Lecture 20

§1 Transformation of multivariate distributions

1. Theorem: Transformation of multivariate distributions

若 $Y_i=g_i(x_1,x_2,\cdots,x_n)$，$g_i$ 为满足以下条件的一系列函数

1° 给定一组 $y_1,y_2,\cdots,y_n$，可以通过 equations $y_i=g_i(x_1,x_2,\cdots,x_n)$ 的 inverse transformations $x_i=h_i(y_1,y_2,\cdots,y_n)$ 唯一求解出一组 $x_1,x_2,\cdots,x_n$.

即由 X's 到 Y's 的 transformation 是 one-to-one correspondence (一一对应) 的.

(每个 g_i，可以由 $x_1,\cdots,x_n$ 唯一确定一组 $y_1,\cdots,y_n$，且由不同 $x_1,\cdots,x_n$ 确定的 $y_1,\cdots,y_n$ 不同)

2° 所有 g_i 均在任意 $(x_1,x_2,\cdots,x_n)$ 处有连续偏导，且 $n\times n$ 的 Jacobian determinant 不为0，即

$$J_0(x_1,x_2,\cdots,x_n)=\begin{vmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n}\end{vmatrix}\neq 0 \quad \text{at all points } (x_1,x_2,\cdots,x_n)$$

*有时上述雅可比行列式 $J_0(x_1,x_2,\cdots,x_n)$ 的计算较为复杂，可以考虑 inverse transformation 的 Jacobian determinant $J(y_1,y_2,\cdots,y_n)$. 即要求

所有 h_i 均在任意 $(y_1,y_2,\cdots,y_n)$ 处有连续偏导，且 $n\times n$ 的 Jacobian determinant 不为0，即

$$J(y_1,y_2,\cdots,y_n)=\begin{vmatrix}\frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n}\\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n}\end{vmatrix}\neq 0 \quad \text{at all points } (y_1,y_2,\cdots,y_n)$$

注：

$$J_0(x_1,x_2,\cdots,x_n)^{-1}=\begin{vmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n}\end{vmatrix}^{-1}=\begin{vmatrix}\frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n}\\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n}\end{vmatrix}=J(y_1,y_2,\cdots,y_n)$$

若满足上述条件，则 $Y_1,Y_2,\cdots,Y_n$ 的 joint pdf 为

$$f_Y(y_1,y_2,\cdots,y_n)=f_X(x_1,x_2,\cdots,x_n)\cdot|J_0(x_1,x_2,\cdots,x_n)|^{-1}$$

或 $$f_Y(y_1,y_2,\cdots,y_n)=f_X(x_1,x_2,\cdots,x_n)\cdot|J(y_1,y_2,\cdots,y_n)|$$

其中 $x_i=h_i(y_1,y_2,\cdots,y_n)$ for $i=1,2,\cdots,n$

例1: 若随机变量 X_1,X_2 有连续型联合分布，分布函数为

$$f_X(x_1,x_2)=\begin{cases}\frac{1}{2}(x_1+x_2)e^{-x_1-x_2}, & x_1>0,\ x_2>0\\ 0, & \text{otherwise}\end{cases}$$

令 $Y_1=X_1+X_2$，$Y_2=X_1-X_2$，求 $f_Y(y_1,y_2)$.

显然，transformation $Y_1=X_1+X_2$，$Y_2=X_1-X_2$ 为一一对应：

Inverse transformation $X_1=\frac{Y_1+Y_2}{2}$，$X_2=\frac{Y_1-Y_2}{2}$.

则 Jacobian determinant 为

$$J(y_1,y_2)=\begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{vmatrix}=\begin{vmatrix}\frac12 & \frac12\\ \frac12 & -\frac12\end{vmatrix}=-\frac12\neq 0$$

因此 Joint pdf of Y_1, Y_2 为

$$f_Y(y_1,y_2)=f_X(x_1,x_2)\cdot|J|=\frac12 y_1 e^{-y_1}\cdot\left|-\frac12\right|=\frac14 y_1 e^{-y_1}$$ （此处要把 x_1, x_2 替换为 y_1, y_2）

with support

$$x_1>0,\ x_2>0 \iff y_1>0,\ -y_1<y_2<y_1$$

因此，

$$f_Y(y_1,y_2)=\begin{cases}\frac14 y_1 e^{-y_1}, & y_1>0,\ -y_1<y_2<y_1\\ 0, & \text{otherwise}\end{cases}$$

例2：若 X, Y, Z independently follow $Exp(1)$，$U=X+Y$，$V=X+Z$，$W=Y+Z$，求 Joint pdf of U, V, W

$f(x,y,z)=e^{-x}e^{-y}\cdot e^{-z}$，$x>0, y>0, z>0$

① considering one-to-one correspondence

$$X=\frac12(U+V-W)$$
$$Y=\frac12(U+W-V)$$
$$Z=\frac12(V+W-U)$$

② considering Jacobian determinant $\neq 0$

$$J(x,y,z)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}1&1&0\\1&0&1\\0&1&1\end{vmatrix}=-2\neq 0$$

$$\Rightarrow f(u,v,w)=f(x,y,z)\cdot|J(x,y,z)|^{-1}$$
$$=e^{-(x+y+z)}\cdot\frac12$$
$$=\frac12 e^{-\frac12(u+v+w)},\ u>0, v>0, w>0$$

例3：Let $X\sim\Gamma(\alpha,\lambda)$ and $Y\sim\Gamma(\beta,\lambda)$ be two independent gamma random variables. Consider the transformation $U=\frac{X}{X+Y}$，$V=X+Y$. Find the pdf of U, V.

$$f_{XY}(x,y)=\begin{cases}\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot x^{\alpha-1}y^{\beta-1}e^{-\lambda(x+y)}, & x>0, y>0\\ 0, & \text{otherwise}\end{cases}$$

① considering one-to-one correspondence

$$X=U\cdot V,\quad Y=V(1-U)$$

② considering Jacobian determinant $\neq 0$

The support of U, V is given by

$$\begin{cases}x>0\\ y>0\end{cases}\iff\begin{cases}uv>0\\ v(1-u)>0\end{cases}\iff\begin{cases}0<u<1\\ v>0\end{cases}$$

$$\frac{\partial x}{\partial u}=v,\quad \frac{\partial x}{\partial v}=u,\quad \frac{\partial y}{\partial u}=-v,\quad \frac{\partial y}{\partial v}=1-u$$

$$J(u,v)=\begin{vmatrix}v & u\\ -v & 1-u\end{vmatrix}=v\neq 0$$

$$\Rightarrow f_{UV}(u,v) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot x^{\alpha-1} y^{\beta-1} e^{-\lambda(x+y)} \cdot |v|$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot (uv)^{\alpha-1} [v(1-u)]^{\beta-1} e^{-\lambda v} \cdot v$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot u^{\alpha-1} (1-u)^{\beta-1} \cdot v^{\alpha+\beta-1} \cdot e^{-\lambda v}$$

$$= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \right] \cdot \left[\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} \cdot v^{\alpha+\beta-1} \cdot e^{-\lambda v} \right], \quad 0<u<1,\ v>0.$$

Therefore U and V are independent, and $U = \frac{X}{X+Y} \sim Beta(\alpha,\beta)$, $V = X+Y \sim \Gamma(\alpha+\beta, \lambda)$.

注：若题目仅要求得出 V 的分布，一种解法是构造一个 U，使得 $(X,Y)\to(U,V)$ 一一对应．其他方法将在下节课提供．

例4: 若随机变量 X,Y 有连续型联合分布，分布函数为

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x^2 y^2}, & x>1,\ y>1 \\ 0, & \text{otherwise} \end{cases}$$

令 $U = XY$, $W = \frac{X}{Y}$, ① 求 $f_{U,W}(u,w)$ ② 求 Marginal pdfs of u, w

① $X = \sqrt{UW}$, $Y = \sqrt{\frac{U}{W}}$

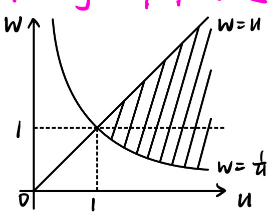
$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{2x}{y}$$

$$f_{U,W}(u,w) = f(x,y)\cdot |J(x,y)|^{-1}$$
$$= \frac{1}{x^2y^2} \cdot \left|-\frac{2x}{y}\right|^{-1}$$
$$= \frac{1}{2x^3y}$$
$$= \frac{1}{2u^2w}$$

由 $X = \sqrt{uw} > 1$, $Y = \sqrt{\frac{u}{w}} > 1$ 可解得:

$u>1$, $\frac{1}{u} < w < u$

② (求 marginal pdf 时建议先画出积分区域)



$$f_U(u) = \int_{\frac{1}{u}}^{u} f(u,w)\,du = \int_{\frac{1}{u}}^{u} \frac{1}{2u^2w}\,dw = \frac{1}{2u^2} \ln w \Big|_{\frac{1}{u}}^{u} = \frac{\ln u}{u^2}, \quad u>1$$

$$u>1,\ \frac{1}{u}<w<u \Rightarrow u > \max(w, \tfrac{1}{w})$$

$$\Rightarrow \begin{cases} u > \frac{1}{w} & 0<w<1 \\ u > w & w \geq 1 \end{cases}$$

① $0<w<1$, $f_W(w) = \int_{\frac{1}{w}}^{\infty} \frac{1}{2u^2w}\,du = \frac{1}{2}$

② $w>1$, $f_W(w) = \int_{w}^{\infty} \frac{1}{2u^2w}\,du = \frac{1}{2w^2}$