Lecture 24

§1. Numerical Integration: Trapezoidal Rule

1. A general idea for approximating the numerical value of an integral

1° In particular, some elementary functions, such as

$$f(x) = \frac{\sin x}{x} \quad \text{and} \quad f(x) = e^{x^2},$$

do not have an elementary antiderivative.

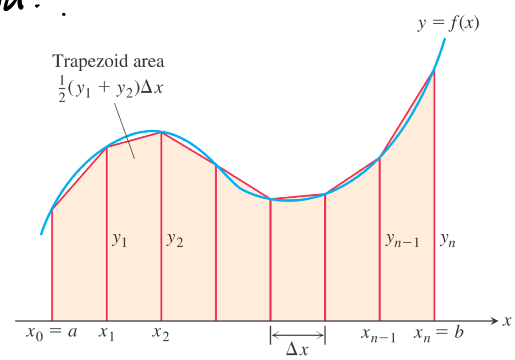
2° Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$

Then:

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx$$

2. Trapezoidal rule

In trapezoidal rule, we approximate each $\int_{x_{k-1}}^{x_k} f(x)$ with the area of a trapezoid.



More specifically, instead of $f(c_k)\cdot\Delta x_k$ in a Riemann sum, we take

$$T_k := \frac{(y_{k-1}+y_k)\cdot\Delta x}{2}$$

where $y_j := f(x_j)$.

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} T_k = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n),$$

where $\Delta x = (b-a)/n$

3. How good is an approximation method

1° Let us define

$$\text{error} := E := \text{approximated value} - \int_a^b f(x)\,dx$$

We want to find $|E|$

2° Let:

E_L = error of left-hand rule

E_R = error of right-hand rule

E_M = error of midpoint rule

E_T = error of trapezium rule

e.g. Consider $I := \int_1^{1.6} \frac{1}{x}\,dx$ with $\Delta x = 0.1$

By trapezium rule:

$$I \approx \frac{0.1}{2} \times \left(\frac{1}{1} + \frac{2}{1.1} + \frac{2}{1.2} + \frac{2}{1.3} + \frac{2}{1.4} + \frac{2}{1.5} + \frac{1}{1.6}\right)$$

$$\approx 0.47051739$$

By analytical solution:

$$I = [\ln x]_1^{1.6}$$

$$= \ln 1.6$$

$$\approx 0.47000363$$

For this example, $|E_M| < |E_T| < |E_R| < |E_L|$

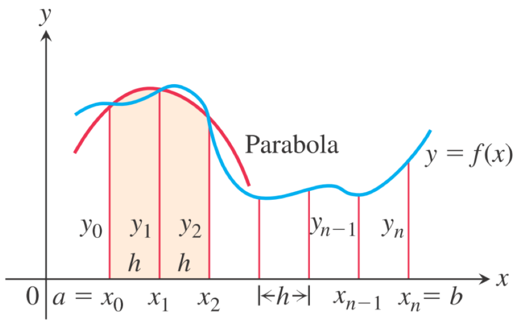
§2 Numerical Integration: Simpson's Rule

1. Simpson's rule

Consider an evenly spaced partition $P := \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$, where n is even.

For $k \in \{1, 3, 5, \cdots, n-1\}$, over the interval $[x_{k-1}, x_{k+1}]$, consider approximating f by the quadratic function $p_k(x) := A_k x^2 + B_k x + C_k$ whose graph passes through the three points (x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1}).

$$\int_a^b f(x)\,dx \approx \sum_{k\in\{1,3,\cdots,n-1\}} \int_{x_{k-1}}^{x_{k+1}} P_k(x)\,dx$$



2. An explicit formula

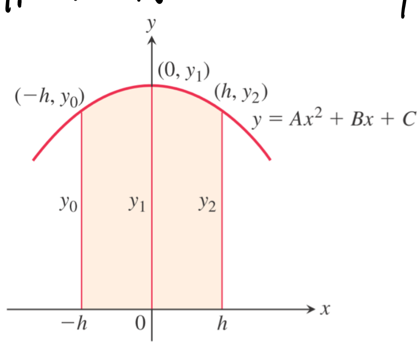
Consider $\int_{x_0}^{x_2} P_1(x)\,dx$.

We may shift the graph so that $x_0 = -\Delta x$, $x_1 = 0$, $x_2 = \Delta x$

$$\int_c^d f(x)\,dx = \int_{c-s}^{d-s} f(x+s)\,dx$$

Let $h := \Delta x$ and write

$$q_1(x) = Ax^2 + Bx + C = P_1(x+h+a)$$



Then
$$\begin{aligned}
\int_{x_0}^{x_2} P_1(x)\,dx &= \int_a^{a+2h} P_1(x)\,dx \\
&= \int_{-h}^{h} q_1(x)\,dx \\
&= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx \\
&= \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_{-h}^{h} \\
&= 2\left(\frac{A}{3}h^3 + Ch\right) \\
&= \frac{h}{3}(2Ah^2 + 6C) \quad ①
\end{aligned}$$

Since $(-h, y_0)$, $(0, y_1)$, (h, y_2) are all on the graph of $y = Ax^2 + Bx + C$, we have:

$$y_0 = Ah^2 - Bh + C$$
$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

$$y_0 + y_2 = 2Ah^2 + 2C$$

$$= 2Ah^2 + 2y_1$$

$$2Ah^2 = y_0 + y_2 - 2y_1$$

By ① we have:

$$\int_{x_0}^{x_2} P_1(x)\,dx = \frac{h}{3}(2Ah^2 + 6C)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

By the same argument, we can more generally state:

$$\int_{x_{k-1}}^{x_{k+1}} P_k(x)\,dx = \frac{h}{3}(y_{k-1} + 4y_k + y_{k+1})$$

Therefore,

$$\int_a^b f(x)\,dx \approx \sum_{k\in\{1,3,\cdots,n-1\}} \int_{x_{k-1}}^{x_{k+1}} P_k(x)\,dx$$

$$= \frac{h_0}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots\cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

where n is even, and the coefficients are $1, 4, 2, 4, 2, 4, 2 \cdots\cdots 4, 2, 4, 1$

e.g. Consider $I := \int_1^{1.6} \frac{1}{x}\,dx$ with $\Delta x = 0.1$

By Simpson's rule:

$$I \approx \frac{0.1}{3} \times \left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{1}{1.6}\right)$$

$$\approx 0.4700064$$

In this example: $|E_S| < |E_M| < |E_T| < |E_R| < |E_L|$

§3 Numerical Integration: Error Bounds

1. Error bounds

Let f be a function that is integrable on $[a,b]$.

Let $\max|f^{(i)}| := \max_{x\in[a,b]} |f^{(i)}(x)|$

Then

$$|E_L| \le \frac{(b-a)^2}{2n} \max|f'|$$
$$|E_R| \le \frac{(b-a)^2}{2n} \max|f'|$$
$$|E_T| \le \frac{(b-a)^3}{12n^2} \max|f''|$$
$$|E_M| \le \frac{(b-a)^3}{24n^2} \max|f''|$$
$$|E_S| \le \frac{(b-a)^5}{180n^4} \max|f^{(4)}|$$

Generally speaking, from most to least accurate:

1° Simpson's rule

2° Mid-point rule

3° Trapezoidal rule

4° Left and right-hand rule

e.g. If we want to approximate $\int_0^1 e^{x^2}dx$ with $|E| < 10^{-5}$, how many sub-intervals do we need at least?

Let $f(x) = e^{x^2}$.

$$f'(x) = e^{x^2} \cdot 2x$$
$$f''(x) = e^{x^2}(4x^2+2) \qquad \max|f''| = 6e$$
$$f'''(x) = e^{x^2}(8x^3+12x)$$
$$f^{(4)} = e^{x^2}(16x^4+48x^2+12) \qquad \max|f^{(4)}| = 76e$$

For Simpson's rule:

$$\frac{1}{180\,n^4} \cdot 76e \le 10^{-5}$$
$$n \geq 18.4$$

Since n is even, we need $n \geq 20$

For Trapezoid rule:

$$\frac{1}{12n^2} \le 10^{-5}$$
$$n \geq 369$$

§4 Improper Integrals — Type I

1. Definition

1° Let $a \in R$. If f is integrable on $[a,b]$ for every $b \in [a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2° Let $b \in R$. If f is integrable on $[a,b]$ for every $a \in (-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3° An improper integral is said to be convergent if the corresponding limit exists, and is said to be divergent if the limit does not exist (as a real number)

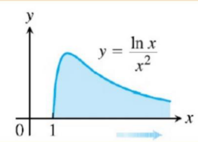
4° We define

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_c^{\infty} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx$$

whenever both improper integrals on the right converge. We may use any real number c in this definition.

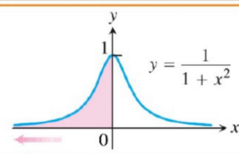
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$



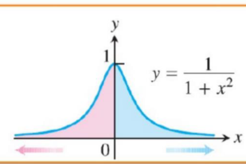
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$



3. Both limits

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2}$$



2. Remarks

1° The definition of $\int_{-\infty}^{\infty} f(x)\,dx$ above also applies to the case where we have $\infty + \infty$, $-\infty - \infty$, or $\alpha \pm \infty$ on the right.

2° But it is not defined if:

① the right-hand side is an interminated form, e.g. $\infty - \infty$.

② one of the terms on the right does not exist.

e.g. Find $\int_1^\infty \frac{1}{x}\,dx$

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx$$
$$= \lim_{b\to\infty} \ln b$$
$$= \infty$$

e.g. Find $\int_1^\infty x^\alpha dx$, $\alpha \in R$

$$\int_1^b x^\alpha dx = \frac{1}{\alpha+1}(b^{\alpha+1}-1) =: F(b)$$

$$\int_1^\infty x^\alpha dx = \lim_{b\to\infty} F(x) = \begin{cases} \infty & \text{if } \alpha \geq -1 \\ -\frac{1}{\alpha+1} & \text{if } \alpha < -1 \end{cases}$$

e.g. Find $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

$$\int_0^b \frac{1}{1+x^2}dx = \arctan b$$
$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{b\to\infty} \arctan b = \frac{\pi}{2}$$
$$\int_a^0 \frac{1}{1+x^2}dx = -\arctan a$$
$$\int_\infty^0 \frac{1}{1+x^2}dx = \lim_{a\to-\infty} -\arctan a = \frac{\pi}{2}$$
$$\int_\infty^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$