Structural Optimization for Large-Scale Problems

Lecture 4: Looking into the Black Box: Smoothing Technique

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Outline

Nonsmooth Optimization

Smoothing technique

Application examples

先前我们假设 oracle 是在一个 black box 中, 但现实中往往我们是对 oracle 有一定的了解的,
因此可以 improve the oracle/ reformulate the problem.

§1之前对于 Nonsmooth convex problem (first-order method) 的讨论结果

Nonsmooth Unconstrained Optimization

(first order)

Problem: $\min \{ f(x) : x \in R^n \} \quad \Rightarrow \quad x^*, \ f^* = f(x^*)$,

where $f(\cdot)$ is a nonsmooth convex function.

Subgradients: $g \in \partial f(x) \Leftrightarrow f(y) \geq f(x) + \langle g, y - x \rangle \quad \forall y \in R^n$.

Main difficulties:

- $g \in \partial f(x)$ is *not* a descent direction at x.
- $g \in \partial f(x^*)$ does not imply $g = 0$.

Example

$$f(x) = \max_{1 \leq j \leq m} \{ \langle a_j, x \rangle + b_j \},$$

$$\partial f(x) = \mathrm{Conv} \ \{ a_j : \ \langle a_j, x \rangle + b_j = f(x) \}.$$

Subgradient methods in Nonsmooth Optimization

Advantages

- Very simple iteration scheme.
- Low memory requirements. $O(\frac{1}{\sqrt{k}})$
- Optimal rate of convergence (uniformly in the dimension).
- Interpretation of the process.

Objections:

- Low rate of convergence. (Confirmed by theory!)
- No acceleration.
- High sensitivity to the step-size strategy.

Lower complexity bounds

Nemirovsky, Yudin 1976

If $f(\cdot)$ is given by a local *black-box*, it is impossible to converge faster than $O\left(\frac{1}{\sqrt{k}}\right)$ uniformly in n. (k is the $\#$ of calls of oracle.)

NB: Convergence is very slow.

Question: We want to find an ϵ-solution of the problem

$$\max_{1 \leq j \leq m} \{\langle a_j, x \rangle + b_j\} \quad \rightarrow \quad \min_x : \; x \in R^n,$$

by a gradient scheme (n and m are big).

What is the worst-case complexity bound?

"Right answer" (Complexity Theory): $O\left(\frac{1}{\epsilon^2}\right)$ calls of oracle.

Our target: A gradient scheme with $O\left(\frac{1}{\epsilon}\right)$ complexity bound.

Reason of speed up: our problem *is not* in a black box.

Complexity of Smooth Minimization

(first order)

Problem: $f(x) \to \min_{x} : x \in R^n$, where f is a convex function and

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L(f)\|x-y\| \text{ for all } x, y \in R^n.$$

(For measuring gradients we use dual norms: $\|s\|_* = \max_{\|x\|=1} \langle s, x \rangle$.)

Rate of convergence: Optimal method gives $O\left(\frac{L(f)}{k^2}\right)$.

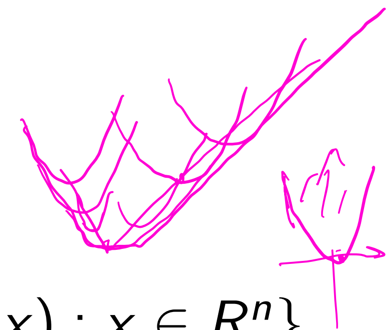
Complexity: $\boxed{O\left(\sqrt{\frac{L(f)}{\epsilon}}\right)}$. The difference with $O\left(\frac{1}{\epsilon^2}\right)$ is very big.

如果能把nonsmooth的问题转化为smooth的，提升会很大

§2 Smoothing technique

Smoothing for convex functions

用Fenchel dual表示这种 idea



For function f define its Fenchel conjugate:

$$\boxed{f_*(s) = \max_{x \in R^n}[\langle s, x\rangle - f(x)].}$$

It is a closed convex function with $\mathrm{dom}\, f_* = \mathrm{Conv}\{f'(x) : x \in R^n\}$.

Moreover, under very mild conditions $(f_*(s))_* \equiv f(x)$.

Define $\boxed{f_\mu(x) = \max_{s \in \mathrm{dom}\, f_*}[\langle s, x\rangle - f_*(s) - \underbrace{\tfrac{\mu}{2}\|s\|_*^2}_{\text{扰动(用于smooth)}}],}$

$(f_*(s))_*(x) = \max_{s \in \mathrm{dom}\, f_*}[\langle x, s\rangle - f_*(s)]$
$= f(x)$

where $\|\cdot\|_*$ is a Euclidean norm.

Note: $f'_\mu(x) = s_\mu(x)$, and $x = f'_*(s_\mu(x)) + \mu s_\mu(x)$. Therefore,

$$\|x_1 - x_2\|^2 = \|f'_*(s_1) - f'_*(s_2)\|^2 + 2\mu\langle f'_*(s_1) - f'_*(s_2), s_1 - s_2\rangle$$
$$+\mu^2\|s_1 - s_2\|^2 \;\geq\; \mu^2\|s_1 - s_2\|^2.$$

Thus, $f_\mu \in C^{1,1}_{1/\mu}$ and

$$f(x) \geq f_\mu(x) \geq f(x) - \tfrac{1}{2}\mu D^2,$$

where $D = \mathrm{Diam}(\mathrm{dom}\, f_*)$.

Main questions

1. Given by a non-smooth convex $f(\cdot)$, can we form its computable smooth ϵ-approximation $f_\epsilon(x)$ with

$$L(f_\epsilon) = O\left(\frac{1}{\epsilon}\right)?$$

If yes, we need only $O\left(\sqrt{\frac{L(f_\epsilon)}{\epsilon}}\right) = O\left(\frac{1}{\epsilon}\right)$ iterations.

2. Can we do this in a systematic way?

Conclusion: We need a convenient *model* of our problem.

Adjoint problem

$$f_\mu(x) = \max_{s \in \mathrm{dom}\, f_*} [\langle s, x\rangle - f_*(s) - \tfrac{\mu}{2}\|s\|_*^2]$$

$\hat{\phi}(\mu)$ 类似于 $f_*(s)$

Primal problem: Find $f^* = \min_x \{f(x) : \ x \in Q_1\}$, where $Q_1 \subset E_1$ is convex closed and bounded.

原本就 smooth　　不一定 smooth

Objective: $f(x) = \hat{f}(x) + \max_u \{\langle Ax, u\rangle_2 - \hat{\phi}(u) : \ u \in Q_2\}$, where

- $\hat{f}(x)$ is differentiable and convex on Q_1. (原本就 smooth 的部分)
- $Q_2 \subset E_2$ is a closed convex and bounded.
- $\hat{\phi}(u)$ is continuous convex function on Q_2. (有多种选择)
- linear operator $A : E_1 \to E_2^*$.

Adjoint problem: $\max_u \{\phi(u) : \ u \in Q_2\}$, where

$$\phi(u) = -\hat{\phi}(u) + \min_x \{\langle Ax, u\rangle_2 + \hat{f}(x) : \ x \in Q_1\}.$$

NB: Adjoint problem is not unique! (取决于 $\hat{\phi}$, E_2 的选取)

$$\min\{f(x)\} = \min_x\{\hat{f}(x) + \max_\mu\{\langle Ax, \mu\rangle_2 - \hat{\phi}(\mu)\}\}$$
$$= \max_\mu\{\min_x\{\hat{f}(x) + \langle Ax, \mu\rangle_2 - \hat{\phi}(\mu)\}\}$$
$$= \max_\mu\{-\hat{\phi}(\mu) + \min_x\{\hat{f}(x) + \langle Ax, \mu\rangle_2\}\} := \max_\mu\{\phi(\mu)\}$$

Example (不同的 adjoint problem)

Consider $f(x) = \max_{1 \le j \le m} |\langle a_j, x\rangle_1 - b_j|$.

1. $Q_2 = E_1^*$, $A = I$, $\underline{\hat{\phi}(u) \equiv f_*(u)} = \max_x \{\langle u, x\rangle_1 - f(x) : \; x \in E_1\}$

$$= \min_{s \in R^m} \left\{ \sum_{j=1}^m s_j b_j : \; u = \sum_{j=1}^m s_j a_j, \; \sum_{j=1}^m |s_j| \le 1 \right\}.$$

2. $E_2 = R^m$, $\underline{\hat{\phi}(u) = \langle b, u\rangle_2}$, $f(x) = \max_{1 \le j \le m} |\langle a_j, x\rangle_1 - b_j|$

$$= \max_{u \in R^m} \left\{ \sum_{j=1}^m u_j [\langle a_j, x\rangle_1 - b_j] : \; \sum_{j=1}^m |u_j| \le 1 \right\}.$$

3. $E_2 = R^{2m}$, $\underline{\hat{\phi}(u) \text{ is a linear,}}$ Q_2 is a simplex

$$f(x) = \max_{u \in R^{2m}} \{ \sum_{j=1}^m (u_j^1 - u_j^2)[\langle a_j, x\rangle_1 - b_j] : \sum_{j=1}^m (u_j^1 + u_j^2) = 1, \; u \ge 0 \}.$$

NB: Increase in dim E_2 decreases the complexity of representation.

Smooth approximations

Prox-function: $d_2(u)$ is continuous and *strongly convex* on Q_2:

$$d_2(v) \geq d_2(u) + \langle \nabla d_2(u), v - u \rangle_2 + \tfrac{1}{2}\sigma_2 \|v - u\|_2^2.$$

Assume: $d_2(u_0) = 0$ and $d_2(u) \geq 0 \ \forall u \in Q_2$.

Fix $\mu > 0$, the *smoothing* parameter, and define

$$f_\mu(x) = \max_u \{ \langle Ax, u \rangle_2 - \hat{\phi}(u) - \mu d_2(u) : \ u \in Q_2 \}.$$

加上了一个扰动 (通常很 simple)

Denote by $u(x)$ the solution of this problem.

Theorem: $f_\mu(x)$ is convex and differentiable for $x \in E_1$.

Its gradient $\nabla f_\mu(x) = A^* u(x)$ is Lipschitz continuous with

smooth

$$L(f_\mu) = \frac{1}{\mu \sigma_2} \|A\|_{1,2}^2,$$

where $\|A\|_{1,2} = \max_{x,u} \{ \langle Ax, u \rangle_2 : \ \|x\|_1 = 1, \ \|u\|_2 = 1 \}$.

NB: 1. For any $x \in E_1$ we have $f_0(x) \geq f_\mu(x) \geq f_0(x) - \mu D_2$,

where $D_2 = \max_u \{ d_2(u) : \ u \in Q_2 \}$.

2. All norms are very important.

Optimal method

（用于解决 smooth 后的问题）

Problem: $\min_x \{f(x) : x \in Q_1\}$ with $f \in C^{1,1}(Q_1)$.

Prox-function: strongly convex $d_1(x)$, $d_1(x^0) = 0$, $d_1(x) \geq 0$, $x \in Q_1$.

Gradient mapping:

$$T_L(x) = \arg\min_{y \in Q_1} \left\{ \langle \nabla f(x), y - x \rangle_1 + \tfrac{1}{2} L \|y - x\|_1^2 \right\}.$$

Method. For $k \geq 0$ **do:**

1. Compute $f(x^k), \nabla f(x^k)$.

2. Find $y^k = T_{L(f)}(x^k)$.

3. Find $z^k = \arg\min_{x \in Q_1} \left\{ \frac{L(f)}{\sigma} d_1(x) + \sum_{i=0}^{k} \frac{i+1}{2} \langle \nabla f(x^i), x \rangle_1 \right\}$.

4. Set $x^{k+1} = \frac{2}{k+3} z^k + \frac{k+1}{k+3} y^k$.

Convergence: $\boxed{f(y^k) - f(x^*) \leq \frac{4L(f) d_1(x^*)}{\sigma_1 (k+1)^2},}$ where x^* is the optimal solution.

$O(\frac{1}{k^2})$

Applications

smooth 的部分

Smoothed problem: $\bar{f}_\mu(x) = \hat{f}(x) + f_\mu(x) \quad \rightarrow \quad \min : \; x \in Q_1.$

Lipschitz constant: $\boxed{L_\mu = L(\hat{f}) + \frac{1}{\mu\sigma_2}\|A\|_{1,2}^2.}$

Denote $D_1 = \max\limits_x\{d_1(x) : \; x \in Q_1\}$.

Theorem: Let us choose $N \geq 1$. Define

$$\mu = \mu(N) = \frac{2\|A\|_{1,2}}{N+1} \cdot \sqrt{\frac{D_1}{\sigma_1\sigma_2 D_2}}.$$

After N iterations set $\hat{x} = y^N \in Q_1$ and

$$\hat{u} = \sum_{i=0}^{N} \frac{2(i+1)}{(N+1)(N+2)} \, u(x^i) \in Q_2.$$

duality gap

Then $0 \leq f(\hat{x}) - \phi(\hat{u}) \leq \frac{4\|A\|_{1,2}}{N+1} \cdot \sqrt{\frac{D_1 D_2}{\sigma_1\sigma_2}} + \frac{4L(\hat{f})D_1}{\sigma_1\cdot(N+1)^2}.$

Corollary. Let $L(\hat{f}) = 0$. For getting an ϵ-solution, we choose

$$\boxed{\mu = \frac{\epsilon}{2D_2}}, \quad L = \frac{D_2}{2\sigma_2} \cdot \frac{\|A\|_{1,2}^2}{\epsilon}, \quad \boxed{N \geq 4\|A\|_{1,2}\sqrt{\frac{D_1 D_2}{\sigma_1\sigma_2}} \cdot \frac{1}{\epsilon}}.$$

Example: Equilibrium in matrix games (1)

Denote $\Delta_n = \{x \in R^n : \; x \geq 0, \; \sum\limits_{i=1}^n x^{(i)} = 1\}$. Consider the problem

$\min\limits_{x \in \Delta_n} \max\limits_{u \in \Delta_m} \{\langle Ax, u\rangle_2 + \langle c, x\rangle_1 + \langle b, u\rangle_2\}$.

Minimization form:

$$\text{dual}\left\{\begin{array}{ll} \min\limits_{x \in \Delta_n} f(x), & f(x) = \langle c, x\rangle_1 + \max\limits_{1 \leq j \leq m} [\langle a_j, x\rangle_1 + b_j], \\ \max\limits_{u \in \Delta_m} \phi(u), & \phi(u) = \langle b, u\rangle_2 + \min\limits_{1 \leq i \leq n} [\langle \hat{a}_i, u\rangle_2 + c_i], \end{array}\right.$$

where a_j are the rows and $\hat{a}_i$ are the columns of A.

1. Euclidean distance: Let us take

$$\|x\|_1^2 = \sum_{i=1}^n x_i^2, \quad \|u\|_2^2 = \sum_{j=1}^m u_j^2,$$

$$d_1(x) = \tfrac{1}{2}\|x - \tfrac{1}{n} e_n\|_1^2, \quad d_2(u) = \tfrac{1}{2}\|u - \tfrac{1}{m} e_m\|_2^2.$$

Then $\|A\|_{1,2} = \lambda_{\max}^{1/2}(A^T A)$ and $f(\hat{x}) - \phi(\hat{u}) \leq \frac{4\lambda_{\max}^{1/2}(A^T A)}{N+1}$.

Example: Equilibrium in matrix games (2)

2. Entropy distance. Let us choose

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad d_1(x) = \ln n + \sum_{i=1}^{n} x_i \ln x_i,$$

$$\|u\|_2 = \sum_{j=1}^{m} |u_j|, \quad d_2(u) = \ln m + \sum_{j=1}^{m} u_j \ln u_j.$$

LM: $\sigma_1 = \sigma_2 = 1$. (Hint: $\langle d_1''(x)h, h\rangle = \sum_{i=1}^{n} \frac{h_i^2}{x_i} \to \min_{x \in \Delta_n} = \|h\|_1^2$.)

Moreover, since $D_1 = \ln n$, $D_2 = \ln m$, and

$$\|A\|_{1,2} = \max_x \{ \max_{1 \le j \le m} |\langle a_j, x\rangle| : \|x\|_1 = 1\} = \max_{i,j} |A_{i,j}|,$$

we have $f(\hat{x}) - \phi(\hat{u}) \le \frac{4\sqrt{\ln n \ln m}}{N+1} \cdot \max_{i,j} |A_{i,j}|$.

NB: 1. Usually $\max_{i,j} |A_{i,j}| << \lambda_{\max}^{1/2}(A^T A)$.

2. We have $\bar{f}_\mu(x) = \langle c, x\rangle_1 + \mu \ln \left(\frac{1}{m} \sum_{j=1}^{m} e^{[\langle a_j, x\rangle + b_j]/\mu} \right)$.

Example 2: Continuous location problem

Problem: p cities with populations m_j, $j = 1, \dots, m$, are located at positions

$$c_j \in R^n, \quad j = 1, \dots, p.$$

Goal: Construct a service center at point x^*, which minimizes the total distance to the center.

That is: Find $f^* = \min\limits_x \left\{ f(x) = \sum\limits_{j=1}^{p} m_j \|x - c_j\|_1 : \ \|x\|_1 \leq \bar{r} \right\}.$

Primal space:

$$\|x\|_1^2 = \sum_{i=1}^{n} (x^{(i)})^2, \quad d_1(x) = \tfrac{1}{2}\|x\|_1^2, \quad \sigma_1 = 1, \quad D_1 = \tfrac{1}{2}\bar{r}^2.$$

Adjoint space: $E_2 = (E_1^*)^p, \quad \|u\|_2^2 = \sum\limits_{j=1}^{p} m_j (\|u_j\|_1^*)^2,$

$$Q_2 = \{u = (u_1, \ldots, u_p) \in E_2 : \ \|u_j\|_1^* \le 1, \ j = 1, \ldots, p\},$$

$$d_2(u) = \tfrac{1}{2}\|u\|_2^2, \quad \sigma_2 = 1, \quad D_2 = \tfrac{1}{2}P.$$

with $P \equiv \sum\limits_{j=1}^{p} m_j$, the total size of population.

Operator norm: $\|A\|_{1,2} = P^{1/2}$.

Rate of convergence: $f(\hat{x}) - f^* \le \frac{2P\bar{r}}{N+1}$.

$$f_\mu(x) = \sum_{j=1}^{p} m_j \psi_\mu(\|x - c_j\|_1), \quad \psi_\mu(\tau) = \begin{cases} \frac{\tau^2}{2\mu}, & \tau \le \mu, \\ \tau - \frac{\mu}{2}, & \mu \le \tau. \end{cases}$$

Example 3: Variational inequalities (linear operator)

Consider $B(w) = Bw + c$: $E \to E^*$, which is *monotone*:

$$\langle Bh, h \rangle \geq 0 \quad \forall h \in E.$$

Problem: Find $w^* \in Q: \quad \langle B(w^*), w - w^* \rangle \geq 0 \quad \forall w \in Q,$

where Q is a bounded convex closed set.

Merit function: $\psi(w) = \max\limits_{v} \{ \langle B(v), w - v \rangle : \ v \in Q \}.$

- $\psi(w)$ is convex on E_1.
- $\psi(w) \geq 0$ for all $w \in Q$.
- $\psi(w) = 0$ if and only if w solves VI-problem.
- $\langle B(v), v \rangle$ is a *convex* function. Thus, ψ is *exactly* in our form.

Primal smoothing: $\psi_\mu(w) = \max\limits_{v} \{ \langle B(v), w - v \rangle - \mu d_2(v) : \ v \in Q \}.$

Dual smoothing: $\phi_\mu(v) = \min\limits_{w} \{ \langle B(v), w - v \rangle + \mu d_1(w) : \ w \in Q \}.$

(Looks better.)

Example 4: Piece-wise linear functions

1. Maximum of absolute values. Consider

$$\min_x \left\{ f(x) = \max_{1 \le j \le m} |\langle a_j, x \rangle_1 - b^{(j)}| : \; x \in Q_1 \right\}.$$

For simplicity choose $\|x\|_1^2 = \sum\limits_{i=1}^{n} (x^{(i)})^2$, $d_1(x) = \frac{1}{2}\|x\|^2$.

It is convenient to choose $E_2 = R^{2m}$,

$$\|u\|_2 = \sum_{j=1}^{2m} |u^{(j)}|, \quad d_2(u) = \ln(2m) + \sum_{j=1}^{2m} u^{(j)} \ln u^{(j)}.$$

Denote by A the matrix with the rows a_j. Then

$$f(x) = \max_u \{ \langle \hat{A}x, u \rangle_2 - \langle \hat{b}, u \rangle_2 : \; u \in \Delta_{2m} \},$$

where $\hat{A} = \begin{pmatrix} A \\ -A \end{pmatrix}$ and $\hat{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$.

Thus, $\sigma_1 = \sigma_2 = 1$, $D_2 = \ln(2m)$, $D_1 = \frac{1}{2}\bar{r}^2$, $\bar{r} = \max\limits_x \{\|x\|_1 : x \in Q_1\}$.

Operator norm: $\|\hat{A}\|_{1,2} = \max\limits_{1 \le j \le m} \|a_j\|_1^*$.

$\boxed{\textbf{Complexity: } 2\sqrt{2}\, \bar{r} \max\limits_{1 \le j \le m} \|a_j\|_1^* \sqrt{\ln(2m)} \cdot \frac{1}{\epsilon}.}$

Approximation: for $\xi(\tau) = \frac{1}{2}[e^{\tau} + e^{-\tau}]$ define

$$\bar{f}_\mu(x) = \mu \ln \left(\frac{1}{m} \sum_{j=1}^{m} \xi \left(\frac{1}{\mu} [\langle a_j, x \rangle + b^{(j)}] \right) \right).$$

Piece-wise linear functions: Sum of absolute values

$$\min_x \left\{ f(x) = \sum_{j=1}^m |\langle a_j, x\rangle_1 - b^{(j)}| : \; x \in Q_1 \right\}.$$

Let us choose $\quad E_2 = R^m,\; Q_2 = \{u \in R^m : \; |u^{(j)}| \le 1,\, j = 1, \dots, m\},$

and $\; d_2(u) = \frac{1}{2}\|u\|_2^2 = \frac{1}{2}\sum_{j=1}^m \|a_j\|_1^* \cdot (u^{(j)})^2.$

Then $f_\mu(x) = \sum_{j=1}^m \|a_j\|_1^* \cdot \psi_\mu\left(\frac{|\langle a_j, x\rangle_1 - b^{(j)}|}{\|a_j\|_1^*}\right),$

$$\|A\|_{1,2}^2 = P \equiv \sum_{j=1}^m \|a_j\|_1^*.$$

On the other hand, $D_2 = \frac{1}{2}P$ and $\sigma_2 = 1$. Thus, we get the following complexity bound: $\quad \boxed{\frac{1}{\epsilon} \cdot \sqrt{\frac{8D_1}{\sigma_1}} \cdot \sum_{j=1}^m \|a_j\|_1^*.}$

NB: The bound and the scheme allow $m \to \infty$.

Computational experiments

Test problem: $\min\limits_{x \in \Delta_n} \max\limits_{u \in \Delta_m} \langle Ax, u \rangle_2$.

Entries of A are uniformly distributed in $[-1, 1]$.

Goal: Test of computational stability. **Computer:** 2.6GHz.

Complexity of iteration: $2mn$ operations.

Results for $\epsilon = 0.01$. Table 1

$m \backslash n$	100	300	1000	3000	10000
100	808 $0''$	1011 $0''$	1112 $3''$	1314 $12''$	1415 $44''$
300	910 $0''$	1112 $2''$	1415 $10''$	1617 $35''$	1819 $135''$
1000	1112 $2''$	1213 $8''$	1415 $32''$	1718 $115''$	2020 $451''$

Number of iterations: $40 - 50\%$ of predicted values.

Results for $\epsilon = 0.001$. **Table 2**

$m \backslash n$	100	300	1000	3000	10000
100	6970 $2''$	8586 $8'$	9394 $29''$,	10000 $91''$	10908 $349''$
300	7778 $8''$	10101 $27''$	12424 $97''$	14242 $313''$	15656 $1162''$
1000	8788 $30''$	11010 $105''$	13030 $339''$	15757 $1083''$	18282 $4085''$

Results for $\epsilon = 0.0001$. **Table 3**

$m \backslash n$	100	300	1000	3000
100	67068 $25''$	72073 $80''$	74075 $287''$	80081 $945''$
300	85086 $89''$, 42%	92093 $243''$	101102 $914''$	112113 $3302''$
1000	97098 $331''$	100101 $760''$	116117 $2936''$	139140 $11028''$

Comparing the bounds

Smoothing + FGM: $2 \cdot 4 \cdot \frac{mn}{\epsilon} \sqrt{\ln n \ln m}$.

Short-step p.-f. method ($n \geq m$): $\left(7.2\sqrt{n} \ln \frac{1}{\epsilon}\right) \cdot \frac{m(m+1)}{2}\, n$.

Right digits

m	n	2	3	4	5
100	100	g	g	b	b
300	300	g	g	b	b
300	1000	g	g	b	b
300	3000	g	g	$=$	b
300	10000	g	g	g	b
1000	1000	g	g	g	b
1000	3000	g	g	g	b
1000	10000	g	g	g	$=$

g - S+FGM, b - barrier method