# DDA2001: Sampling Problem Set

This problem set is for **Midterm examination**.

1. Similar questions to the following exercises with minor changes.
2. Only in Midterm, not in assignments, quiz or final.

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1. **Simple $X$, Complicated $g$**: Give pseudocode to calculate the following integration using sampling,

   (a) $\int_0^2 \exp[x + \cos(x)]\mathrm{d}x$.

   (b) $\int_0^\infty \exp[-x^2 - x + \cos(x)]dx$; (Hint: recall the pdf of exponential distribution with parameter 1 is $f(x) = e^{-x}, x > 0$.)

   (c) $\int_{-\infty}^\infty \exp[\cos(x) - 2(x-1)^2]dx$; (Hint: Recall the pdf of the Normal distribution with mean $\mu$ and variance $\sigma^2$ is $f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}$.)

***Solution:***

(a) Since

$$\int_0^2 \exp[x + \cos(x)]\mathrm{d}x = \int_0^2 \underbrace{2\exp[x + \cos(x)]}\underbrace{\frac{1}{2}}\mathrm{d}x,$$

where $\int_0^2 1/2\mathrm{d}x = 1$, we can calculate the integration as follows,

Step 1: construct a random variable $X$, which is uniformly distributed in $[0, 2]$;

Step 2: generate $N$ samples of $X$, denoted as $X_1, \cdots, X_N$, where $N$ is sufficiently large;

Step 3: calculate

$$\frac{1}{N}\sum_{i=1}^{N} 2\exp\left[X_i + \cos(X_i)\right]$$

to approximate

$$\mathbb{E}\left[2\exp[X + \cos(X)]\right] = \int_0^2 \exp[x + \cos(x)]\mathrm{d}x.$$

(b) Since

$$\int_0^\infty \exp[-x^2 - x + \cos(x)]dx = \int_0^\infty \underbrace{\exp[-x^2 + \cos(x)]}\underbrace{\exp(-x)}dx,$$

where $\int_0^\infty \exp(-x)dx = 1$, we can calculate the integration as follows,

Step 1: construct an exponentially distributed random variable $X$;

Step 2: generate $N$ samples of $X$, denoted as $X_1, \cdots, X_N$, where $N$ is sufficiently large,

Step 3: calculate

$$\frac{1}{N}\sum_{i=1}^{N}\exp[-X_i^2+\cos(X_i)]$$

to approximate

$$\mathbb{E}\left\{\exp[-X^2+\cos(X)]\right\} = \int_0^{\infty}\exp[-x^2-x+\cos(x)]dx.$$

(c) Recall the probability density function of the normal distribution with mean $\mu$ and variance $\sigma^2$ is given by

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right],$$

where $-\infty < x < \infty$.

Since we have $\int_{-\infty}^{\infty}\frac{2}{\sqrt{2\pi}}\exp[-2(x-1)^2]dx = 1$, we can calculate the integration as follows:

Step 1: construct a normally distributed random variable $X$ with mean $\mu = 1$ and variance $\sigma^2 = 1/4$;

Step 2: generate $N$ samples of $X$, denoted as $X_1, \cdots, X_N$, where $N$ is sufficiently large;

Step 3: calculate

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\sqrt{2\pi}}{2}\exp(\cos(X))$$

to approximate

$$\mathbb{E}\left[\frac{\sqrt{2\pi}}{2}\exp(\cos(X))\right] = \int_{-\infty}^{\infty}\exp[\cos(x)-2(x-1)^2]dx.$$

2. **Simple $X$, Complicated $g$**: Please find an algorithm to estimate $\pi$. (**Hint:** $\pi$ is the area for unit disc).

***Solution:*** If we pick randomly in the dotted square, then the probability that it will land in some part within the square will be $\dfrac{\text{area of the part}}{\text{area of the square}}$.
Therefore, we can draw a unit disc at the center of a $2\times 2$ square, and randomly pick dot within the square, then the probability that this dot is within the disc equals to $\frac{\pi}{4}$.
The formally procedure is as follows:
Choose a large enough constant $N$.

Step 1: Generate random points $(X_i, Y_i)$ within unit square by drawing $X_i, Y_i \overset{i.i.d}{\sim} Unif(-1,1), \quad i = 1, 2, ..., N$.

Step 2: For $i = 1, 2, ..., N$:
· If $X_i^2 + Y_i^2 \leq 1$, set $U_i$=1;
· Otherwise, set $U_i = 0$.

Step 3: Calculate $4 * \frac{1}{N}\sum_{i=1}^{N} U_i$ to approximate $\pi$.

$\square$

3. **Simple $g$, Complicated $X$-know how $X$ is generated**: Toss a fair coin continuously. Give the pseudocode to find the expected steps such that you get $n$ consecutive heads.

***Solution:***

Repeat $N$ times ($N$ is large enough) the following process:

Step 1: Initialize *steps* $= 0$, list=[ ](record the history of tossing)

Step 2: Draw a Bernoulli random variable $X$ with probability 0.5. Set *steps* $\longleftarrow$ *steps*$+1$.
· If $X = 1$, add an element 1 to the end if list
· If $X = 0$, add an element 0 to the end if list

Step 3: Check whether to continue:
· If all the last $n$ elements in list are 1, stop.
· Otherwise, go back to Step 2.

Finally compute the mean value of *steps* of $N$ times. $\square$

4. **Simple $g$, Simple $X$-know CDF**:
(1) Give the pseudocode for generating a Bernoulli ($p$) random variable $X$ by Inverse Transform Method (ITM).
(2) Give the pseudocode for generating a Binomial ($n, p$) random variable $X$ by Inverse Transform Method (ITM).

***Solution:***

(1)

Step 1: Generate $U \sim Unif(0, 1)$.

Step 2: Output $X = 0$ if $U \leq 1-p$; $X = 1$ if $U > 1-p$.

(2)

Step 1: Generate $n$ iid random variables $U_1, \ldots, U_n \sim Unif(0, 1)$.

Step 2: For each $1 \leq i \leq n$:
· If $U_i \leq 1-p$, set $Y_i = 0$ ;
· If $U_i > 1-p$, set $Y_i = 1$
(This yields $n$ iid Bernoulli ($p$) random variables).

Step 3: Output $X = \sum_{i=1}^{n} Y_i$

$\square$

5. **Simple $g$, Simple $X$-know CDF**: Suppose discrete random variable $X$ takes value $x_i$ with probability $p_i$, $i = 1, 2, ..., n.(\sum_{i=1}^{n} p_i = 1)$. Generate this random variable by Inverse Transform Method (ITM).

***Solution:***

Step 1: Generate $U \sim Unif(0, 1)$.

Step 2: If $U \in [0, p_1]$, output $x_1$;
If $U \in (p_1, p_1 + p_2]$, output $x_2$;
...
If $U \in (\sum_{i=1}^{n-1} p_i, \sum_{i=1}^{n} p_i]$, output $x_n$;

$\square$

6. **Complicated $X$-know complicated CDF** Use the uniform probability density function $f(x) = 1$ for $x \in (0, 1)$ to generate a random variable having the cumulative distribution function:

$$F(x) = 1 - \exp(-x^2),\ x \geq 0$$

by Inverse Transform Method (ITM).

***Solution:*** The inverse function of $F(x)$ is $F^{-1}(u) = \sqrt{-\ln(1-u)}$.
Hence, the inverse transform method is as follows:

Step 1: Draw $U \sim Unif(0, 1)$;

Step 2: Output $X = \sqrt{-\ln(1-U)}$.

$\square$

7. **Complicated $X$-know complicated CDF** The double exponential density is defined as

$$g(x) = e^{-|x|}/2, \quad -\infty < x < +\infty.$$

Generate a random variable with probability density function $g(x)$ by Inverse Transform Method(ITM).

***Solution:***

We first compute the cumulative distribution function. When $x \leq 0$, $F(x) = \int_{-\infty}^{x} e^t/2dt = e^x/2$. When $x > 0$, $F(x) = 1/2 + \int_0^x e^{-t}/2dt = 1 - e^{-x}/2$. So,

$$F(x) = \begin{cases} e^x/2 & x \leq 0 \\ 1 - e^{-x}/2 & x > 0 \end{cases}$$

Then, we have

$$F^{-1}(u) = \begin{cases} \ln(2u) & u \leq 1/2 \\ -\ln 2(1-u) & u > 1/2 \end{cases}$$

We generate the random variable $X$ as follows:

Step 1: Generate $U \sim Uniform(0, 1)$;

Step 2: If $U \leq 1/2$, output $X = \ln(2U)$;
Otherwise, output $X = -\ln 2(1-U)$;

$\square$

8. **Simple $g$, Complicated $X$-know complicated CDF** Use the uniform probability density function $f(x) = 1$ for $x \in (0, 1)$ to generate $E[e^{cos(X)}]$, where random variable $X$ having the cumulative distribution function:

$$F(x) = 1 - \exp(-x/\lambda), \ x \geq 0$$

by Inverse Transform Method (ITM).

***Solution:*** The inverse function of $F(x)$ is $F^{-1}(u) = -\lambda \ln(1-u)$.
Hence, the inverse transform method is as follows:
Choose a large enough constant $N$.

Step 1: Draw $U_i \overset{i.i.d}{\sim} Unif(0, 1), \quad i = 1, 2, ..., N;$

Step 2: Output $X_i = -\lambda \ln(1 - U_i), \quad i = 1, 2, ..., N;$

Step 3: Calculate $\frac{1}{N}\sum_{i=1}^{N} e^{cos(X_i)}$ to approximate $E[e^{cos(X)}]$.

□

9. **Complicated $X$-know complicated PDF** Using the uniform probability density function $f(x) = 1$ for $x \in (0, 1)$ as the envelope function to generate a random variable having the probability density function:

$$g(x) = 6x(1-x), 0 < x < 1$$

by Acceptance/Rejection Method (ARM).

***Solution:*** By differentiating the ratio $\dfrac{g(x)}{f(x)} = 6x(1-x)$ with respect to $x$ and setting the resultant derivative equal to zero, we obtain the maximal value of this ratio at x = 1/2.
Hence,

$$c \leq \min_{0<x<1} \frac{f(x)}{g(x)} = \frac{1}{6 \times \frac{1}{2}(\frac{1}{2})} = \frac{2}{3},$$

Therefore, we can choose $c = \frac{1}{2}$ and

$$\frac{cg(x)}{f(x)} = 3x(1-x).$$

The rejection method is as follows:

Step 1: Draw $X \sim Unif(0, 1)$;

Step 2: If $X = x$, accept with probability $3x(1-x)$;
Otherwise, go to Step 1.

□

10. **Complicated $X$-know complicated PDF** Using the uniform probability density function $f(x) = 1$ for $x \in (0, 1)$ as the envelope function to generate a random variable having the beta density

$$g(x) = 20x(1-x)^3, 0 < x < 1$$

by Acceptance/Rejection Method (ARM).

***Solution:*** By differentiating the ratio $\frac{g(x)}{f(x)} = 20x(1-x)^3$ with respect to $x$ and setting the resultant derivative equal to zero, we obtain the maximal value of this ratio at $x = 1/4$.
Hence

$$c \leq \min_{0<x<1} \frac{f(x)}{g(x)} = \frac{1}{20 \times \frac{1}{4}(\frac{3}{4})^3} = \frac{64}{135},$$

Therefore, we can choose $c = \frac{1}{10}$ and

$$\frac{cg(x)}{f(x)} = 2x(1-x)^3.$$

The rejection method is as follows:

Step 1: Draw $X \sim Unif(0,1)$;

Step 2: If $X = x$, accept with probability $2x(1-x)^3$;
otherwise,go to Step 1.

□

11. **Complicated $X$-know complicated PDF** Suppose you want to generate a random variable $X$ having the probability density function

$$g(x) = xe^{-x}, \quad x \geq 0.$$

Let's try the Acceptance/Rejection Method with $f(x) = \frac{1}{2}e^{-\frac{1}{2}x}, x \geq 0$ (which is the lambda).

(a) Find the constant $c$ such that

$$c \leq \frac{f(x)}{g(x)}$$

for all $x \geq 0$.

(b) Write the Acceptance Rejection Method in detail to generate a $X$ having the probability density function $g(x)$.

***Solution:***

(a) Denote $h(x) = \frac{f(x)}{g(x)} = \frac{1}{2x}e^{\frac{1}{2}x}, \quad x \geq 0$, then we have

$$h'(x) = e^{\frac{1}{2}x}(\frac{1}{4x} - \frac{1}{2x^2}).$$

Set $h'(x) = 0$ and then we can obtain the minimal value of $h(x)$ at $x = 2$.
Hence

$$c \leq \min_{0 \leq x} \frac{f(x)}{g(x)} = \frac{e}{4}.$$

(b) We can choose $c = \frac{1}{2}$ and we have

$$\frac{cg(x)}{f(x)} = xe^{-\frac{1}{2}x}.$$

The rejection method is as follows:

Step 1: Draw $X \sim Exp(\frac{1}{2})$;

Step 2: If $X = x$, accept with probability $xe^{-\frac{1}{2}x}$;
Otherwise, go to Step 1.

$\square$