Lecture 26

§1 Solving First-order Differential Equations: Separable Equation

1. Definition of separable equation

A differential equation of the form

$$\frac{dy}{dx} = g(x)f(y)$$

is said to be separable.

e.g. $y' = e^x e^y$ is separable.

$y' = e^x + e^y$ is not separable

2. Solve separable equation

Suppose that $y' = g(x)f(y)$ and f is not the zero function.

Then

$$\int \frac{1}{f(y)} \cdot y' \, dx = \int g(x) \, dx$$

$$\int \frac{1}{f(y)} \, dy = \int g(x) \, dx$$

After integration, sometimes we may solve y explicitly in terms of x.

e.g. Solve $y' = e^{x+y}$

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

$$\int \frac{1}{e^y} \, dy = \int e^x \, dx$$

$$-e^{-y} + C_1 = e^x + C_2$$

$$e^{-y} = -e^x + C$$

$$y = -\ln(C - e^x)$$

e.g. Solve $y' = x^2 y$

1° Note that $y=0$ (zero function) is a solution

2° Suppose $y \neq 0$ for some x. Then

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln|y| = \frac{1}{3}x^3 + K$$

$$|y| = e^K \cdot e^{\frac{1}{3}x^3}$$

$$y = \pm e^K \cdot e^{\frac{1}{3}x^3}$$

$$y = C \cdot e^{\frac{1}{3}x^3}, \quad C = \pm e^K$$

Since the zero function is also a solution, the general solution is:

$$y = C \cdot e^{\frac{1}{3}x^3}, \quad C \in R$$

e.g. Solve the IVP: $y' = (x/y)^2$, $y(0) = 2$

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{1}{3}y^3 + C_1 = \frac{1}{3}x^3 + C_2$$

$$y = (x^3 + C)^{\frac{1}{3}}$$

Since $y(0) = 2$, we have

$$2 = (0 + C)^{\frac{1}{3}}$$

$$C = 8$$

$\therefore$ Particular solution is $y = (x^3 + 8)^{\frac{1}{3}}$

§2 Solving First-order Differential Equations: Linear Equation

1. Definition of linear equation

A first-order linear equation is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

If $\frac{dy}{dx} = F(x,y)$, then

$$F(x,y) = -P(x)y + Q(x)$$

which is linear in y.

2. Solve first-order linear equation

1° Consider solving $y' + \frac{y}{x} = 2$, for $x > 0$

Multiplying both sides by x yields:

$$xy' + y = 2x$$
$$(xy)' = 2x$$
$$xy = x^2 + C$$
$$y = x + \frac{C}{x}, \quad C \in R$$

This example was solved by realizing that the left-hand side is the derivative of some function $v(x) \cdot y$.

2° Consider solving $y' + P(x) y = Q(x)$

Consider multiplying both sides by $v(x)$, (v is not the zero function):

$$v(x) y' + v(x) \cdot P(x) y = v(x) \cdot Q(x)$$

If we can choose $v(x)$ such that the left-hand side is $(v(x) y)'$, then

$$(v(x) \cdot y)' = v(x) Q(x)$$
$$\int (v(x) \cdot y)' dx = \int v(x) Q(x) dx$$
$$v(x) \cdot y = \int v(x) Q(x) dx$$
$$y = \frac{1}{v(x)} \int v(x) Q(x) dx$$

Any function $v(x)$ (not the zero function) that satisfies

$$\frac{d(v(x) \cdot y)}{dx} = v(x) \left(\frac{dy}{dx} + P(x) y \right)$$

is called an integrating factor

3° Finding an integrating factor

$$(v(x) y)' = v(x) y' + v(x) P(x) y$$

$$v(x)y' + v'(x)y = v(x)y' + v(x)P(x)y$$
$$v'(x)y = v(x)P(x)y$$

Assume $y \neq 0$:

$$v'(x) = v(x)P(x)$$

Put another way, with $v := v(x)$

$$\frac{dv}{dx} = v \cdot P(x)$$
$$\int \frac{1}{v} dv = \int P(x)dx$$
$$\ln|v| = \int P(x)dx$$
$$v = \pm e^{\int P(x)dx}$$

Since we only need one integrating factor, we can choose "+" in the equation above, and pick $\int P(x)dx$ to be any particular antiderivative of $P(x)$.

$$v = e^{\int P(x)dx}$$

4° In summary, to solve $y' + P(x)y = Q(x)$:

① Let $v(x) = e^{\int P(x)dx}$ (an integrating factor), where $\int P(x)dx$ is any antiderivative of $P(x)$.

② The general solution is given by:

$$y = \frac{1}{v(x)} \int v(x)Q(x)dx$$

e.g. Solve the IVP:

$$xy' + 2y = x^2 - x + 1, \quad x > 0, \quad y(1) = \frac{1}{2}$$

In standard form:

$$y' + \frac{2}{x}y = x - 1 + \frac{1}{x}$$

Integrating factor:

$$v(x) = e^{\int \frac{2}{x}dx} = e^{\ln 2x} = x^2$$

General solution:

$$y = \frac{1}{x^2}\int x^2\cdot(x-1+\frac{1}{x^2})\,dx$$
$$= \frac{1}{x^2}\cdot(\frac{1}{4}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2+C)\,dx$$
$$y(1)=\frac{1}{2} \Rightarrow \frac{1}{2}=\frac{1}{4}-\frac{1}{3}+\frac{1}{2}+C$$
$$C=\frac{1}{12}$$

Particular solution is:

$$y=\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{2}+\frac{1}{12x^2}$$

e.g. Solve the D.E.:

$$\cos(x)\,y'+\sin(x)\,y = 2\cos^3(x)\sin(x)-1, \quad 0\le x<\frac{\pi}{2}$$

In standard form:

$$y'+\tan x\cdot y = 2\cos^2(x)\sin x-\frac{1}{\cos x}$$

Integrating factor:

$$\nu(x)=e^{\int \tan x\,dx}=e^{\ln(\sec x)}=\sec x$$

General solution is:

$$y=\cos x\int \sec x\cdot(2\cos^2(x)\sin x-\frac{1}{\cos x})\,dx$$
$$=\cos x\int 2\cos x\sin x-\sec^2 x\,dx$$
$$=\cos x\,(-\frac{1}{2}\cos(2x)-\tan x+C_0)$$
$$=-\cos x\,(\frac{1}{2}\cos(2x)-\tan x+C)$$

§3 Applications: Population Growing Models

1. Malthusian growth model

If $P=P(t)$ is the population at time t, then

$$\frac{dP}{dt}=kP$$

Solve for P:

$$\int \frac{1}{kP}\,dP = \int dt$$

$$\frac{1}{k}\ln P = t + A \qquad (P>0)$$

$$P = e^{kA}e^{kt}$$

$$P = Ce^{kt} \qquad (C>0)$$

If $P(0)=P_0$, then $P_0 = Ce^0 = C$

Hence the particular solution is $P = P_0 e^{kt}$

where P_0 is the "initial population".

2. Logistic growth model

In this model,

$$\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right) \qquad (k>0)$$

where $M>0$ is the carrying capacity.

Let us assume $0<P<M$

$$\frac{dP}{dt} = \frac{kP(M-P)}{M}$$

$$\int \frac{M}{P(M-P)}\,dP = \int k\,dt$$

$$\int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP = kt$$

$$\ln P - \ln(M-P) = kt + A$$

$$\frac{P}{M-P} = e^{A}\cdot e^{kt}$$

$$\frac{M-P}{P} = e^{-A}e^{-kt}$$

$$P = \frac{M}{1+Ce^{-kt}} \qquad (C>0)$$

This is the general solution called the logistic function.

If the initial condition is $P(0)=P_0$, then

$$P_0 = \frac{M}{1+C}$$

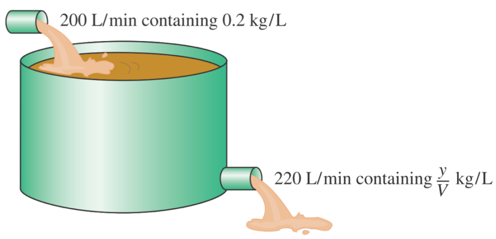
$$C = \frac{M-P_0}{P_0}$$

§4 Mixture Problems

Consider a container satisfying the following conditions:

- It initially contains 10000 L of solution, having 50 kg of salts dissolved in it.
- The solution leaks out of the contain at a rate 220 L/min.
- A solution with salt, whose concentration is 0.2 kg/L, is pumped into the container at a rate of 200L/min.

Suppose that the newly added solution is **instantly** well mixed with the solution that was already in the container. **What is the amount of salt in the container 20 minutes after the initial time?**



Solution:

Let $y(t) :=$ mass of salt t minutes since beginning (units: kg)

Then $y(0) = 50$

Want to find $y(20)$

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= (200)(0.2) - (220)\frac{y(t)}{V(t)}$$

$$= 40 - (220)\cdot\frac{y(t)}{10000 - 20t}$$

$$= 40 - \frac{11y(t)}{500 - t}$$

Hence IVP is:

$$\begin{cases} y' = 40 - \dfrac{11y(t)}{500-t} \\ y(0) = 50 \end{cases}$$

$$y' + \frac{11}{500-t}y(t) = 40$$

$$\int P(t)dt = -11\ln(500-t) + K \qquad (\text{assume } 0 < t < 500)$$

$$v(t) = e^{\int P(t)dt}$$

$$= e^{-11\ln(500-t)}$$

$$= (500-t)^{-11}$$

General solution:

$$y = (500-t)^{11} \int 40(500-t)^{-11}\,dt$$

$$= (500-t)^{11}\left(\frac{40}{-10}(-1)(500-t)^{-10} + C\right)$$

$$= 4(500-t) + C(500-t)^{11}$$

$$y(0) = 50 \implies 50 = 2000 + C(500)^{11}$$

$$C = -\frac{1950}{500^{11}}$$

$$y = 4(500-t) - \frac{1950}{500^{11}}(500-t)^{11}$$

$$y(20) = 4(480) - \frac{1950}{500^{11}}(480)^{11}$$

$$= 1920 - 1950\left(\frac{24}{25}\right)^{11}$$

$$\approx 675.43\,\text{kg}$$