§1 Proof of Theorem 4.1.2

1. Theorem 4.1.2

THEOREM 4.1.2 —The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

2. Proof

THEOREM 2.2.5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

Suppose that f has a local maximum at c (the case where c gives a minimum is similar)

Then there exists a with $a>0$

such that $f(c) \geq f(x)$ for all $x \in (c-a, c+a)$

Let $g(x) = \dfrac{f(x)-f(c)}{x-c}$ for $x \in (c-a, c+a) \setminus \{c\}$

1° For $x \in (c-a, a)$,

$f(x)-f(c) \le 0$ and $x-c<0$

so $g(x) \geq 0$

By Theorem 2.2.5 (one-side version) we have

$$\lim_{x\to c^-} g(x) \geq \lim_{x\to c^-} 0 = 0$$

2° For $x \in (c, c+a)$,

$f(x)-f(c) \le 0$ and $x-c>0$,

so $g(x) \le 0$

By Theorem 2.2.5 (one-side version) we have

$$\lim_{x\to c^+} g(x) \le \lim_{x\to c^+} 0 = 0$$

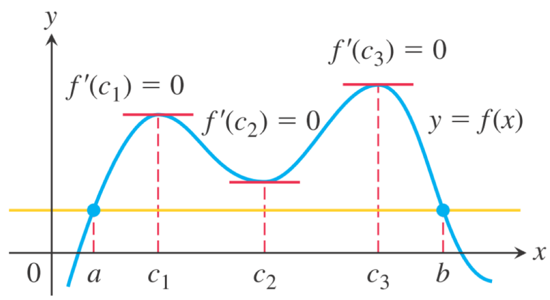
3° Now $0 \le \lim_{x\to c^-} g(x) = f'_-(c) = f'(c) = f'_+(c) = \lim_{x\to c^+} g(x) \le 0$

so $f'(c)=0$

§2 Rolle's Theorem

1. Rolle's Theorem

Suppose that a function f is continuous on $[a,b]$ and differentiable at every point in (a,b), and it satisfies $f(a)=f(b)$
Then there exists c in (a,b) such that $f'(c)=0$



2. Proof of Rolle's Theorem

By the Extreme Value Theorem, there exists $c_1, c_2 \in [a,b]$ such that $f(c_1)=m$ and $f(c_2)=M$, where m and M are absolute minimum and maximum respectively.

Note that m and M are local extrema as well.

1° If $m=M$:

then $f(x)=M$ for all $x\in[a,b]$,

so $f'(c)=0$ for any $c\in(a,b)$.

2° Suppose $m<M$:

Either $f(a)=f(b)\neq m$ or $f(a)=f(b)\neq M$

① Suppose $f(a)=f(b)\neq m$

Since $f(c_1)=m$, we have $c_1\in(a,b)$

Since $f'(c_1)$ is defined, by Theorem 4.1.2 $f'(c_1)=0$

② Suppose $f(a)=f(b)\neq M$

Since $f(c_2)=M$, we have $c_2\in(a,b)$

Since $f'(c_2)$ is defined, by Theorem 4.1.2 $f'(c_2)=0$

e.g. Show that the equation $x^3+3x+1=0$ has exactly one real solution.

Let $f(x) = x^3+3x+1$

$f(-1) = -1-3+1 = -3$

$f(0) = 1$

By IVT, $f(x_0)=0$ for $x_0 \in (-1,0)$

If we have point $f(x_1)=0$,

By Rolle's Theorem we must have $f'(c)=0$, $c \in (x_1, x_2)$

Since $f'(x) = 3x^2+3 > 0 \ \forall x \in R$

There is no point where $f'(x)=0$

$\therefore$ Only one point where $f(x)=0$

§3 Mean Value Theorem (MVT) (中值定理)

1. Mean Value Theorem (MVT)

Suppose that a function f is continuous on $[a,b]$ and differentiable at every point in (a,b). Then there exists c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

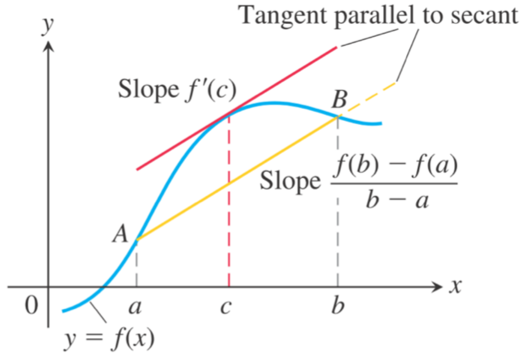


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to the secant joining A and B.

2. Proof

Define h on $[a,b]$ by

$$h(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$$

Note that $h(a) = h(b) = 0$

h is continuous on $[a,b]$ and differentiable on (a,b), since f is.

By Rolle's Theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$

$$h(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \quad \forall x \in (a,b)$$

$$\Rightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

3. Physical consequence of the MVT

Suppose that $f(t)$ represents the distance travelled until time t.

Then the mean value theorem implies that if we pick two moments $t = a$ and $t = b$, there has to be some moment $t = c$ in between at which the instantaneous speed is equal to the average speed between $t = a$ and $t = b$.

4. Corollary 1 of the MVT

If $f'(x) = 0$ at each point x of an open interval (a,b), then $f(x) = C$ for all $x \in (a,b)$, where C is a constant.

Proof:

Consider 2 points, $x_1, x_2 \in (a,b)$

MVT tells us that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{with } c \in (x_1, x_2)$$

Then if $f'(c) = 0$:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

$$f(x_2) = f(x_1) = C$$

Hence if $f'(x) = 0$ for all $x \in (a, b)$:

$$f(x) = C$$

5. Corollary 2 of the MVT

> If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

Proof:

Let $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x) = 0$

Then, from Corollary 1:

$$h(x) = C$$

Therefore: $f(x) = g(x) + C$

e.g. If $f'(x) = \sin x$, what is $f(x)$?

We know if $g(x) = -\cos x$, then

$$g'(x) = \sin x = f'(x), \ \forall x$$

Hence $f(x) = g(x) + C = -\cos x + C$

e.g. Prove that $|\sin x - \sin y| \le |x - y|$ for all $x, y \in R$

1° For $x = y$, both sides $= 0$, hence the statement is true in this case.

2° Suppose $x \neq y$. Consider $y < x$.

Let $f(z) = \sin z$

Since $\sin z$ is differentiable and continuous for all z, by MVT there exists c in (y, x) such that

$$\cos c = \frac{\sin x - \sin y}{x - y}$$

$$(x-y)\cos c = \sin x - \sin y$$

$$|x-y||\cos c| = |\sin x - \sin y|$$

$$0 \leq \cos c \leq 1$$

$$|x-y||\cos c| \leq |x-y|$$

$$|\sin x - \sin y| \leq |x-y|$$

QED

§4 Monotonicity

1. Definition

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1° If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be increasing on I

2° If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be decreasing on I

A function that is increasing or decreasing on I is said to be monotonic on I.

2. Corollary 4.33

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).
(i) If $f'(x)>0$ at each point $x\in(a,b)$, then f is increasing on $[a,b]$.
(ii) If $f'(x)<0$ at each point $x\in(a,b)$, then f is decreasing on $[a,b]$.

Proof of 4.3.3 (i):

Let x_1 and x_2 be in $[a,b]$ with $x_1<x_2$.

Then f is continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2)

By MVT, there exists $c\in(x_1,x_2)$ such that

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)$$

Since $x_2-x_1>0$ and $f'(c)>0$, we have

$$f(x_2)-f(x_1)>0$$

i.e. $f(x_2)>f(x_1)$, so f is increasing on $[a,b]$.

e.g. Prove that $f(x)=\sqrt{x}$ is increasing on $[0,\infty)$

$f'(x)=\frac{1}{2\sqrt{x}}>0$ for all $x>0$

By Corollary 4.3.3, f is increasing on $[0,b]$ for all $b\in R$

Let x_1 and x_2 be in $[0,\infty)$ with $x_1<x_2$

Since f is increasing on $[0,x_2]$ and $0\le x_1<x_2$, we have $f(x_1)<f(x_2)$.

By definition, f is increasing on $[0,+\infty)$