Lecture 20

§1 Inverse function theorem

Purpose: Given $y=f(x)$, $x\in\mathbb{R}^n$, $y\in\mathbb{R}^n$. Want to solve x in terms of y: $x=g(y)$

1. Theorem: Inverse function theorem

Let Ω be open in $\mathbb{R}^n$, $f:\Omega\to\mathbb{R}^n$ be C^1-smooth.

Suppose $\exists\, a\in\Omega$ s.t. Jacobian $(f'(a))_{n\times n}$ is non-singular. Let $b=f(a)$

Then

(i) $\exists$ open sets U & V in $\mathbb{R}^n$, s.t. $a\in U$, $b\in V$

f is one-to-one on U & $f(U)=V$

(ii) (由(i)可知 $(f|_U)^{-1}$ 在 V 上 locally 存在)

Let $g(y)=(f|_U)^{-1}(y)$, $y\in V$. Then

g is also C^1-smooth on V, and $(g'(y))_{n\times n}=(f'(x))^{-1}_{n\times n}$, x & y are related by $y=f(x)$

f 被称为 local diffeomorphism from U to V

注: 对于 function of one variable $g(x)=f^{-1}(x)$, $g'(x)=\frac{1}{f'(x)}$

2. Theorem: contraction mapping theorem (Preparations for proving Inverse function theorem)

Let A be a closed subset of $\mathbb{R}^n$.

Suppose $f: A\to A$ satisfies: $\forall\, x,y\in A$, $|f(x)-f(y)|\le\lambda|x-y|$ for some constant $\lambda\in(0,1)$

Then $\exists\,!$ (unique) $x_\infty\in A$, s.t. $f(x_\infty)=x_\infty$ (fixed point of f)

注: Broswer's fixed point theorem:

$f:[a,b]\to[a,b]$, $\exists\, x_\infty$, s.t. $f(x_\infty)=x_\infty$

证明:

Pick $x_0\in A$, let $x_1=f(x_0)$, $x_2=f(x_1)$, ------, $x_{k+1}\overset{\text{def}}{=}f(x_k)$ (*), $\forall\, k\ge 1$

W.T.S. $x_k\to$ some x_∞ as $k\to\infty$

Just need to show $\sum_{k=0}^{\infty}(x_{k+1}-x_k)$ converges

Just need to show $\sum_{k=0}^{\infty}|x_{k+1}-x_k|$ converges

$$
\begin{aligned}
|x_{k+1}-x_k| &= |f(x_k)-f(x_{k-1})| \\
&\le \lambda|x_k-x_{k-1}| \\
&= \lambda|f(x_{k-1})-f(x_{k-2})| \\
&\le \lambda^2|x_{k-1}-x_{k-2}| \\
&\cdots\cdots \\
&\le \lambda^k|x_1-x_0|
\end{aligned}
$$

By geometric series $0<\lambda<1$, $\sum_{k=0}^{\infty}|x_{k+1}-x_k|$ converges

$\because$ A is closed and $x_k\in A$

$\therefore$ $x_\infty\in A$

$\therefore\ 0 \le |f(x_k) - f(x_\infty)| \le \lambda |x_k - x_\infty| \to 0$ as $k \to \infty$

$\therefore\ f(x_k) \to f(x_\infty)$ as $k \to \infty$

By $(*)$, $\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(x_k)$

$\Rightarrow x_\infty = f(x_\infty)$

Suppose f has another fixed point y_∞ (证明 uniqueness)

$\Rightarrow \underbrace{|f(x_\infty) - f(y_\infty)|}_{\|} \le \lambda |x_\infty - y_\infty|$

$|x_\infty - y_\infty| \le \lambda |x_\infty - y_\infty|$

$\Rightarrow |x_\infty - y_\infty| = 0$

$x_\infty = y_\infty$ (contradiction)

3. Prove of Inverse function theorem

1° Discussion: solve for x from $y = f(x)$

$$y = f(x) = f(a) + f'(a)_{n\times n}(x-a) + o(|x-a|), \quad x \approx a$$

$\Rightarrow y - f(a) - o(|x-a|) = f'(a)_{n\times n}(x-a)$

$\Rightarrow A^{-1}(y - f(a) - o(|x-a|)) = x - a$ (令 $A = f'(a)_{n\times n}$)

$\Rightarrow x = a + A^{-1}(y - f(a) - o(|x-a|))$

$= a + A^{-1}[y - f(a) - f(x) + f(a) + A(x-a)]$

$= A^{-1}(y - f(x)) + x$

令 $A^{-1}(y - f(x)) + x = \phi(x)$，可考虑使用 contraction mapping theorem

2° 证明:

Take small & closed ball $\breve{U}$ centered at a.

$\breve{U} \subset \Omega$. Fix $y \in R^n$.

Claim 1: If $\breve{U}$ is small enough, then ϕ satisfies $\forall x_1, x_2 \in \breve{U}$, $|\phi(x_2) - \phi(x_1)| \le \frac{1}{2}|x_2 - x_1|$.

The smallness of $\breve{U}$ is independent of y

$|\phi(x_2) - \phi(x_1)| \overset{MVT}{=} M|x_2 - x_1|$, $M \ge \|\phi'(x)\|$, $\forall x \in \breve{U}$

$\phi'(x) = (A^{-1}(y - f(x)) + x)'$

$= -A^{-1} f'(x) + I$

$\to -A^{-1} f'(a) + I$, as $x \to a$ (由于 f C^1-smooth)

$= -A^{-1} \cdot A + I$

$= 0$

$\therefore\ \phi'(x) \to 0_{n\times n}$ as $x \to a$

$\Rightarrow$ If $\breve{U}$ small enough, then $\|\phi'(x)\| \le \frac{1}{2}$, $\forall x \in U$

Take $M = \frac{1}{2}$ $\Rightarrow$ Claim 1

Now take V to be a small nbhd of $b = f(a)$

Claim 2: $\forall$ fixed $y \in V$, ϕ maps $\breve{U} \to \breve{U}$

W.T.S. $\forall x \in U$, $|\phi(x) - a| \le$ radius of U

$$
\begin{aligned}
|\phi(x)-a| &= |A^{-1}(y-f(x)) + x - a| \\
&= |A^{-1}[y-(f(a)+A(x-a)+o(|x-a|))] + x-a| \\
&= |A^{-1}[y-f(a)] - A^{-1}o(|x-a|)| \\
&\le |A^{-1}[y-f(a)]| + |A^{-1}o(1)\cdot|x-a|| \\
&\le \|A^{-1}\|\cdot|y-f(a)| + \|A^{-1}\|\cdot o(1)\cdot|x-a| \\
&\le \|A^{-1}\|\cdot(\text{radius of } V) + \|A^{-1}\|\cdot o(1)\cdot(\text{radius of } \check{U}) \\
&\le \tfrac{1}{2}\text{ radius of } \check{U} \quad (\text{if } V \text{ is taken small enough}) \ (*)
\end{aligned}
$$

By Claim of 1 and 2, we can use contraction mapping theorem to ϕ on $\check{U}$

$\Rightarrow \exists!\, x$ (x_∞) $\in \check{U}$ s.t. $\phi(x) = A^{-1}(y-f(x)) + x = x$

$\Rightarrow A^{-1}(y-f(x)) = 0$

$\Rightarrow y = f(x)$

In summary: $\forall y \in V$, $\exists!\, x \in \check{U}$ s.t. $f(x) = y$

Claim 3: This $x \notin \partial \check{U}$

By $(*)$, $|\phi(x) - a| \le \frac{1}{2}$ radius of $\check{U}$

$\therefore$ $|x-a| \le \frac{1}{2}$ radius of $\check{U}$

$\therefore$ $x \notin \partial\check{U}$

$\because$ f is C^1-smooth

$\therefore$ f is continuous on Ω

$\therefore$ $f^{-1}(V)$ is open in R^n (由于 V open)

$\therefore$ $f^{-1}(V) \cap \mathring{\check{U}}$ is open

Let $U = f^{-1}(V) \cap \mathring{\check{U}}$

W.T.S. f 1-1 on U

$\forall x_1, x_2 \in U \subset \check{U}$. if $f(x_1) = f(x_2) =$ some y

$$
\Rightarrow \begin{aligned}
\phi(x_1) &= A^{-1}(y - f(x_1)) + x_1 \\
\phi(x_2) &= A^{-1}(y - f(x_2)) + x_2
\end{aligned}
$$

By Claim 1, $|\phi(x_2) - \phi(x_1)| = |x_2 - x_1| \le \frac{1}{2}|x_2 - x_1|$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1 on U

W.T.S. $f(U) = V$

Recall: $\forall y \in V$, $\exists!\, x \in \check{U}$ s.t. $f(x) = y$

$\Rightarrow x \in f^{-1}(V)$

$\Rightarrow x \in \check{U} \cap f^{-1}(V)$

But by Claim 3, $x \notin \partial\check{U}$

$\Rightarrow x \in \mathring{U} \cap f^{-1}(V) = U$

$\Rightarrow f(U) = V$

Let $g = (f|_U)^{-1} : V \to U$

Let $x+h = g(y+k) \in U$

Claim 4: $|h| \leq 2\|A^{-1}\||k|$

Recall from Claim 1:

$|\phi(x+h) - \phi(x)| \leq \frac{1}{2}|h|$

$\Rightarrow |A^{-1}(y - f(x+h)) + x+h - A^{-1}(y - f(x)) - x| \leq \frac{1}{2}|h|$

$\Rightarrow |h + A^{-1}(f(x) - f(x+h))| \leq \frac{1}{2}|h|$

$\Rightarrow |h + A^{-1}(y - (y+k))| \leq \frac{1}{2}|h|$

$\Rightarrow |h - A^{-1}k| \leq \frac{1}{2}|h|$

$\Rightarrow |h| - |A^{-1}k| \leq \frac{1}{2}|h|$

$\Rightarrow |h| \leq 2\|A^{-1}\||k|$

$\forall y, y+k \in V$, W.T.S. $g(y+k) = g(y) + (f'(x))^{-1}k + o(|k|)$ as $k \to 0$ where $f(x) = y$ (#)

$\because f$ is differentiable at x

$\therefore f(x+h) - f(x) = f'(x)h + o(|h|)$ as $h \to 0$

$\Rightarrow y+k-y = f'(x)h + o(|h|)$

$\Rightarrow h = (f'(x))^{-1}(k - o(|h|))$

$= (f'(x))^{-1}(k - o(|k|))$ as $k \to 0$ (由 Claim 4)

(#) $\iff x+h = x + (f'(x))^{-1}k + o(|k|)$ as $k \to 0$

$\iff h = (f'(x))^{-1}k + o(|k|)$ as $k \to 0$

$\therefore$ (#) 得证

Also by (#), $g'(y) = (f'(x))^{-1} = (f'(g(y)))^{-1}$, $y \in V$

W.T.S. g' is continuous in $y \in V$

Just need to show $(f'(x))^{-1}$ is continuous in $x \in U$

(由 f is C^1-smooth 可知, $f'(x)$ continuous in $x \in \Omega$)

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $\quad A^{-1} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{11} & \cdots & b_{nn} \end{bmatrix}$

$b_{ij} = \dfrac{(-1)^{i+j} A_{ji}}{\det A}$ → (j,i) minor of A $= \dfrac{(-1)^{i+j} \cdot \text{sum of products of elements of } A}{\text{sum of products of elements of } A}$

$\therefore (f'(x))^{-1}$ is continuous in $x \in U$