Lecture 13

§1 Properties of Definite Integrals

1. Definition 1

For $a<b$, we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. Definition 2

$\int_a^a f(x)\,dx = 0$

3. Properties of definite integrals

1° Order of Integral: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2° Zero Width Integral: $\int_a^a f(x)\,dx = 0$

3° Constant Multiple: $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

4° Sum and Difference: $\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5° Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

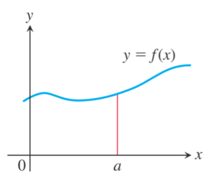
6° Max-Min Inequality: If f has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then

$\min f\cdot(b-a) \le \int_a^b f(x)\,dx \le \max f\cdot(b-a)$

7° Domination: $f(x)\ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

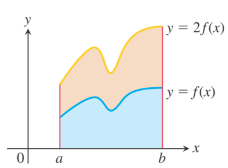
$f(x)\ge 0$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$

Graphs:



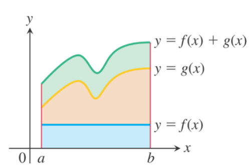
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



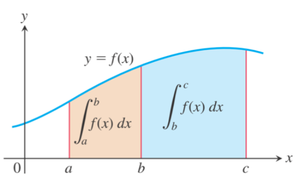
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



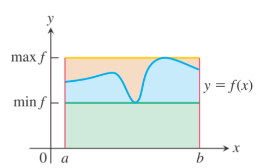
(c) *Sum: (areas add)*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



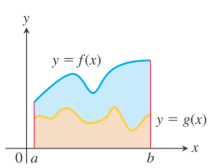
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f\cdot(b-a) \le \int_a^b f(x)\,dx \le \max f\cdot(b-a)$$



(f) *Domination:*

$f(x)\ge g(x)$ on $[a,b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

* Knowing that Addictivity (Property 5) works for $b \in [a,c]$, we can see that it works even if $b \in [a,c]$

e.g. $\int_3^5 f(x)\,dx + \int_5^6 f(x)\,dx = \int_3^6 f(x)\,dx$

Then $\int_3^5 f(x)\,dx = \int_3^6 f(x)\,dx - \int_5^6 f(x)\,dx$

$\int_3^5 f(x)\,dx = \int_3^6 f(x)\,dx + \int_6^5 f(x)\,dx$

* All properties 3-6 and 7(i) works: even if f is not always ≥ 0 on $[a,b]$, and even if f is not continuous

4. Proof of Property 6:

Suppose $\int_a^b f(x)\,dx$ exists. Let

$$M = \max_{x\in[a,b]} f(x) \qquad m = \min_{x\in[a,b]} f(x)$$

Then every Riemann sum satisfied:

$$m\sum_{k=1}^{n} \Delta x_k = \sum_{k=1}^{n} m\Delta x_k \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq \sum_{k=1}^{n} M\Delta x_k = M\cdot\sum_{k=1}^{n}\Delta x_k$$

So $m \lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta x_k \leq \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x_k \leq M \lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta x_k$

Hence $m\int_a^b dx \leq \int_a^b f(x)\,dx \leq M\int_a^b dx$

Note that $\int_a^b dx = (b-a)$

5. Definition 3

If $y=f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the area under the curve $y=f(x)$ over $[a,b]$ is the integral of f from a to b.

$$A = \int_a^b f(x)\,dx$$

Note: If $f(x)<0$ for some $x\in[a,b]$, then this definition doesn't hold.

§2 Average values of a function

1. Definition

If f is integrable on $[a,b]$, then its average value on $[a,b]$, also called its mean, is

$$\operatorname{av} f(x) = \frac{1}{b-a}\int_a^b f(x)$$

§3 MVT for Definite Integrals

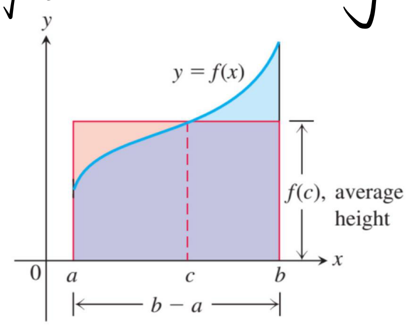
1. Theorem 5.4.3 — The Mean Value Theorem for Definite Integrals

If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

2. Geometric interpretation

If f is continuous, then there is a point $c \in [a,b]$ such that $f(c)$ is the average height.



3. Proof

Since f is continuous, there exist x_1 and x_2 in $[a,b]$ such that

$$f(x_1) = m = \min_{x\in[a,b]} f(x)$$

$$f(x_2) = M = \max_{x\in[a,b]} f(x)$$

Let us assume that $x_1 \neq x_2$.

By min-max inequality,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

By IVT,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \text{ for some } c \text{ between } x_1 \text{ and } x_2$$

so $c \in [a,b]$

4. Consequence

If f is continuous on $[a,b]$ and $\int_a^b f(x)\,dx = 0$

then $f(c)=0$ for some $c \in [a,b]$

§4 Fundamental Theorem of Calculus (FTC)

1. Theorem 5.4.4 — The Fundamental Theorem of Calculus, Part 1

> If f is continuous on $[a,b]$, then
> $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and
> $F(x) = \int_a^x f(t)\,dt$ is differentiable on (a,b) and
> its derivative is $f(x)$:
> $$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

2. Proof of FTC ($F'(x) = f(x)$)

Suppose f is continuous on $[a,b]$.

Let $F: [a,b] \to R$, $F(x) = \int_a^x f(t)\,dt$

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \quad ①$$

Look at $F'_+(x)$ first. For $h>0$, by MVT for integrals,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c) \text{ for some } c \in [x, x+h] \quad ②$$

As $h \to 0^+$, $c \to x^+$, so from ① and ②

$$F'_+(x) = \lim_{h\to 0^+} f(c) = \lim_{c\to x^+} f(c) = f(x)$$

For $h<0$,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = -\frac{1}{h}\int_{x+h}^{x} f(t)\,dt = f(c),\ c\in[x+h,x]$$

So,

$$F_-'(x) = \lim_{h\to 0^-} f(c) = \lim_{c\to x^-} f(c) = f(x)$$

Hence $F'(x)=f(x)\ \forall x\in(a,b)$

3. Proof of FTC (continuity and differentiability)

We have show that

$$F'(x)=f(x)\quad \forall x\in(a,b)$$

Hence, F is differentiable on interval (a,b).

Hence, F is continuous on interval (a,b).

The argument above can be used to show that

$$F_+'(a)=f(a)\quad\text{and}\quad F_-'(b)=f(b)$$

$\therefore$ F is one-sided differentiable at $x=a$ and $x=b$

$\therefore$ F is one-sided continuous at $x=a$ and $x=b$

$\therefore$ F is continuous on $[a,b]$

e.g. Use the Fundamental Theorem to find dy/dx if

(a) $y=\int_a^x (t^3+1)\,dt$

(b) $y=\int_x^5 3t\sin t\,dt$

(c) $y=\int_1^{x^2} \cos t\,dt$

(d) $y=\int_{1+3x^2}^{4} \frac{1}{2+t}\,dt$

(a) $y=\int_a^x (t^3+1)\,dt$

Let $F(x)=y=\int_a^x (t^3+1)\,dt$ and $f(t)=t^3+1$

Then $dy/dt = F'(x)$

By FTC I, $F'(x)=f(x)=x^3+1$

Hence $dy/dx = x^3+1$

(b) $y=\int_x^5 3t\sin t\,dt$

Let $F(x)=\int_5^x 3t\sin t\,dt$ and $f(t)=3t\sin t$

$y=-F(x)$

$dy/dx=(-F(x))'=-F'(x)=-f(x)=-3x\sin x$

(c) $y=\int_1^{x^2}\cos t\,dt$

Let $F(u)=\int_1^u \cos t\,dt$ and $u=x^2$

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=f(u)\cdot 2x=\cos(u)\cdot 2x=\cos x^2\cdot 2x$

(d) $y=\int_{1+3x^2}^4 \frac{1}{2+t}\,dt$

Let $F(u)=\int_4^u \frac{1}{2+t}$, $u=1+3x^2$

$y=-F(u)$

$\frac{dy}{dx}=-f(u)\cdot 6x=-\frac{6x}{2+u}=\frac{-2x}{1+x^2}$

Remark:

If $F(u)=\int_a^u f(t)\,dt$, then

$$\int_a^{g(x)} f(t)\,dt=(F\circ g)(x)$$

which means

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt=(F\circ g)'(x)=F'(g(x))\,g'(x)$$

By FTC I:

$$F'(g(x))\,g'(x)=f(g(x))\,g'(x)$$

Hence:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt=f(g(x))\,g'(x)$$

4. Theorem 5.4.4 (Continued) – The Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx=F(b)-F(a)$$

Proof:

Let $G(x)=\int_a^x f(t)\,dt$

By FTC 1, G is an antiderivative of f on (a,b).

Since F is also an antiderivative of f, $\exists c$ such that

$$G(x)=F(x)+c,\quad \forall x\in(a,b)$$

$G(a)=\lim_{x\to a^+}G(x)=\lim_{x\to a^+}(F(x)+c)=F(a)+c$

Similarly, $G(b)=F(b)+c$

$$\int_a^b f(x)\,dx=G(b)$$

$$\int_a^b f(x)\,dx=G(b)-G(a)$$

$$\int_a^b f(x)\,dx=(F(b)+c)-(F(a)+c)$$

$$=F(b)-F(a)$$

* By FTC 2, in order to calculate $\int_a^b f(x)\,dx$, it suffices to find any antiderivative F of f.

Notation:

We write $F(x)\Big|_{x=a}^{b}$ or $F(x)\Big|_a^b$ or $[F(x)]_a^b$ to mean $F(b)-F(a)$

e.g. $\int_0^\pi \cos x\,dx = \sin x\Big|_0^\pi = \sin\pi-\sin 0=0$

e.g. $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1=\frac{1}{3}-0=\frac{1}{3}$