Lecture 17

§1 Fluid Forces

1. Fluid pressure

For static water, the pressure p at depth h is given by

$$p = wh$$

where w is the weight-density (容重) of fluid.

2. Fluid forces against horizontal plate

For a container with a horizontal base, the total force applied by the fluid to the base is

$$F = pA = whA$$

where A is the base area.

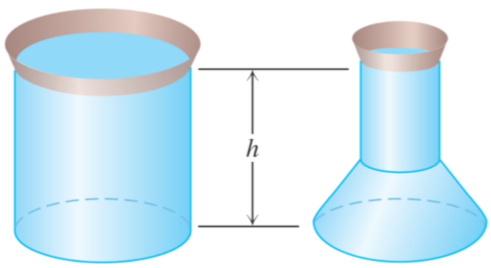


FIGURE 6.41 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

3. Fluid forces against horizontal plate

If a flat plate is submerged vertically, the pressure against it depends on the depth of the portion of the plate.

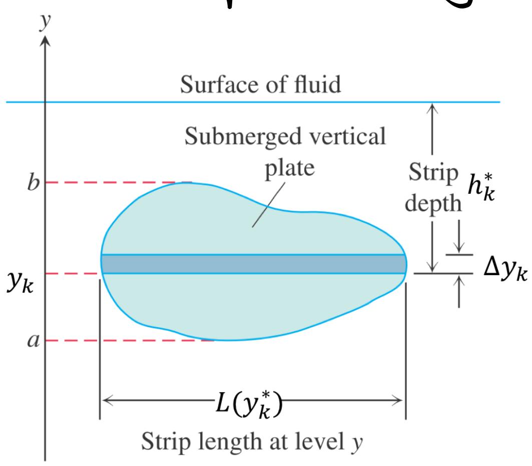
1° Devide the plate into horizontal thin slice S_k:

Width of $S_k = \Delta y_k$

Depth of $S_k = h_k^*$

Length of $S_k = L(y_k^*)$

Area of $S_k \approx L(y_k^*) \cdot \Delta y_k$



2° Force exerted on S_k:

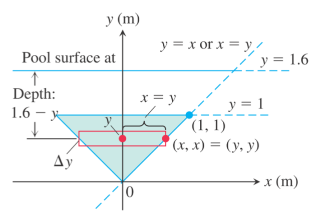
$F_k = whA \approx wh_k^* \cdot L(y_k^*) \cdot \Delta y_k$

3° Total force exerted on plate is approximately:

$F \approx \sum_{k=1}^{n} wh_k^* L(y_k)^* \cdot \Delta y_k$

$= \int_a^b wh(y) L(y)\, dy$

e.g. **EXAMPLE 6** A flat isosceles right-triangular plate with base 2 m and height 1 m is submerged vertically, base up, 0.6 m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.



$L(y) = 2y$

$w = \rho g = (1000\,kg/m^3)(9.8\,m/s^2) = 9.8 \times 10^3\,N/m^3$

$h = 1.6 - y$

$F = \int_0^1 (9.8\times10^3)(1.6-y)(2y)\, dy$

$= 9.8\times10^3 \times \left[-\frac{2}{3}y^3 + 1.6y^2\right]_0^1$

$\approx 9147\,N$

§2 Differentiation of inverse functions

1. Definition of one-to-one, onto, and bijective

Let $f: D \to Y$ be a function

1° f is one-to-one (or injective) (单射) if $f(x_1) \neq f(x_2)$ for all distinct x_1 and x_2 in D (that is, $x_1 \neq x_2$).

2° f is onto (or surjective) (满射) if, for every $y \in Y$, there exists $x \in D$ such that $f(x) = y$.

3° f is bijective (双射/一一对映) if it is both one-to-one and onto.

A bijective function is called a bijection.

e.g. Is the following function one-to-one, onto or bijective?

(1) $f: [0,1] \to R$, $f(x) = x^2$ one-to-one

(2) $f: [-1,1] \to R$, $f(x) = x^2$ none of the above

2. Consider $f: D \to \text{range}(f)$

1° $f: D \to \text{range}(f)$ is always onto (surjective) where $\text{range}(f) = \{f(x) : x \in D\}$

i.e. Every function is onto its range.

2° Therefore, $f: D \to \text{range}(f)$ is bijective if and only if it is one-to-one

3. Definition of inverse function (反函数)

Let $f: D \to \text{range}(f)$ be one-to-one (and hence bijective).

The inverse function of f is the function $f^{-1}: \text{range}(f) \to D$ defined by

$$f^{-1}(y_0) = x_0, \text{ where } f(x_0) = y_0$$

Note:

1° Inverse of f is only defined when $f: D \to \text{range}(f)$ is one-to-one

2° Since for all $x_0 \in D$ there exists $y_0 \in \text{range}(f)$ such that $f^{-1}(y_0) = x_0$, we have $\text{range}(f^{-1}) = D$

3° By definition,

$$\forall x_0 \in D, \quad (f^{-1} \circ f)(x_0) = f^{-1}(f(x_0)) = x_0$$

$$\forall y_0 \in \text{range}(f), \quad (f \circ f^{-1})(y_0) = f(f^{-1}(y_0)) = y_0$$

So $(f^{-1} \circ f)$ and $(f \circ f^{-1})$ are both identity functions

4° If f is monotonic on D, the f is one-to-one on D, so $f^{-1}: \text{range}(f) \to D$ must exist.

e.g. Find the inverse function:

(1) $f: [0,2] \to [0,4]$, $f(x) = x^2$

$f^{-1}: [0,4] \to [0,2]$, $f^{-1} = \sqrt{y}$

(2) $f: R \setminus \{0\} \to R \setminus \{0\}$, $f(x) = \frac{1}{x}$

$f^{-1} = f$

4. Facts (without proof)

1° If f is continuous and f^{-1} exists, the f^{-1} is also continuous

2° If f is continuous and its domain is an interval, then range(f^{-1}) is also an interval.

3° If f is one-to-one and continuous on an interval I, then f^{-1} is monotonic on I.

5. Theorem: Derivative rule for inverses

Let $f: I \to Y$ be bijective, where I is an interval (and $Y = \text{range}(f)$). If f is differentiable and f' is never zero on I, then f^{-1} is differentiable and

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

for every $y_0 \in Y$.

In other words, if $f(x_0) = y_0$, then

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$

Proof 1:

Let $y_0 \in Y$ and let $x_0 \in I$ be such that $f(x_0) = y_0$

For any $y_0 + h \in Y$, there is a unique $x_0 + k \in I$ such that

$$f(x_0+k) = y_0 + h \quad ①$$

Note that

$$\lim_{h\to 0} \frac{f^{-1}(y_0+h) - f^{-1}(y_0)}{h} = \lim_{h\to 0} \frac{x_0+k-x_0}{f(x_0+k)-y_0} \quad ②$$

By ①, $x_0 + k = f^{-1}(y_0+h)$. By Fact 1, f^{-1} is also continuous, so

$$\begin{aligned}\lim_{h\to 0} f^{-1}(y_0+h) &= f^{-1}(\lim_{h\to 0}(y_0+h)) \\ &= f^{-1}(y_0) \\ &= x_0\end{aligned}$$

This means that as $h \to 0$, $x_0 + k \to x_0$, so $k \to 0$

By ②,

$$\begin{aligned}(f^{-1})'(y_0) &= \lim_{k\to 0} \frac{k}{f(x_0+k)-f(x_0)} \\ &= \frac{1}{f'(x_0)} \\ &= \frac{1}{f'(f^{-1}(y_0))}\end{aligned}$$

Finally, if y_0 is an endpoint of Y, then one may consider one-sided derivatives instead.

Proof 2:

In the theorem above, if we assume the differentiability of f^{-1}, then the fomula can be easily derived from the chain rule:

$$\because \quad f^{-1}(f(x)) = x, \quad \forall x \in I$$

$$\therefore \quad (f^{-1})'(f(x)) \cdot f'(x) = 1$$

$$\therefore \quad (f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

6. Differentiation of arcsin

The sine function is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and has range $[-1, 1]$, so it has an inverse function

$$\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Let $y_0 \in (-1, 1)$ and let $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be such that

$$\sin x_0 = y_0$$

Then, by inverse differentiation

$$\arcsin'(y_0) = \frac{1}{\sin'(x_0)} = \frac{1}{\cos x_0} = \frac{1}{\sqrt{1 - \sin^2 x_0}} = \frac{1}{\sqrt{1 - y_0^2}}$$

If $y_0 = \pm 1$, then $x_0 = \pm \frac{\pi}{2}$ and so $\sin' x_0 = \cos x_0 = 0$, and the rule doesn't apply, as the denominator is 0.

Expressing using independent variable x, we have

$$\arcsin' x = \frac{1}{\sqrt{1 - x^2}} \quad \text{for all } x \in (-1, 1)$$

based on the assumption that sin has domain $(-\frac{\pi}{2}, \frac{\pi}{2})$

7. Differentiation of ln

If $\exp : R \to (0, \infty)$ is the function $\exp(x) = e^x$, then it has an inverse function $\ln : (0, \infty) \to R$ that is called the natural logarithmic function.

Let $y_0 \in (0, \infty)$. By inverse differentiation, if $\exp(x_0) = y_0$, then:

$$\ln'(y_0) = \frac{1}{\exp'(x_0)} = \frac{1}{\exp(x_0)} = \frac{1}{y_0}$$

In a different notation:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \text{for all } x \in (0, \infty)$$

§3 Natural Exponential and Logarithmic Function: Revisited

We are going to define exp and ln in an alternative way and show that they still satisfy the familiar properties.

1. Definition of ln

Define ln with domain $(0, +\infty)$ by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

2. Properties of natural logarithm

1° $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

2° $\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$

Hence, ln is differentiable, and hence continuous on $(0, \infty)$

Since $\ln' x = \frac{1}{x} > 0 \ \forall x \in (0, +\infty)$, ln is increasing on $(0, \infty)$

3° Note that

$$\ln 4 = \int_1^4 \frac{1}{t}\,dt = \int_1^2 \frac{1}{t}\,dt + \int_2^3 \frac{1}{t}\,dt + \int_3^4 \frac{1}{t}\,dt$$
$$\geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$$

By IVT, since ln is increasing, there is unique number x_0 such that $\ln(x_0) = 1$

3. Definition of exp

Define e to be the unique number in $(0, \infty)$ such that:

$$\ln(e) = 1$$