

Lecture 20

§1 Inverse function theorem

Purpose: Given $y = f(x)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. Want to solve x in terms of y : $x = g(y)$

1. Theorem: Inverse function theorem

Let Ω be open in \mathbb{R}^n , $f: \Omega \rightarrow \mathbb{R}^n$ be C^1 -smooth.

Suppose $\exists a \in \Omega$ s.t. Jacobian $(f'(a))_{n \times n}$ is non-singular. Let $b = f(a)$

Then

(i) \exists open sets U & V in \mathbb{R}^n , s.t. $a \in U$, $b \in V$

f is one-to-one on U & $f(U) = V$

(ii) (由(i)可知 $(f|_U)^{-1}$ 在 V 上 locally 存在)

Let $g(y) = (f|_U)^{-1}(y)$, $y \in V$. Then

g is also C^1 -smooth on V , and $(g'(y))_{n \times n} = (f'(x))_{n \times n}^{-1}$, x & y are related by $y = f(x)$

f 被称为 local diffeomorphism from U to V

注: 对于 function of one variable $g(x) = f^{-1}(x)$, $g'(x) = \frac{1}{f'(x)}$

2. Theorem: contraction mapping theorem (Preparations for proving Inverse function theorem)

Let A be a closed subset of \mathbb{R}^n .

Suppose $f: A \rightarrow A$ satisfies: $\forall x, y \in A$, $|f(x) - f(y)| \leq \lambda |x - y|$ for some constant $\lambda \in (0, 1)$

Then \exists ! (unique) $x_\infty \in A$, s.t. $f(x_\infty) = x_\infty$ (fixed point of f)

注: Brouwer's fixed point theorem:

$f: [a, b] \rightarrow [a, b]$, $\exists x_\infty$ s.t. $f(x_\infty) = x_\infty$

证明:

Pick $x_0 \in A$, let $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_{k+1} \stackrel{\text{def}}{=} f(x_k)$ (*), $\forall k \geq 0$

W.T.S. $x_k \rightarrow$ some x_∞ as $k \rightarrow \infty$

Just need to show $\sum_{k=0}^{\infty} (x_{k+1} - x_k)$ converges

Just need to show $\sum_{k=0}^{\infty} |x_{k+1} - x_k|$ converges

$$|x_{k+1} - x_k| = |f(x_k) - f(x_{k-1})|$$

$$\leq \lambda |x_k - x_{k-1}|$$

$$= \lambda |f(x_{k-1}) - f(x_{k-2})|$$

$$\leq \lambda^2 |x_{k-1} - x_{k-2}|$$

$$\leq \lambda^k |x_1 - x_0|$$

By geometric series $0 < \lambda < 1$, $\sum_{k=0}^{\infty} |x_{k+1} - x_k|$ converges

$\therefore A$ is closed and $x_k \in A$

$\therefore x_\infty \in A$

$$\therefore 0 \leq |f(x_k) - f(x_\infty)| \leq \lambda |x_k - x_\infty| \rightarrow 0 \text{ as } k \rightarrow \infty$$

$$\therefore f(x_k) \rightarrow f(x_\infty) \text{ as } k \rightarrow \infty$$

$$\text{By } (*), \lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} f(x_k)$$

$$\Rightarrow x_\infty = f(x_\infty)$$

Suppose f has another fixed point y_∞ (证明 uniqueness)

$$\Rightarrow |f(x_\infty) - f(y_\infty)| \leq \lambda |x_\infty - y_\infty|$$

$$\stackrel{||}{|x_\infty - y_\infty|} \leq \lambda |x_\infty - y_\infty|$$

$$\Rightarrow |x_\infty - y_\infty| = 0$$

$$x_\infty = y_\infty \text{ (contradiction)}$$

3. Prove of Inverse function theorem

1° Discussion: solve for x from $y = f(x)$

$$y = f(x) = f(a) + f'(a)_{n \times n}(x-a) + o(|x-a|), \quad x \approx a$$

$$\Rightarrow y - f(a) - o(|x-a|) = f'(a)_{n \times n}(x-a)$$

$$\Rightarrow A^{-1}(y - f(a) - o(|x-a|)) = x - a \quad (\text{令 } A = f'(a)_{n \times n})$$

$$\Rightarrow x = a + A^{-1}(y - f(a) - o(|x-a|))$$

$$= a + A^{-1}[y - f(a) - f(x) + f(a) + A(x-a)]$$

$$= A^{-1}(y - f(x)) + x$$

令 $A^{-1}(y - f(x)) + x = \phi(x)$, 可考虑使用 contraction mapping theorem

2° 证明:

Take small & closed ball \tilde{U} centered at a .

$\tilde{U} \subset \Omega$. Fix $y \in \mathbb{R}^n$.

Claim 1: If \tilde{U} is small enough, then ϕ satisfies $\forall x_1, x_2 \in \tilde{U}, |\phi(x_1) - \phi(x_2)| \leq \frac{1}{2} |x_2 - x_1|$.

The smallness of \tilde{U} is independent of y

$$|\phi(x_2) - \phi(x_1)| \stackrel{MVT}{=} M |x_2 - x_1|, \quad M \geq \|\phi'(x)\|, \quad \forall x \in \tilde{U}$$

$$\phi'(x) = (A^{-1}(y - f(x)) + x)'$$

$$= -A^{-1}f'(x) + I$$

$$\rightarrow -A^{-1}f'(a) + I, \text{ as } x \rightarrow a \text{ (由于 } f \text{ } C^1\text{-smooth)}$$

$$= -A^{-1}A + I$$

$$= 0$$

$$\therefore \phi'(x) \rightarrow 0_{n \times n} \text{ as } x \rightarrow a$$

$$\Rightarrow \text{If } \tilde{U} \text{ small enough, then } \|\phi'(x)\| \leq \frac{1}{2}, \quad \forall x \in \tilde{U}$$

$$\text{Take } M = \frac{1}{2} \Rightarrow \text{Claim 1}$$

Now take V to be a small nbhd of $b = f(a)$

Claim 2: \forall fixed $y \in V$, ϕ maps $\tilde{U} \rightarrow \tilde{U}$

W.T.S. $\forall x \in U, |\phi(x) - a| \leq \text{radius of } U$

$$|\phi(x) - a| = |A^{-1}(y - f(x)) + x - a|$$

$$= |A^{-1}[y - (f(a) + A(x-a) + o(|x-a|))] + x - a|$$

$$= |A^{-1}[y - f(a)] - A^{-1}o(|x-a|)|$$

$$\leq |A^{-1}[y - f(a)]| + |A^{-1}o(1)| \cdot |x-a|$$

$$\leq \|A^{-1}\| \cdot |y - f(a)| + \|A^{-1}\| \cdot o(1) \cdot |x-a|$$

$$\leq \|A^{-1}\| \cdot (\text{radius of } V) + \|A^{-1}\| \cdot o(1) \cdot (\text{radius of } \tilde{U})$$

$$\leq \frac{1}{2} \text{ radius of } \tilde{U} \quad (\text{if } V \text{ is taken small enough}) \quad (*)$$

By Claim of 1 and 2, we can use contraction mapping theorem to ϕ on \tilde{U}

$$\Rightarrow \exists! x \text{ (x_{\infty})} \in \tilde{U} \text{ s.t. } \phi(x) = A^{-1}(y - f(x)) + x = x$$

$$\Rightarrow A^{-1}(y - f(x)) = 0$$

$$\Rightarrow y = f(x)$$

In summary: $\forall y \in V, \exists! x \in \tilde{U} \text{ s.t. } f(x) = y$

Claim 3: This $x \notin \partial \tilde{U}$

$$\text{By } (*), |\phi(x) - a| \leq \frac{1}{2} \text{ radius of } \tilde{U}$$

$$\therefore |x - a| \leq \frac{1}{2} \text{ radius of } \tilde{U}$$

$$\therefore x \notin \partial \tilde{U}$$

$\therefore f$ is C^1 -smooth

$\therefore f$ is continuous on Ω

$\therefore f^{-1}(V)$ is open in \mathbb{R}^n (by V open)

$\therefore f^{-1}(V) \cap \overset{\circ}{\tilde{U}}$ is open

$$\text{Let } U = f^{-1}(V) \cap \overset{\circ}{\tilde{U}}$$

W.T.S. f 1-1 on U

$\forall x_1, x_2 \in U \subset \tilde{U}$, if $f(x_1) = f(x_2) = \text{some } y$

$$\Rightarrow \phi(x_1) = A^{-1}(y - f(x_1)) + x_1$$

$$\phi(x_2) = A^{-1}(y - f(x_2)) + x_2$$

$$\text{By Claim 1, } |\phi(x_2) - \phi(x_1)| = |x_2 - x_1| \leq \frac{1}{2} |x_2 - x_1|$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow f$ is 1-1 on U

W.T.S. $f(U) = V$

Recall: $\forall y \in V, \exists! x \in \tilde{U} \text{ s.t. } f(x) = y$

$$\Rightarrow x \in f^{-1}(V)$$

$$\Rightarrow x \in \tilde{U} \cap f^{-1}(V)$$

But by Claim 3, $x \notin \partial \tilde{U}$

$$\Rightarrow x \in \dot{U} \cap f^{-1}(V) = U$$

$$\Rightarrow f(U) = V$$

$$\text{Let } g = (f|_U)^{-1} : V \rightarrow U$$

$$\text{Let } x+h = g(y+k) \in U$$

$$\text{Claim 4: } |h| \leq 2 \|A^{-1}\| |k|$$

Recall from Claim 1:

$$|\phi(x+h) - \phi(x)| \leq \frac{1}{2} |h|$$

$$\Rightarrow |A^{-1}(y - f(x+h)) + x+h - A^{-1}(y - f(x)) - x| \leq \frac{1}{2} |h|$$

$$\Rightarrow |h + A^{-1}(f(x) - f(x+h))| \leq \frac{1}{2} |h|$$

$$\Rightarrow |h + A^{-1}(y - (y+k))| \leq \frac{1}{2} |h|$$

$$\Rightarrow |h - A^{-1}k| \leq \frac{1}{2} |h|$$

$$\Rightarrow |h| - |A^{-1}k| \leq \frac{1}{2} |h|$$

$$\Rightarrow |h| \leq 2 \|A^{-1}\| |k|$$

$$\forall y, y+k \in V, \text{ W.T.S. } g(y+k) = g(y) + (f'(x))^{-1}k + o(|k|) \text{ as } k \rightarrow 0 \text{ where } f(x)=y \text{ (}\# \text{)}$$

$\therefore f$ is differentiable at x

$$\therefore f(x+h) - f(x) = f'(x)h + o(|h|) \text{ as } h \rightarrow 0$$

$$\Rightarrow y+k - y = f'(x)h + o(|h|)$$

$$\Rightarrow h = (f'(x))^{-1}(k - o(|h|))$$

$$= (f'(x))^{-1}(k - o(|k|)) \text{ as } k \rightarrow 0 \text{ (由 Claim 4)}$$

$$(\#) \Leftrightarrow x+h = x + (f'(x))^{-1}k + o(|k|) \text{ as } k \rightarrow 0$$

$$\Leftrightarrow h = (f'(x))^{-1}k + o(|k|) \text{ as } k \rightarrow 0$$

$\therefore (\#)$ 得证

$$\text{Also by } (\#), g'(y) = (f'(x))^{-1} = (f'(g(y)))^{-1}, y \in V$$

W.T.S. g' is continuous in $y \in V$

Just need to show $(f'(x))^{-1}$ is continuous in $x \in U$

(由 f is C^1 -smooth 可知, $f'(x)$ continuous in $x \in \Omega$)

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \quad A^{-1} = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}$$

$$b_{ij} = \frac{(-1)^{i+j} A_{ji}}{\det A} \rightarrow \text{(j,i) minor of } A = \frac{(-1)^{i+j} \cdot \text{sum of products of elements of } A}{\text{sum of products of elements of } A}$$

$\therefore (f'(x))^{-1}$ is continuous in $x \in U$