Lecture 21

§1 Implicit function theorem

Purpose: Give a system $f(x,y)=0 \Rightarrow \begin{cases} f_1(x_1,\cdots,x_m,y_1,\cdots,y_n)=0 \\ f_2(x_1,\cdots,x_m,y_1,\cdots,y_n)=0 \\ \vdots \\ f_n(x_1,\cdots,x_m,y_1,\cdots,y_n)=0 \end{cases}$

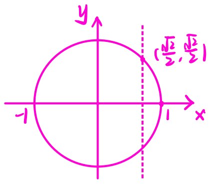
Want to solve $(y_1,\cdots,y_n)$ in terms of $(x_1,\cdots,x_m)$ from equations.

e.g.1 $x^2+y^2+1=0$: Not possible to solve for y in terms of x

Moral of the story: need at least one point (a,b) s.t. $f(a,b)=0$

e.g.2 $x^2+y^2-1=0$: $y^2=1-x^2 \Rightarrow y=\pm\sqrt{1-x^2}$, $-1\le x\le 1$

(lack of uniqueness. In order to determine y uniquely, need to be told near where we are supposed to solve for y)



near $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$, y is unique

near $(1,0)$, no uniqueness for y

e.g.3 $y+\sin y\ln x+xe^x-e-1=0$. Find $\frac{dy}{dx}\Big|_{x=1,y=1}$

$$\frac{d}{dx}(y+\sin y\ln x+xe^x-e-1)=0$$

$$\Rightarrow \frac{dy}{dx}+\cos y\frac{dy}{dx}\cdot\ln x+\sin y\frac{1}{x}+(x+1)e^x=0$$

More generally, $f(x,y)=0$. Assume $\exists(a,b)\in\mathbb{R}^2$ s.t. $f(a,b)=0$, want to find $\frac{dy}{dx}\Big|_{\substack{x=a\\y=b}}$

$$\frac{d}{dx}f(x,y)\ \left(=\frac{dz}{dx}\right)=0 \xRightarrow{\text{chain rule}} \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}=f_x(x,y)+f_y(x,y)\frac{dy}{dx}=0$$

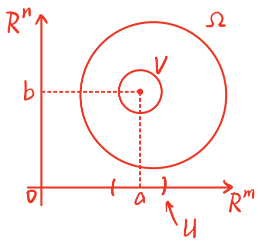
$$\Rightarrow \frac{dy}{dx}\Big|_{\substack{x=a\\y=b}}=\frac{-f_x(x,y)}{f_y(x,y)}\Big|_{\substack{x=a\\y=b}} \quad \text{provided } f_y(x,y)\neq 0$$

1. Theorem: Implicit function theorem

Let Ω be open in $\mathbb{R}^{m+n}=\mathbb{R}^m\times\mathbb{R}^n$, let $f:\Omega\subset\mathbb{R}^{m+n}\to\mathbb{R}^n$ be C^1-smooth.

- Write a generic point in $\mathbb{R}^{m+n}$ as (x,y), $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$
- Write f as $f(x,y)$, suppose $\exists(a,b)\in\Omega$ s.t. ① $f(a,b)=\vec{0}$

  ② $\left(D_yf(x,y)\Big|_{\substack{x=a\\y=b}}\right)_{n\times n}$ is nonsingular



Then $\exists$ open set U in $\mathbb{R}^m$, V in $\mathbb{R}^{m+n}$, s.t.

- $a\in U$, $(a,b)\in V$
- $\forall x\in U$, $\exists!\, y\in\mathbb{R}^n$ s.t. $(x,y)\in V$, and $f(x,y)=\vec{0}$

Let $y=g(x)$, $x\in U$. $g(x)$ is called the function implicitly defined by $f(x,y)=0$

$g: \mathcal{U} \to R^n$ satisfies

- $g(a) = b$
- g is C^1-smooth
- $f(x, g(x)) = 0, \quad \forall x \in \mathcal{U}$
- $(Dg(x))_{n\times m} = -(D_y f(x,y))^{-1}_{n\times n} (D_x f(x,y))_{n\times m}$

证明:

Going to use Inverse Function Theorem

Let $F(x,y) = \begin{bmatrix} x \\ f(x,y) \end{bmatrix}_{(m+n)\times 1}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $y = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$

$\Rightarrow F: \Omega \in R^{m+n} \to R^{m+n}$ is C^1-smooth (由于 f is C^1-smooth)

- $F(a,b) = \begin{bmatrix} a \\ 0 \end{bmatrix}$

$DF(a,b) = \begin{bmatrix} I_{m\times m} & 0_{m\times n} \\ D_x f(a,b)_{n\times m} & D_y f(a,b)_{n\times n} \end{bmatrix}_{(m+n)\times(m+n)}$

(Q: $DF(a,b)$ non-singular?)

$\because$ $\det DF(a,b) = \det I_{m\times m} \cdot \det D_y f(a,b)_{n\times n} \neq 0$

$\therefore$ $DF(a,b)$ non-singular

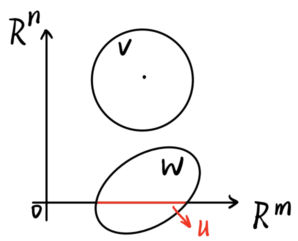
By Inverse function theorem,

$\exists$ open sets of R^{m+n}: $(a,b) \in V$, $\begin{pmatrix} a \\ 0 \end{pmatrix} \in W$ s.t.

$F: V \to W$ is 1-1 and onto. Let $G = (F|_V)^{-1}$.

Then G is C^1-smooth on V

Let $\mathcal{U} = \{x \in R^m \mid (x,0) \in W\} \Rightarrow a \in \mathcal{U}$



Claim: $\mathcal{U}$ is open in R^m

$\forall x_0 \in \mathcal{U}$

$\because$ $(x_0, 0) \in W$ and W open

$\therefore$ nbhd $N_r(x_0, 0) \subset W$

$\Rightarrow$ nbhd $N_r(x_0)$ (nbhd of x_0 in R^m) $\times \{0\} \subset W$

$\Rightarrow N_r(x_0) \subset \mathcal{U}$

$\Rightarrow \mathcal{U}$ is open in R^m

Now $\forall x \in \mathcal{U}$, $(x,0) \in W \Rightarrow \exists! (\tilde{x}, y) \in V$ s.t. $F(\tilde{x}, y) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\because\ F(\tilde{x}, y) = \begin{bmatrix} \tilde{x} \\ f(\tilde{x}, y) \end{bmatrix}$

$\therefore\ \begin{bmatrix} \tilde{x} \\ f(\tilde{x}, y) \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\therefore\ \tilde{x} = x,\ f(x, y) = 0$

Denote this y by $g(x)$, $x \in U$

Then $f(a, b) = 0 \Rightarrow \begin{cases} f(x, g(x)) = 0 \\ g(a) = b \end{cases}$

(Q: g C^1-smooth on U?)

observe $F(x, g(x)) = \begin{bmatrix} x \\ f(x, g(x)) \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ g(x) \end{bmatrix} = F^{-1}\begin{bmatrix} x \\ 0 \end{bmatrix}$

$\because\ F^{-1}$ is C^1-smooth, $\forall x \in U$

$\therefore\ g$ is C^1-smooth on U

(Q: $Dg(x) = ?$)

Observe $f(x, y) = 0$, $y = g(x)$, $x \in U$

$\Rightarrow D_x(f(x, g(x)))$ $(\neq (D_x f)(x))$ $= (D_x f)(x, g(x)) + D_y f(x, g(x))_{n \times n} \cdot (D_x g(x))_{n \times m}$

$\Rightarrow D_y f(x, g(x))\, D_x g(x) = -D_x f(x, g(x))$

$\Rightarrow Dg(x) = -(D_y f(x, g(x)))^{-1} D_x f(x, g(x))$

Q.E.D.

例1: $\begin{cases} e^{xu}\cos(yv) = u + \frac{\sqrt{2}}{2} \\ e^{xu}\sin(yv) = \frac{2\sqrt{2}v}{\pi} \end{cases}$ solve for (u, v) in terms of x & y near $(x, y, u, v) = (\underbrace{1, 1}_{a}, \underbrace{0, \tfrac{\pi}{4}}_{b})$

Let $f(x, y, u, v) = \begin{bmatrix} e^{xu}\cos(yv) - u - \frac{\sqrt{2}}{2} \\ e^{xu}\sin(yv) - \frac{2\sqrt{2}v}{\pi} \end{bmatrix}$

$f(1, 1, 0, \frac{\pi}{4}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$D_{(u,v)} f = \begin{bmatrix} xe^{xu}\cos(yv) - 1 & -e^{xu} y \sin(yv) \\ xe^{xu}\sin(yv) & ye^{xu}\cos(yv) - \frac{2\sqrt{2}}{\pi} \end{bmatrix}$

$D_{(u,v)} f(1, 1, 0, \frac{\pi}{4}) = \begin{vmatrix} \frac{\sqrt{2}}{2} - 1 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{\pi} \end{vmatrix} \neq 0$

By the Implicit function theorem, can solve for (u, v) in terms of $(x, y) \approx (1, 1)$