# Lecture 5 (2021.9.23)

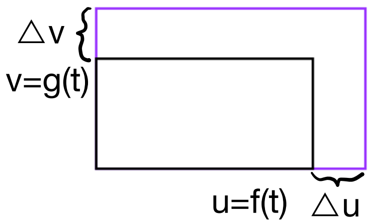
## §1 Product and quotient rule

### 1. Product rule

If $f$ and $g$ are differentiable at $x$, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

### 2. Intuitive proof of product rule




Q: How fast is the area changing at time $t$?

i.e

$$(uv)' = \frac{d}{dt}(uv) = ?$$

$$\Delta(uv) = (f\cdot g)(t+\Delta t) - (f\cdot g)(t)$$

$$\Delta(uv) = f(t+\Delta t)\cdot \Delta v + g(t)\,\Delta u$$

$$\frac{\Delta(uv)}{\Delta t} = f(t+\Delta t)\cdot\frac{\Delta v}{\Delta t} + g(t)\frac{\Delta u}{\Delta t}$$

$$\frac{d}{dt}(uv) = \lim_{\Delta t\to 0}\left[f(t+\Delta t)\cdot\frac{\Delta v}{\Delta t} + g(t)\frac{\Delta u}{\Delta t}\right]$$

$$= \left[\lim_{\Delta t\to 0} f(t+\Delta t)\right]\cdot\left[\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}\right] + g(t)\left[\lim_{\Delta t\to 0}\frac{\Delta u}{\Delta t}\right]$$

$$= f(t)g'(t) + g(t)f'(t)$$

### 3. Quotient rule

If $f$ and $g$ are differentiable at $x$, and $g(x)\neq 0$, then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

## §2 $n^{th}$ derivatives

### 1. Definition

Let $f: D\to R$ be a function

1° If $f'$ is differentiable at $c$, then we call $(f')'(c)$ the second derivative of $f$ at $c$, which we denote by $f''(c)$ or $f^{(2)}c$

2° If $f''$ is differentiable at $c$, then we call $(f'')'(c)$ the third derivative of $f$ at $c$, which we denote by $f'''(c)$ or $f^{(3)}c$

3° More generally, we can define the $n^{th}$ derivative of $f$ at $c$ to be $(f^{(n-1)})'(c)$, which we denote by $f^{(n)}(c)$.

2. Notation:

if $y=f(x)$, then

1° Second derivatives:

$y''$ $\quad f''(x)$ $\quad \frac{d}{dx}\left(\frac{dy}{dx}\right)$ $\quad \frac{d^2}{dx^2}y$ $\quad \frac{d^2y}{dx^2}$

2° $n^{th}$ derivatives:

$y^{(n)}$ $\quad f^{(n)}(x)$ $\quad \frac{d^n}{dx^n}y$ $\quad \frac{d^ny}{dx^n}$

## §3 Trigonometric functions

1. Proof of $\sin' x = \cos x$, $\cos' x = -\sin x$

$$\sin' x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cosh+\cos x\sinh-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}$$

$$= \sin x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)+\cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$

$$= \sin x\left(\lim_{h\to 0}\frac{-2\sin^2(\frac{h}{2})}{h}\right)+\cos x$$

$$= \sin x\cdot\left(\lim_{h\to 0}\frac{\sin(\frac{h}{2})}{\frac{h}{2}}\cdot\lim_{h\to 0}(-\sin\tfrac{h}{2})\right)+\cos x$$

$$= \cos x$$

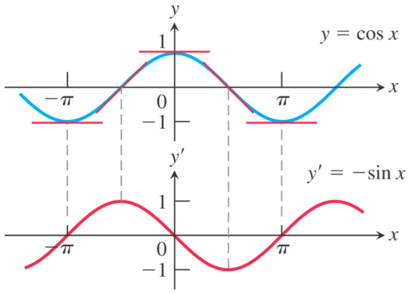



**FIGURE 3.20** The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

## 2. The derivatives of the other trigonometric functions:

**The derivatives of the other trigonometric functions:**

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

# §4 Derivative in the "real world"

## 1. Motion along a line

Suppose an object is moving along a line, whose position is $s = f(t)$ at time $t$.

Note that $s$ can be negative.

1° Displacement: The displacement of the object over a time interval $[a,b]$ is $f(a)-f(b)$

2° Velocity

**DEFINITION** **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time $t$ is $s = f(t)$, then the body's velocity at time $t$ is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

① Velocity can be negative

② velocity $>0$, moving forwards ($s\uparrow$ as $t\uparrow$)

③ velocity $<0$, moving backwards ($s\downarrow$ as $t\uparrow$)

3° Speed

① Speed is always non-negative

**DEFINITION** **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

4° Acceleration and Jerk

**DEFINITIONS** **Acceleration** is the derivative of velocity with respect to time. If a body's position at time $t$ is $s = f(t)$, then the body's acceleration at time $t$ is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

**Jerk** is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

2. Costs and production

Suppose the cost of producing $x$ units of good is $c(x)$

Then the marginal cost of production is the derivative of cost w.r.t production

$$\text{marginal cost at } x_0 \text{ units of production} = c'(x_0) = \lim_{h\to 0}\frac{c(x_0+h)-c(x_0)}{h}$$

§5 Euler constant e

$z = 1-\frac{x+1}{x}$ $\quad z = -\frac{1}{x}$

1. Definition

The euler constant e is defined by $e \overset{\text{def}}{=} \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 2.71828\ldots$

2. ☆ $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$

Proof: Note that $e = \lim_{y\to 0^+}(1+y)^{\frac{1}{y}}$ by setting $y=\frac{1}{x}$

It can be shown that $\lim_{y\to 0^-}(1+y)^{\frac{1}{y}} = e$

Let $z$ satisfy $1+y = \frac{1}{1+z}$ (say $-1<y<0$)

Then $y = \frac{1}{1+z} - 1 = -\frac{z}{1+z}$, $\quad \frac{1}{y} = -\frac{1+z}{z}$

$$(1+y)^{\frac{1}{y}} = \left(\frac{1}{1+z}\right)^{-\frac{1+z}{z}} = (1+z)^{\frac{1+z}{z}}$$

$$= (1+z)^{\frac{1}{z}(1+z)}$$

As $y\to 0^-$, $z = -\frac{y}{1+y} \to 0^+$

So $\lim_{y\to 0^-}(1+y)^{\frac{1}{y}} = \lim_{z\to 0^+}(1+z)^{\frac{1}{z}(1+z)} = \lim_{z\to 0^+}(1+z)^{\frac{1}{z}} = e$

What we know so far:

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad (\text{by definition})$$

$$e=\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} \quad (\text{by setting } y=\tfrac{1}{x})$$

$$e=\lim_{x\to 0^-}(1+x)^{\frac{1}{x}} \quad (\text{proven above})$$

e.g. Suppose $P_0$ dollars is deposited into an account with annual interest rate $r$, compounded $n$ times a year, where $n$ is very big.

What would the balance $P$ be after $t$ years (approximately)?

$$\begin{aligned} P &\approx \lim_{n\to\infty} P_0\left(1+r\cdot\frac{1}{n}\right)^{nt} \\ &= \lim_{n\to\infty} P_0\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}\cdot rt} \\ &= P_0\left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt} \\ &= P_0 e^{rt} \end{aligned}$$

## §6 Natural exponential function $e^x$

### 1. The derivative of $e^x$

Let $f(x)=e^x$, $f'(x)=e^x$

### 2. Proof

Two special limits: ① $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$ ② $\lim_{x\to 0}\frac{e^x-1}{x}=1$

① 
$$\begin{aligned} \lim_{x\to 0}\frac{\ln(1+x)}{x} &= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} \\ &= \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right) \\ &= \ln e = 1 \end{aligned}$$

② $\lim_{x\to 0}\frac{e^x-1}{x}$

Let $y=e^x-1 \Rightarrow x=\ln(1+y)$ and $y\to 0$ as $x\to 0$

$$\begin{aligned} \lim_{x\to 0}\frac{e^x-1}{x} &= \lim_{y\to 0}\frac{y}{\ln(1+y)} \\ &= \frac{1}{\lim_{y\to 0}\frac{\ln(1+y)}{y}} \end{aligned}$$

$$= 1$$

Now, for $f(x) = e^x$,

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x$$

Therefore: $(e^x)' = e^x$