Lecture 19

§1 Queueing system

1. M/M/1 queue

① State space: $X(t) = t$ 时刻 system 中的 customers 数

② Stationary distribution: 当 $\lambda < \mu$ 时，有

$$\pi_i = (1-\rho)\rho^i, \quad i = 0, 1, 2, \cdots$$

其中，$\rho = \frac{\lambda}{\mu}$ 被称为 load / utilization

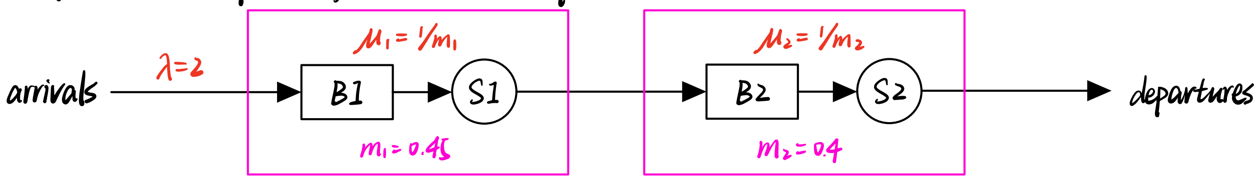
③ 每个 customer 的平均等待时间为

$$W_q = m\frac{\rho}{1-\rho}$$

其中 $m = 1/\mu$ 为 mean service times.

2. 2-station tandem queue (Station k = buffer k + server k)

其中一个 M/M/1 queue feeds another queue:



① 求每个 server 的 utilization:

- $\rho_1 = \frac{\lambda}{\mu_1} = \lambda \cdot m_1 = 0.9$
- $\rho_2 = \frac{\lambda}{\mu_2} = \lambda \cdot m_2 = 0.8$ (in the long run 是这样的: arrival rate 为 λ)

② 求 system 的 throughput:

λ

③ 2与3个 customers 分别位于 upstream 与 downstream stations 的 proportion of time.

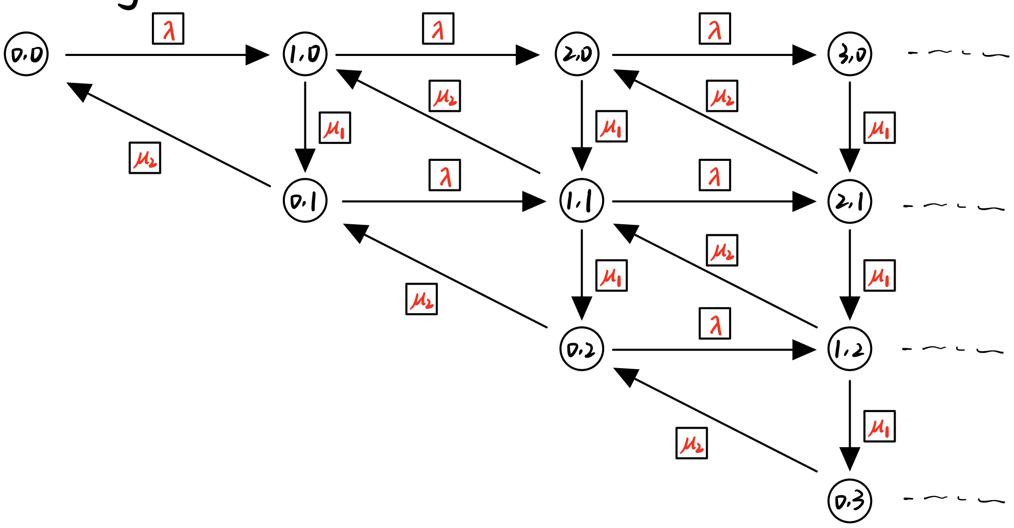
$\pi(2,3)$

3. Stationary distribution 的求解方法

① State space:

State $= (i_1, i_2)$ 其中 $i_k =$ station k 中的 customers 数

② Rate diagram:



③ 求解方法：Guess & Check

1° 由于 CTMC irreducible $\Rightarrow$ 至多有一个 stationary distribution

因此我们仅需"guess"出一个满足

$$\vec{\pi} G = \vec{0} \quad \& \quad \sum_i \pi_i = 1$$

的 π 即可

2° Guess 的思路：

考虑 2-station tandem queue 可被拆解为 2个 independent M/M/1 queue，即

$$\pi_{(i_1,i_2)} = \pi_{(i_1)} \cdot \pi_{(i_2)} = (1-\rho_1)\rho_1^{i_1}(1-\rho_2)\rho_2^{i_2}$$

其中 $i_1, i_2 \in \{0, 1, \cdots\}$，$\rho_k = \frac{\lambda}{\mu_k}$，$k = 1, 2$.

3° Check 的思路：

(1) check 是否 rate out = rate in

(2) check 是否 sums 为 1

e.g. check $\pi(2,3)$ 是否满足 rate out = rate in

W.T.S. $(\lambda+\mu_1+\mu_2)\pi(2,3) = \lambda\pi(1,3) + \mu_2\pi(2,4) + \mu_1\pi(3,2)$

$\Longleftrightarrow$ $(\lambda+\mu_1+\mu_2)\rho_1^2(1-\rho_1)\rho_2^3(1-\rho_2)$ (若$\pi_{(i_1,i_2)}$可被拆解，则 $\pi_{(i_1,i_2)} = (1-\rho_1)\rho_1^{i_1}(1-\rho_2)\rho_2^{i_2}$)

$= \lambda\rho_1(1-\rho_1)\rho_2^3(1-\rho_2) + \mu_2\rho_1^2(1-\rho_1)\rho_2^4(1-\rho_2) + \mu_1\rho_1^3(1-\rho_1)\rho_2^2(1-\rho_2)$

$\Longleftrightarrow$ $(\lambda+\mu_1+\mu_2)\rho_1^2\rho_2^3$

$= \lambda\rho_1\rho_2^3 + \mu_2\rho_1^2\rho_2^4 + \mu_1\rho_1^3\rho_2^2$

$\Longleftrightarrow$ $\lambda\rho_1^2\rho_2^3 + \mu_1\rho_1^2\rho_2^3 + \mu_2\rho_1^2\rho_2^3$

$= \lambda\rho_1\rho_2^3 + \mu_2\rho_1^2\rho_2^4 + \mu_1\rho_1^3\rho_2^2$

注意到 $\rho_i = \frac{\lambda}{\mu_i}$ 与 $\lambda = \rho_i\mu_i$，有

- $\lambda\rho_1^2\rho_2^3 = (\rho_2\mu_2)\rho_1^2\rho_2^3 = \mu_2\rho_1^2\rho_2^4$
- $\mu_1\rho_1^2\rho_2^3 = \mu_1 \cdot \frac{\lambda}{\mu_1} \cdot \rho_1 \cdot \rho_2^3 = \lambda\rho_1\rho_2^3$
- $\mu_2\rho_1^2\rho_2^3 = \mu_2 \cdot (\frac{\lambda}{\mu_2})\rho_1^2\rho_2^2 = \mu_1\rho_1\rho_1^2\rho_2^2 = \mu_1\rho_1^3\rho_2^2$

完成证明

Theorem

The stationary distribution π of the 2-station tandem queue is given by

$$\pi_{(i_1,i_2)} = \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_1}\right)^{i_1}\left(1-\frac{\lambda}{\mu_2}\right)\left(\frac{\lambda}{\mu_2}\right)^{i_2}$$
$$= (1-\rho_1)\rho_1^{i_1}(1-\rho_2)\rho_2^{i_2}$$

for $i_1, i_2 \in \{0, 1, \ldots\}$, $\rho_k = \lambda/\mu_k$, $k = 1, 2$.