Lecture 5

§1 Cauchy sequence

1. **Theorem**: Bolzano - Weierstrass theorem（有界序列必有收敛子序列）

If $\{a_n\}$ is bdd, then it contains a convergent subsequence.

证明:

$\because$ $\{a_n\}$ is bdd

$\therefore$ $\overline{\lim_{n\to\infty}} a_n$ is a finite number

$\because$ $\overline{\lim_{n\to\infty}} a_n \in E$

$\therefore$ $\exists$ subseq of $\{a_n\}$ converging to $\overline{\lim_{n\to\infty}} a_n$

Q.E.D.

2. **Definition**: Cauchy sequences（柯西序列）

We say $\{a_n\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists N$, s.t. $|a_n - a_m| < \varepsilon$, as long as $n, m > N$

注: ① $|a_n - a_m| < \varepsilon$ 可以被替换为 $|a_n - a_m| \le c\varepsilon$

② Motivation: 在不知道极限具体数值的情况下判断极限是否存在

例: $\lim_{n\to\infty}(a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n)$

3. **Theorem**: Cauchy $\Longleftrightarrow$ 收敛

$\{a_n\}$ convergent $\Longleftrightarrow$ $\{a_n\}$ is Cauchy

证明:

proof of "$\Rightarrow$":

$\forall \varepsilon > 0$, $\exists N$, s.t. if $n \ge N$, then $|a_n - l| < \varepsilon$, where $l = \lim_{n\to\infty} a_n$

whenever $n, m \ge N$, $|a_m - l| < \varepsilon$,

$$
\begin{aligned}
|a_n - a_m| &= |a_n - l + l - a_m| \\
&\le |a_n - l| + |a_m - l| \\
&< 2\varepsilon
\end{aligned}
$$

proof of "$\Leftarrow$":

（先证明$\{a_n\}$有界）

In def of Cauchyness, take $\varepsilon = 1$

$\exists N_1$, s.t. if $n, m \ge N_1$, $|a_n - a_m| < 1$

$\Rightarrow |a_n - a_{N_1}| < 1$

$\Rightarrow |a_n| - |a_{N_1}| < 1$

$\Rightarrow |a_n| < |a_{N_1}| + 1$, $\forall n \ge N_1$

$\therefore$ $\{a_n\}$ is bdd

（再由 Bolzano-Weierstrass theorem, 存在一子序列极限为 l）

By Bolzano-Weierstrass theorem, $\exists$ subseq $\{a_{n_k}\}_{k=1}^{\infty} \to$ some l as $k \to \infty$

$\because \{a_n\}$ is Cauchy

$\therefore \forall \varepsilon > 0, \exists N_\varepsilon$, s.t. if $n, m \geq N_\varepsilon$, $|a_n - a_m| < \varepsilon$

$\because a_{n_k} \to l$ as $k \to \infty$

$\therefore \exists K_\varepsilon$, s.t. if $k \geq K_\varepsilon$, then $|a_{n_k} - l| < \varepsilon$

(已知存在一子序列极限为l，而n足够大时，a_n与子序列中的项差值小于ε)

$\because n_k \to \infty$ as $k \to \infty$

$\therefore \exists \overline{K_\varepsilon}$ s.t. if $k \geq \overline{K_\varepsilon}$, then $n_k \geq N_\varepsilon$

Now as long as $k \geq \max(K_\varepsilon, \overline{K_\varepsilon})$, we have

$$|a_n - a_{n_k}| < \varepsilon$$

Thus,
$$\begin{aligned} |a_n - l| &= |a_n - a_{n_k} + a_{n_k} - l| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - l| \\ &< 2\varepsilon \end{aligned}$$

Q.E.D.

例 1: $\{a_n\}$ bdd, $|q| < 1$, $\lim_{n\to\infty}(a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n)$ exists?

$\forall n, m \geq 1$ (WLOG (without loss of generality), assume $n \geq m$)

$$\begin{aligned} & |(a_0 + a_1 q + \cdots + a_n q^n) - (a_0 + a_1 q + \cdots + a_m q^m)| \\ = & |a_{m+1} q^{m+1} + \cdots + a_n q^n| \\ \leq & |a_{m+1}||q|^{m+1} + \cdots + |a_n||q|^n \\ \leq & M(|q|^{m+1} + \cdots + |q|^n) \\ \leq & M|q|^{m+1} \frac{1 - |q|^{n-m}}{1 - |q|} \\ < & M \frac{|q|^{m+1}}{1 - |q|} \\ < & \varepsilon, \quad \text{whenever } m > \frac{\ln\left(\frac{\varepsilon(1-|q|)}{M}\right)}{\ln|q|} - 1 \end{aligned}$$

Q.E.D.