Lecture 5: Continuous Random Variable

§1 Probability density function (PDF) (概率密度方程)

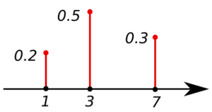
1、Probability density function

1° 适用于 continuous random variable

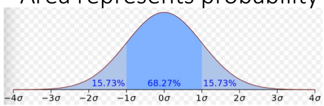
2° $f(\omega)$ 表示每个 outcome $\omega \in \Omega$ 的 probability density

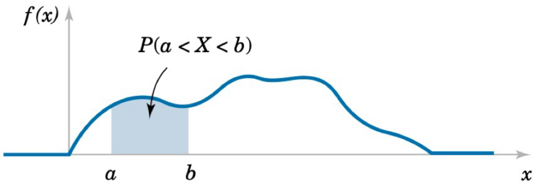
3° Area 表示 probability

- Discrete:
  - ✓ Probability mass function.
  - ✓ f(ω): gives the probability for each outcome ω ∈ Ω



- Continuous
  - ✓ Probability density function.
  - ✓ f(ω): gives the probability density for each outcome ω ∈ Ω
  - ✓ Area represents probability





Probability determined from the area under *f(x)*

2、Formal definition

对于一个 continuous random variable X，probability density function $f(\cdot)$ 满足：

1° $f(x) \geq 0$ for all x

2° $\int_{-\infty}^{\infty} f(x)\,dx = 1$

3° $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b.

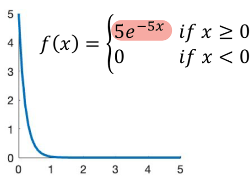
注：① 对于不在 sample space 中的 x 值，$f(x) = 0$

② 对任意 x，$P(X = x) = 0$

③ $f(x)$ 的值越大，表示 x 附近的值更可能被观测到.

例：Example: interarrival time

- Imagine in a town where buses do not operate on a fixed schedule, but they arrive according to some specific processes.
- Let X be the time (in hour) that you have to wait if you show up at a bus stop at an arbitrary time.
- From experience, the random variable X has density function



* $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Area $= \int_0^{\infty} \lambda \cdot e^{-\lambda x}\,dx = -\int_0^{\infty} e^{-\lambda x}\,d(-\lambda x) = -e^{-\lambda x}\Big|_0^{\infty} = \lim_{a \to 0} e^{-\lambda a} - \lim_{b \to \infty} e^{-\lambda b} = 1$

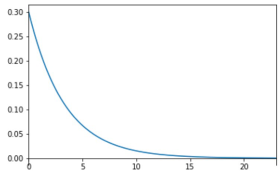
例: Example: lifetime

- Let X be the lifetime of a lightbulb (in years).
- From experience, the random variable X has density function:

$$f(x)=\begin{cases}0.3e^{-0.3x} & if\ x\ge 0\\ 0 & if\ x<0\end{cases}$$

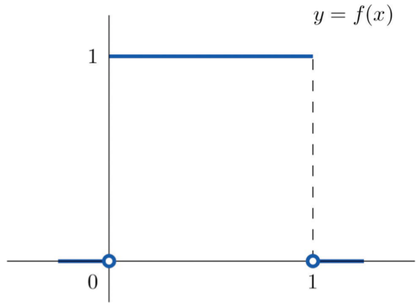
- What is the probability that this lightbulb lasts between 3 and 5 years?

$P(3\le X\le 5)=\int_3^5 0.3\cdot e^{-0.3x}dx$
$=-e^{-0.3x}\Big|_3^5=e^{-0.9}-e^{-1.5}$



3. Uniform distribution (均匀分布)

- Let X follows a uniform distribution on [0,1].
  - f(X)= $\begin{cases}1, & 0\le X\le 1\\ 0, & otherwise\end{cases}$
  - P(a ≤ X ≤ b)= $\int_a^b f(x)dx$
- For example
  - P(0.5 ≤ X ≤ 0.7) = 0.2
  - P (-1 ≤ X ≤ 0.7) = 0.7
  - P(0.5 ≤ X ≤ 1.7) = 0.5



4. Cumulative distribution functions (CDF)

对于连续型随机变量，CDF被定义为：

$F(x)=P(X\le x)=\int_{-\infty}^{x}f(u)du$

并满足：

1° $0\le F(x)\le 1$

2° If $x\le y$, then $F(x)\le F(y)$

3° PDF $\overset{积分}{\underset{求导}{\rightleftharpoons}}$ CDF

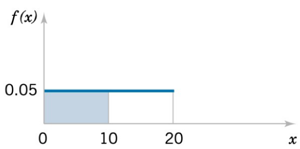
$F(x)=P(X\le x)=\int_{-\infty}^{x}f(u)du$

integration

probability density function (PDF) ⇄ cumulative distribution function (CDF)

differentiation

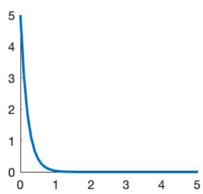
$F'(x)=f(x)$



例:
- X is the interarrival time (in hour) that you have to wait if you show up at a bus stop at an arbitrary time.

$$f(x)=\begin{cases}5e^{-5x} & if\ x\ge 0\\ 0 & if\ x<0\end{cases}$$

- CDF? $F(x)=\int_{-\infty}^{x}f(x)dx=\int_0^x 5e^{-5x}dx$
$=-e^{-5x}\Big|_0^x$
$=1-e^{-5x}$



§2 Mean and Variance

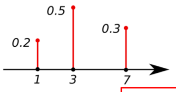
1. Mean

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

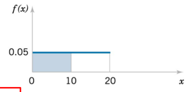
2. Variance

$$Var[X] = \int_{-\infty}^{\infty} (x - E(x))^2 \cdot f(x)\,dx$$

- Discrete:
  - ✓ Probability mass function.



- Continuous
  - ✓ Probability density function.



$\boxed{\textbf{Summation} \leftrightarrow \textbf{Integration}}$

Discrete:

- Mean

  E[X]= $\sum x\, f(x)$

- Variance

  Var[X]= $\sum (x - \mathrm{E}[X])^2\, f(x)$

Continuous:

- Mean

  E[X]= $\int_{-\infty}^{\infty} x f(x) dx$

- Variance

  Var[X]=$\int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx$

例:

Calculate E[X^2]

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx$$

- Let X follows a uniform distribution on [0,1].
  - f(X)=$\begin{cases} 1, & 0 \le X \le 1 \\ 0, & otherwise \end{cases}$
  - P(a ≤ X ≤ b)= $\int_a^b f(x)dx$
- E[X^2]= $\int_0^1 x^2 f(x)dx = \int_0^1 x^2 dx = 1/3$
- Var[X] = 1/3-1/2*1/2=1/12