Lecture 21

§1 Exercise : convex sets

例: Convex sets --- exercise 1

- Let C be a convex set, with $x_1, x_2, x_3 \in C$, and let $\theta_1, \theta_2, \theta_3$ satisfy $\theta_i \geq 0$, $\theta_1 + \theta_2 + \theta_3 = 1$. Show that $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$.

We will show that $y = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$. At least one of the θ_i is not equal to one; without loss of generality we can assume that $\theta_1 \neq 1$. Then we can write

$$y = \theta_1 x_1 + (1 - \theta_1)(\mu_2 x_2 + \mu_3 x_3)$$

where $\mu_2 = \frac{\theta_2}{1-\theta_1}$ and $\mu_3 = \frac{\theta_3}{1-\theta_1}$. Note that $\mu_2, \mu_3 \geq 0$ and

$$\mu_2 + \mu_3 = \frac{\theta_2 + \theta_3}{1 - \theta_1} = \frac{1 - \theta_1}{1 - \theta_1} = 1.$$

Since C is convex and $x_2, x_3 \in C$, we conclude that $\mu_2 x_2 + \mu_3 x_3 \in C$. Since this point and x_1 are in C, y $\in C$.

例: Convex sets --- exercise 2

- Consider a discrete random variable X with pmf $f(x_i) = p_i$, $i = 1, \ldots, n$. Let P be the set that contains all such well-defined probability distributions, i.e.,
$$P = \{(p_1, \ldots, p_n) | p_1 + \cdots + p_n = 1, p_i \geq 0\}.$$
- Is P a convex set? Why?

P is a convex set.

To show this, consider any $(p_1, \ldots, p_n) \in P$ and $(q_1, \ldots, q_n) \in P$. Take any $\theta \in [0,1]$. Then

$$\theta(p_1, \ldots, p_n) + (1-\theta)(q_1, \ldots, q_n) = (\theta p_1 + (1-\theta)q_1, \ldots, \theta p_n + (1-\theta)q_n).$$

Note that

$$\theta p_1 + (1-\theta)q_1 + \cdots + \theta p_n + (1-\theta)q_n = 1$$

and

$$\theta p_i + (1-\theta)q_i \geq 0.$$

Therefore $(\theta p_1 + (1-\theta)q_1, \ldots, \theta p_n + (1-\theta)q_n) \in P$, and hence P is convex.

例: Convex sets --- exercise 3

- Continuing exercise 2, consider the following condition on the random variable X
$$\alpha \leq \mathbb{E}g(X) \leq \beta.$$

Show that the subset of P that satisfies this condition is a convex set.

Recall that $\mathbb{E}g(X) = p_1 g(x_1) + \cdots + p_n g(x_n)$. Therefore the subset of P we consider is

$$\{(p_1, \ldots, p_n) | \alpha \leq p_1 g(x_1) + \cdots + p_n g(x_n) \leq \beta, p_1 + \cdots + p_n = 1, p_i \geq 0\}.$$

This is an intersection of two convex sets P and

$$\{(p_1, \ldots, p_n) | \alpha \leq p_1 g(x_1) + \cdots + p_n g(x_n) \leq \beta\}.$$

Therefore the result follows.

§2 Exercise: convex functions

例:

Convex functions --- exercise 1

Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, and $a,\ b \in \mathbf{dom}\, f$ with $a < b$.

Show:

(a) $f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$ for all $x \in [a, b]$.

(b) $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$ for all $x \in [a, b]$.

(c) $f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$.

(a) Let $\lambda = (b-x)/(b-a)$

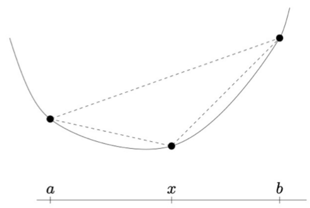
$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b) \Rightarrow f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

(b) $$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \xrightarrow{\text{subtracting f(a) from both sides}} \frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$$

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \xrightarrow{\text{subtracting f(b) from both sides}} \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

(c) $$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \xrightarrow{x \to a} f'(a) \leq \frac{f(b)-f(a)}{b-a}$$

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x} \xrightarrow{x \to b} \frac{f(b)-f(a)}{b-a} \leq f'(b)$$



例:

Convex functions --- exercise 2

A family of concave utility functions. For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\, u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbf{R}_{++}$).

Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

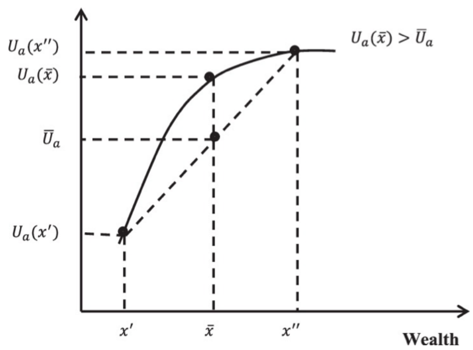
These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation*.

Solution: $u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0.$

$$u'_\alpha(x) = x^{\alpha-1},$$

$$u''_\alpha(x) = (\alpha - 1)x^{\alpha-2}. \leq 0$$

By Second-order condition (SOC), the utility function is concave



例: Convex functions --- exercise 3

- Show the following function is convex

$f(x) = \sum_{i=1}^{r} |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

Solution: Write f as

$$f(x) = \sum_{i=1}^{r} |x|_{[i]} = \max_{1 \le i_1 < i_2 < \cdots < i_r \le n} |x_{i_1}| + \cdots + |x_{i_r}|$$

which is the pointwise maximum of $n!/(r!(n-r)!)$ convex functions.

§3 Exercise: quasi convex

例: Quasi-convex - Exercise

Example 3.30 *Length of a vector.* We define the *length* of $x \in \mathbf{R}^n$ as the largest index of a nonzero component, *i.e.*,

$$f(x) = \max\{i \mid x_i \neq 0\}.$$

(We define the length of the zero vector to be zero.) This function is quasiconvex on $\mathbf{R}^n$, since its sublevel sets are subspaces:

$$f(x) \le \alpha \iff x_i = 0 \text{ for } i = \lfloor \alpha \rfloor + 1, \ldots, n.$$

例: Quasi-convex - Exercise

Example 3.32 *Linear-fractional function.* The function

$$f(x) = \frac{a^T x + b}{c^T x + d},$$

with $\mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$, is quasiconvex, and quasiconcave, *i.e.*, quasilinear. Its α-sublevel set is

$$\begin{aligned} S_\alpha &= \{x \mid c^T x + d > 0,\ (a^T x + b)/(c^T x + d) \le \alpha\} \\ &= \{x \mid c^T x + d > 0,\ a^T x + b \le \alpha(c^T x + d)\}, \end{aligned}$$

which is convex, since it is the intersection of an open halfspace and a closed halfspace. (The same method can be used to show its superlevel sets are convex.)