Lecture 22

§1 Trigonometric Integrals

1. $\int \sin^m x \cdot \cos^n x \, dx$ $(m, n \in \mathbb{N} := \{0, 1, 2 \cdots\cdots\})$

1° If m or n is odd, consider one of them, and take out all the even powers.

e.g. Evaluate $\int \sin^5 x \, dx$

$$\int \sin^5 x \, dx = \int (\sin^2 x)^2 \cdot \sin x \, dx$$
$$= \int (1-\cos^2 x)^2 \sin x \, dx \quad (\text{using } \sin^2 x + \cos^2 x = 1)$$
$$= \int -(1-\cos^2 x)^2 \, d(\cos x)$$
$$= \int -\cos^4 x + 2\cos^2 x - 1 \quad d(\cos x)$$
$$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$$

e.g. Evaluate $\int \sin^5 x \cos^7 x \, dx$

$$\int \sin^5 x \cdot \cos^7 x \, dx = \int (\sin^2 x)^2 \cdot \cos^7 x \cdot \sin x \, dx$$
$$= \int (1-\cos^2 x)^2 \cos^7 x \cdot \sin x \, dx$$

Let $u = \cos x$, then $dx = du/(-\sin x)$

$$\int (1-\cos^2 x)^2 \cos^7 x \cdot \sin x \, dx = -\int u^7 (1-u^2)^2 \, du$$
$$= -\int u^{11} - 2u^9 + u^7 \, du$$
$$= -\frac{1}{12}u^{12} + \frac{1}{5}u^{10} - \frac{1}{8}u^8 + C$$
$$= -\frac{1}{12}\cos^{12} x + \frac{1}{5}\cos^{10} x - \frac{1}{8}\cos^8 x + C$$

e.g. Evaluate $\int \cos^7 x \, dx$

$$\int \cos^7 x = \int (\cos^2 x)^3 \cdot \cos x \, dx$$
$$= \int (1-\sin^2 x)^3 \cdot \cos x \, dx$$

Let $u = \sin x$, then $dx = du/\cos x$

$$\int (1-\sin^2 x)^3 \cdot \cos x \, dx = \int (1-u^2)^3 \, du$$

$$= \int 1 - 3u^2 + 3u^4 - u^6 \, du$$
$$= u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C$$
$$= -\frac{1}{7}\sin^7 x + \frac{3}{5}\sin^5 x - \sin^3 x + \sin x + C$$

2° If both m and n is even, then use half angle identities:

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

e.g. Evaluate $\int \sin^2 x \cos^4 x \, dx$

$$\int \sin^2 x \cos^4 x \, dx = \int \sin^2 x (\cos^2 x)^2$$
$$= \int \left(\frac{1-\cos^2 2x}{2}\right) \cdot \left(\frac{1+\cos^2 2x}{2}\right)^2 dx$$
$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx$$

1° $\int (1+\cos 2x) \, dx = x + \frac{1}{2}\sin 2x + C_1$

2° $\int \cos^2 2x \, dx = \int \frac{1+\cos 4x}{2} dx = \frac{1}{2}x + \frac{1}{8}\sin 4x + C_2$

3° $\int \cos^3 2x \, dx = \int (1-\sin^2 2x) \cos 2x \, dx$

$$= \int (1-u^2)\frac{1}{2} du$$
$$= \frac{1}{2}u - \frac{1}{6}u^3 + C_3$$
$$= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C_3$$

Finally, combining the simplifying:

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right) + C$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

2. $\int \tan^m x \, dx \quad (m \in \mathbb{N} := \{0, 1, 2, 3 \cdots\})$

Recall that $\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$

Use $\tan^2 + 1 = \sec^2 x$ to obtain a reduction formula.

Assume $m \geq 2$, then

$$\int \tan^m x \, dx = \int (\tan^{m-2} x) \cdot (\tan^2 x) \, dx$$

$$= \int (-\tan^{m-2}x)\cdot(\sec^2 x - 1)\,dx$$
$$= \int (\tan^{m-2}x)\cdot(\sec^2 x)\,dx - \int (\tan^{m-2}x)\,dx$$

Let $u = \tan x$, $\frac{du}{dx} = \sec^2 x$

$$\int (\tan^{m-2}x)\cdot(\sec^2 x)\,dx - \int(\tan^{m-2}x)\,dx = \int u^{m-2}\,du - \int(\tan^{m-2}x)\,dx$$
$$= \frac{\tan^{m-1}x}{m-1} - \int(\tan^{m-2}x)\,dx$$

To summarize:

$$\int \tan^m x\,dx = \frac{\tan^{m-1}x}{m-1} - \int(\tan^{m-2}x)\,dx$$

e.g. Evaluate $\int \tan^4 x\,dx$

$$\int \tan^4 x\,dx = \int \tan^2 x\cdot\tan^2 x\,dx$$
$$= \int \tan^2 x(\sec^2 x - 1)\,dx$$
$$= \int \tan^2 x\sec^2 x\,dx - \int \tan^2 x\,dx$$
$$= \int u^2\,du - \int \sec^2 x - 1\,dx$$
$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

3. $\int \sec^n x\,dx \quad (n \in \mathbb{N} := \{0, 1, 2, 3, \cdots\})$

Recall that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

Use integration by parts to obtain a reduction formula.

Assume $n \geq 3$.

$$\int \sec^n x\,dx = \int \sec^{n-2}x\cdot\sec^2 x\,dx$$
$$= \sec^{n-2}x\cdot\tan x - \int(n-2)(\sec^{n-2}x)\cdot(\tan^2 x)\,dx$$
$$= \sec^{n-2}x\cdot\tan x - (n-2)\int(\sec^{n-2}x)(\sec^2 x - 1)\,dx$$
$$= \sec^{n-2}x\cdot\tan x - (n-2)\int(\sec^n x)\,dx + (n-2)\int(\sec^{n-2}x)\,dx$$

$$(n-1)\int \sec^n x\,dx = \sec^{n-2}x\tan x + (n-2)\int(\sec^{n-2}x)\,dx$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\cdot\tan x}{n-1} + \frac{n-2}{n-1}\int(\sec^{n-2}x)\,dx$$

To summarize:

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x \cdot \tan x}{n-1} + \frac{n-2}{n-1}\int (\sec^{n-2}x)\,dx$$

e.g. Evaluate $\int \sec^3 x\,dx$

$$\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$$
$$= \sec x \cdot \tan x - \int \sec x \cdot \tan^2 x\,dx$$
$$= \sec x \cdot \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$
$$2\int \sec x\,dx = \sec x \cdot \tan x + \ln|\sec x + \tan x| + C$$
$$\int \sec x\,dx = \frac{1}{2}\sec x \cdot \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

4. $\int \tan^m x \cdot \sec^n x\,dx \quad (m, n \in \mathbb{Z}_+ := \{1, 2, 3 \cdots\cdots\})$

1° If n is even, then take out a copy of $\sec^2 x$ and express everything in terms of $\tan x$

$$\int (\tan^m x)(\sec^{2k}x)\,dx = \int (\tan^m x)(\sec^{2k-2}x)\cdot(\sec^2 x)\,dx$$
$$= \int (\tan^m x)(\tan x + 1)^{k-1}\,d(\tan x)$$

e.g. Evaluate $\int \tan^4 x \cdot \sec^4 x\,dx$

$$\int \tan^4 x \cdot \sec^4 x\,dx = \int \tan^4 \sec^2 x \cdot \sec^2 x\,dx$$
$$= \int \tan^4 (1 + \tan^2 x)\,d(\tan x)$$
$$= \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$$

2° If m is odd and n is odd, then take out a copy of $\tan x \cdot \sec x$ and express the rest in terms of $\sec x$.

$$\int (\tan^{2r+1}x)\cdot(\sec^{2r+1}x)\,dx = \int (\tan^{2r}x)(\sec^{2r}x)\cdot(\tan x \cdot \sec x)\,dx$$
$$= \int (\sec^2 x - 1)^r (\sec^{2r}x)\,d(\sec x)$$

3° If m is even and n is odd, then use $\tan^2 x = \sec^2 x - 1$ to convert the integrand into sums of powers of $\sec x$.

$$\int (\tan^{2r}x)(\sec^n x)\,dx = \int (\sec^2 x-1)^r (\sec^n x)\,dx$$

then use integration by parts or reduction formula to solve

$$\int \sec^t x\,dx$$

5. Integrals with square roots

Trig identities involving squares, such as

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

may help remove square root signs in a trigonometric integral.

e.g. Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx &= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx \\
&= \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx \quad (\cos 2x > 0) \\
&= \frac{\sqrt{2}}{2}\cdot \sin 2x \Big|_0^{\frac{\pi}{4}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

6. Product of sine and cosine functions

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

These identities can be used to evaluate integrals of the form

1° $\int \sin mx \cdot \sin nx\,dx$

2° $\int \sin mx \cdot \cos nx\,dx$

3° $\int \cos mx \cdot \cos nx\,dx$

e.g. Evaluate $\int \sin 4x \cdot \cos 8x\,dx$

$$\begin{aligned}
\int \sin 4x \cdot \cos 8x\,dx &= \frac{1}{2}\int \sin(-4x) + \sin(12x)\,dx \\
&= \frac{1}{8}\cos 4x - \frac{1}{12}\cos(12x) + C
\end{aligned}$$

§2 Trigonometric Substitutions

Trigonometric substitutions can be very useful when integrating functions involving $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$.

Integrand	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$
$\sqrt{x^2+a^2}$	$x=a\tan\theta$	$\tan^2\theta+1=\sec^2\theta$

e.g. Evaluate $\int\frac{dx}{\sqrt{9+x^2}}$

Let $x=3\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$dx=3\sec^2\theta\,d\theta$

$$
\begin{aligned}
\int\frac{dx}{\sqrt{9+x^2}} &= \int\frac{3\sec^2\theta\,d\theta}{\sqrt{9+9\tan^2\theta}} \\
&= \int\frac{3\sec^2\theta\,d\theta}{3\sqrt{1+\tan^2\theta}} \\
&= \int\frac{3\sec^2\theta\,d\theta}{3\sqrt{\sec^2\theta}} \\
&= \int\sec\theta\,d\theta \quad (\sec\theta>0) \\
&= \ln|\sec\theta+\tan\theta|+C \\
&= \ln\left|\sqrt{1+\tan^2\theta}+\tan\theta\right|+C \\
&= \ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|+C
\end{aligned}
$$

Note:

For the substitution $x=f(\theta)$, the domain of f is chosen so that f is injective (one-to-one) and covers all the possible value of x.

e.g. Evaluate $\int\frac{dx}{x^2\sqrt{x^2-4}}$

Let $x=2\sec\theta$, then $dx=2\sec\theta\tan\theta\,d\theta$

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^2\theta \cdot 2\sqrt{\tan^2\theta}}$$

$$= \frac{1}{4}\int \frac{\tan\theta\, d\theta}{\sec\theta \cdot |\tan\theta|}$$

$$= \begin{cases} \frac{1}{4}\int \cos\theta\, d\theta, & \text{for } x>2 \\ -\frac{1}{4}\int \cos\theta\, d\theta, & \text{for } x<-2 \end{cases}$$

$$= \begin{cases} \frac{1}{4}\sin\theta + C, & \text{for } x>2 \\ -\frac{1}{4}\sin\theta + C, & \text{for } x<-2 \end{cases} \quad ①$$

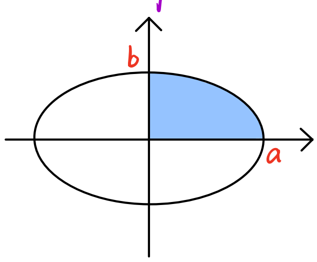
Now $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{\sec^2\theta}}$

$$= \sqrt{\frac{x^2-4}{x^2}} = \begin{cases} \frac{1}{x}\sqrt{x^2-4} & \text{for } x>2 \\ -\frac{1}{x}\sqrt{x^2-4} & \text{for } x<-2 \end{cases} \quad ②$$

Combining ① and ② we get:

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{1}{4x}\sqrt{x^2-4} + C$$

e.g. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let area $A = 4A_1$, where A_1 is the shaded area.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2-x^2)$$

$y = \frac{b}{a}\sqrt{a^2-x^2}$, for $y>0$

$$A_1 = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta \quad \text{for substitution } x = a\sin\theta$$

$$= ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{4}\pi$$

$$A = 4A_1 = ab\pi$$