Lecture 13

在这一个 chapter 中，我们将学习 ReLU ANN 对于某类 one-dimensional functions 的 approximation results

In learning problems ANNs are heavily used with the aim to approximate certain target functions. In this chapter we review basic ReLU ANN approximation results for a class of one-dimensional target functions (see Section 3.3). ANN approximation results for multi-dimensional target functions are treated in Chapter 4 below.

In the scientific literature the capacity of ANNs to approximate certain classes of target functions has been thoroughly studied; cf., for instance, [9, 22, 44, 106, 107] for early universal ANN approximation results, cf., for example, [15, 23, 88, 165, 194, 228] and the references therein for more recent ANN approximation results establishing rates in the approximation of different classes of target functions, and cf., for instance, [61, 90, 134, 189] and the references therein for approximation capacities of ANNs related to solutions of PDEs (cf. also ???? in ?? of these lecture notes for machine learning methods for PDEs). This chapter is based on Ackermann et al. [3, Section 4.2] (cf., for example, also Hutzenthaler et al. [110, Section 3.4]).

§1 Modulus of continuity

1. **Definition:** Modulus of continuity (3.1.1)

令 ① 函数的 domain 组成的 set : $A \subseteq \mathbb{R}$

② 函数 : $f: A \to \mathbb{R}$

则称函数 $w_f: [0,\infty) \to [0,\infty)$ 为 the modulus of continuity of f .

若对 $\forall$ (interval length) $h \in [0,\infty)$, 有

$$w_f(h) = \sup\Big(\big\{ |f(x)-f(y)| : \underbrace{(x,y \in A \text{ with } |x-y| \le h)}_{\text{限制 } x \text{ 和 } y \text{ 均在 domain 内且距离不超过 } h} \big\} \cup \{0\}\Big)$$

$$= \sup\Big(\big\{ r \in \mathbb{R} : \underbrace{(\exists\, x \in A,\ y \in A \cap [x-h, x+h] :}_{\text{限制 } x \text{ 和 } y \text{ 均在 domain 内且距离不超过 } h}\ r = |f(x)-f(y)|)\big\} \cup \{0\}\Big)$$

注: ① $w_f(h)$ 衡量了在长度不超过 h 的区间上的函数值差值的 maximum .

② $\cup \{0\}$ 是为了防止前面的集合为空集

eg. ① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$, $\forall x \in \mathbb{R}$ $\Rightarrow$ $w_f(h) = h$, $\forall h \in [0,\infty)$

② $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, $\forall x \in \mathbb{R}$ $\Rightarrow$ $w_g(1) = \infty$

③ $m: [-10,10] \to \mathbb{R}$, $m(x) = |x|$, $\forall x \in [-10,10]$ $\Rightarrow$ $w_m(h) = \min\{h, 10\}$, $\forall h \in [0,\infty)$

④ $n: [0,10] \to \mathbb{R}$, $n(x) = x^2$, $\forall x \in [0,10]$ $\Rightarrow$ $w_n(h) = \begin{cases} 10^2 - (10-h)^2 = 20h - h^2 , & h \in [0,10] \\ 100 , & h \in (10,\infty) \end{cases}$

2. **Property:** Modulus of continuity 的 elementary properties (3.1.2)

令 ① 函数的 domain 组成的 set : $A \subseteq \mathbb{R}$

② 函数 : $f: A \to \mathbb{R}$

则 ① w_f 为 non-decreasing

② f 为 uniformly continuous $\iff$ $\lim_{h \searrow 0} w_f(h) = 0$

③ f 为 globally bounded $\iff$ $w_f(\infty) < \infty$

④ $|f(x) - f(y)| \le w_f(|x-y|)$

⑤ $w_f(0) = 0$

证明: ②

$\lim\limits_{h\to 0} \omega_f(h) = 0$

$\iff \forall \varepsilon > 0, \exists \delta > 0$, s.t. $\omega_f(h) \le \varepsilon$ as long as $0 \le h \le \delta$

$\iff \forall \varepsilon > 0, \exists \delta > 0$, s.t. $\omega_f(\delta) \le \varepsilon$

$\iff \forall \varepsilon > 0, \exists \delta > 0$, s.t. $\underline{\forall x, y \in A}$, $|f(x) - f(y)| \le \varepsilon$ as long as $|x - y| \le \delta$

在 s.t. 后，故为 globally cont.

证明: ③

$\omega_f(\infty) < \infty$

$\iff \sup(\{|f(x) - f(y)| : (x, y \in A \text{ with } |x-y| \le \infty)\} \cup \{0\})$

$\iff \bigcup_{x,y\in A}\{|f(x) - f(y)|\}$ is bounded

$\iff \bigcup_{x\in A}\{f(x)\}$ is bounded

$\iff f$ is bounded

3. Property: Modulus of continuity 的 subadditivity (3.1.3)

令 ① 函数的 domain 下界：$a \in [-\infty, \infty]$

② 函数的 domain 上界：$b \in [a, \infty]$

③ 函数：$f : ([a, b] \cap \mathbb{R}) \to \mathbb{R}$

④ 两 interval length：$h, \eta \in [0, \infty]$

定义域为 compact set

则 ① $\omega_f(h + \eta) \le \omega_f(h) + \omega_f(\eta)$

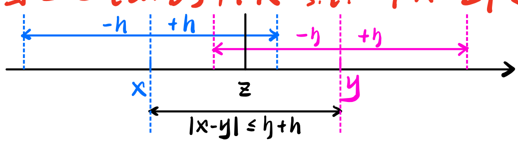
证明:

WLOG，令 $0 \le \eta \le h \le \infty$

注意到，$\forall x, y \in [a, b] \cap \mathbb{R}$ with $|x + y| \le h + \eta$，有

$[x - h, x + h] \cap [y - \eta, y + \eta] \cap [a, b] \ne \emptyset$

$\iff \exists z \in [a, b] \cap \mathbb{R}$ s.t. $|x - z| \le h$ and $|y - z| \le \eta$



因此，$\forall x, y \in [a, b] \cap \mathbb{R}$ with $|x + y| \le h + \eta$，有

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f(z)| + |f(y) - f(z)| && \text{(三角不等式)} \\ &\le \omega_f(|x - z|) + \omega_f(|y - z|) && \text{(3.1.2 / ④)} \\ &\le \omega_f(h) + \omega_f(\eta) && \text{(3.1.2 / ①)} \end{aligned}$$

$\Rightarrow \omega_f(h + \eta) \le \omega_f(h) + \omega_f(\eta)$ (3.1.1 定义)

4. Property: Lipschitz continuous functions 的 moduli of continuity 的性质 (3.1.4)

令 ① 函数的 domain 组成的 set：$A \subseteq \mathbb{R}$

② Lipschitz 系数：$L \in [0, \infty)$

③ 函数：$f : A \to \mathbb{R}$，满足 $|f(x) - f(y)| \le L|x - y|$，$\forall x, y \in A$ (Lipschitz-L 连续)

④ interval length: $h \in [0, \infty)$

则 ① $\omega_f(h) \le Lh$

证明:

$$\begin{aligned}
\omega_f(h) &= \sup(\{|f(x)-f(y)| : (x, y \in A \text{ with } |x-y| \le h)\} \cup \{0\}) \quad \text{(3.1.1 定义)} \\
&\le \sup(\{L|x-y| : (x, y \in A \text{ with } |x-y| \le h)\} \cup \{0\}) \quad \text{(Lipschitz-L 连续定义)} \\
&\le \sup(\{Lh, 0\}) \\
&= Lh
\end{aligned}$$