Lecture 18

§1 证明：Error of standard linear approximation

1. 定理回顾

选定一个起始点 (x_0,y_0)，进行线性近似.

则近似的 error 为 $E(x,y) := f(x,y) - L(x,y)$

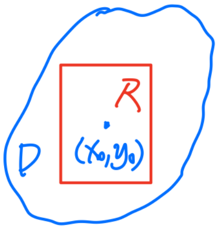
令：① $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ 在一个 open region D 上均连续

② 在某个以 (x_0,y_0) 为中心的长方形区域 R 内 $(R \subseteq D)$，存在 M，使得

$|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ $(=|f_{yx}|)$ are all bounded above by M

则：

$$|E(x,y)| \le \frac{1}{2} M (|x-x_0| + |y-y_0|)^2 \text{ for all } (x,y) \in R.$$



2. 证明

- 令 $h := \Delta x$ and $k := \Delta y$ be "small changes"

  定义 $F(t) := f(x(t), y(t)) = f(x_0+th, y_0+tk)$，$t \in [0,1]$

  (用单一变量 t 表示长方形区域内点的函数值)

- 由 Taylor's theorem，因为 F 关于 t 在 [0,1] 上连续 & 可微，

$$F(1) = F(0) + F'(0)\cdot(1-0) + \frac{1}{2}F''(c)(1-0)^2$$
$$= F(0) + F'(0) + \frac{1}{2}F''(c), \quad c \in (0,1)$$

  ($F(1)$ 表示 $f(x_0+h, y_0+k)$，即所求的值)

- 由 chain rule，可求出 $F'(t)$ 与 $F''(t)$

$$F'(t) = f_x \cdot x'(t) + f_y \cdot y'(t)$$
$$= f_x \cdot h + f_y \cdot k$$
$$F''(t) = h\cdot(f_{xx}\cdot x'(t) + f_{xy}\cdot y'(t)) + k\cdot(f_{yx}\cdot x'(t) + f_{yy}\cdot y'(t))$$
$$= f_{xx}\cdot h^2 + 2f_{xy}\cdot hk + f_{yy}\cdot k^2$$

- 代入 $F(1)$ 表达式，得：

$$f(x_0+h, y_0+k) = f(x_0,y_0) + f_x(x_0,y_0)\cdot h + f_y(x_0,y_0)\cdot k$$
$$+ \frac{1}{2}(f_{xx}\cdot h^2 + 2f_{xy}\cdot hk + f_{yy}\cdot k^2)\Big|_{(x,y)=(x_0+ch,\, y_0+ck)}$$

  经比对，得出 error 即为 $\frac{1}{2}F''(c)$

- In short notation and by triangle inequality，

$$|E(x,y)| = \frac{1}{2}\left| f_{xx}\cdot(x-x_0)^2 + 2f_{xy}\cdot(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2 \right|$$

  (evaluated at (x_0+ch, y_0+ck))

$$\le \frac{1}{2}\Big(\underbrace{|f_{xx}|}_{\le M}\cdot(x-x_0)^2 + 2\underbrace{|f_{xy}|}_{\le M}|x-x_0||y-y_0| + \underbrace{|f_{yy}|}_{\le M}\cdot(y-y_0)^2\Big)$$

$$\le \frac{1}{2}M\cdot(|x-x_0| + |y-y_0|)^2$$

* 注：$f_{xx}\cdot h^2 + 2f_{xy}\cdot hk + f_{yy}\cdot k^2$ 也可写作 $(h\cdot\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f$

§2 证明：The second derivative test

1、定理回顾

Theorem (Second Derivative Test)

Let f be a function whose second partial derivatives are all continuous on a open ~~ball~~ disk centered at (a,b). Suppose that $\nabla f(a,b)=\vec{0}$ (so (a,b) is a critical point of f). Let

$$H := H(a,b) := f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2.$$

- *If $H>0$ and $f_{xx}(a,b)>0$, then f has a local minimum at (a,b).*
- *If $H>0$ and $f_{xx}(a,b)<0$, then f has a local maximum at (a,b).*
- *If $H<0$, then f has no local extremum at (a,b); that is, (a,b) is a saddle point ~~d~~ of f.*

2、证明

- 令 $h:=\Delta x$ and $k:=\Delta y$ be "small changes" from (a,b)
- 由 §1 中的证明可知：

$$f(a+h,b+k) = f(a,b) + f_x(a,b)\cdot h + f_y(a,b)\cdot k + \frac{1}{2}\left(f_{xx}\cdot h^2 + 2f_{xy}\cdot hk + f_{yy}\cdot k^2\right)\Big|_{(x,y)=(a+ch,\,b+ck)}$$

for some $c\in(0,1)$

- 由 $\nabla f(a,b)=\langle 0,0\rangle$ 得：

$$f(a+h,b+k)-f(a,b) = \frac{1}{2}\left(f_{xx}\cdot h^2 + 2f_{xy}\cdot hk + f_{yy}\cdot k^2\right)\Big|_{(x,y)=(a+ch,\,b+ck)} =: Q(c,h,k) =: Q$$

Case 1: $H(a,b) = (f_{xx}f_{yy}-f_{xy}^2)(a,b) > 0$, $f_{xx}(a,b)>0$

- 由 countinuity of second partials ,

$\exists\, \delta>0$ s.t. $\forall (x_0,y_0)\in B_\delta(a,b)$, $H(x_0,y_0)>0$ and $f_{xx}(x_0,y_0)>0$

- 令 (h,k) 满足 $0<\sqrt{h^2+k^2}<\delta$, 则

$$Q = \frac{1}{2}\left(f_{xx}\cdot h^2 + 2f_{xy}\cdot hk + f_{yy}\cdot k^2\right)\Big|_{(x,y)=(a+ch,\,b+ck)}$$

$x_0 := a+ch$, $y_0 := b+ck$

- 两侧同乘 $f_{xx}(x_0,y_0)$，得：

$$\begin{aligned} f_{xx}(x_0,y_0)\cdot Q &= \frac{1}{2}\left(f_{xx}^2\cdot h^2 + 2f_{xx}f_{xy}hk + f_{xx}f_{yy}\cdot k^2\right)\Big|_{(x,y)=(x_0,y_0)} \\ &= \frac{1}{2}\Big[(f_{xx}\cdot h + f_{xy}\cdot k)^2 + \underbrace{(f_{xx}f_{yy}-f_{xy}^2)}_{H(x_0,y_0)>0}k^2\Big]\Big|_{(x,y)=(x_0,y_0)} \\ &> 0 \end{aligned}$$

- Since $f_{xx}(x_0,y_0)>0$, we have $Q>0$. So

$f(a+h,b+k) > f(a,b)$ $\forall (h,k)$ with $0<\sqrt{h^2+k^2}<\delta$,

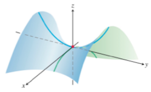
So (a,b) is a local minimum of f.

Case 2: $H(a,b) = (f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0$, $f_{xx}(a,b) < 0$

类似的，有 $Q<0$，(a,b) 为极大值

Case 3: $H(a,b) = (f_{xx}f_{yy} - f_{xy}^2)(a,b) < 0$

- We show that $\exists$ direction $\vec{w}$ and $\vec{v}$ such that f has a local min at (a,b) while restricted to $\vec{w}$ and has a local max at (a,b) while restricted to $\vec{v}$. This would show that (a,b) is a saddle point of f.



- Let $\vec{u} := \langle h, k \rangle$ be a unit vector. Consider

$$F(t) := f(a+th,\, b+tk).$$

Then $F'(0) = (f_x h + f_y k)\big|_{(x,y)=(a,b)} = 0$, and

$$F''(0) = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \big|_{(x,y)=(a,b)} .$$

- Define $g(t) := \underbrace{f_{xx}}_{A} t^2 + \underbrace{2f_{xy}}_{B} t + \underbrace{f_{yy}}_{C}$ ← Evaluated at (a,b). Suppose $f_{xx}(a,b) \neq 0$.

Then $B^2 - 4AC = 4f_{xy}^2 - 4f_{xx}f_{yy} = -4(f_{xx}f_{yy} - f_{xy}^2) > 0$.

This means that $g(t)$ has two distinct real roots, and $g(t)$ is sometimes >0 and sometimes <0 (depending on t). ②

- If $k \neq 0$, then $g(\frac{h}{k}) = f_{xx}\frac{h^2}{k^2} + 2f_{xy}\frac{h}{k} + f_{yy}$

$\Rightarrow k^2 g(\frac{h}{k}) = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 = F''(0)$. ③

- By ② and ③, we may pick different combinations of (h,k) so that $F''(0) > 0$ on one and $F''(0) < 0$ on the other.

Concave up, local min in one direction

Concave down, local max in another

- If $f_{xx}(a,b) = 0$, then $g(t) = 2f_{xy} + f_{yy}$ with $f_{xy}(a,b) \neq 0$.

The rest is similar. Sometimes >0 and Sometimes <0. $\square$

§3 二元函数的 Taylor's Theorem

1. 关于 t 的 n 阶导数

若 f 所有的 n 阶偏导均在包含 (a,b) 的开区域内连续，则

$$F^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f \,\Big|_{(a+th,\, b+tk)}$$

e.g. $F^{(4)}(t) = h^4 f_{xxxx} + 4h^3k f_{xxxy} + 6h^2k^2 f_{xxyy} + 4hk^3 f_{xyyy} + k^4 f_{yyyy}$

*2. 高阶微分

$$d^n f = \left(dx\cdot\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n f$$

3. 二元函数的 Taylor's Theorem

若 $f(x,y)$ 在一个包含点 (a,b) 的开区域 R 内有连续的 $n+1$ 项偏导.

则对 R 中的任意点，有：

$$f(a+h, b+k) = \left(\sum_{i=1}^{n} \frac{1}{i!}\cdot\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f \,\Big|_{(a,b)}\right) + \frac{1}{(n+1)!}\left(h\cdot\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f \,\Big|_{(a+ch,\, b+ck)}$$

for some $c \in (0,1)$

例: **EXAMPLE 4** Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?

Sol: $f(0,0) = \sin x \sin y \,\big|_{(0,0)} = 0$

$f_x(0,0) = \cos x \sin y \,\big|_{(0,0)} = 0$

$f_y(0,0) = \sin x \cos y \,\big|_{(0,0)} = 0$

$f_{xx}(0,0) = -\sin x \sin y \,\big|_{(0,0)} = 0$

$f_{xy}(0,0) = \cos x \cos y \,\big|_{(0,0)} = 1$

$f_{yy}(0,0) = -\sin x \sin y \,\big|_{(0,0)} = 0$

- $f(x,y) \approx f(0,0) + (x f_x + y f_y)\big|_{(0,0)} + \frac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})\big|_{(0,0)}$
  $= xy$
- $E(x,y) = \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xxy} + y^3 f_{yyy})\big|_{(cx,cy)}$
- Computing all third-order partials shows that their absolute values are all ≤ 1
- Hence $|E(x,y)| \leq \frac{1}{6}(0.1^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3) \leq 0.00134$