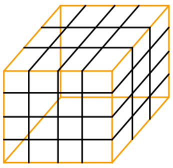
Lecture 20

§1 Triple integrals on rectangular solid

1. Riemann sum (motivation: density and mass)



Let f be a three-variable function defined on the rectangular solid

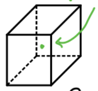
$$R := [a, b] \times [c, d] \times [r, s].$$

Then the triple integral of f over R is defined by

$$\iiint_R f(x, y, z)\, dV := \lim_{\|P\| \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk},$$

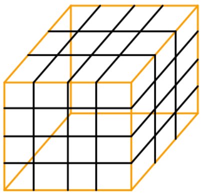
provided that the limit exists. The Riemann sum above is defined similarly to that defined ~~on Page 6.~~ for double integrals.

$(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$



R_{ijk}

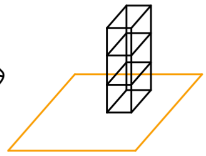
One way to think of triple integrals is that, if f is the density function, then $\iiint_R \underbrace{f(x, y, z)}_{\text{density at } (x,y,z), \text{ say } kg/m^3}\, dV$ is the mass of the solid R.



→

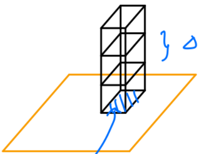
Projection of solid onto xy-plane

→



Consider a "pile".

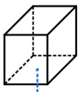
→



$\} \Delta z_k$

Area $= \Delta A_{ij}$
$= \Delta x_i \Delta y_j$

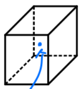
→



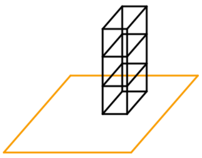
Fix (x_{ij}^*, y_{ij}^*) in the base region

→

R_{ijk}

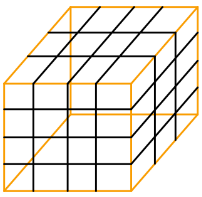


Pick height z_{ijk}^*.
Sample point $(x_{ij}^*, y_{ij}^*, z_{ijk}^*)$



Mass of this pile is

$$m_{ij} \approx \sum_{k=1}^{n} f(x_{ij}^*, y_{ij}^*, z_{ijk}^*) \Delta z_k \Delta A_{ij}$$



Mass of "Tofu" is

$$m = \sum_{i,j} m_{ij}$$
$$\approx \sum_{i,j} \left(\sum_k f(x_{ij}^*, y_{ij}^*, z_{ijk}^*) \Delta z_k \right) \Delta A_{ij}$$

· Taking limit as $\|P\| \to 0$ yields

$$m = \iint_A \left(\int_r^s f(x,y,z)\, dz \right) dA$$

If f is cts

$$= \int_a^b \int_c^d \int_r^s f(x,y,z)\, dz\, dy\, dx. \quad \text{(two-variable Fubini's thm)}$$

2. Fubini's Theorem (Triple integrals)

Theorem (Fubini's Theorem) (Triple integrals)
If f is continuous on the rectangular solid

$$R = \underset{x}{[a, b]} \times \underset{y}{[c, d]} \times \underset{z}{[r, s]},$$

then

$$\iiint_R f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

And order does not matter.

注: 技巧:

Double: $\int_a^b \int_c^d f(x)g(y)\,dy\,dx = \left(\int_b^a f(x)dx\right)\cdot\left(\int_c^d g(y)\,dy\right)$

Triple: $\int_a^b \int_c^d \int_r^s f(x)g(y)h(z)\,dz\,dy\,dx = \left(\int_b^a f(x)dx\right)\cdot\left(\int_c^d g(y)dy\right)\cdot\left(\int_r^s h(z)dz\right)$

例: **Exercise**
Evaluate $\iiint_R xyz^2\, dV$, where $R = [0, 1] \times [-1, 2] \times [0, 3]$.

Sol: $I = \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dz\,dy\,dx$

$= \left(\int_0^1 x\,dx\right)\left(\int_{-1}^2 y\,dy\right)\left(\int_0^3 z^2\,dz\right)$

$= \frac{1}{2} \times \frac{3}{2} \times 9$

$= \frac{27}{4}$

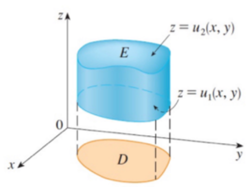
§2 Triple integrals (General cases)

1. Triple integrals (General cases) (三重积分)

Suppose that E has the form

$$\{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\},$$

where D is the projection of E onto the xy-plane.



Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA.$$

Note that in the inner integral on the right-hand side, x and y are held fixed. If D is a type-I region, then E can be written as

$$\{(x, y, z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

By Fubini's theorem for double integrals, the equation above becomes

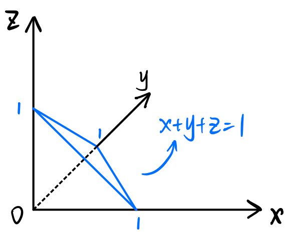
$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

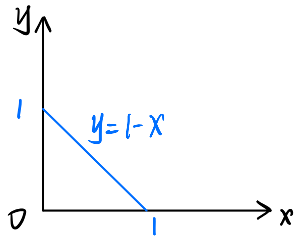
A similar strategy can be used when D is of type II or when E has other similar forms.

例: Example

Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Sol:





$$\iiint_E z\,dV = \iint_D \left(\int_0^{1-x-y} z\,dz\right) dA$$
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2\,dy\,dx$$
$$= \cdots\cdots$$
$$= \frac{1}{24}$$

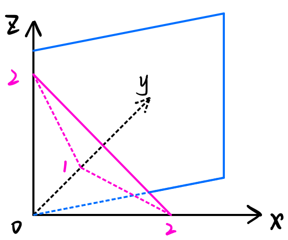
2、Volumn and average value (density)

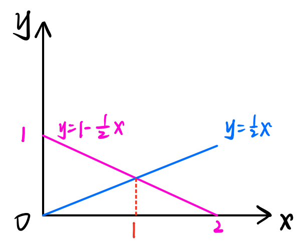
> **Definition**
> The volume $V(E)$ of a solid E in $\mathbb{R}^3$ is defined by
> $$V(E) := \iiint_E dV := \iiint_E 1\,dV.$$
> The average value of a function f over E is
> $$\frac{1}{V(E)} \iiint_E f(x, y, z)\,dV.$$

例: Exercise

Use a triple integral to find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Sol:





Project on xy-plane

$$V = \iiint_E 1\,dV$$
$$= \iint_D \left(\int_0^{2-x-2y} dz\right) dA$$
$$= \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \int_0^{2-x-2y} dz\,dy\,dx$$
$$= \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} 2-x-2y\,dy\,dx$$
$$= \int_0^1 (2-x)(1-x) - (1-\tfrac{1}{2}x)^2 + (\tfrac{1}{2}x)^2\,dx$$
$$= \int_0^1 x^2-2x+1\,dx$$
$$= \frac{1}{3}x^3 - x^2 + x \Big|_0^1$$
$$= \frac{1}{3}$$

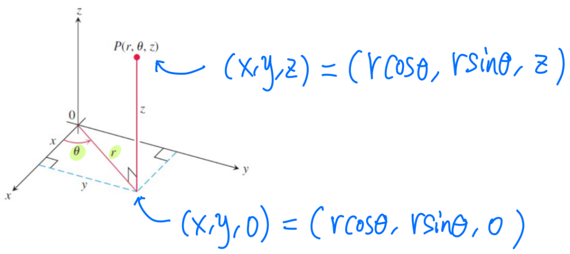
§3 Cylindrical coordinates

1. Cylindrical coordinates

Given any point P in the xyz-space, we can represent the point by (r, θ, z), where

- r and θ are polar coordinates of the projection of P onto the xy-plane, and;
- z is the z-coordinate of P in the Cartesian coordinates.

The coordinate system above in (r, θ, z) is called the cylindrical coordinate system.



· Cylindrical coordinates 有助于计算此类三重积分:

$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

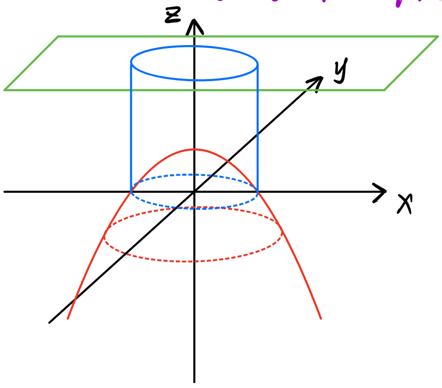
其中 D(积分区域E在 xy-平面上的投影)易于用极坐标表示.

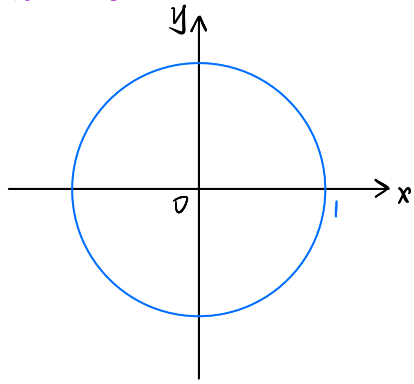
· 若D由 $\alpha \le \theta \le \beta$ $(\beta - \alpha \le 2\pi)$, $0 \le h_1(\theta) \le r \le h_2(\theta)$ 表示,则

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right) dA$$
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\, r\,dr\,d\theta$$

注: 将 xyz 转变为 cylindrical coordinates 之后, 需要将 dV 转变为 $r\,dz\,dr\,d\theta$

例: A solid E lies inside the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at a point $P(x,y)$ is $K\sqrt{x^2+y^2}$ for some constant K. Find mass(E).





Sol: Project on xy-plane

$$\text{Mass}(E) = \iint_D \left(\int_{1-x^2-y^2}^{4} K\sqrt{x^2+y^2}\,dz\right) dA$$
$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} Kr\,dz\,r\,dr\,d\theta$$
$$= 2\pi\int_0^1 Kr^2(3+r^2)\,dr$$
$$= \frac{12}{5}\pi K$$

例: **Exercise**

Use cylindrical coordinates to evaluate the integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$$

Sol: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\,dz\,r\,dr\,d\theta$

$= 2\pi\int_0^2 2r^3-r^4\,dr$

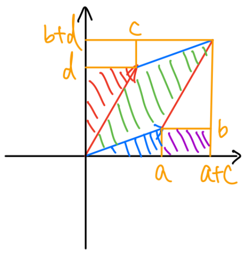
$= \frac{16}{5}\pi$

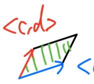
§4 Change of variable formula (Double Integrals)

1. Geometric properties of determinants

1° Fact: $\left|\det\begin{bmatrix} a & c \\ b & d\end{bmatrix}\right|$ 为 parallelogram (spanned by $\langle a,b\rangle$ and $\langle c,d\rangle$) 的面积

Consider the following figure:



Area of 

$= (a+c)(b+d) - 2(\tfrac{1}{2}ab) - 2(\tfrac{1}{2}cd) - 2(cb)$

$= ab + ad + bc + cd - ab - cd - 2cb$

$= ad - cb = \det\begin{bmatrix} a & c \\ b & d\end{bmatrix}$

2° Fact: 可以从行列式的同一行或同一列中提出相同的系数

$$\det\begin{bmatrix} \lambda a & \mu c \\ \lambda b & \mu d\end{bmatrix} = \lambda\det\begin{bmatrix} a & \mu c \\ b & \mu d\end{bmatrix} = \lambda\mu\det\begin{bmatrix} a & c \\ b & d\end{bmatrix}$$

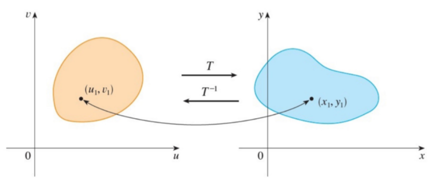
2. Change of variable formula

Suppose that

$$x = g(u,v) \quad\text{and}\quad y = h(u,v),$$

where g and h have continuous partial derivatives. We may understand this as a transformation T that maps a point in the uv-plane to a point in the xy-plane:

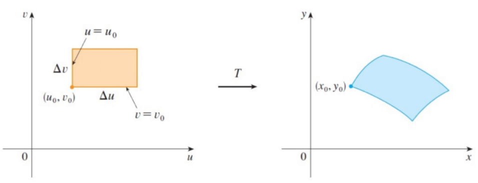
$$T(u,v) := (g(u,v), h(u,v)) = (x,y).$$



Let R be a region in the uv-plane, and let D be the image of R under the transformation T. That is,

$$D = T(R) := \{T(u,v) : (u,v)\in R\}.$$

Partition R into rectangles R_{ij}, and let D_{ij} be the image of R_{ij} under T.



Let $A(D_{ij})$ and $A(R_{ij})$ be the area of D_{ij} and R_{ij}, respectively.
When the norm of the partition is small,

$$A(D_{ij}) \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| A(R_{ij}) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u_i \Delta v_j,$$

absolute value of the Jacobian

where

Jacobian, which is a det

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

$\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

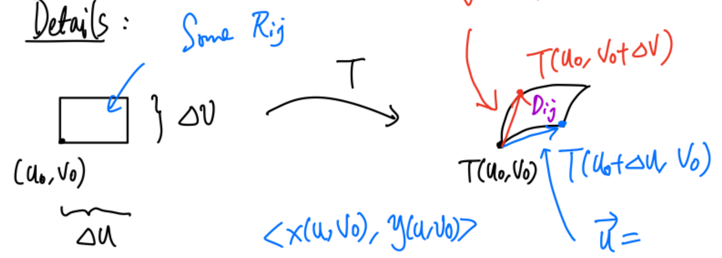
(雅可比行列式)

The determinant $\frac{\partial(x,y)}{\partial(u,v)}$ above is called the Jacobian of the transformation.

Details:

$\vec{v} = T(u_0, v_0+\Delta v) - T(u_0, v_0) = \vec{r}_2(v_0+\Delta v) - \vec{r}_2(v_0)$

$\approx \vec{r}_2{}'(v_0)\Delta v$



$\langle x(u,v_0), y(u,v_0)\rangle$

$\vec{u} = T(u_0+\Delta u, v_0) - T(u_0, v_0)$

$= \vec{r}_1(u_0+\Delta u) - \vec{r}_1(u_0)$

$\approx \vec{r}_1{}'(u_0)\Delta u$

Fix v_0, let $\vec{r}_1(u) := T(u, v_0)$

Fix u_0, let $\vec{r}_2(v) := T(u_0, v)$

$\langle x(u_0,v), y(u_0,v)\rangle$

Hence $A(D_{ij}) \approx \text{Area}(\vec{v}, \vec{u}) = \left|\det\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right|$

$= \left|\det\begin{bmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{bmatrix}\right| = \Delta u \Delta v \left|\underbrace{\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}_{\frac{\partial(x,y)}{\partial(u,v)}}\right|$

Evaluated at (u_0, v_0) "Jacobian of T"

3. Theorem

In terms of differentials, one can memorize the above as

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

Theorem (Change of Variables in Double Integrals)

Let f be continuous on a region D in the xy-plane. Suppose that

- *the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ maps a region R in the uv-plane onto D, and;*
- *the transformation is one-to-one in the interior of R, and;*
- *all partial derivatives of g and h are continuous.*

Then

$$\iint_D f(x,y)\, dA = \iint_R f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

例: Polar transformation

If a change of variables is given by the transformation

$$x = g(r,\theta) = r\cos\theta \quad \text{and} \quad y = h(r,\theta) = r\sin\theta,$$

then it follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \geqslant 0,$$

which gives

$$\iint_D f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

This is consistent with the formula we stated in ~~Week 9~~ last week.

To satisfy the one-to-one condition, here we restrict to $r \geq 0$, i.e., domain of T is $\{(r,\theta):\ r\geq 0,\ \theta \text{ is in an interval with length} = 2\pi\}$

例:

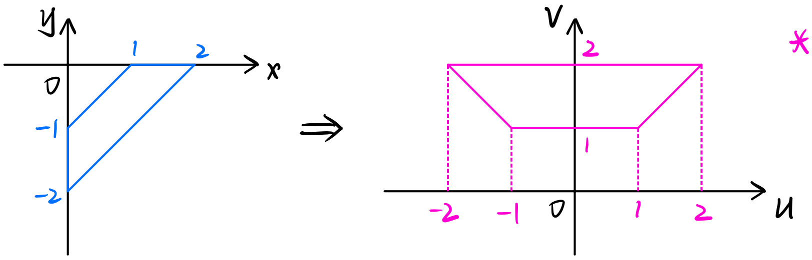
Example

Evaluate the following integrals.

(a) $\iint_D e^{(x+y)/(x-y)}dA$, where D is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

(b) $\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}e^{\sqrt{xy}}\,dx\,dy$.

(a)



* 转换方法

① 转换端点，之后连接

② 转换曲线，利用线性规划

- Suppose $u = x+y$, $v = x-y$

$\Rightarrow x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

- $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

So $\iint_D e^{\frac{x+y}{x-y}}dA = \iint_R e^{u/v}\left|-\frac{1}{2}\right|du\,dv$

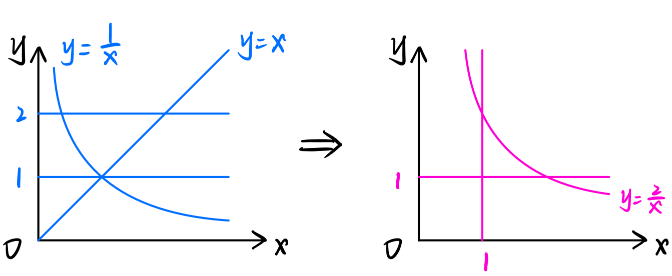
$= \frac{1}{2}\int_1^2\int_{-v}^{v} e^{\frac{u}{v}}du\,dv$

$= \frac{1}{2}\int_1^2 \left(ve^{\frac{u}{v}}\Big|_{-v}^{v}\right)dv$

$= \frac{1}{2}\int_1^2 (ve - ve^{-1})\,dv$

$= \frac{3}{4}(e - e^{-1})$

(b)



- Suppose $u=\sqrt{xy}$, $v=\sqrt{\frac{y}{x}}$

  $\Rightarrow y=uv$, $x=\frac{u}{v}$

- $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v} > 0$

- Boundary

D	R
$x=y$	$v=1$
$xy=1$	$u=1$
$y=2$	$v=\frac{2}{u}$

- $I = \iint_D \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dA$

  $= \int_1^2 \int_1^{\frac{2}{u}} ve^u \cdot \frac{2u}{v} dvdu$

  $= \int_1^2 (\frac{2}{u}-1) e^u \cdot 2u \, du$

  $= 2e(e-2)$

※ 三重积分的计算技巧

1. 利用对称性化简

   - 三元函数奇偶性
     - 若 $f(x,y,z)$ 满足 $f(-x,y,z) = f(x,y,z)$，则称 $f(x,y,z)$ 是关于 x 的偶函数.
     - 若 $f(x,y,z)$ 满足 $f(-x,y,z) = -f(x,y,z)$，则称 $f(x,y,z)$ 是关于 x 的奇函数.

     同理得另外两个变量奇偶性的定义

   - 三重积分中关于对称性的结论
     - 若积分区域 Ω 关于 yOz 坐标面对称，且 $f(x,y,z)$ 是关于 x 的奇函数，

       则 $\iiint_\Omega f(x,y,z) dxdydz = 0$

     - 若积分区域 Ω 关于 yOz 坐标面对称，且 $f(x,y,z)$ 是关于 x 的偶函数，

       则 $\iiint_\Omega f(x,y,z) dxdydz = 2\iiint_{\Omega_1} f(x,y,z) dxdydz$

       其中 Ω_1 为 Ω 位于 yOz 坐标面一侧（左侧，$x>0$）的部分

2. 可利用质心公式化简

   $\iiint_\Omega x \, dxdydz = \bar{x} \cdot \iiint_\Omega dxdydz = \bar{x} \cdot V_\Omega$（若 $\bar{x}$ 与 V_Ω 显而易见，可使用此方法）