Lecture 3

§1 关于极限的 basic facts（接上次）

1. Fact 5（子序列的极限）

Suppose $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = l$. $n_1 < n_2 < \cdots < n_k < \cdots \to \infty$. Then for any subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, we have $\lim_{k\to\infty} a_{n_k} = l$

证明:

$\because \lim_{n\to\infty} a_n = l$

$\therefore \forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $|a_n - l| < \varepsilon$ as long as $n \geq N_\varepsilon$ (*)

$\because n_k \to \infty$ as $k \to \infty$

$\therefore \exists K_\varepsilon$, s.t. $n_k \geq N_\varepsilon$ as long as $k \geq K_\varepsilon$

By (*), $|a_{n_k} - l| < \varepsilon$, if $k \geq K_\varepsilon$

$\therefore \lim_{k\to\infty} a_{n_k} = l$

Q.E.D.

2. Fact 6（sub-subsequence 的极限）

Suppose $\{a_n\}$ satisfies:

For any subseq of $\{a_n\}$, $\exists$ sub-subseq $\{a_{n_k}\}_{k=1}^{\infty}$, s.t. $\lim_{k\to\infty} a_{n_k}$ exists and $\lim_{k\to\infty} a_{n_k} = l$, where l is independent of the choice of subseq.

Then $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = l$.

证明:

Suppose otherwise. Then by rigorous definition of $\lim_{n\to\infty} a_n \neq l$,

$\exists \varepsilon_0 > 0$ and subseq $\{a_{n_k}\}_{k=1}^{\infty}$, s.t. $|a_{n_k} - l| \geq \varepsilon$, $\forall k \geq 1$ (*)

By assumption, $\exists$ sub-subseq $\{a_{n_{k_m}}\}_{m=1}^{\infty}$ s.t. $\lim_{m\to\infty} a_{n_{k_m}} = l$

$\because \{a_{n_{k_m}}\}_{m=1}^{\infty} \subset \{a_{n_k}\}_{k=1}^{\infty}$

$\therefore$ By (*), $|a_{n_{k_m}} - l| \geq \varepsilon$, $\forall m \geq 1$

$\therefore \lim_{m\to\infty} a_{n_{k_m}} \neq l$ (Contradiction)

Q.E.D.

注: 由 Fact 6 可知, the Converse of Fact 5 也是正确的. 即:

若 $\{a_n\}$ 的任一子序列极限均为 l, 则 $\{a_n\}$ 极限存在且为 l

§2 Preparation: 关于 boundness 的定义与性质

1. Definition: boundness（有界性）

Let S be a subset of $\mathbb{R}$,

1° If $\exists$ constant M s.t. $x \leq M$, $\forall x \in S$, then M is called the upper-bound of S (上界).
We say S is bdd (bounded) from above.

2° If $\exists$ constant M s.t. $x \geq M$, $\forall x \in S$, then M is called the lower-bound of S.
We say S is bdd from below.

3° If S is both bdd from above & below, then we say S is bdd.

2. Definition: least upper (greatest lower) bound (上确界/下确界)

Suppose S is bounded from above (below).
A real number s is said to be the least upper (greatest lower) bound of S, if:

1° s is an upper (lower) bound of S, i.e. $\forall x \in S$, $x \leq (\geq) s$

2° $\forall \varepsilon > 0$, $s - \varepsilon < x_\varepsilon$ $(s + \varepsilon > x_\varepsilon)$, for some $x_\varepsilon \in S$.

e.g. $S = \{1 - \frac{1}{n}\}_{n=1}^{\infty}$: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\cdots$

$\sup S = 1$

注: ① 对于序列 S, 上确界用 $\sup S$ 表示, 下确界用 $\inf S$ 表示.

② $\forall$ upper bound M of S, we have $M \geq \sup S$

$\forall$ lower bound m of S, we have $m \leq \inf S$

3. Axiom: The least upper bound axiom (上确界公理)

If S is bdd from above, then $\sup S$ exists.
若S有上界, 则上确界一定存在.

4. Theorem: 下确界定理

If S is bdd from below, then $\inf S$ exists.
若S有下界, 则下确界一定存在.

证明:

Let $T = -S$, then T is bounded above

By axiom, $\sup T$ exists, it satisfies:

· $\forall -s \in T$ $(s \in S)$, we have

$$-s \leq \sup T \Rightarrow s \geq -\sup T, \ \forall s \in S$$

· $\forall \varepsilon > 0$, $\exists -s_\varepsilon \in T$ $(s_\varepsilon \in S)$, s.t.

$$\sup T - \varepsilon < -s_\varepsilon \Rightarrow s_\varepsilon > -\sup T + \varepsilon$$

$\therefore -\sup T$ is $\inf S$

注: By-product:

$\sup(-S) = -\inf S$

$\inf(-S) = -\sup S$

§3 关于极限的 basic facts (接上)

1. Fact 7 (单调有界序列极限存在: Monotone Convergence Theorem)

Suppose $\{a_n\}$ is monotone and is bdd. Then $\lim\limits_{n\to\infty} a_n$ exists as a finite number.

证明:

Only consider the case of monotone increase.

(先利用上确界公理证出上确界为实数)

Let $l = \sup\{a_n\}_{n=1}^{\infty}$

$\because$ $\{a_n\}$ is bdd from above

$\therefore$ l exists & l is finite

(随后利用上确界的定义证出极限存在)

By definition of Supremum,

$\forall n, \ l > a_n$

$\forall \varepsilon > 0, \ l - \varepsilon < a_N$ for some N

$\because$ a_n is monotone increasing.

$\therefore$ $l - \varepsilon < a_N \leq a_n \leq l < l + \varepsilon$

$\therefore$ $|a_n - l| < \varepsilon$, for $n \geq N$

$\therefore$ $\lim_{n\to\infty} a_n = l$

Q.E.D.

注: 若 a_n 单调递增且无界, 则 $\lim_{n\to\infty} a_n = +\infty$

证明:

$\because$ a_n is not bdd

$\therefore$ $\forall M$, $\exists$ some n_M s.t. $a_{n_M} > M$

$\because$ a_n is monotone increasing

$\therefore$ $a_n \geqslant a_{n_M} > M$ whenever $n \geqslant n_M$

例1: $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$, ... $\lim_{n\to\infty} a_n$ exists?

(Observe: a_n is monotone increasing

a_n is bdd: $\sqrt{2+\sqrt{2+\sqrt{2}\cdots}} < 2$)

proof by mathematical induction:

- $a_1 = \sqrt{2} < 2$
- assume $a_n < 2$, want to show $a_{n+1} < 2$

  $a_{n+1} = \sqrt{2 + a_n} < \sqrt{2+2} = 2$

By M.C.T., $\lim_{n\to\infty} a_n$ exists as a finite number.

$\lim_{n\to\infty} (a_{n+1} \cdot a_{n+1}) = \lim_{n\to\infty} (2 + a_n)$, $\forall n$

$(\lim_{n\to\infty} a_n)(\lim_{n\to\infty} a_n) = 2 + \lim_{n\to\infty} a_n$

$\lim_{n\to\infty} a_n = 2$ or -1

Since $a_n > 0$, $\lim_{n\to\infty} a_n = 2$

例 2: Prove that $\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists as a finite #

(先证明 $(1+\frac{1}{n})^n$ 递增)

Claim 1: $\{(1+\frac{1}{n})^n\}$ is $\uparrow$

$$
\begin{aligned}
(1+\tfrac{1}{n})^n &= 1+n\cdot\tfrac{1}{n}+\tfrac{n(n-1)}{2!}(\tfrac{1}{n})^2+\cdots+\tfrac{n(n-1)\cdots 1}{n!}\cdot(\tfrac{1}{n})^n \\
&= 1+1+\tfrac{1}{2!}\cdot 1\cdot(1-\tfrac{1}{n})+\cdots+\tfrac{1}{n!}\cdot 1\cdot(1-\tfrac{1}{n})\cdot(1-\tfrac{2}{n})\cdots(1-\tfrac{n-1}{n}) \\
&< 1+1+\tfrac{1}{2!}\cdot 1\cdot(1-\tfrac{1}{n+1})+\cdots+\tfrac{1}{n!}\cdot 1\cdot(1-\tfrac{1}{n+1})(1-\tfrac{2}{n+1})\cdots(1-\tfrac{n-1}{n+1}) \quad (*) \\
&< (*)+\tfrac{1}{(n+1)!}\cdot 1\cdot(1-\tfrac{1}{n+1})\cdot(1-\tfrac{2}{n+1})\cdots\cdots(1-\tfrac{n}{n+1}) \\
&= (1+\tfrac{1}{n+1})^{n+1}
\end{aligned}
$$

$\Rightarrow$ Claim 1

Claim 2: $\{(1+\frac{1}{n})^n\}$ is bdd (from above)

$$
\begin{aligned}
(1+\tfrac{1}{n})^n &= 1+n\cdot\tfrac{1}{n}+\tfrac{n(n-1)}{2!}(\tfrac{1}{n})^2+\cdots+\tfrac{n(n-1)\cdots 1}{n!}\cdot(\tfrac{1}{n})^n \\
&= 1+1+\tfrac{1}{2!}\cdot 1\cdot(1-\tfrac{1}{n})+\cdots+\tfrac{1}{n!}\cdot 1\cdot(1-\tfrac{1}{n})\cdot(1-\tfrac{2}{n})\cdots(1-\tfrac{n-1}{n}) \\
&< 1+1+\tfrac{1}{2!}\cdot 1\cdot(1-\tfrac{1}{n+1})+\cdots+\tfrac{1}{n!}\cdot 1\cdot(1-\tfrac{1}{n+1})(1-\tfrac{2}{n+1})\cdots(1-\tfrac{n-1}{n+1}) \\
&< 1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+\cdots+\tfrac{1}{n!} \\
&< 1+1+\tfrac{1}{2}+\tfrac{1}{2^2}+\cdots+\tfrac{1}{2^{n-1}} \\
&= 1+\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} \\
&< 3
\end{aligned}
$$

$\Rightarrow$ Claim 2

Now by MCT, $\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists as a finite #