Lecture 24

§1 Mode of convergence

令 $X_1, X_2, \cdots, X_n$ 为一系列随机变量(不一定独立)，X 为另一个随机变量.

令 $F_{X_n}(x)$ 为 X_n 的分布函数，$F_X(x)$ 为 X 的分布函数.

1. Definition: limiting distribution (极限分布)

当样本容量 n 很大(趋向无穷)时，sample mean 的 probabilistic behavior 被称为 limiting distribution of the sample mean.

2. Definition: converges in distribution / converges in law / weak convergence (依分布收敛)

X_n 被称为 converges in distribution to X 若

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for all points at which $F_X(x)$ is continuous.

表示为 $X_n \xrightarrow{d} X$ 或 $X_n \xrightarrow{L} X$

例：令 $U_1, U_2, \cdots \sim U(0,1)$,

(1) 定义 X_n 为 $U_1, U_2, \cdots, U_n$ 中的最大值. 求 $\lim_{n\to\infty} F_{X_n}(x)$

(Step 1: 先求 X_n 的分布函数)

$F_{X_n}(x) = 0$, for $x \le 0$

$F_{X_n}(x) = 1$, for $x \ge 1$

$$\begin{aligned} F_{X_n}(x) &= Pr(X \le x) \\ &= Pr(U_1 \le x, U_2 \le x, \cdots, U_n \le x) \\ &= Pr(U_1 \le x)\, Pr(U_2 \le x) \cdots Pr(U_n \le x) \\ &= x^n , \quad \text{for } 0 < x < 1 \end{aligned}$$

(Step 2: 求 $\lim_{n\to\infty} F_{X_n}(x)$)

Therefore,

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 , & \text{if } x < 1 \\ 1 , & \text{if } x \ge 1 \end{cases}$$

(*Step 3: 求出与 $\lim_{n\to\infty} X_n$ 匹配的概率分布)

On the other hand, consider a random variable which is degenerated at 1, i.e. $Pr(X=1)=1$.

The distribution of X is

$$F_X(x) = Pr(X \le x) = \begin{cases} 0 , & \text{if } x < 1 \\ 1 , & \text{if } x \ge 1 \end{cases}$$

Hence, $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ and thereby $X_n \xrightarrow{d} X$. We may also write

$X_n \xrightarrow{d} 1$ $\quad (Pr(X=1)=1)$

as X is degenerated at 1

(2) 定义 $Y_n = n(1-X_n)$，求 $\lim_{n\to\infty} F_{Y_n}(y)$

$$F_{Y_n}(y) = Pr(Y_n \le y) = Pr(n(1-X_n)\le y) = Pr(X_n \ge 1-\frac{y}{n})$$
$$= 1 - F_{X_n}(1-\frac{y}{n})$$
$$= \begin{cases} 0 & , \text{ if } y\le 0 \\ 1-(1-\frac{y}{n})^n & , \text{ if } 0<y<n \\ 1 & , \text{ if } y\ge n \end{cases}$$

Therefore,

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & , \text{ if } y\le 0 \\ 1-e^{-y} & , \text{ if } 0<y<\infty \end{cases}$$

which is the distribution function of $Exp(1)$. Hence $Y_n = n(1-X_n)$ converges in distribution to an exponential random variable with parameter $\lambda = 1$, i.e.

$$Y_n = n[1-\max(U_1, U_2, \cdots, U_n)] \xrightarrow{d} Exp(1)$$

3. **Definition:** converges in probability（依概率收敛）

X 被称为 converge in probability to X 若

对于 $\forall \varepsilon > 0$，$\lim_{n\to\infty} Pr(|X_n - X| \ge \varepsilon) = 0$

表示为 $X_n \xrightarrow{P} X$

例：令 $U_1, U_2, \cdots \sim U(0,1)$，定义 X_n 为 $U_1, U_2, \cdots, U_n$ 中的最大值。证明 $X_n \xrightarrow{P} 1$

Obviously, if $\varepsilon > 1$, $Pr(|X_n - 1| \ge \varepsilon) = 0$

For any $0<\varepsilon\le 1$,

$$Pr(|X_n-1|\ge\varepsilon) = Pr(1-X_n\ge\varepsilon) = Pr(X_n\le 1-\varepsilon) = F_{X_n}(1-\varepsilon) = (1-\varepsilon)^n$$

Therefore, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} Pr(|X_n-1|\ge\varepsilon) = 0$$

and hence

$$X_n \xrightarrow{P} 1$$

4. **Definition:** Converges almost surely / strong convergence（几乎必然收敛）

X_n 被称为 converge almost surely to X 若

$$Pr(\lim_{n\to\infty} X_n = X) = 1$$

表示为 $X_n \xrightarrow{a.s.} X$

例：令 $\Omega = [0,1]$ 为样本空间，w 为从 Ω 中 uniformly drawn 的一点。定义 $X_n(w) = w + w^n$，$X(w) = w$。证明 $X_n \xrightarrow{a.s.} X$

$$\lim_{n\to\infty} X_n(w) = \begin{cases} X(w) & , \text{ if } 0\le w<1 \\ X(w)+1 & , \text{ if } w=1 \end{cases}$$

Since the convergence occurs on the set $[0,1)$ and $Pr(w\in[0,1)) = 1$, in other words, $Pr(w=1) = 0$, then

$$Pr(\lim_{n\to\infty} X_n = X) = 1$$

that is

$$X_n \xrightarrow{a.s.} X$$

注：① 上述三种 convergence mode 的关系为

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$$

② 对几乎必然收敛的理解：

$\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $x \in \Omega$ except those $\omega \in S$ where $S \subset \Omega$ and $\Pr(S) = 0$

e.g. **Example 11.4.**

Let $\Omega = [0,1]$ be the sample space and ω be a point uniformly drawn from Ω. Define

$$\begin{array}{rclrclrcl}
X(\omega) & = & \omega, & & & & & & \\
X_1(\omega) & = & \omega + \mathbf{1}_{[0,1]}(\omega), & & & & & & \\
X_2(\omega) & = & \omega + \mathbf{1}_{[0,1/2]}(\omega), & X_3(\omega) & = & \omega + \mathbf{1}_{[1/2,1]}(\omega), & & & \\
X_4(\omega) & = & \omega + \mathbf{1}_{[0,1/3]}(\omega), & X_5(\omega) & = & \omega + \mathbf{1}_{[1/3,2/3]}(\omega), & X_6(\omega) & = & \omega + \mathbf{1}_{[2/3,1]}(\omega), \\
& \vdots & & & \vdots & & & \vdots &
\end{array}$$

Obviously, as $n \to \infty$, $\Pr(|X_n(\omega) - X(\omega)| \geq \epsilon)$ is equal to the probability of an interval of ω values whose length tends to 0.

Hence,

$$X_n \xrightarrow{p} X.$$

However, for every ω, the value $X_n(\omega)$ alternates between the values ω and $\omega + 1$ infinitely often.

Thus there is no value of $\omega \in \Omega$ for which $X_n(\omega)$ converges to $X(\omega)$, i.e., X_n does not converge to X almost surely.

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e.g. **Example 11.5.**

To check convergence in distribution, nothing needs to be known about the joint distribution of X_n and X, whereas this distribution must be defined to check convergence in probability.

For example, if $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$, then

$$X_n \xrightarrow{d} X_1,$$

but X_n does not converge in probability to X_1.

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补：一些性质：

① 在离散概率空间中，依概率收敛 $\Longleftrightarrow$ 几乎必然收敛

② 依分布收敛蕴含依概率收敛当且仅当依分布收敛的极限为常数

③ 连续映射定理：

若 $X_n \xrightarrow{P} X$，f 为连续函数，则 $f(X_n) \xrightarrow{P} f(X)$

*注：对收敛模式的理解：

① 依概率收敛：

$\forall \varepsilon$，$\exists N$，s.t. 当 $n \geq N$ 时，$\Pr(|X_n - X| < \varepsilon) = 1$

随着 n 增大，随机变量 X_n 落在 $(X-\varepsilon, X+\varepsilon)$ 外的概率趋近于0（还是可能落在外面的）

② 几乎必然收敛：

X_n 可能还由另一变量 ω 决定，随着 n 增大，随机变量 X_n 不会落在 $(X-\varepsilon, X+\varepsilon)$ 外

(除去某些 ω_0，但这些 ω_0 构成的集合的测度为0)