Lecture 3 (10.4, 10.5, 10.6)

比较审敛法，绝对收敛，交错数列

§1 Comparison Test (針对正项级数)

1. Direct comparison test (直接审敛法)

> **Theorem ((Direct) Comparison Test)**
>
> *Let $\sum a_n$ and $\sum b_n$ be series. Suppose that there exists an integer N such that $a_n \geq b_n \geq 0$ for all n satisfying $n \geq N$.*
>
> (i) *If $\sum a_n$ converges, then $\sum b_n$ converges.*
>
> (ii) *If $\sum b_n$ diverges, then $\sum a_n$ diverges.*

证明: (i)
- Suppose $\sum_{n=N}^{\infty} a_n = L$. For $n \geqslant N$, let $S_n := \sum_{k=N}^{n} b_k = b_N + \cdots + b_n$
- Then for all $n \geqslant N$,
  $0 \leqslant S_n = \sum_{k=N}^{n} b_k \leqslant \sum_{k=N}^{n} a_k \leqslant L$
- Hence $\{S_n\}$ is bounded, implying that $\sum_{n=N}^{\infty} b_n$ converges by the Monotonic Sequence Theorem.
- (ii) equivalent to (i)

e.g. Determine the convergence of $\sum_{n=8}^{\infty} \frac{1}{3^n + n^{\frac{1}{3}}}$

Sol: $\forall\, n \geqslant 8$, $0 < \frac{1}{3^n + n^{\frac{1}{3}}} < \frac{1}{3^n}$

Since $\sum_{n=0}^{\infty} (\frac{1}{3})^n$ converges (geometric series with $|r|<1$)

Series converges.

e.g. Determine the convergence of $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Since $0 < \frac{1}{n} < \frac{\ln n}{n}$, $\forall\, n \geqslant 3$, and $\sum \frac{1}{n}$ diverges.

So $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

2. Limit Comparison Test

> **Theorem (Limit Comparison Test)**
>
> *Let $\sum a_n$ and $\sum b_n$ be series. Suppose that there exists an integer N such that $a_n > 0$ and $b_n > 0$ for all n satisfying $n \geq N$.*
>
> (i) *If $\lim_{n\to\infty} (a_n/b_n) = L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.* ($L \in \mathbb{R}$)
>
> (ii) *If $\lim_{n\to\infty} (a_n/b_n) = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*
>
> (iii) *If $\lim_{n\to\infty} (a_n/b_n) = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

证明: (ii) Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} < \varepsilon < 1$, $\lim_{n\to\infty} a_n < \lim_{n\to\infty} \varepsilon b_n < \lim_{n\to\infty} b_n$

Since $\sum b_n$ converges,

$\sum a_n$ converges.

e.g. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{5/4}}$ converges or not?

Sol: Compare with $\frac{1}{n^{4.5/4}} = \frac{1}{n^{9/8}}$

$\frac{\ln n}{n^{5/4}} / \frac{1}{n^{9/8}} = \frac{\ln n}{n^{\frac{1}{8}}}$

Since $\lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{8}}} = \lim_{n\to\infty} \frac{1/n}{\frac{1}{8} n^{-\frac{7}{8}}} = \lim_{n\to\infty} 8 \frac{1}{n^{\frac{1}{8}}} = 0$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{8}}}$ converges (p-series with $p = \frac{9}{8}$)

$\sum_n \frac{\ln n}{n^{5/4}}$ converges

* $\sum_n \frac{\ln n}{n^p}$ converges for $p > 1$

compare $\frac{\ln n}{n^p}$ with $\frac{1}{n^q}$ where $1 < q < p$

e.g. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ converges or not?

Sol: Compare $\sin\frac{1}{n}$ with $\frac{1}{n}$

Since $\lim_{n\to\infty} \sin\frac{1}{n} / \frac{1}{n} = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\sum_n \sin\frac{1}{n}$ diverges

§2 Absolute Convergence (绝对收敛)

1. 定义

> **Definition**
> A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

2. 性质

> Theorem (Absolute Convergence Test)
> If $\sum |a_n|$ converges, then $\sum a_n$ converges. In other words, if $\sum a_n$ converges absolutely, then it converges.

	$\sum \lvert a_n \rvert$	$\sum a_n$	例
绝对收敛	收敛	收敛	$\sum (-1)^{n+1} \cdot \frac{1}{n^2}$
条件收敛	发散	收敛	$\sum (-1)^{n+1} \cdot \frac{1}{n}$

证明: Since $a_n + |a_n| = \begin{cases} 0 & , \text{ if } a_n < 0 \\ 2|a_n| & , \text{ if } a_n > 0 \end{cases}$

$0 \le a_n + |a_n| \le 2|a_n|, \ \forall n$

If $\sum |a_n|$ converges, then $\sum (a_n + |a_n|)$ converges by comparison test.

This means $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ also converges.

* 逆定理为否命题

§3 Ratio Test (比值审敛法)

Theorem (Ratio Test)

Let $\sum a_n$ be a series, and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \in \mathbb{R}\cup\{\infty\}.$$

Then:

(i) *if $L<1$, the series converges absolutely.*
(ii) *if $L>1$ or $L=\infty$, the series diverges.*
(iii) *if $L=1$, the test is inconclusive.*

证明: (ii) If $L>1$ or $L=\infty$, then $\exists N$ s.t. $\forall n\geq N$, $\left|\frac{a_{n+1}}{a_n}\right|>1$.

i.e. $|a_{n+1}|>|a_n|$, so $|a_N|<|a_{N+1}|<|a_{N+2}|<\cdots\cdots$

This means that $\lim_{n\to\infty} a_n\neq 0$, so $\sum a_n$ diverges.

(i) If $L<1$, let $r\in(L,1)$. Then $\exists N$ s.t. $\forall n\geq N$, $\left|\frac{a_{n+1}}{a_n}\right|<r$.

This means $|a_{N+1}|<r|a_N|$

$|a_{N+2}|<r|a_{N+1}|<r(r|a_N|)=r^2|a_N|$

$\cdots\cdots$

$|a_{N+k}|<\cdots\cdots<r^k|a_N|$

Since $0<|a_{N+k}|\leq r^k|a_N|$ $\forall k\geq 0$,

and $\sum_{k=0}^{\infty}|a_N|\cdot r^k$ converges $(0\leq L<r<1)$, we have

$\sum_{k=0}^{\infty}|a_{N+k}|=\sum_{n=N+k}^{\infty}|a_n|$ converges.

e.g. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Since $\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{n+1}{n+1}\cdot\frac{n^n}{(n+1)^n}=\left(\frac{n}{n+1}\right)^n=\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}<1$,

$\sum_n\frac{n!}{n^n}$ converges

e.g. $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$

Since $\frac{2^{n+1}+5}{3^{n+1}}\cdot\frac{3^n}{2^n+5}=\frac{2^{n+1}+5}{3(2^n+5)}=\frac{1+5\cdot2^{-n-1}}{3(2^{-1}+5\cdot2^{-n-1})}\to\frac{2}{3}<1$ as $n\to\infty$

$\sum_n\frac{2^n+5}{3^n}$ converges

§4 Root Test (根值审敛法)

Theorem (Root Test)

Let $\sum a_n$ be a series, and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = L \in \mathbb{R}\cup\{\infty\}.$$

Then:

(i) *if $L<1$, the series converges absolutely.*
(ii) *if $L>1$ or $L=\infty$, the series diverges.*
(iii) *if $L=1$, the test is inconclusive.*

证明：(ii) If $L>1$ or $L=\infty$, then $\exists N$ s.t. $\forall n \geq N$, $\sqrt[n]{|a_n|}>1$.

so $|a_n|>1^n=1$

so $\lim_{n\to\infty} a_n \neq 0$

so $\sum_n a_n$ diverges.

(i) If $L<1$, let $r\in(L,1)$. Then $\exists N$ s.t. $\forall n\geq N$, $\sqrt[n]{|a_n|}<r$

so $|a_n|<r^n$

Since $0\leq|a_n|<r^n$, $\forall n\geq N$, $\sum_{n=N}^{\infty} r^n$ converges

so $\sum|a_n|$ converges.

e.g. $\sum_{n=1}^{\infty} a_n$, where $a_n=\begin{cases} n/2^n, & \text{if } n \text{ is odd} \\ 1/2^n, & \text{if } n \text{ is even}\end{cases}$

Since $\sqrt[n]{|a_n|}=\begin{cases}\frac{\sqrt[n]{n}}{2}, & \text{if } n \text{ is odd} \\ \frac{1}{2}, & \text{if } n \text{ is even}\end{cases}$

$\frac{1}{2}\leq\sqrt[n]{|a_n|}\leq\frac{\sqrt[n]{n}}{2}$, $\forall n$

Since $\frac{\sqrt[n]{n}}{2}\to\frac{1}{2}$, by Sandwich theorem, $\sqrt[n]{|a_n|}\to\frac{1}{2}<1$.

so $\sum a_n$ converges.

e.g. $\sum_{n=1}^{\infty}\frac{2^n}{n^3}$

$\sqrt[n]{|a_n|}=\frac{2}{(\sqrt[n]{n})^3}\to 2$ as $n\to\infty$, so series diverges.

§5 Alternating Series (交错级数)

1. 定义

Definition

An alternating series is a series of the form $\sum(-1)^{n+1}u_n$, where $u_n>0$ for all n.

注：起始 index 不一定为 1（第一项可为负项）

2. 莱布尼茨定理

10.6.15

THEOREM ~~15~~—The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n\geq u_{n+1}$ for all $n\geq N$, for some integer N.
3. $u_n\to 0$.

* Fact：若 $\{a_1,a_3,a_5\cdots\cdots\}$ 与 $\{a_2,a_4,a_6\cdots\cdots\}$ 均收敛于 L，则 $\lim_{n\to\infty} a_n=L$

* 若交错级数由负项开始，可用 -1 乘以级数，再使用定理.

证明: For notational simplicity, assume $N=1$

Let S_n be the partial sum. Consider $b_k := S_{2k}$

- For any $k \geq 1$, $b_k = S_{2k} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2k-1} - u_{2k})$,
  so $\{b_k\}$ is nondecreasing, since $u_{2i-1} - u_{2i} \geq 0$, $\forall i$.
- $\{b_k\}$ is bounded, since
  $$0 \leq u_1 - (u_2 - u_3) - (u_4 - u_5) \cdots\cdots - (u_{2k-2} - u_{2k-1}) - u_{2k}$$
  $$\leq u_1$$
- By Monotonic Sequence Theorem, $\{b_k\} = \{S_{2k}\}$ converges,
  say $S_{2k} \to L$.
- Consider odd sequence $\{S_{2k-1}\}$. Then
  $$S_{2k-1} = S_{2k} + u_{2k} \text{ (negative)}$$
  $$\lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} u_{2k} = L + 0 = L.$$
- So $\lim_{n\to\infty} S_n = L$,
  so series converges.

e.g. Alternating p-series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$, $p>0$

$\{\frac{1}{n^p}\}$ positive and $\lim_{n\to\infty} \frac{1}{n^p} = 0$, $\frac{1}{n^p}$ is nonincreasing

so the series converges.

e.g. $\sum (-1)^{n+1} \frac{n^2}{n^3+1}$

$f(x) := \frac{x^2}{x^3+1}$

- $a_n = \frac{n^2}{n^3+1} > 0$ and $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$
- $f'(x) = \frac{2x(x^3+1) - x^2 \cdot 3x^2}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2} < 0$ for $x \geq 2$
  so a_n decreases for $n \geq 2$
- so the series converges.