Lecture 17

§1 关于 convex optimization 的 terminologies

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \le 0, i = 1, \dots, m \\ &&& h_i(x) = 0, i = 1, \dots, n \end{aligned}$$

1. Feasible set (可行集)

满足所有 constraints 的所有点的集合

2. Global minimizer (最小值点)

若对 feasible set 内的任意 y，有 $f(x^*) \le f(y)$，则 x^* 为 global minimizer

3. Local minimizer (极小值点)

- 令 S 表示 feasible set
- 令 $B(x, \varepsilon) = \{y : \|y - x\| \le \varepsilon\}$
- 若存在一个 ε 使得对 $\forall y \in S \cap B(x, \varepsilon)$，有 $f(x) \le f(y)$. 则 x 被称作 local minimizer of the optimization problem.

注：global minimizer 同时也是 local minimizer

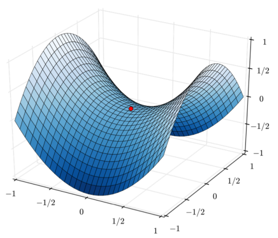
4. saddle point (鞍点)

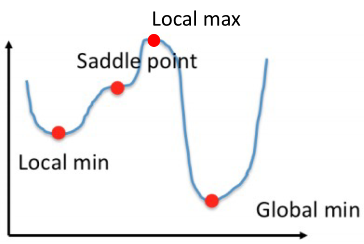
若点 x 满足：

① $\left.\dfrac{d f(x + t\vec{e})}{dt}\right|_{t=0} = 0$ for any $\vec{e}$

② 不是 local minimizer / maximizer.

则 x 被称作 saddle point.





§2 Convex optimization definition

1. convex optimization 的性质

任一 local minimizer 同时也是 global minimizer

2. convex optimization 的前提

1° feasible set 为一个 convex set (凸集)

2° objective function 为一个 convex function (凸函数)

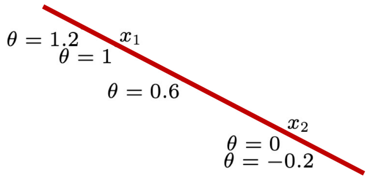
§3 Convex set

1. preparation - Line

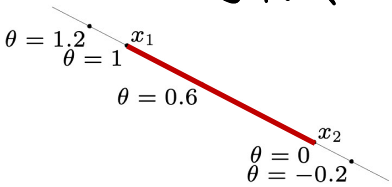
- 令 $x_1 \ne x_2$ 为 R^n 空间内两点，则形式为

$x = \theta x_1 + (1-\theta) x_2$, where $\theta \in R$

的点,构成了经过 x_1 与 x_2 的 line



- 若 $\theta \in [0,1]$, 则构成 x_1 与 x_2 之间的 line segment (线段)



2. Convex set

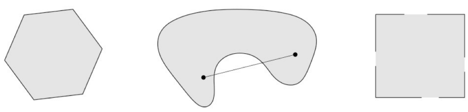
对于集合 C, 若 C 内任意两点形成的线段均落在 C 内, 即

$$\forall x_1, x_2 \in C, \ 0 \le \theta \le 1 \Rightarrow \theta x_1 + (1-\theta) x_2 \in C$$

则 C 为一个 convex set.

Examples

- Which set is a convex set?



A ✓ B ✗ C ✗

3. Convex set examples

1° empty set $\emptyset$, singleton set $\{x_0\}$, complete space R

2° Line $\{(x,y): y = ax+b\}$

3° Halfspace, e.g. $\{(x,y): y \le ax+b\}$

4° Balls $\{(x,y): (x-x_0)^2 + (y-y_0)^2 \le r\}$ $(r \ge 0)$; Ellipsoids

5° Polyhedron: intersection of a finite set of halfspaces

- $\{(x,y): y \le a_i x + b_i \text{ for all } i\}$

证明:

2° For any (x_1, y_1) and (x_2, y_2) in the line

- $y_1 = ax_1 + b$

  $y_2 = ax_2 + b$

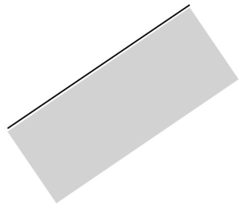
- The for any $\theta \in [0,1]$

  - $\theta y_1 = a\theta x_1 + \theta b$ and $(1-\theta) y_2 = a(1-\theta) x_2 + (1-\theta) b$

    $\theta (x_1, y_1) + (1-\theta)(x_2, y_2) = (\theta x_1 + (1-\theta) x_2, \ \theta y_1 + (1-\theta) y_2)$

  - $\theta y_1 + (1-\theta) y_2 = a[\theta x_1 + (1-\theta) x_2] + b$

3° For any (x_1, y_1) and (x_2, y_2) in the line



- $y_1 \le a x_1 + b$

  $y_2 \le a x_2 + b$

- Then for any $\theta \in [0, 1]$

  - $\theta y_1 \le a \theta x_1 + \theta b$ and $(1-\theta) y_2 \le a (1-\theta) x_2 + (1-\theta) b$

    $\theta (x_1, y_1) + (1-\theta)(x_2, y_2) = (\theta x_1 + (1-\theta) x_2, \theta y_1 + (1-\theta) y_2)$

  - $\theta y_1 + (1-\theta) y_2 \le a [\theta x_1 + (1-\theta) x_2] + b$

5° 引理 (Lemma):

若 S_1 与 S_2 均为 convex sets，则 $S_1 \cap S_2$ 也为 convex set.

证明:

- Given any two points x_1 and x_2 in $S_1 \cap S_2$

  Let x be a point on the line segment between x_1 and x_2.

- As S_1 is convex set, x is within S_1

  As S_2 is convex set, x is within S_2

- Thus, x is within $S_1 \cap S_2$

由引理易证得 5°.