Lecture 5

§1 幂级数的运算

1. addition / subtraction

设 $\sum_{n=0}^{\infty} a_n x^n$ 和 $\sum_{n=0}^{\infty} b_n x^n$ 分别在 $(-R, R)$ 及 $(-R', R')$ 内收敛

则 $\sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ 在区间 $(-R, R)$ 及 $(-R', R')$ 中较小的区间内成立

若 $a_n \pm b_n = 0$，则 $\sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ 在 R 上成立

2. multiplication

设 $\sum a_n (x-c)^n$ 的收敛半径为 R_a，$\sum b_n (x-c)^n$ 的收敛半径为 R_b.

令 $R := \min\{R_a, R_b\}$, Then

$$\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n (x-c)^n\right) = \sum_{n=0}^{\infty} c_n (x-c)^n$$

for all x with $|x-c| < R$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots\cdots + a_n b_0 = \sum_{k=0}^{n} a_k \cdot b_{n-k}$$

Idea:

$$(a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots\cdots) \cdot (b_0 + b_1 (x-c) + b_2 (x-c)^2 + \cdots\cdots)$$

To get $(x-c)^n$, we can get it from $a_k (x-c)^k$, $b_{n-k} (x-c)^{n-k}$, $0 \le k \le n$.

3. substitution

Substitution 可在收敛域 (interval of convergence) 内进行.

e.g. Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $x \in (-1, 1)$, we have

$\sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$ valid for $|3x| < 1$, i.e., $x \in (-\frac{1}{3}, \frac{1}{3})$

4. differentiation and integration

Theorem (Term-by-Term Differentiation and Integration)

Suppose that $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence R, with $R > 0$. Define f on $(a-R, a+R)$ by

$$\boxed{f(x) := \sum_{n=0}^{\infty} c_n (x-a)^n.}$$

Then on $(a-R, a+R)$, the function f is differentiable and has an antiderivative, with

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$.

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$.

若 $\sum_{n=0}^{\infty} c_n (x-a)^n$ 在 $(a-R, a+R)$ 上收敛于 $f(x)$

则 $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ 在 $(a-R, a+R)$ 上收敛于 $f'(x)$

$\sum_{n=1}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$ 在 $(a-R, a+R)$ 上收敛于 $f(x)$ 的 antiderivative

注：在 term-by-term differentiation / integration 之后，端点 $a \pm R$ 处的敛散性不能确定.

e.g. $\sum \frac{x^n}{n^2}$

$\because \frac{|x|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x|^n} = \left(\frac{n}{n+1}\right)^2 \cdot |x| \to |x| \Rightarrow R = 1$

- At $x=1$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ cvgs
- But $\sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ does not converge at $x=1$

例：求 $\sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n \cdot x^{n+1}$

$\sum_{n}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}$ for $|x|<1$

$\Rightarrow \int \frac{1}{1+x} dx = C + \sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n (x)^{n+1}$ for $|x|<1$

$\Rightarrow \ln(1+x) = C + \sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n (x)^{n+1}$ for $|x|<1$

For $x=0$, $0 = \ln 1 = C + 0 \Rightarrow C=0$

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n x^{n+1} = \ln(1+x)$ for $x \in (-1,1]$

例：求 $f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

- By Ratio test, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+3}}{2n+3} \cdot \frac{2n+1}{|x|^{2n+1}} = \frac{2n+1}{2n+3} \cdot |x|^2 \longrightarrow |x|^2$

  so series converges when $|x|^2<1 \Rightarrow |x|<1$

  diverges when $|x|^2>1 \Rightarrow |x|>1$

  f is defined on $[-1,1]$
- $f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}$, for $|x| \le 1$

  $f(x) = \int f'(x) dx = \int \frac{1}{1+x^2} dx = \arctan x + C$, $C = f(0) - \int f'(0)$

  $f(0) = 0 = \arctan 0 + C \Rightarrow C=0$
- $f(x) = \arctan x$, $x \in [-1,1]$.

例：求 $\sum_{n=1}^{\infty} n \cdot x^n$

- $\sum_{n=1}^{\infty} n \cdot x^n = x \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1}$
- 令 $S(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1}$,

取决于中心点

$\int_0^x S(t) dt = \int_0^x \sum_{n=1}^{\infty} n \cdot t^{n-1} dt = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$

$S(x) = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$
- $\sum_{n=1}^{\infty} n \cdot x^n = x \cdot S(x) = \frac{x}{(1-x)^2}$, $|x|<1$

例：求 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n$

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1}$
- 令 $S(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1}$

  $S'(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$

  $S''(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$

  $S'(x) = \int_0^x \frac{1}{1-t} dt + \underbrace{S'(0)}_{=0} = -\ln(1-x)$

  $S(x) = \int_0^x -\ln(1-t) dt + \underbrace{S(0)}_{=0} = x + (1-x)\ln(1-x)$
- $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n = \begin{cases} 1 + \left(\frac{1}{x} - 1\right) \ln(1-x), & x \in [-1,0) \cup (0,1) \\ 0, & x=0 \\ 1 \ (\text{裂项}), & x=1 \end{cases}$$

也可先裂项，括号打开后求解

例：求 $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{2n+1} x^{2n+1}$

- 令 $S(x)=\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{2n+1} x^{2n+1}$
- $\int_0^x S(x)dx = \sum_{n=1}^{\infty}(-1)^n \cdot \frac{n+1}{2n+1} \cdot \frac{1}{2n+2} \cdot x^{2n+2}$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{2n+1} x^{2n+2}$$

$$= \frac{1}{2}x\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1} x^{2n+1}$$

- 令 $S_1(x)=\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1} x^{2n+1}$
- $S_1'(x)=\sum(-1)^n \cdot x^{2n} = \sum(-x^2)^n = \frac{-x^2}{1+x^2}$

$S_1(x)=\int_0^x \frac{-x^2}{1+x^2}dx + \underbrace{S_1(0)}_{=0} = -x+\arctan x$

$\int_0^x S(x)dx = \frac{1}{2}x(-x+\arctan x)$

- $S(x) = -x+\frac{1}{2}\arctan x + \frac{x}{2(1+x^2)}$

✡ 也可先将 $\frac{n+1}{2n+1}$ 化为 $\frac{1}{2}+\frac{\frac{1}{2}}{2n+1}$

例：求 $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-1\right)\cdot x^{2n}$

- $\sum\left(\frac{1}{2n+1}-1\right)\cdot x^{2n} = \frac{1}{x}\cdot\underbrace{\sum\frac{1}{2n+1}x^{2n+1}}_{S(x)} - \underbrace{\sum x^{2n}}_{f(x)}$

- $S'(x)=\sum x^{2n} = \frac{x^2}{1-x^2}$

$S(x)=\int_0^x S'(t)dt + S(0) = \frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)-x$

- $f(x)=\frac{x^2}{1-x^2}$

$$\sum\left(\frac{1}{2n+1}-1\right)\cdot x^{2n} = \begin{cases} \frac{1}{2x}\cdot\ln\frac{1+x}{1-x}-\frac{1}{1-x^2}, & x\in(-1,0)\cup(0,1) \\ 0, & x=0 \end{cases}$$

例：求 $\sum \frac{(-1)^{n-1}\cdot x^{2n+1}}{n(2n-1)}$

- $\sum \frac{(-1)^{n-1}\cdot x^{2n+1}}{n(2n-1)} = x\cdot\underbrace{\sum\frac{(-1)^{n-1}\cdot x^{2n}}{n(2n-1)}}_{S(x)}$

- $S'(x)=\sum(-1)^{n-1}\cdot 2\cdot\frac{x^{2n-1}}{2n-1}$

$S''(x)=\sum(-1)^{n-1}\cdot 2\cdot x^{2n-2} = 2\sum(-x^2)^{n-1} = 2\cdot\frac{1}{1+x^2}$

$S'(x)=\int_0^x 2\cdot\frac{1}{1+t^2}dt + S'(0) = 2\arctan x$

$S(x)=\int_0^x 2\arctan t\, dt + S(0) = 2x\cdot\arctan x - \ln(1+x^2)$

- $\sum\frac{(-1)^{n-1}x^{2n+1}}{n(2n-1)} = 2x^2\arctan x - x\cdot\ln(1+x^2)$ ， $x\in[-1,1]$

✡ 也可先将 $\frac{1}{n(2n-1)}$ 化为 $2\cdot\left(\frac{1}{2n-1}-\frac{1}{2n}\right)$

方法归纳：

① 先导再积

② 先积再导

③ 提 $x^1, x^2, x^{-1}, x^{-2}\cdots\cdots$

④ 先求两次导，再求两次积分

⑤ 先求两次积分，再求两次导

④ 拆项（裂项 / 将分母因式分解后拆项……）

§2 Taylor Series（泰勒级数）

1. 泰勒级数 & 麦克劳林级数

Definition

Let f be a function such that for all $n \in \mathbb{N}$, $f^{(n)}$ exists on some open interval containing a. The Taylor series of f centered at a is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

or "Taylor series generated by f"

The Maclaurin series of f is the Taylor series of f centered at 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

若函数 $f(x)$ 可以被一个幂级数表示，

假设 $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

- $f(a) = \sum_{n=0}^{\infty} c_n(a-a)^n = c_0 \Rightarrow c_0 = f(a)$
- $f'(a) = \sum_{n=1}^{\infty} n c_n(a-a)^{n-1} = c_1 \Rightarrow c_1 = \frac{f'(a)}{1!}$
- $f''(a) = \sum_{n=2}^{\infty} n(n-1)\cdot c_n(a-a)^{n-2} = 2\cdot 1\cdot c_2 \Rightarrow c_2 = \frac{f''(a)}{2!}$

……

- $f^{(k)}(a) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\cdot c_n(a-a)^{n-k} \Rightarrow c_k = \frac{f^{(k)}(a)}{k!}$

Hence, if $f(x)$ has a power series representation $\sum_{n=0}^{\infty} c_n(x-a)^n$, it must be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$$

注：上述过程仅证明了 $f(x)$ 用幂级数展开后的形式为 $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$，
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$ 的充要条件是 $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$ 存在且收敛（余项的极限 → 0）

e.g. The Maclaurin series of $f(x) := e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

The Maclaurin series of $f(x) := \cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

2. 泰勒多项式（Taylor polynomial）

Taylor polynomial 是 Taylor series 的 partial sums

Definition

Let f be a function such that for all $n \in \{0, 1, \ldots, N\}$, $f^{(n)}$ exists on some open interval containing a. The Taylor polynomial of f (of order n) centered at a is the polynomial

$$\boxed{P_n(x)} := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

注：1° 对于 $f(x) = \cos x$, $a = 0$. 因为 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots\cdots$

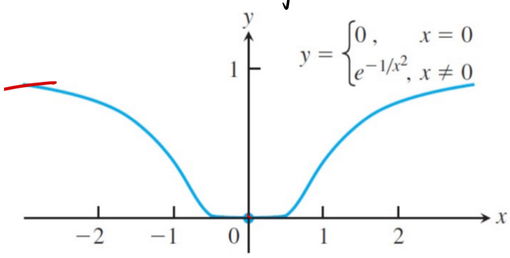
有 $P_5(x) = P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$. $P_n(x)$ 的次数可能小于 n.

2° f 的泰勒级数在 $x=a$ 处必收敛于 $f(a)$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(a-a)^n=\frac{f^{(0)}(a)}{0!}=f(a)$$

3° $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ 可能对任意 $x\neq a$ 都不收敛于 $f(x)$. 不是所有函数都有泰勒展开形式.

e.g. $f(x):=\begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$



f 无穷可导，但 $f^{(n)}(0)=0$，$\forall n\geq 0$

因此泰勒级数 $=0$，但 $f(x)\neq 0$，$\forall x\neq 0$

3. 常见的麦克劳林展开式

① $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots+\frac{x^n}{n!}+\cdots\cdots \quad (-\infty<x<+\infty)$

② $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\cdots \quad (-\infty<x<+\infty)$

③ $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots\cdots+x^n+\cdots\cdots \quad (-1<x<1)$

④ $\frac{1}{1+x} = 1-x+x^2-x^3+\cdots\cdots+(-1)^n x^n+\cdots\cdots \quad (-1<x<1)$

⑤ $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots+(-1)^{n-1}\cdot\frac{x^n}{n}+\cdots\cdots \quad (-1<x\leq 1)$

⑥ $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\cdots+(-1)^n\cdot\frac{x^{2n}}{(2n)!}+\cdots\cdots \quad (-\infty<x<+\infty)$

⑦ $a^x = 1+x\ln a+\frac{x^2(\ln a)^2}{2!}+\frac{x^3(\ln a)^3}{3!}+\cdots\cdots+\frac{x^n(\ln a)^n}{n!}+\cdots\cdots \quad (-\infty<x<+\infty)$

⑧ $\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots\cdots+(-1)^n x^{2n}+\cdots\cdots \quad (-1<x<1)$

⑨ $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\cdots+(-1)^n\cdot\frac{x^{2n+1}}{2n+1} \quad (-1\leq x\leq 1)$

4. 将函数展开为幂级数

例：将 $f(x)=(1-x)\ln(1+x)$ 展开成 x 的幂级数

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\cdots\cdots$$

$$\begin{aligned}f(x)&=(1-x)\cdot\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\cdots\cdots\right)\\&=x+\left(-\frac{1}{2}-1\right)x^2+\left(\frac{1}{3}+\frac{1}{2}\right)x^3+\left(-\frac{1}{4}-\frac{1}{3}\right)x^4+\cdots\cdots\\&=x+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-1)x^n}{n(n-1)} \quad (-1<x\leq 1)\end{aligned}$$

例：将 $\sin x$ 展开成 $(x-\frac{\pi}{4})$ 的幂级数

$$\begin{aligned}\sin x&=\sin\left(x-\frac{\pi}{4}+\frac{\pi}{4}\right)\\&=\sin\frac{\pi}{4}\cos\left(x-\frac{\pi}{4}\right)+\cos\frac{\pi}{4}\sin\left(x-\frac{\pi}{4}\right)\\&=\frac{\sqrt{2}}{2}\left[\cos\left(x-\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)\right]\\&=\frac{\sqrt{2}}{2}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{(x-\frac{\pi}{4})^2}{2!}-\frac{(x-\frac{\pi}{4})^3}{3!}+\cdots\cdots\right]\\&=\cdots\cdots\end{aligned}$$

例：将 $f(x)=\frac{1}{x^2+4x+3}$ 展成 $(x-1)$ 的幂级数

$$f(x)=\frac{1}{(x+1)(x+3)}$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1+x}-\frac{1}{3+x}\right)$$

$$\frac{1}{1+x}=\frac{1}{2+(x-1)}$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1+\frac{x-1}{2}}\right)$$

$$=\frac{1}{2}\cdot\left[1-\frac{x-1}{2}+\left(\frac{x-1}{2}\right)^2-\left(\frac{x-1}{2}\right)^3+\cdots\cdots\right] \quad \left(-1<\frac{x-1}{2}<1 \Rightarrow -1<x<3\right)$$

$$\frac{1}{3+x}=\frac{1}{4+(x-1)}$$

$$=\frac{1}{4}\left(\frac{1}{1+\frac{x-1}{4}}\right)$$

$$=\frac{1}{4}\left[1-\frac{x-1}{4}+\left(\frac{x-1}{4}\right)^2-\left(\frac{x-1}{4}\right)^3+\cdots\cdots\right] \quad \left(-1<\frac{x-1}{4}<1 \Rightarrow -3<x<5\right)$$

$$f(x)=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2^{n+2}}-\frac{1}{2^{2n+3}}\right)(x-1)^n \quad (-1<x<3)$$

例：将 $f(x)=\ln x$ 展成 $(x-2)$ 的幂级数

$$f(x)=\ln(2+(x-2))$$

$$=\ln 2\left(1+\frac{x-2}{2}\right)$$

$$=\ln 2+\ln\left(1+\frac{x-2}{2}\right)$$

$$=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{x-2}{2}\right)^{n+1} \quad (0<x\le 4)$$