Lecture 19

§1 General Powers and General Exponential Functions

1. Definition of general power

For any $a \in (0,\infty)$ and $x \in R$, define

$$a^x \overset{def}{=\!=} e^{x\ln a}$$

In particular, when $a = e$ we have

$$a^x = e^{x\ln a} = e^{x\ln e} = e^x$$

2. Definition of two types of function

1° General exponetial function with base a:

Fix $a \in (0,\infty)$. Define $f(x) = a^x$, $D = R$

2° General power function:

Fix $a \in R$. Define $f(x) = x^a$, $D = (0,\infty)$. Here

$x^a = e^{a\ln x}$ (by definition)

3. Algebraic property 4 for ln and exp also holds for irrational value.

1° $\ln(x^a) = a\ln x$, $\forall x \in (0,\infty)$, $\forall a \in R$

2° $(e^x)^a = e^{ax}$, $\forall x \in R$, $\forall a \in R$

Proof:

1° holds because $e^{\ln(x^a)} = x^a = e^{a\ln x}$

2° can be shown using 1°

4. Algebraic properties for general exponential functions

For any $a \in (0,\infty)$ and any real numbers x, x_1, x_2 and r:

1. $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$
2. $a^{-x} = \frac{1}{a^x}$
3. $a^{x_1}/a^{x_2} = a^{x_1-x_2}$

4. $(a^x)^r = a^{rx}$

Proof of 1:

$$a^{x_1+x_2} = e^{(x_1+x_2)\ln a}$$
$$= e^{x_1\ln a + x_2\ln a}$$
$$= e^{x_1\ln a}\cdot e^{x_2\ln a}$$
$$= a^{x_1}\cdot a^{x_2}$$

5. Value of e

Theorem of e

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

Proof:

1. $(1+x)^{\frac{1}{x}} = e^{(\frac{1}{x})\cdot\ln(1+x)}$, $\forall\, x\in(-1,1)\setminus\{0\}$
2. By 1 and continuity of exp, we have
$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)}$$
$$= e^{\lim_{x\to 0}\frac{1}{x}\ln(1+x)}$$
3. Note that
$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x}$$
$$= \ln'(1)$$
$$= \frac{1}{x}\Big|_{x=1}$$
$$= 1$$
4. By step 2 and step 3 above
$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e^1 = e$$
By looking at values of $(1+x)^{\frac{1}{x}}$ with x close to 0, we get
$e \approx 2.71828182845$

6. General power function: derivative

$\frac{d}{dx} x^a = a x^{a-1}$ for $x \in (0, \infty)$

Proof:

$$\frac{d}{dx} x^a = \frac{d}{dx} e^{a \ln x}$$
$$= e^{a\ln x} a \frac{d}{dx} \ln x$$
$$= e^{a \ln x} \frac{a}{x}$$
$$= x^a \frac{a}{x}$$
$$= a \cdot x^{a-1}$$

e.g. If $f(x) = x^x$ for $x \in (0, \infty)$, what is $f'(x)$

$$f(x) = x^x = e^{x \ln x}$$
So $f'(x) = e^{x\ln x} (\ln x + x \cdot \frac{1}{x})$
$$= x^x (1 + \ln x)$$

7. General exponential functions: derivative

$\frac{d}{dx} a^x = a^x \ln a$

Proof:

For a fixed $a \in (0, \infty)$, let $f(x) = a^x$.
The $f(x) = e^{x \ln a}$, so
$$f'(x) = e^{x\ln a} \cdot \ln a = a^x \ln a$$

Note:

Provided that $a \neq 1$:
$$\int a^x dx = \frac{a^x}{\ln a} + C \text{ , on } R$$

7. General exponential functions: graph

Note that

$$\frac{d}{dx} a^x \begin{cases} > 0, \ \forall x, & \text{if } a > 1 \\ < 0, \ \forall x, & \text{if } 0 < a < 1 \end{cases}$$

So $f(x)=a^x$ is

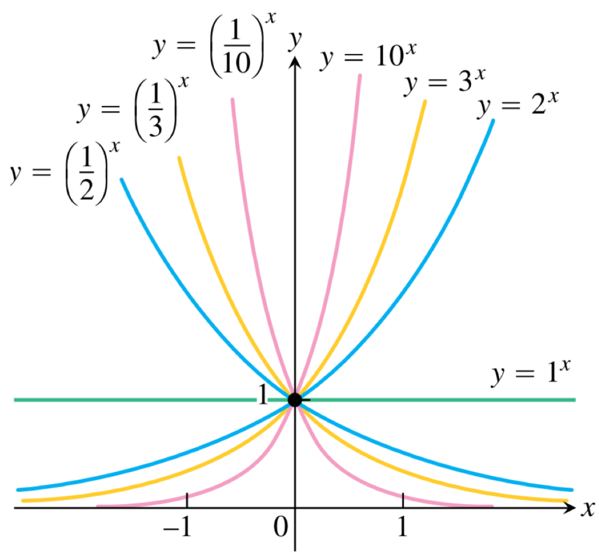
increasing on R if $a>1$

decreasing on R if $0<a<1$

Since

$$f''(x)=(\ln a)^2 a^x > 0,\ \forall x,\ a\neq 1$$

$y=a^x$ is always concave up on R.



§2 General Logarithmic Function

1. Definition of general logarithmic function

Fix $a\in(0,\infty)\setminus\{1\}$. The function f_a given by $f_a(x)=a^x$ is monotonic, so it is one-to-one on domain R.

From the definition $a^x=e^{x\ln a}$ and the fact that range$(\exp)=(0,\infty)$, it follows that range of f_a is also $(0,\infty)$.

Hence, $f_a: R\to(0,\infty)$ has an inverse function $\log_a:(0,\infty)\to R$ called the logarithmic function with base a

By definition, $a^{\log_a x}=x,\ \forall x\in(0,\infty)$

$$\log_a a^x = x,\ \forall x\in R$$

Since $a^{\log_a x} = x$, by taking $\ln$ on both sides, we have:

$$\ln a^{\log_a x} = \ln x$$

$$\log_a x \cdot \ln a = \ln x$$

so, $$\log_a x = \frac{\ln x}{\ln a}$$

2. Algebraic properties for $\log_a$

For any $a \in (0,\infty)\setminus\{1\}$ and any $x, y \in \mathbb{R}_{>0}$

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a \frac{1}{x} = -\log_a x$
4. $\log_a(x^r) = r\log_a x$, for any $r \in \mathbb{R}$

Proof of 1:

$$\log_a xy = \frac{\ln(xy)}{\ln a} = \frac{\ln x + \ln y}{\ln a} = \log_a x + \log_a y$$

3. Derivative of $\log_a$

From the identity $\log_a x = \frac{\ln x}{\ln a}$, we can see that

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

§3 L'Hôpital's Rule

1. Theorem (L'Hôpital's Rule or L'Hospital's Rule)

Let $c \in \mathbb{R}$. Suppose that f and g are differentiable on $D := (c-a, c+a)\setminus\{c\}$ for some $a > 0$, and that $g'(x) \neq 0$ for all $x \in D$.

Suppose that one of the following conditions holds:

$$\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$$

$\lim_{x\to c} f(x) \in \{-\infty, \infty\}$ and $\lim_{x\to c} g(x) \in \{-\infty, \infty\}$.

Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists (or is ∞ or $-\infty$).

Remarks:

(i) L'Hôpital's rule is also valid if we replace "$\lim_{x\to c}$" with "$\lim_{x\to c^+}$" or "$\lim_{x\to c^-}$".

(ii) If D is changed to an unbounded interval, then L'Hopital's rule is also valid if we replace "$\lim_{x\to c}$" with "$\lim_{x\to \infty}$" or "$\lim_{x\to -\infty}$".

e.g. (a) Show that $\lim_{x\to 1} \frac{x^3+3x-4}{x^4-6x^2+5} = -\frac{3}{4}$

Since $\lim_{x\to 1} x^3+3x-4 = 0$

$\lim_{x\to 1} x^4-6x^2+5 = 0$

L'Hôpital's rule applies.

$$\lim_{x\to 1} \frac{x^3+3x-4}{x^4-6x^2+5} = \lim_{x\to 1} \frac{3x^2+3}{4x^3-12x} = \frac{3+3}{4-12} = -\frac{3}{4}$$

(b) Show that $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{1}{6}$

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

(c) Show that $\lim_{x\to 0} \frac{\ln^3(x+1)}{e^x-x-1} = 0$

$$\lim_{x\to 0} \frac{\ln^3(x+1)}{e^x-x-1} = \lim_{x\to 0} \frac{3\ln^2(x+1)\cdot\frac{1}{x+1}}{e^x-1} = \lim_{x\to 0} \frac{3\ln^2(x+1)}{e^x-1} \cdot \lim_{x\to 0} \frac{1}{x+1} = \lim_{x\to 0} \frac{6\ln(x+1)}{e^x(x+1)}$$

$$= 0$$

(d) Show that $\lim_{x\to 2} \frac{\sin(\pi x)}{(x-2)^2}$ does not exist.

$$\lim_{x\to 2^+} \frac{\sin(\pi x)}{(x-2)^2} = \lim_{x\to 2^+} \frac{\pi\cos(\pi x)}{2(x-2)} = \infty$$

$$\lim_{x\to 2^-} \frac{\sin(\pi x)}{(x-2)^2} = \lim_{x\to 2^-} \frac{\pi\cos(\pi x)}{2(x-2)} = -\infty$$

so the limit doesn't exist.

(e) Show that $\lim_{x\to\infty} \frac{\ln x}{x^\pi} = 0$

$$\lim_{x\to\infty} \frac{\ln x}{x^\pi} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\pi x^{\pi-1}} = \lim_{x\to\infty} \frac{1}{\pi x^\pi} = 0$$

(f) Show that $\lim_{x\to\infty} x^2 e^{-\sqrt{x}} = 0$. ("$\infty \cdot 0$")

$$\begin{aligned}
\lim_{x\to\infty} x^2 e^{-\sqrt{x}} &= \lim_{x\to\infty} \frac{x^2}{e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{2x}{e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}}} \\
&= \lim_{x\to\infty} \frac{4x^{\frac{3}{2}}}{e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{12x}{e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{24x^{\frac{1}{2}}}{e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{24}{e^{\sqrt{x}}} \\
&= 0
\end{aligned}$$

(g) Show that $\lim_{x\to 0} (1-2x)^{\frac{3}{x}} = e^{-6}$ ("1^∞")

$$\begin{aligned}
\lim_{x\to 0} (1-2x)^{\frac{3}{x}} &= \lim_{x\to 0} e^{\frac{3\ln(1-2x)}{x}} \\
&= e^{\lim_{x\to 0}\frac{3\ln(1-2x)}{x}} \\
&= e^{\lim_{x\to 0}\frac{-6}{1-2x}} \\
&= e^{-6}
\end{aligned}$$

(h) Show that $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = 0$ ("$\infty - \infty$")

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} &= \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} \\
&= \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} \\
&= \lim_{x\to 0^+} \frac{-\sin x}{2\cos x + x\sin x}
\end{aligned}$$

$= 0$

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PAPER CODE	EXAMINER	DEPARTMENT	TEL
MTH019		Mathematical Sciences	

2019-20 1st SEMESTER FINAL EXAMINATION (Year 1)
CALCULUS (BUSINESS)

Time allowed: 120 minutes

Total marks: 100; Total questions: 20.

INSTRUCTIONS TO CANDIDATES

1. Total marks available are 100 and there are 20 questions in total.
2. All the solutions and answer should be written in the booklet; otherwise 0 marks will be given.
3. You have **120 minutes** to answer the questions.
4. The University approved calculator - **Casio FX82ES/83ES** can be used.
5. **For parts I and II, marks are given solely for correct answers. For part III, the process of solution should be provided. Otherwise, 0 marks will be given.**
6. Please write down your name in the lower left corner of the booklet cover; and in the lower right corner of the booklet cover, please write down your student ID number clearly, and also write down **subgroup number** and **your teacher's name** above the line of 'ID No.'.

Do not open this question booklet until instructed to do so

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I. Fill in the blanks. (3 marks for each question, 24 marks in total)

1. $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$ _______________

2. If the function $f(x) = \begin{cases} \dfrac{\sin x}{3x}, & x < 0 \\ e^x + c, & x \geq 0 \end{cases}$ is continuous at $x = 0$, then $c =$ _______________

3. If $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b), then by the Mean Value Theorem for Derivatives, there is at least a point c in (a, b), such that _______________

4. Let $f(x) = x^4 + 2x^3$. Then $f(x)$ attains local minimum at $x =$ _______________

5. $\displaystyle\int_{-3}^{3} \left(x^2 + \frac{6x^5}{x^6+4}\right) dx =$ _______________

6. If $f(x) = \displaystyle\int_1^x \frac{1}{1+\ln t}\, dt$, then $f'(2) =$ _______________

7. The particular solution of $xy' + y = 1$ that satisfies $y = 2$ when $x = 1$ is _______________

8. The length of the curve determined by the parametric equations $x = e^t \cos t$, $y = e^t \sin t$ for $0 \leq t \leq 1$ is _______________

II. Choose the correct answer. (3 marks for each question, 18 marks in total).

MCQ

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MCQ

III. Calculations and comprehensive problems. (58 marks in total)

15. (10 marks) Basic Computations about Derivatives.

(1) Evaluate $f'(x)$ and $f''(x)$, where $f(x) = e^{5x} + \sin(x^2)$.

(2) Evaluate $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x}$.

16. (10 marks) Compute the following integrals.

(1) $\displaystyle\int \frac{x+3x^5}{x^2}\,dx$.

(2) $\displaystyle\int_0^1 x \arctan x\,dx$.

17. (10 marks) As shown in the Figure 1, the curve C is the graph of the function $f(x) = 4x^2 - x^3$ and L is the tangent line to the curve C at the point $x = 3$. Let S be the region bounded by the curve C, the line L and the x-axis and let R be the region bounded by the curve C and the x-axis.

(1) Find an equation for the tangent line L.

(2) Find the area of S.

(3) Find the volume of the solid of revolution generated by revolving R about the y-axis.

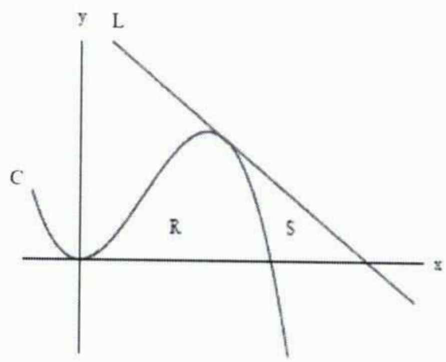

Figure 1 Graph of L, S and R

18. (10 marks) We want to make a rectangular box whose base length is 3 times the base width (See Figure 2). The material used in the top and bottom costs $\$10/\mathrm{m}^2$ and the material used in the sides costs $\$6/\mathrm{m}^2$. If the volume of the box is 60 m^3, determine the values of l, w and h that will minimize the cost to make the box and find the minimum cost.

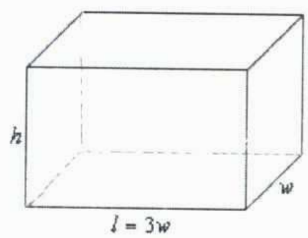

Figure 2 The box

19. (12 marks) The graph of a differentiable function f is shown below for $-3 \leq x \leq 3$ (see Figure 3). The graph of f has horizonal tangent lines at $x = -1$, $x = 1$ and $x = 2$. The area of regions A, B, C and D are $5, 4, 5$ and 3 respectively. Let g be the antiderivative of f such that $g(3) = 7$.

(1) On what open intervals contained in $(-3, 3)$ is the graph of g concave up? Justify your answer.

(2) Let h be the function defined by $h(x) = 3f(2x+1) + 4$. Find the value of $\int_{-2}^{1} h(x)\, dx$.

(3) Find the value of $\lim_{x \to 0} \frac{g(x)+1}{2x}$, or state that it does not exist. Show the work that leads to your answer.

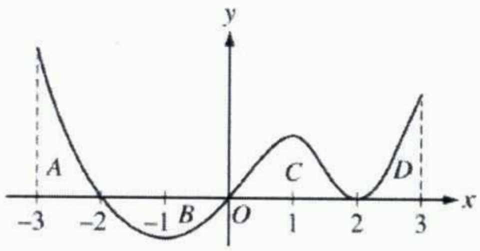

Figure 3 Graph of f

20. (6 marks) Suppose that $f(x)$ is continuous and increasing on $[a, \infty)$. Show that for all $x > a$,

$$\int_a^x t f(t)\, dt > \frac{a+x}{2} \int_a^x f(t)\, dt.$$

End of Exam Paper