Lecture 6

# §1 Taylor's Theorem (带拉格朗日型余项)

## 1、Taylor's Theorem (泰勒定理)(带拉格朗日型余项)

> **Taylor's Theorem**
>
> Suppose:
>
> - $f, f', \dots, f^{(n)}$ are continuous on $[a,x]$ (or $[x,a]$);
> - $f^{(n+1)}$ exists on $(a,x)$ (or $(x,a)$).
>
> Then there exists $c \in (a,x)$ (or $c \in (x,a)$) such that
>
> $$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)\,:\,\text{Remainder}}.$$

注：1° 带入 $n=0$，有 $f(x)=f(a)+f'(c)(x-a)$. 说明MVT为特殊情况

## 2、证明函数f的泰勒级数收敛于f

1° 对于给定的x，只要 $R_n(x)\to 0$ as $n\to\infty$，

则 $f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

因此 $R_n(x)\to 0$ 是函数f的泰勒级数收敛于f(对于该x)的充分条件

2° 证明

Suppose $\exists$ constant $M$ such that $|f^{(n)}(t)| \le M$, $\forall t \in [a,x]$, $\forall n$.

( $M$为$|f^{(n+1)}(t)|$在$[a,x]$上的一个上界 )

Then $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\cdot |x-a|^{n+1} \le M \cdot \frac{|x-a|^{n+1}}{(n+1)!} \to 0$ as $n\to\infty$

So $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

例: Show that the Maclaurin series of $f(x) := \sin x$ converges $f(x)$ for all $x \in \mathbb{R}$

Maclaurin series for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

① When $x=0$, LHS = RHS

② Fix $x \neq 0$. $f^{(n)}(c) = \{\pm \sin c, \pm \cos c\}$

$|f^{(n)}(c)| \le \underset{M}{1}$ for $\forall c \in [0,x]$

$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\cdot |x-a|^{n+1} \le \frac{|x-a|^{n+1}}{(n+1)!} \to 0$ as $n\to\infty$

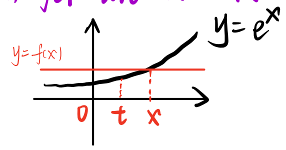
So the Maclaurin series of $f(x) := \sin x$ converges $f(x)$ for all $x \in \mathbb{R}$

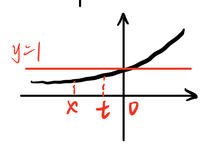
例: Show that the Maclaurin series of $f(x) := e^x$ converges $f(x)$ for all $x \in \mathbb{R}$

① When $x=0$, LHS = RHS

② Fix $x>0$, $|f^{(n)}(t)| = |e^t| = e^t \le \underset{M}{e^x}$ for $\forall t \in [0,x]$

③ Fix $x<0$, $|f^{(n)}(t)| = |e^t| = e^t \le \underset{M}{1}$ for $\forall t \in [x,0]$



## 3. 泰勒多项式的性质

$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ 为函数在f在a处的 n阶泰勒多项式，

则 $P_n(x)$的 n阶导数与f(x)的 n阶导数在a处的斜率相同.

证明: $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$

$P_n'(x) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}\cdot k(x-a)^{k-1} = f'(a) + \frac{f''(a)}{1!}(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots\cdots$

$\cdots\cdots$

$P_n^{(i)}(x) = \sum_{k=i}^{n} \frac{f^{(k)}(a)}{k!} k(k-1)\cdots\cdots(k-i)(x-a)^{k-i} = f^{(i)}(a) + \frac{f^{(i+1)}(a)}{1!}(x-a) + \frac{f^{(i+2)}(a)}{2!}(x-a)^2 + \cdots\cdots$

$P_n^{(i)}(a) = f^{(i)}(a)$

## 4. Taylor's Theorem (带拉格朗日型余项)的证明

对于 fixed b:

> **Taylor's Theorem**
>
> Suppose:
>
> - $f, f', \cdots, f^{(n)}$ are continuous on $[a,b]$ (or $[b,a]$);
> - $f^{(n+1)}$ exists on $(a,b)$ (or $(b,a)$).
>
> Then there exists $c \in (a,b)$ (or $c \in (b,a)$) such that
>
> $$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

- Define $P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ on $[a,b]$

  Define $F(x) := f(x) - P_n(x) - K(x-a)^{n+1}$ (K为需要求的数，使得 $F(b)=0$)
- For $k \in \{0, 1, 2, \cdots\cdots, n\}$,

  $F^{(k)}(a) = f(a) - P_n(a) - K(a-a)^{n+1} = 0$
- Since $F(a) = F(b) = 0$, $F$ cts on $[a,b]$, differentiable on $(a,b)$

  by Rolle's Theorem, $\exists c_1 \in (a,b)$, s.t. $F'(c_1) = 0$
- Since $F'(a) = F'(c_1) = 0$, $F'$ cts on $[a,c_1]$, differentiable on $(a,c_1)$

  by Rolle's Theorem, $\exists c_2 \in (a,c_1)$, s.t. $F''(c_2) = 0$

  $\cdots\cdots$
- $\exists c_{n+1} \in (a, c_n) \subseteq (a,b)$ s.t. $F^{(n+1)}(c_{n+1}) = 0$
- Note that $F^{(n+1)}(x) = f^{(n+1)}(x) - \underbrace{P_n^{(n+1)}(x)}_{=0} - K\cdot(n+1)!$

  $= f^{(n+1)}(x) - K(n+1)!$
- Substituting $x = c := c_{n+1}$ yields: $0 = f^{(n+1)}(c) - K(n+1)!$

  i.e. $K = \frac{f^{(n+1)}(c)}{(n+1)!}$
- Since $F(b) = 0$, we have $f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$, as desired.

## §2 Taylor's Theorem (带佩亚诺型余项)

### 1. Big-O and Little-O (as $x\to a$)

**Def:** For functions $f$ and $g$, we write

- $f(x)=o(g(x))$ as $x\to a$, if $\lim_{x\to a}\frac{f(x)}{g(x)}=0$.
  Little-O
- $f(x)=O(g(x))$ as $x\to a$, if $\exists M\in\mathbb{R}_{\geq 0}$ and $\exists\delta>0$ s.t.
  Big-O $\quad\forall x\in(a-\delta,a+\delta)\setminus\{a\}$, $|f(x)|\leq M|g(x)|$.

$\Leftrightarrow\left|\frac{f(x)}{g(x)}\right|\leq M$ if $g$ is never zero

### 2. Taylor's Theorem (O and o version)

**Taylor's Theorem (O and o version)**

Suppose $f, f', \dots, f^{(n)}$ are continuous on a neighbourhood $I$ of $a$, and $f^{(n+1)}$ exists and is bounded on $I$. Then, as $x\to a$,

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+O((x-a)^{n+1})=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+o((x-a)^n). \quad (*)$$

证明: Suppose that on $I=(a-\delta,a+\delta)$, $|f^{(n+1)}|$ is bounded by $M$.

Let $x\in I\setminus\{a\}$, and let $R_n(x)$ be its error term in Taylor's Theorem.

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+R_n(x)$$

Then

$$\left|\frac{R_n(x)}{(x-a)^{n+1}}\right|=\frac{|f^{(n+1)}(c)|}{(n+1)!}\leq\frac{M}{(n+1)!}$$

Since this holds for all $x\in I\setminus\{a\}$, we have $R_n(x)=O((x-a)^{n+1})$ as $x\to a$

On the other hand, since

$$\left|\frac{R_n(x)}{(x-a)^{n}}\right|=\frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|\leq\frac{M}{(n+1)!}|x-a|\to 0 \text{ as } x\to a$$

by Sandwich Theorem, $\lim_{x\to a}\frac{R_n(x)}{(x-a)^n}=0$, $R_n(x)=o((x-a)^n)$ as $x\to a$

### 3. Taylor's Theorem (带佩亚诺型余项)

**Theorem** If $f$ can be represented by its Taylor series with center $x=a$ on some neighbourhood $I$ of $a$, then for any fixed $n\in\mathbb{Z}_{\geq 0}$, as $x\to a$,

$$f(x)=\underbrace{\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)}+O((x-a)^{n+1})=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+o((x-a)^n).$$

证明: Suppose $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$, $\forall x\in I$.

Then for $\forall n\in\mathbb{Z}_{\geq 0}$, $f(x)=P_n(x)+R_n(x)$, where

$$R_n(x)=\sum_{k=n+1}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Now $\lim\limits_{x\to a}\frac{R_n(x)}{(x-a)^{n+1}} = \lim\limits_{x\to a}\sum\limits_{k=n+1}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^{k-n-1}$

$= \sum\limits_{k=n+1}^{\infty}\lim\limits_{x\to a}\frac{f^{(k)}(a)}{k!}(x-a)^{k-n-1}$ (an un-proven property of power series)

$= \frac{f^{(k)}(a)}{k!}$ ①

and $\lim\limits_{x\to a}\frac{R_n(x)}{(x-a)^{n}} = \lim\limits_{x\to a}\sum\limits_{k=n+1}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^{k-n}$

$= \sum\limits_{k=n+1}^{\infty}\lim\limits_{x\to a}\frac{f^{(k)}(a)}{k!}(x-a)^{k-n}$

$= 0$ ②

By ①, $\left|\frac{R_n(x)}{(x-a)^{n+1}}\right|$ is bounded in some neighbourhood of $a$,

so $R_n(x) = O((x-a)^{n+1})$ as $x\to a$

By ②, $R_n(x) = o((x-a)^n)$ as $x\to a$.

注: 1° 若 $m>0$ 且 $f(x)=O(x^m)$，则 $\lim\limits_{x\to 0}f(x)=0$

2° $f(x)=o(g(x)) \Rightarrow f(x)=O(g(x))$

$f(x)=O(g(x)) \not\Rightarrow f(x)=o(g(x))$

## *4. 高阶无穷小量的运算

1° $\lim\limits_{x\to 0}\frac{o(x)}{x}=0$

2° $o(x^m)\pm o(x^n)=o(x^p)$，其中 $p=\min\{m,n\}$ (低阶吸收高阶)

3° $o(x^m)\cdot o(x^n)=o(x^{m+n})$

4° $x^m\cdot o(x^n)=o(x^{m+n})$

5° $[o(x^m)]^n=o(x^{mn})$

6° $\frac{o(x^m)}{x^n}=o(x^{m-n})$，要求 $m\geqslant n$

注: $\frac{o(x^m)}{o(x^n)}\neq o(x^{m-n})$

## 5. 利用 Taylor's Theorem (带佩亚诺型余项) 计算极限

1° 泰勒公式展开阶数的确定

① 分式展开到分子分母同阶

② 加减展开到系数不为0的最低次幂

例: 求 $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

$\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim\limits_{x\to 0}\frac{x-\sin x}{x\sin x}$

$= \lim\limits_{x\to 0}\frac{x-(x+o(x))}{x\cdot(x+o(x))}$

$= \lim\limits_{x\to 0}\frac{-o(x)}{x^2+o(x^2)}$

$= 0$

例: 求 $\lim\limits_{x\to 0}\frac{\cos x-1+\frac{1}{2}x\cdot\sin x}{[\ln(1+x)]^2}$

$\lim\limits_{x\to 0}\frac{\cos x-1+\frac{1}{2}x\cdot\sin x}{[\ln(1+x)]^2} = \lim\limits_{x\to 0}\frac{1-\frac{x^2}{2!}+\frac{x^4}{4!}-o(x^5)-1+\frac{1}{2}x\cdot(x+\frac{x^3}{3!}+o(x^4))}{[x+o(x)]^4}$

$= \lim\limits_{x\to 0}\frac{-\frac{1}{24}x^4-\frac{1}{2}o(x^5)}{[x-o(x)]^4}$

$= \lim\limits_{x\to 0}\frac{-\frac{1}{24}-\frac{1}{2}o(x)}{[1-o(1)]^4}$

$= -\frac{1}{24}$