Lecture 24

§1 Riemann integrable on a set

1. Definition: Riemann integrable on a set

S = subset of $\mathbb{R}^n$, bdd.

Let $f(x)$ be defined on S and bdd on S. Take a large closed rectangle $Q \supset S$.

Let $f_S(x) = \begin{cases} f(x) & , x \in S \\ 0 & , x \notin S \end{cases}$

若 $f_S(x)$ is Riemann integrable on Q, 则 $f(x)$ is Riemann integrable on S.

令 $\int_S f(x)\,dx = \int_Q f_S(x)\,dx$

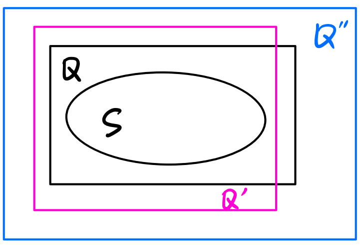
2. Question 1

Issue: 令 Q' 为 $\mathbb{R}^n$ 中的另一个 closed rectangle, $Q' \supset S$

Q: 若 $\int_Q f_S(x)dx$ 存在, 则 $\int_{Q'} f_S(x)\,dx$ 是否存在?

$\int_Q f_S(x)dx \overset{?}{=} \int_{Q'} f_S(x)\,dx$

A: Yes; Yes.



证明:

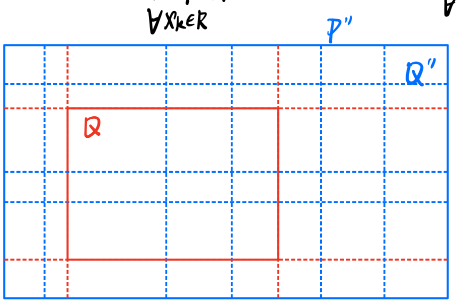
Take large closed rectangle $Q'' \supset Q', Q$

W.T.S. $\int_Q f_S(x)dx$ exists, then $\int_{Q''} f_S(x)dx$ exists

and $\int_Q f_S(x)dx = \int_{Q''} f_S(x)dx$

Observe: $\forall$ partition P'' of Q'', it induces naturally a partition P of Q;

Moreover, $\sum_{\substack{R \in P'' \cup P \\ \forall x_k \in R}} f_S(x_k)|R| = \sum_{\substack{R \in P \\ \forall x_k \in R}} f_S(x_k)|R|$ (#)



Observe: as $\|P''\| \to 0$, we have $\|P\| \to 0$ & $\|P'' \cup P\| \to 0$

Then RHS of (#) $\to \int_Q f_S(x)dx$ as $\|P''\| \to 0$

LHS of (#) $\to \int_{Q''} f_S(x)dx$

$\Rightarrow \int_{Q''} f_S(x)dx$ exists & $= \int_Q f_S(x)dx$

3. Question 2

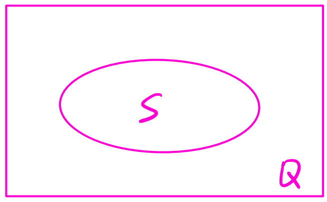
Q: If $f(x)$ continuous on S, is it R-integrable of S? (It depends)

* Special Q: Is constant function $\mathbb{1}$ always integrable on S?

A: define

$$\mathbb{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

Take $Q > S$, $\int_Q \mathbb{1}_S(x)\,dx$ exists?



Let D = set of discontinuous point of $\mathbb{1}_S(x)$

- $\mathbb{1}_S(x)$ continuous in $\mathring{S}$
- $\mathbb{1}_S(x)$ continuous in $Q \setminus S$
- $\mathbb{1}_S(x)$ discontinuous at every point $\in \partial S$

$\Rightarrow D = \partial S$

$\Rightarrow \int_Q \mathbb{1}_S(x)\,dx$ exists $\Longleftrightarrow |\partial S| = 0$

Definition:

若 $|\partial S| = 0$，则称 S 为 rectifiable

Definition:

若 S 为 bdd & rectifiable，则 $\int_S \mathbb{1}\,dx$ exists 且被称为 the volume / measure of S

To answer Q,

Theorem:

If S is bdd & rectifiable & f is continuous on S, then $f(x)$ is R-integrable in S

证明:

令 $f_S(x) = \begin{cases} f(x), & x \in S \\ 0, & x \notin S \end{cases}$, $Q > S$

Observe: ① $f_S(x) = f(x)$ in $\mathring{S}$ & continuous at every point $\in \mathring{S}$

② $f_S(x) \equiv 0$ in $D \setminus \bar{S}$ & continuous at every point $\in D \setminus \bar{S}$

③ set D of discontinuous points $\subset \partial S$

$\Rightarrow |D| = 0$

$\Rightarrow \int_Q f_S(x)\,dx$ exists

$\Rightarrow f(x)$ is R-integrable in S

4. Question 3

Q: User-friendly criterion for S to be rectifiable?

Definition:

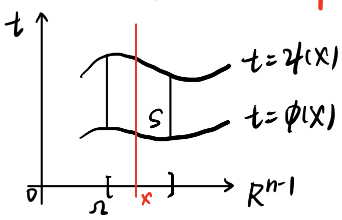
令 Ω 为 compact, rectifiable subset in R^{n-1} $(n \geq 1)$

- ψ & ϕ are continuous on Ω

· $\psi \geq \phi$ on Ω

Let $S = \{(x,t) \in R^{n-1} \times R \mid \phi(x) \leq t \leq \psi(x),\ x \in \Omega\}$

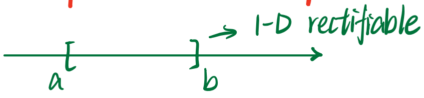
则 S 被称为 a simple region in R^n

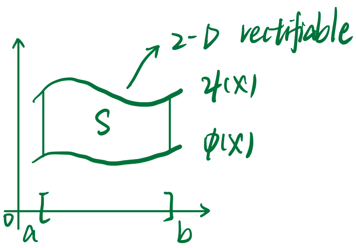


Theorem:

S is simple $\Rightarrow$ S compact & rectifiable

e.g.





证明:

W.T.S. $|\partial S| = 0$

∂S = graph of $\phi(x)$ & $\psi(x)$ and $L = \{(x,t) \in R^{n-1} \times R \mid x \in \partial\Omega,\ \phi(x) \leq t \leq \psi(x)\}$

(想利用 $|E_i| = 0,\ \forall i \geq 1 \Rightarrow |\bigcup_{i=1}^{\infty} E_i| = 0$)

Claim 1: $|L| = 0$

since Ω is rectifiable, $|\partial\Omega| = 0$ ($\partial\Omega$ as a subset of R^{n-1})

$\xRightarrow{\text{by def}}$ $\exists$ closed rectangles $\{Q_i\}_{i=1}^{\infty}$ in R^{n-1} s.t.

- $\bigcup_{i=1}^{\infty} \mathring{Q}_i \supset \partial\Omega$
- $\sum_{i=1}^{\infty} |Q_i| < \varepsilon$

$\because$ ϕ & ψ continuous on Ω, which is compact.

$\therefore$ $\exists$ big $M > 0$ s.t. $-M \leq \phi(x) \leq \psi(x) \leq M,\ \forall x \in \Omega$

Observe

- $\bigcup_{i=1}^{\infty}$ interior of $Q_i \times [-M-1, M+1] \supset L$
- $\sum_{i=1}^{\infty} |Q_i \times [-M-1, M+1]| = \sum_{i=1}^{\infty} |Q_i||2M+2| < (2M+2)\varepsilon \Rightarrow |L| = 0$

Claim 2: $|$Graph of $t = \phi(x)$ over $\Omega| = 0$

$\because$ ϕ is continuous on Ω & Ω is compact

$\therefore$ ϕ is uniformly continuous on Ω, i.e.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x, y \in \Omega$, $|x-y| < \delta$, we have $|\phi(x) - \phi(y)| < \varepsilon$ (#)

Take large closed rectangle Q in R^{n-1} s.t. $Q \supset \Omega$

Take partition P of Q with $\|P\|$ so small that diameters of subrectangles determined by $P < \delta$.

$\forall R \in P$ s.t. $R \cap \Omega \neq \phi \Rightarrow$ can take $x_R \in R \cap \Omega$

Then in (#1), take $x = x_R$, y arbitrary point in $R \cap \Omega$

$\Rightarrow |\phi(x) - \phi(y)| < \varepsilon$, $\forall y \in R \cap \Omega$

$\phi(x) - \varepsilon < \phi(y) < \phi(x) + \varepsilon$

$\Rightarrow$ graph of ϕ over $\Omega \subset \bigcup_{\substack{R \in P \\ R \cap \Omega \neq \emptyset}} R \times (\phi(x_R) - \varepsilon, \phi(x_R) + \varepsilon)$

interior of

$\subset \bigcup_{\substack{R_{fat} \in P \\ R \cap \Omega \neq \emptyset}} R_{fat} \times (\phi(x_R) - \varepsilon, \phi(x_R) + \varepsilon)$ $(|R_{fat}| = 2|R|)$

$\sum |R_{fat} \times [\phi(x_R) - \varepsilon, \phi(x_R) + \varepsilon]| = \sum 2^n |R| \cdot 2\varepsilon$

$\leq 2^n |Q| \cdot 2\varepsilon$

$\Rightarrow$ Claim 2

Q.E.D.

§2 Fubini's theorem on simple region

1. **Theorem:** Fubini's theorem on simple region

Suppose S is as given before (hence rectifiable in $\mathbb{R}^n$) & $f: S \to \mathbb{R}$ continuous, then

$\int_S f(x,t)\,dx\,dt = \int_\Omega \left(\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right) dx$

证明:

Take a large closed rectangle $Q \subset \mathbb{R}^n$ s.t. $Q \supset S$, $Q = Q_1 \times [a,b]$, $Q_1 \subset \mathbb{R}^{n-1}$, $\Omega \subset Q_1$

$\because$ S is rectifiable

$\therefore$ $\int_S f(x,t)\,dx\,dt$ exists

i.e. $\int_Q f_S(x,t)\,dx\,dt$ exists

Fix $x_0 \in Q_1$.

If $x_0 \notin \Omega$, then $f_S(x_0,t) \equiv 0$, $\forall t \in [a,b] \Rightarrow \int_{[a,b]} f_S(x_0,t)\,dt = 0$

If $x_0 \in \Omega$, then $f_S(x_0,t)$ is continuous on $[a,b]$, possibly except at $t = \phi(x_0)$, $t = \psi(x_0)$

$\Rightarrow \int_{[a,b]} f_S(x_0,t)\,dt$ exists $= \int_{\phi(x_0)}^{\psi(x_0)} f(x_0,t)\,dt$

By ole Fubini,

$\int_{Q_1} \left(\int_{[a,b]} f_S(x,t)\,dt\right) dx$ exists & $= \int_Q f_S(x,t)\,dx\,dt = \int_S f(x,t)\,dx\,dt$

$\|$

$\int_\Omega \left(\int_{\phi(x_0)}^{\psi(x_0)} f(x,t)\,dt\right) dx$