Lecture 19

§1 Taylor's theorem

1、Definition: n-th order Taylor's polynomial

Goal: 用 polynomials locally near x_0 来近似 $f(x)$

- Use 1st order polynomial $p_1(x)$ (linearization of f at x_0)

  $f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$ as $x \to x_0$ if f differentiable at x_0

- Use 2nd order polynomial $p_2(x)$

  $f(x) \approx p_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2$

  Demand: $f(x_0) = p_2(x_0) \Rightarrow c_0 = f(x_0)$

  $f'(x_0) = p_2'(x_0) \Rightarrow c_1 = f'(x_0)$

  $f''(x_0) = p_2''(x_0) \Rightarrow c_2 = \frac{f''(x_0)}{2}$

- Better approximation by higher-order polynomial $p_n(x)$?

  $p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

  这被称为 n-th order Taylor's polynomial of $f(x)$ at x_0

Question: What can we say about $\text{error}(x) = f(x) - p_{n-1}(x)$, $n \geq 1$

特例: $n=1 \Rightarrow p_0(x) = f(x_0)$

$\text{error} = f(x) - f(x_0) \overset{M.V.T.}{=} f'(c)(x-x_0)$

2、Theorem: Taylor's theorem (带拉格朗日型余项)

Let $n \geq 1$, suppose $f^{(n-1)}$ exists & continuous on $[a,b]$, and $f^{(n)}$ exists on (a,b), $x_0 \in [a,b]$.

Then $f(x) - p_{n-1}(x) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n$, $\forall x \in [a,b]$, for some c between x and x_0.

其中 $\frac{f^{(n)}(c)}{n!}(x-x_0)^n$ 被称为 Cauchy's remainder

注: 另一种表示形式为 $f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$

证明:

- If $x = x_0$, done
- Suppose $x \neq x_0$, fix x, x_0, Define $M = \frac{f(x) - p_{n-1}(x)}{(x-x_0)^n}$

  W.T.S. $M = \frac{f^{(n)}(c)}{n!}$ for some c between x and x_0

  W.L.O.G. Let $x_0 < x$

  Define $g(t) = f(t) - p_{n-1}(t) - M(t-x_0)^n$

  - $g^{(n-1)}(t)$ exists & continuous on closed interval $[x_0, x]$

    $g^{(n)}(t)$ exists on (x_0, x)

  - $g(x) = 0$
  - $g(x_0) = 0 \overset{M.V.T.}{\underset{\text{on } [x_0,x]}{\Longrightarrow}} \overset{0}{g(x)} - \overset{0}{g(x_0)} = g'(x_1)(x-x_0)$ for some $x_1 \in (x_0, x) \Rightarrow g'(x_1) = 0$

- $g'(x_0)=0 \xRightarrow[\text{on } [x_0,x_1]]{\text{M.V.T.}} \overset{0}{g'(x_1)} - \overset{0}{g'(x_0)} = g''(x_2)(x_1-x_0)$ for some $x_2 \in (x_0, x_1) \Rightarrow g''(x_2)=0$

$\vdots$

- $g^{(n-1)}(x_0)=0 \xRightarrow[\text{on } [x_0,x_{n-1}]]{\text{M.V.T.}} \exists\, x_n \in (x_0, x_{n-1})$ s.t. $g^{(n)}(x_n)=0 \Rightarrow f^{(n)}(x_n) - 0 - M\cdot n!$

$\therefore\ M = \frac{f^{(n)}(c)}{n!}$, $c = x_n \in (x_0, x_{n-1}) \subset (x_0, x)$

3. **Theorem:** Taylor's theorem (带佩亚诺型余项)

Suppose $f^{(n)}$ exists for some $n \geq 1$. Then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o(x-x_0)^n \quad \text{as } x \to x_0$$

其中 $o(x-x_0)^n$ 被称为 Peano's remainder

证明:

W.T.S. $\dfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)-\cdots-\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^n} = o(1)$ as $x \to x_0$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)-\cdots-\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^n}$$

$$\overset{\text{L'H}}{=} \lim_{x\to x_0} \frac{f'(x)-f'(x_0)-f''(x_0)(x-x_0)-\cdots-\frac{f^{(n)}(x_0)}{(n-1)!}(x-x_0)^{n-1}}{n(x-x_0)^{n-1}}$$

$$\overset{\text{L'H}}{=} \lim_{x\to x_0} \frac{f''(x)-f''(x_0)-f'''(x_0)(x-x_0)-\cdots-\frac{f^{(n)}(x_0)}{(n-2)!}(x-x_0)^{n-2}}{n(n-1)(x-x_0)^{n-2}}$$

$\cdots\cdots$

$$\overset{\text{L'H}}{=} \lim_{x\to x_0} \frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{n(n-1)\cdots 2(x-x_0)}$$

$$= \frac{1}{n(n-1)\cdots 2} \lim_{x\to x_0} \frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{x-x_0}$$

linear approximation (Let $g = f^{(n-1)}$)

$$= \frac{1}{n(n-1)\cdots 2} \lim_{x\to x_0} \frac{o(x-x_0)}{x-x_0}$$

$$= 0$$

$$\therefore\ \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)-\cdots-\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^n} = o(1) \quad \text{as } x \to x_0$$

e.g. $e^x = e^0 + (e^x)'|_{x=0}(x-0) + \cdots + \frac{(e^x)^{(n-1)}|_{x=0}}{(n-1)!}(x-0)^{n-1} + \frac{(e^x)^{(n)}|_{x=c}}{n!}(x-0)^n$

$= 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^c}{n!}x^n$, $\forall x \in \mathbb{R}, n \geq 1$, $c \in (0, x)$

$\left|\frac{e^c x^n}{n!}\right| \leq \frac{e^{|x|}|x|^n}{n!} = e^{|x|}\frac{|x|\cdot|x|\cdots|x|}{n\cdot(n-1)\cdots 1} \to 0$ as $n \to \infty$

$\therefore\ e^x = 1 + x + \cdots + \frac{x^n}{n!} + \cdots \quad \forall x \in R$

例 1: Find $\lim_{x\to\infty}[(x^3 - x^2 + \frac{x}{2})e^{\frac{1}{x}} - x^3]$

$$\lim_{x\to\infty}[(x^3 - x^2 + \frac{x}{2})e^{\frac{1}{x}} - x^3]$$

$$\overset{y=\frac{1}{x}}{=\!=} \lim_{y\to 0}[(y^{-3} - y^{-2} + \frac{y^{-1}}{2})e^y - y^{-3}]$$

$$= \lim_{y\to 0}[(y^{-3} - y^{-2} + \frac{y^{-1}}{2})(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + o(y^3)) - y^{-3}]$$

$$= \lim_{y\to 0}[y^{-3} - y^{-2} + \frac{y^{-1}}{2} + y^{-2} - y^{-1} + \frac{1}{2} + \frac{y^{-1}}{2} - \frac{1}{2} + \frac{y}{4} + (\frac{1}{3!} - \frac{y}{3!} + \frac{y}{3!2})(1 + o(1)) - y^{-3}]$$

$$= \frac{1}{6}$$

§2 关于 Taylor's theorem 的 facts

1. Theorem: second derivative test for local extreme

Suppose $f''(x_0)$ exists.

(i) Necessary conditions:

If $f(x_0)$ is local max, then $f''(x_0) \le 0$

If $f(x_0)$ is local min, then $f''(x_0) \ge 0$

(ii) Sufficient conditions:

If $f'(x_0) = 0$, and $f''(x_0) < 0$, then $f(x_0)$ is local max

If $f'(x_0) = 0$, and $f''(x_0) < 0$, then $f(x_0)$ is local max

证明: (proof of (i))

证法一: (bad proof)

$\because\ f(x_0)$ is local max

$\therefore \begin{cases} f'(x_0) = 0 \\ f(x_0) \ge f(x) \quad \forall x \approx x_0 \end{cases}$

Take $x > x_0$, then $\overset{>0}{f(x_0) - f(x)} \overset{M.V.T.}{=} \overset{<0}{f'(z)}\overset{<0}{(x_0 - x)}$, z between x and x_0

Now, $f''(x_0) = \lim_{z\to x_0^+} \frac{\overset{\le 0}{f'(z)} - \overset{=0}{f'(x_0)}}{\underset{>0}{z - x_0}} \le 0$

证法二: (better proof)

By Peano, $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + o((x - x_0)^2)$

Argue by contradiction,

If $f''(x_0) > 0$, then $f(x) = f(x_0) + (\frac{f''(x_0)}{2!} + o(1))(x - x_0)^2$

$\because\ \frac{f''(x_0)}{2!} + o(1) > 0$ if x very close to x_0

$\therefore\ f(x_0)$ is a local min (contradiction)

2. Theorem: Inverse function theorem

Let Ω be open in R^n, $f: \Omega \to R^n$ be C^1-smooth.

Suppose $\exists\, a \in \Omega$ s.t. $(f'(a))_{n\times n}$ is non-singular

Then $\exists$ open sets U & V in R^n, s.t.

- $a \in U$, $f(a) \in V$
- f is one-to-one on U & $f(U) = V$
- Let g be the inverse of $f|_U$, then g is also C^1-smooth on V,

  and $(g'(y))_{n\times n} = (f'(x))^{-1}_{n\times n}$

f 被称为 diffeomorphism from U to V

注：对于 function of one variable $g(x) = f^{-1}(x)$, $g'(x) = \frac{1}{f'(x)}$