Lecture 27

§1 Automonous Equations and Phase Line Analysis : General Idea

1. Definition

An autonomous equation is a differential equation of the form

$$\frac{dy}{dx}=f(y)$$

2. Solve autonomous equation

Suppose K is a root of f, i.e. $f(K)=0$

Consider the constant function $y=y(x)=K$

$$\frac{dy}{dx}=0 \quad \forall x$$

$$f(y)=f(y(x))=f(K)=0 \quad \forall x$$

Hence the constant function $y=K$ is a solution to $\frac{dy}{dx}=f(y)$.

3. Definition

Given an autonomous equation $\frac{dy}{dx}=f(y)$,

for any root K of f:

K is called an equilibrium value

The constant function $y=K$ is called an equilibrium solution to autonomous equation.

4. Phase line analysis

We can analyse the behaviour of the solutions

by drawing all equilibrium points on the y-axis

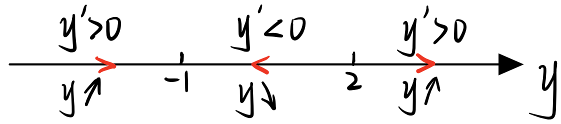
and analysing the signs of derivatives on the intervals separates by the equilibrium points.

e.g. Consider $\frac{dy}{dx}=y^2-y-2$

$$y'=(y+1)(y-2)$$

equilibrium values are -1 and 2

phase analysis:

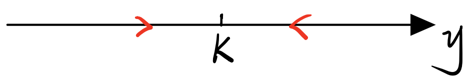


§2 Automonous Equations and Phase Line Analysis: Stable and Unstable Equilibrium

1. Stable equilibrium

Like K be an equilibrium value of $\frac{dy}{dx} = f(y)$.

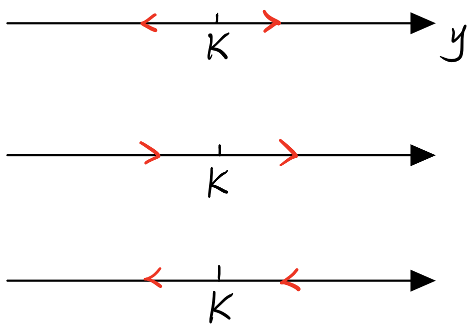
Consider a phase line analysis:



In the case, $y = K$ is a stable equilibrium.

2. Unstable equilibrium

For other cases of $y = K$ we may have an unstable equilibrium.



e.g. For $\frac{dy}{dx} = y^2 - y - 2$

$y = -1$ is a stable equilibrium solution

$y = 2$ is an unstable equilibrium solution

§3 Automonous Equations and Phase Line Analysis : Newton's Law of Cooling

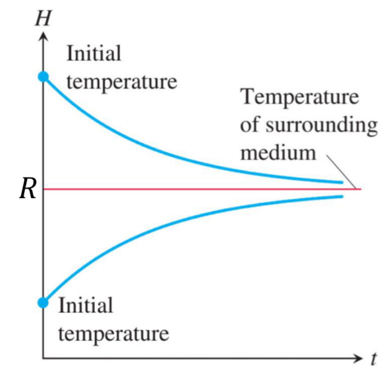
1. Newton's law of cooling (heating)

Newton's law of cooling (or heating) states that the rate of change of the temperature of an object is proportional to the difference of temperatures between the object and its surroundings. In other words, if $H(t)$ is the temperature of an object at time t, then H satisfies the differential equation

$$\frac{dH}{dt} = k(H - R), \qquad (k < 0)$$

where k is some negative constant (why?) and R is the surrounding temperature, which is a constant.

- $H = R$ is the equilibrium solution:



e.g. Example

The body of a murder victim is found at noon in a room with a constant temperature of 20°C. At noon the temperature of the body is 35°C; two hours later the temperature of the body is 33°C.

(a) Find the temperature, H, of the body as a function of t, the time in hours since it was found.

(b) Assuming that the body had the normal temperature 37°C at the time of murder, estimate the time of the murder.

(a) IVP is: $\begin{cases} \dfrac{dH}{dt} = k(H-20) \\ H(0) = 35 \\ H(2) = 33 \end{cases}$

$$\int \frac{1}{H-20}\, dH = \int k\, dt$$

$$\ln|H-20| = kt + C$$

$$|H-20| = e^{kt+C} = Ae^{kt}$$

Since $H(0)=35$, consider $H>20$:

$$H-20=Ae^{kt}$$

$$\begin{cases} 35-20=A \\ 33-20=Ae^{2k} \end{cases}$$

$$\begin{cases} A=15 \\ k=\frac{1}{2}\ln\frac{13}{15} \end{cases}$$

$$H=15\left(\frac{13}{15}\right)^{\frac{t}{2}}+20$$

§4 Euler's Method

1. Euler's method

The main idea of Euler's method is to start with (x_0,y_0) and approximate the solution curve $y=y(x)$ by generating a sequence of points (x_0,y_0), (x_1,y_1), (x_2,y_2) ...... using tangent line approximation.

2. How to use Euler's method

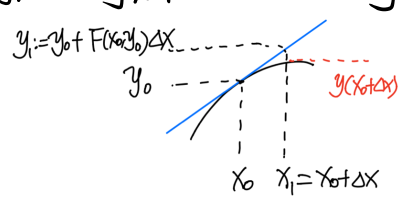
1° Start with (x_0,y_0). Pick a small Δx

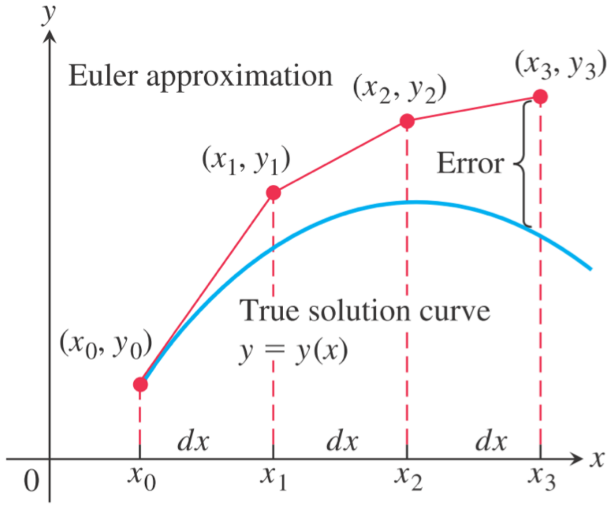
2° Set $\begin{cases} x_1:=x_0+\Delta x \\ y_1:=y_0+F(x_0,y_0)\Delta x \end{cases}$

3° Now set $x_2:=x_1+\Delta x$ and $y_2:=y_1+F(x_1,y_1)\Delta x$

In general, for $k\geq 1$, set

$$\begin{cases} x_k:=x_0+k\Delta x \\ y_k:=y_{k-1}+F(x_{k-1},y_{k-1})\Delta x \end{cases}$$





e.g. Given $y' = 1+y$ and $y(0)=1$, approximating $y(0.3)$ with $\Delta x = 0.1$

$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0.1 \qquad y_1 = 1 + (1+1)(0.1) = 1.2$

$x_2 = 0.2 \qquad y_2 = 1.2 + (1+1.2)(0.1) = 1.42$

$x_3 = 0.3 \qquad y_3 = 1.42 + (1+1.42)(0.1) = 1.662$

$y(0.3) = y(x_3) \approx y_3 = 1.662$

exact solution:

$v(x) = e^{\int(-1)dx} = e^{-x}$

$y = e^{x} \cdot \int e^{-x} dx$

$= e^{x}(-e^{-x} + C)$

$= -1 + Ce^{x}$

$y(0) = 1 \Rightarrow C = 2$

$y(0.3) = -1 + 2e^{0.3} = 1.6997$