Lecture 4: Elementary Probility Theory (countinued) & Discrete Random Variables

§1 Discrete random variable and probability distributions

1. random variable (随机变量)

The outcome of the experiment is called the random variable

Examples 1:

Roll a fair dice twice, and let the random variable **X** denote the summation of the two numbers.

- What is the range (possible values) of the random variable? (think about the sample space)
- How can you describe the event that the summation of larger than 10 using the random variable X?

Examples 2:

A group of 10,000 people are tested for a gene called Ifi202 that has been found to increase the risk for lupus. The random variable **X** is the number of people who carry the gene.

- What is the range (possible values) of the random variable? (think about the sample space) $\Omega = \{0, 1, 2, \cdots, 10000\}$
- How can you describe the event that more than half of the people carry the gene using the random variable X? $\{X \geq 5000\}$

2. discrete probability function (离散型概率函数)

被称为 probability mass function (概率质量函数)

对于一个离散型随机变量 X (可能值为 $x_1, x_2, \cdots\cdots, x_n$)，概率质量函数满足

1° $f(x_i) \geq 0$ for all $x_1, x_2, \cdots\cdots, x_n$

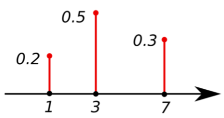
2° $\sum_{i=1}^{n} f(x_i) = 1$

3° $f(x_i) = P(X = x_i)$ for all $x_1, x_2, \cdots\cdots, x_n$

3. probability distributions (概率分布)

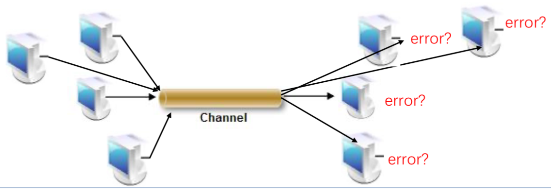
一个随机变量 X 的 probability distribution 描述了与 X 的可能值相关的概率

- The probability distribution of a random variable X is a description of the probabilities associated with the possible values of X.
- For discrete random variable, the distribution is just a list of values, e.g., {0.2,0.5,0.3}.



Probability Distributions - Example

- There is a chance that a bit transmitted through a digital transmission channel is received in error.
- Let X equals the number of bits in error in the next four bits transmitted.
- Q: What is the possible values for X?

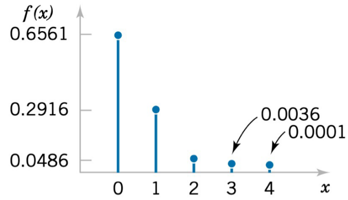


- The expert gives the following probability distribution on X.

$\boxed{P(X=0)} = 0.6561$ $\quad P(X=1)=0.2916 \quad P(X=2)=0.0486$

$P(X=3)=0.0036 \quad P(X=4)=0.0001$

Probability that no error received.



Probability distribution for bits in error

Exercise

- The sample space of a random experiment is {a,b,c,d,e,f}, and each outcome is equally likely. A random variable X is defined as follows:

outcome	a	b	c	d	e	f
x	0	0	1.5	1.5	2	3

- Q: What is the probability mass function f of X? $f(0)=\frac{1}{3}$, $f(1.5)=\frac{1}{3}$, $f(2)=\frac{1}{6}$, $f(3)=\frac{1}{6}$
- Use the probability mass function to determine:

(a) $P(X=1.5)$ $\frac{1}{3}$ (b) $P(0.5<X<2.7)$ $\frac{1}{2}$
(c) $P(X>3)$ 0 (d) $P(0 \le X < 2)$ $\frac{5}{6}$
(e) $P(X=0$ or $X=2)$ $\frac{1}{2}$

4. Cumulative distribution function (CDF) ((累积)分布函数)

1° 一个离散型随机变量X的(累积)分布函数(CDF),被记作$F(x)$

2° $F(x)=P(X\le x)=\sum_{x_i\le x} f(x_i)$

3° 性质:

- $0\le F(x)\le 1$
- If $x\le y$, then $F(x)\le F(y)$

注: 即使随机变量X只能取正整数,累积分布函数在非整数处仍有定义,例: $F(1.5)=F(0)+F(1)$

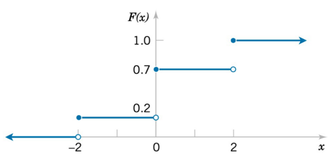
4° 可利用 cumulative distribution function 求 probability mass function

Example

$$F(x)=P(X\le x)=\sum_{x_i\le x} f(x_i)$$

- Find the probability mass function of X form the following cumulative distribution function

$$F(x)=\begin{cases} 0 & x<-2 \\ 0.2 & -2\le x<0 \\ 0.7 & 0\le x<2 \\ 1 & 2\le x \end{cases}$$



$f(-2)=0.1 \quad f(0)=0.5 \quad f(2)=0.3$

§2 Mean (期望/均值) and Variance (方差)

1. Example: investment

Maximize average return?

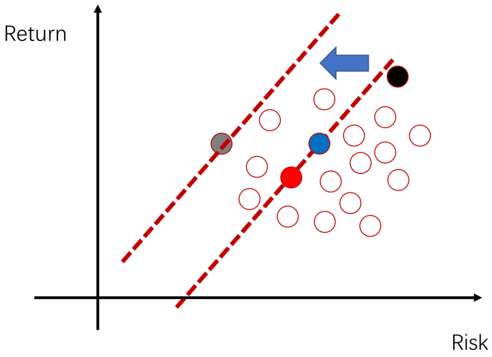
No! $\underbrace{\text{Also reduce the chance of losing too much after one investment.}}_{\text{Risk}}$

Real objective:

Maximize: Average Return - Risk

Markowitz Model

The first model you need to learn to work in finance.



Objective: Return – Risk → $a\ Return - b\ Risk$

Suppose you choose blue

When choosing grey:
same return but smaller risk

Best choice: move the line with slope 1 to left side until there is no dot at the left side of the line →

$a\ Return - b\ Risk = Z$

$Return = \frac{Z}{a} + \frac{b}{a} Risk$

$slope = \frac{b}{a}$,

moving to the left side (for Z_{max})

Each dot represents a stock (股票)

2. mean / expectation : 计算 return

$E[X] = \Sigma_x x \cdot P(X=x) = \Sigma_x x \cdot f(x)$

3. variance : 计算 risk (how far is a random variable from its mean, on average?)

$Var[X] = E[(X-E[X])^2] = \Sigma_x (x-E[X])^2 \cdot f(x)$

Game 1:
- If heads, You pay me \$1.
- If Tails, I pay you \$1.

- Return -> Mean

$$E[X] = \Sigma_x x\, P(X=x) = \Sigma_x x\, f(x) = 1\times\frac{1}{2} + (-1)\times\frac{1}{2} = 0$$

- Risk -> Variance

$$Var[X] = \Sigma_x (x - E[X])^2 f(x) = 1^2\times\frac{1}{2} + (-1)^2\times\frac{1}{2} = 1$$

3. Some useful formulas

1. Formula 1 : Linearity

$E[\Sigma_i X_i] = \Sigma_i E[X_i]$

例: An Example

Toss a coin:
- If Head, you earn 2 dollar
- If Tail, you lose 1 dollar

Suppose you toss n times, how much you will earn on average?

Events	HH⋯H	⋯	⋯	TT⋯T
Earnings: $\Sigma_i X_i$	2n	⋯	⋯	-n
Probability	$\left(\frac{1}{2}\right)^n$	⋯	⋯	$\left(\frac{1}{2}\right)^n$

Difficult to calculate directly!

An Example

Toss a coin:
- If Head, you earn 2 dollar
- If Tail, you lose 1 dollar

Suppose you toss n times, how much you will earn on average?

Linearity: $E[\Sigma_i X_i] = \Sigma_i E[X_i]$

- For each toss, you win: 2*0.5-1*0.5=0.5
- In total, you win 0.5*n.

例: Hat Check

n people go to a party and leave their hat with a hat-check person. At the end of the party, she returns hats randomly since she doesn't care about her job. Let X be the number of people who get their original hat back. What is $E[X]$?

Brute Force: $\Omega_X = \{0,1,2,\dots,n-2,n\}$.

$$p_X(n) = \frac{1}{n!}$$

$$p_X(0) = ???$$

Too hard → Use linearity!

Quick question: does it matter where you are in line?

If first in line, $P(\text{get hat back}) = \frac{1}{n}$, because there are n in total.
If last in line, $P(\text{get hat back}) = \frac{1}{n}$, because there is 1 left, and the chance it is yours is $\frac{1}{n}$.

For $i = 1,\dots,n$, let $X_i = \begin{cases} 1, & \text{if } i^{th} \text{ person got hat back} \\ 0, & \text{otherwise} \end{cases}$. Then $X = \sum_{i=1}^{n} X_i$.

We will use linearity of expectation.

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1) = P(i^{th} \text{ person got hat back}) = \frac{1}{n}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] \overset{\text{Linearity}}{=} \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

2、Formula 2

$E[g(X)] = \Sigma_x g(x) \cdot P(X=x) = \Sigma_x g(x) f(x)$

注: $E[g(x)] \neq g(E[x])$ 例外: linearity

- Toss a coin: head as 1, tail as -1.
- Then $E[X^2] = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$ ← $\boxed{g(x) = x^2}$
- But $(E[X])^2 = 0$

3、计算 variance (利用 formula 2)

$$\begin{aligned} Var[X] &= E[(X-E[X])^2] \\ &= E[X^2] - 2E[X] \cdot E[X] + (E[X])^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

e.g. Variance (Example)

Let X be the outcome of a fair 6-sided die roll. What is $Var(X)$?

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$$

$$E[X^2] = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + \cdots + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$