Some Terminologies 套利

- In finance, arbitrage is the activity of buying shares or currency in one financial market and selling it at a profit in another.
- 期权
  Options are financial instruments that convey the right, but not the obligation, to engage in a future transaction on some underlying security. For example, buying a call option provides the right to buy a specified quantity of a security at a set strike price at some time (exercise date) on or before expiration, while buying a put option provides the right to sell. In this section, we discuss only call option.
- Option, costing c (the price of the option) per share, give us the option for purchasing shares of the stock at time t for the fixed price of K per share.

A Simple Example in Options Pricing

- Consider one stock at two times: present time $t = 0$ and future time $t = 1$;
- Suppose that the price of the stock at present time is $X_0 = \$100$, and suppose we know that in the future $X_1 = \$200$ or $X_1 = \$50$ per share.
- Assume that the price of the option is c, which provides the right to buy a share of the stock at the fixed price \$150/per unit share at time $t = 1$.
- If the stock rises to \$200 then you would exercise the option at time 1 and realize a gain of \$200-150 = \$50 for each option.

  行使\$150/股的期权，并以\$200/股卖出
- On the other hand, if the price of the stock is \$50, then the option is worthless at time 1.

  放弃行使\$150/股的期权

Cont'd

- At time 0, if we purchase x shares of stock at price \$100 and y shares of option at price c, then the original cost is (x, y 可以为负, 表示卖出)

portfolio

$$100x + yc;$$

(x, y 符号通常相反: 可理解为对冲)

卖(买)股票的钱 买(卖)期权的钱

- and the value at time 1 is

$$200x + y(200 - 150) = 200x + 50y$$

if the price is \$200, or

$$50x$$

if the price is \$50.

- If we choose y such that

$$200x + 50y = 50x,$$

i.e. $y = -3x$ (Note that buy x shares $\equiv$ sell $-x$ shares). Thus, with $y = -3x$, the value of our holding at time 1 is

$$\text{value} = 50x,$$

whatever price of the stock at time 1.

- Further, we gain on the transaction is

$$50x - (100x + yc) = 3cx - 50x = (3c - 50)x.$$

(可根据c选择x的正负)

- Thus, if $3c = 50$, then the gain is 0; otherwise, we can guarantee a positive gain (no matter what the price of the stock at time 1).

  For example, if $c = 10$, we purchase $x = -2$ shares (i.e. sell 2 shares) of stock and purchase $y = -3 * (-2) = 6$ shares of option at time 0, then the gain is $(3 * 10 - 50) * (-2) = 40$;

  if $c = 20$, we purchase $x = 2$ shares of stocks and purchase $y = -3 * 2 = -6$ shares (i.e. sell 6 shares) of option, then the gain is $(3 * 20 - 50) * 2 = 20$.

- A sure win betting scheme is called an arbitrage.

  Thus, as we have just seen, the only option cost c that does not result in an arbitrage is $c = 50/3$.

The Black-Scholes option pricing formula

The option pricing problem

- Price process of a stock Y_t, $0 \leq t \leq T$.
- We know the present price $Y_0 = y_0$.
- Note that for any future price Y_t, its present value is $e^{-rt}Y_t$, where r is the discount factor (fixed interest rate).
- pay at time 0 (maturity date)
  Suppose c is the cost of an option to purchase one share at time T at the fixed price K (*call*).
  strike price
- We want to determine value of c for which there is no betting strategy that leads to a sure win.

Pricing of a call option

- Stock price follows a Geometric BM:

$$Y_t = Y_0 e^{X_t} = Y_0 e^{mt+\sigma B_t}, \quad t \geq 0.$$

可以用 martingale 使结论更完备

- We assume the risk-neutral condition: $\boxed{m + \frac{1}{2}\sigma^2 = r}$.

- Consider the wager of purchasing an option. At maturity time T,

$$\text{the worth of option at time } T = \begin{cases} Y_T - K, & \text{if } Y_T \geq K, \\ 0, & \text{if } Y_T < K. \end{cases}$$
$$= (Y_T - K)^+.$$

T时刻 $1在0时刻的价值

- Hence, the present value of the worth of the option is $e^{-rT}(Y_T - K)^+$. In order for purchasing the option to have expected return 0, we must have that

$$E[e^{-rT}(Y_T - K)^+] = c. \qquad (*)$$

The solution

- To solve Equation (*), notice that $m + \frac{\sigma^2}{2} = r$, and

$$Y_T = Y_0 e^{mT + \sigma B_T},$$
$$mT + \sigma B_T \sim \mathcal{N}(mT, \sigma^2 T), \quad \text{with density f.}$$
$$\frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(y-mT)^2}{2\sigma^2 T}}, \quad y \in \mathbb{R}$$

$= E[(Y_0 e^{X_T} - K)^+]$, 其中 $X_T = mT + \sigma B_T \sim N(mT, \sigma^2 T)$

- we have

$$\begin{aligned} ce^{rT} &= E[(Y_T - K)^+] = \int_{-\infty}^{\infty} (y_0 e^y - K)^+ \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(y-mT)^2}{2\sigma^2 T}} dy \\ &= y_0 e^{rT} \Phi(\sigma\sqrt{T} - a) - K\Phi(-a), \end{aligned}$$

where [see details below]

$$a = \frac{1}{\sigma\sqrt{T}} \left(\log \frac{K}{y_0} - mT \right).$$

technical details

Note that $(y_0 e^y - K \geq 0 \Leftrightarrow y \geq \log \frac{K}{y_0}$, we have

$$
\begin{aligned}
ce^{rT} &= \int_{\log \frac{K}{y_0}}^{\infty} (y_0 e^y - K) \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(y-mT)^2}{2\sigma^2 T}} dy \\
&\qquad (\text{Let } w = \frac{y - mT}{\sigma\sqrt{T}}, y = mT + \sigma\sqrt{T} w, \text{denote } a = \frac{\log \frac{K}{y_0} - mT}{\sigma\sqrt{T}}) \\
&= \int_a^{\infty} y_0 e^{mT + \sigma\sqrt{T} w} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw - \int_a^{\infty} K \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw.
\end{aligned}
$$

The second integral equals $K\Phi(-a)$ and the first integral is

$$
\begin{aligned}
& y_0 e^{mT} \int_a^{\infty} e^{\frac{\sigma^2}{2} T} \frac{1}{\sqrt{2\pi}} e^{-\frac{(w - \sigma\sqrt{T})^2}{2}} dw \\
=\ & y_0 e^{(m + \frac{\sigma^2}{2})T} P\{N(\sigma\sqrt{T}, 1) > a\} = y_0 e^{(m + \frac{\sigma^2}{2})T} \left(1 - \Phi\left(a - \sigma\sqrt{T}\right)\right) \\
=\ & y_0 e^{\alpha T} \Phi(\sigma\sqrt{T} - a).
\end{aligned}
$$

So

$$
ce^{rT} = y_0 e^{(m + \frac{\sigma^2}{2})T} \Phi(\sigma\sqrt{T} - a) - K\Phi(-a) .
$$

The solution (cont'd)

Hence we have

$$c = \underbrace{y_0}_{\text{initial price}} \Phi(\sigma\sqrt{T} + b) - Ke^{-rT}\Phi(b), \quad (BS)$$

where

$$b = -a = \frac{mT - \log\frac{K}{y_0}}{\sigma\sqrt{T}} = \frac{rT - \frac{\sigma^2 T}{2} - \log\frac{K}{y_0}}{\sigma\sqrt{T}}.$$

若 $K=y_0$, $r=0$, 则 $c=y_0\Phi(\sigma\sqrt{T}+b)-y_0\Phi(b)$, $b=\frac{mT}{\sigma\sqrt{T}}=\frac{m}{\sigma}\sqrt{T}$

- Equation (BS) is known as the Black-Scholes option cost valuation.
- The value of c depends on the initial price of the stock y_0, the option exercise time t, the option exercise price K, the discount factor r, and the value σ^2, but not m. 因为 risk neutral condition, m 被表示为 σ 和 r
  可以用 B-S equation 估计 σ^2 (implied volatility, can be computed in real time)
- If the option itself can be traded, then the formula of Equation (BS) can be used to set its price in such a way so that no arbitrage is possible.
- If at time s the price of the stock is $Y_s = x_s$, then the price of a (T, K) option ($s < T$)— that is, an option to purchase one unit of the stock at time t for a price K — should be set by replacing T by $T - s$ and y_0 by Y_s in Equation (BS).

Estimation the volatility parameter σ^2

Question: if we observe $\{Y_t\}$, a BM with drift coefficient m and variance parameter σ^2, during the time interval $t \in [0, T]$, how can we estimate σ^2?

One solution:

- divide $(0, T]$ into N bins of binwidth h:

$$[0, T] = (0, h] + (h, 2h] + \cdots + ((N-1)h, Nh] \ .$$

- the N increments $W_i = Y_{ih} - Y_{(i-1)h}$, $1 \le i \le N$ are i.i.d. $\mathcal{N}(mh, \sigma^2 h)$.
- One natural estimator of the variance $\sigma^2 h$ is the sample variance

$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(W_i - \overline{W})^2, \quad \overline{W} = \frac{1}{N}\sum_{i=1}^{N} W_i \ ;$$

- Since $\dfrac{(N-1)S^2}{\sigma^2 h} \sim \chi^2_{N-1}$, $E\chi^2_k = k$ and $\mathrm{var}\chi^2_k = 2k$, it follows that

$$E\left(\frac{S^2}{h}\right) = \sigma^2, \qquad \mathrm{var}\left(\frac{S^2}{h}\right) = \frac{2\sigma^4}{N-1} \to 0.$$

- So a "good" estimator of σ^2 is $\boxed{\widehat{\sigma}^2 = \dfrac{S^2}{h}.}$