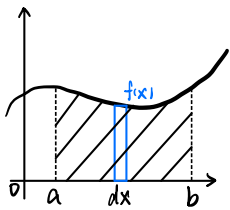


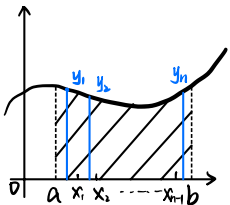
## Lecture 22

### Recall:

1.  $f(x) \geq 0$  on  $[a, b]$



$$\text{Area} = \int_a^b f(x) dx$$

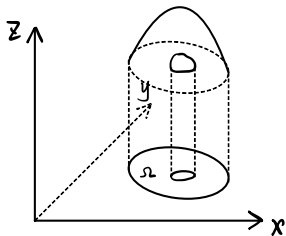


$$\text{partition } P = \{x_0, x_1, \dots, x_n\}$$

$\|P\|$ : norm of  $P$  = longest subinterval

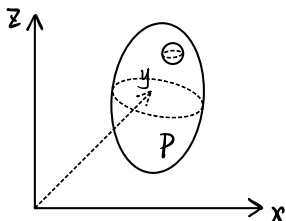
$$\text{Area} = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(y_i) \Delta x_i \quad (\text{Riemann sum})$$

2.  $z = f(x, y) \geq 0, (x, y) \in \Omega$



$$\text{Volume } V = \iint_{\Omega} f(x, y) dA$$

3. density function  $f(x, y, z) = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} \approx \frac{\Delta m}{\Delta V}$



$$\Delta m = f(x, y, z) \Delta V$$

$$\text{total mass} = \iiint_P f(x, y, z) dV$$

### Goal:

Q1: Given set  $S \subset \mathbb{R}^n$ , how to define rigorously  $\int_S f(x) dx$

Q2: Necessary & sufficient condition on  $f(x)$  so that  $\int_S f(x) dx$  exists.

Q3: Generalization of Fubini's theorem ( $\iint_{[a,b] \times [c,d]} f(x, y) dy dx = \int_a^b (\int_c^d f(x, y) dy) dx$ )

## §1 Integration theory

### 1. Jargons

1° Closed rectangle in  $\mathbb{R}^n$

$$Q = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$

$$= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \dots, a_n \leq x_n \leq b_n\}$$

$$\text{Volume of } Q \stackrel{\text{def}}{=} (b_1 - a_1)(b_2 - a_2) \dots (b_n - a_n)$$

## 2° Partition

Let  $p_1$  be a partition of  $[a_1, b_1]$ .

Let  $p_2$  be a partition of  $[a_2, b_2]$ .

⋮

Let  $p_n$  be a partition of  $[a_n, b_n]$ .

则称  $n$ -tuple  $(p_1, p_2, \dots, p_n)$  为一个 partition of  $\mathcal{Q}$

若  $I_j$  为一个 subinterval determined by  $p_j$ ,  $j=1, \dots, n$ . 则 the rectangle  $R \stackrel{\text{def}}{=} I_1 \times I_2 \times \dots \times I_n$  被称为一个 subrectangle determined by  $P$ . 记作  $R \leq P$ .  $|R|$  表示 rectangle 的 volume

3° norm of  $\|P\| =$  所有 subrectangle  $\leq P$  的最大的边

## 2. Definition: Riemann integrable

令  $f(x)$  为 real-valued & bdd on  $\mathcal{Q}$ , 则称  $f$  为 Riemann integrable on  $\mathcal{Q}$ .

若  $\forall \varepsilon > 0, \exists \delta > 0$ , s.t. as long as  $\|P\| < \delta$ , we have  $|\sum_{R \in P} f(x_R) |R| - A| < \varepsilon$  for some constant  $A$  for any  $x_R \in R$ .

其中  $\sum_{R \in P} f(x_R) |R|$  被称为 Riemann sum.

在这种情况下, 记  $\lim_{\|P\| \rightarrow 0} \sum_{R \in P} f(x_R) |R| = A$ . 其中  $A$  被称作 integral of  $f$  on  $\mathcal{Q}$ , 记作  $\int_{\mathcal{Q}} f(x) dx$

例:  $\mathcal{Q} = [0, 1] \times [0, 1]$

Dirichlet function  $D = \begin{cases} 1, & \text{if } (x, y) \text{ irrational} \\ 0, & \text{if } (x, y) \text{ rational} \end{cases}$

判断  $D$  是否 Riemann integrable

$$\sum_{R \in P} D(x_R, y_R) |R| = \begin{cases} \sum_{R \in P} |R| = 1111, & \text{if always take } (x_R, y_R) \text{ irrational} \\ 0, & \text{if always take } (x_R, y_R) \text{ rational} \end{cases}$$

$\Rightarrow$  "A" does not exist

$\Rightarrow D(x, y)$  not Riemann integrable on  $\mathcal{Q}$

Q: When is  $f$  R-integrable on  $\mathcal{Q}$ ?

Agony: 不知道  $A$  的值

Discussion:  $|\sum_{R \in P} f(x_R) |R| - A| < \varepsilon$

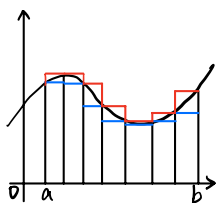
$$\Leftrightarrow A - \varepsilon < \sum_{R \in P} f(x_R) |R| < A + \varepsilon$$

$$\Rightarrow \begin{cases} A - \varepsilon \leq \sum_{R \in P} \inf_{x \in R} f(x) |R| & \text{记 } \inf_{x \in R} f(x) = m_R(f) \\ \sum_{R \in P} \sup_{x \in R} f(x) |R| \leq A + \varepsilon & \text{记 } \sup_{x \in R} f(x) = M_R(f) \end{cases}$$

$$\Rightarrow A - \varepsilon \leq \sum_{R \in P} m_R(f) |R| \leq \sum_{R \in P} M_R(f) |R| \leq A + \varepsilon$$

Define  $L(f; P) = \sum_{R \in P} m_R(f) |R|$  lower sum determined by  $P$

$U(f; P) = \sum_{R \in P} M_R(f) |R|$  upper sum determined by  $P$



由此可推出一些 Facts

### 3. Fact 1

$f$  is R-integrable on  $\mathbb{R}$

$$\iff \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 \leq U(f; P) - L(f; P) < \varepsilon, \text{ whenever } \|P\| < \delta$$

证明:

(见作业)

#### 4. Fact 2: (more user-friendly)

$f$  is  $R$ -integrable on  $Q$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \text{ partition } P' \text{ s.t. } (0 \leq) U(f; P') - L(f; P') < \varepsilon \quad (*)$$

证明:

① proof of " $\Rightarrow$ ":

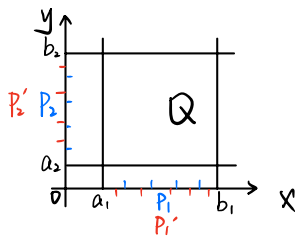
trivial by Fact 1.

② proof of " $\Leftarrow$ ":

(W.T.S. Fact 1 holds)

Define the 3-rd partition  $P'' = P \cup P' = (P_1 \cup P'_1, P_2 \cup P'_2, \dots, P_n \cup P'_n)$

$P''$  被称为 common refinement of  $P$  &  $P'$ .



Observe: can obtain  $P''$  from  $P$  as follows

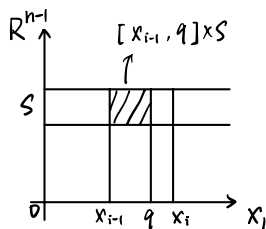
$$P \xrightarrow{\text{add 1 pt}} P^1 \xrightarrow{\text{add 1 pt}} P^2 \xrightarrow{\text{add 1 pt}} \dots \xrightarrow{\text{add 1 pt}} P^k = P''$$

Suppose  $x_{i-1} \leq q \leq x_i$  for some  $i$ ,

WLOG, assume  $P' = (P_1 \cup \{q\}, P_2, \dots, P_n)$

$$\begin{aligned} 0 \leq L(f; P') - L(f; P) &= \sum_{S \in (P_1, \dots, P_n)} \{ m_{[x_{i-1}, q] \times S}(f) \cdot |[x_{i-1}, q] \times S| \\ &\quad + m_{[q, x_i] \times S}(f) \cdot |[q, x_i] \times S| \\ &\quad - \sum_{S \in (P_1, \dots, P_n)} \{ m_{[x_{i-1}, x_i] \times S}(f) \cdot |[x_{i-1}, x_i] \times S| \} \end{aligned}$$

(前两项 > 最后一项)



$\therefore |f| \leq \text{constant } M \text{ on } Q$

$$\begin{aligned} \therefore L(f; P') - L(f; P) &\leq 2M \sum_{S \in (P_1, \dots, P_n)} |S| (x_i - x_{i-1}) \\ &\leq 2M \|P\| \sum_{S \in (P_1, \dots, P_n)} |S| \quad (x_i - x_{i-1} \leq \|P\|) \\ &= 2M \|P\| (b_2 - a_2) \dots (b_n - a_n) \\ &\leq 2M \|P\| \cdot (\text{width of } Q)^{n-1} \end{aligned}$$

$$\therefore 0 \leq L(f; P') - L(f; P) \leq 2M \|P\| \cdot (\text{width of } Q)^{n-1}$$

$$\text{Similarly, } 0 \leq L(f; P^2) - L(f; P') \leq 2M \|P\| \cdot (\text{width of } Q)^{n-1}$$

$$\dots$$

$$0 \leq L(f; P^k) - L(f; P^{k-1}) \leq 2M \|P\| \cdot (\text{width of } Q)^{n-1}$$

$$\therefore 0 \leq L(f; p'') - L(f; p) \leq 2Mk \|p\| \cdot (\text{width of } Q)^{n-1} \quad (*)$$

$$\text{Similarly, } 0 \geq U(f; p'') - U(f; p) \geq -2Mk \|p\| \cdot (\text{width of } Q)^{n-1} \quad (\#)$$

(\*) - (#) 得:

$$\begin{aligned} 0 &\leq L(f; p'') - L(f; p) - U(f; p'') + U(f; p) \leq 4Mk \|p\| \cdot (\text{width of } Q)^{n-1} \\ \Rightarrow U(f; p) - L(f; p) &\leq 4Mk \|p\| \cdot (\text{width of } Q)^{n-1} + U(f; p'') - L(f; p'') \\ &< 4Mk \|p\| \cdot (\text{width of } Q)^{n-1} + \varepsilon \quad (\text{由} (*) ) \\ &< 2\varepsilon \end{aligned}$$

if  $\|p\| < \delta$ , which is taken s.t.  $4Mk \|p\| \cdot (\text{width of } Q)^{n-1} < \varepsilon$

注: 在 refine partition 之后,

$$U(f, p) \geq U(f, p \cup p') \quad L(f, p) \leq L(f, p \cup p')$$

