

Lecture 7

§1 关于级数的 basic facts (接上)

1. Fact 9 (Root test)

Let $\alpha = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ (也可换成 $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$),

- (i) if $\alpha < 1$, then $\sum |a_n|$ converges
- (ii) if $\alpha > 1$, then $\sum |a_n|$ diverges
- (iii) if $\alpha = 1$, inconclusive (root test doesn't apply)

证明:

① proof of (i):

(思路: 取 $\alpha < \beta < 1$, 则 $|a_n|^{\frac{1}{n}} < \beta$, \forall large n . 则 $|a_n| < \beta^n$, 由 comparison test 可证得)

Take $\beta \in (\alpha, 1)$

$$\therefore \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \alpha < \beta$$

$$\therefore \exists N, \text{ s.t. } |a_n|^{\frac{1}{n}} < \beta, \forall n \geq N$$

$$\therefore |a_n| < \beta^n, \forall n \geq N$$

$$\therefore \sum \beta^n \text{ (geometric series) converges } (0 < \beta < 1)$$

$$\therefore \text{By C.T. } \sum |a_n| \text{ converges.}$$

② proof of (ii)

(思路: 可以取 $|a_n|^{\frac{1}{n}}$ 的一个子序列, 其极限大于 1, 因此当 $n \rightarrow \infty$ 时, 存在 $a_n \rightarrow \infty \neq 0$)

$$\therefore \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \alpha$$

$$\therefore \exists \text{ subseq } \{|a_{n_k}|^{\frac{1}{n_k}}\}_{k=1}^{\infty}, \text{ s.t. } \lim_{k \rightarrow \infty} |a_{n_k}|^{\frac{1}{n_k}} = \alpha$$

$$\therefore \forall \varepsilon, \exists K, \text{ s.t. } \alpha - \varepsilon < |a_{n_k}|^{\frac{1}{n_k}} < \alpha + \varepsilon, \text{ if } k \geq K$$

$$\text{Take } \varepsilon = \frac{\alpha - 1}{2}, \text{ then } \alpha - \varepsilon = \frac{\alpha + 1}{2} > 1$$

$$\therefore |a_{n_k}| > (\alpha - \varepsilon)^{n_k}, \forall k \geq K$$

$$\therefore \lim_{k \rightarrow \infty} (\alpha - \varepsilon)^{n_k} = \infty$$

$$\therefore \lim_{k \rightarrow \infty} a_{n_k} = \infty$$

$$\therefore \lim_{n \rightarrow \infty} a_n \neq 0$$

$$\therefore \sum a_n \text{ diverges}$$

③ proof of (iii)

$\sum \frac{1}{n}$ divergent

$$\alpha = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} \frac{1}{e^{\ln n \cdot \frac{1}{n}}} = 1$$

$\sum \frac{1}{n^2}$ convergent

$$\alpha = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{2}{n}}} = 1$$

2. Fact 10 (Ratio test)

- (i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum |a_n|$ converges
- (ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum |a_n|$ diverges
- (iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, inconclusive

证明:

① proof of (i)

(思路: 取 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$, 则 $|a_{n+p}| < \alpha^p |a_n|$, 由比较审敛法证明)

Take $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$, then $\left| \frac{a_{n+1}}{a_n} \right| < \alpha$, $\forall n \geq \text{some } N$

$$\therefore |a_{n+1}| < \alpha |a_n|$$

$$\therefore |a_{N+1}| < \alpha |a_N|$$

$$|a_{N+2}| < \alpha |a_{N+1}|$$

$$|a_{N+p}| < \alpha |a_{N+p-1}|$$

$$\therefore |a_{N+p}| < \alpha^p |a_N|$$

$$\therefore \sum_{p=1}^{\infty} |a_{N+p}| < \sum_{p=1}^{\infty} \alpha^p \cdot |a_N|$$

$$\therefore \sum_{n=N+1}^{\infty} |a_n| \text{ converges}$$

$$\therefore \sum |a_n| \text{ converges}$$

② proof of (ii)

(思路: 若 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, 则 n 足够大时, $\left| \frac{a_{n+1}}{a_n} \right| > 1$, 因此 $|a_n| \uparrow$, 因此 $|a_n|$ 的极限不为 0, 级数发散)

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

$$\therefore \left| \frac{a_{n+1}}{a_n} \right| > 1, \forall n \geq \text{some } N$$

$$\therefore |a_n| \text{ is increasing}$$

$$\therefore \lim_{n \rightarrow \infty} |a_n| = l \text{ exists (may be } +\infty)$$

$$\text{Then } l \neq 0, \lim_{n \rightarrow \infty} a_n \neq 0$$

$$\therefore \text{By divergent test, } \sum a_n \text{ diverges}$$

例 1: 判断级数 $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$ 的敛散性

(先考虑 ratio test)

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{\frac{1}{3^k}}{\frac{1}{2^k}} = \left(\frac{2}{3}\right)^k, & n = 2k-1 \\ \frac{\frac{1}{2^{k+1}}}{\frac{1}{3^k}} = \frac{1}{2} \left(\frac{3}{2}\right)^k, & n = 2k \end{cases}$$

$$\text{Then } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty, \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0$$

(Ratio test fails)

(再考虑 root test)

$$|a_n|^{\frac{1}{n}} = \begin{cases} \frac{1}{2^{\frac{1}{2k-1}}} & , n=2k-1 \\ \frac{1}{3^{\frac{1}{2k}}} & , n=2k \end{cases}$$

$$\therefore E = \{x \mid \exists \text{ subseq } |a_{n_k}|^{\frac{1}{n_k}} \rightarrow x \text{ as } k \rightarrow \infty\}$$

$$= \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\}$$

$$\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \sup E = \frac{1}{\sqrt{2}} = \alpha < 1$$

$\therefore \sum |a_n|$ converges

3. **Theorem** (根值审敛法较比值审敛法适用范围更广)

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

证明:

$$(\text{先证明 } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}})$$

$$\text{Let } \beta = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \geq 0$$

Case 1: $\beta = 0$

$$\text{then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

Case 2: $\beta = \infty$

Case 3: $0 < \beta < \infty$

take $\gamma \in (0, \beta)$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \beta > \gamma$$

$$\therefore \exists N, \text{ s.t. } \left| \frac{a_{n+1}}{a_n} \right| > \gamma \text{ if } n \geq N$$

$$\therefore |a_{N+p}| > \gamma^p |a_N|, \forall p \geq 1$$

$$\therefore |a_k| > \gamma^{k-N} |a_N|, \forall k \geq N+1$$

$$\therefore |a_k|^{\frac{1}{k}} > \gamma^{\frac{k-N}{k}} |a_N|^{\frac{1}{k}}, \forall k \geq N+1$$

$$\therefore \lim_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} \geq \gamma, \forall \gamma \in (0, \beta)$$

$$\therefore \lim_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} \geq \beta$$

$$(\text{再证明 } \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|)$$

$$\text{Let } \alpha = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Case 1: $\alpha = \infty$

$$\text{then } \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$$

Case 2: $0 \leq \alpha < \infty$

take $\delta \in (\alpha, \infty)$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < \delta$$

$$\therefore \exists N, \text{ s.t. } \left| \frac{a_{n+1}}{a_n} \right| < \delta, \text{ if } n \geq N$$

$$\therefore |a_{N+p}| < \delta^p |a_N|, \forall p \geq 1$$

$$\therefore |a_k| < \delta^{k-N} |a_N|, \forall k \geq N+1$$

$$\therefore |a_k|^{\frac{1}{k}} < \delta^{\frac{k-N}{k}} |a_N|^{\frac{1}{k}}, \forall k \geq N+1$$

$$\therefore \overline{\lim}_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} \leq \delta, \forall \delta > \alpha$$

$$\therefore \overline{\lim}_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} \leq \alpha$$

Q.E.D.

Corollary:

If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists (may be $+\infty$), then $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ also exists and $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$