_a^b f(x(t), y(t), z(t))\,\underbrace{|\mathbf{r}'(t)|}_{\text{Speed of movement}}\,dt \\ &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt.\end{aligned}$$

- If $f \equiv 1$, then $\int_C f\,ds$ just gives the arc length of C.

对于 R^2 平面内的曲线:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\,|\vec{r}'(t)|\,dt = \int_a^b f(\vec{r}(t))\,|r'(t)|\,dt$$

注: 1° 若曲线 L 由 $y=y(x)$ $(a\le x\le b)$ 给出，其中 $y=y(x)$ 在 $[a,b]$ 上有连续的导数. 又假定 $f(x,y)$ 在 L 上连续，则有

$$\int_L f(x,y)\,ds = \int_a^b f(x,f(x))\sqrt{1+[y'(x)]^2}\,dx$$

2° 若曲线 L 由参数方程给出

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} \quad \alpha\le t\le\beta$$

其中 $\varphi(t)$ 与 $\psi(t)$ 在 $[a,b]$ 上有连续的一阶导数. 若 $f(x,y)$ 在 L 上连续，则有:

$$\int_L f(x,y)\,ds = \int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$$

3° 若 L 为一空间曲线，由参数方程给出

$$\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases} \quad (\alpha\le t\le\beta)$$

其中 $x(t), y(t), z(t)$ 在 $[a,b]$ 上有连续的导数. 若 $f(x,y,z)$ 在 L 上连续，则有:

$$\int_L f(x,y,z)\,ds = \int_\alpha^\beta f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

例: Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of $x^2+y^2=1$

- Parametrize C:

$$x=\cos\theta, \quad y=\sin\theta, \quad \vec{r}(\theta)=\langle\cos\theta,\sin\theta\rangle, \quad 0\le\theta\le\pi$$

- Compute

$$\begin{aligned}\int_C (2+x^2y)\,ds &= \int_0^\pi (2+\cos^2\theta\sin\theta)\cdot|\vec{r}\,'(\theta)|\,d\theta \\ &= \int_0^\pi (2+\cos^2\theta\sin\theta)\cdot\sqrt{(-\sin\theta)^2+(\cos\theta)^2}\,d\theta \\ &= \int_0^\pi 2+\cos^2\theta\sin\theta\,d\theta \\ &= 2\pi+\frac{2}{3}\end{aligned}$$

注: Remark

If C is a piecewise smooth curve, that is, if C is a union of finitely many smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds.$$