Lecture 18

§1 关于 Differentiation 的 facts（接上）

1. Fact 11: 偏导连续 $\Rightarrow$ total differentiability

Suppose f: open $\Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ has continuous partial derivatives (i.e. $\frac{\partial f_i}{\partial x_j}(x_1,\cdots,x_n)$ exists & continuous on Ω, $i=1,\cdots,m$, $j=1,\cdots,n$)

Then $\forall c \in \Omega$, f is (totally) differentiable at c.

证明:

$$\text{W.T.S. } f(c+h) = f(c) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(c) & \frac{\partial f_1}{\partial x_2}(c) & \cdots & \frac{\partial f_1}{\partial x_n}(c) \\ \frac{\partial f_2}{\partial x_1}(c) & \frac{\partial f_2}{\partial x_2}(c) & \cdots & \frac{\partial f_2}{\partial x_n}(c) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(c) & \frac{\partial f_m}{\partial x_2}(c) & \cdots & \frac{\partial f_m}{\partial x_n}(c) \end{bmatrix}_{m\times n} h + o(h) \quad \text{as } h \to 0$$

Suffice to show:

$$f_i(c+h) = f_i(c) + \nabla f_i(c)\cdot h + o(h) \quad \text{as } h\to 0 \quad i=1,2,\cdots,n$$

To save notation, write f_i as f

$$\begin{aligned}
& f(c+h) - f(c) - \nabla f(c) h \\
= \ & f(c_1+h_1, c_2+h_2, \cdots, c_n+h_n) - f(c_1,c_2,\cdots,c_n) \\
& - \frac{\partial f}{\partial x_1}(c_1,c_2,\cdots c_n)\cdot h_1 - \frac{\partial f}{\partial x_2}(c_1,c_2,\cdots c_n)\cdot h_2 - \cdots - \frac{\partial f}{\partial x_n}(c_1,c_2,\cdots c_n)\cdot h_n \\
= \ & f(c_1+h_1, c_2+h_2, \cdots, c_n+h_n) - f(c_1, c_2+h_2, \cdots, c_n+h_n) \\
& + f(c_1, c_2+h_2, \cdots, c_n+h_n) - f(c_1, c_2, c_3+h_3, \cdots, c_n+h_n) \\
& + f(c_1, c_2, c_3+h_3, \cdots, c_n+h_n) - f(c_1, c_2, c_3, c_4+h_4, \cdots, c_n+h_n) \\
& \qquad \cdots\cdots \\
& + f(c_1, c_2, \cdots, c_{n-1}, c_n+h_n) - f(c_1, c_2, \cdots, c_n) \\
& - \frac{\partial f}{\partial x_1}(c_1,c_2,\cdots c_n)\cdot h_1 - \frac{\partial f}{\partial x_2}(c_1,c_2,\cdots c_n)\cdot h_2 - \cdots - \frac{\partial f}{\partial x_n}(c_1,c_2,\cdots c_n)\cdot h_n \\
\overset{\text{M.V.T.}}{=} \ & \frac{\partial f}{\partial x_1}(\tilde{c}_1, c_2+h_2, \cdots, c_n+h_n)\cdot h_1 \\
& + \frac{\partial f}{\partial x_2}(c_1, \tilde{c}_2, c_3+h_3, \cdots, c_n+h_n)\cdot h_2 \\
& + \cdots \\
& + \frac{\partial f}{\partial x_n}(c_1, c_2, \cdots, c_{n-1}, \tilde{c}_n) h_n \\
& - \frac{\partial f}{\partial x_1}(c_1,c_2,\cdots c_n)\cdot h_1 - \frac{\partial f}{\partial x_2}(c_1,c_2,\cdots c_n)\cdot h_2 - \cdots - \frac{\partial f}{\partial x_n}(c_1,c_2,\cdots c_n)\cdot h_n
\end{aligned}$$

($\tilde{c}_i$ between c_i+h_i and c_i)

$$\begin{aligned}
= \ & \left[\frac{\partial f}{\partial x_1}(\tilde{c}_1, c_2+h_2, \cdots, c_n+h_n) - \frac{\partial f}{\partial x_1}(c_1,c_2,\cdots c_n)\right]\cdot h_1 \\
& + \left[\frac{\partial f}{\partial x_2}(c_1, \tilde{c}_2, c_3+h_3, \cdots, c_n+h_n) - \frac{\partial f}{\partial x_2}(c_1,c_2,\cdots c_n)\right]\cdot h_2 \\
& + \cdots \\
& + \left[\frac{\partial f}{\partial x_n}(c_1, c_2, \cdots, c_{n-1}, \tilde{c}_n) - \frac{\partial f}{\partial x_n}(c_1,c_2,\cdots c_n)\right]\cdot h_n \\
= \ & o(1)\cdot h_1 + o(1)\cdot h_2 + \cdots + o(1)\cdot h_n \quad \text{as } h\to 0
\end{aligned}$$

(由于偏导连续，$\frac{\partial f}{\partial x_1}(\tilde{c}_1, c_2+h_2, \cdots, c_n+h_n) - \frac{\partial f}{\partial x_1}(c_1, c_2, \cdots, c_n) \to 0$ as $h \to \infty$)

$= o(1)|h| + o(1)|h| + \cdots + o(1)|h|$ as $h \to 0$ (由于 $|h_i| \le |h|$)

$= o(1)|h|$ as $|h| \to 0$

Therefore, $f(c+h) - f(c) - Df(c) \cdot h = o(|h|)$ as $h \to 0$

$\therefore$ f total differentiable at c

Q.E.D.

2. **Definition:** C^k- smooth

- 我们称 f: open $D \subset R^n \to R^m$ 为 C^1-smooth on D.
  若所有 $\frac{\partial f}{\partial x_i}$ 存在且连续 on Ω
- 我们称 f: open $D \subset R^n \to R^m$ 为 C^2-smooth on D.
  若 f 为 C^1-smooth on D，且所有 second partials $\frac{\partial^2 f}{\partial x_i \partial x_m}$ 存在且连续 on Ω
- 类似地，可以定义 C^k-smooth ($k \ge 1$)，C^∞-smooth
- $C^1(D)$ = real-valued functions which are C^1-smooth
  注：$C^1(D)$ 可理解为 vector space；$f \in C^1(D)$，$g \in C^1(D)$，$c_1 f + c_2 g \in C^1(D)$

3. **Lemma:**

Let $f(x,y)$ be real-valued & defined near $(x_0, y_0) \in R^2$ ($f(x_0,y_0)$ may not defined) & $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A$ (finite #).

Assume $\forall y \approx y_0$, $y \ne y_0$, $\lim_{x \to x_0} f(x,y)$ exists as a finite, denoted as $g(y)$.

Then $\lim_{y \to y_0} g(y)$ exists and equals to A

即 $\lim_{y\to y_0}(\lim_{x \to x_0} f(x,y)) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$

证明:

$\because \lim_{(x,y)\to(x_0,y_0)} f(x,y) = A$

$\therefore$ $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. as long as $0 < |(x,y) - (x_0,y_0)| < \delta$, $|f(x,y) - A| < \varepsilon$

Now fix $y \approx y_0$, $y \ne y_0$, s.t. $|y - y_0| < \frac{\delta}{2}$

Then $|f(x,y) - A| < \varepsilon$ for all x close to x_0, $x \ne x_0$

Then sending $x \to x_0$,

$$\lim_{x\to x_0} |f(x,y) - A| \le \lim_{x \to x_0} \varepsilon$$

$$\Rightarrow |g(y) - A| \le \varepsilon$$

$$\Rightarrow \lim_{y\to y_0} g(y) = A$$

Q.E.D.

4. **Fact 12:** Clairaut's theorem (克劳莱定理)

Let f: open $D \subset R^n \to R$, $f \in C^1(D)$, Suppose $\frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})$ exists and continuous on D.

Then $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})$ also exists and equals to $\frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})$ in D

证明:

WLOG, assume f is a function of $(x,y) \in D \subset \mathbb{R}^2$

Know: $\forall c \in D$, $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(c)$ exists $= A$

$$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(c) = \frac{d}{dy}(\frac{\partial f}{\partial x}(c_1, y))\Big|_{y=c_2}$$

$$= \lim_{k \to 0} \frac{\frac{\partial f}{\partial x}(c_1, c_2+k) - \frac{\partial f}{\partial x}(c_1, c_2)}{k} = A$$

W.T.S. $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(c)$ exists $= A$

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(c) = \frac{d}{dx}(\frac{\partial f}{\partial y}(x, c_2))\Big|_{x=c_1}$$

$$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(c_1+h, c_2) - \frac{\partial f}{\partial y}(c_1, c_2)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\lim_{k \to 0}\left(\frac{f(c_1+h, c_2+k) - f(c_1+h, c_2)}{k} - \frac{f(c_1, c_2+k) - f(c_1, c_2)}{k}\right)\right]$$

$$= \lim_{h \to 0}\lim_{k \to 0} \frac{1}{hk}\left[f(c_1+h, c_2+k) - f(c_1+h, c_2) - f(c_1, c_2+k) + f(c_1, c_2)\right]$$

Let $F(h,k) = \frac{1}{hk}\left[f(c_1+h, c_2+k) - f(c_1+h, c_2) - f(c_1, c_2+k) + f(c_1, c_2)\right]$ defined for $(h,k) \approx 0$, $h \neq 0$, $k \neq 0$

$F(h,k)$ satisfies:

- For each fixed $h \approx 0$, $h \neq 0$

$$\lim_{k \to 0} F(h,k) = \frac{1}{h}\left[\frac{\partial f}{\partial y}(c_1+h, c_2) - \frac{\partial f}{\partial y}(c_1, c_2)\right]$$

- $$\lim_{\substack{(h,k)\to(0,0)\\ h\neq 0, k\neq 0}} F(h,k) \overset{\text{M.V.T.}}{=} \lim_{\substack{(h,k)\to(0,0)\\ h\neq 0, k\neq 0}} \frac{1}{hk}\left[\frac{\partial f}{\partial x}(\tilde{c}_1, c_2+k)h - \frac{\partial f}{\partial x}(\tilde{c}_1, c_2)h\right]$$

$$= \lim_{\substack{(h,k)\to(0,0)\\ h\neq 0, k\neq 0}} \frac{1}{k}\left[\frac{\partial f}{\partial x}(\tilde{c}_1, c_2+k) - \frac{\partial f}{\partial x}(\tilde{c}_1, c_2)\right]$$

$$\overset{\text{M.V.T.}}{=} \lim_{\substack{(h,k)\to(0,0)\\ h\neq 0, k\neq 0}} \frac{1}{k}\frac{\partial f}{\partial y}(\frac{\partial}{\partial x})(\tilde{c}_1, \tilde{c}_2)\cdot k$$

$$= \lim_{\substack{(h,k)\to(0,0)\\ h\neq 0, k\neq 0}} \frac{\partial f}{\partial y}(\frac{\partial}{\partial x})(\tilde{c}_1, \tilde{c}_2)$$

$$= \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(c_1, c_2) = A$$

$\therefore$ By lemma, $\lim_{h \to 0}(\lim_{k \to 0} F(h,k))$ exists & $= A$

$\therefore$ $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(c)$ exists & $= A$

注: 有时 $\frac{\partial^2 f}{\partial x \partial y}(c) \neq \frac{\partial^2 f}{\partial y \partial x}(c)$

e.g. $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$

$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$

$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$