Lecture 19

§1 Maxima convolution 的 ANN representation

1. **Lemma**: Maxima convolution 的 ANN representation (4.2.9)

令 ① 维度: $d \in \mathbb{N}$

② mesh points (插值点) 的个数: $K \in \mathbb{N}$

③ Lipschitz parameter: $L \in [0, \infty)$

④ mesh points (插值点): $\mathcal{x}_1, \mathcal{x}_2, \cdots, \mathcal{x}_K \in \mathbb{R}^d$

⑤ 插值函数值: $\mathcal{y} = (y_1, y_2, \cdots, y_K) \in \mathbb{R}^K$

⑥ Maxima convolution 的 ANN representation: $\Phi \in \mathbf{N}$, 其满足

$$\Phi = M_K \bullet \underbrace{A_{-LI_K, \mathcal{y}}}_{\text{构造}(y_i - L\|\mathcal{x}-x_i\|_1)_{i\in\{1,\cdots,K\}}} \bullet \underbrace{P_K(L_d \bullet A_{I_d, -x_1}, L_d \bullet A_{I_d, -x_2}, \cdots, L_d \bullet A_{I_d, -x_K})}_{\text{构造}(\|\mathcal{x}-x_1\|_1, \|\mathcal{x}-x_2\|_1, \cdots, \|\mathcal{x}-x_K\|_1)} \bullet \underbrace{T_{d,K}}_{\text{将}\mathcal{x}\in\mathbb{R}^d\text{复制}K\text{份}}$$

则 ① $I(\Phi) = d$

② $D(\Phi) = 1$

③ $H(\Phi) = \lceil \log_2(K) \rceil + 1$

④ $D_1(\Phi) = 2dK$

⑤ $D_i(\Phi) \le 3\lceil \frac{K}{2^{i-1}} \rceil$, $i \in \{2, 3, 4, \cdots\}$

⑥ $\|T(\Phi)\|_\infty \le \max\{1, L, \max_{k\in\{1,2,\cdots,K\}} \|x_k\|_\infty, 2\|\mathcal{y}\|_\infty\}$

⑦ $(R_r^N(\Phi))(\mathcal{x}) = \max_{k\in\{1,2,\cdots,K\}}(y_k - L\|\mathcal{x} - x_k\|_1)$

证明:

Proof of Lemma 4.2.9. Throughout this proof, let $\Psi_k \in \mathbf{N}$, $k \in \{1, 2, \ldots, K\}$, satisfy for all $k \in \{1, 2, \ldots, K\}$ that $\Psi_k = \mathbb{L}_d \bullet \mathbf{A}_{\mathrm{I}_d, -\mathfrak{x}_k}$, let $\Xi \in \mathbf{N}$ satisfy

$$\Xi = \mathbf{A}_{-L\,\mathrm{I}_K, \mathfrak{y}} \bullet \mathbf{P}_K(\Psi_1, \Psi_2, \ldots, \Psi_K) \bullet \mathbb{T}_{d,K}, \tag{4.56}$$

and let $|||\cdot|||\colon \bigcup_{m,n\in\mathbb{N}} \mathbb{R}^{m\times n} \to [0, \infty)$ satisfy for all $m, n \in \mathbb{N}$, $M = (M_{i,j})_{i\in\{1,\ldots,m\}, j\in\{1,\ldots,n\}} \in \mathbb{R}^{m\times n}$ that $|||M||| = \max_{i\in\{1,\ldots,m\}, j\in\{1,\ldots,n\}} |M_{i,j}|$. Observe that (4.55) and Proposition 2.1.2 ensure that $\mathcal{O}(\Phi) = \mathcal{O}(\mathbb{M}_K) = 1$ and $\mathcal{I}(\Phi) = \mathcal{I}(\mathbb{T}_{d,K}) = d$. This proves items (i) and (ii). Moreover, observe that the fact that for all $m, n \in \mathbb{N}$, $\mathfrak{W} \in \mathbb{R}^{m\times n}$, $\mathfrak{B} \in \mathbb{R}^m$ it holds that $\mathcal{H}(\mathbf{A}_{\mathfrak{W},\mathfrak{B}}) = 0 = \mathcal{H}(\mathbb{T}_{d,K})$, the fact that $\mathcal{H}(\mathbb{L}_d) = 1$, and Proposition 2.1.2 assure that

$$\mathcal{H}(\Xi) = \mathcal{H}(\mathbf{A}_{-L\,\mathrm{I}_K,\mathfrak{y}}) + \mathcal{H}(\mathbf{P}_K(\Psi_1, \Psi_2, \ldots, \Psi_K)) + \mathcal{H}(\mathbb{T}_{d,K}) = \mathcal{H}(\Psi_1) = \mathcal{H}(\mathbb{L}_d) = 1. \tag{4.57}$$

Proposition 2.1.2 and Proposition 4.2.7 hence ensure that

$$\mathcal{H}(\Phi) = \mathcal{H}(\mathbb{M}_K \bullet \Xi) = \mathcal{H}(\mathbb{M}_K) + \mathcal{H}(\Xi) = \lceil \log_2(K) \rceil + 1 \tag{4.58}$$

(cf. Definition 4.2.6). This establishes item (iii). Next observe that the fact that $\mathcal{H}(\Xi) = 1$, Proposition 2.1.2, and Proposition 4.2.7 assure that for all $i \in \{2, 3, 4, \ldots\}$ it holds that

$$\mathbb{D}_i(\Phi) = \mathbb{D}_{i-1}(\mathbb{M}_K) \le 3\left\lceil \tfrac{K}{2^{i-1}} \right\rceil. \tag{4.59}$$

This proves item (v). Furthermore, note that Proposition 2.1.2, Proposition 2.2.4, and Proposition 4.2.2 assure that

$$\mathbb{D}_1(\Phi) = \mathbb{D}_1(\Xi) = \mathbb{D}_1(\mathbf{P}_K(\Psi_1, \Psi_2, \dots, \Psi_K)) = \sum_{i=1}^{K} \mathbb{D}_1(\Psi_i) = \sum_{i=1}^{K} \mathbb{D}_1(\mathbb{L}_d) = 2dK. \tag{4.60}$$

This establishes item (iv). Moreover, observe that (2.2) and Lemma 4.2.8 imply that

$$\begin{aligned} \Phi = \big(& (\mathcal{W}_{1,\Xi}, \mathcal{B}_{1,\Xi}), (\mathcal{W}_{1,\mathbb{M}_K} \mathcal{W}_{2,\Xi}, \mathcal{W}_{1,\mathbb{M}_K} \mathcal{B}_{2,\Xi}), \\ & (\mathcal{W}_{2,\mathbb{M}_K}, 0), \dots, (\mathcal{W}_{\mathcal{L}(\mathbb{M}_K),\mathbb{M}_K}, 0) \big). \end{aligned} \tag{4.61}$$

Next note that the fact that for all $k \in \{1, 2, \dots, K\}$ it holds that $\mathcal{W}_{1,\Psi_k} = \mathcal{W}_{1,\mathbf{A}_{\mathrm{I}_d,-\mathfrak{x}_k}} \mathcal{W}_{1,\mathbb{L}_d} = \mathcal{W}_{1,\mathbb{L}_d}$ assures that

$$\begin{aligned} \mathcal{W}_{1,\Xi} = \mathcal{W}_{1,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} \mathcal{W}_{1,\mathbb{T}_{d,K}} &= \begin{pmatrix} \mathcal{W}_{1,\Psi_1} & 0 & \cdots & 0 \\ 0 & \mathcal{W}_{1,\Psi_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_{1,\Psi_K} \end{pmatrix} \begin{pmatrix} \mathrm{I}_d \\ \mathrm{I}_d \\ \vdots \\ \mathrm{I}_d \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{W}_{1,\Psi_1} \\ \mathcal{W}_{1,\Psi_2} \\ \vdots \\ \mathcal{W}_{1,\Psi_K} \end{pmatrix} = \begin{pmatrix} \mathcal{W}_{1,\mathbb{L}_d} \\ \mathcal{W}_{1,\mathbb{L}_d} \\ \vdots \\ \mathcal{W}_{1,\mathbb{L}_d} \end{pmatrix}. \end{aligned} \tag{4.62}$$

Lemma 4.2.3 hence demonstrates that $|||\mathcal{W}_{1,\Xi}||| = 1$. In addition, note that (2.2) implies that

$$\mathcal{B}_{1,\Xi} = \mathcal{W}_{1,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} \mathcal{B}_{1,\mathbb{T}_{d,K}} + \mathcal{B}_{1,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} = \mathcal{B}_{1,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} = \begin{pmatrix} \mathcal{B}_{1,\Psi_1} \\ \mathcal{B}_{1,\Psi_2} \\ \vdots \\ \mathcal{B}_{1,\Psi_K} \end{pmatrix}. \tag{4.63}$$

Furthermore, observe that Lemma 4.2.3 implies that for all $k \in \{1, 2, \dots, K\}$ it holds that

$$\mathcal{B}_{1,\Psi_k} = \mathcal{W}_{1,\mathbb{L}_d} \mathcal{B}_{1,\mathbf{A}_{\mathrm{I}_d,-\mathfrak{x}_k}} + \mathcal{B}_{1,\mathbb{L}_d} = -\mathcal{W}_{1,\mathbb{L}_d} \mathfrak{x}_k. \tag{4.64}$$

This, (4.63), and Lemma 4.2.3 show that

$$\boxed{\|\mathcal{B}_{1,\Xi}\|_\infty = \max_{k \in \{1,2,\dots,K\}} \|\mathcal{B}_{1,\Psi_k}\|_\infty = \max_{k \in \{1,2,\dots,K\}} \|\mathcal{W}_{1,\mathbb{L}_d} \mathfrak{x}_k\|_\infty = \max_{k \in \{1,2,\dots,K\}} \|\mathfrak{x}_k\|_\infty} \tag{4.65}$$

(cf. Definition 3.3.4). Combining this, (4.61), Lemma 4.2.8, and the fact that $\boxed{|||\mathcal{W}_{1,\Xi}||| = 1}$ shows that

$\hookrightarrow |||\mathcal{W}_{i,\mathbb{M}_K}||| = 1,\ i \in N$

$$\begin{aligned} \|\mathcal{T}(\Phi)\|_\infty &= \max\{|||\mathcal{W}_{1,\Xi}|||, \|\mathcal{B}_{1,\Xi}\|_\infty, |||\mathcal{W}_{1,\mathbb{M}_K} \mathcal{W}_{2,\Xi}|||, \|\mathcal{W}_{1,\mathbb{M}_K} \mathcal{B}_{2,\Xi}\|_\infty, 1\} \\ &= \max\big\{1, \max\nolimits_{k \in \{1,2,\dots,K\}} \|\mathfrak{x}_k\|_\infty, |||\mathcal{W}_{1,\mathbb{M}_K} \mathcal{W}_{2,\Xi}|||, \|\mathcal{W}_{1,\mathbb{M}_K} \mathcal{B}_{2,\Xi}\|_\infty \big\} \end{aligned} \tag{4.66}$$

(cf. Definition 1.3.5). Next note that Lemma 4.2.3 ensures that for all $k \in \{1, 2, \dots, K\}$ it holds that $\mathcal{B}_{2,\Psi_k} = \mathcal{B}_{2,\mathbb{L}_d} = 0$. Hence, we obtain that $\mathcal{B}_{2,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} = 0$. This implies that

$$\mathcal{B}_{2,\Xi} = \mathcal{W}_{1,\mathbf{A}_{-L\,\mathrm{I}_K,\mathfrak{y}}} \mathcal{B}_{2,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} + \mathcal{B}_{1,\mathbf{A}_{-L\,\mathrm{I}_K,\mathfrak{y}}} = \mathcal{B}_{1,\mathbf{A}_{-L\,\mathrm{I}_K,\mathfrak{y}}} = \mathfrak{y}. \tag{4.67}$$

In addition, observe that the fact that for all $k \in \{1, 2, \dots, K\}$ it holds that $\mathcal{W}_{2,\Psi_k} = \mathcal{W}_{2,\mathbb{L}_d}$ assures that

$$\begin{aligned} \mathcal{W}_{2,\Xi} &= \mathcal{W}_{1,\mathbf{A}_{-L\,\mathrm{I}_K,\mathfrak{y}}} \mathcal{W}_{2,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} = -L \mathcal{W}_{2,\mathbf{P}_K(\Psi_1,\Psi_2,\dots,\Psi_K)} \\ &= -L \begin{pmatrix} \mathcal{W}_{2,\Psi_1} & 0 & \cdots & 0 \\ 0 & \mathcal{W}_{2,\Psi_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_{2,\Psi_K} \end{pmatrix} = \begin{pmatrix} -L\mathcal{W}_{2,\mathbb{L}_d} & 0 & \cdots & 0 \\ 0 & -L\mathcal{W}_{2,\mathbb{L}_d} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -L\mathcal{W}_{2,\mathbb{L}_d} \end{pmatrix}. \end{aligned} \tag{4.68}$$

Item (v) in Lemma 4.2.3 and Lemma 4.2.8 hence imply that

$$\boxed{|||\mathscr{W}_{1,\mathbb{M}_K}\mathscr{W}_{2,\Xi}||| = L|||\mathscr{W}_{1,\mathbb{M}_K}||| \le L.} \tag{4.69}$$

Moreover, observe that (4.67) and Lemma 4.2.8 show that

$$\boxed{\|\mathscr{W}_{1,\mathbb{M}_K}\mathscr{B}_{2,\Xi}\|_\infty \le 2\|\mathscr{B}_{2,\Xi}\|_\infty = 2\|\mathfrak{y}\|_\infty.} \tag{4.70}$$

Combining this with (4.66) and (4.69) establishes item (vi). Next observe that Proposition 4.2.2 and Lemma 2.3.3 show that for all $x \in \mathbb{R}^d$, $k \in \{1, 2, \ldots, K\}$ it holds that

$$(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\Psi_k))(x) = \big(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbb{L}_d) \circ \mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbf{A}_{\mathrm{I}_d,-\mathfrak{x}_k})\big)(x) = \|x - \mathfrak{x}_k\|_1. \tag{4.71}$$

This, Proposition 2.2.3, and Proposition 2.1.2 imply that for all $x \in \mathbb{R}^d$ it holds that

$$\big(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbf{P}_K(\Psi_1, \Psi_2, \ldots, \Psi_K) \bullet \mathbb{T}_{d,K})\big)(x) = \big(\|x - \mathfrak{x}_1\|_1, \|x - \mathfrak{x}_2\|_1, \ldots, \|x - \mathfrak{x}_K\|_1\big). \tag{4.72}$$

(cf. Definitions 1.2.4 and 1.3.4). Combining this and Lemma 2.3.3 establishes that for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\Xi))(x) &= \big(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbf{A}_{-L\,\mathrm{I}_K,\mathfrak{y}}) \circ \mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbf{P}_K(\Psi_1, \Psi_2, \ldots, \Psi_K) \bullet \mathbb{T}_{d,K})\big)(x) \\ &= \big(\mathfrak{y}_1 - L\|x - \mathfrak{x}_1\|_1, \mathfrak{y}_2 - L\|x - \mathfrak{x}_2\|_1, \ldots, \mathfrak{y}_K - L\|x - \mathfrak{x}_K\|_1\big).\end{aligned} \tag{4.73}$$

Proposition 2.1.2 and Proposition 4.2.7 hence demonstrate that for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\Phi))(x) &= \big(\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbb{M}_K) \circ \mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\Xi)\big)(x) \\ &= (\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\mathbb{M}_K))\big(\mathfrak{y}_1 - L\|x - \mathfrak{x}_1\|_1, \mathfrak{y}_2 - L\|x - \mathfrak{x}_2\|_1, \ldots, \mathfrak{y}_K - L\|x - \mathfrak{x}_K\|_1\big) \\ &= \max\nolimits_{k \in \{1,2,\ldots,K\}}(\mathfrak{y}_k - L\|x - \mathfrak{x}_k\|_1).\end{aligned} \tag{4.74}$$

This establishes item (vii). The proof of Lemma 4.2.9 is thus complete. □

§2 Constructive ANN approximation results (构造一个ANN来达到目标近似精度)

1. Proposition: ANN approximation through maxima convolutions (4.3.1)

令 ① 维度: $d \in N$

② mesh points (插值点) 的个数: $K \in N$

③ Lipschitz parameter: $L \in [0, \infty)$

④ 被近似的函数的定义域: $E \subseteq R^d$

⑤ mesh points (插值点): $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_K \in E$

⑥ 被近似的函数: $f: E \to R$, 且满足

$|f(x) - f(y)| \le L \cdot \|x - y\|_1$, $\forall x, y \in E$ (先前只要求 $y \in$ mesh point set)

⑦ 插值函数值: $\mathfrak{y} \in R^K$, 且满足:

$\mathfrak{y} = (f(\mathfrak{x}_1), f(\mathfrak{x}_2), \cdots, f(\mathfrak{x}_K))$

⑧ Maxima convolution 的 ANN representation: $\Phi \in N$, 其满足

$$\Phi = M_K \bullet \underbrace{A_{-LI_K, \mathfrak{y}}}_{\text{构造}(y_i - L\|x - x_i\|_1)_{i \in \{1,\cdots,K\}}} \bullet \underbrace{P_K(L_d \bullet A_{I_d, -\mathfrak{x}_1}, L_d \bullet A_{I_d, -\mathfrak{x}_2}, \cdots, L_d \bullet A_{I_d, -\mathfrak{x}_K})}_{\text{构造}(\|x - x_1\|_1, \|x - x_2\|_1, \cdots, \|x - x_K\|_1)} \bullet \underbrace{T_{d,K}}_{\text{将 } x \in R^d \text{ 复制K份}}$$

则 ①* $\sup_{x \in E} |(R_r^N(\Phi))(x) - f(x)| \le 2L[\sup_{x \in E}(\min_{k \in \{1,2,\cdots,K\}} \|x - \mathfrak{x}_k\|_1)]$

注: $2\sup_{x \in E}(\min_{k \in \{1,2,\cdots,K\}} \|x - \mathfrak{x}_k\|_1) = \sup_{i,j \in \{1,2,\cdots,K\}}(\|\mathfrak{x}_i - \mathfrak{x}_j\|_1)$ 插值点的最大间距

证明:

Proof of Proposition 4.3.1. Throughout this proof, let $F\colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that

$$F(x) = \max_{k\in\{1,2,\ldots,K\}}(f(\mathfrak{x}_k) - L\|x - \mathfrak{x}_k\|_1). \tag{4.77}$$

Observe that Corollary 4.1.4, (4.77), and the assumption that for all $x, y \in E$ it holds that $|f(x) - f(y)| \le L\|x - y\|_1$ establish that

$$\sup_{x\in E}|F(x) - f(x)| \le 2L\big[\sup_{x\in E}\big(\min_{k\in\{1,2,\ldots,K\}}\|x - \mathfrak{x}_k\|_1\big)\big]. \tag{4.78}$$

Moreover, note that Lemma 4.2.9 ensures that for all $x \in E$ it holds that $F(x) = (\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\Phi))(x)$. Combining this and (4.78) establishes (4.76). The proof of Proposition 4.3.1 is thus complete. □

2. **Corollary:** 给定error tolerances时, ANN的近似结果(implicit)和asymptotic parameter bounds (explicit) (4.3.11)

令 ① 被近似的函数维度: $d \in \mathbb{N}$

② 被近似的函数各维度定义域下、上限: $a \in \mathbb{R}$, $b \in [a, \infty)$

③ 被近似的函数的 Lipschitz constant: $L \in [0, \infty)$

被近似的函数: $f: [a,b]^d \to \mathbb{R}$, 且满足:

$$|f(x) - f(y)| \le L\|x - y\|_1, \quad \forall x, y \in [a,b]^d$$

则存在 $C \in \mathbb{R}$, s.t. 对 $\forall \varepsilon \in (0,1]$, 存在 $F \in \mathbf{N}$, 其满足:

注意不是∞

① $\mathcal{R}_r^N(F) \in C(\mathbb{R}^d, \mathbb{R})$

② $\sup_{x\in[a,b]^d} |(\mathcal{R}_r^N(F))(x) - f(x)| \le \varepsilon$

③ $\mathcal{P}(F) \le C \cdot \varepsilon^{-2d}$

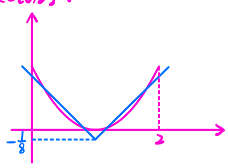
例1: *Exercise* 4.3.1. Prove or disprove the following statement: There exists $\Phi \in \mathbf{N}$ such that $\mathcal{I}(\Phi) = 2$, $\mathcal{O}(\Phi) = 1$, $\mathcal{P}(\Phi) < 20$, and

$$\sup_{v=(x,y)\in[0,2]^2} \left|x^2 + y^2 - 2x - 2y + 2 - (\mathcal{R}_{\mathfrak{r}}^{\mathbf{N}}(\Phi))(v)\right| \le \tfrac{3}{8}. \tag{4.79}$$

Intuition: 注意到 $f(x,y) = x^2 + y^2 - 2x - 2y + 2 = (x-1)^2 + (y-1)^2$, 因此我们可以在两个维度上分别用 $G(x)$ 与 $G(y)$ 近似, 并确保:

$$\begin{aligned}\sup_{(x,y)\in[0,2]^2} |f(x,y) - G(x) - G(y)| &\le \sup_{(x,y)\in[0,2]^2} |(x-1)^2 - G(x)| + |(y-1)^2 - G(y)| \\ &= \sup_{x\in[0,2]} 2\cdot|(x-1)^2 - G(x)| \\ &\le \frac{3}{8}\end{aligned}$$

$$\Rightarrow \sup_{x\in[0,2]} |(x-1)^2 - G(x)| \le \frac{3}{16}$$



令 $\Phi \in \mathbf{N}$ with $\mathcal{I}(\Phi) = 2$, $\mathcal{O}(\Phi) = 1$, $\mathcal{P}(\Phi) = 17 < 20$, s.t.

$$\Phi = \left(\left(\begin{bmatrix}1 & 0\\ -1 & 0\\ 0 & 1\\ 0 & -1\end{bmatrix}, \begin{bmatrix}-1\\ 1\\ -1\\ 1\end{bmatrix}\right), \left(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}, \begin{bmatrix}-\frac{1}{4}\end{bmatrix}\right)\right)$$

then it holds that for $\forall x, y \in [0,2]$,

$$(\mathcal{R}_r^N(\Phi))(x,y) = \max\{x-1, 0\} + \max\{-(x-1), 0\} + \max\{y-1, 0\} + \max\{-(y-1), 0\} - \frac{1}{4}$$

$$= |x-1| - \frac{1}{8} + |y-1| - \frac{1}{8}$$

$$\Rightarrow \sup_{(x,y)\in[0,2]^2} \left| (x-1)^2 + (y-1)^2 - (R_r^N(\Phi))(x,y) \right| = \sup_{(x,y)\in[0,2]^2} \left| (x-1)^2 + (y-1)^2 - |x-1| + \frac{1}{8} - |y-1| + \frac{1}{8} \right|$$

$$\le 2 \cdot \sup_{x\in[0,2]} \Big| \underbrace{(x-1)^2 - |x-1| + \frac{1}{8}}_{\text{关于 1 symmetric}} \Big|$$

$$= 2 \cdot \sup_{x\in[0,1]} \left| x^2 - x + \frac{1}{8} \right|$$

$$= 2 \cdot \max\left\{ \left|0^2 - 0 + \frac{1}{8}\right|, \left|1^2 - 1 + \frac{1}{8}\right|, \left|\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{1}{8}\right| \right\}$$

$$= \frac{1}{4}$$

$$\le \frac{3}{8}$$