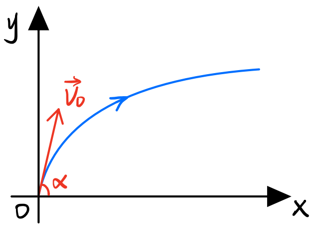
Lecture 12

§1 Ideal projectile

- 物体由原点处以发射角α射出 $(0<\alpha<\frac{\pi}{2})$
- 仅有重力$\vec{F}$作用于物体.



1. algebraic formula for the position $\vec{r}$ of the object

- Initial velocity v_0:

$$\vec{v_0} = v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\cdot\vec{j}$$

- Acceleration due to gravity:

$$\vec{a} = -g\cdot\vec{j}$$

- By FTC 2:

$$\vec{v}(t) - \vec{v}(0) = \int_0^t \vec{a}(u)\,du = \int_0^t -g\vec{j}\,du = -gt\vec{j}$$

$$\vec{v}(t) = \vec{v_0} - gt\vec{j}$$

- By FTC 2:

$$\vec{r}(t) - \vec{r}(0) = \int_0^t \vec{v}(u)\,du = \int_0^t \vec{v_0} - gu\vec{j}\,du = \vec{v_0}t - \frac{1}{2}gt^2\vec{j}$$

$$\vec{r}(t) = \vec{v_0}t - \frac{1}{2}gt^2\vec{j}$$

- So, we have:

$$\vec{r}(t) = v_0\cos\alpha\,t\cdot\vec{i} + (v_0\sin\alpha\cdot t - \frac{1}{2}gt^2)\cdot\vec{j}$$

- In parametric form, we have:

$$x = v_0\cos\alpha\,t$$

$$y = v_0\sin\alpha\cdot t - \frac{1}{2}gt^2 \quad , \ t \geq 0$$

- We can write y as an explicit function of x:

$$y = \tan\alpha\ x - \frac{1}{2}g\frac{x^2}{v_0^2\cos^2\alpha}$$

2. Max height

$$y = v_0\sin\alpha\cdot t - \frac{1}{2}gt^2$$

$$y' = v_0\sin\alpha - gt$$

So $y'=0$ if and only if $t = \frac{v_0\sin\alpha}{g}$,

$$y_{max} = \frac{v_0^2\sin^2\alpha}{2g}$$

3. Flight time

由 $v_0 \sin\alpha \cdot t - \frac{1}{2} g t^2 = 0$ 得:

$$t = \frac{2 v_0 \sin\alpha}{g}$$

4. Range

由 $R = v_0 \cos\alpha \cdot t$ 得:

$$R = \frac{v_0^2 \cdot \sin 2\alpha}{g}$$

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

Maximum height: $\quad y_{\max} = \dfrac{(v_0 \sin\alpha)^2}{2g}$

Flight time: $\quad t = \dfrac{2v_0 \sin\alpha}{g}$

Range: $\quad R = \dfrac{{v_0}^2}{g}\sin 2\alpha.$

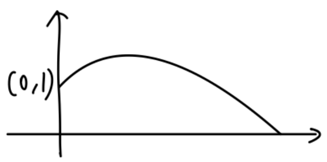
例:

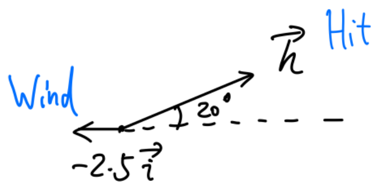
EXAMPLE ~~5~~ 13.2.5 A baseball is hit when it is 1 m above the ground. It leaves the bat with initial speed of 50 m/s, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-2.5\mathbf{i}$ (m/s) to the ball's initial velocity (2.5 m/s = 9 km/h).

(a) Find a vector equation (position vector) for the path of the baseball.

(b) How high does the baseball go, and when does it reach maximum height?

(c) Assuming that the ball is not caught, find its range and flight time.





Solution

(a) Using Equation (3) and accounting for the gust of wind, the initial velocity of the baseball is

$$\begin{aligned}\mathbf{v}_0 &= (v_0 \cos\alpha)\mathbf{i} + (v_0 \sin\alpha)\mathbf{j} - 2.5\mathbf{i} \\ &= (50\cos 20°)\mathbf{i} + (50 \sin 20°)\mathbf{j} - (2.5)\mathbf{i} \\ &= (50\cos 20° - 2.5)\mathbf{i} + (50\sin 20°)\mathbf{j}.\end{aligned}$$

The initial position is $\mathbf{r}_0 = 0\mathbf{i} + 1\mathbf{j}$. Integration of $d^2\mathbf{r}/dt^2 = -g\mathbf{j}$ gives

$$\frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$$

A second integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0.$$

Substituting the values of $\mathbf{v}_0$ and $\mathbf{r}_0$ into the last equation gives the position vector of the baseball.

$$\begin{aligned}\mathbf{r} &= -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0 \\ &= -4.9t^2\mathbf{j} + (50\cos 20° - 2.5)t\mathbf{i} + (50 \sin 20°)t\mathbf{j} + 1\mathbf{j} \\ &= (50\cos 20° - 2.5)t\mathbf{i} + \left(1 + (50\sin 20°)t - 4.9t^2\right)\mathbf{j}.\end{aligned}$$

(b) The baseball reaches its highest point when the vertical component of velocity is zero, or

$$\frac{dy}{dt} = 50\sin 20^\circ - 9.8t = 0.$$

Solving for t we find

$$t = \frac{50\sin 20^\circ}{9.8} \approx 1.75 \text{ s}.$$

Substituting this time into the vertical component for $\mathbf{r}$ gives the maximum height

$$\begin{aligned} y_{\max} &= 1 + (50\sin 20^\circ)(1.75) - 4.9(1.75)^2 \\ &\approx 15.9 \text{ m}. \end{aligned}$$

That is, the maximum height of the baseball is about 15.9 m, reached about 1.75 s after leaving the bat.

(c) To find when the baseball lands, we set the vertical component for $\mathbf{r}$ equal to 0 and solve for t:

$$\begin{aligned} 1 + (50\sin 20^\circ)t - 4.9t^2 &= 0 \\ 1 + (17.1)t - 4.9t^2 &= 0. \end{aligned}$$

The solution values are about $t = 3.55$ s and $t = -0.06$ s. Substituting the positive time into the horizontal component for $\mathbf{r}$, we find the range

$$\begin{aligned} R &= (50\cos 20^\circ - 2.5)(3.55) \\ &\approx 157.8 \text{ m}. \end{aligned}$$

Thus, the horizontal range is about 157.8 m, and the flight time is about 3.55 s. ■

§2 Arc length

1、定义

Definition

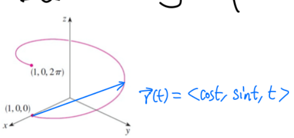
Let $\vec{r}: [a,b] \to \mathbb{R}^n$ be a continuously differentiable vector function (that is, $\vec{r}\,'$ is continuous). The arc length (or length) of a curve parametrized by $\vec{r}$ is defined to be the number L, where

$$L := \int_a^b |\vec{r}\,'(t)|\, dt,$$

assuming that the curve is traced exactly once (i.e. $\vec{r}$ is one-to-one). In particular, for a space curve ($n = 3$) parametrized by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, its length is given by

$$\int_a^b \sqrt{\underbrace{(f'(t))^2}_{dx/dt} + \underbrace{(g'(t))^2}_{dy/dt} + \underbrace{(h'(t))^2}_{dz/dt}}\, dt.$$

e.g. The length of a part of the helix below is given by



$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\; dt = 2\pi\sqrt{2}.$$

2. The arc length function

一条曲线 parametrized by 一个定义在区间 $t \in I$ 上的 one-to-one continuously differentiable 的函数 $\vec{r}(t)$. 给定一个值 $t_0 \in I$，则 arc length function $s: I \to \mathbb{R}$ 被定义为:

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(u)|\, du$$

注: 1° $s(t)$ 表示点 $P(t_0)$ 到点 $P(t)$ 的沿曲线的 signed distance

2° 若$t>t_0$，则 $s(t)>0$
若$t<t_0$，则 $s(t)<0$

3° 若t表示时间，则根据FTC：
$s'(t)=\frac{ds}{dt}=|\vec{r}'(t)|$，为运动的速率

4° 若对于$\forall t$，$\vec{r}'(t)\neq 0$，
则对于$\forall t$，$s'(t)>0$，s随着t的增加而增加

5° 一条曲线可以通过 arc length parameter s 来进行 re-parametrized.
(i.e. 曲线上的每一点，均可以用该点到一定点P_0的 signed distance 来表示)

6° 适合用于研究曲线的几何性质，而非运动性质

例：Example

Re-parametrize the helix $\vec{r}(t)=\langle\cos(t),\sin(t),t\rangle$ with respect to arc length measured from $(1,0,0)$ in the direction of increasing t.

- Base point：$P_0=(1,0,0)$
  $\Rightarrow t_0=0$
- $s=s(t)=\int_0^t|\vec{r}'(u)|du=\int_0^t\sqrt{\sin^2u+\cos^2u+1}\,du=\sqrt{2}t$
  $\Rightarrow t=\frac{s}{\sqrt{2}}$
- $\vec{g}(s):=\vec{r}(\frac{s}{\sqrt{2}})=<\cos\frac{s}{\sqrt{2}},\sin\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}}>$，$s\in(-\infty,+\infty)$
  is a new parametrization

§3 Tangent, curvature and normal

1. Tangent

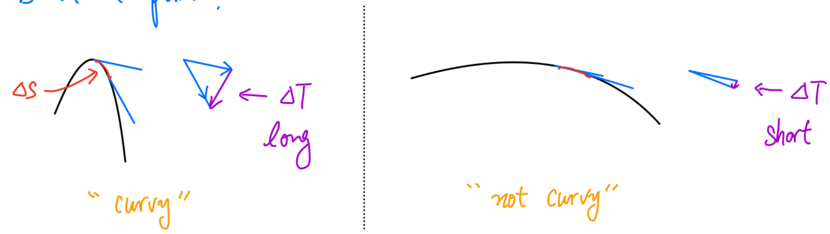
令$\vec{r}$表示R^2平面内一条曲线的 position vector，$\vec{r}$用 arc length parameter s 表示 (with a fixed base point). 且$\vec{r}$同时有一个 smooth parametrization $\vec{r}(t)$. 则 What is $d\vec{r}/ds$？

- 因为s随着t的增加而增加，所以s与s(t)有 one-to-one (单射) 的关系.
  因为s(t)可导，则 $\frac{ds}{dt}=\frac{ds(t)}{dt}=|\vec{r}'(t)|$
  因为$\vec{r}$ is smooth，则对于$\forall t$，$|\vec{r}'(t)|\neq 0$
  则 $\frac{dt}{ds}=\frac{1}{ds/dt}=\frac{1}{|\vec{r}'(t)|}$
- 根据链式法则：
  $$\frac{d\vec{r}}{ds}=\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

因此，unit tangent vector $\vec{T}(t)=\frac{d\vec{r}}{ds}=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

2. Curvature (曲率)

Q: What would be a reasonable way to capture how "curvy" a curve is at a point?



> **Def**: The curvature of a (smooth) curve is the scalar κ defined by
> $$\kappa := \left|\frac{d\vec{T}}{ds}\right|$$

注: 1° The bigger κ is, the sharper turn the curve would make at a point.

2° 用 t 来计算 κ:

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right| = \frac{1}{|\vec{r}\,'(t)|}\cdot\left|\frac{d\vec{T}}{dt}\right|$$

例: 计算下列曲线的曲率:

C_1: $\vec{r}_1(t) = \langle t, t\rangle$, $t\in\mathbb{R}$

C_2: $\vec{r}_2(t) = \langle a\cos t, a\sin t\rangle$, $t\in\mathbb{R}$, $a>0$

C_3: $\vec{r}_3(t) = \langle t, t^2\rangle$, $t\in\mathbb{R}$

C_1: $\vec{r}_1\,'(t) = \langle 1, 1\rangle$

$T_1(t) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$\left|\frac{dT_1}{dt}\right| = 0$

$\kappa = 0$ at all points

C_2: $\vec{r}_2\,'(t) = \langle -a\sin t, a\cos t\rangle$

$\vec{T}_2(t) = \langle -\sin t, \cos t\rangle$

$\left|\frac{d\vec{T}_2}{dt}\right| = |\langle -\cos t, -\sin t\rangle| = 1$

$\kappa = \frac{1}{|\vec{r}_2\,'|}\left|\frac{d\vec{T}_2}{dt}\right| = \frac{1}{a}$ at all points

C_3: $\vec{r}_3\,'(t) = \langle 1, 2t\rangle$

$|\vec{T}_3(t)| = \langle \frac{1}{\sqrt{4t^2+1}}, \frac{2t}{\sqrt{4t^2+1}}\rangle$

$$\frac{d\vec{T}_3}{dt} = \left\langle -\frac{1}{2}(1+4t^2)^{-\frac{3}{2}}\cdot 8t,\ \frac{1}{1+4t^2}\cdot\left(2\sqrt{1+4t^2} - 2t\cdot\frac{1}{2}(1+4t^2)^{-\frac{1}{2}}\cdot 8t\right)\right\rangle$$
$$= (1+4t^2)^{-\frac{3}{2}}\langle -4t, 2\rangle$$

$$\kappa(t) = \frac{1}{|\vec{r}_3\,'(t)|}\left|\frac{d\vec{T}_3}{dt}\right| = \frac{1}{\sqrt{1+4t^2}}\cdot(1+4t^2)^{-\frac{3}{2}}\cdot\sqrt{16t^2+4} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}$$

注: κ 在 $t=0$ 处最大，说明 the curve turns the most sharply at $(0,0)$

* 在平面直角坐标系中，

$\tan\theta = y' \Rightarrow \theta = \arctan y' \Rightarrow \frac{d\theta}{dx} = \frac{d}{dx}\arctan y'$

$$\kappa = \frac{d\theta}{dx}\cdot\frac{dx}{ds} = \frac{y''}{(1+y'^2)^2}\cdot\frac{1}{(1+y'^2)^{\frac{1}{2}}} = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}$$

3. Principal unit normal (单位法向量)

- 令 R^2 平面内的一条曲线用 arc length parameter s 表示
- $K=\left|\frac{d\vec{T}}{ds}\right|$，现在研究向量 $\frac{d\vec{T}}{ds}$
- $\frac{d\vec{T}}{ds}$ 指向曲线弯曲的方向
- 因为 $|\vec{T}|=1$，所以 $\vec{T}\cdot\vec{T}=1$，$\vec{T}'\cdot\vec{T}+\vec{T}\cdot\vec{T}'=0$，所以 $\frac{d\vec{T}}{ds}$ is orthogonal to $\vec{T}$ at every point.

> Def: At a point (on a curve) where $K\neq 0$, the principal unit normal vector is
> $$\vec{N}:=\frac{d\vec{T}}{ds}\Big/\left|\frac{d\vec{T}}{ds}\right|=\frac{1}{K}\frac{d\vec{T}}{ds}.$$

- 用 t 来计算 $\vec{N}$：

$$\vec{N}=\frac{d\vec{T}/ds}{|d\vec{T}/ds|}=\frac{\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}}{\left|\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}\right|}=\frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

4. osculating circle / circle of curvature (曲率圆)

1° 定义

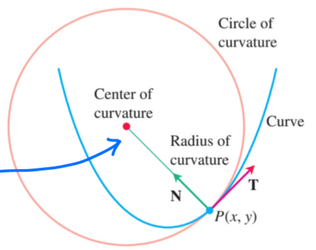
> Def: Given a plane curve C and a point P on C at which $K\neq 0$, the osculating circle at P is the circle that
> - shares the same tangent line at P as C, and;
> - has the same curvature as C at P, and;
> - has center that lies on the side to which $\vec{N}$ points.

2° radius of curvature (曲率半径)

- 一个(条)曲率 $K\neq 0$ 的点(曲线)，它的曲率半径 $\rho=\frac{1}{K}$

  * 对于一个圆，其曲率 $K=\frac{2\pi}{2\pi R}=\frac{1}{R}$
- 对于 $\vec{r}$ 处的点，其曲率圆的圆心为 $\vec{r}+\frac{1}{K}\vec{N}$

$\vec{r}+\frac{1}{K}\vec{N}$



3° K 与 $\vec{N}$ 同样可以在空间中被定义，计算公式相同

例: Find the osculating circle of the graph of $y=x^2$ at $(0,0)$

Sol: · By previous exercise, $K=\frac{2}{(1+4t^2)^{\frac{3}{2}}}$

$K=2$ at $(0,0)$ $\Rightarrow$ $\rho=\frac{1}{2}$

· $\frac{d\vec{T}}{dt}=(1+4t^2)^{-\frac{3}{2}}\langle -4t, 2\rangle=\langle 0,2\rangle$ at $(0,0)$

$\vec{N} = \langle 0, 1 \rangle$

- Hence center of curvature is given by

$$\langle 0, 0 \rangle + \frac{1}{2}\langle 0, 1 \rangle = \langle 0, \frac{1}{2} \rangle$$

- Circle is $(x-0)^2 + (y-\frac{1}{2})^2 = \frac{1}{4}$

例: For the helix given by $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$, find K

$\vec{r}'(t) = \langle -a\sin t, a\cos t, b \rangle$

$\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\sin t, a\cos t, b \rangle$

$\vec{T}'(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\cos t, -a\sin t, 0 \rangle$

$K = \frac{1}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} \cdot |\langle -a\cos t, -a\sin t, 0 \rangle|$

$= \frac{a}{a^2+b^2}$

* $b\uparrow \Rightarrow K\downarrow$

$a=0 \Rightarrow K=0$ (line)

$b=0 \Rightarrow K=\frac{1}{a}$ (circle)