Lecture 18

§1 Oscillation

1. Oscillation (振动)

1° 物体往复运动

2° 具有 periodic manner (周期性)

2. 物理量

1° Frequency f (频率)

单位时间内完成 full oscillation (cycle) (全振动) 的次数

SI-unit: Hz

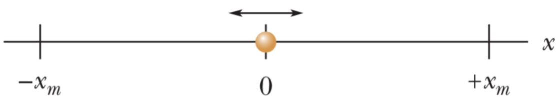
$1 \text{ hertz} = 1\text{ Hz} = 1 \text{ oscillation per second} = 1\ s^{-1}$

2° Period T (周期)

完成一次 full cycle 的时间，仅与振子质量和 k 有关

SI-unit: second

$T = \frac{1}{f}$



§2 Simple harmonic motion (SHM)

1. Simple harmonic motion (SHM) (简谐运动)

运动是一个 sinusoidal (正弦的) function of time t

(可以被写作 sine 或 cosine function)

2. Displacement (or position) function

$x(t) = x_m \cos(\omega t + \phi)$

$x(t) = x_m \sin(\omega t + \phi)$

3. 基本物理量

$$\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos\overbrace{(\underbrace{\omega}_{\text{Angular frequency}} \underbrace{t}_{\text{Time}} + \underbrace{\phi}_{\text{Phase constant or phase angle}})}^{\text{Phase}}$$

1° Angular frequency ω (角频率)

单位: rad/s

$\omega = \frac{2\pi}{T} = 2\pi f$

2° Amplitude x_m (振幅)

离开 equilibrium point (平衡点) 的最大位移

3° Phase $\omega t+\phi$ (相位)

描述周期性运动某时刻的状态

反映运动的步调

单位: rad 或 °

4° Phase constant ϕ (相位常量/初相)

单位: rad

* x_m 与 ϕ 取决为质点在 $t=0$ 时的位移和速度 (initial conditions)

Examples

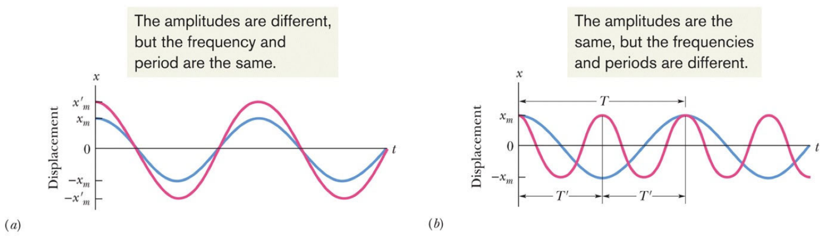
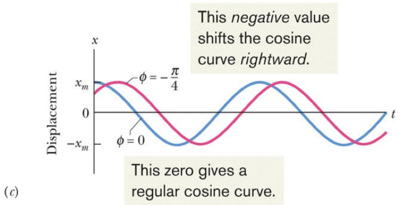


Figure 15-5 In all three cases, the blue curve is obtained from Eq. 15-3 with $\phi = 0$. (*a*) The red curve differs from the blue curve *only* in that the red-curve amplitude x'_m is greater (the red-curve extremes of displacement are higher and lower). (*b*) The red curve differs from the blue curve *only* in that the red-curve period is $T' = T/2$ (the red curve is compressed horizontally). (*c*) The red curve differs from the blue curve *only* in that for the red curve $\phi = -\pi/4$ rad rather than zero (the negative value of ϕ shifts the red curve to the right).



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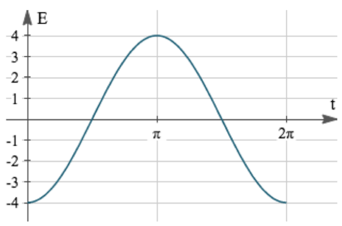
例:

Problem

Checkpoint 1

A particle undergoing simple harmonic oscillation of period T (like that in Fig. 15-2) is at $-x_m$ at time $t = 0$. Is it at $-x_m$, at $+x_m$, at 0, between $-x_m$ and 0, or between 0 and $+x_m$ when (a) $t = 2.00T$, (b) $t = 3.50T$, and (c) $t = 5.25T$?

Answer: (a) at $-x_m$ (b) at x_m (c) at 0

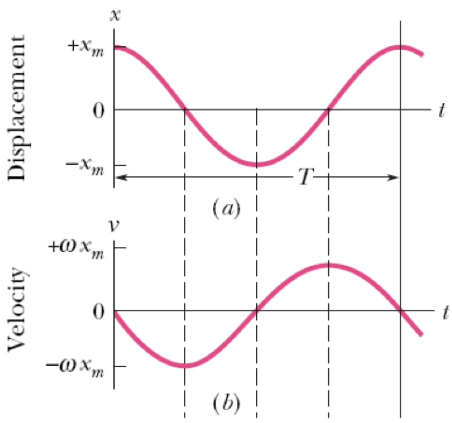


4. Velocity

$$v(t)=\frac{dx(t)}{dt}=\frac{d}{dt}[x_m\cos(\omega t+\phi)]$$

$$v(t)=-\omega x_m\sin(\omega t+\phi)$$

$$=\omega x_m\cos(\omega t+\phi+\frac{\pi}{2})$$



注: ① 速度的相位比位置方程的快 $\frac{\pi}{2}$ ($+\frac{\pi}{2}$),

将 cosine 函数变为 $-$sine 函数

② velocity amplitude: $v_m=\omega\cdot x_m$

在 $x=0$ 时取到 maximum speed.

5. Acceleration

$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}[-\omega x_m \sin(\omega t + \phi)]$

$a(t) = -\omega^2 x_m \cdot \cos(\omega t + \phi)$

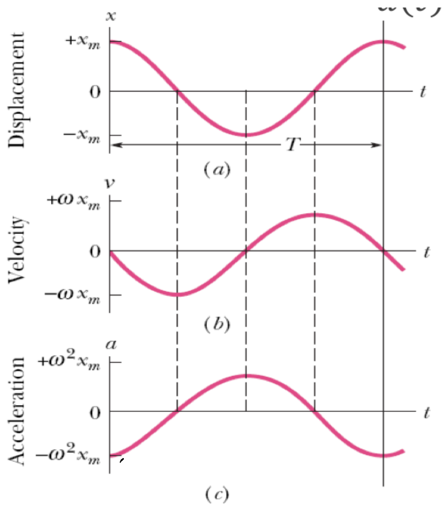
$\quad = -\omega^2 \cdot x(t)$

注: ① 加速度方向始终与位移方向相反

② Acceleration amplitude: $a_m = \omega^2 \cdot x_m$

在 $x = \pm x_m$ 时取到 maximum acceleration

③ 加速度的相位比位置方程的快π (+π)



例:

Problem

Checkpoint 2

Which of the following relationships between a particle's acceleration a and its position x indicates simple harmonic oscillation: (a) $a = 3x^2$, (b) $a = 5x$, (c) $a = -4x$, (d) $a = -2/x$? For the SHM, what is the angular frequency (assume the unit of rad/s)?

Answer: (c) where the angular frequency is 2 rad/s.

6. Force

由牛二: $F = ma = -m\omega^2 \cdot x$

由胡克定律: $F = -kx$

由此推出: $k = m\omega^2$

$\omega = \sqrt{\frac{k}{m}}$

$T = 2\pi\sqrt{\frac{m}{k}}$

$k\uparrow \Rightarrow \omega\uparrow \Rightarrow T\downarrow$

$m\uparrow \Rightarrow \omega\downarrow \Rightarrow T\uparrow$

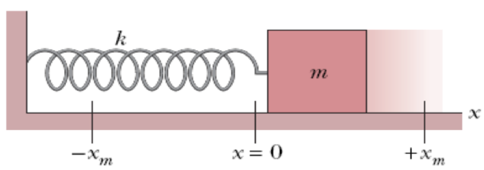


Figure 15-7 A linear simple harmonic oscillator. The surface is frictionless. Like the particle of Fig. 15-2, the block moves in simple harmonic motion once it has been either pulled or pushed away from the $x = 0$ position and released. Its displacement is then given by Eq. 15-3.

Linear simple harmonic oscillator, where *linear* indicates that F is proportional to x.

Additional Information

- Newton's 2nd Law & Hooke's Law

$$F = ma = -kx$$
$$ma + kx = 0$$
$$m\frac{d^2x}{dt^2} + kx = 0$$

- Solution: Let $x = Ae^{i(\omega t + \phi)}$

$$\frac{dx}{dt} = i\omega Ae^{i(\omega t+\phi)}$$
$$\frac{d^2x}{dt^2} = -\omega^2 Ae^{i(\omega t+\phi)}$$

- Put it back to Newton's and Hooke's Law

$$-\omega^2 Ae^{i(\omega t+\phi)} \times m + kAe^{i(\omega t+\phi)} = 0$$

$$\omega = \pm\sqrt{\frac{k}{m}} \rightarrow \omega = \sqrt{\frac{k}{m}}$$

例: **Problem**

Checkpoint 3

Which of the following relationships between the force F on a particle and the particle's position x gives SHM: (a) $F = -5x$, (b) $F = -400x^2$, (c) $F = 10x$, (d) $F = 3x^2$?

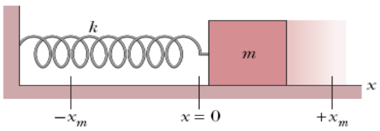
Answer: only (a) is simple harmonic motion

例: **Problem**

At $t = 0$, the displacement $x(0)$ of the block in a linear oscillator like that of Fig. 15-7 is -8.50 cm. (Read $x(0)$ as "x at time zero.") The block's velocity $v(0)$ then is -0.920 m/s, and its acceleration $a(0)$ is $+47.0\ \text{m/s}^2$.

(a) What is the angular frequency ω of this system?

(b) What are the phase constant ϕ and amplitude x_m?



Solution:

With the block in SHM, Eqs. 15-3, 15-6, and 15-7 give its displacement, velocity, and acceleration, respectively, and each contains ω.

Calculations: Let's substitute $t = 0$ into each to see whether we can solve any one of them for ω. We find

$$x(0) = x_m \cos\phi, \qquad (15\text{-}15)$$

$$v(0) = -\omega x_m \sin\phi, \qquad (15\text{-}16)$$

and

$$a(0) = -\omega^2 x_m \cos\phi. \qquad (15\text{-}17)$$

In Eq. 15-15, ω has disappeared. In Eqs. 15-16 and 15-17, we know values for the left sides, but we do not know x_m and ϕ. However, if we divide Eq. 15-17 by Eq. 15-15, we neatly eliminate both x_m and ϕ and can then solve for ω as

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0\ \text{m/s}^2}{-0.0850\ \text{m}}}$$

$$= 23.5\ \text{rad/s}. \qquad \text{(Answer)}$$

Calculations: We know ω and want ϕ and x_m. If we divide Eq. 15-16 by Eq. 15-15, we eliminate one of those unknowns and reduce the other to a single trig function:

$$\frac{v(0)}{x(0)} = \frac{-\omega x_m \sin\phi}{x_m \cos\phi} = -\omega \tan\phi.$$

Solving for $\tan\phi$, we find

$$\tan\phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920\ \text{m/s}}{(23.5\ \text{rad/s})(-0.0850\ \text{m})}$$

$$= -0.461.$$

This equation has two solutions:

$$\phi = -25^\circ \quad \text{and} \quad \phi = 180^\circ + (-25^\circ) = 155^\circ.$$

Normally only the first solution here is displayed by a calculator, but it may not be the physically possible solution. To choose the proper solution, we test them both by using them to compute values for the amplitude x_m. From Eq. 15-15, we find that if $\phi = -25^\circ$, then

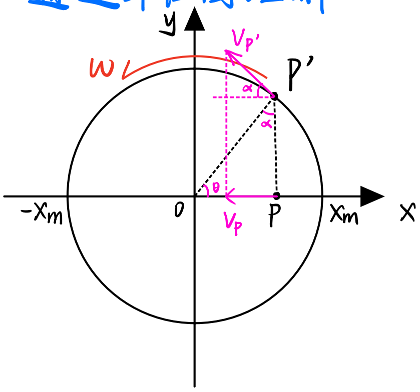
$$x_m = \frac{x(0)}{\cos\phi} = \frac{-0.0850\ \text{m}}{\cos(-25^\circ)} = -0.094\ \text{m}.$$

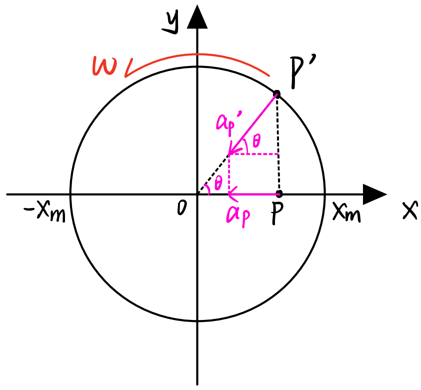
We find similarly that if $\phi = 155^\circ$, then $x_m = 0.094$ m. Because the amplitude of SHM must be a positive constant, the correct phase constant and amplitude here are

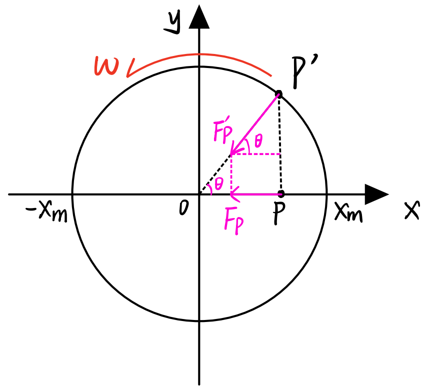
$$\phi = 155^\circ \quad \text{and} \quad x_m = 0.094\ \text{m} = 9.4\ \text{cm}. \qquad \text{(Answer)}$$

The angular constant $\emptyset$ can also be presented in Radian Measure

*7. 通过单位圆理解







$$\begin{cases} \theta = \omega t + \phi \\ V_{p'} = -\omega \cdot X_m \\ V_p = V_{p'} \cdot \cos\alpha = V_{p'} \cdot \sin\theta \end{cases}$$

$$V_p = -\omega X_m \cdot \sin(\omega t + \phi)$$

$$\begin{cases} \theta = \omega t + \phi \\ a_{p'} = -\omega^2 \cdot x_m \\ a_p = a_{p'} \cdot \cos\theta \end{cases}$$

$$a_p = -\omega^2 X_m \cdot \cos(\omega t + \phi) = -\omega^2 x(t)$$

$$\begin{cases} \theta = \omega t + \phi \\ F_{p'} = -m\omega^2 \cdot X_m \\ F_p = F_{p'} \cdot \cos\theta \end{cases}$$

$$F_p = -m\omega^2 X_m \cdot \cos(\omega t + \phi) = -m\omega^2 x(t)$$

8. 能量

1° Potential energy

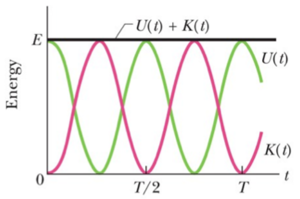
$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}k X_m^2 \cdot \cos^2(\omega t + \phi)$$

2° Kinetic energy

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 X_m^2 \sin^2(\omega t + \phi) = \frac{1}{2}k X_m^2 \sin^2(\omega t + \phi)$$

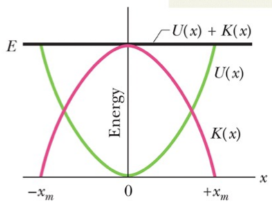
3° Mechanical energy

$$\begin{aligned} E &= U(t) + K(t) \\ &= \frac{1}{2}k X_m^2 [\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] \\ &= \frac{1}{2}k X_m^2 \\ &= \frac{1}{2}m V_m^2 \end{aligned}$$



(a)

As *time* changes, the energy shifts between the two types, but the total is constant.



(b)

As *position* changes, the energy shifts between the two types, but the total is constant.

- The potential energy is maximum when the block is at full extension of the spring
- The kinetic energy is maximum when the block is moving through the mid-point of the oscillation
- The total energy is constant. It depends only on the initial condition: i.e., initial displacement and velocity.

9. Rules to distinguish SHM

1° 质点的 restoring force 为

$F = -kx$

2° 质点的 dynamical equation (动力学方程) 为

$\frac{d^2x}{dt^2} + \omega^2 x = 0$

3° 质点的位移为

$x(t) = x_m \cdot \cos(\omega t + \phi)$

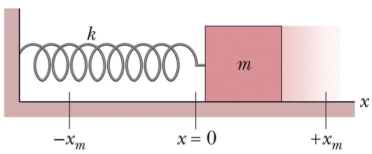
4° 质点的机械能守恒且取决于 amplitude 的平方

$E = U + K = \frac{1}{2} k x_m^2$

例: **Problem**

Checkpoint 4

In Fig. 15-7, the block has a kinetic energy of 3 J and the spring has an elastic potential energy of 2 J when the block is at $x = +2.0$ cm. (a) What is the kinetic energy when the block is at $x = 0$? What is the elastic potential energy when the block is at (b) $x = -2.0$ cm and (c) $x = -x_m$?



Answer: (a) 5 J (b) 2 J (c) 5 J

§3 Angular simple harmonic motion

1. An angular simple harmonic oscillator (角简谐振子)

1° Torsion pendulum (扭摆) with a suspension wire

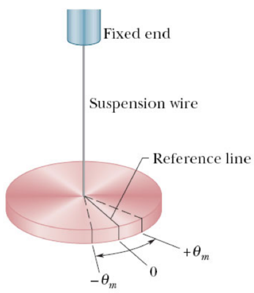
2° 将 disk 旋转过角度 θ, 释放后会 oscillate about rest position

3° Restoring torque (回复力矩)

$\tau = -\kappa\theta$

4° Torsion constant (扭转常数) κ (kappa)

取决于长度、材质、直径



2. Angular simple harmonic motion (角简谐振动)

$T = 2\pi\sqrt{\frac{I}{\kappa}}$

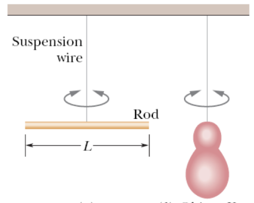
I: oscillating disk 的转动惯量

通过测量不同物体的T，可以由已知物体的I求出未知物体的I

例:

Problem

Fig. (a) shows a thin rod whose length L is 12.4 cm and whose mass m is 135 g, suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call object X, is then hung from the same wire, as in Fig. (b), and its period T_b is found to be 4.76 s. What is the rotational inertia of object X about its suspension axis?



(a) (b) Object X

Solution:

$$I_a = \tfrac{1}{12}mL^2 = (\tfrac{1}{12})(0.135\text{ kg})(0.124\text{ m})^2$$
$$= 1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2.$$

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

The constant κ, which is a property of the suspension wire, is the same for both figures; only the periods and the rotational inertias differ.

$$I_b = I_a\frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2)\frac{(4.76\text{ s})^2}{(2.53\text{ s})^2}$$
$$= 6.12 \times 10^{-4}\text{ kg}\cdot\text{m}^2. \qquad \text{(Answer)}$$

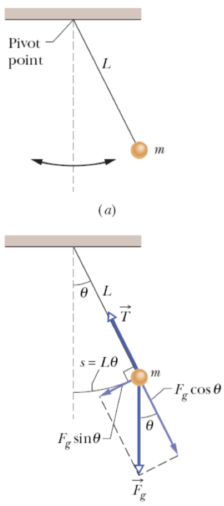
§4 The simple pendulum

1. The simple pendulum

$\alpha = -\frac{mgL}{I}\sin\theta \approx -\frac{mgL}{I}\cdot\theta = -\omega^2\theta(t)$

$T = 2\pi\sqrt{\frac{L}{g}}$

The Simple Pendulum



(a) (b)

- In a simple pendulum, a particle of mass **m** is suspended from one end of an unstretchable massless string of length **L** that is fixed at the other end.
- The restoring torque acting on the mass when its angular displacement is θ, is:

$$\tau = -L(F_g\sin\theta) = I\alpha$$
$$\alpha = -\frac{mgL}{I}\sin\theta \approx -\frac{mgL}{I}\theta$$

- SHM in angular form

$$\alpha(t) = -\omega^2\theta(t)$$

- Therefore, it is a SHM if it swings through only a small angle.

Simple Pendulum

- Angular Frequency

$$\alpha(t) = -\omega^2\theta(t) = -\frac{mgL}{I}\theta(t)$$

$$\omega = \sqrt{\frac{mgL}{I}}$$

- Period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL}}$$

Small-Angle Swinging

- For Simple Pendulum

$$I = mL^2$$

- Thus,

$$T = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

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Small Angle Approximation

- In the small-angle approximation we can assume that $\theta << 1$ and use the approximation $\sin\theta \cong \theta$. Let us investigate up to what angle θ is the approximation reasonably accurate?

θ (degrees)	θ (radians)	$\sin\theta$
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259 (1% off)
20	0.349	0.342 (2% off)

Conclusion: If we keep $\theta < 10°$ we make less than 1 % error

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Summary

- Frequency

$$1\ hertz = 1\ Hz = 1\ oscillation\ per\ second = 1\ s^{-1}$$

- Period

$$T = \frac{1}{f}$$

- Simple Harmonic Motion

$$x(t) = x_m\cos(\omega t + \emptyset)$$

$$\omega = \frac{2\pi}{T} = 2\pi f$$

$$v(t) = -\omega x_m\sin(\omega t + \emptyset)$$

$$v_m = \omega x_m$$

$$a(t) = -\omega^2 x_m\cos(\omega t + \emptyset) = -\omega^2 x(t)$$

$$a_m = \omega^2 x_m$$

Summary

- The Linear Simple Harmonic Oscillator

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Energy in SHM

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2cos^2(\omega t + \emptyset)$$

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2sin^2(\omega t + \emptyset)$$

$$E = U + K = \frac{1}{2}kx_m^2$$

Summary

- Pendulum
  - Torsion Pendulum

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

  - Simple Pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$