Lecture 6 (2021.9.28)

§1 Chain rule

1. Contents

If $y=f(x)$ is differentiable at $x=x_0$,

and $z=g(y)$ is differentiable at $y=y_0$,

then $g\circ f$ is differentiable at $x=x_0$, and

$(g\circ f)'(x_0) = g'(f(x_0))\cdot f'(x_0)$ or $\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$ (more simple)

2. Proof

When Δx is small, $\frac{\Delta y}{\Delta x}\approx\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}\Big|_{x=x_0}=f'(x_0)$

So: $\Delta y\approx f'(x_0)\Delta x$

Δy can be as small as we want by choosing a small enough Δx

(i.e. $\Delta y\to 0$ as $\Delta x\to 0$)

Since Δy is small, $\frac{\Delta z}{\Delta y}\approx\lim\limits_{\Delta y\to 0}\frac{\Delta z}{\Delta y}=\frac{dz}{dy}\Big|_{y=y_0}=g'(y_0)=g'(f(x_0))$

So: $\Delta z\approx g'(f(x_0))\Delta y\approx g'(f(x_0))\cdot f'(x_0)\,\Delta x$

Hence: $\frac{\Delta z}{\Delta x}\approx g'(f(x_0))\cdot f'(x_0)$

It turns out that as we take a limit as $\Delta x\to 0$, the "$\approx$" become "$=$"

3. Quotient rule proof

Assume f and g are differentiable at $x=x_0$, with $g(x_0)\neq 0$.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x_0)}\right)\Big|_{x=x_0}=\frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right)\Big|_{x=x_0}$$

$$=f(x_0)\frac{d}{dx}\left(\frac{1}{g(x)}\right)\Big|_{x=x_0}+f'(x_0)\cdot\frac{1}{g(x_0)}$$

$$=f(x_0)(-1)\frac{1}{(g(x_0))^2}\cdot g'(x_0)+f'(x_0)\cdot\frac{1}{g(x_0)}$$

$$=\frac{-f(x_0)g'(x_0)+f'(x_0)g(x_0)}{}$$

$$= \frac{\quad}{(g(x_0))^2}$$

§2 Implicit differentiation

1. Implicitly defined function

If x and y has the form $f(x,y)=0$, we say that y is an implicitly defined function of x

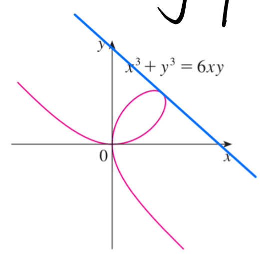
e.g. $x^2+y^2=1$

2. Implicit differentiation

Q: What's the slope of the tangent line to the graph of

$$x^3+y^3-6xy=0$$

at the point $(3,3)$?



A: Treat y as a function of x. Apply $\frac{d}{dx}$ to both sides:

$$x^3+y^3-6xy=0$$

$$3x^2+3y^2\frac{dy}{dx}-6\left(x\frac{dy}{dx}+y\right)=0$$

$$3x^2-6y+\frac{dy}{dx}(3y^2-6x)=0$$

$$\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}$$

$$\left.\frac{dy}{dx}\right|_{(3,3)}=\frac{6(3)-3(3)^2}{3(3)^2-6(3)}=-1$$

* Note:

In order for the approach above to work, y has to be a function of x at the required point "locally". (i.e. on a small scale), and it cannot have a vertical tangent.

e.g. It doesn't work at $(0,0)$, since no matter how close you zoom in at $(0,0)$, the curve is not the graph of an x-y function.

e.g. Find $\frac{d^2y}{dx^2}$ given $2x^3+3y^2=8$

Treat y as a function of x. Write $\frac{dy}{dx}$ as y'

Apply $\frac{d}{dx}$ to both sides:

$$2x^3+3y^2=8$$

$$6x^2+6y\cdot y'=0$$

$$y'=-\frac{x^2}{y} \quad (y\neq 0)$$

Apply $\frac{d}{dx}$ again:

$$y''=\frac{-y(2x)+x^2y'}{y^2}$$

$$=\frac{-2xy-x^4/y}{y^2}$$

$$=\frac{-2xy^2-x^4}{y^3}$$

§3 Normal line

1. Definition

The normal line to a curve at (x_0,y_0) is the line perpendicular ($\perp$) to the tangent line to the curve at (x_0,y_0).

2. Remark:

For the graph of a differentiable function $y=f(x)$, the slope of the normal line at (x_0,y_0) is $-\frac{1}{f'(x)}$, provided that $f'(x)\neq 0$.

§4 Related rates (application of chain rule)

1. Related rates

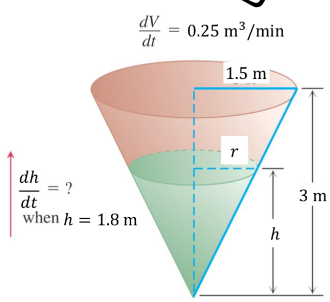
The chain rule $\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$

states a relation between three rates of change.
Given 2 of them, you can find the remaining one.

e.g. Water runs into a conical tank at the rate of 0.25 m³/min.

The tank stands point down and has a height of 3 m and a base radius of 1.5 m.

How fast is the water level rising when the water is 1.8 m deep?



$$r = \frac{1}{2}h$$

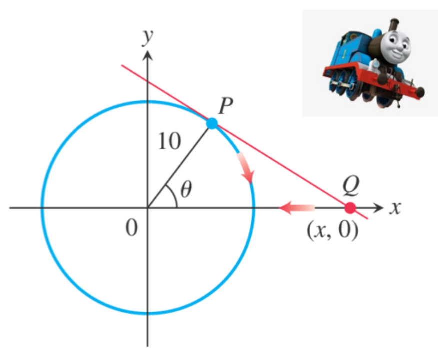
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{4}\pi h^2$$

$$\left.\frac{dV}{dh}\right|_{h=1.8} = 0.81\pi$$

$$\left.\frac{dh}{dt}\right|_{h=1.8} = \frac{\left.\frac{dV}{dt}\right|_{h=1.8}}{\left.\frac{dV}{dh}\right|_{h=1.8}} = \frac{25}{81\pi}$$

e.g. **EXAMPLE 3.8.4** A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is (0, 10) on the y-axis, and its final destination is the point (10, 0) on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 m from the center of the circle?



$$\frac{d\theta}{dt} = \frac{\Delta\theta}{\Delta t} = \frac{-\frac{\pi}{2}}{30} = -\frac{\pi}{60}$$

$$x = 10\sec\theta$$

$$\frac{dx}{dt} = \frac{d\theta}{dt}\cdot\frac{dx}{d\theta} = 10\cdot\sec\theta\cdot\tan\theta$$

When $x = 20$, $\theta = \frac{\pi}{3}$

$$\frac{dx}{dt} = -\frac{\sqrt{3}\pi}{3}$$