Lecture 1 Infinite Sequences (10.1)

无穷序列

§1 定义

An infinite sequence, or simply sequence, is a list $\{a_1, a_2, a_3, \cdots\cdots, a_n, \cdots\cdots\}$

注意: 1° 一个序列 (sequence) 可被看作一个定义域为一组正整数 $\{1,2,3,\cdots\cdots\}$ 的函数.

i.e. $a_1 = f(1)$, $a_2 = f(2)$, $\cdots\cdots$

2° 定义域 $\{1,2,3,\cdots\cdots\}$ 也被称为序列 $\{a_1, a_2, a_3, \cdots\cdots\}$ 的 index set.

3° index set 无需从 1 开始.

e.g. $\{a_0, a_1, a_2 \cdots\cdots\}$ 和 $\{b_3, b_4, b_5 \cdots\cdots\}$ 也是序列.

4° 我们用 $\{a_n\}_{n=1}^{\infty}$ 表示 $\{a_1, a_2, a_3, \cdots\cdots\}$, 也可简写为 $\{a_n\}$

e.g. $\{(-1)^{n+1} n^2\}_{n=1}^{\infty} = \{1, -4, 9, -16, 25, -36 \cdots\cdots\}$

5° 一个序列可以 be defined recursively

e.g. the rule $F_0 := 0$, $F_1 := 1$, $F_i := F_{i-1} + F_{i-2}$ (for $i \geq 2$)

define $\{F_n\}_{n=1}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13 \cdots\cdots\}$

(called the Fibonacci sequence (斐波那契数列))

§2 序列极限和敛散性 limits and convergence

1. 序列极限与收敛序列的定义

Let $\{a_n\}$ be a sequence.

- If $L \in \mathbb{R}$ satisfies the condition

  $\forall \varepsilon > 0, \exists N := N_\varepsilon \in \mathbb{Z}$ such that $\forall n > N$, $|a_n - L| < \varepsilon$,

  then we say that $\lim_{n\to\infty} a_n = L$, and call L the limit of $\{a_n\}$.
- A sequence having a limit $L \in \mathbb{R}$ is said to be convergent.
- A sequence is said to be divergent if it is not convergent.

注意: 1° N 的选择通常取决于 ε (ε 越小, N 越大)

2° "$\forall n > N$" 可被替换为 "$\forall n \geq N$"

例: Use definition, show that $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0

Sol. Let $\varepsilon > 0$ be arbitrary.

If we pick N to satisfy $N > \frac{1}{\varepsilon}$ (由 $|\frac{1}{n} - 0| < \varepsilon$ 得出),

then for all $n > \lfloor \frac{1}{\varepsilon} \rfloor + 1$, we have

$|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon$

so $\lim_{n\to\infty} \frac{1}{n} = 0$ i.e. $\{\frac{1}{n}\}$ converges to 0

例: Show that $\{a_n\}_{n=1}^{\infty}$ converges using definition, where $a_n := \frac{n}{2n+1}$

Proof 1: It suffices to show that $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

- Let $\varepsilon > 0$ be fixed. Let $N := \max\{1, \lceil \frac{\frac{1}{\varepsilon}-2}{4} \rceil\}$, where $\lceil x \rceil$ denotes the smallest integer m such that $m \geq x$

  * $\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon$

  $\Rightarrow \left|\frac{-1}{2(2n+1)}\right| < \varepsilon$

  $\Rightarrow n > \frac{\frac{1}{\varepsilon}-2}{4}$

  * 为防止对于较大的 ε，出现 $N<0$ 的情况，N 取 $\max\{1, \lceil \frac{\frac{1}{\varepsilon}-2}{4} \rceil\}$.

- For all n if $n > N$, then a_n is defined, and

  $n > \lceil \frac{\frac{1}{\varepsilon}-2}{4} \rceil \geq \frac{\frac{1}{\varepsilon}-2}{4}$

  $\Rightarrow 4n > \frac{1}{\varepsilon} - 2 \Rightarrow 4n+2 > \frac{1}{\varepsilon}$

  $\Rightarrow \frac{1}{2(2n+1)} < \varepsilon \Rightarrow \left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon$

- By definition, $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

Proof 2: ......

$\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon$

$\Rightarrow \frac{1}{2\cdot(2n+1)} < \varepsilon$

$\Rightarrow \cancel{\frac{1}{4n} < \varepsilon}$

$\Rightarrow N > \frac{1}{4\varepsilon}$

......

例: Show that $\{(-1)^n\}_{n=0}^{\infty}$ diverges.

Proof: Suppose not.

Then $\lim_{n\to\infty} (-1)^n = L$ for some $L \in \mathbb{R}$

- Let $a_n := (-1)^n$
- By definition, for $\varepsilon = 1$, $\exists N$ s.t. $\forall n > N$, $|a_n - L| < 1$

  For even n, $|1 - L| < 1$, i.e., $0 < L < 2$

  For odd n, $|-1-L| < 1$, i.e., $-2 < L < 0$
- Hence $L \in (-2,0) \cap (0,2) = \varnothing$, contradiction.

注意: 1° 对于无穷序列，增加/减少有限多项不改变敛散性

e.g. $\{7, 48, 56, -\pi, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots\}$ converge

2. Divergence to infinity 的定义

Let $\{a_n\}$ be a sequence of real numbers.

- We write $\lim_{n\to\infty} a_n = \infty$ or simply $a_n \to \infty$ (as $n\to\infty$), if for every $M \in \mathbb{R}$, there exists N such that $a_n > M$ for all n satisfying $n > N$.
- We write $\lim_{n\to\infty} a_n = -\infty$ or simply $a_n \to -\infty$ (as $n\to\infty$), if for every $M \in \mathbb{R}$, there exists

N such that $a_n < M$ for all n satisfying $n > N$.

1° If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ is said to diverge to ∞

If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ is said to diverge to $-\infty$

2° $\{(-1)^n n\}$ diverges, but not to $\pm\infty$

§3 序列极限的性质 properties of sequence limits

1、性质 1：序列极限的运算

THEOREM ~~1~~ 10.1.1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

2、性质 2：序列的复合

THEOREM ~~3~~ 10.1.3—The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Note that $\{f(a_n)\}$ is a sequence, and the theorem is saying that the sequence limit of $\{f(a_n)\}$ is $f(L)$.

证明 (optional)：

Let $\varepsilon > 0$ be arbitrary.

- Since f is continuous at L,
  $\exists\, \delta > 0$ such that
  if $|x - L| < \delta$, then $|f(x) - f(L)| < \varepsilon$.
- Since $a_n \to L$,
  $\exists\, N$ such that
  for all $n > N$, $|a_n - L| < \delta$.
- Combining ① and ②, we have
  $\forall\, n > N$, $|f(a_n) - f(L)| < \varepsilon$
- By definition, $f(a_n) \to f(L)$ as $n \to \infty$

例: Find $\lim_{n\to\infty} 2^{\frac{1}{n}}$

Sol: The function $f(x) = \frac{1}{x}$ is continuous

Since $\frac{1}{n} \to 0$ as $n \to \infty$,

$2^{\frac{1}{n}} = f(\frac{1}{n}) \to f(0) = 2^0 = 1$ as $n \to \infty$

$\therefore \lim_{n\to 0} 2^{\frac{1}{n}} = 1$

3、性质3：夹逼定理

THEOREM ~~2~~ 10.1.2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

例：Proof $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Sol: Since $0 < \frac{n!}{n^n} = \frac{n\cdot(n-1)\cdots\cdots 2\cdot 1}{n\cdot n\cdots\cdots n\cdot n} = \frac{n}{n}\cdot\frac{n-1}{n}\cdots\cdots\frac{2}{n}\cdot\frac{1}{n} < \frac{1}{n}$, $\forall n \ge 1$

and $\frac{1}{n} \to 0$ as $n \to \infty$,

we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

例：Proof that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Since $-|a_n| \le a_n \le |a_n|$, $\forall n$

and $\lim_{n\to\infty}(-|a_n|) = -0 = 0$

$\lim_{n\to\infty}(|a_n|) = 0$

$\lim_{n\to\infty} a_n = 0$

$*$ $\lim_{n\to\infty} |a_n| = L \neq 0 \not\Rightarrow \lim_{n\to\infty} a_n = L$

4、性质4：连接序列与函数

THEOREM ~~4~~ 10.1.4 Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$

Seq. limit　　function limit.

证明：Suppose that $\lim_{x\to\infty} f(x) = L$.

Let $\varepsilon > 0$. Then $\exists M \in R$, $\forall x \in R$, if $x > M$ then $|f(x) - L| < \varepsilon$

For all integers n with $n > N := \max\{n_0, \lceil M \rceil\}$,

$|a_n - L| = |f(n) - L| < \varepsilon$.

Hence $\lim_{n\to\infty} a_n = L$.

例：Find $\lim_{n\to\infty} \frac{\ln n}{n}$

Sol. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

§5 常见数列极限 Some Common Limits

THEOREM ~~5~~ 10.1.5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

证明: 2. $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$

6. Since $\frac{-|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$ for all x and $n>0$,

it suffice to show that $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$

choose N s.t. $N \ge |x|$. Then for all $n > N$

$$0 \le \frac{|x|^n}{n!} = \underbrace{\frac{|x|^N}{N!}}_{\le |x|^N} \cdot \underbrace{\frac{|x|}{N+1} \cdot \frac{|x|}{N+2} \cdots\cdots}_{<1} \frac{|x|}{n} \le |x|^N \cdot \frac{|x|}{n} \to 0 \text{ as } n\to\infty.$$

Hence $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ by Sandwich Theorem.

§6 递推序列 recursively defined sequence

例: Given the sequence $\{2, 2+\frac{1}{2}, 2+\frac{1}{2+\frac{1}{2}} \cdots\cdots\}$ converges, find its limit.

Sol: The sequence is given by $a_1 = 2$, $a_{n+1} = 2 + \frac{1}{a_n}$ for $n \ge 1$.

- Let $L := \lim_{n\to\infty} a_n$. Then $L = \lim_{n\to\infty} a_{n+1}$
- $a_{n+1} = 2 + \frac{1}{a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = 2 + \frac{1}{\lim_{n\to\infty} a_n} \Rightarrow L = 2 + \frac{1}{L} \Rightarrow L = 1 \pm \sqrt{2}$
- Since $a_n > 0$ for all n, $L \ge 0$.

So $L = 1 + \sqrt{2}$

§7 有界序列

1. 定义

DEFINITIONS A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \ge m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

注意: 1° $\{a_n\}$有界的充要条件是 $\exists\, x\in\mathbb{R}, K\in\mathbb{R}$, 使 $|a_n - x| \le K$ 对任意n恒成立.

2. 性质

If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.

证明：Let $L := \lim_{n\to\infty} a_n$.

Then $\exists N$ s.t. $|a_n - L| < 1$ for all $n > N$

Let $K := \max\{|a_1 - L|, |a_2 - L|, \cdots\cdots, |a_N - L|, 1\}$

Then $|a_n - L| \le K$ for all $n \ge 1$. Hence $\{a_n\}$ is bounded.