Lecture 2 Infinite Series, Integral Test (10.1, 10.2, 10.3)

无穷级数，积分审敛法

§1 单调有界序列 bounded monotonic sequences

1. monotonic 定义

DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** (or weakly increasing) if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le \ldots$. The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** (or monotone) if it is either nondecreasing or nonincreasing.

任意常序列既 nondecreasing 又 noneincreasing

e.g. $\{(-1)^n\}$ is not monotonic

2. 性质：单调有界序列必收敛

Another name: Monotone Convergence Theorem

THEOREM ~~6~~ 10.1.6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

证明: Suppose $\{a_n\}$ is nondecreasing.

- Since $\{a_n\}$ is bounded, it's bounded above.
  Let M be the least upper bound for $\{a_n\}$.
- Let ε be arbitrary. Then $M-\varepsilon$ is not an upper bound for $\{a_n\}$.
  so $\exists N$ s.t. $a_N > M-\varepsilon$.
- Since $\{a_n\}$ is nondecreasing, $a_n \ge a_N > M-\varepsilon \quad \forall n>N$.
- Also, $a_n \le M$, $\forall n$
- Therefore, $\forall n>N$, $M-\varepsilon < a_n < M$, so $|a_n - M| < \varepsilon$.
  So $\lim_{n\to\infty} a_n = M$.

§2 无穷级数 infinite series

1. 无穷级数与部分和

Definition

Given a sequence $\{a_n\}$ of real numbers ($n \ge 1$), define

$$s_k := \sum_{n=1}^{k} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} a_n := \lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n.$$

The symbol $\sum_{n=1}^{\infty} a_n$ is called an infinite series, or simply a series, of the sequence ~~(a_n)~~ $\{a_n\}$. The number a_n is called the n-th term of the series, and the number s_k is called the k-th partial sum.

The series is said to be convergent if the limit exists (as a real number), and is said to be divergent otherwise.

例: 几何级数 (geometric series): $\sum_{n=1}^{\infty} r^{n-1} = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots\cdots$

调和级数 (harmonic series): $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$

1° 通常用 $\sum_n a_n$ 或 $\sum a_n$ 或 $\sum_n a_n$ 表示级数.
有时 initial index 会是除 1 以外的数.
2° 序列 $\{S_n\}=\{S_1, S_2, \cdots\}$ 被称为 sequence of partial sum.
3° 定义 $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k = S$
若 $\lim_{n\to\infty} S_n$ 存在, $\sum a_n$ 收敛 (convergent), 或 converges to S
若 $\lim_{n\to\infty} S_n$ 不存在, $\sum a_n$ 发散 (divergent), 或 diverges
特别地, 若 $\lim_{n\to\infty} S_n = \infty$, 则 $\sum a_n$ diverges to infinity.

例: 分析几何级数 $\sum_{n=1}^{\infty} a\cdot r^{n-1} = a + ar + ar^2 \cdots\cdots$ 敛散性

① when $r \neq 1$, then

$S_k = a\cdot\frac{1-r^k}{1-r}$

so, $\lim_{k\to\infty} S_k = \begin{cases} \frac{a}{1-r}, & \text{if } |r|<1 \\ \infty, & \text{if } |r|>1 \end{cases}$

② when $r=1$, then

$\sum_{n=1}^{\infty} a r^{n-1} = \sum_{n=1}^{\infty} a$, diverges.

Hence, geometric series $\sum_n a r^{n-1}$ $\begin{cases} \text{converges if } |r|<1 \\ \text{diverges if } |r|\geq 1 \end{cases}$

例: 分析 $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1\cdots\cdots$ 的敛散性

For odd n: $S_n = 1$

For even n: $S_n = 0$

Hence, $\lim_{n\to\infty} S_n$ does not exist, $\sum a_n = \sum (-1)^n$ diverges.

例: 分析裂项和 (telescoping series) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ 敛散性.

Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$S_k = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots\cdots + (\frac{1}{k}-\frac{1}{k+1}) = 1-\frac{1}{k+1}$,

so $\sum_n \frac{1}{n(n+1)} = \lim_{k\to\infty}(1-\frac{1}{k+1}) = 1$

§3 无穷级数的性质

1. 性质一: 运算法则

THEOREM ~~8~~ 5.2.8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k\sum a_n = kA$ (any number k).

证明 1: $\sum_{n=1}^{\infty}(a_n+b_n) = \lim_{k\to\infty}\sum_{n=1}^{k}(a_n+b_n) = \lim_{k\to\infty}(\sum_{n=1}^{k}a_n + \sum_{n=1}^{k}b_n)$

$= (\lim_{k\to\infty}\sum_{n=1}^{k}a_n) + (\lim_{k\to\infty}\sum_{n=1}^{k}b_n) = \sum_{n=1}^{\infty}a_n + \sum_{n=1}^{\infty}b_n$

2. 性质二

1、两个收敛级数相加(减)后仍收敛.

2、相加(减)后收敛的两级数未必收敛.

3、性质三

去掉、加上或改变有限项，敛散性不变，和的大小改变

4、性质四

1、$\sum u_n$ 收敛，任意加括号得级数也收敛，且和不变

2、加括号后收敛，原级数未必收敛

3、加括号后发散，原级数发散

5、性质五

1、$\sum_{n=1}^{\infty} u_n$ 收敛于 s，则 $\sum_{n=1}^{\infty} ku_n$ 收敛于 ks

2、若 $\sum a_n$ 收敛，但 $\sum b_n$ 发散，则 $\sum (a_n+b_n)$ 发散

6、性质六：The n^{th} Term Test

Theorem (The n-th Term Test)

If a series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$; in other words, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

证明：Suppose that a series $\sum a_n$ converges to S. Then

$$\lim_{n\to\infty} S_n = S = \lim_{n\to\infty} S_{n-1}$$

Since $a_n = S_n - S_{n-1}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

* $a_n \to 0$，级数未必收敛.

例：1. The series $\sum_{n=1}^{\infty} \frac{e^n-2}{4e^n+5}$ diverges since $\lim_{n\to\infty} \frac{e^n-2}{4e^n+5} = \frac{1}{4} \neq 0$

2. The series $\sum_{n=1}^{\infty} \sin\frac{n\pi}{2}$ diverges since $\lim_{n\to\infty} \sin\frac{n\pi}{2}$ doesn't exist.

§4 正项数列审敛法 Convergence Tests

1、正项数列 (series with nonnegative terms)

Theorem

Let $\{a_n\}$ be a sequence such that $a_n \geq 0$ for all n, and let $\{s_k\}$ be the corresponding sequence of partial sums. Then $\sum a_n$ converges if and only if $\{s_k\}$ is bounded.

2、审敛法一：The n^{th} term test

Theorem (The n-th Term Test)

If a series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$; in other words, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

3. 审敛法二：Integral test

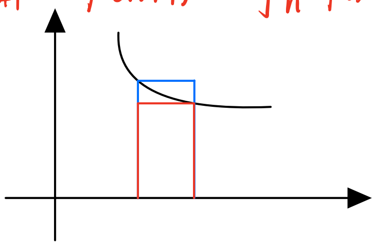
> **Theorem (Integral Test)** $(N \geq 1)$
>
> *Suppose that $a_n = f(n) \geq 0$ for all n satisfying $n \geq N$ (with N being a fixed integer), where f is a nonincreasing continuous function on $[N, \infty)$. Then $\sum a_n$ converges if and only if the improper integral $\int_N^\infty f(x)\,dx$ converges.*

integrable is enough.

证明：Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges

- For any $n \geq N$, since f is nonincreasing,

$$a_{n+1} = f(n+1) \leq \int_n^{n+1} f(x)\,dx \leq f(n) = a_n$$



- Hence $\sum_{n=N+1}^{N+1+k} a_n \leq \int_N^{N+k} f(x)\,dx \leq \sum_{n=N}^{N+k} a_n$
- If $\int_N^\infty f(x)\,dx$ converges, then

$$\sum_{n=N+1}^{N+1+k} a_n \leq \int_N^{N+k} f(x)\,dx \leq \int_N^{\infty} f(x)\,dx$$

  so the partial sums of $\sum a_n$ is bounded.
  Hence $\sum a_n$ converges.
- If $\int_N^\infty f(x)\,dx$ diverges,
  Since $\int_N^{N+k} f(x)\,dx \leq \sum_{n=N}^{N+k} a_n$,
  the partial sums of $\sum a_n$ is unbounded.
  Hence $\sum a_n$ diverges.

例：分析 p 级数 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ 的敛散性

① If $p \leq 0$,
then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so series diverges.

② If $p > 0$
then $f(x) = \frac{1}{x^p}$ is positive and decreasing on $[1, \infty)$
Since $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $0<p\leq 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges}, & p>1. \\ \text{diverges}, & p\leq 1. \end{cases}$$

4. 审敛法三：比较审敛法(一)

$\sum u_n, \sum v_n$ 是正项级数，$0 \leq u_n \leq v_n$，则

$$\begin{cases} \sum v_n \text{收敛} \Rightarrow \sum u_n \text{收敛} \\ \sum u_n \text{发散} \Rightarrow \sum v_n \text{发散} \end{cases}$$

5. 审敛法四：比较审敛法（二）

$\sum u_n$，$\sum v_n$ 是正项级数

1° $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l \Rightarrow \sum u_n$ 与 $\sum v_n$ 敛散性相同

2° $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=+\infty \Rightarrow$ 若 $\sum v_n$ 发散，则 $\sum u_n$ 发散

3° $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=0 \Rightarrow$ 若 $\sum v_n$ 收敛，则 $\sum u_n$ 收敛

6. 审敛法五：比值审敛法

$\sum u_n$ 是正项级数，$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\rho$，则

$\begin{cases}\rho<1 \Rightarrow \text{收敛}\\ \rho>1 \Rightarrow \text{发散}\\ \rho=1 \Rightarrow \text{此方法失效}\end{cases}$

7. 审敛法六：根值审敛法（柯西判别法）

$\sum u_n$ 是正项级数，$\lim\limits_{n\to\infty}\sqrt[n]{u_n}=\rho$，则

$\begin{cases}\rho<1 \Rightarrow \text{收敛}\\ \rho>1 \Rightarrow \text{发散}\\ \rho=1 \Rightarrow \text{此方法失效}\end{cases}$

e.g. $\sum\limits_{n=1}^{\infty}\frac{2+(-1)^n}{2^n}$：$\sqrt[n]{\frac{2+(-1)^n}{2^n}}=\frac{1}{2}(2+(-1)^n)^{\frac{1}{n}}=\frac{1}{2}\cdot e^{\frac{1}{n}\cdot\ln(2+(-1)^n)}\to\frac{1}{2}\Rightarrow$ 收敛

§5 Integral test: Approximation

Suppose that we have a convergent series $\sum a_n$ satisfying the conditions in the integral test, and suppose that $\sum a_n = S$. Consider the error term $R_N := S - s_N$. Then

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \ldots = \sum_{n=N+1}^{\infty} a_n.$$

By the argument in the proof of the integral test, it follows that

$$\int_{N+1}^{\infty} f(x)\,dx \le R_N \le \int_N^{\infty} f(x)\,dx.$$

If we add s_N to the inequality above, we get

$$\boxed{s_N + \int_{N+1}^{\infty} f(x)\,dx \le S \le s_N + \int_N^{\infty} f(x)\,dx.} \qquad (1)$$

This gives an interval I in which S lies, where $S = \sum a_n$. Hence, if we take the midpoint of I as an approximation of $\sum a_n$, the error is at most half the length of I, which is

$$\boxed{\frac{1}{2}\left(\int_N^{\infty} f(x)\,dx - \int_{N+1}^{\infty} f(x)\,dx\right).}$$

Note that this error is at most $a_N/2$. (Why?)

$$S_N + \int_{N+1}^{\infty} f(x)dx \le S \le S_N + \int_N^{\infty} f(x)dx$$

$$\text{error} = \frac{1}{2}\left(\int_N^{\infty} f(x)dx - \int_{N+1}^{\infty} f(x)dx\right)$$

Example

Approximate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the method on the previous slide, with $N = 10$. Find an upper bound for the error.

Solution

- An easy computation shows that $\int_N^\infty (1/x^2)\,dx = 1/N$.
- By Inequality (1) on the previous slide, we have

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}.$$

- $s_{10} = 1 + 1/4 + 1/9 + \ldots + 1/100 \approx 1.54977$, so

$$1.64068 \le S \le 1.64977.$$

  Then we take the midpoint of the interval, which is 1.645225. The error of this is at most $0.5 \times (1.64977 - 1.64068)$, which is 0.004545.

Fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.644934\ldots$. Proof of this is not elementary.