Lecture 12

§1 Formulae for Finite Sums

1. Notation of finite sums

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots\cdots + a_n$$

The variable i is a "dummy variable", meaning it can be changed to a different symbol without changing the meaning.

2. Identities of finite sums

1° Finite sums satisfy linearity:

$$\sum_{i=1}^{n} (ka_i + tb_i) = k\sum_{i=1}^{n} a_i + t\sum_{i=1}^{n} b_i$$

2° Useful identities

① $\sum_{k=1}^{n} k = 1+2+\cdots\cdots+k = \frac{1}{2}n(n+1)$

② $\sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots\cdots+k^2 = \frac{1}{6}n(n+1)(2n+1)$

③ $\sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots\cdots+k^3 = \frac{1}{4}n^2(n+1)^2$

Proof:

Strategy 1 (To prove identity ①)

$$\begin{array}{cccccc} & 1 & 2 & \cdots\cdots & n-1 & n \\ + & n & n-1 & \cdots\cdots & 2 & 1 \\ \hline & n+1 & n+1 & \cdots\cdots & n+1 & n+1 \end{array}$$

$$2\sum_{k=1}^{n} k = (n+1)n \Longrightarrow \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

Strategy 2 (To find $\sum_{k=1}^{n} k^m$ for a fixed value of m)

Start with n^{m+1} and use formulae for small m.

e.g. Find $\sum_{k=1}^{n} k$

Start with n^2. Let $S_n = \sum_{k=1}^{n} k$

$$n^2 = (n^2-(n-1)^2) + ((n-1)^2-(n-2)^2) + \cdots\cdots + (2^2-1^2) + (1^2-0^2)$$

$$= \sum_{k=1}^{n} (k^2-(k-1)^2)$$

$$= \sum_{k=1}^{n} (k^2-k^2+2k-1)$$

$$= \sum_{k=1}^{n} (2k-1)$$

$$= 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= 2S_n - n$$

$$S_n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

e.g. Prove the formula $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

$$n^3 = (n^3-(n-1)^3) + ((n-1)^3-(n-2)^3) + \cdots\cdots + (2^3-1^3) + (1^3-0^3)$$

$$= \sum_{k=1}^{n} (k^3-(k-1)^3)$$

$$= \sum_{k=1}^{n} (k^3-k^3+3k^2-3k+1)$$

$$= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 3\sum_{k=1}^{n} k^2 - \frac{3}{2}n(n+1) + n$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}(n^3+\frac{3}{2}n^2+\frac{1}{2}n)$$

$$= \frac{1}{6}(2n^3+3n^2+n)$$

$$= \frac{1}{6}n(n+1)(2n+1)$$

§2 A Little More on Riemann Sums

1. Definition of norm

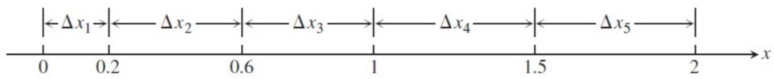
Let $P = \{x_0, x_1, \cdots\cdots, x_n\}$ be a partition of $[a,b]$. The norm of P, denoted by $\|P\|$, is defined by

$$\|P\| = \max_{k:1\le k\le n} \Delta x_k$$

That is, $\|P\|$ is the length of the largest subinterval given by P.

Example

The partition P of $[0,2]$ represented by the following figure has norm $\|P\| = 0.5$.



§3 Definite integrals: Integrability

1. Definition of definite integral

Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that a number J is the definite integral of f over $[a,b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon>0$ there is a corresponding number $\delta>0$ such that for every partition $P=\{x_0, x_1, \cdots\cdots, x_n\}$ of $[a,b]$ with $\|P\|<\delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - J\right| < \varepsilon$$

With the definition above:

We write

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J$$

2. Notation

If the limits J exists, we say that f is intergrable on $[a,b]$ and write the limit J as

$$\int_a^b f(x)\,dx$$

This symbol is called the definite integral or Riemann integral of f

over $[a,b]$ (or from a to b)

3. Theorem 5.3.1 — Integrability of Continuous Functions

If a function f is continuous over the interval $[a,b]$, or if f has at most finitely many jump discontinuities or removable discontinuities, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a,b]$

Proof:

For each $[x_{k-1}, x_k]$

Define $M_k = \max\{f(x_k^*) : x_k^* \in [x_{k-1}, x_k]\}$

$m_k = \min\{f(x_k^*) : x_k^* \in [x_{k-1}, x_k]\}$

Define $U_P(f) = \sum_{k=1}^{n} M_k \cdot \Delta x_k$

$L_P(f) = \sum_{k=1}^{n} m_k \cdot \Delta x_k$

Thus for any choice of $c_k \in [x_{k-1}, x_k]$, $m_k \le f(c_k) \le M$

So $L_P(f) = \sum_{k=1}^{n} m_k \cdot \Delta x_k \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le \sum_{k=1}^{n} M_k \cdot \Delta x_k = U_P(f)$

As $\|P\| \to 0$, $U_P(f) - L_P(f) \to 0$

More formally,

for any given ε, choose all Δx small enough such that

$M_k - m_k < \frac{\varepsilon}{a-b}$, $\forall k$ (This is possible for continuous functions)

$$U_P(f) - L_P(f) = \sum_{k=1}^{n}(M_k - m_k)\Delta x_k < \frac{\varepsilon}{b-a}\sum_{k=1}^{n}\Delta x_k$$

$$\frac{\varepsilon}{b-a}\sum_{k=1}^{n}\Delta x_k = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

$$U_P(f) - L_P(f) < \varepsilon$$

$\lim_{\|P\|\to 0}(U_P(f) - L_P(f)) = 0$

$\lim_{\|P\|\to 0} U_P(f) = \lim_{\|P\|\to 0} L_P(f)$

Since all Reimann sums $S_P = S_P(f)$ satisfy

$$L_P(f) \le S_P(f) \le U_P(f)$$

Then $\lim_{\|P\|\to 0} S_P(f) = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$ must exist.

§4 Definite integrals: Computation using Riemann sums

Suppose we know that f is integrable on $[a,b]$. Then we can compute the limit of Riemann sums by choosing any sequence of partitions P such that $\|P\|\to 0$.

In particular, we may choose $\Delta x_k = \Delta x = \frac{b-a}{n}$

i.e. P divided $[a,b]$ into n subintervals of equal length.

Then, taking $n\to\infty$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x$$

Here $c_k \in [x_{k-1}, x_k]$ can be chosen in anyway (you may choose it to make the computation convenient).

e.g. Evaluate $\int_0^1 x^2\,dx$ using Riemann sums.

Consider the partition $P=\{x_0,\cdots,x_n\}$ of $[0,1]$, with $x_k=\frac{k}{n}$

Hence, $\Delta x = \frac{1}{n}$

We may choose $c_k = x_k$ for $k=1,\cdots\cdots,n$

i.e. c_k is the right endpoint of $[x_{k-1}, x_k]$

Then
$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} x_k^2\,\Delta x \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} \frac{k^2}{n^2}\cdot\frac{1}{n} \\ &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n} k^2 \\ &= \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\ &= \frac{1}{3}\end{aligned}$$

§5 Definite integrals: Nonintegrability

If f is not integrable on $[a,b]$

Intuitively, this happens when the upper sum and the lower sum do not converge to the same number J.

i.e. $U_P(f) - L_P(f)$ doesn't converge to 0 as $\|P\| \to 0$.

More formally, there exists $\varepsilon > 0$ such that no matter how small a given $\delta > 0$ is, there is a partition P of $[a,b]$ with $\|P\| < \delta$, such that $U_P(f) - L_P(f) > \varepsilon$

e.g. Consider the Dirichlet function.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Here $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\}$

Proof:

For any interval $[x_{k-1}, x_k]$, we have some value $c_1, c_2 \in [x_{k-1}, x_k]$ such that $f(c_1) = 1$, and $f(c_2) = 0$.

Then for any partition P of $[a,b]$ with $\|P\| < \delta$

$$L_P(f) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} (0) \Delta x_k = 0$$

$$U_P(f) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} (1) \Delta x_k = b-a$$

$$U_P(f) - L_P(f) = b-a > \varepsilon$$

Integral doesn't exist.