Lecture 22

§1 Variations on Brownian motion

1. **Definition:** Brownian motion with drift （带漂移的布朗运动）

一个 Brownian motion process $\{X_t\}_{t\geq 0}$ with drift coefficient $m \in R$ 与 variance parameter $\sigma^2 > 0$ 被定义为：

$$X_t = \sigma B_t + mt \quad , \quad t \geq 0$$

其中 $\{B_t\}_{t\geq 0}$ 为一个 standard Brownian motion.

注：① mt 为 deterministic part，σB_t 为 random part，σ 被称为 volatility.

2. **Proposition:** Brownian motion with drift 的性质

一个 Brownian motion process $\{X_t\}_{t\geq 0}$ with drift coefficient 与 variance parameter 满足：

① $X_0 = 0$

② Independent increment

③ Stationary Gaussian increment：对 $\forall\, t > s$，$X_t - X_s$ 服从正态分布：

$$X_t - X_s \sim N[m(t-s),\ \sigma^2(t-s)]$$

特别的，有

$$X_t \sim N(mt,\ \sigma^2 t)$$

证明：③

$$X_t - X_s = \sigma(B_t - B_s) + m(t-s)$$

由于 $B_t - B_s \sim N(0, t-s)$，有

$$X_t - X_s \sim N[m(t-s),\ \sigma^2(t-s)]$$

3. **Definition:** Geometric Brownian motion （几何布朗运动）

一个 geometric Brownian motion $\{Y_t\}_{t\geq 0}$ 被定义为

$$Y_t = Y_0\, e^{X_t} = Y_0\, e^{mt + \sigma B_t} \quad , \quad t \geq 0$$

其中 $\{X_t\}_{t\geq 0}$ 为一个 Brownian motion with drift m and variance σ^2，
且独立于 initial value Y_0

4. Geometric Brownian motion 的 conditional mean

给定 past history till time $s < t$，计算 t 时刻的 conditional mean：

- $$\begin{aligned} E[Y_t \mid Y_u, 0 \leq u \leq s] &= E[Y_0 e^{X_t} \mid Y_0, \{X_u, 0 \leq u \leq s\}] \\ &= E[Y_0 e^{X_t - X_s + X_s} \mid Y_0, \{X_u, 0 \leq u \leq s\}] \\ &= Y_0 e^{X_s} E[e^{X_t - X_s} \mid Y_0, \{X_u, 0 \leq u \leq s\}] \quad (Y_0, X_s \text{ 给定}) \\ &= Y_s E[e^{X_t - X_s}] \quad (\{X_t\}_{t\geq 0} \text{ 满足 independent increment}) \end{aligned}$$

- 注意到若 $X \sim N(\mu, \sigma^2)$，则其 m.g.f. 为

$$E[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

由于 $X_t - X_s \sim N(m(t-s),\ \sigma^2(t-s))$，

令 $\mu = m(t-s)$, $\sigma^2 = \sigma^2(t-s)$, $t=1$, 有

$$E[e^{X_t - X_s}] = e^{m(t-s)} e^{\frac{1}{2}\sigma^2(t-s)}$$
$$= e^{(m+\frac{1}{2}\sigma^2)(t-s)}$$

· 综上,

$$E[Y_t \mid Y_u, 0 \le u \le s] = Y_s \cdot e^{(m+\frac{1}{2}\sigma^2)(t-s)}$$

5. Geometric Brownian motion 的应用: 模拟股票

Geometric Brownian motion 是对股票的 simplest model, 被用于 Black-Shole formula 中:

$$Y_t = Y_0 e^{X_t} = Y_0 e^{mt+\sigma B_t}, \quad t \ge 0$$

$$\iff \log Y_t = \log Y_0 + mt + \sigma B_t$$

即表示 log-price process 服从 Brownian motion

注: 此处 m 与 σ 均为定值, 但在现实中, (m, σ) 会变化且满足一些条件 (smile of volatisity)

6. Geometric Brownian motion 的应用: 确定 risk-neutual market 的 interest rate

- Let $r > 0$ be the fixed interest rate; (time t 时的 \$1 $\iff$ time 0 时的 \$$\underbrace{e^{-rt}}_{<1}$)
- Consider the wager of observing the stock for a time s and then purchasing (or selling) one share with the intention of selling (or purchasing) it at a later time t $(s < t)$.

  买进时的价值 = 买进时的货币价值 × 股价
- The present value of the amount paid for the stock is $e^{-rs}Y_s$, whereas the present value of the amount received is $e^{-rt}Y_t$.

  卖出时的价值 = 卖出时的货币价值 × 股价
- In a risk neutral market, no such arbitrage is possible. So the expected return of this wager must be zero, i.e., 无法套利!

$$E(e^{-rt}Y_t \mid Y_u, 0 \le u \le s) = e^{-rs}Y_s.$$

卖出时的期望价值 = 买进时的价值

Therefore,

$$Y_s e^{\left(m+\frac{1}{2}\sigma^2\right)(t-s)} \cdot e^{-rt} = e^{-rs}Y_s, \quad 0 < s < t.$$

Thus we must have

$$\boxed{m + \frac{1}{2}\sigma^2 = r}.$$ risk-neutual market 的 interest rate

7. Definition: Planar Brownian motion (平面布朗运动)

Definition

A standard planar or 2-d Brownian motion is a two component process $\{(X_t, Y_t)\}_{t\ge0}$ where $\{X_t\}_{t\ge0}$ and $\{Y_t\}_{t\ge0}$ are two independent standard Brownian motions.

It is easily checked that a planar motion starts from the origin $(0,0)$ and has independent, stationary and Gaussian increments.

A simulated planar motion path: $t \in [0,1]$ (left) and $t \in [0,5]$ (right).