Structural Optimization for Large-Scale Problems

Lecture 2: Universal Gradient Methods

P15

Yurii Nesterov

Minicourse: November 15, 16, 22, 23, 2024 (SDS, Shenzhen)

Outline

Smooth and nonsmooth convex functions

Optimization methods

Uniformly convex functions and application example

Composite minimization and Bregman distances

Universal gradient methods

Numerical experiments

§1 先前的结论和缺陷

Smooth convex functions

- Gradient represents a first-order model of the objective:

$$f(x) + \langle \nabla f(x), h \rangle \le f(x+h) \le f(x) + \langle \nabla f(x), h \rangle + o(\|h\|).$$

- For $f \in C^{1,1}$, we can ensure monotonic decrease of the objective:

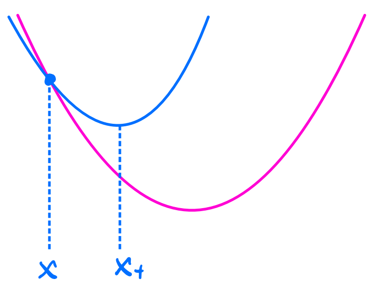
$$x_+ = \arg\min_{y \in Q} \{ f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2} L \|y - x\|^2 \},$$

① 有 descent lemma (确保 monotonic decrease in function value)

$$\boxed{f(x_+) \le \min_{y \in Q} \{ f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2} L \|y - x\|^2 \}.}$$

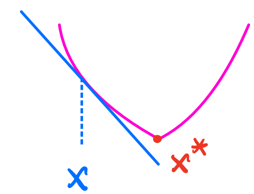
- At unconstrained optimum, the gradient vanishes. ② 可以令 step size 为 const.

  Consequently, in the gradient method $x_+ = x - h \nabla f(x)$, the stepsize $h > 0$ can be constant.



Nonsmooth convex functions

- Subgradient represents a zero-order model of the objective:
$$f(x) + \langle \nabla f(x), h \rangle \le f(x+h) \le f(x) + \langle \nabla f(x), h \rangle + O(\|h\|).$$
- For $f \in C^{0,0}$, we cannot ensure monotonicity. ① 函数值不一定始终下降
- At unconstrained optimum, the gradient does not vanish. ② 不能令 step size 为 const.
- The most useful property of subgradient is
$$\boxed{\langle \nabla f(x), x - x^* \rangle \ge 0,}$$



where x^* is the optimal solution. ③ 可以确定 argument 接近 x^* 的方向

$\langle \nabla f(x), x-y \rangle \ge f(x) - f(y)$

$\Rightarrow$ 取 $y = x^*$ 得证

Optimization methods

Smooth functions ($f \in C^{1,1}$):

- Primal gradient method: $x_{k+1} = \pi_Q(x_k - \frac{1}{L}\nabla f(x_k))$.
- Dual gradient methods:

$$x_{k+1} = \arg\min_{x \in Q} \left\{ \sum_{i=0}^{k} \langle \nabla f(x_i), x - x_i \rangle + \frac{1}{2} L \|x - x_0\|^2 \right\}.$$

(Both are not optimal.) $O(\frac{1}{k})$ 而不是 $O(\frac{1}{k^2})$

Nonsmooth functions ($f \in C^{0,0}$). Primal subgradient schemes:

- $x_{k+1} = \pi_Q(x_k - h_k \nabla f(x_k))$, $h_k > 0$, $h_k \to 0$, $\sum_{k=0}^{\infty} h_k = \infty$.
- $x_{k+1} = \pi_Q\left(x_k - \frac{f(x_k) - f^*}{\|\nabla f(x_k)\|^2} \nabla f(x_k)\right)$. (optimal stepsize)

(Both are optimal.)

按理说简单的问题更该 optimal

$\Rightarrow$ 用 smoothness 来区分 problem class 太粗糙了

§2 新的 problem class: Hölder's continuity

Intermediate problem classes

For finite-dimensional linear vector space E, define a norm $\|\cdot\|$.

$g \in E^*$ 在 $x \in E$ 处的值

Then in the dual space E^*, we have $\|g\|_* \stackrel{\mathrm{def}}{=} \max\limits_{\|x\| \leq 1} \langle g, x \rangle$. (conjugate norm)

包含E中所有 linear functions

确保 C-S inequality

Hölder continuity of the gradients: for some $\nu \in [0,1]$ and all $x, y \in Q$ we have

$$\boxed{\|\nabla f(x) - \nabla f(y)\|_* \leq M_\nu(f) \|x - y\|^\nu.}$$

Notation: $f \in C^{1,\nu}(Q)$.

(descent lemma)

Main property: $\boxed{f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{M_\nu}{1+\nu} \|x - y\|^{1+\nu}}$ for all $x, y \in Q$.

用于确定 algorithm

Proof: Denote $h = y - x$. Then

$$f(y) - f(x) - \langle \nabla f(x), h \rangle = \int_0^1 \langle \nabla f(x + \tau h) - \nabla f(x), h \rangle d\tau$$

$$\leq \|h\| \int_0^1 \|\nabla f(x + \tau h) - \nabla f(x)\|_* d\tau \leq M_\nu \|h\|^{1+\nu} \int_0^1 \tau^\nu d\tau. \qquad \square$$

Examples

1. $\nu = 1$: functions with Lipschitz-continuous gradients. If $f \in C^2$, and the metric is Euclidean, then

$$\nabla^2 f(x) \preceq M_1(f) I, \quad x \in Q.$$

2. $\nu = 0$: functions with bounded variation of subgradients:

$$\|\nabla f(x) - \nabla f(y)\|_* \le M_0(f), \quad x, y \in Q.$$

NB: Addition of linear function does not change the constant $M_0(f)$.

3. Functions with $\nu \in (0, 1)$ are often obtained by duality.

利用 duality 将 p-uniformly convex function f (convexity para σ_p)
转化为 Hölder continuous function f^* (Fenchel dual)

$\nu = \frac{1}{p-1}$ $\quad M_\nu(f^*) = \left(\frac{p}{2\sigma_p}\right)^{\frac{1}{p-1}}$

§3 将 uniformly convex function 转化进新的 problem class

Uniformly convex functions

Def: Let $f(x) \in C^1$. It is *p-uniformly convex of degree* $p \geq 2$ if

$$\boxed{f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{p}\sigma_p \|y - x\|^p \text{ for all } x, y \in E,}$$

where $\sigma_p = \sigma_p(f)$ is the *parameter* of uniform convexity. p-uniformly convex 的性质

Adding such f to a convex function does not change the parameter.
If $p = 2$, then f is *strongly convex*.

Lemma 1. *Let* $\boxed{\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \sigma \|x - y\|^p, \forall x, y \in E}$.
Then function f is p-uniformly convex on E with parameters σ.

p-uniformly convex 的判定

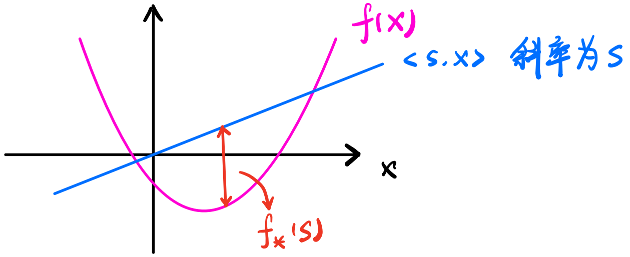
Proof.

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle = \int_0^1 \langle \nabla f(x + \tau(y - x)) - \nabla f(x), y - x \rangle d\tau$$

$$= \int_0^1 \frac{1}{\tau} \langle \nabla f(x + \tau(y - x)) - \nabla f(x), \tau(y - x) \rangle d\tau$$

$$\geq \int_0^1 \sigma \tau^{p-1} \|y - x\|^p d\tau \;=\; \frac{1}{p}\sigma \|y - x\|^p. \qquad \square$$

Duality



For $f(x) \in C^1$ define its *Fenchel dual*: $\boxed{f_*(s) = \sup\limits_{x\in E}[\langle s, x\rangle - f(x)].}$

NB: $\nabla f_*(s) = x_f(s) = \arg\max\limits_{x\in E}[\langle s, x\rangle - f(x)], \qquad \nabla f(x_f(s)) = s.$

Lemma 2. If f is p-uniformly convex, then $f_* \in C^{1,\nu}$ with

$\frac{1}{\nu} + 1 = p$

$$\nu = \frac{1}{p-1}, \qquad M_\nu(f_*) = \left(\frac{p}{2\sigma_p}\right)^{\frac{1}{p-1}}.$$

p-uniformly convex

$\downarrow$ dual

Hölder continuity

Proof. For two points s_1 and s_2, denote $x_i = x_f(s_i)$. Then

$$f(x_{3-i}) \geq f(x_i) + \langle \nabla f(x_i), x_{3-i} - x_i\rangle + \tfrac{1}{p}\sigma_p\|x_{3-i} - x_i\|^p, \ i = 1, 2.$$

Adding these inequalities, we get

$$\tfrac{2}{p}\sigma_p\|x_1 - x_2\|^p \leq \langle s_1 - s_2, x_1 - x_2\rangle \leq \|s_1 - s_2\|_*\|x_1 - x_2\|.$$

□

Example

1. Consider $f(\tau) = \frac{1}{3}|\tau|^3$, $\tau \in \mathbb{R}$. Then $\nabla f(\tau) = \tau|\tau|$. Note that

$$
\begin{aligned}
(\nabla f(\tau_1) - \nabla f(\tau_2))(\tau_1 - \tau_2) &= \left|\tau_1|\tau_1| - \tau_2|\tau_2|\right| \cdot |\tau_1 - \tau_2| \\
&\geq \frac{1}{2}|\tau_1 - \tau_2|^3.
\end{aligned}
$$

(3-uniformly convex with para $\frac{1}{2}$)

Hence, $f_*(\xi) = \max_\tau \left[\xi\tau - \frac{1}{3}|\tau|^3\right] = \frac{2}{3}|\xi|^{\frac{3}{2}} \in C^{1,1/2}$, and

$$M_{1/2} = \left[\frac{3}{2 \cdot \frac{1}{2}}\right]^{1/2} = \sqrt{3}.$$

2. Consider $F(x) = \frac{1}{3}\sum_{i=1}^n \alpha_i |x^{(i)}|^3$. Then for $\|h\|_\alpha^3 \stackrel{\text{def}}{=} \sum_{i=1}^n \alpha_i |h^{(i)}|^3$

$$\langle \nabla F(x) - \nabla F(y), x - y\rangle \geq \frac{1}{2}\|x - y\|_\alpha^3 \quad (\alpha > 0).$$

Therefore the dual function $F_*(s) = \frac{2}{3}\sum_{i=1}^n \frac{1}{\sqrt{\alpha_i}}|s^{(i)}|^{3/2}$ is in $C^{1,1/2}$ with

$= \frac{2}{3}\left(\|s\|_\alpha^*\right)^{\frac{3}{2}}$

$M_{1/2} = \sqrt{3}$. Note that $\|s\|_\alpha^* = \left[\sum_{i=1}^n \frac{1}{\sqrt{\alpha_i}}|s^{(i)}|^{3/2}\right]^{2/3}$ (Check!)

Application Example: Gas Network

将 primal problem 转化为 dual problem，接着可以构造 $f_*(s)$ 的形式，这部分是 $f(x)$ 的 Fechel dual. 若 $f(x)$ 为 p-uniformly convex，则 dual problem 的 objective 为 Holder cont.

Given:

- Structure of pipe lines.
- Length and diameter of each pipe.
- Positions and required volumes for sources and sinks.

Goal: Compute the flows in the pipes and pressure at the nodes.

Equilibrium principle: the flows minimize the dispersed energy.

$$\min_{f\in\mathbb{R}^n}\left\{\frac{1}{3}\sum_{i=1}^n \alpha_i|f_i|^3 : \; Af = d\right\}.$$

Duality:

$$\min_{f\in\mathbb{R}^n}\max_{y\in\mathbb{R}^m}\left\{\frac{1}{3}\sum_{i=1}^n \alpha_i|f_i|^3 + \langle y, d - Af\rangle\right\}$$ (dual problem)

$$= \max_{y\in\mathbb{R}^m}\underline{\min_{f\in\mathbb{R}^n}\left\{\frac{1}{3}\sum_{i=1}^n \alpha_i|f_i|^3 - \langle A^T y, f\rangle + \langle y, d\rangle\right\}}$$

F-dual: $\boxed{f_*(s) = \sup_{x\in E}[\langle s, x\rangle - f(x)].}$

$= -\sup_{f\in\mathbb{R}^n}\{\langle A^Ty, f\rangle - \frac{1}{3}\sum_{i=1}^n \alpha_i|f_i|^3\}$

$$= \max_{y\in\mathbb{R}^m}\left\{\langle d, y\rangle - \frac{2}{3}\left(\|A^T y\|_\alpha^*\right)^{3/2}\right\}.$$ (Dual objective is in $C^{1,1/2}$.)

$:= -f_*(A^Ty)$,

加上这一项不改变 para

即 $f(x) = \frac{1}{3}\sum_{i=1}^n \alpha_i|x_i|^3$ 的 Fenchel dual 的相反数

Structure of Holder constants

(在研究方法前，首先介绍 M_ν 的性质)

Hölder constant 的定义

Define $\boxed{M_\nu \equiv M_\nu(f) = \sup\limits_{\substack{x,y \in Q,\\ x \neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|_*}{\|x-y\|^\nu}}, \quad \nu \geq 0.$

Since $\ln M_\nu = \sup\limits_{\substack{x,y \in Q,\\ x \neq y}} [\ \ln \|\nabla f(x) - \nabla f(y)\|_* - \nu \ln \|x - y\|\ ]$,

M_ν is a *log-convex* function of ν.

- For certain $\nu \in [0,1]$, M_ν can be infinite. (某些 M_ν 可能为 ∞)
- If M_0 and M_1 are finite, then $M_\nu \leq M_0^{1-\nu} M_1^\nu$, $0 \leq \nu \leq 1$. (若 M_0 和 M_1 bdd，则 M_ν bdd，$\forall \nu$)
- If $M_\nu < \infty$, then $\|\nabla f(x) - \nabla f(y)\|_* \leq M_\nu \|x - y\|^\nu$, $x, y \in Q$.

Therefore, (descent lemma)

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{M_\nu}{1+\nu} \|x - y\|^{1+\nu}, \quad x, y \in Q.$$

$\boxed{\textbf{Assumption: } \hat{M}(f) \stackrel{\text{def}}{=} \inf\limits_{0 \leq \nu \leq 1} M_\nu(f) < +\infty.}$

后面只考虑这种情况

§4 minimization method

Composite Minimization and Bregman distances

Problem: $\boxed{\min_{x\in Q}\left[\ \tilde{f}(x) \stackrel{\text{def}}{=} f(x)+\Psi(x)\ \right]}$, where

- Q is a simple closed convex set,
- Ψ is a *simple* closed convex function (e.g. squared Euclidean norm, l_1-norm, barrier functions, indicator of convex set, etc.).
- f is assumed to be subdifferentiable on Q.

Prox-function $d(x)$: a differentiable ① strongly convex function:

$d(y) \geq d(x) + \langle \nabla d(x), y-x\rangle + \frac{1}{2}\|x-y\|^2, \quad x, y \in \text{rint}\, Q.$

② 最小值为0

Let $d(x)$ attain its minimum on Q at x_0, and $d(x_0)=0$.

Thus, $\quad d(x) \geq \frac{1}{2}\|x-x_0\|^2, \quad x \in Q.$

Prox-function defines the *Bregman distance*: (一种类似于 $\|\cdot\|$ 的 distance)

$$\xi(x,y) \stackrel{\text{def}}{=} d(y) - d(x) - \langle \nabla d(x), y-x\rangle.$$

Clearly, ① $\xi(x,x)\equiv 0$, and ② $\xi(x,y) \geq \frac{1}{2}\|x-y\|^2, \quad x,y\in Q.$

Bregman Mapping

先前为 $\frac{1}{2}\|\cdot\|^2$

For any $x \in Q$ we can define the *Bregman mapping* $\mathcal{B}_M(x) =$

$$\arg\min_{y \in Q} \left\{ \psi_M(x,y) \stackrel{\text{def}}{=} f(x) + \langle \nabla f(x), y - x \rangle + M\xi(x,y) + \Psi(y) \right\}.$$

Assumption: This point is easily computable. (因为 Ψ simple)

First-order optimality condition for the auxiliary optimization problem: $\forall y \in Q$

不可微时考虑 subgradient

$$\langle \nabla f(x) + M(\nabla d(\mathcal{B}_M(x)) - \nabla d(x)) + \nabla \Psi(\mathcal{B}_M(x)), y - \mathcal{B}_M(x) \rangle \geq 0.$$

$\frac{\partial}{\partial y} \psi_M(x,y) \big|_{\mathcal{B}_M(x)}$

Denote $\psi_M^*(x) = \psi_M(x, \mathcal{B}_M(x))$.

(To be compared with $\tilde{f}(\mathcal{B}_M(x))$.) 想证明 $\tilde{f}(\mathcal{B}_M(x)) \leq \psi_M^*(x)$

$\tilde{f}(x) = f(x) + \Psi(x)$ 在 x 处的 local model 的最小值

Main Lemma

Lemma: If $M \geq \left[\frac{1}{\delta}\right]^{\frac{1-\nu}{1+\nu}} M_\nu^{\frac{2}{1+\nu}}$ with $\delta > 0$, then for $x, y \in Q$

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2} M \|y - x\|^2 + \tfrac{\delta}{2}.$$

Therefore, $\tilde{f}(\mathcal{B}_M(x)) \leq \psi_M^*(x) + \frac{\delta}{2}$. 允许有误差时, $\tilde{f}(B_M(x)) \leq \psi_M^*(x)$

Proof: For $\tau, s > 0$, we have $\frac{1}{p}\tau^p + \frac{1}{q}s^q \geq \tau s$, with $\frac{1}{p} + \frac{1}{q} = 1$.

Taking $p = \frac{2}{1+\nu}$, $q = \frac{2}{1-\nu}$, and $\tau = t^{1+\nu}$, we get

$$t^{1+\nu} \leq \frac{1+\nu}{2s} t^2 + \frac{1-\nu}{2} s^{\frac{1+\nu}{1-\nu}}.$$

Denote $\delta = \frac{1-\nu}{1+\nu} M_\nu s^{\frac{1+\nu}{1-\nu}}$. Then $s = \left[\frac{1+\nu}{1-\nu} \cdot \frac{\delta}{M_\nu}\right]^{\frac{1-\nu}{1+\nu}}$. Therefore,

$$\frac{M_\nu}{1+\nu} t^{1+\nu} \leq \frac{1}{2s} M_\nu t^2 + \frac{\delta}{2} = \frac{1}{2}\left[\frac{1-\nu}{1+\nu} \cdot \frac{1}{\delta}\right]^{\frac{1-\nu}{1+\nu}} M_\nu^{\frac{2}{1+\nu}} t^2 + \frac{\delta}{2} \leq \frac{1}{2} M t^2 + \frac{\delta}{2}.$$

Proof (continued)

Further, denoting $x_+ = \mathcal{B}_M(x)$, we obtain:

$$\begin{aligned} f(x_+) &\leq f(x) + \langle \nabla f(x), x_+ - x \rangle + \frac{M_\nu}{1+\nu} \|x_+ - x\|^{1+\nu} \\ &\leq f(x) + \langle \nabla f(x), x_+ - x \rangle + \frac{M}{2} \|x_+ - x\|^2 + \frac{\delta}{2} \\ &\leq f(x) + \langle \nabla f(x), x_+ - x \rangle + M\xi(x, x_+) + \frac{\delta}{2}. \end{aligned}$$

类似 $x+y \geq 2\sqrt{xy}$

$(\xi(x,y) \geq \frac{1}{2}\|x-y\|^2)$

Therefore, $\tilde{f}(x_+) = f(x_+) + \Psi(x_+) \leq \psi_M^*(x) + \frac{\delta}{2}$. □

Universal Primal Gradient Method (PGM)

Initialization. Choose $L_0 > 0$ and accuracy $\epsilon > 0$.

For $k \geq 0$ **do:**

1. Find the smallest $i_k \geq 0$ such that

   对M进行"line search"

   $$\tilde{f}\left(\mathcal{B}_{2^{i_k} L_k}(x_k)\right) \leq \psi^*_{2^{i_k} L_k}(x_k) + \frac{1}{2}\epsilon.$$

2. Set $x_{k+1} = \mathcal{B}_{2^{i_k} L_k}(x_k)$, and $L_{k+1} = 2^{i_k - 1} L_k$.

   多除以2，给 L_k 下降的余地

PGM: convergence

Denote $\quad \gamma(M,\epsilon) \stackrel{\mathrm{def}}{=} \left[\frac{1}{\epsilon}\right]^{\frac{1-\nu}{1+\nu}} M^{\frac{2}{1+\nu}}, \qquad$ and

$$S_k = \sum_{i=1}^{k+1} \frac{1}{L_k}, \quad \tilde{f}_k^* = \frac{1}{S_k} \sum_{i=0}^{k} \frac{1}{L_{i+1}} \tilde{f}(x_i).$$

Theorem: Let $M_\nu(f) < \infty$ and $L_0 \le \gamma(M_\nu, \epsilon)$.

Then for all $k \ge 0$ we have $L_{k+1} \le \gamma(M_\nu, \epsilon)$. Moreover, for all $y \in Q$

$$\tilde{f}_k^* \le \frac{1}{S_k} \sum_{i=0}^{k} \frac{1}{L_{i+1}} [f(x_i) + \langle \nabla f(x_i), y - x_i \rangle] + \Psi(y) + \frac{\epsilon}{2} + \frac{2}{S_k} \xi(x_0, y).$$

Therefore, $\boxed{\tilde{f}_k^* - \tilde{f}(x^*) \le \frac{\epsilon}{2} + \frac{2\gamma(M_\nu,\epsilon)}{k+1} \xi(x_0, x^*)}$.

除了k,还取决于ε,因此要想找出k,需要解不等式

Proof, page 1

Let us fix $y \in Q$. Denote $r_k(y) \stackrel{\mathrm{def}}{=} \xi(x_k, y)$. Then (by FOOC)

$$\begin{aligned} r_{k+1}(y) &= d(y) - d(x_{k+1}) - \langle \nabla d(x_{k+1}), y - x_{k+1} \rangle \\ &\leq d(y) - d(x_{k+1}) - \langle \nabla d(x_k), y - x_{k+1} \rangle \\ &\quad + \tfrac{1}{2L_{k+1}} \langle \nabla f(x_k) + \nabla \Psi(x_{k+1}), y - x_{k+1} \rangle. \end{aligned}$$

Note that

$$\begin{aligned} & d(y) - d(x_{k+1}) - \langle \nabla d(x_k), y - x_{k+1} \rangle \\ = \;& d(y) - d(x_k) - \langle \nabla d(x_k), x_{k+1} - x_k \rangle - \xi(x_k, x_{k+1}) \\ & - \langle \nabla d(x_k), y - x_{k+1} \rangle \quad = \quad r_k(y) - \xi(x_k, x_{k+1}). \end{aligned}$$

Proof, page 2

Thus, $r_{k+1}(y) - r_k(y) \;\leq$

$$\frac{1}{2L_{k+1}} \langle \nabla f(x_k) + \nabla \Psi(x_{k+1}), y - x_{k+1} \rangle - \xi(x_k, x_{k+1})$$

$$= \frac{1}{2L_{k+1}} \langle \nabla \Psi(x_{k+1}), y - x_{k+1} \rangle - \frac{1}{2L_{k+1}} \Big(\langle \nabla f(x_k), x_{k+1} - x_k \rangle$$

$$+ 2L_{k+1} \xi(x_k, x_{k+1}) \Big) + \frac{1}{2L_{k+1}} \langle \nabla f(x_k), y - x_k \rangle$$

$$\leq \frac{1}{2L_{k+1}} \left(\Psi(y) - \Psi(x_{k+1}) + f(x_k) - f(x_{k+1}) + \tfrac{\epsilon}{2} + \langle \nabla f(x_k), y - x_k \rangle \right).$$

Hence, $\quad \frac{1}{2L_{k+1}} \tilde{f}(x_{k+1}) + r_{k+1}(y)$

$$\leq \frac{1}{2L_{k+1}} \left(f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \Psi(y) + \tfrac{\epsilon}{2} \right) + r_k(y).$$

Summing up these inequalities, we obtain

$$\tilde{f}_k^* \quad \leq \frac{1}{S_k} \sum_{i=0}^{k} \frac{1}{L_{i+1}} \left[f(x_i) + \langle \nabla f(x_i), y - x_i \rangle \right] + \Psi(y) + \tfrac{\epsilon}{2} + \tfrac{2}{S_k} r_0(y). \Box$$

Consequences

$k+1 \geq 4 \cdot \left(\frac{M}{\varepsilon}\right)^{\frac{2}{1+\nu}} \cdot \xi(x_0, x^*)$

Complexity: $\quad \frac{\epsilon}{2} + \frac{2\gamma(M_\nu,\epsilon)}{k+1}\xi(x_0, x^*) \leq \epsilon \quad$ with

$\gamma(M,\epsilon) = \left[\frac{1}{\epsilon}\right]^{\frac{1-\nu}{1+\nu}} M^{\frac{2}{1+\nu}}$. Hence, we need

$$\boxed{4\xi(x_0, x^*) \inf_{0\leq\nu\leq 1} \left(\frac{M_\nu}{\epsilon}\right)^{\frac{2}{1+\nu}} \text{ iterations.}}$$

Stopping criterion.

Assume we have a bound $\quad \xi(x_0, x^*) \leq D$. $\quad x_0$到x^*的距离$\leq D$

Denote $\ell_k^p(y) \stackrel{\text{def}}{=} \frac{1}{S_k}\sum_{i=0}^{k} \frac{1}{L_{i+1}}\left[f(x_i) + \langle \nabla f(x_i), y - x_i\rangle\right]$, and define

$$\hat{f}_k = \min_{y\in Q}\{\ell_k^p(y) + \Psi(y) : \ \xi(x_0, y) \leq D\}.$$

Then $\tilde{f}_k^* - \tilde{f}(x^*) \leq \tilde{f}_k^* - \hat{f}_k \leq \frac{2\gamma(M_\nu,\epsilon)}{k+1} D$.

Thus, we have implementable $\boxed{\text{stopping criterion } \tilde{f}_k^* - \hat{f}_k \leq \epsilon.}$

Number of calls of oracle （考虑 line search）

Denote by $N(k)$, the total number of computations of function values in PGM after k iterations. Note that

$$L_{k+1} = \tfrac{1}{2} 2^{i_k} L_k.$$

Therefore, $i_k - 1 = \log_2 \frac{L_{k+1}}{L_k}$. Hence, for any $\nu \in [0,1]$, we have

$$\begin{aligned} N(k) &= \sum_{j=0}^{k} (i_j + 1) \;=\; 2(k+1) + \log_2 L_{k+1} - \log_2 L_0 \\ &\leq \; 2(k+1) + \tfrac{1-\nu}{1+\nu} \log_2 \tfrac{1}{\epsilon} + \tfrac{2}{1+\nu} \log_2 M_\nu - \log_2 L_0. \end{aligned}$$

Finally, we come to the following upper bound:

$$\boxed{N(k) \leq 2(k+1) - \log_2 L_0 + \inf_{0 \leq \nu \leq 1} \left[\tfrac{1-\nu}{1+\nu} \log_2 \tfrac{1}{\epsilon} + \tfrac{2}{1+\nu} \log_2 M_\nu \right].}$$

Thus in average, PGM needs <u>two</u> computations of function values per iteration.

$O(\underbrace{\frac{1}{\gamma_1(f)}}_{\text{问题的 condition number}} \cdot \ln \frac{1}{\varepsilon})$

Universal Dual Gradient Method (DGM)

Initialization. Choose $L_0 > 0$. Define $\phi_0(x) = \xi(x_0, x)$.

For $k \geq 0$ **do:**

1. Find the smallest $i_k \geq 0$ such that for point
$x_{k,i_k} = \arg\min_{x \in Q} \left\{ \phi_k(x) + \frac{1}{2^{i_k} L_k} [f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \Psi(x)] \right\}$

we have $\quad \tilde{f}(x_{k,i_k}) \leq \psi^*_{2^{i_k} L_k}(x_{k,i_k}) + \frac{\epsilon}{2}$.

2. Set $x_{k+1} = x_{k,i_k}$, $L_{k+1} = 2^{i_k - 1} L_k$, and

$\phi_{k+1}(x) = \phi_k(x) + \frac{1}{2L_{k+1}} [f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \Psi(x)]$.

Convergence of DGM

Theorem. For all $k \geq 0$ and $\nu \in [0,1]$ we have

$$\tilde{f}_k^* - \tilde{f}(x^*) \leq \frac{\epsilon}{2} + \frac{2\gamma(M_\nu,\epsilon)}{k+1}\xi(x_0,x^*).$$

Complexity: $4\xi(x_0,x^*) \inf\limits_{0\leq\nu\leq 1} \left(\frac{M_\nu}{\epsilon}\right)^{\frac{2}{1+\nu}}$ iterations. (不是number of oracles数)

Average # of calls: 2 per iteration.

NB: for $\nu \in (0,1]$ the complexity is not optimal!

Universal Fast Gradient Method (FGM)

Choose $L_0 > 0$. Define $\phi_0(x) = \xi(x_0, x)$, $y_0 = x_0$, $A_0 = 0$.
For $k \geq 0$ **do:**
1. Find $v_k = \arg\min\limits_{x \in Q} \phi_k(x)$.
2. Find the smallest $i_k \geq 0$ such that a_{k+1,i_k}, computed from equation $a_{k+1,i_k}^2 = \frac{1}{2^{i_k} L_k}(A_k + a_{k+1,i_k})$ and used in the definitions

$$A_{k+1,i_k} = A_k + a_{k+1,i_k}, \ \ \tau_{k,i_k} = \frac{a_{k+1,i_k}}{A_{k+1,i_k}}, \ \ x_{k+1,i_k} = \tau_{k,i_k} v_k + (1 - \tau_{k,i_k}) y_k,$$

$$\hat{x}_{k+1,i_k} = \arg\min_{y \in Q} \left\{ \xi(v_k, y) + a_{k+1,i_k}[\langle \nabla f(x_{k+1,i_k}), y \rangle + \Psi(y)] \right\},$$

$y_{k+1,i_k} = \tau_{k,i_k} \hat{x}_{k+1,i_k} + (1 - \tau_{k,i_k}) y_k$, ensures the following relation:

$$\begin{aligned} f(y_{k+1,i_k}) \quad &\leq \quad f(x_{k+1,i_k}) + \langle \nabla f(x_{k+1,i_k}), y_{k+1,i_k} - x_{k+1,i_k} \rangle \\ &\quad\quad + 2^{i_k - 1} L_k \| y_{k+1,i_k} - x_{k+1,i_k} \|^2 + \tfrac{\epsilon}{2} \tau_{k,i_k}. \end{aligned}$$

3. Set $x_{k+1} = x_{k+1,i_k}$, $y_{k+1} = y_{k+1,i_k}$, $a_{k+1} = a_{k+1,i_k}$, $\tau_k = \tau_{k,i_k}$.
Define $A_{k+1} = A_k + a_{k+1}$, $L_{k+1} = 2^{i_k - 1} L_k$, and
$\phi_{k+1}(x) = \phi_k(x) + a_{k+1}[f(x_{k+1}) + \langle \nabla f(x_{k+1}), x - x_{k+1} \rangle + \Psi(x)]$.

Convergence of FGM

Theorem. For all $k \geq 0$ we have

$$A_k \left(\tilde{f}(y_k) - \tfrac{\epsilon}{2} \right) \leq \phi_k^* \stackrel{\mathrm{def}}{=} \min_{x \in Q} \phi_k(x),$$

where $A_k \geq \left[\frac{1}{2^{2+4\nu} M_\nu^2} \, \epsilon^{1-\nu} \, k^{1+3\nu} \right]^{\frac{1}{1+\nu}}$.

Consequently, for all $k \geq 1$ we have

$$\tilde{f}(y_k) - \tilde{f}(x^*) \leq \left[\frac{2^{2+4\nu} M_\nu^2}{\epsilon^{1-\nu} \, k^{1+3\nu}} \right]^{\frac{1}{1+\nu}} \xi(x_0, x^*) + \tfrac{\epsilon}{2}.$$

Complexity: $k \leq \inf\limits_{0 \leq \nu \leq 1} \left[\left(\frac{2^{\frac{3+5\nu}{2}} M_\nu}{\epsilon} \right)^{\frac{2}{1+3\nu}} \xi(x_0, x^*)^{\frac{1+\nu}{1+3\nu}} \right].$

It is optimal! (Note quasi-convexity in ν.)

Calls per iteration: four.

Numerical experiments

1. Matrix game: $\min\limits_{x \in \Delta_n} \max\limits_{y \in \Delta_m} \langle x, Ay \rangle$

$$= \min_{x \in \Delta_n} \left\{ \psi_p(x) \stackrel{\mathrm{def}}{=} \max_{1 \le j \le m} \langle x, Ae_j \rangle \right\} = \max_{y \in \Delta_m} \left\{ \psi_d(y) \stackrel{\mathrm{def}}{=} \min_{1 \le i \le n} \langle e_i, Ay \rangle \right\}.$$

It can be posed as a minimization problem

$$\min_{x \in \Delta_n, y \in \Delta_m} \{ \psi_{pd}(x, y) = \psi_p(x) - \psi_d(y) \}$$

with optimal value zero. We generate $A_{i,j} \in [-1, 1]$ randomly.

For $\mathcal{F} = \{ z = (x, y) : \; x \in \Delta_n, \; y \in \Delta_m \}$, natural prox-function is the *entropy*:

$$\eta(z) = \sum_{i=1}^{n} z^{(i)} \ln z^{(i)}.$$

It is strongly convex in ℓ_1-norm (good for measuring simplexes).

Entropy Setup ($n = 896$, $m = 128$)

Eps	$\text{FGM}_{Entropy}$			$\text{PGM}_{Entropy}$		
2^{-5}	516	$6.0E{-}2$	$1.3E2$	722	$8.2E{-}2$	8.0
2^{-6}	1127	$2.9E{-}2$	$2.6E2$	2065	$5.2E{-}2$	$1.6E1$
2^{-7}	1937	$1.6E{-}2$	$2.0E2$	5675	$3.4E{-}2$	$3.2E1$
2^{-8}	4684	$7.9E{-}3$	$2.0E3$	15731	$2.3E{-}2$	$6.4E1$
2^{-9}	8129	$3.8E{-}3$	$8.2E3$	44829	$1.5E{-}2$	$1.3E2$
2^{-10}	17556	$2.1E{-}3$	$4.1E3$	122959	$1.0E{-}2$	$2.6E2$

FGM: $O\left(\frac{1}{\epsilon}\right)$.

PGM: $O\left(\frac{1}{\epsilon^{1.57}}\right)$.

Continuous Steiner problem ($n = 256,\ m = 512$)

$$\min_{x \in Q} f(x) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{m} \|x - a_i\|. \qquad \text{(Euclidean norms)}$$

Eps		FGM_{Euclid}			PGM_{Euclid}	
2^{-5}	205	$3.1E{-2}$	$2.6E2$	9925	$3.1E{-2}$	$2.6E2$
2^{-6}	307	$1.5E{-2}$	$5.1E2$	19895	$1.5E{-2}$	$5.1E2$
2^{-7}	277	$6.8E{-3}$	$2.6E2$	39803	$7.8E{-3}$	$2.6E2$
2^{-8}	611	$3.9E{-3}$	$5.1E2$	77138	$3.9E{-3}$	$5.1E2$
2^{-9}	827	$1.9E{-3}$	$5.1E2$	155038	$2.0E{-3}$	$2.6E2$
2^{-10}	1226	$9.8E{-4}$	$2.6E2$		out of time	
2^{-11}	1655	$4.8E{-4}$	$2.6E2$			
2^{-12}	2385	$2.4E{-4}$	$5.1E2$			
2^{-13}	3388	$1.2E{-4}$	$5.1E2$			

FGM: $O(\frac{1}{\epsilon^{1/2}})$, **PGM:** $O(\frac{1}{\epsilon})$.