Lecture 7

§1 Binomial Series (二项级数)

1. Definition

The Maclaurin series of $f(x)=(1+x)^\alpha$ is $\sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$

> Def:
> - For any $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}:=\{0,1,2,\dots\}$,
>   "α choose n" $\rightarrow \binom{\alpha}{n}:=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$
>   is a binomial coefficient. By default, $\binom{\alpha}{0}=1$.
> - The series $\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$ is a binomial series.

注: 1° 当 $\alpha\in\mathbb{N}$ 时, $\binom{\alpha}{n}$ is the "usual" binomial coefficient that has a counting meaning (α 选 n).

2° 当 $\alpha\in\mathbb{N}$ 且 $n>\alpha$ 时 $\binom{\alpha}{n}=0$

2. 敛散性

1° 当 $\alpha=m\in\mathbb{N}$ 时, binomial series 是一个 finite sum:

$$\sum_{n=0}^{m}\binom{m}{n}x^n$$

对 $\forall x\in\mathbb{R}$ 都收敛

2° 当 $\alpha\notin\mathbb{N}$ 时, binomial series 是一个 infinite sum.

通过比值审敛法:

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\cdot\frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)}\right|\cdot|x|$$

$$=\frac{|\alpha-n|}{n+1}|x|\longrightarrow|x| \text{ as } n\to\infty$$

收敛半径为 1.

> Fact: $\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n=(1+x)^\alpha$ for $x\in(-1,1)$.

例: 求二项级数 $\sqrt{1+x}$ 的系数

For $\alpha=\frac{1}{2}$, $\binom{\frac12}{0}=1$, $\binom{\frac12}{1}=\frac12$, $\binom{\frac12}{2}=\frac{\frac12(\frac12-1)}{2!}=-\frac18$,

and for $n\geq 2$:

$$\binom{\frac12}{n}=\frac{\frac12(\frac12-1)\cdots(\frac12-n+1)}{n!}=\frac{\frac12(-\frac12)\cdots(-\frac{2n-3}{2})}{n!}$$

$$=(-1)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-3)}{2^n\,n!}$$

§2 非初等积分 (nonelementary integrals) 的近似

一些 elementary integral 的麦克劳林级数是交错级数.

e.g. $\sin x$, $\cos x$, $\ln(1+x)$, etc.

我们可以用交错级数的近似来估算非初等函数的值.

例: Estimate $\int_0^1 \sin(x^2)\,dx$ with an error <0.001

Sol. · $\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} \cdots\cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{y^{2n+1}}{(2n+1)!}$, $\forall y \in \mathbb{R}$

$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n+2}}{(2n+1)!}$, $\forall x \in \mathbb{R}$

$$\begin{aligned}\cdot\ \int_0^1 \sin(x^2)\,dx &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,dx \\ &= \sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,dx \\ &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n+1)! \cdot (4n+3)} \cdot x^{4n+3} \right) \Big|_{x=0}^{1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot (4n+3)}\end{aligned}$$

· $u_n = \frac{1}{(2n+1)! \cdot (4n+3)}$ is decreasing and $\to 0$ as $n \to \infty$

· when $n=2$, $u_n = u_2 = \frac{1}{5! \cdot 11} \approx 0.00076 < 0.001$, so

$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{3! \cdot 7} \approx 0.310$

§3 关于 arctan

在 Lecture 5 中, 有 $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\cdots \quad \forall x \in (-1,1)$

证: This holds for the endpoint $x \pm 1$

· For any $y \in \mathbb{R}$, we have

$$(1-y)(1+y+y^2+\cdots\cdots+y^n) = 1 - y^{n+1}$$

Substituting $y = -t^2$ in yields

$$(1+t^2)(1-t^2+t^4-t^6+\cdots\cdots(-t^2)^n) = 1-(-t^2)^{n+1}$$

$$\Rightarrow \sum_{k=0}^{n} (-1)^k t^{2k} = \frac{1}{1+t^2} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$$

$$\Rightarrow \frac{1}{1+t^2} = \sum_{k=0}^{n} (-1)^k t^{2k} + \frac{(-1)^n t^{2n+2}}{1+t^2}$$

$$\begin{aligned}\cdot\ \text{so } \arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} t^{2k+1} \Big|_{t=0}^{x} + \int_0^x \frac{(-1)^n t^{2n+2}}{1+t^2}\,dt \\ &= \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1}}_{P_{2n+1}(x)} + \underbrace{\int_0^x \frac{(-1)^n t^{2n+2}}{1+t^2}\,dt}_{R_{2n+1}(x)}\end{aligned}$$

问题转化为: $\lim_{n\to\infty} R(1)$ 是否为 0 & $\lim_{n\to\infty} R(-1)$ 是否为 0 ?

· For $x=1$

$$\begin{aligned}|R_n(1)| &= \left| \int_0^1 \frac{(-1)^n t^{2n+2}}{1+t^2}\,dt \right| \\ &\le \int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt \\ &< \int_0^1 t^{2n+2}\,dt \\ &= \frac{t^{2n+3}}{2n+3} \Big|_{t=0}^{1} \\ &= \frac{1}{2n+3} \to 0 \text{ as } n \to \infty\end{aligned}$$

· For $x=-1$

$$\begin{aligned}|R_n(-1)| &= \left| \int_0^{-1} \frac{(-1)^n t^{2n+2}}{1+t^2}\,dt \right| \\ &= \left| \int_{-1}^{0} \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\,dt \right|\end{aligned}$$

$$\le \int_{-1}^{0} \frac{t^{2n+2}}{1+t^2}\,dt$$
$$< \int_{-1}^{0} t^{2n+2}\,dt$$
$$= \left.\frac{t^{2n+3}}{2n+3}\right|_{t=-1}^{0}$$
$$= \frac{1}{2n+3} \longrightarrow 0 \text{ as } n\to\infty$$

Hence, $\lim\limits_{n\to\infty} R(1) = \lim\limits_{n\to\infty} R(-1) = 0$, and

$$\arctan x = \lim_{n\to\infty} P_n(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\cdots$$

also holds for $x=\pm 1$, and for all $x \in [-1,1]$.

注: $x=1 \Rightarrow \frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \cdots\cdots$

§4 常用的 Maclaurin series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

§5 Euler's Formula

1. complex number (复数)

1° 形式: $x+yi$ $(x, y \in \mathbb{R}, i^2=-1)$

2° 乘法: $(x+yi)(a+bi) = (ax-by) + (bx+ay)i$

3° complex exponential function $\exp: \mathbb{C} \to \mathbb{C}$ is defined by

$$\exp(z) := 1 + z + \frac{z^2}{2!} + \cdots\cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

2. Euler's Formula (欧拉公式)

For $\theta \in \mathbb{R}$, we have

$$\exp(i\theta) = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots\cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\cdots\right)$$
$$= \cos\theta + i\sin\theta$$

计 $\theta = \pi$, we have $e^{i\pi} = -1 + 0$, or

$$e^{i\pi} + 1 = 0$$

§6 Parametric Plane Curves

1. Parametric equation (参数方程) 定义

Definition

If a plane curve has the form

$$\{(x, y) : x = f(t), y = g(t), t \in I\},$$

where I is an interval, then the curve is called a parametric curve, and the equations

$$x = f(t),\ y = g(t),\ t \in I$$

t is called the parameter

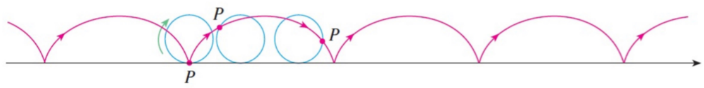
are called parametric equations of the curve.

注: 1° 若一条曲线为 $y=f(x)$ 的图像, $x\in I$, 则它也是一条参数曲线, 参数方程为 $x=t,\ y=f(t),\ t\in I$.

2° 一条曲线可以有多种参数方程表示方法.

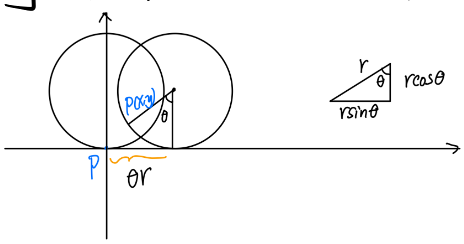
2. Cycloid (摆线)

The curve traced out by a point P on a disk as the disk rolls along a straight line is called a cycloid (see the figure below). Assume that the disk has radius r and rolls along the x-axis. If one position of P is the origin, find parametric equations for the cycloid.



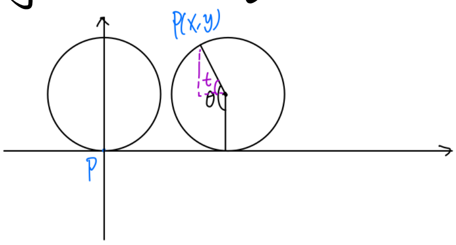
① For $\theta \in [0, \frac{\pi}{2}]$:

$$x = \theta r - r\sin\theta = r(\theta - \sin\theta)$$
$$y = r - r\cos\theta = r(1 - \cos\theta)$$



② For $\theta \in [\frac{\pi}{2}, \pi]$:

$$x = \theta r - r\cos t = \theta r - r\cdot\sin\theta = r(\theta - \sin\theta)$$
$$y = r + r\sin t = r + r(-\cos\theta) = r(1-\cos\theta)$$



Using similar arguments for $\theta \in [\pi, \frac{3\pi}{2}]$ and $\theta \in [\frac{3\pi}{2}, 2\pi]$, one can show that one arc of the cycloid can be parametrized by

$$x = r(\theta - \sin\theta),\quad y = r(1-\cos\theta),\quad 0 \le \theta \le 2\pi$$

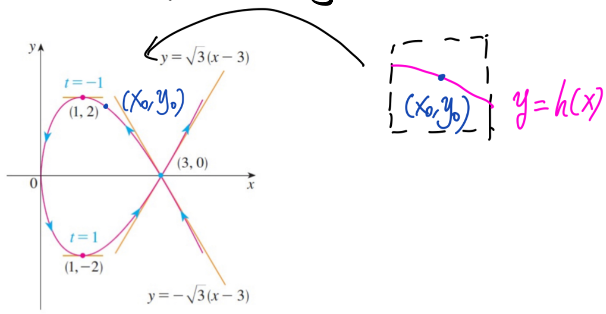
Since changing θ by $2k\pi$ changes the x-value by $2k\pi r$ and the y-value should remain the same by curve nature, the full cycloid is parametrized by $x=r(\theta-\sin\theta)$, $y=r(1-\cos\theta)$, $\theta\in\mathbb{R}$

3. 参数曲线的切线

若 $x=f(t)$, $y=g(t)$, y是关于 x 的可导函数，则

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt} \quad (\text{if } \frac{dx}{dt}\neq 0)$$

例: 对于曲线 $x=t^2$, $y=t^3-3t$, $t\in\mathbb{R}$



$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)}=\left.\frac{3t^2-3}{2t}\right|_{t=t_0} \quad \text{where } (x_0,y_0)=(f(t_0),g(t_0))$$

4. 二次求导

若 $x=f(t)$, $y=g(t)$, y是关于 x 的二次可导函数，则

$$\frac{d^2y}{dx^2}=\frac{d}{dx}y'=\frac{dy'/dt}{dx/dt}, \quad \text{where } y'=\frac{dy}{dx}$$

注: 切勿将一次求导的结果直接对 t 求导

例: 对上述例子，

$$\frac{d^2y}{dx^2}=\frac{d\left(\frac{3t^2-3}{2t}\right)/dt}{dx/dt}=\frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t}=\frac{3(t^2+1)}{4t^3}\begin{cases}>0 & \text{if } t>0 \ (\text{concave up})\\ <0 & \text{if } t<0 \ (\text{concave down})\end{cases}$$

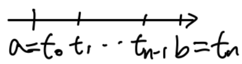
5. Arc length

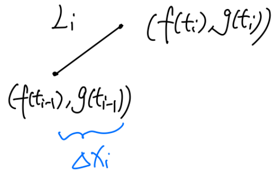
1° 证明

若 ① 曲线 C 为 $x=f(t)$, $y=g(t)$, $t\in[a,b]$.

② f' 与 g' 连续.

③ C is traversed exactly once. (不两次通过无穷多个点，反例: $\begin{cases}x=\cos 2t\\ y=\sin 2t\\ t\in[0,2\pi]\end{cases}$)





- $L_i=\sqrt{\Delta x^2+\Delta y^2}$

$\Delta x_i=x_i-x_{i-1}=f(t_i)-f(t_{i-1})\overset{MVT}{=\!=}f'(c_i)\Delta t_i$, where $c_i\in(t_{i-1},t_i)$

$\Delta y_i=y_i-y_{i-1}=g(t_i)-g(t_{i-1})\overset{MVT}{=\!=}g'(d_i)\Delta t_i$, where $d_i\in(t_{i-1},t_i)$

- Length of C: $L\approx\sum_{i=1}^{n}L_i=\sum_{i=1}^{n}\sqrt{f'(c_i)+g'(d_i)^2}\,\Delta t_i$

· It can be shown formally using upper and lower sums that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{f'(c_i)^2+g'(d_i)^2}\,\Delta t_i = \int_a^b\sqrt{f'(t)^2+g'(t)^2}\,dt$$

2° 定义

Definition

Let C be a curve given by $x=f(t)$, $y=g(t)$ and $t\in[a,b]$, where f' and g' are continuous on $[a,b]$. If C is traversed exactly once as t increases from a to b (except that $t=a$ and $t=b$ may give the same point), then the length L of C is defined by

$$L := \int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b\sqrt{(f'(t))^2+(g'(t))^2}\,dt.$$

例: Find the length of the parametric curve given by

$x=\sin 2t$, $y=\cos 2t$, $0\le t\le 2\pi$

Sol: 曲线为单位圆，但 traced twice.

$$L=\int_0^{\pi}\sqrt{f'(t)^2+g'(t)^2}\,dt$$
$$=\int_0^{\pi}\sqrt{4\cos^2(2t)+4\sin^2(2t)}\,dt$$
$$=2\int_0^{\pi}dt$$
$$=2\pi$$

6. Area under curves

We know that the area under a the graph of a nonnegative function $y=h(x)$ from $x=\alpha$ to $x=\beta$ is given by $A=\int_\alpha^\beta h(x)\,dx$. If the graph is traced out exactly once by the parametric equations

$$x=f(t),\quad y=g(t),\quad a\le t\le b,$$

where f' and g' are continuous, then by substitution, we have

$$A=\int_a^b g(t)f'(t)\,dt \quad \left(\text{or } A=\int_b^a g(t)f'(t)\,dt\right).$$

1° 若 $x\uparrow$ as $t\uparrow$, $A=\int_a^b g(t)f'(t)\,dt$

2° 若 $x\downarrow$ as $t\uparrow$, $A=\int_b^a g(t)f'(t)\,dt$

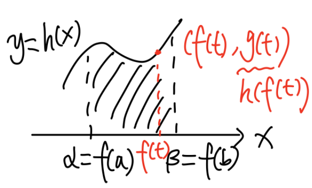
证明思路 1:

$$\Delta x_k = x_k - x_{k-1} = f(t_k)-f(t_{k-1}) = f'(c_k)\,\Delta t_k$$
$$A\approx\sum_{k=1}^{n} y_k\, f'(c_k)\,\Delta t_k = \sum_{k=1}^{n} g(c_k)\,f'(c_k)\,\Delta t_k$$

Taking limit as $n\to\infty$ motivates the definition

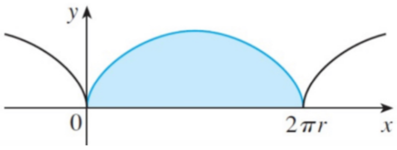
证明思路 2:

$$A=\int_\alpha^\beta h(x)\,dx$$
$$=\int_a^b h(f(t))\,f'(t)\,dt$$
$$=\int_a^b g(t)\,f'(t)\,dt$$



例: Find the area under one arch of the cycloid

$x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$, $\theta \in [0, 2\pi]$



$$A = \int_0^{2\pi r} y\, dx$$

$$= \int_0^{2\pi} r(1-\cos\theta) \cdot r(1-\cos\theta)\, d\theta$$

$$= r^2 \int_0^{2\pi} \cos^2\theta - 2\cos\theta + 1\, d\theta$$

$$= r^2 \int_0^{2\pi} \frac{1+\cos 2\theta}{2} + 1\, d\theta$$

$$= r^2 \cdot 3\pi$$

$$= 3\pi r^2$$