Lecture 23

类似于 UMA confidence bound，可以基于同样的 intuition 定义 UMA confidence set.

§1 Uniformly most accurate (UMA) confidence set

1. Definition: Uniformly most accurate (UMA) confidence set

一个 confidence region $\hat{C}(X)$ 被称为 θ 的 UMA confidence set at level $(1-\alpha)$，若

① $\hat{C}(X)$ 是 θ 的一个 $(1-\alpha)$-confidence set，即

$\inf_{\theta\in\Theta} P_\theta(\theta\in\hat{C}(X)) \geq 1-\alpha$ （当任意 θ 为真时，$\hat{C}(X)$ 包含住 θ 的概率(关于 X)恒大于等于 $1-\alpha$）

② $\hat{C}(X)$ 恒 minimize 错误估计 θ 的概率，即

对 $\forall\theta\in\Theta$ 与 $\forall\theta'\neq\theta$，有

$$P_\theta(\theta'\in\hat{C}(X)) \leq P_\theta(\theta'\in\hat{C}_1(X))$$

其中 $\hat{C}_1(X)$ 为任意其他 $(1-\alpha)$-confidence set

（考虑所有的 θ 的 $(1-\alpha)$-confidence sets，当任意 θ 为真时，对于任意 θ 的错误估计 $\theta'\neq\theta$，$\hat{C}(X)$ 包含住这一错误估计 θ' 的概率(关于 X) 恒是最小的.）

2. Theorem: UMA confidence set 与 UMP test

若 ① 对于 $\forall\theta'\in\Theta$，存在 a level α UMP test $\phi_{\theta'}$ for testing

$$H_0: \theta=\theta' \quad \text{v.s.} \quad H_1: \theta\neq\theta'$$ （是一个关于 X 的 rejection region）

② $\phi_{\theta'}$ 的 acceptance region 为：

$$A_{\theta'} = \{X\in\Omega: \phi_{\theta'}(X)\neq 1\}, \quad \theta'\in\Theta$$

则 ① UMA confidence set $\hat{C}(X)$ 存在

② $\hat{C}(X)$ 满足：

$$\hat{C}(X) = \{\hat{\theta}\in\Theta: X\in A_{\hat{\theta}}\}$$ （是一个关于 $\hat{\theta}$ 的区间）

（$\hat{C}(X)$ 相当于将 acceptance region 的 X fix，令 θ unfix，得到一个关于 θ 的 set）

证明:

Proof. Notice that $A_{\theta'}$ is the acceptance region of a level α test, therefore,

$$\mathbb{P}_{\theta'}\left(\theta'\in\hat{C}(X)\right) = \mathbb{P}_{\theta'}\left(X\in A_{\theta'}\right) = 1-\mathbb{E}_{\theta'}\phi_{\theta'} \geq 1-\alpha.$$

Hence $\hat{C}(X)$ indeed has confidence level $(1-\alpha)$. Meanwhile, for any other confidence region $\hat{C}_1(X)$ with confidence level $(1-\alpha)$, and for arbitrary $\theta_0\neq\theta'$,

$$\boxed{\phi_{1,\theta'} \triangleq 1-\mathbb{1}\Big(X\in A_{1,\theta'}\Big), \quad \text{with } A_{1,\theta'} = \Big\{x: \theta'\in\hat{C}_1(x)\Big\}}$$

is the test inverting from the confidence region $\hat{C}_1(X)$ for testing (3.1), which is of level α since

$$\mathbb{E}_{\theta'}\phi_{1,\theta'} = 1-\mathbb{P}_{\theta'}\Big(X\in A_{1,\theta'}\Big) = 1-\mathbb{P}_{\theta'}\Big(\theta'\in\hat{C}_1(x)\Big) \leq \alpha.$$

Because $\phi_{\theta'}$ is a UMP test of (3.1), so $\phi_{\theta'}$ has smaller type-II error compare to $\phi_{1,\theta'}$, which means

$$\mathbb{P}_{\theta'}\Big(\theta_0\in\hat{C}(X)\Big) = 1-\mathbb{E}_{\theta'}\phi_{\theta_0} \leq 1-\mathbb{E}_{\theta'}\phi_{1,\theta_0} = \mathbb{P}_{\theta'}\Big(\theta_0\in\hat{C}_1(X)\Big),$$

thus we conclude that $\hat{C}(X)$ is a UMA confidence set. □

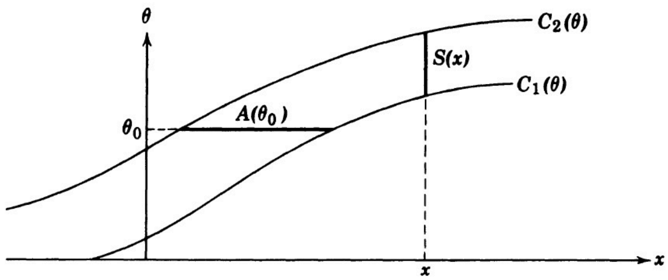

Figure 1: An illustrational sketch of inverting hypothesis testing and confidence interval. Picture from Lehmann, Romano and Casella (1986).

例 1:

• *Example* 3.3 (♣ **UMA Confidence Set for Exponential Distribution Location Parameter**). Let $\{X_1, \cdots, X_n\}$ be a random sample from the exponential distribution $\text{Exp}(a, b)$,

$$f_{X_1}(x) = \frac{1}{b} \exp\Big(-\frac{x-a}{b}\Big) \cdot \mathbb{1}(x \geq a),$$

with both (a, b) being unknown. We are seeking a UMA confidence set for the location parameter a.

PART 1: 准备工作
证明 $X_{(1)}$ 与 $Y:=\Sigma(X_i-X_{(1)})$ 独立
1.1 证明 $X_{(1)}$ 是 a 的 sufficient statistic

Answer. Despite the fact of not knowing b, we fix b. Then the likelihood is

$$f(x|a, b) = \frac{e^{na/b}}{b^n} \exp\Big(-\frac{\sum_{i=1}^n x_i}{b}\Big) \cdot \mathbb{1}(x_{(1)} \geq a),$$

so by Factorization theorem, we have $X_{(1)}$ is a sufficient statistic of a. Since

1.2 求出 $X_{(1)}$ 的分布
证明 $X_{(1)}$ 是 a 的 complete statistic

$$\mathbb{P}(X_{(1)} \geq x) = \Big(\mathbb{P}(X_i \geq x)\Big)^n = \exp\Big(-\frac{n(x-a)}{b}\Big) \cdot \mathbb{1}(x > a) + \mathbb{1}(x \leq a),$$

hence $X_{(1)} \sim \text{Exp}(a, b/n)$. For arbitrary function φ satisfy $\mathbb{E}_a \varphi(X_{(1)}) = 0$ for all $a \in \mathbb{R}$, we have

$$
\begin{aligned}
& \int_a^\infty \varphi(x) \cdot \frac{n}{b} \exp\Big(-\frac{n(x-a)}{b}\Big) dx = 0 \quad \text{for all } a \in \mathbb{R}, \\
\Leftrightarrow \quad & \int_a^\infty \varphi(x) \cdot \exp\Big(-\frac{nx}{b}\Big) dx = 0 \quad \text{for all } a \in \mathbb{R}, \\
\Leftrightarrow \quad & \varphi(a) \cdot \exp\Big(-\frac{na}{b}\Big) = 0 \quad \text{for all } a \in \mathbb{R} \text{ except a measure zero set.} \\
\Leftrightarrow \quad & \varphi(a) = 0 \quad \text{for all } a \in \mathbb{R} \text{ except a measure zero set.} \\
\Leftrightarrow \quad & \mathbb{P}\Big(\varphi(X_{(1)}) = 0\Big) = 1.
\end{aligned}
$$

1.3 证明Y是a的 ancillary statistic (见Assignment 2/4.1)

Hence $X_{(1)}$ is also a complete statistic of a for every given b. Meanwhile, $Y \triangleq \sum_{i=1}^n [X_i - X_{(1)}]$ is ancillary of a, so $X_{(1)}$ is independent of Y according to Basu's theorem. Now for the hypothesis testing problem

PART 2:

求出 $H_0: a=a_0$ vs. $H_1: a<a_0$ 的 UMP

2.1 研究 subhypothesis, 写出 likelihood ratio $\lambda(X)$

$$H_0 : a = a_0, \quad v.s. \quad H_1 : a = a_1, \quad \text{for some } a_1 < a_0. \tag{3.2}$$

Notice the likelihood ratio is given by

$$\boxed{\lambda(x) = \frac{f(x|a_1, b)}{f(x|a_0, b)} = \begin{cases} +\infty & \text{when } a_1 < x_{(1)} \le a_0, \\ e^{n(a_1-a_0)/b} & \text{when } a_0 < x_{(1)}. \end{cases}}$$

2.2. 利用 $\lambda(X)$ 的形式, 求出 level-α test 的 power 的上界

Thus, for arbitrart level α test ϕ of (3.2), which leads to

$$\begin{aligned} \beta(a_1) &= \mathbb{E}_{a_1}\phi = \int_{a_1}^{\infty} \phi \cdot f(x|a_1, b)dx \\ &= \int_{a_1}^{a_0} \underbrace{\phi}_{\le 1} \cdot f(x|a_1, b)dx + \int_{a_0}^{\infty} \phi \cdot \boxed{\frac{f(x|a_1, b)}{f(x|a_0, b)}} \cdot f(x|a_0, b)dx \\ &\le \underbrace{\int_{a_1}^{a_0} f(x|a_1, b)dx}_{} + e^{n(a_1-a_0)/b} \underbrace{\int_{a_0}^{\infty} \phi \cdot f(x|a_0, b)dx}_{=\alpha} \\ &\le 1 - e^{n(a_1-a_0)/b} + \alpha \cdot e^{n(a_1-a_0)/b} \triangleq \beta_1^*. \end{aligned}$$

$= \lambda(X)$ 在 $X > a_0$ 时为 constant $e^{n(a_1-a_0)/b}$

$= 1 - \int_{a_0}^{+\infty} \frac{f(x|a_1,b)}{f(x|a_0,b)} \cdot f(x|a_0,b)\, dx$

$= 1 - e^{n(a_1-a_0)/b}$

2.3 研究一个 test ϕ_0, 确保是 level-α

Meanwhile, consider

$$\boxed{\phi_0 = \mathbb{1}\left(\frac{X_{(1)} - a_0}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} \le C_1\right) + \mathbb{1}\left(\frac{X_{(1)} - a_0}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} > C_2\right)}.$$

where C_1, C_2 are determined by $\mathbb{E}_{a_0}\phi_0 = \alpha$, i.e.,

$$\begin{aligned} \alpha &= \underbrace{\mathbb{P}_{a_0}\left(\frac{X_{(1)} - a_0}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} < C_1\right)}_{} + \underbrace{\mathbb{P}_{a_0}\left(\frac{X_{(1)} - a_0}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} > C_2\right)}_{} \\ &= 1 - \mathbb{E}\exp\left(-\frac{nC_1Y}{b}\right) + \mathbb{E}\exp\left(-\frac{nC_2Y}{b}\right). \end{aligned}$$

$E[P_{a_0}(X_{(1)} \le C_1Y + a_0 | Y)]$ $E[P_{a_0}(X_{(1)} > C_1Y + a_0 | Y)]$

*Since Y actually follow $Y \sim (b/2) \cdot \chi^2_{2n-2}$, so

$$\alpha = 1 - (1 + nC_1)^{-(n-1)} + (1 + nC_2)^{-(n-1)}.$$ (*) (可令 $C_1=0$. C_2 s.t. (*) holds)

2.4 证明 ϕ_0 的 power 取到 upper bound, 从而为 UMP

We directly calculate the the power of ϕ_0,

$$\begin{aligned} \beta_{\phi_0}(a_1) = \mathbb{E}_{a_1}\phi_0 &= 1 - \mathbb{E}\left[\mathbb{P}_{a_1}\left(\frac{X_{(1)} - a_1}{Y} > C_1 + \frac{a_0 - a_1}{Y}\Big|Y\right)\right] \\ &\quad + \mathbb{E}\left[\mathbb{P}_{a_1}\left(\frac{X_{(1)} - a_1}{Y} > C_2 + \frac{a_0 - a_1}{Y}\Big|Y\right)\right] \\ &= 1 - e^{n(a_1-a_0)/b} \underbrace{\left(\mathbb{E}\exp\left(-\frac{nC_1Y}{b}\right) - \mathbb{E}\exp\left(-\frac{nC_2Y}{b}\right)\right)}_{=1-\alpha} = \beta_1^*, \end{aligned}$$

(attain the upper bound)

2.5 说明 ϕ_0 与 a_1 无关

which is ϕ_0 attains the upper bound of the power, hence ϕ_0 is a UMP for testing (3.2), and since ϕ_0 does not depend on the specific value of a_1, so ϕ_0 is also a UMP for testing

$$H_0 : a = a_0, \quad v.s. \quad H_1 : a < a_0.$$

PART 3:

求出 $H_0: a=a_0$ v.s. $H_1: a>a_0$ 的 UMP

3.1 研究 subhypothesis，写出 likelihood ratio $\lambda(x)$

Similarly, for the hypothesis testing problem

$$H_0 : a = a_0, \quad v.s. \quad H_1 : a = a_1, \quad \text{for some } a_1 > a_0. \tag{3.3}$$

Notice the likelihood ratio is given by

$$\boxed{\lambda(x) = \frac{f(x|a_1, b)}{f(x|a_0, b)} = \begin{cases} e^{n(a_1-a_0)/b} & \text{when } a_1 < x_{(1)}, \\ 0 & \text{when } a_0 < x_{(1)} \le a_1. \end{cases}}$$

3.2 利用 NP Lemma，写出 UMP ϕ_1 的形式

According to Neyman Pearson Lemma, the UMP test of (3.3) is given by

$$\phi_1 = \mathbb{1}\left(\lambda(X) > k\right) + \gamma \cdot \mathbb{1}\left(\lambda(X) = k\right)$$

where the critical value k and tunning parameter γ are determined by

$$\mathbb{E}_{a_0}\phi_1 = \alpha = \mathbb{P}_{a_0}\left(\lambda(X) > k\right) + \gamma \cdot \mathbb{P}_{a_0}\left(\lambda(X) = k\right).$$

3.3 讨论 n 的取值，求出 ϕ_1 的 power

Therefore, if $\exp(-n(a_1 - a_0)/b) \ge \alpha$, $\longrightarrow P(a_1 < X_{(1)})$

$$\phi_1 = \alpha e^{\frac{n(a_1-a_0)}{b}} \cdot \mathbb{1}\left(X_{(1)} > a_1\right), \quad \Rightarrow \quad \beta_{\phi_1}(a_1) = \alpha e^{\frac{n(a_1-a_0)}{b}},$$

if $\exp(-n(a_1 - a_0)/b) < \alpha$, $\longrightarrow P(a_0 < X_{(1)} \le a_1)$

$$\phi_1 = \mathbb{1}\left(X_{(1)} > a_1\right) + \frac{\alpha - e^{-\frac{n(a_1-a_0)}{b}}}{1 - e^{-\frac{n(a_1-a_0)}{b}}} \cdot \mathbb{1}\left(a_0 < X_{(1)} \le a_1\right), \quad \Rightarrow \quad \beta_{\phi_1}(a_1) = 1.$$

3.4 研究 ϕ_0，找出一组 C_1, C_2，使得 ϕ_0 为 size-α

Define two constants

$$k_1 = \mathbb{E}\exp\left(-\frac{nC_1Y}{b}\right), \quad k_2 = \mathbb{E}\exp\left(-\frac{nC_2Y}{b}\right),$$

and we seek for k_1, k_2 satisfying

$$k_1 - k_2 = 1 - \alpha, \quad k_1 \ge \alpha, \quad k_2 \ge \alpha.$$

For instance, we can pick $k_1 = 1$ and $k_2 = \alpha$, which corresponding to $C_1 = 0$ and $(1 + nC_2)^{-(n-1)} = \alpha$. We now directly calculate the size of ϕ_0,

$$\mathbb{E}_{a_0}\phi_0 = \mathbb{P}_{a_0}\left(\frac{X_{(1)} - a_0}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} < 0\right) + \mathbb{P}_{a_0}\left(\frac{X_{(1)} - a_0}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} > C_2\right) = \alpha,$$

3.5 求出 ϕ_0 的 power，说明 ϕ_0 与 ϕ_1 power 相同，因此 ϕ_0 为 UMP

and the power of ϕ_0,

$$\mathbb{E}_{a_1}\phi_0 = \mathbb{P}_{a_1}\left(\frac{n(X_{(1)} - a_1)}{b} > \frac{nC_2Y}{b} + \frac{n(a_0 - a_1)}{b}\right)$$

$$
\begin{aligned}
&= e^{\frac{n(a_1-a_0)}{b}} \cdot \mathbb{E}\exp\left(-\frac{nC_2Y}{b}\right) \cdot \mathbb{1}\left(e^{\frac{n(a_0-a_1)}{b}} \geq \alpha\right) + \mathbb{1}\left(e^{\frac{n(a_0-a_1)}{b}} < \alpha\right) \\
&= \alpha \cdot e^{\frac{n(a_1-a_0)}{b}} \cdot \mathbb{1}\left(e^{\frac{n(a_0-a_1)}{b}} \geq \alpha\right) + \mathbb{1}\left(e^{\frac{n(a_0-a_1)}{b}} < \alpha\right),
\end{aligned}
$$

which means ϕ_0 has the same power as ϕ_1, hence ϕ_0 is a UMP for testing (3.3), and since ϕ_0 does not depend on the specific value of a_1, so ϕ_0 is also a UMP for testing

3.6 说明 ϕ_0 与 a_1 无关

$$
H_0 : a = a_0, \quad v.s. \quad H_1 : a > a_0.
$$

PART 4:
利用 duality 将 UMP 转化为 UMA

Overall, we conclude that ϕ_0 is a UMP test for testing

$$
H_0 : a = a_0, \quad v.s. \quad H_1 : a \neq a_0, \tag{3.4}
$$

Notice that, for each specific a_0, the acceptence of this UMP test ϕ_0 is

$$
A_{a_0} = \left\{ x \in \Omega : 0 \leq \frac{x_{(1)} - a_0}{\sum_{i=1}^n \left[x_i - x_{(1)}\right]} \leq \frac{1}{n}\left[\alpha^{-\frac{1}{n-1}} - 1\right] \right\},
$$

事实上，满足条件的 C_1, C_2 仅有 $C_1 = 0$, $C_2 = \frac{1}{n}[\alpha^{-\frac{1}{n-1}} - 1]$

accordingly, the confidence set obtained using the duality between testing and interval estimation, i.e., Theorem.??, is given by

$$
\boxed{\hat{C}(X) = \left\{ a \in \mathbb{R} : 0 \leq \frac{X_{(1)} - a}{\sum_{i=1}^n \left[X_i - X_{(1)}\right]} \leq \frac{1}{n}\left[\alpha^{-\frac{1}{n-1}} - 1\right] \right\},}
$$

and accordingly to theorem.3.2, $\hat{C}(X)$ is a confidence set for the location parameter a. □