Lecture 12

§1 Kinetic energy of rolling

1. Kinetic energy of rolling

1° $K = \frac{1}{2} I_P \omega^2$

$I_P = I_{com} + Mh^2 = I_{com} + MR^2$

$K = \frac{1}{2}(I_{com} + MR^2)\omega^2$

$= \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M v_{com}^2$

2° 理解：

旋转体的动能由两部分组成

① Rotation about the COM

② Translation of the COM

2. 功能关系：施加 force 以创造 translation

· 对于自行车的被动轮：

$$a_{com} = \frac{F_{applied}}{m} \cdot \frac{1}{(1 + \frac{I_{com}}{mR^2})}$$

· 最初自行车停止，动能改变量

$$\Delta K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} m v_{com}^2$$

$$= \frac{1}{2} I_{com}(2\alpha\Delta\theta) + \frac{1}{2} m (2 a_{com}\Delta S)$$

$$= \frac{1}{2} I_{com} \cdot (2 \cdot \frac{a_{com}}{R} \cdot \frac{\Delta S}{R}) + \frac{1}{2} m (2 a_{com} \Delta S)$$

$$= (\frac{I_{com}}{mR^2} + 1) m a_{com} \Delta S$$

$$= F_{applied} \Delta S$$

1° applied force 所做的功，全被转移进纯滚动的动能中

2° 该过程无热量产生

3° 若无摩擦力，轮子将做平动

$$F_{net} = m a_{com}$$
$$F_{applied} - f_s = F_{net} = m a_{com}$$
$$\tau_{net} = f_s R = I\alpha = I\frac{a_{com}}{R}$$
$$a_{com} = \frac{F_{applied}}{m} \frac{1}{(1 + \frac{I}{mR^2})}$$

3、功能关系：施加 torque 以创造 rotation

· 对于自行车的主动轮：

$$a_{com} = \frac{\tau_{applied}}{mR} \frac{1}{(1+\frac{I_{com}}{mR^2})}$$

· 最初自行车停止，动能改变量

$$\Delta K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} m v_{com}^2$$

$$= \frac{1}{2} I_{com}(2\alpha\Delta\theta) + \frac{1}{2} m(2a_{com}\Delta S)$$

$$= \frac{1}{2} I_{com}\cdot(2\cdot\frac{a_{com}}{R}\cdot\frac{\Delta S}{R}) + \frac{1}{2} m(2a_{com}\Delta S)$$

$$= (\frac{I_{com}}{mR^2}+1) m a_{com}\Delta S$$

$$= \frac{\tau_{applied}}{R}\Delta S$$

$$= \tau_{applied}\cdot\Delta\theta$$

1° applied torque 所做的功，全被转移进纯滚动的动能中

2° 若无摩擦力，轮子将做旋转无平动运动.

Example: Frictional Force for Driving Wheel

$$F_{net} = ma_{com}$$
$$f_s = F_{net} = ma_{com}$$
$$\tau_{net} = \tau_{applied} - f_s R = I\alpha = I\frac{a_{com}}{R}$$
$$a_{com} = \frac{\tau_{applied}}{mR}\frac{1}{(1+\frac{I}{mR^2})}$$

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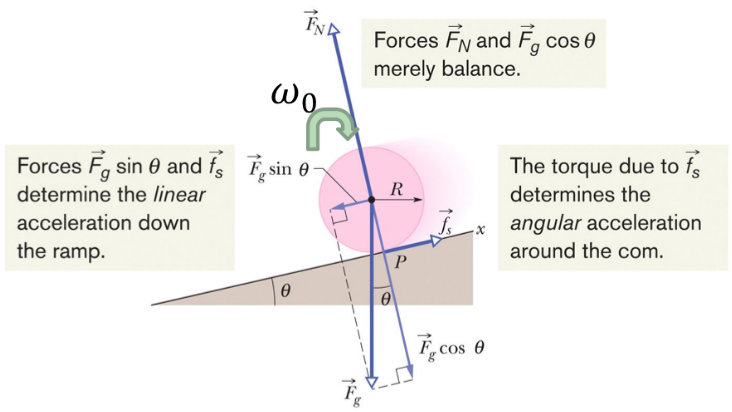
4. 功能关系：沿斜面滚动（向上）



1° f_s 的方向：沿斜面向上

2° 无机械能损耗

$$\Delta U = -\Delta K$$

$$\Delta K = mgh$$

3° 最大高度

$$mgh = \frac{1}{2} I_{com} \cdot \omega_0^2 + \frac{1}{2} m \cdot v_{com}^2$$
$$= \frac{1}{2}(\omega_0 R)^2 \left(\frac{I_{com}}{R^2} + m\right)$$

$$h = \frac{(\omega_0 R)^2}{2g}\left(1 + \frac{I_{com}}{mR^2}\right)$$

4° 若斜面光滑

$$h' = \frac{(\omega_0 R)^2}{2g}$$

$h > h'$，因为斜面光滑时，rotational kinetic energy 无法转化为势能

例：

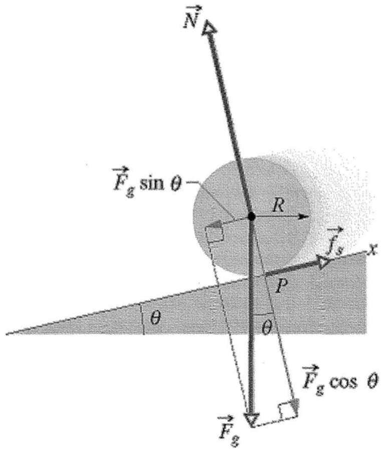
Checkpoint 2

Disks A and B are identical and roll across a floor with equal speeds. Then disk A rolls up an incline, reaching a maximum height h, and disk B moves up an incline that is identical except that it is frictionless. Is the maximum height reached by disk B greater than, less than, or equal to h?

Answer: The maximum height reached by B is less than that reached by A. For A, all the kinetic energy becomes potential energy at h. Since the ramp is frictionless for B, all of the rotational K stays rotational, and only the translational kinetic energy becomes potential energy at its maximum height.

$$\Delta h = h' - h = \frac{(\omega_0 R)^2}{2g} \times \frac{I_{com}}{mR^2} = \frac{I_{com}(\omega_0 R)^2}{2mgR^2}$$

例：沿斜面下滚



A uniform solid ball of mass M = 6.00 kg and radius R, rolls smoothly from rest down a ramp at angle θ= 30.0°

(a) The ball descends a vertical height h =1.20 m to reach the bottom of the ramp. What is its speed at the bottom?

(b) What are the magnitude and direction of the friction force on the ball as it rolls down the ramp?

(a)
$$\begin{cases} mgh = \frac{1}{2} m v_{com}^2 + \frac{1}{2} I \omega^2 \\ v_{com} = \omega \cdot R \\ I = \frac{2}{5} m R^2 \end{cases}$$

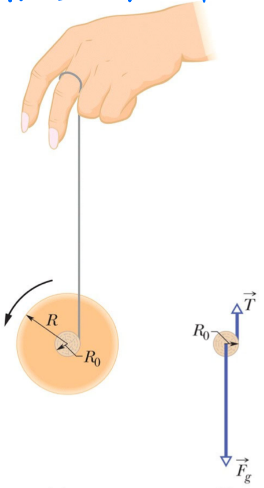
$$v_{com} = \sqrt{\frac{10}{7} \cdot gh} = 4.1\ m/s$$

(b)
$$\begin{cases} ma = mg\sin\theta - f_s \\ I\alpha = f_s \cdot R \\ a = \alpha \cdot R \end{cases}$$

$f_s = \frac{2}{7} mg\sin\theta = 8.4\,N$

A positive f_s points up plane

5. 功能关系：The yo-yo



(a) (b)

- As a yo-yo moves down a string, it loses potential energy *mgh* but gains rotational and translational kinetic energy
- To find the linear acceleration of a yo-yo accelerating down its string:
  1. Rolls down a "ramp" of angle 90°
  2. Rolls on an axle instead of its outer surface, R_0
  3. Slowed by tension T rather than friction

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分析：

① 可以理解为沿倾角90°的斜面下滑.

② 沿着外表面内侧，半径为R_0的轮轴滚动.

③ 阻力主要为张力T

1° 情况一：无滑滚动

$$\begin{cases} mg - T = m a_{com} \\ T\cdot R_0 = I\cdot\alpha \\ a_{com} = \alpha\cdot R_0 \end{cases}$$

$$\begin{cases} T = \dfrac{Img}{I+mR_0^2} \\ \alpha = \dfrac{R_0 mg}{I+mR_0^2} \\ a_{com} = \dfrac{R_0^2 mg}{I+mR_0^2} \end{cases}$$ downward

注：$a_{com} = \alpha R_0$

$a_{com} = \frac{dv_{com}}{dt}$

$v_{com} = \frac{ds}{dt} = \frac{d\theta}{dt}\cdot R_0 \Rightarrow a_{com} = \alpha R_0$

2° 情况二：有滑滚动

半径为R的yo-yo，有角速度ω_0，在$t_0=0s$时，它与平面接触.

① 接触时：有滑滚动，摩擦力为f_k

平动：

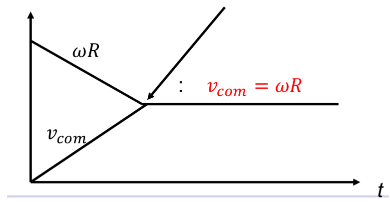
$$\begin{cases} f_k = \mu_k mg = m a_{com} \\ v_{com} = a_{com}\cdot t \quad (a_{com} \neq \alpha\cdot R) \end{cases}$$

旋转：

$$\begin{cases} f_k R = \mu_k mg R = I_{com}\alpha \\ \omega = \omega_0 - \alpha t \end{cases}$$

② 当 $V_{com} = \omega R$ 时，开始做无滑滚动

$$t = \frac{\omega_0 R}{\mu_k g \left(1 + \frac{mR^2}{I_{com}}\right)}$$



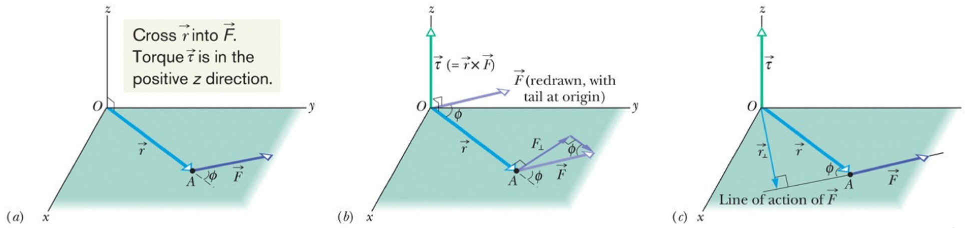
§2 Torque

- Apply the torque to an individual particle that moves along any path relative to a fixed point
- 路线不一定要为圆.

$\vec{\tau} = \vec{r} \times \vec{F}$

$\tau = r \cdot F \cdot \sin\phi$

- Direction determined by right-hand rule



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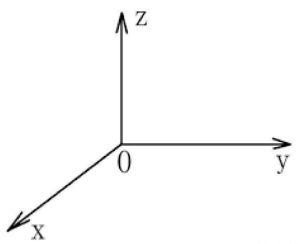
例:

Checkpoint 3

The position vector $\vec{r}$ of a particle points along the positive direction of a z axis. If the torque on the particle is (a) zero, (b) in the negative direction of x, and (c) in the negative direction of y, in what direction is the force causing the torque?



Answer: (a) along the z direction (b) in the +y (included) and z plane (c) in the -x (included) and z plane

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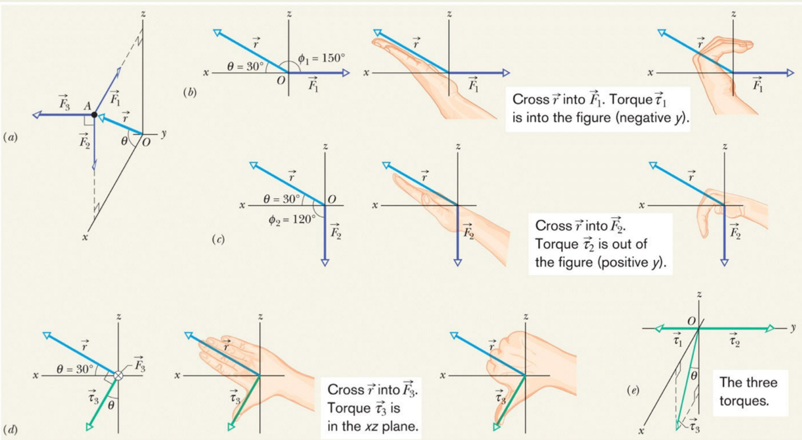
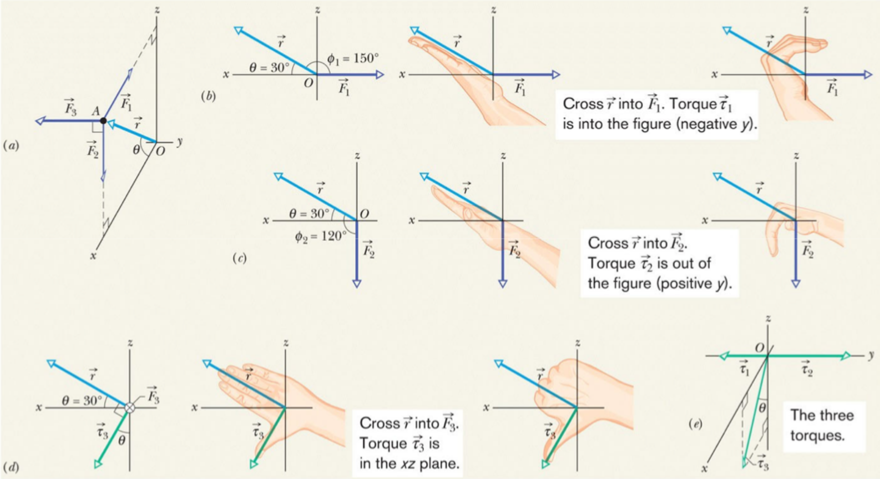
例: In Fig. 11-11*a*, three forces, each of magnitude 2.0 N, act on a particle. The particle is in the *xz* plane at point *A* given by position vector $\vec{r}$, where $r = 3.0$ m and $\theta = 30°$. What is the torque, about the origin *O*, due to each force?





$$\tau_1 = rF_1 \sin\phi_1 = (3.0\text{ m})(2.0\text{ N})(\sin 150°) = 3.0\text{ N}\cdot\text{m},$$

$$\tau_2 = rF_2 \sin\phi_2 = (3.0\text{ m})(2.0\text{ N})(\sin 120°) = 5.2\text{ N}\cdot\text{m},$$

and $$\tau_3 = rF_3 \sin\phi_3 = (3.0\text{ m})(2.0\text{ N})(\sin 90°) = 6.0\text{ N}\cdot\text{m}. \qquad \text{(Answer)}$$

Summary

- Rolling Smoothly

$$v_{com} = \frac{s}{\Delta t} = \frac{ds}{dt} = \frac{d\theta}{dt}R = \omega R$$

$$a_{com} = \frac{dv_{com}}{dt} = \frac{d\omega}{dt}R = \alpha R$$

$$K = \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv_{com}^2$$

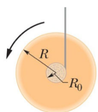
Summary

- Rolling on a slope

$$a_{com,x} = \frac{g\sin\theta}{1 + I_{com}/MR^2}$$

- Rolling for Yo-Yo

$$a_{com,x} = \frac{g}{1 + I_{com}/MR_0^2}$$



Summary

- Torque as a Vector

$$\vec{\tau} = \vec{r} \times \vec{F} \;\;\&\;\; \tau = rF\sin\emptyset$$