Lecture 25

§1 Improper Integrals : Type 2 : Discontinuity

1. Definition

1° If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx := \lim_{c\to b^-}\int_a^c f(x)\,dx$$

2° If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx := \lim_{c\to a^+}\int_c^b f(x)\,dx$$

3° If f is discontinuous at c, where $a<c<b$ and is continuous on $[a,b]\setminus\{c\}$, then

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided that both improper integrals on the right converge.

Remarks:

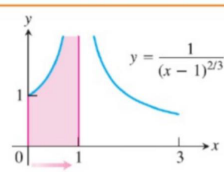
In the definition above, $\int_a^b f(x)\,dx$ may also be defined if the right-hand side is $\infty+\infty$, $-\infty-\infty$, or $a\pm\infty$ (where $a\in\mathbb{R}$)

But it is undefined for "$\infty-\infty$"

Like Type 1 improper integrals, Type 2 ones can be viewed as areas under the graph for non-negative integrands.

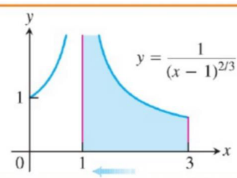
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}}$$



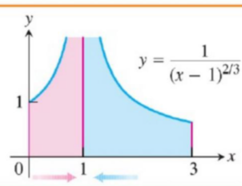
5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to 1^+}\int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



e.g. Suppose $p>0$. Find $\int_0^1 \frac{1}{x^p}\,dx$

$\lim_{x\to 0^+} \frac{1}{x^p} = \infty$

i.e. we have an essential discontinuity at $x=0$. Continuous on $(0,1]$

$$\int_a^1 \frac{1}{x^p}\,dx = \left[\frac{1}{-p+1}\cdot x^{-p+1}\right]_a^1, \quad p\neq 1$$

$$= \frac{1}{1-p}\left(1-\frac{1}{a^{p-1}}\right)$$

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \frac{1}{1-p}\left(1-\frac{1}{a^{p-1}}\right)$$

$$= \begin{cases} \infty, & \text{if } p>1 \\ \frac{1}{1-p}, & \text{if } 0<p<1 \end{cases}$$

If $p=1$: $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}(\ln 1 - \ln a) = \infty$

Hence:

$$\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{converges to } \frac{1}{1-p}, & \text{if } 0<p<1 \\ \text{diverges to } \infty, & \text{if } p>1 \end{cases}$$

Combined with Type 1 improper integrals, this means that:

- If $0<p<1$: $\int_0^1 \frac{1}{x^p}dx$ converges, $\int_1^\infty \frac{1}{x^p}dx$ diverges.
- If $p>1$: $\int_0^1 \frac{1}{x^p}dx$ diverges, $\int_1^\infty \frac{1}{x^p}dx$ converges.
- If $p=1$: Both $\int_0^1 \frac{1}{x^p}dx$ and $\int_1^\infty \frac{1}{x^p}dx$ diverge.

e.g. Find $\int_0^3 \frac{1}{x-1}\,dx$

$\frac{1}{x-1}$ is continuous on $[0,3]\setminus\{1\}$

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{a\to 1^-}(\ln|a-1| - \ln|-1|) = -\infty$$

$$\int_1^3 \frac{1}{x-1}\,dx = \lim_{a\to 1^+}(\ln|2| - \ln|a-1|) = \infty$$

So $\int_1^3 \frac{1}{x-1}\,dx$ is undefined according to definition

§2 Improper Integrals: Convergence Tests

1. **Theorem (Direct Comparison Test)**

> Suppose that $a \in \mathbb{R}$, and suppose that f and g are continuous functions on $[a, \infty)$. If there exists $c \in [a, \infty)$ such that $0 \le f(x) \le g(x)$ for all $x \in [c, \infty)$, then
>
> (i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
>
> (ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

2. Idea (not a complete proof of (i))

Since $\int_a^\infty g(x)\,dx = \int_a^c g(x)\,dx + \int_c^\infty g(x)\,dx$

then $\int_a^\infty g(x)\,dx$ converges if and only if $\int_c^\infty g(x)\,dx$ converges.

Similarly $\int_a^\infty f(x)\,dx$ converges if and only if $\int_c^\infty f(x)\,dx$ converges.

Let $F(t) := \int_c^t f(x)\,dx$ and $G(t) := \int_c^t g(x)\,dx$, for $t \ge c$

Since $0 \le f(x) \le g(x)$ for all $x \in [c, \infty)$, we have

$$0 \le F(t) \le G(t), \quad \forall t \in [c, \infty)$$

Assume that $\int_c^\infty g(x)\,dx$ converges. Then for some $L \in \mathbb{R}$:

$$\lim_{t \to \infty} G(t) = L$$

Then $F(t)$ is bounded above by L and has a "least upper bound", say A

Then $\lim_{t\to\infty} F(t) = A$, so $\int_c^\infty f(x)\,dx = A$ (convergent)

e.g. Determine whether $\int_0^\infty e^{-x^2}\,dx$ converges or not.

Since $e^{x^2} > e^x > 0$ for all $x \in [1, \infty)$, we have

$$0 < e^{-x^2} < e^{-x}, \quad \forall x \in [1, \infty)$$

$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}(-e^{-b} + e^0) = 1$ (converges)

so, $\int_0^\infty e^{-x^2}\,dx$ also converges.

e.g. Consider $\int_2^\infty \frac{\sqrt[3]{x^7+2}}{x^3\ln x}\,dx$

$$\frac{\sqrt[3]{x^7+2}}{x^3\ln x} \approx \frac{x^{\frac{7}{3}}}{x^2\cdot x\ln x} = \frac{x^{\frac{1}{3}}}{x\ln x} > \frac{1}{x\ln x} > 0 \text{ for } x>1$$

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{u}\,du$$
$$= \lim_{b\to\infty}\left(\ln u\Big|_{\ln 2}^{\ln b}\right)$$
$$= \lim_{b\to\infty}\left(\ln(\ln b)-\ln(\ln 2)\right)$$
$$= \infty$$

So $\int_2^\infty \frac{1}{x\ln x}$ diverges.

Hence $\int_2^\infty \frac{\sqrt[3]{x^7+2}}{x^3\ln x}\,dx$ is divergent.

3. Theorem (Limit Comparison Test)

Suppose that $a\in\mathbb{R}$, and suppose that f and g are positive continuous functions on $[a,\infty)$. If

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$$

for some $L\in\mathbb{R}_+ := (0,\infty)$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or diverge.

4. Proof

Since $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$, $\exists M\in\mathbb{R}$ such that $\forall x\in[M,\infty)$

$$\frac{L}{2} \le \frac{f(x)}{g(x)} \le \frac{3L}{2}$$

$$0 < \frac{L}{2}g(x) \le f(x) \le \frac{3L}{2}g(x)$$

If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty \frac{L}{2}g(x)\,dx$ converges, so

$\int_a^\infty g(x)\,dx = \frac{2}{L}\int_a^\infty \frac{L}{2}g(x)\,dx$ also converges.

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty \frac{3L}{2}g(x)\,dx$ diverges, so

$\int_a^\infty g(x)\,dx = \frac{2}{3L}\int_a^\infty \frac{3L}{2}g(x)\,dx$ also diverges.

e.g. Consider $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$

(1) $0 < \frac{1-e^{-x}}{x} < \frac{1}{x}$

$\int_1^\infty \frac{1}{x}dx$ diverges, but no conclusion can be drawn by direct comparison

(2) $\lim_{x\to\infty} \frac{(1-e^{-x})/x}{1/x} = \lim_{x\to\infty}(1-e^{-x}) = 1$

By limit comparison test (comparing with $\int_1^\infty \frac{1}{x}dx$)

$\int_1^\infty \frac{1-e^{-x}}{x}dx$ diverges.

Remark:

If f and g are positive and continuous on $[a,\infty)$, and $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$, then

(1) If $L=0$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

(2) If $L=\infty$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

§3 First-order Differential Equation

1. Definition of first-order differential equation

A first-order differential equation is an equation of the form

$$\frac{dy}{dx} = F(x,y)$$

where y is the dependent variable and x is the independent variable.

A solution to a first-order differential equation is a function $y=y(x)$ defined on some interval I such that

$$y'(x) = F(x, y(x)), \quad \forall x\in I$$

A general solution to a first-order differential equation is the collection of all solutions to the equation.

A first-order initial value problem (IVP) is a first-order differential equation with an initial value condition:

$$\begin{cases} \frac{dy}{dx} = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

A particular solution is a solution that satisfies the IVP.

2. Slope field

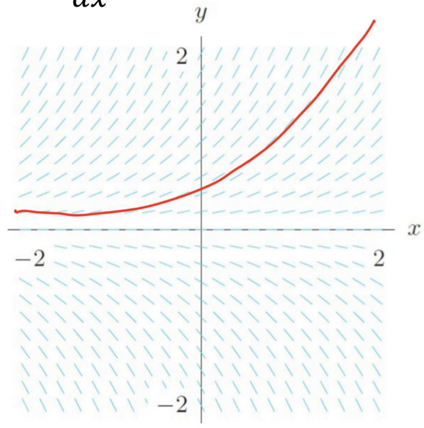
Suppose $y=f(x)$ is a solution satisfying $f(x_0)=y_0$ and $f'(x_0)=F(x_0,y_0)$

1° Plot many points on the plane

2° For each plotted point (x_0,y_0), draw a short line segment with slope $F(x_0,y_0)$

This method generates a diagram called a slope field.

E.g. A slope field of $\frac{dy}{dx}=y$.



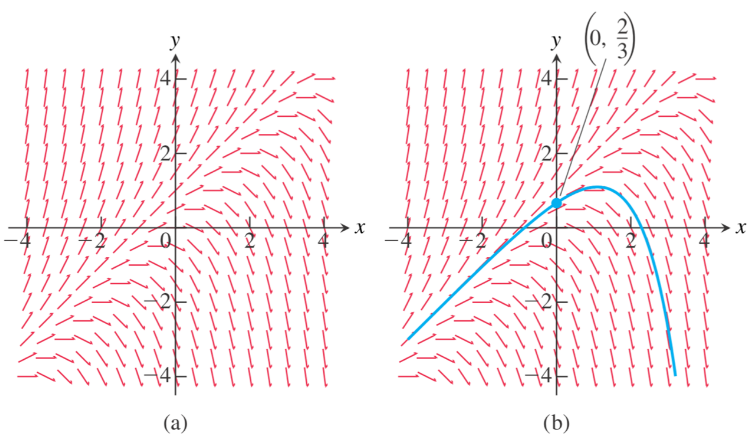


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).