Lecture 3

§1 Introduction to Markov Chain

1. 一个例子

A wanderer or drunkard and an endless street divided into blocks. In each of unit of time, say 5 minutes, he walks one block from street corner to corner, and each corner he may choose to go ahead with probability p or turn back with probability $1-p$.

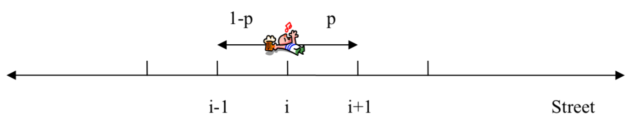
A mathematical representation of the random walk. Let X_0 be the starting point and Y_n be the nth step taken:

$$Y_n = \begin{cases} +1, & \text{with probability } p, \\ -1, & \text{with probability } 1-p. \end{cases}$$

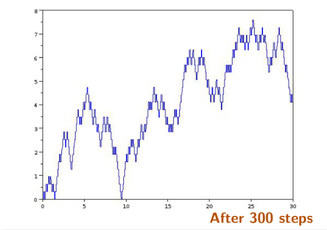
Then $\{Y_n, n=1,2,\ldots\}$ are i.i.d. rvs, and the position at time n (or after n steps) is just

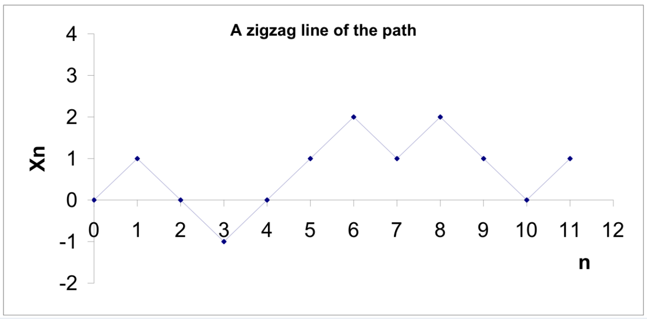
$$X_n = \underbrace{X_0}_{\text{initial state}} + Y_1 + \cdots + Y_n.$$

Starting point: $X_0 = i_0$. At time n, assume $X_n = i$, where will he go next step?



Step n	Yn		Xn
0		Starting point	0
1	1	Go ahead	1
2	-1	Turn back	0
3	-1	Turn back	-1
4	1	Go ahead	0
5	1	Go ahead	1
6	1	Go ahead	2
7	-1	Turn back	1
8	1	Go ahead	2
9	-1	Turn back	1
10	-1	Turn back	0
11	1	Go ahead	1





- 我们有一个 discrete-time stochastic process $\{X_n\}_{n\geq 0}$，其有 discrete state space $E=\{0,\pm1,\pm2,\cdots,\pm i,\cdots\}=\mathbb{Z}$
- 注意到，"$n+1$ 时刻的位置"仅取决于"n 时刻的位置"

2. Definition: Markov chain (马氏链/马尔可夫链)

一个有 discrete state space 的 stochastic process $\{X_n\}_{n\geq 0}$ 被称为 Markov Chain (MC)，若

$$P[X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_1=i_1, X_0=i_0] = P[X_{n+1}=j \mid X_n=i]$$

for all states $i_0, i_1, \cdots, i_{n-1}, i, j$ and all $n \geq 0$

- 这种性质也被称为 Markov property (马氏性)
- 我们仅考虑 homogeneous MC (齐次马氏链)，即 $P[X_{n+1}=j \mid X_n=i]=p_{ij}$ 与时间 n 独立

注：一种理解方式：$P(\text{future} \mid \text{current}, \text{past}) = P(\text{future} \mid \text{current})$

由贝叶斯公式：$P(A|BC) = \dfrac{P(AB|C)}{P(B|C)}$，可以得出

$$P(\underbrace{\text{future}}_{A} \mid \underbrace{\text{current}}_{C}, \underbrace{\text{past}}_{B}) = \frac{P(\text{future}, \text{past} \mid \text{current})}{P(\text{past} \mid \text{current})} = P(\text{future} \mid \text{current})$$

$$\Rightarrow P(\text{future}, \text{past} \mid \text{current}) = P(\text{future} \mid \text{current}) \cdot P(\text{past} \mid \text{current})$$

表示给定现在的信息后，未来的状态和过去的信息独立

Past, Present and Future states

- Termnology: $\underset{\text{future}}{X_{n+1}}$ $\underset{\text{present}}{X_n}$ $\underbrace{X_{n-1}, \dots, X_1, X_0}_{\text{past}}$
- Markov Property: 'future' states depend only on 'present' states and not on 'past' states:

$$P(futr \mid pres, past) = P(futr \mid pres) \; ;$$

3. Definition: transition probability (转移概率)

① One-step transition probabilities (单步转移概率) 为

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

表示当前位于 state i，下一步转移至 state j 的概率

- 对任意 i，the family $p_{i\cdot} = \{j \mapsto p_{ij}\}$（$j$ 向 p_{ij} 的映射）表示给定 $X_n = i$，X_{n+1} 的条件分布
- 对 $\forall i, j$，有 $p_{ij} \geq 0$
- 对 $\forall i$，有 $\sum_{j=0}^{\infty} p_{ij} = 1$

注：若 state space $E = \{0, \dots, n\}$，则 $\sum_{j=0}^{n} p_{ij} = \sum_{j=0}^{n} P(X_{n+1} = j \mid X_n = i) = P(X_{n+1} \in \{1, \dots, n\} \mid X_n = i) = 1$

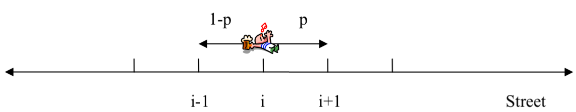
② One-step transition matrix（单步转移概率矩阵）of the Markov Chain 为矩阵

$$P = (p_{ij}) = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0j} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i0} & p_{i1} & \cdots & p_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

注：若矩阵 A 满足 $\sum_{j=0}^{\infty} p_{ij} = 1$，$\forall i$，则 A 被称为 stochastic matrix，满足 A 的一个特征值为 1

e.g. Drunkard's random walk

Starting point: $X_0 = i_0$. At time n, assume $X_n = i$, where will he go next step?



- State space 为 $E = \{\cdots, -n, \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots, n, \cdots\}$
- 单步转移概率矩阵为

$$P = (p_{ij}) = \begin{bmatrix} & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1-p & 0 & p & 0 & \cdots & \\ \cdots & 0 & 1-p & 0 & p & \cdots & \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \end{bmatrix}$$

e.g.

- Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions.

  Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

- If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the preceding is a two-state Markov chain whose transition probabilities are given by

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}.$$

e.g.

- On any given day Gary is either cheerful (C), so-so (S), or glum (G).

  If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1.

  If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3.

  If he is feeling glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.

- Letting X_n denote Gray's mood on the nth day, then $\{X_n, n \geq 0\}$ is a three-state Markov chain (state $0{=}C$, state $1{=}S$, state $2 = G$) with transition probability matrix

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

e.g. 通过选取合适的 state 将 process 转化成 Markov Chain

- Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it

(i) has rained for the past two days,	then it will rain tomorrow with probability	*0.7;*
(ii) rained today but not yesterday,		*0.5;*
(iii) rained yesterday but not today,		*0.4;*
(iv) has not rained in the past two days,		*0.2.*

- If we let the two states as in Example 2.1, then the preceding model is NOT a Markov chain:

$$P(\text{it will rain tomorrow} \mid \text{it rains today, it rained yesterday})$$
$$P(\text{it will rain tomorrow} \mid \text{it rains today, it didn't rain yesterday})$$
$$\neq P(\text{it will rain tomorrow} \mid \text{it rains today})$$

- Cont'd. But if we introduce $X_n = (\text{weather on day } n, \text{weather on day } n-1)$

State	Rains? Today	Rains? Yesterday
0	Y	Y
1	Y	N
2	N	Y
3	N	N

令 0 表示 rain，1 表示 no rain，
则 $E = \{(0,0), (1,0), (0,1), (1,1)\}$

Then the preceding would then present a four-state Markov chain having a transition probability matrix

$$P = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \\ \end{array}$$

$$P = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix} \begin{array}{c} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array}.$$

(How to derive e.g. $p_{33} = 0.8$?) $= P(X_{n+1} = (1,1) \mid X_n = (1,1)) = P(\text{tmr } 1 \mid \text{today } 1, \text{yes } 1)$
$= 1 - 0.2 = 0.8$

例 1: 考虑随机变量序列 $Y_0, Y_1, \cdots$ 表示每次抛硬币的结果，正面为 0，反面为 1，且概率均为 $\frac{1}{2}$。
设 $X_n = Y_n + Y_{n-1}$，$n \geq 1$，判断序列 $\{X_n\}$ 是否为马氏链。

不是！

$P(X_3=2 \mid X_2=1, X_1=0) = \frac{1}{2}$

$P(X_3=2 \mid X_2=1, X_1=2) = 0$

注：$X_i=0$ 与 $X_i=2$ 均只包含一种内部情况 $(1,1)$ 与 $(0,0)$，但 $X_i=1$ 包含两种内部情况 $(1,0)$ 与 $(0,1)$. 由于 X_1 与 X_2 的结果限定了 X_2 的内部情况，进一步限定了 X_3 的结果，所以 $\{X_i\}$ 不为马氏链.

4. 两种常见的马氏链模型

① random walk model（状态空间无界 & 无吸收态）

Example 2.5. A random walk model. A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \ldots$ is said to be a random walk if for some number $0 < p < 1$,

$$p_{i,i} = 0, \quad p_{i,i+1} = p, \quad p_{i,i-1} = 1 - p;$$

The transition probability matrix $P =$

$$\begin{pmatrix} \cdots & p_{0,-1} = 1-p & p_{00} = 0 & p_{01} = p & p_{02} = 0 & \cdots \\ \cdots & 0 & p_{10} = 1-p & p_{11} = 0 & p_{12} = p & \cdots \\ \cdots & 0 & 0 & p_{21} = 1-p & p_{22} = 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

② Gambling model（状态空间有界 & 有吸收态）

Example 2.6. A Gambling Model At each play of game, the gambler either wins \$1 with probability p, or loses \$1 with probability $1-p$. The gambler quits playing either when he goes broke, or when he attains a fortune of \$$N$.

- It is easy to see that the gambler's fortune is a Markov Chain having transition probabilities and matrix

$$p_{i,i+1} = p, \quad p_{i,i-1} = 1-p, \quad i = 1, \ldots, N-1, p_{00} = 1 = p_{NN}\ ;$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1-p & 0 & p & 0 & 0 & \cdots & 0 \\ 0 & 1-p & 0 & p & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- States 0 and N are called absorbing states, since once entered they are never left.
  This is a random walk on a finite state space $\{0, 1, \ldots, N\}$ with absorbing barriers 0 and N.

e.g.
- Bonus Malus (Latin for Good-Bad) system, is used to determine annual automobile insurance premiums in most of Europe and Asia.
- Each policyholder is given a positive integer valued state and the annual premium is a function of this state (along, of course, with the type of car being insured and the level of insurance).
- A policyholder's state changes from year to year in response to the number of claims made by that policyholder. Because lower numbered states correspond to lower annual premiums, a policyholder's state will usually decrease if he or she had no claims in the preceding year, and will generally increase if he or she had at least one claim.
- Thus no claims is good and typically results in a decreased premium, while claims are bad and typically results in a higher premium.

- For a given Bonus Malus, let $s_i(k)$ denote the next state of a policyholder who was in sate i in the previous year and who made a total of k claims in that year.

- If we suppose that the number of yearly claims made by a particular policyholder is a Poisson random variable with parameter λ, then the successive states of this policyholder will constitute a Markov chain with transition probabilities

$$p_{i,j} = \sum_{k:s_i(k)=j} e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k:s_i(k)=j} a_k, \quad j \geq 0$$

- Whereas there are usually many states (20 or so is not typical), the following table specifies a hypothetical Bonus Malus system having four states.

$P_{i,j} = P(X_{t+1}=j \mid X_t=i) = P(\text{make } k \text{ claims} : S_i(k)=j)$

$= \sum_{k:S_i(k)=j} P(\text{make } k \text{ claims})$

		Next state if			
State	Annual premium	0 claims	1 claim	2 claim	$\geq$ 3 claims
1	200	1	2	3	4
2	250	1	3	4	4
3	400	2	4	4	4
4	600	3	4	4	4

- The transition matrix of the successive states of this policyholder is

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ a_0 & 0 & a_1 & 1-a_0-a_1 \\ 0 & a_0 & 0 & 1-a_0 \\ 0 & 0 & a_0 & 1-a_0 \end{pmatrix}.$$