Lecture 16

§1 Gravitational inside Earth

1. Newton's shell theorem

1° 一个均匀 shell 对其内部一点，施加的 gravitational force 为 0

Newton's shell theorem can also be applied to a situation in which a particle is located *inside* a uniform shell, to show the following:

> A uniform shell of matter exerts no net gravitational force on a particle located inside it.

2° 对于地球内部一点（距球心 r，该心内侧球体质量为 M_{ins}）：

① inner mass

$$\rho = \frac{M_{ins}}{\frac{4}{3}\pi r^3} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$M_{ins} = \frac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3$$

② gravitational force

$$F = \frac{GmM_{ins}}{r^2} = \frac{GmM}{R^3} \cdot r$$

in vector form:

$$\vec{F} = -K\vec{r}$$

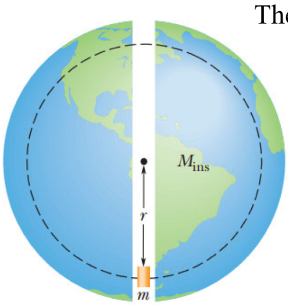

Figure 13-7 A capsule of mass m falls from rest through a tunnel that connects Earth's south and north poles. When the capsule is at distance r from Earth's center, the portion of Earth's mass that is contained in a sphere of that radius is M_{ins}.

The magnitude of the gravitation force is

$$F = \frac{GmM_{ins}}{r^2}.$$

We assume that the density if uniform

$$\rho = \frac{M_{ins}}{\frac{4}{3}\pi r^3} = \frac{M}{\frac{4}{3}\pi R^3}.$$

The inner mass is

$$M_{ins} = \tfrac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3.$$

We have

$$F = \frac{GmM}{R^3} r.$$

In the vector form

$$\vec{F} = -K\vec{r},$$

2

2. Gravitational potential energy（重力势能）

1° 靠近地表时

- gravitational force is conservative
- distance ↓ , potential energy ↑
- difference = mgh

2° 远离地表时

- reference 为无穷远处. At $r \to \infty$, $U \to 0$
- 对于由两点组成的系统, gravitational potential energy 为

$$U = -\frac{GMm}{r}$$

Proof

The work done by the gravitational force is

$$W = \int_R^\infty \vec{F}(r) \cdot d\vec{r}.$$

For the dot product,

$$\vec{F}(r) \cdot d\vec{r} = F(r)\, dr \cos\phi,$$

Since $\phi = 180^o$,

$$\vec{F}(r) \cdot d\vec{r} = -\frac{GMm}{r^2} dr,$$

We have

$$W = -GMm \int_R^\infty \frac{1}{r^2} dr = \left[\frac{GMm}{r}\right]_R^\infty$$

$$= 0 - \frac{GMm}{R} = -\frac{GMm}{R},$$

Proof

The mechanical energy is conservative,

$$U_\infty - U = -W.$$

Taking $U_\infty = 0$, we have

$$U = W = -\frac{GMm}{R}.$$

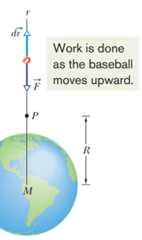

Figure 13-9 A baseball is shot directly away from Earth, through point P at radial distance R from Earth's center. The gravitational force $\vec{F}$ on the ball and a differential displacement vector $d\vec{r}$ are shown, both directed along a radial r axis.

5

- 对于三点系统：

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

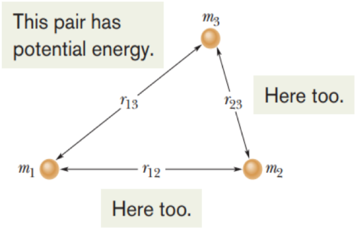

Figure 13-8 A system consisting of three particles. The gravitational potential energy *of the system* is the sum of the gravitational potential energies of all three pairs of particles.

3° Path independence

$$\Delta U = U_f - U_i = -W$$

4° 势能与力

$$\Delta U = U_f - U_i = -W = -F(r)\Delta r$$

因此有：

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right)$$
$$= -\frac{GMm}{r^2}$$

3. Escape speed （脱离速度）

- $U = -\frac{GMm}{R}$
- $K + U = \frac{1}{2}Mv^2 + \left(-\frac{GMm}{R}\right) = 0$
- $v = \sqrt{\frac{2GM}{R}}$

§2 Planets and satellites: Kepler's Laws （开普勒定律）

1. 第一定律：Law of orbits

所有行星绕太阳运动的轨道都是椭圆，太阳在椭圆的一个焦点上.

1. THE LAW OF ORBITS: All planets move in elliptical orbits with Sun at one focus

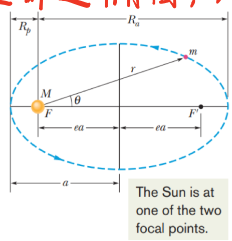

Figure 13-12 A planet of mass m moving in an elliptical orbit around the Sun. The Sun, of mass M, is at one focus F of the ellipse. The other focus is F', which is located in empty space. The semimajor axis a of the ellipse, the perihelion (nearest the Sun) distance R_p, and the aphelion (farthest from the Sun) distance R_a are also shown.

2. 第二定律：Law of areas

对于一个行星，其与太阳连线在相等时间间隔内扫过面积相等

The instantaneous rate at which area is being swept out is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega,$$

The angular momentum is

$$L = rp_\perp = (r)(mv_\perp) = (r)(m\omega r)$$
$$= mr^2\omega,$$

Based on the above two equations

$$\frac{dA}{dt} = \frac{L}{2m}.$$

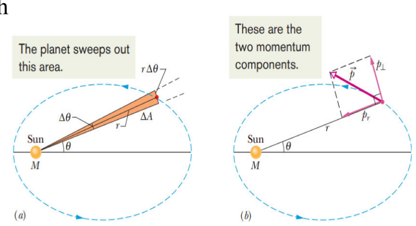

Figure 13-13 (a) In time Δt, the line r connecting the planet to the Sun moves through an angle $\Delta\theta$, sweeping out an area ΔA (shaded). (b) The linear momentum $\vec{p}$ of the planet and the components of $\vec{p}$.

Law of conservation of angular momentum

3、第三定律：Law of periods

所有行星绕太阳一周的恒星时间（T_i）的平方与它们轨道半长轴的立方成正比

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

For a circular motion,

$$\frac{GMm}{r^2} = (m)(\omega^2 r).$$

As we know,

$$T = 2\pi/\omega$$

We have

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \quad \text{(law of periods).}$$

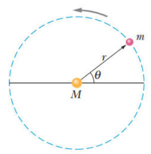

Figure 13-14 A planet of mass m moving around the Sun in a circular orbit of radius r.

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

Table 13-3 Kepler's Law of Periods for the Solar System

Planet	Semimajor Axis a (10^{10} m)	Period T (y)	T^2/a^3 (10^{-34} y^2/m^3)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

§3 Satellites: Orbits and energy

1、机械能（圆轨道）

- 势能：$U = -\frac{GMm}{r}$
- 动能：$\frac{GMm}{r^2} = m\frac{v^2}{r}$

  $K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$

  $K = -\frac{U}{2}$
- 机械能：$E = K + U$

  $= -\frac{GMm}{2r}$

  $E = -K$

* 对于椭圆轨道：

$E = -\frac{GMm}{2a}$ （a：semimajor axis）

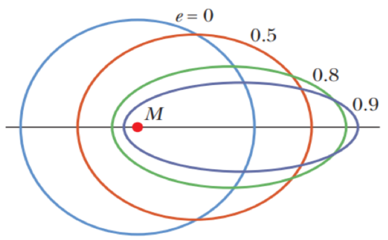

Figure 13-15 Four orbits with different eccentricities e about an object of mass M. All four orbits have the same semimajor axis a and thus correspond to the same total mechanical energy E.

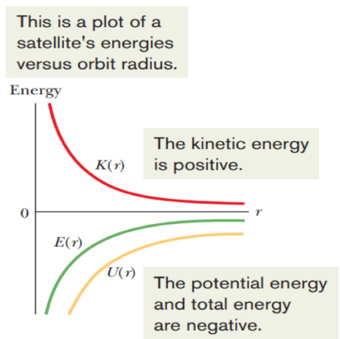

Figure 13-16 The variation of kinetic energy K, potential energy U, and total energy E with radius r for a satellite in a circular orbit. For any value of r, the values of U and E are negative, the value of K is positive, and $E = -K$. As $r \to \infty$, all three energy curves approach a value of zero.

例:

A playful astronaut releases a bowling ball, of mass $m = 7.20$ kg, into circular orbit about Earth at an altitude h of 350 km.

(a) What is the mechanical energy E of the ball in its orbit?

Calculations: The orbital radius must be

$$r = R + h = 6370\text{ km} + 350\text{ km} = 6.72 \times 10^6\text{ m},$$

in which R is the radius of Earth. Then, from Eq. 13-40 with Earth mass $M = 5.98 \times 10^{24}$ kg, the mechanical energy is

$$\begin{aligned} E &= -\frac{GMm}{2r} \\ &= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})(7.20\text{ kg})}{(2)(6.72 \times 10^6\text{ m})} \\ &= -2.14 \times 10^8\text{ J} = -214\text{ MJ}. \qquad \text{(Answer)} \end{aligned}$$

(b) What is the mechanical energy E_0 of the ball on the launchpad at the Kennedy Space Center (before launch)? From there to the orbit, what is the change ΔE in the ball's mechanical energy?

$$\begin{aligned} U_0 &= -\frac{GMm}{R} \\ &= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})(7.20\text{ kg})}{6.37 \times 10^6\text{ m}} \\ &= -4.51 \times 10^8\text{ J} = -451\text{ MJ}. \end{aligned}$$

The kinetic energy K_0 of the ball is due to the ball's motion with Earth's rotation. You can show that K_0 is less than 1 MJ, which is negligible relative to U_0. Thus, the mechanical energy of the ball on the launchpad is

$$E_0 = K_0 + U_0 \approx 0 - 451\text{ MJ} = -451\text{ MJ}. \qquad \text{(Answer)}$$

The *increase* in the mechanical energy of the ball from launchpad to orbit is

$$\begin{aligned} \Delta E = E - E_0 &= (-214\text{ MJ}) - (-451\text{ MJ}) \\ &= 237\text{ MJ}. \qquad \text{(Answer)} \end{aligned}$$

Summary

- A uniform shell of matter exerts no *net* gravitational force on a particle located inside it.
- The gravitational force $\vec{F}$ on a particle inside a uniform solid sphere, at a distance r from the center, is due only to mass M_{ins} in an "inside sphere" with that radius r:

$$M_{ins} = \tfrac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3,$$

where ρ is the solid sphere's density, R is its radius, and M is its mass. We can assign this inside mass to be that of a particle at the center of the solid sphere and then apply Newton's law of gravitation for particles. We find that the magnitude of the force acting on mass m is

$$F = \frac{GmM}{R^3} r.$$

Summary

- The gravitational potential energy $U(r)$ of a system of two particles, with masses M and m and separated by a distance r, is the negative of the work that would be done by the gravitational force of either particle acting on the other if the separation between the particles were changed from infinite (very large) to r. This energy is

$$U = -\frac{GMm}{r} \quad \text{(gravitational potential energy).}$$

- If a system contains more than two particles, its total gravitational potential energy U is the sum of the terms representing the potential energies of all the pairs. As an example, for three particles, of masses m_1, m_2, and m_3,

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right).$$

- An object will escape the gravitational pull of an astronomical body of mass M and radius R (that is, it will reach an infinite distance) if the object's speed near the body's surface is at least equal to the escape speed, given by

$$v = \sqrt{\frac{2GM}{R}}.$$

Summary

- The motion of satellites, both natural and artificial, is governed by Kepler's laws:

1. *The law of orbits.* All planets move in elliptical orbits with the Sun at one focus.
2. *The law of areas.* A line joining any planet to the Sun sweeps out equal areas in equal time intervals. (This statement is equivalent to conservation of angular momentum.)
3. *The law of periods.* The square of the period T of any planet is proportional to the cube of the semimajor axis a of its orbit. For circular orbits with radius r,

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \quad \text{(law of periods),}$$

where M is the mass of the attracting body—the Sun in the case of the solar system. For elliptical planetary orbits, the semimajor axis a is substituted for r.

Summary

- When a planet or satellite with mass m moves in a circular orbit with radius r, its potential energy U and kinetic energy K are given by

$$U = -\frac{GMm}{r} \quad \text{and} \quad K = \frac{GMm}{2r}.$$

The mechanical energy $E = K + U$ is then

$$E = -\frac{GMm}{2r}.$$

For an elliptical orbit of semimajor axis a,

$$E = -\frac{GMm}{2a}.$$