Lecture 21

§1 Limits of Products and Quotients

1. An alternative view of limits of products and quotients

Suppose $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=L$. Then

1° $\lim\limits_{x\to a} f(x)h(x) = \lim\limits_{x\to a} g(x)\frac{f(x)}{g(x)}\cdot h(x)$

$= L\lim\limits_{x\to a} g(x)\cdot h(x)$

provided the limits above exist.

2° $\lim\limits_{x\to a}\frac{f(x)}{h(x)} = \lim\limits_{x\to a}\frac{f(x)}{g(x)}\cdot\frac{g(x)}{h(x)}$

$= L\lim\limits_{x\to a}\frac{g(x)}{h(x)}$

provided the limits above exist.

To summarise, for $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=L$,

$\lim\limits_{x\to a} f(x)\cdot h(x) = L\lim\limits_{x\to a} g(x)\cdot h(x)$

$\lim\limits_{x\to a}\frac{f(x)}{h(x)} = L\lim\limits_{x\to a}\frac{g(x)}{h(x)}$

2. Note

1° Replacing $f(x)$ with $L\cdot g(x)$ in the product or quotient will keep the limit unchanged.

2° This does not work with sums or differences, in general.

3° $\lim\limits_{x\to a}$ can be replaced with $\lim\limits_{x\to\infty}$ or $\lim\limits_{x\to-\infty}$

4° If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=1$, we may write $f(x)\sim g(x)$ as $x\to a$.

e.g. $\sin x\sim x$ as $x\to 0$, since $\lim\limits_{x\to 0}\frac{\sin x}{x}=\lim\limits_{x\to 0}\cos x=1$

$\tan x\sim x$ as $x\to 0$, since $\lim\limits_{x\to 0}\frac{\tan x}{x}=\lim\limits_{x\to 0}\sec^2 x=1$

3. When $x\to 0$.

1° $\sin x\sim x$ $\quad$ $\tan x\sim x$ $\quad$ $1-\cos x\sim\frac{x^2}{2}$

$\arcsin x\sim x$ $\quad$ $\arctan x\sim x$

2° $\ln(1+x) \sim x$ $\qquad$ $\log_a(1+x) \sim \frac{x}{\ln a}$

$e^x - 1 \sim x$ $\qquad$ $a^x - 1 \sim x \cdot \ln a$

$(1+x)^\mu - 1 \sim \mu x$ $\qquad$ $\sqrt[n]{1+x} - 1 \sim \frac{x}{n}$

$\sqrt{1+x} - 1 \sim \frac{x}{2}$

3° $x - \sin x \sim \frac{x^3}{6}$ $\qquad$ $\arcsin x - x \sim \frac{x^3}{6}$

$\tan x - x \sim \frac{x^3}{3}$ $\qquad$ $x - \arctan x \sim \frac{x^3}{3}$

$\tan x - \sin x \sim \frac{x^3}{2}$ $\qquad$ $x - \ln(1+x) \sim \frac{x^2}{2}$

e.g. Compute $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x - \sin x \cos x}{x^3 \cos x}$$

$$= \lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3 \cos x}$$

$$= \lim_{x\to 0} \frac{x \cdot (\frac{1}{2}x^2)}{x^3 \cos x}$$

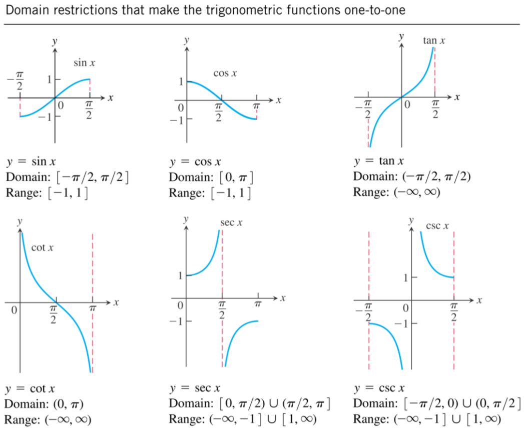
$$= \frac{1}{2}$$

§2 More on Inverse Trig Function

1. Covention on domains

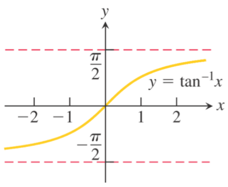
We make the following convention on the domains before inverting a trig function

Domain restrictions that make the trigonometric functions one-to-one



e.g. $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $\arctan: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



If $\arctan x_0 = y_0$ then $\tan y_0 = x_0$

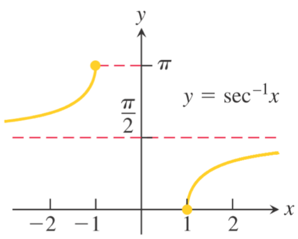
By derivative rule for inverse function:

$$\arctan'(x_0) = \frac{1}{\tan' y_0} = \frac{1}{\sec^2 y_0} = \frac{1}{1+\tan^2 y_0} = \frac{1}{1+x_0^2}$$

So: $$\arctan'(x) = \frac{1}{1+x^2}$$

e.g. $\sec : [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \to [1, \infty) \cup (-\infty, -1]$

$\text{arcsec}: [1, \infty) \cup (-\infty, -1] \to [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$



If $\text{arcsec}\, x_0 = y_0$, $x_0 \neq \pm 1$, then

$$\text{arcsec}'(x_0) = \frac{1}{\sec'(y_0)} = \frac{1}{\sec y_0 \tan y_0} = \frac{1}{x_0 \tan y_0}$$

Since $\sec^2 y_0 = 1 + \tan^2 y_0$

$$\tan y_0 = \begin{cases} \sqrt{\sec y_0^2 - 1} & \text{if } y_0 \in (0, \frac{\pi}{2}) \\ -\sqrt{\sec y_0^2 - 1} & \text{if } y_0 \in (\frac{\pi}{2}, 1) \end{cases}$$

$$\tan y_0 = \begin{cases} \sqrt{x_0^2 - 1} & \text{if } x_0 > 1 \\ -\sqrt{x_0^2 - 1} & \text{if } x_0 < -1 \end{cases}$$

Then

$$\text{arcsec}' x_0 = \begin{cases} \frac{1}{x_0\sqrt{x_0^2-1}}, & \text{if } x_0 > 1 \\ \frac{-1}{x_0\sqrt{x_0^2-1}}, & \text{if } x_0 < -1 \end{cases}$$

So $\text{arcsec}' x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

2. Some identities

Since $\sin y = \cos(y - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - y)$

If $x = \sin y$ then

$$\arcsin x = y \quad \text{and} \quad \arccos x = \frac{\pi}{2} - y$$

Which give the identity:

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

Similarly,

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

$$\operatorname{arcsec} x + \operatorname{arccsc} x = \frac{\pi}{2}$$

3. Derivative of trigonometric functions

1° $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

2° $\cos : (0, \pi) \to (-1, 1)$

$$\arccos' x = -\frac{1}{\sqrt{1-x^2}}$$

3° $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to R$

$$\arctan' x = \frac{1}{1+x^2}$$

4° $\cot : (0, \pi) \to R$

$$\operatorname{arccot}' x = -\frac{1}{1+x^2}$$

5° $\sec : (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \to (-\infty, -1) \cup (1, \infty)$

$$\operatorname{arcsec}' x = \frac{1}{|x|\sqrt{x^2-1}}$$

6° $\csc : (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}) \to (-\infty, -1) \cup (1, \infty)$

$$\operatorname{arccsc}' x = \frac{-1}{|x|\sqrt{x^2-1}}$$

§3 Integration by Parts

1. Formula (Integration by parts)

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\, dx$$

If we consider $u = f(x)$ and $v = g(x)$, then the formula above can be written as:

$$\int u\, dv = uv - \int v\, du$$

e.g. Find $\int x\cos x\, dx$

Let $f(x) = x$, $f'(x) = 1$

$g'(x) = \cos x$, $g(x) = \sin x$

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx$$

$$= x\sin x + \cos x + C$$

e.g. Find $\int x^2 e^x\, dx$

Let $f(x) = x^2$, $f'(x) = 2x$

$g'(x) = e^x$, $g(x) = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$$

Let $f(x) = 2x$, $f'(x) = 2$

$g'(x) = e^x$, $g(x) = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + \int 2e^x dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

e.g. Find $\int e^x \sin x\, dx$

$$\int e^x \sin x\, dx = -e^x \cos x - \int e^x(-\cos x)\, dx$$

$$= -e^x\cos x + \int e^x \cos x\, dx$$

$$= -e^x\cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$2\int e^x \sin x\, dx = e^x\sin x - e^x \cos x$$

$$\int e^x \sin x = \frac{1}{2} e^x (\sin x - \cos x) + C$$

e.g. Find $\int \ln x \, dx$

$$\begin{aligned}\int \ln x \, dx &= \int \ln x \cdot 1 \, dx \\ &= x \cdot \ln x - \int \frac{1}{x} \cdot x \, dx \\ &= x \ln x - x + C\end{aligned}$$

2. Reduction formulae

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \cdot \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Proof:

$$\begin{aligned}I &= \int \sin^n x \, dx \\ &= \int \sin^{n-1} x \cdot \sin x \, dx \\ &= -\sin^{n-1} x \cdot \cos x - \int (n-1) \sin^{n-2} x \cdot \cos x \cdot (-\cos x) \, dx \\ &= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx \\ &= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot (1 - \sin^2 x) \, dx \\ &= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x - (n-1) \int \sin^n x \, dx \\ &= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x - (n-1) I\end{aligned}$$

$$nI = -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x$$

$$I = -\frac{1}{n} \cos x \cdot \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

e.g. Evaluate $\int \sin^3 x \, dx$

$$\begin{aligned}\int \sin^3 x \, dx &= -\frac{1}{3} \cos x \sin^2 x + \frac{2}{3} \int \sin x \, dx \\ &= -\frac{1}{3} \cos x \sin^2 x - \frac{2}{3} \cos x + C\end{aligned}$$

3. Definite integrals

$$\int_a^b f(x) g'(x) \, dx = f(x) g(x) \Big|_a^b - \int_a^b f'(x) g(x) \, dx$$

e.g. Evaluate $\int_0^1 \arctan x \, dx$

$$\int_0^1 \arctan x\, dx = \int_0^1 \arctan x \cdot 1\, dx$$
$$= \arctan x \cdot x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$
$$= \arctan x \cdot x \Big|_0^1 - \frac{1}{2}\ln|1+x^2| \Big|_0^1$$
$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

4. Tabular integration

Consider finding $\int x^{10}\cos x\, dx$.

Suppose we have $f_i'(x) = f_{i+1}(x)$

$g_i(x) = g_{i+1}'(x)$

Then
$$\int f_0(x)g_0(x)dx = f_0(x)g_1(x) - \int f_1(x)g_1(x)\, dx$$
$$= f_0(x)g_1(x) - f_1(x)g_2(x) + \int f_2(x)g_2(x)dx$$
$$= \cdots\cdots$$
$$= f_0(x)g_1(x) - f_1(x)g_2(x) + f_2(x)g_3(x) \cdots\cdots \pm \int f_m(x)g_m(x)\, dx$$

is particularly effective for solving $\int x^n g(x)\, dx$

e.g. Evaluate $\int x^3 3^x\, dx$

i	Sign	$f_i(x)$	$g_i(x)$
0	+	x^3	3^x
1	−	$3x^2$	$(\ln 3)^{-1}3^x$
2	+	$6x$	$(\ln 3)^{-2}3^x$
3	−	6	$(\ln 3)^{-3}3^x$
4	+	0	$(\ln 3)^{-4}3^x$
5	−		

$$\int x^3 3^x dx = x^3 \frac{3^x}{\ln 3} - 3x^2 \frac{3^x}{(\ln 3)^2} + 6x \frac{3^x}{(\ln 3)^3} - 6 \frac{3^x}{(\ln 3)^4} + C$$