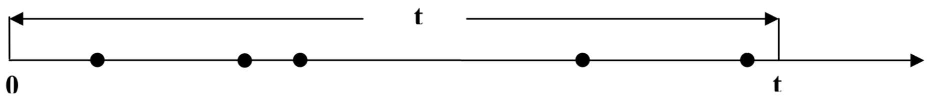
Lecture 10

§1 Poisson process

1. 一个 Introductory example：phone call

Introductory Example: Phone Calls



- Consider the arrivals of telephone calls at a telephone exchange. We assume that we count the arrivals of calls, from the beginning of some time and we view this beginning time as $t = 0$.
- Let N_t denote the number of calls arrived by time t.
- Then N_t is a random variable and all the possible values of this random variable are $\{0, 1, 2, \ldots\}$, i.e. non-negative integer valued random variable.
- Also for each $t \geq 0$, we have a random variable N_t and thus we have a random process $\{N_t, t \geq 0\}$.

For this process we see that

1. The time parameter $t \in [0, \infty)$ is continuous;
2. The state space $E = \{0, 1, 2, \ldots\}$ is discrete.

2. Definition: Counting process（计数过程）

一个随机过程 $\{N_t : t \geq 0\}$ 被称为 counting process，若

① N_t 为非负整数，$\forall t \geq 0$

② $N_t - N_s$ 表示 interval $(s, t]$ 内 events 的发生次数，$\forall s < t$

注：第②条表示 $t \mapsto N_t$ 为 increasing，即 $N_s \leq N_t$ 若 $s \leq t$

e.g. Counting process 的例子：

① N_t = time interval $(0, t]$ 中的 phone calls 数

② N_t = time interval $(0, t]$ 中 insurance company 的 claims 数

③ N_t = time interval $(0, t]$ 中 people 的出生数

3. Property: Independent increment（独立增加）（泊松过程需要满足的性质一）

在 disjoint time intervals 内的 events 发生数相互独立

e.g. 在 example ①中，phone calls 数满足 independent

在 example ③中，t 时刻出生的人数会影响 $(t, t+s]$ interval 内出生的人数

4. Property: Stationary increment（平稳增加）（泊松过程需要满足的性质二）

$N_{t+u} - N_{s+u}$ 与 $N_t - N_s$ 有相同的分布，$\forall t > s > 0$，$u \geq 0$

也可以表述为：任意 interval of time 内，事件发生数的分布仅取决于 interval 的 length

e.g. 在 example ①中，若以 days 为 unit of time，则满足 stationary increment 为 reasonable

注：若以 hours 为 unit of time，则可能不满足（白天和半夜的来电数分布不一样）

在 example ③中，即使出生率为 constant，也不满足 stationary increment

5. **Definition:** Poisson process （泊松过程）

The counting process $\{N_t : t \geq 0\}$ 被称为一个 Poisson process having rate λ，若其满足：

① $N_0 = 0$

② process 满足 independent increment

③ process 满足 stationary increment，且任意 interval of length t 内的事件发生数服从均值为 λt 的 Poisson distribution，即

$$P\{N_{t+s} - N_s = n\} = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \cdots, \quad \forall s, t \geq 0$$

由③可推出，对 $\forall t \geq 0$，t 时间前的事件发生数 N_t 满足：

$$N_t = N_t - N_0 \sim Poi(\lambda t)$$

且有 $E[N_t] = \lambda t$

注：$\lambda = \frac{\lambda t}{t} = E[\frac{N(t+s) - N(s)}{t}]$ 被称为 arrival rate，表示单位时间内的平均发生次数

§2 Poisson process 的性质：与 Bernulli distribution 的关系

1. **Definition:** Laudau's $o(\cdot)$

Definition. *A function $f(x)$ is said to be $o(g(x))$ (small o of $g(x)$) in the neighbourhood of a, denoted as $x \sim a$, if*

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = 0 .}$$

In other words, $f(x)$ is much (infinitely) smaller than $g(x)$ when $x \sim a$.

Examples. Most of time, $g(x) = x^n$, a power of x.

- $4x^4 = o(x^3)$ when $x \sim 0$;
- $\log(x) = o(x)$ and $x^k = o(e^x)$, when $x \sim \infty$.
- Link to Taylor expansions:

$$\boxed{\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + o(x^2), \quad x \sim 0, \\ (1+x)^s &= 1 + sx + \frac{s(s-1)}{2}x^2 + o(x^2), \quad x \sim 0 . \end{aligned}}$$

2. **Theorem:** interval 足够小时的 Poisson process ： Poisson process $\longrightarrow$ Bernulli distribution

令 N_t 为 Poisson process $\{N_t : t \geq 0\}$ with rate λ 在 $(0, h]$ 上的 number of points，则对于 small $h \sim 0$，有

① $P\{N_h = 0\} = 1 - \lambda h + o(h)$ $\quad (e^{-\lambda h} = 1 - \lambda h + o(h))$

② $P\{N_h = 1\} = \lambda h + o(h)$ $\quad (e^{-\lambda h} \cdot \lambda h = \lambda h + o(h))$

③ $P\{N_h \geq 2\} = o(h)$ $\quad (1 - P\{N_h = 0\} - P\{N_h = 1\} = 1 - [1 + o(h)] = o(h))$

即，当 interval length h 非常小时，几乎没有 points：

- 有 1 point 的概率与 h 同阶，
- 有 ≥ 2 points 的概率与 $o(h)$ 同阶，

换句话说，up to an error of $o(h)$，N_h 可视作一个 $ber(\lambda h)$ r.v.

3. **Theorem:** Poisson approximation: Bernulli distribution $\longrightarrow$ Poisson distribution

令 $X_1, \cdots, X_n$ 为 a sequence of $Ber(p)$ r.v.'s，并假设 p 与 n 相关，满足 $pn \to \lambda > 0$.

则 $S_n = X_1 + \cdots + X_n$ converges to $Poi(\lambda)$，when $n \to \infty$

证明:

$Ber(p)$ 的 m.g.f. 为

$$M_{X_i}(t) = 1 - p + pe^t \quad \text{for all } t \in R$$

因此，

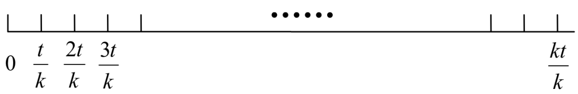
$$
\begin{aligned}
M_{S_n}(t) &= \prod_{i=1}^{n} (1 - p + pe^t) \\
&= (1 - p + pe^t)^n \\
&\to \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t\right)^n \\
&= \left(1 + \frac{(e^t - 1)\lambda}{n}\right)^n \\
&\to e^{(e^t - 1)\lambda} \quad \text{as } n \to \infty \qquad \left(\left(1 + \frac{x}{n}\right)^n \to e^x\right)
\end{aligned}
$$

为 $Poi(\lambda)$ 的 m.g.f.

4. Poisson process 与 Bernulli distribution 的关系

Now Consider a P.P. (N_t) and for a fixed $t > 0$,

- a partition of $(0, t]$ on k equal bins $((i-1)h, ih]$ with bin-width $h = t/k$, $1 \le i \le k$.



- The bin-width $h = t/k$ is small for large k, so that the counts in the bins

$$X_i = N_{ih} - N_{(i-1)h}, \qquad 1 \le i \le k,$$

could be considered as i.i.d. $b(\lambda h)$;

- Clearly, $N_t = X_1 + \cdots + X_k$;
- Letting $k \to \infty$ and $k \cdot \lambda h = k \cdot \lambda t/k \to \lambda t$, by the theorem, $N_t \sim \mathscr{P}(\lambda t)$.