Lecture 20

§1 More on L'Hôpital's Rule

1. Apply with care!

1° Make sure to stop at the right step ("$\frac{\infty}{\infty}$" or "$\frac{0}{0}$").

2° L'Hôpital's rule has its limition.

e.g. $\lim\limits_{x\to\infty} \frac{x^2+\sin x}{x^2}$ and $\lim\limits_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}}$

2. The proof for special case

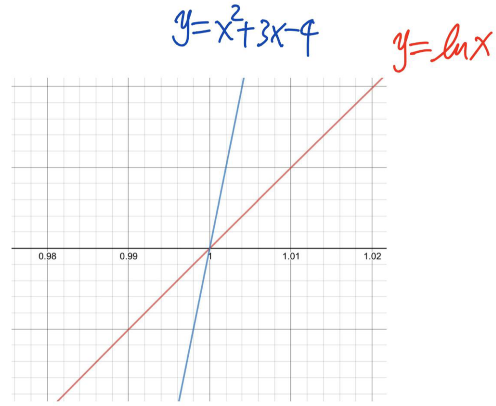
f and g are differentiable on $(c-a, c+a)$ for some $a>0$, $g'(x)\neq 0$ $\forall x\in(c-a, c+a)\setminus\{c\}$; and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$

Intuition (not proof):

Consider $\lim\limits_{x\to 1} \frac{x^2+3x-4}{\ln x}$

$$= \lim_{x\to 1} \frac{2x+3}{1/x}$$

$$= 5$$



For x near c:

$$\frac{f(x)}{g(x)} \approx \frac{\Delta f}{\Delta g} = \frac{\Delta f/\Delta x}{\Delta g/\Delta x} \approx \frac{f'(x)}{g'(x)}$$

§2 Cauchy's Mean Value Theorem.

1. Theorem 7.5.6 - Cauchy's Mean Value Theorem (柯西中值定理)

Suppose functions f and g are continuous on $[a,b]$ and differentiable throughout (a,b) and also suppose $g'(x)\neq 0$ throughout (a,b). Then there exists a number c in (a,b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

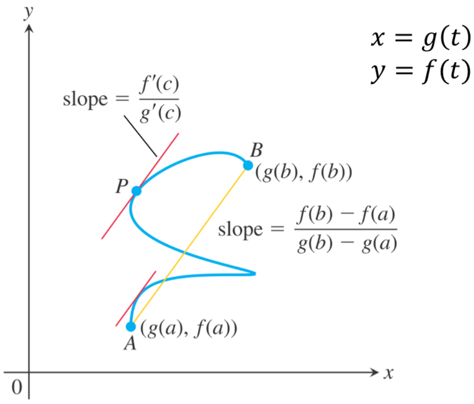


FIGURE 7.20 There is at least one point P on the curve C for which the slope of the tangent to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

$*$ slope $= \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{f'(t)}{g'(t)}$

2. Proof of Cauchy's Mean Value Theorem

1° Note that $g(b) \neq g(a)$, since by MVT

$$g(b) = g(a) + g'(c_0)(b-a) \text{ for some } c_0 \in (a,b)$$

and we assume $g'(c_0) \neq 0$

2° Define h by

$$h(t) = f(a) + \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(t)-g(a)) - f(t)$$

and note that $h(a) = 0 = h(b)$.

Since h is continuous on $[a,b]$ and differentiable on (a,b), by Rolle's theorem there exists $c \in (a,b)$ such that $h'(c) = 0$.

3° But

$$h'(t) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(t) - f'(t)$$

so $h'(c) = 0$ means

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

§3 Proof of L'Hôpital's Rule

We will prove it for the special case where:
f and g are differentiable on $(c-a, c+a)$ for some $a>0$, $g'(x)\neq 0$ $\forall x\in(c-a,c+a)\setminus\{c\}$; and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

For $\lim_{x\to c^+}$:

Pick any $k\in(c, c+a)$ and consider Cauchy's MVT on $[c, x]$.
There exists $x_0\in(c,x)$ such that

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x)-f(c)}{g(x)-g(c)}$$

Since f and g are differentiable at c, they are continuous at c, so

$$f(c) = \lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x) = g(c)$$

Hence

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f(x)}{g(x)}$$

As $x\to c^+$, $x_0\to c^+$ as well, so

$$\lim_{x\to c^+}\frac{f(x)}{g(x)} = \lim_{x\to c^+}\frac{f'(x_0)}{g'(x_0)}$$

$$= \lim_{x_0\to c^+}\frac{f'(x_0)}{g'(x_0)}$$

$$= \lim_{x\to c^+}\frac{f'(x)}{g'(x)}$$

Proof for $\lim_{x\to c^-}$ is similar.

Hence the rule holds for $\lim_{x\to c}$

§4 Relative Rates of Growth

1. An example

Suppose we have n numbers $x_1, x_2, \dots, x_n$ listed in increasing order.

We want to find a target number T, and we know $T = x_k$ for some $k \in \{1,2,\dots,n\}$.

Method 1: Linear search / sequential search

1. Set $i{:=}1$.
2. If $x_i = T$, return i and stop.
3. Else, increase i by 1 and go back to step 2.

In the worst case scenario, where $x_n = T$, n loops of this algorithm are required,

with each loop containing two steps (setting i and checking x_i).

Method 2: Binary search

1. Let $L \coloneqq 1$ and $R \coloneqq n$.
2. Let $M \coloneqq \lfloor (L+R)/2 \rfloor$.
3. If $x_M = T$, return M and stop.
4. Else, if $x_M < T$, set $L \coloneqq M+1$ and go back to step 2.
5. Else, set $R \coloneqq M-1$ and go back to step 2.

E.g. Use binary search to find 48 from the list below:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
2	3	5	7	11	16	42	48	57	61	71	97

Here, 4 loops are required, and each loop contains a constant number of steps.

Note that $4 = \lceil \log_2 12 \rceil$, where $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is **ceiling function**.

(E.g. $\lceil 34 \rceil = 34$, $\lceil 34.00001 \rceil = 35$, $\lceil 34.999 \rceil = 35$

In general, for a list of n numbers, the binary search requires at most $\lceil \log_2 n \rceil$ times a constant number of steps to find T.

In the theory of computational complexity:

- Linear search is said to have class $O(n)$
- Binary search is said to have class $O(\log_2 n)$

(Notation will be formally defined later).

2. Definition

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1° f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$$

We also say that g grows slower than f as $x\to\infty$.

2° f and g grow at the same rate as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$$

where L is infinite and positive

3. Some examples

1° kf vs f

If k is a positive constant, then

$$\lim_{x\to\infty}\frac{kf(x)}{f(x)} = k$$

so kf and f always grows at the same rate.

2° x^x vs b^x

For any fixed $b\in(0,\infty)$

$$\lim_{x\to\infty}\frac{x^x}{b^x} = \lim_{x\to\infty}\left(\frac{x}{b}\right)^x$$

$$= \lim_{x\to\infty} e^{x\ln\frac{x}{b}}$$

$$= \infty$$

so x^x grows faster than b^x

3° b^x vs a^x

If $b>a>0$, then

$$\lim_{x\to\infty}\frac{b^x}{a^x} = \lim_{x\to\infty}\left(\frac{b}{a}\right)^x$$

$$= \lim_{x\to\infty} e^{x\ln\frac{b}{a}}$$

$$= \infty$$

so b^x grows faster than a^x for $b>a>0$

4° a^x vs x^n

If $a>1$ and $n\in\mathbb{Z}_+$ then

$$\lim_{x\to\infty}\frac{a^x}{x^n}=\lim_{x\to\infty}\frac{a^x\ln a}{nx^{n-1}}$$
$$=\lim_{x\to\infty}\frac{a^x(\ln a)^2}{n(n-1)x^{n-2}}$$
$$=\cdots\cdots$$
$$=\lim_{x\to\infty}\frac{a^x(\ln a)^n}{n!}$$
$$=\infty$$

so a^x grows faster that x^n for $a>1$ and $n\in\mathbb{Z}_+$

5° x^n vs $\ln x$

For $n\in\mathbb{Z}_+$:

$$\lim_{x\to\infty}\frac{x^n}{\ln x}=\lim_{x\to\infty}\frac{nx^{n-1}}{1/x}$$
$$=\lim_{x\to\infty}nx^n$$
$$=\infty$$

so x^n grows faster than $\ln x$ for $n\in\mathbb{Z}_+$

6° $\log_a$ VS $\log_b$

For $a>1$ and $b>1$:

$$\lim_{x\to\infty}\frac{\log_a x}{\log_b x}=\lim_{x\to\infty}\frac{\ln x/\ln a}{\ln x/\ln b}$$
$$=\frac{\ln b}{\ln a}>0$$

so Log functions with base >1 all grow at the same rate.

4. Transitive property

Growing at the same rate is a transitive relation

If f and g grow at the same rate,

and g and h grow at the same rate,

then so do f and h

Proof:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L_1>0 \text{ and } \lim_{x\to\infty}\frac{g(x)}{h(x)}=L_2>0$$

$$\lim_{x\to\infty}\frac{f(x)}{h(x)}=\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\cdot\frac{g(x)}{h(x)}\right)=L_1L_2>0$$

So f and h grow at the same rate.

e.g. Consider $\sqrt{x^2+2021}$ and $(98\sqrt{x}-1)^2$

$$\lim_{x\to\infty}\frac{\sqrt{x^2+2021}}{x}=\lim_{x\to\infty}\sqrt{\frac{x^2+2021}{x^2}}=1$$

$$\lim_{x\to\infty}\frac{(98\sqrt{x}-1)^2}{x}=\lim_{x\to\infty}\left(\frac{98\sqrt{x}-1}{\sqrt{x}}\right)^2=98^2$$

So $\sqrt{x^2+2021}$ and $(98\sqrt{x}-1)^2$ grow at the same rate.

5. Definitions of little-oh and big-oh notation

1° A function f is of smaller order than g as $x\to\infty$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$.
We indicate this by writing $f=o(g)$ ("f is little-oh of g")

2° Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is of at most the order of g as $x\to\infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)}\le M$$

for x sufficiently large. We indicate this by writing $f=O(g)$ ("f is big-oh of g")

e.g. Show that $\log_2 x=O(x)$

$$\lim_{x\to\infty}\frac{\log_2 x}{x}=\lim_{x\to\infty}\frac{1}{x\ln 2}=0$$

Hence $\log_2 x=O(x)$