Lecture 14

§1 Elasticity

1. Elasticity (弹性)

1° 绝对的刚体在现实中是不存在的，物体在受力时会发生形变

2° 用于描述受力时 how a real body deforms (形变).

3° 是物体在被 stretched (拉伸) 或 compressed (压缩) 时恢复原有形态的能力

4° 固体受力时 change dimensions 的三种情况

(a) Tensile stress (张应力)

(b) Shearing stress (剪应力/切应力)

(c) Hydraulic stress (液压应力)

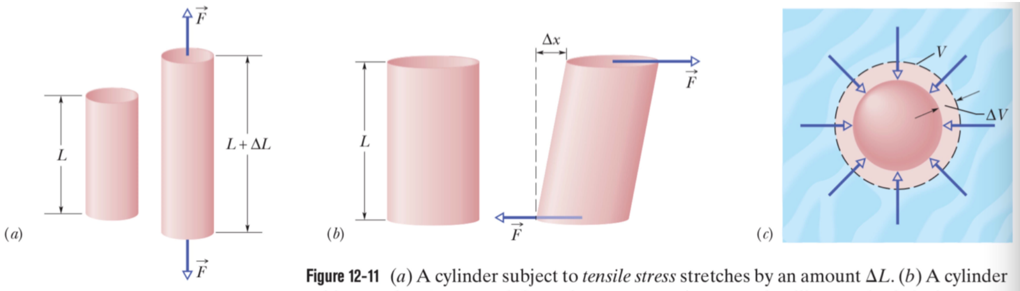


Figure 12-11 (a) A cylinder subject to *tensile stress* stretches by an amount ΔL. (b) A cylinder subject to *shearing stress* deforms by an amount Δx, somewhat like a pack of playing cards would. (c) A solid sphere subject to uniform *hydraulic stress* from a fluid shrinks in volume by an amount ΔV. All the deformations shown are greatly exaggerated.

2. Stress (应力) 和 strain (应变)

1° Stress 应力:

① 描述单位面积上的受力情况：$\frac{F}{A}$

② stress 为张量，pressure (压强) 为标量.

对于理想流体，其的方向性不存在，这时的应力也被称为压力 (压强)

③ 单位：N/m^2

2° strain 应变:

① 描述物体在外力作用下的相对形变

② 是一个比例，dimensionless

3° stress 与 strain 成正比，比例为 modulus of elasticity (弹性系数)

stress = Modulus × strain

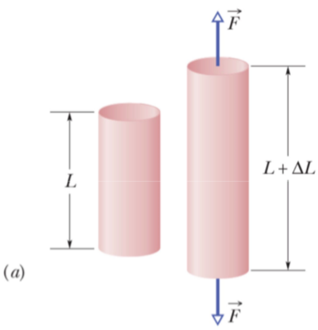
3. Tension (张力) 与 compression (压力)

1° stress : $\frac{F}{A}$

2° strain : $\frac{\Delta L}{L}$

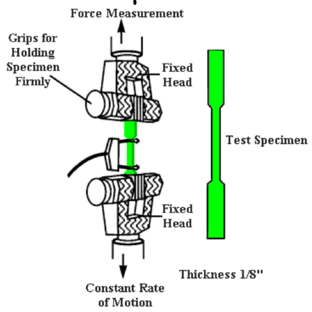
3° Young's modulus : E (杨氏模量)

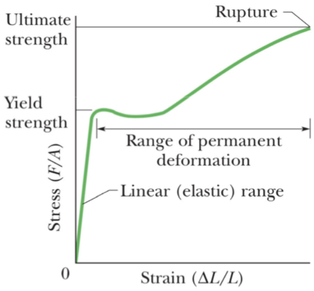
4° $\frac{F}{A} = E \cdot \frac{\Delta L}{L}$



5° 实验：stress vs strain 曲线

- stress < yield stress (抗拉强度)
  ① 线性关系
  ② 停止施力后形变恢复 (elastic deformation (弹性形变))
- stress > yield stress
  ① permanent deformation





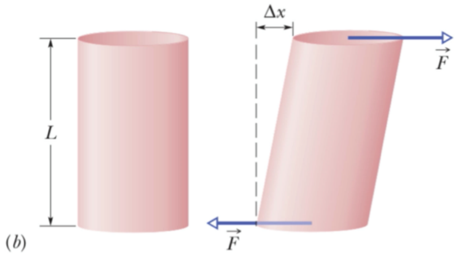
4. Shearing

1° stress : $\frac{F}{A}$

2° strain : $\frac{\Delta x}{L}$

3° shear modulus : G (剪切模量)

4° $\frac{F}{A} = G \cdot \frac{\Delta x}{L}$



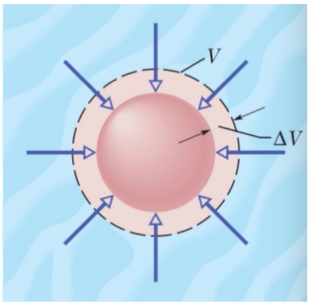
5. Hydraulic stress

1° stress : p (液体压强)

2° strain : $\frac{\Delta V}{V}$

3° Bulk modulus : B (体积弹性模量)

4° $p = B \cdot \frac{\Delta V}{V}$ $\qquad B = -V\frac{dP}{dV}$



例: **Examples**

- In Pacific Ocean, at its average depth of about 4000 m, $p = 4.0 \times 10^7 N/m^2$
- For water, $B = 2.2 \times 10^9 N/m^2 \rightarrow \frac{\Delta V}{V} = 1.8\%$
- For Steel, $B = 1.6 \times 10^{11} N/m^2 \rightarrow \frac{\Delta V}{V} = 0.025\%$
- In general, solids – with their rigid atomic lattices – are less compressible than liquids, in which the atoms or molecules are less tightly couple to their neighbors.

Table 12-1 Some Elastic Properties of Selected Materials of Engineering Interest

Material	Density ρ (kg/m^3)	Young's Modulus E ($10^9\ N/m^2$)	Ultimate Strength S_u ($10^6\ N/m^2$)	Yield Strength S_y ($10^6\ N/m^2$)
Steel[a]	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50[b]	—
Concrete[c]	2320	30	40[b]	—
Wood[d]	525	13	50[b]	—
Bone	1900	9[b]	170[b]	—
Polystyrene	1050	3	48	—

[a]Structural steel (ASTM-A36).
[b]In compression.
[c]High strength
[d]Douglas fir.

例: **Problem**

One end of a steel rod of radius $R = 9.5$ mm and length $L = 81$ cm is held in a vise. A force of magnitude $F = 62$ kN is then applied perpendicularly to the end face (uniformly across the area) at the other end, pulling directly away from the vise. What are the stress on the rod and the elongation ΔL and strain of the rod?

Solution:

$$\text{stress} = \frac{F}{A} = \frac{F}{\pi R^2} = \frac{6.2 \times 10^4\ \text{N}}{(\pi)(9.5 \times 10^{-3}\ \text{m})^2}$$
$$= 2.2 \times 10^8\ \text{N/m}^2. \qquad \text{(Answer)}$$

The yield strength for structural steel is $2.5 \times 10^8\ \text{N/m}^2$, so this rod is dangerously close to its yield strength.

We find the value of Young's modulus for steel in Table 12-1. Then from Eq. 12-23 we find the elongation:

$$\Delta L = \frac{(F/A)L}{E} = \frac{(2.2 \times 10^8\ \text{N/m}^2)(0.81\ \text{m})}{2.0 \times 10^{11}\ \text{N/m}^2}$$
$$= 8.9 \times 10^{-4}\ \text{m} = 0.89\ \text{mm}. \qquad \text{(Answer)}$$

For the strain, we have

$$\frac{\Delta L}{L} = \frac{8.9 \times 10^{-4}\ \text{m}}{0.81\ \text{m}}$$
$$= 1.1 \times 10^{-3} = 0.11\%. \qquad \text{(Answer)}$$

例 **Problem**

- A table has three legs that are 1.00 m in length and a fourth leg that is longer by d = 0.50 mm, so that the table wobbles slightly. A heavy steel cylinder with mass M = 290 kg is placed upright on the table (with a mass much less than M) so that all four legs are compressed and the table no longer wobbles. The legs are wooden cylinders with cross-sectional area A = 1.0 cm^2. The Young's modulus E for the wood is 1.3 x 10^{10} N/m^2. Assume that the tabletop remains level and that the legs do not buckle. What are the magnitudes of the forces on the legs from the floor?

Solution

- For the three legs with same length:

$$F_1 = F_2 = F_3 = A \times E \frac{\Delta L}{L}$$

- The forth leg:

$$F_4 = A \times E \frac{\Delta L + d}{L + d} \approx A \times E \frac{\Delta L + d}{L} = F_3 + A \times E \frac{d}{L}$$

- For the system:

$$F_1 + F_2 + F_3 + F_4 = Mg$$

$$4F_3 + A \times E \frac{d}{L} = Mg$$

$$F_3 = \frac{Mg}{4} - \frac{dAE}{4L}$$

$$= \frac{(290\text{ kg})(9.8\text{ m/s}^2)}{4} - \frac{(5.0 \times 10^{-4}\text{ m})(10^{-4}\text{ m}^2)(1.3 \times 10^{10}\text{ N/m}^2)}{(4)(1.00\text{ m})}$$

$$= 548\text{ N} \approx 550\text{ N}$$

$$F_4 = Mg - 3F_3 = (290\text{ kg})(9.8\text{ m/s}^2) - 3(548\text{ N})$$

$$\approx 1200\text{ N}$$

§2 Equilibrium

1. Stability

1° 对于一些建筑物而言，即使受到力与力矩，建筑仍被认为是 stable 的.

2° Stability:

对于刚体，考虑 force 和 torque 的 equalibrium

对于非刚体，还要考虑 elasticity

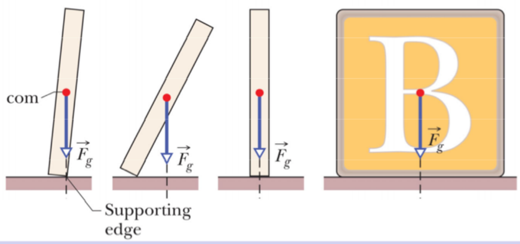
2. Equilibrium (平衡)

$\vec{P}$ = constant & $\vec{L}$ = constant

3. static equilibrium (静态平衡)

$\vec{P} = 0$ & $\vec{L} = 0$

- Stable/ Unstable static equilibrium



4. equilibrium 的条件

1° translational equilibrium: balance of forces

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} = 0$$

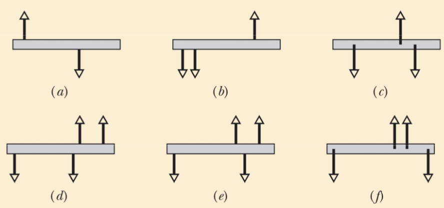
2° rotational equilibrium: balance of torques

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0$$ (measured about any possible point)

例

Checkpoint 1

The figure gives six overhead views of a uniform rod on which two or more forces act perpendicularly to the rod. If the magnitudes of the forces are adjusted properly (but kept nonzero), in which situations can the rod be in static equilibrium?

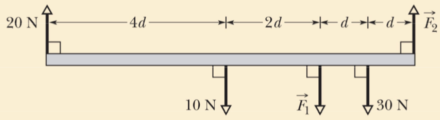


Answer: (c), (e), and (f)

例：

Checkpoint 2

The figure gives an overhead view of a uniform rod in static equilibrium. (a) Can you find the magnitudes of unknown forces $\vec{F}_1$ and $\vec{F}_2$ by balancing the forces? (b) If you wish to find the magnitude of force $\vec{F}_2$ by using a balance of torques equation, where should you place a rotation axis to eliminate $\vec{F}_1$ from the equation? (c) The magnitude of $\vec{F}_2$ turns out to be 65 N. What then is the magnitude of $\vec{F}_1$?



Answer: (a) No. (b) at site of $\vec{F}_1$, perpendicular to plane of figure; (c) 45 N

5. The center of gravity (重心)

若作用于一个物体的重力F_g等效于作用在一点上，则该点为 center of gravity.

若作用于一个物体各个部分的重力g相同，则重心(cog)与质心(com)相同.

$x_{com} = x_{cog}$

Proof

- Consider individual point

$$\tau_i = x_i F_{gi}$$

- The net torque:

$$\tau_{net} = \sum \tau_i = \sum x_i F_{gi}$$

- Consider the effective cog point

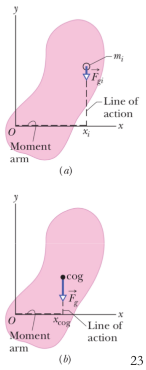
$$\tau = x_{cog} F_g$$

$$\tau = x_{cog} \sum F_{gi}$$

- Consider the effective cog point

$$\sum x_i F_{gi} = x_{cog} \sum F_{gi}$$

$$x_{cog} = \frac{1}{M} \sum x_i m_i = x_{com}$$



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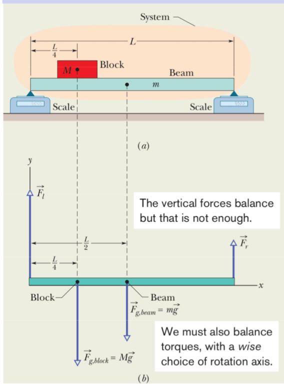
例：

Problem



A uniform beam, of length L and mass **m =1.8 kg**, is at rest with its ends on two scales. A uniform block, with mass **M = 2.7 kg,** is at rest on the beam, with its center a distance ***L/4*** from the beam's left end. What do the scales read?

SOLUTION:

$$F_l + F_r - Mg - mg = 0$$

Choose the rotation axis at the left end of the bean

$$(0)(F_l) - (L/4)(Mg) - (L/2)(mg) + (L)(F_r) = 0$$

$$\begin{aligned} F_r &= \tfrac{1}{4}Mg + \tfrac{1}{2}mg \\ &= \tfrac{1}{4}(2.7\text{ kg})(9.8\text{ m/s}^2) + \tfrac{1}{2}(1.8\text{ kg})(9.8\text{ m/s}^2) \\ &= 15.44\text{ N} \approx 15\text{ N} \end{aligned}$$

$$\begin{aligned} F_l &= (M+m)g - F_r \\ &= (2.7\text{ kg} + 1.8\text{ kg})(9.8\text{ m/s}^2) - 15.44\text{ N} \\ &= 28.66\text{ N} \approx 29\text{ N} \end{aligned}$$

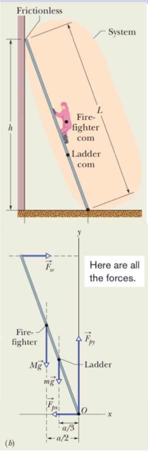
- *Strategy in the solution:*

$$\vec{\tau}_{net} = 0 \;\&\; \vec{F}_{net} = 0$$

- With two unknowns, chose the rotational axis to eliminate one force for the torque equation
- With only one unknown left, it can be solved by the balance of force equation.

例:

Problem



A ladder of length ***L* = 12 m** and mass ***m* = 45 kg** leans against a slick (frictionless) wall. Its upper end is at height ***h* = 9.3 m** above the pavement on which the lower end rests (the pavement is not frictionless). The ladder's center of mass is ***L/3*** from the lower end. A firefighter of mass ***M* =72 kg** climbs the ladder until her center of mass is ***L/2*** from the lower end. What then are the magnitudes of the forces on the ladder from the wall and the pavement?

SOLUTION:

Choose the rotation axis at O

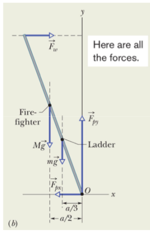
$$-(h)(F_w)+(a/2)(Mg)+(a/3)(mg)+(0)(F_{px})+(0)(F_{py})=0$$

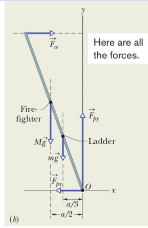
$$a=\sqrt{L^2-h^2}=7.58\text{ m}$$

$$F_w=\frac{ga(M/2+m/3)}{h}$$

$$=\frac{(9.8\text{ m/s}^2)(7.58\text{ m})(72/2\text{ kg}+45/3\text{ kg})}{9.3\text{ m}}$$

$$=407\text{ N}\approx 410\text{ N}$$





$$F_w-F_{px}=0$$

$$F_{px}=F_w=410\text{ N}$$

$$F_{py}-Mg-mg=0$$

$$F_{py}=(M+m)g=(72\text{ kg}+45\text{ kg})(9.8\text{ m/s}^2)$$

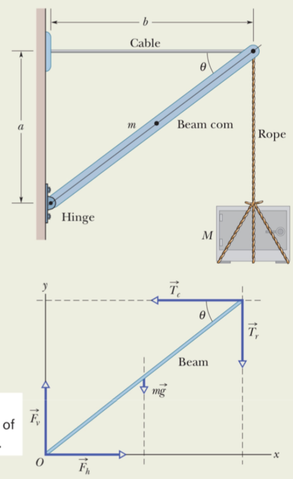
$$=1146.6\text{ N}\approx 1100\text{ N}$$

Note that F_{px} is the static friction from the pavement.

例:

Problem

Figure 12-6 (*a*) A heavy safe is hung from a boom consisting of a horizontal steel cable and a uniform beam. (*b*) A free-body diagram for the beam.



Here is the *wise* choice of rotation axis.

A safe, of mass **M =430 kg**, is hanging by a rope from a boom with dimensions ***a*=1.9 m** and ***b*=2.5 m**. The boom consists of a hinged beam and a horizontal cable that connects the beam to a wall. The uniform beam has a mass ***m*=85 kg**; the mass of the cable and rope are negligible.

a) What is the Tension T_c in the cable?

b) Find the magnitude of net force on the beam from the hinge.

SOLUTION:

Take the rotation axis at O.

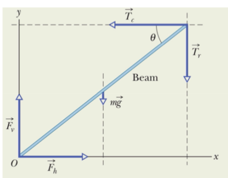
$$(a)(T_c)-(b)(T_r)-(\tfrac{1}{2}b)(mg)=0$$

Since T_r = Mg, we have :

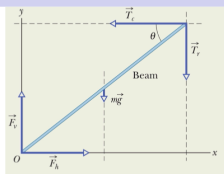
$$T_c=\frac{gb(M+\frac{1}{2}m)}{a}$$

$$=\frac{(9.8\text{ m/s}^2)(2.5\text{ m})(430\text{ kg}+85/2\text{ kg})}{1.9\text{ m}}$$

$$=6093\text{ N}\approx 6100\text{ N}$$



SOLUTION:



$$F_h-T_c=0$$

$$F_h=T_c=6093\text{ N}$$

$$F_v-mg-T_r=0$$

$$F_v=(m+M)g=(85\text{ kg}+430\text{ kg})(9.8\text{ m/s}^2)$$

$$=5047\text{ N}$$

$$F=\sqrt{F_h^{\,2}+F_v^{\,2}}$$

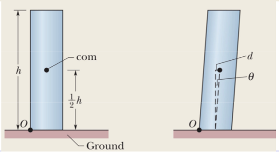
$$=\sqrt{(6093\text{ N})^2+(5047\text{ N})^2}\approx 7900\text{ N}$$

Note that the force $\vec{F}$ does not point along the beam.

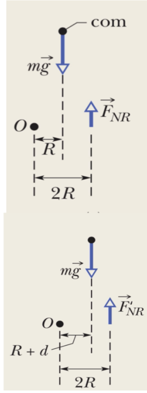
例:

Problem

Let's assume that the Tower of Pisa is a uniform hollow cylinder of radius R = 9.8 m and height h = 60 m. The center of mass is located at height $h/2$, along the cylinder's central axis. In Fig. 12-8*a*, the cylinder is upright. In Fig. 12-8*b*, it leans rightward (toward the tower's southern wall) by θ = 5.5°, which shifts the com by a distance d. Let's assume that the ground exerts only two forces on the tower. A normal force $\vec{F}_{NL}$ acts on the left (northern) wall, and a normal force $\vec{F}_{NR}$ acts on the right (southern) wall. By what percent does the magnitude F_{NR} increase because of the leaning?



SOLUTION:



- Without leaning

$$F_{NR} = \frac{1}{2}mg$$

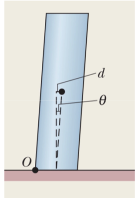
- With leaning

$$-(R+d)mg + 2RF'_{NR} = 0$$

$$d = \frac{1}{2}h \times tan\theta$$

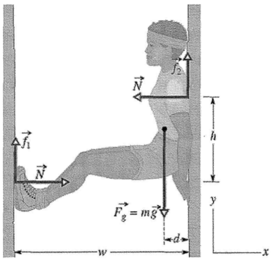
$$F'_{NR} = \frac{(R+d)}{2R}mg$$

$$\frac{F_{NR}}{F'_{NR}} = \frac{R+d}{R} = 1 + \frac{h\,tan\theta}{2R} = 1.29$$



例:

Problem



A rock climber with mass **m = 55 kg** rests during a "chimney climb", pressing only with her shoulders and feet against the walls of a fissure of width ***w* = 1.0 m**. Her center of mass is a horizontal distance ***d* = 0.20 m** from the wall against which her shoulders are pressed. The coefficient of static friction between her shoes and the wall is $\mu_1 = \mathbf{1.1}$, and between her shoulders and the wall is $\mu_2 = \mathbf{0.70}$. To rest, the climber wants to minimize her horizontal push on the walls. The minimum occurs when her feet and her shoulders are both on the verge of sliding.

a) What is that minimum horizontal push on the walls?

b) For that push, what must be the vertical distance h between her feet and her shoulders if she is to be stable?

SOLUTION: (a)

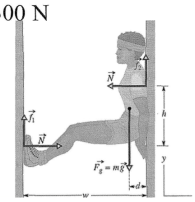
$$f_1 + f_2 - mg = 0$$

The frictional forces are at their maximum values.

$$f_1 = \mu_1 N \quad \text{and} \quad f_2 = \mu_2 N$$

$$N = \frac{mg}{\mu_1 + \mu_2} = \frac{(55\text{ kg})(9.8\text{ m/s}^2)}{1.1 + 0.70} = 229\text{ N} \approx 300\text{ N}$$

The minimum horizontal push is 300 N.



SOLUTION: (b)

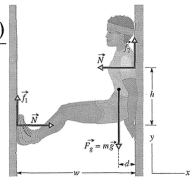
Choose the rotational axis at the shoulder.

$$-(w)(f_1) + (h)(N) + (d)(mg) + (0)(f_2) + (0)(N) = 0$$

$$h = \frac{f_1 w - mgd}{N} = \frac{\mu_1 N w - mgd}{N} = \mu_1 w - \frac{mgd}{N}$$

$$= (1.1)(1.0\text{ m}) - \frac{(55\text{ kg})(9.8\text{ m/s}^2)(0.20\text{ m})}{299\text{ N}}$$

$$= 0.739\text{ m} \approx 0.74\text{ m}$$



Summary

- Static Equilibrium

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} = 0 \;\&\; \vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0$$

- If the forces acting on the xy plane,

$$F_{net,x} = 0,\; F_{net,y} = 0,\;\; \tau_{net,z} = 0$$

- Center of Gravity (g is the same)

$$x_{cog} = x_{com}$$

- Elastic Modulus

$$Stress = Modulus \times strain$$

  - Tension and Compression

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

Summary

- Shearing

$$\frac{F}{A} = G\frac{\Delta x}{L}$$

- Hydraulic stress

$$p = B\frac{\Delta V}{V}$$