Lecture 19

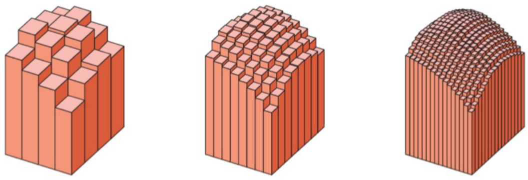
§1 Double integrals (二重积分) on rectangles

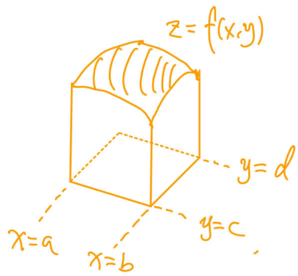
1. Riemann sum (黎曼求和)

Let f be a two-variable nonnegative function defined on the rectangle

$$R := [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

To approximate the volume of the solid lying between the xy-plane and the graph of f, we can use rectangular solids.





More precisely:

- Take a partition $\{x_0, x_1, \ldots, x_m\}$ of $[a, b]$ and a partition $\{y_0, y_1, \ldots, y_n\}$ of $[c, d]$; they together form a partition of R, and break R into mn rectangles R_{ij}.
- Let ΔA_{ij} be the area of R_{ij}, and let (x_{ij}^*, y_{ij}^*) be a point in R_{ij}.
- Then $f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$ is the volume of the rectangular solid with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.
- The total volume is then approximated by the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}.$$

The sum above is called a Riemann sum of f.

2. 定义

Definition

Let P be a partition of a rectangle R. The norm of P, denoted by $\|P\|$, is the largest width of the sub-rectangles in the partition P.
For a function f defined on the rectangle R, if the limit of the Riemann sums

$$S(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

exists as $\|P\| \to 0$, then f is said to be integrable (on R), and its limit is denoted by the double integral

$$\iint_R f(x, y)\, dA.$$

注: ① "limit of the Riemann sums exists" 是指:

$\exists L \in \mathbb{R}$, 对 $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. if $\|P\| < \delta$, then $|S(P,f) - L| \leq \varepsilon$,

$\forall$ Riemann sums $S(P,f)$

② 连续函数可积

有界，仅在有限个点、线处不连续的f可积

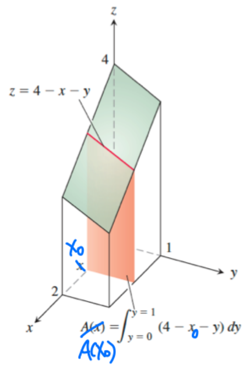
③ 若f为定义在R上的非负函数，则图像与 xy-plane 间的体积定义为:

$$V := \iint_R f(x,y)\, dA$$

3、计算 $\iint_R f(x,y)\,dA$

例：Find the volume under the plane $z=4-x-y$ and over $R: 0\le x\le 2, 0\le y\le 1$ in the xy-plane.

法一：



- Fix $x_0 \in [0,2]$
- 考虑 the thin "plate"，面积为

  $$A(x_0) = \int_0^1 (4-x_0-y_0)\,dy$$

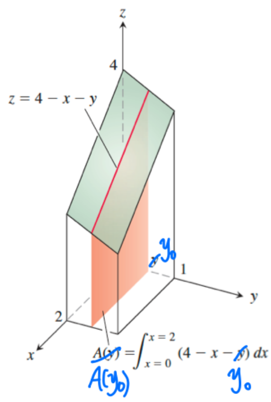
- 若 thin plate 有厚度 Δx，则其体积为 $A(x_0)\cdot\Delta x$，则有

  $$V = \int_0^2 A(x)\,dx$$
  $$= \int_0^2\int_0^1 (4-x-y)\,dy\,dx$$

  ＊ 此处为 iterated integral（累次积分），即将单变量积分重复多次

  $$= \int_0^2 \left[(4y-xy-\tfrac{1}{2}y^2)\Big|_{y=0}^{1} \right] dx$$
  $$= \int_0^2 (4-x-\tfrac{1}{2})\,dx$$
  $$= (\tfrac{7}{2}x-\tfrac{1}{2}x^2)\Big|_{x=0}^{2}$$
  $$= 5$$

法二：



- Fix $y_0 \in [0,1]$
- $A(y_0) = \int_0^2 (4-x-y_0)\,dx$
- $V = \int_0^1 A(y)\,dy$

  $$= \int_0^1\int_0^2 (4-x-y)\,dx\,dy$$
  $$= \int_0^1 \left[(4x-\tfrac{1}{2}x^2-yx)\Big|_{x=0}^{2} \right] dy$$
  $$= \int_0^1 (8-2-2y)\,dy$$
  $$= 5$$

4. Fubini's Theorem（富比尼定理）(for rectangles)

> **Theorem (Fubini's Theorem)** (for rectangles)
> *If f is continuous on* $R := [a,b]\times[c,d]$, *then*
>
> $$\iint_R f(x,y)\,dA = \underbrace{\int_c^d\int_a^b f(x,y)\,dx\,dy}_{\text{iterated integrals}} = \underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx}.$$

注：若f在矩形区域R上连续，

① double integral 可以由 iterated integrals 计算

② iterated integrals 的顺序不重要

例: Evaluate $\iint_R y\sin(xy)\,dA$, where $R=[1,2]\times[0,\pi]$

$$\begin{aligned}\iint_R y\sin(xy)\,dA &= \int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy\\ &= \int_0^\pi \left(-\cos(xy)\Big|_1^2\right)dy\\ &= \int_0^\pi \cos y-\cos 2y\,dy\\ &= \left(\sin y-\tfrac{1}{2}\sin 2y\right)\Big|_0^\pi\\ &= 0\end{aligned}$$

§2 Double integrals on general bounded regions

1. 定义

Essential

Let D be a closed and bounded region in $\mathbb{R}^2$, and suppose that f is a function defined on D. Let R be a rectangle enclosing D, and define a new function F on R by

$D\subseteq R$

$$F(x,y):=\begin{cases} f(x,y), & \text{if } (x,y)\in D;\\ 0, & \text{if } (x,y)\in R\setminus D.\end{cases}$$

We define the double integral of f over D by

$$\iint_D f(x,y)\,dA := \iint_R F(x,y)\,dA.$$

Again, if f is nonnegative and continuous on D, then we define the volume of the solid lying between the xy-plane and the graph of f to be $\iint_D f(x,y)\,dA$.

2. Type I and Type II Regions

Definition

A plane region D is said to be of type I if it has the form

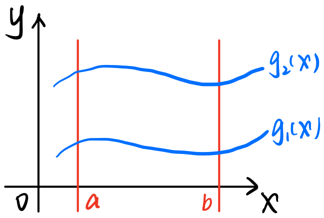
$$\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\},$$

where g_1 and g_2 are continuous functions of x, and D is said to be of type II if it has the form

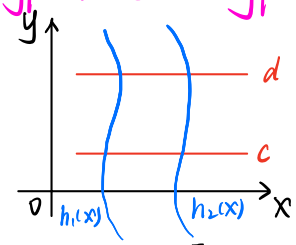
$$\{(x,y): c\le y\le d,\ h_1(y)\le x\le h_2(y)\},$$

where h_1 and h_2 are continuous functions of y.

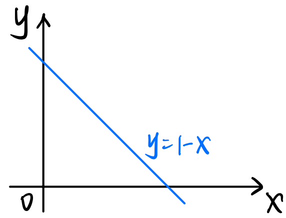
注：一个区域可以既是 type 1，也是 type 2



type I



type II



type I & type II

type I：任意直线 $x=t$，$t\in R$ 均不会穿过区域边界 2次以上

type II：任意直线 $y=t$，$t\in R$ 均不会穿过区域边界 2次以上

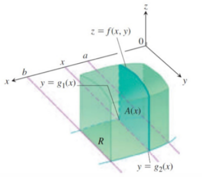
3. Fubini's Theorem (for type I & II regions)

Theorem (Fubini's Theorem) (Type-I Regions)
If f is continuous on a type I region D, where

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$



Theorem (Fubini's Theorem) (Type-II Regions)
If f is continuous on a type II region D, where

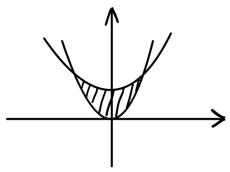
$$D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

例: (a) Evaluate the double integral $\iint_D x + 2y\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

- D:



$-1 \le x \le 1$

$2x^2 \le y \le 1 + x^2$

- $I = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$

$= \int_{-1}^{1} (xy + y^2)\Big|_{2x^2}^{1+x^2}\,dx$

$= \int_{-1}^{1} (1 + 2x^2 - 3x^4)\,dx$

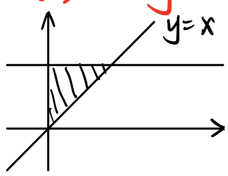
$= \frac{32}{15}$

例: (b) Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

- $I := \int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

$= \iint_D \sin(y^2)\,dA$, $D := \{(x,y) : 0 \le x \le 1,\ x \le y \le 1\}$

- D:



- $I = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$

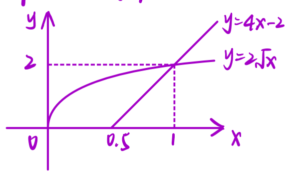
$= \int_0^1 y\sin(y^2)\,dy$

$= -\frac{1}{2}\cos(y^2)\Big|_0^1$

$= \frac{1}{2}(1 - \cos 1)$

注: 可以通过将二次积分在 type I 与 type II 之间改写来简化计算

例: Find $\iint_R (16-x^2-y^2)\,dA$



$$\iint_R (16-x^2-y^2) = \int_0^2 \int_{\frac14 y^2}^{\frac14(y+2)} (16-x^2-y^2)\,dx\,dy$$
$$= \int_0^2 (16-y^2)x - \frac13 x^3 \Big|_{\frac14 y^2}^{\frac14(y+2)} dy$$
$$= \int_0^2 (16-y^2)\cdot\frac14(y+2-y^2) - \frac{1}{192}\left[(y+2)^3 - y^6\right] dy$$

......

4. 二重积分的性质

Theorem

Assuming that the integrals exist, the following hold:

(a) $\iint_D (f(x,y) \pm g(x,y))\,dA = \iint_D f(x,y)\,dA \pm \iint_D g(x,y)\,dA.$

(b) $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA.$

(c) *If* $f(x,y) \geq g(x,y)$ *for all* $(x,y) \in D$*, then*

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$$

In particular, if $f(x,y) \geq 0$ $\forall (x,y) \in D$, then $\iint_D f(x,y)\,dA \geq 0$.

(d) *If* $D = D_1 \cup D_2$*, where* D_1 *and* D_2 *do not overlap except possibly on their boundaries, then*

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

5. Area and average values

The idea of using double integrals to represent volumes motivates the following definitions.

Definition (usually closed and bounded)

The **area** $A(D)$ of a plane region D is defined by

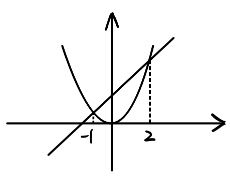
$$A(D) := \iint_D dA := \iint_D 1\,dA.$$

The **average value** of a function f over D is

$$\frac{1}{A(D)}\iint_D f(x,y)\,dA.$$

例: **Example**

Find the area of the region D enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

- 

- $A(D) = \iint_D dA$ $\qquad D: -1 \leq x \leq 2,\ x^2 \leq y \leq x+2$

$$= \int_{-1}^{2}\int_{x^2}^{x+2} 1\,dy\,dx$$
$$= \int_{-1}^{2} (x+2-x^2)\,dx$$
$$= \frac92$$

§3 Regions in polar coordinates

1. Riemann sum

Regions in Polar Coordinates

Consider integrating over the region

$$D = \{(x, y) : 1 \le x^2 + y^2 \le 4,\ y \ge 0\}.$$

It could be tedious to describe D as a union of type I/type II regions, but it can be described very naturally using polar coordinates; it corresponds to

$$R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}.$$

In general, suppose that D is a region on the xy-plane which can be described in polar coordinates by

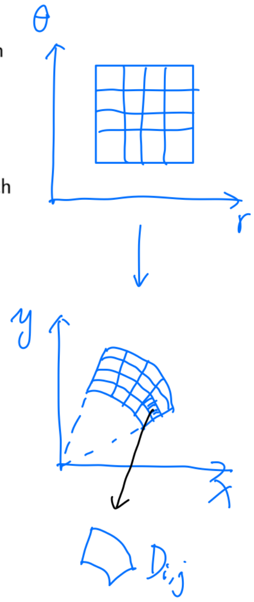
$$0 \le a \le r \le b, \quad \alpha \le \theta \le \beta;$$

- Partition this into polar subrectangles R_{ij} of equal size, which corresponds to a region D_{ij} on the xy-plane.
- Let ΔA_{ij} be the area of D_{ij}.
- Pick a point (r_i^*, θ_j^*) in R_{ij}; we may pick its center, i.e.,

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \quad \text{and} \quad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

- Computing area shows that $\Delta A_{ij} = r_i^* \Delta r \Delta \theta$, which yields

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta \theta.$$



2. Theorem 1

Theorem

If f is continuous on a region D which has a polar description

$$0 \le a \le r \le b, \quad \alpha \le \theta \le \beta,$$

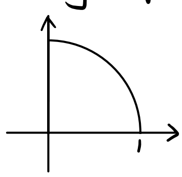
where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta.$$

例: Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy dx.$$

- $R:\ 0 \le x \le 1$
  $0 \le y \le \sqrt{1-x^2}$



- $I = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cdot r dr d\theta$
  $= \int_0^{\frac{\pi}{2}} \left(\frac{1}{4} r^4 \Big|_0^1\right) d\theta$
  $= \frac{\pi}{2} \cdot \frac{1}{4}$
  $= \frac{\pi}{8}$

3. Theorem 2 (General case)

Theorem

If f is continuous on a region D which has a polar description

$$\alpha \le \theta \le \beta, \quad 0 \le h_1(\theta) \le r \le h_2(\theta),$$

where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

注：xy平面内的 dA，在极坐标平面内写作 $r\,dr\,d\theta$

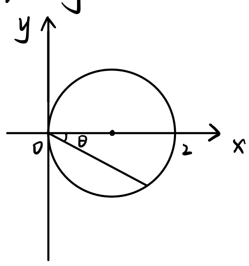
例：Find the area enclosed by one loop of the four-leaved rose given by $r=\cos 2\theta$

$$A(D) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\,dr\,d\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\cos^2 2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \cos^2 2\theta\,d\theta$$
$$= \frac{\pi}{8}$$

例：**Example**

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

- $x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r=0$ or $r = 2\cos\theta$



D: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 2\cos\theta$

- $V = \iint_D (x^2+y^2)\,dA$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}(2\cos\theta)^4\,d\theta$$
$$= 8\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$
$$= \frac{3}{2}\pi$$