Lecture 24

§1 Green's Theorem (For simple closed curve)

1. 边界曲线的方向

设区域D的边界为L，L是由一条或多条简单闭曲线所组成，我们说边界曲线是 positively oriented（正向的），当沿这个方向前进时，区域总落在左侧. 规定了正向的边界曲线记作 L^+.

2. Green's theorem（格林公式）(Circulation version)

Theorem (Green's Theorem) (Circulation Version)

Let C be a ~~positively oriented,~~ counterclockwise piecewise-smooth, simple closed curve in the xy-plane, and let R be the region bounded by C. Assume that M ad N have continuous partial derivatives on an open region containing R. Then

"circulation density"

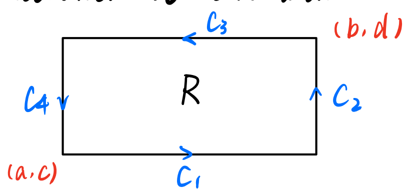
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

C is the boundary of R.

A result about counterclockwise circulation.

证明：

给出某 fluid 的 velocity field $\vec{F}$，考虑沿着矩形区域 $R := [a,b]\times[c,d]$ 的边界的 counterclockwise circulation.



- $\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + \oint_C N\,dy$

- $$\begin{aligned}\oint_C M\,dx &= \int_{C_1} M\,dx + \int_{C_2} M\,dx + \int_{C_3} M\,dx + \int_{C_4} M\,dx \\ &= \int_{C_1} M\,dx - \int_{-C_3} M\,dx \\ &= \int_a^b M(t,c) - M(t,d)\,dt \\ &= \int_a^b \int_d^c \frac{\partial M}{\partial y}(t,y)\,dy\,dt \\ &= \int_a^b \int_d^c \frac{\partial M}{\partial y}(x,y)\,dy\,dx \\ &= -\int_a^b \int_c^d \frac{\partial M}{\partial y}(x,y)\,dy\,dx \\ &= -\iint_R \frac{\partial M}{\partial y}(x,y)\,dA \quad ①\end{aligned}$$

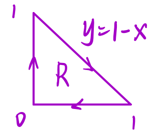
- $$\begin{aligned}\oint_C N\,dy &= \int_{C_2} N\,dy - \int_{-C_4} N\,dy \\ &= \int_c^d N(b,t)\,dt - \int_c^d N(a,t)\,dt \\ &= \int_c^d \int_a^b \frac{\partial}{\partial x} N(x,t)\,dx\,dt \\ &= \int_c^d \int_a^b \frac{\partial}{\partial x} N(x,y)\,dx\,dy \\ &= \iint_R \frac{\partial}{\partial x} N(x,y)\,dA \quad ②\end{aligned}$$

- Combine ① & ②, we have:

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \quad ③$$

· 同样的方法可以证明 ③式对 type-I 和 type-II 区域同样适用.
通过将区域分割为 type-I 和 type-II 区域，可以证明 ③式对任一 bounded region R that has a simple closed boundary 均适用.

例: Find the clockwise circulation of $\vec{F} := \langle x^4, xy \rangle$ along the triange with vertices $(0,0)$, $(1,0)$, $(0,1)$



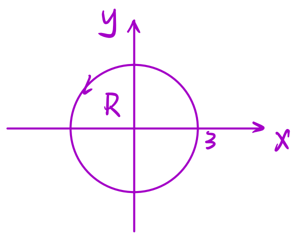
$$\oint \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \int_0^1 \int_0^{1-x} y - 0 \, dy\, dx$$
$$= \int_0^1 \frac{1}{2}(1-x)^2\, dx$$
$$= \frac{1}{6}$$
$$\oint \vec{F}\cdot d\vec{r} = -\frac{1}{6}$$

例: Example

Evaluate

$$\oint_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy,$$

where C is the circle $x^2+y^2=9$, traversed ~~clockwise~~. counterclockwise



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$
$$= 4\iint_R dA$$
$$= 4\cdot \text{Area}(R)$$
$$= 36\pi$$

例: Exercise

Evaluate

$$\oint_C y^2\, dx + 3xy\, dy,$$

where C is the positively oriented boundary of the region R given by $1 \le x^2+y^2 \le 4$, $y \ge 0$.

= Counterclockwise

$$\oint_C y^2 dx + 3xy\, dy = \iint_D 3y - 2y\, dA$$
$$= \iint_D y\, dA$$
$$= \int_0^{\pi}\int_1^2 r^2 \sin\theta\, dr\, d\theta$$
$$= 2 \times \frac{1}{3}(2^3 - 1^3)$$
$$= \frac{14}{3}$$

例: *Question*

- Consider $\mathbf{F}(x,y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.
- Let C be the unit circle $x^2+y^2=1$.
- We shown that $\oint_C \vec{F}\cdot\vec{T}\,ds = 2\pi$ by direct computation.
- On the other hand, one can check that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ always holds.
- By Green's theorem, $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0$.
- Therefore, $2\pi = 0$. WELL DONE

O.K., seriously, what is wrong?

$M(x,y)$, $N(x,y)$ 在原点处无定义，不满足格林公式的成立条件.

3. Green's Theorem (Flux version)

> <u>Theorem</u> (Green's Theorem, Flux Version)
>
> Under the assumptions for Green's theorem (circulation version), except orientation (direction) of C is no longer required,
>
> "flux density"
>
> $$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

4. Circulation density 与 flux density (二维)

1° circulation density (环量面密度 / 方向旋量)

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\text{circulation along a small simple closed curve around a point}}{\text{area of bounded region}}$$

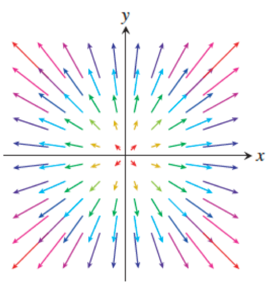
2° outward flux density / divergence (散度)

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\text{outward flux across a small simple closed curve around a point}}{\text{area of bounded region}}$$

例: (a) $\vec{F} = \langle cx, cy \rangle$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$: no rotation at very small scale.

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2c \begin{cases} >0, & \text{if } c>0 \ \text{(expands / diverges)} \\ <0, & \text{if } c<0 \ \text{(compress)} \end{cases}$$

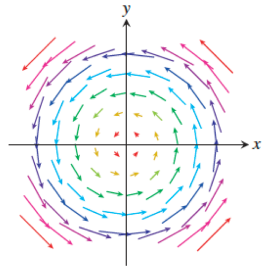


(a)

(b) $\vec{F} = \langle -cy, cx \rangle$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2c \begin{cases} > 0, & \text{if } c > 0 \ (\circlearrowleft) \\ < 0, & \text{if } c < 0 \ (\circlearrowright) \end{cases}$$

$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$: no expansion

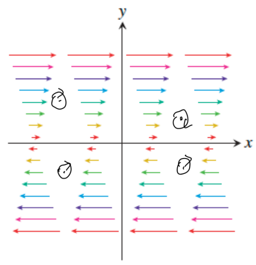


(b)

(c) $\vec{F} = \langle cy, 0 \rangle$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -c \begin{cases} < 0, & \text{if } c > 0 \ (\circlearrowright) \\ > 0, & \text{if } c < 0 \ (\circlearrowleft) \end{cases}$$

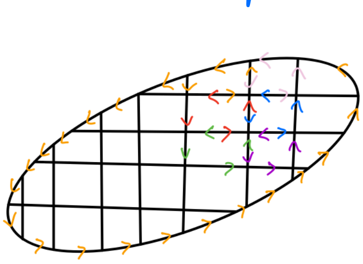
$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$: no expansion



(c)

§2 Green's Theorem (General version)

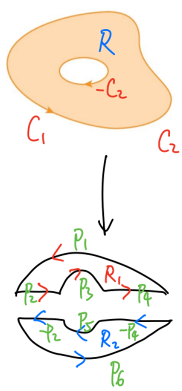
1. Idea (microscopic circulation)



- Sum up all circulations around small regions $\left(\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\right)$
- Adjacent "side" cancelled
- Leaving only the circulation along boundary of the whole region R.

(Idea for flux version is the same.)

2. More general version



Consider the region R on the left.

- Boundary of R is not a simple closed curve — it is the union of two such curves.
- Previous version of Green's theorem cannot be used directly.
- $\iint_R (N_x - M_y)\, dA = \iint_{R_1} + \iint_{R_2}$

$$\overset{\text{Green's}}{=\!=} \left(\int_{P_1} \vec{F} \cdot d\vec{r} + \int_{P_2} + \int_{P_3} + \int_{P_4}\right) + \left(\int_{-P_4} + \int_{P_5} + \int_{-P_2} + \int_{P_6}\right)$$

$$= \left(\int_{P_1} + \int_{P_6}\right) + \left(\int_{P_3} + \int_{P_5}\right) = \oint_{C_1} + \oint_{C_2} = \int_{\substack{\text{Boundary} \\ \text{of } R}} \vec{F} \cdot d\vec{r}$$

3. Green's Theorem (General version)

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{\substack{\text{positively oriented} \\ \text{boundary } C \text{ of } R}} \vec{F} \cdot d\vec{r}$$

C is the union of one or more closed curves