Lecture 9

§1 Intervals of monotonicity

Suppose that $x_1, x_2, \cdots, x_n$ are all critical points of a function f, with $x_1 < x_2 < \cdots < x_n$.

Let x_i and x_{i+1} be two consecutive critical points.

If f' is continuous on $[x_i, x_{i+1}]$, the f' is either entirely positive or entiry netagive on (x_i, x_{i+1}).

By Corollary 4.3.3:

f is increasing on $[x_i, x_{i+1}]$ if $f'(c) > 0$ for some $c \in (x_i, x_{i+1})$

f is decreasing on $[x_i, x_{i+1}]$ if $f'(c) < 0$ for some $c \in (x_i, x_{i+1})$

Similar statement can be made about the intervals $(-\infty, x_1)$ and (x_n, ∞)

e.g. $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$

Critical points at $x_1 = -2$ and $x_2 = 2$

Since $f'(-3) > 0$, $f'(0) < 0$ and $f'(3) > 0$:

§2 First derivative test

1. First derivative test for local extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself.

Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;

2. if f' changes from positive to negative at c, then f has a local maximum at c;

3. if f' doesn't change sign at c, then f has no local extremum at c.

Remarks:

For the conditions above, more formally:

1. There exists $a>0$ such that $f'(x)<0$ for all $x\in(c-a,c)$ and $f'(x)>0$ for all $x\in(c,c+a)$
2. There exists $a>0$ such that $f'(x)>0$ for all $x\in(c-a,c)$ and $f'(x)<0$ for all $x\in(c,c+a)$
3. There exists $a>0$ such that $f'(x)$ is always positive or negative for all $x\in(c-a,c+a)\setminus\{c\}$

2. Proof

Proof for 1:

By Corollary 4.3.3, f is decreasing on $[c-a,c]$ and increasing on $[c,c+a]$.

which means $f(c)<f(x)$ for all $x\in[c-a,c)$

and $f(c)<f(x)$ for all $x\in(c,c+a]$

so $f(c)<f(x)$ for all $x\in(c-a,c+a)\setminus\{c\}$

i.e. f has a local minimum at c

Proof for 2 and 3:

Similar to Proof for 1

e.g. Find all absolute and local extrema for

$$f(x)=x^{4/3}-4x^{1/3}$$

Sol: $f'(x)=\frac{4}{3}x^{\frac{1}{3}}-\frac{4}{3}x^{-\frac{2}{3}}=\frac{4}{3}x^{-\frac{2}{3}}(x-1)$

for all $x \neq 0$. Note that $f'(x)=0 \Rightarrow x=1$.

For $x=0$,

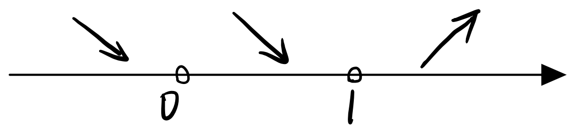
$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{4/3}-4h^{1/3}}{h}=\lim_{h\to 0} h^{\frac{1}{3}}-4h^{-2/3}$$

does not exist.

$f'(0)$ is not defined,

hence we have critical points at $x=0$ and $x=1$.

Since $f'(-1)<0$, $f'(\frac{1}{2})<0$ and $f'(2)>0$, we have



By the first derivative test, the only local extremum occurs at $x=1$, which is a local minimum.

Since f is decreasing on $(-\infty,1)$ and increasing on $(1,+\infty)$, $f(1)$ is also an absolute minimum.

f has no absolute maximum, since otherwise there would have been a local maximum.

$f(1)=1-4=-3$

§3 Concavity and inflection points

1. Definition

The graph of a differentiable function $y=f(x)$ is

(a) concave up on an open interval I if f' is increasing on I

(b) concave down on an open interval I if f' is decreasing on I

The property of being concave up or concave down is called concavity.

2. The second derivative test for concavity

Let $y=f(x)$ be twice-differentiable on an interval I.
1° If $f''>0$ on I, the graph of f over I is cave up.
2° If $f''<0$ on I, the graph of f over I is cave down.

Proof:

Suppose $f''(x)>0$ for all x in an open interval I
By Corollary 4.3.3, f' is increasing on I.
i.e. f is concave up on I
A similar statement can be made for $f''(x)<0$ for all $x\in I$.

e.g. Consider x^3

$f(x)=x^3$, $f'(x)=3x^2$

$$f''(x)=6x\begin{cases} >0 & \text{if } x>0 \\ <0 & \text{if } x<0 \end{cases}$$

So f is concave up on $(0,+\infty)$ and concave down on $(-\infty,0)$

3. Definition of points of inflection

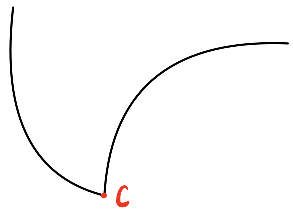
A point $(c,f(c))$ where the graph of a function has a tangent line and where the concavity change is a point of inflection.

Alternative definition:

A point $(c,f(c))$ where f has different concavity on $(c-a,c)$ and on $(c,c+a)$, for some $a>0$.

Remark:

Thomas says not an inflection point



4. How to find an inflection point

1° **A fact:** If $(c, f(c))$ is an inflection point of f and $f''(c)$ exists, then $f''(c) = 0$.

2° **Note:**

$f''(x) = 0$ doesn't imply that $(c, f(c))$ is an inflection point.

e.g. $f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$f''(x) = -\frac{2}{9} x^{-\frac{5}{3}}$

$f'(0)$ doesn't exist, so $f''(0)$ doesn't exist.

$\lim\limits_{x \to 0} f'(x) = \lim\limits_{x \to 0} \frac{1}{3\sqrt[3]{x^2}} = +\infty$

i.e. there is a vertical tangent at $x = 0$

$$f''(x) \begin{cases} > 0 & \text{if } x < 0 \\ < 0 & \text{if } x > 0 \end{cases}$$

$\therefore$ $(0, 0)$ is a point of inflection

e.g. $f(x) = x^4$

$f'(x) = 4x^3 \qquad f''(x) = 12x^2$

$f''(x)$ always exists and $f''(x) = 0$ at $x = 0$

But $(0, 0)$ is not a point of inflection, since $f''(x) > 0$ for all x with $x \neq 0$

i.e. Concavity doesn't change at $(0, 0)$

§4 Second derivative test

1. Theorem 4.4.5 Second derivative test for local extrema

Suppose f'' is continuous on an open interval that contains $x=c$.

1. If $f'(c)=0$ and $f''(c)<0$, then f has a local **maximum** at $x=c$
2. If $f'(c)=0$ and $f''(c)>0$, then f has a local **minimum** at $x=c$
3. If $f'(c)=0$ and $f''(c)=0$, then the test **fails**. The function f may have a local maximum, a local minimum, or neither.

2. Proof of 2

Suppose $f'(c)=0$ and $f''(c)>0$

Then $0<f''(c)=\lim\limits_{x\to c}\dfrac{f'(x)-f'(c)}{x-c}=\lim\limits_{x\to c}\dfrac{f'(x)}{x-c}$

Let $L=f''(c)$. For $\varepsilon=\frac{L}{2}$ there exists $\delta>0$ such that

$$\frac{f'(c)}{x-c}\in\left(\frac{L}{2},\frac{3L}{2}\right)\ \forall x\in(c-\delta,c+\delta)$$

This means that

$$\frac{f'(c)}{x-c}>\frac{L}{2}>0\quad\forall x\in(c-\delta,c+\delta)$$

So $f'(x)<0$ for $x\in(c-\delta,c)$

$f'(x)>0$ for $x\in(c,c+\delta)$

By Corollary 4.3.3, f is decreasing on $[c-\delta,c]$ and increasing on $[c,c+\delta]$.

Hence f has a local minimum at c.

3. Proof of 3

For $f(x)=x^3$, $g(x)=x^4$, $h(x)=-x^4$

we have $f'(x)=3x^2$, $g'(x)=4x^3$, $h'(x)=-4x^3$

and $f''(x)=6x$, $g''(x)=12x^2$, $h''(x)=-12x^2$

so $f'(0)=f''(0)=g'(0)=g''(0)=h'(0)=h''(0)=0$

But at $x=0$: f has no local extremum

g has a local minimum

h has a local maximum

Conclusion: for $f'(0)=f''(0)=0$ anything can happen!

4. Find all local extrema

Step 1 Find all value of x for which $f'(x)=0$

Step 2 Find the value of $f''(x)$ at $f'(x)=0$

Step 3 Apply second derivative test to identify the concavity of local extrema.

Step 4 Find the value of $f(x)$ at the local extrema.

e.g. $f(x)=x^4-4x^3+10$. Find all local extrema

$f'(x)=4x^3-12x^2=4x^2(x-3)$

$f'(x)=0 \Leftrightarrow x=0$ and $x=3$

$f''(x)=12x^2-24x$

$f''(0)=0 \Rightarrow$ second derivative test fails

$f''(3)=36>0$

$\Rightarrow$ $x=3$ gives a local minimum at $f(3)=-17$

Note: For $x=0$, note that $f'(-1)<0$ and $f'(1)<0$,

so it gives no local extrema by first derivative test.