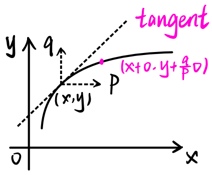
Lecture 1

§1 从 intuitive calculus 到 rigorous calculus 的转变

1、 Nov 13, 1665：牛顿提出 intuitive calculus

x-y 平面上有一条曲线 $y=f(x)$，曲线上一点沿 x 方向的变化率是 p，沿 y 轴方向的变化率是 q，求 $m=\frac{p}{q}$



slope = $\frac{p}{q}$ = "m"

取一个"infinitely small"的量 o，则点 $(x+o, y+o\frac{q}{p})$ 仍在曲线上，因此可得：

$$\begin{cases} y+o\frac{q}{p}=f(x+o) \\ y=f(x) \end{cases}$$

两式作差可得：$o\cdot\frac{q}{p}=f(x+o)-f(x) \Rightarrow m=\frac{q}{p}=\frac{f(x+o)-f(x)}{o}$

e.g. $y=x^2$，求 $(1,1)$ 处的 m

$$m=\frac{p}{q}=\frac{(x+o)^2-x^2}{o}=\frac{2o+o^2}{o}=2+o=2$$

注："infinitely small" = infinitesimal = indivisable

= a "creature" whose absolute value < any positive # but ≠ 0

2、 18 世纪的态度：who cares rigorous !

- machanics was based on Calculus
- Taylor theorem & Laplace transform
- equations of motion of solor system solved
- ……

Euler：求 $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=?$

Lemma (algebra)：$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x_1+1=0$，令 $r_1, r_2, \cdots, r_n$ 为 n 个 roots，

则 $a_1=-(\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_n})$ (Vieta's Theorem)

考虑 $\sin x=0$，$x=0, \pm\pi, \pm2\pi, \cdots$，

由泰勒展开：$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots=0$

$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\cdots=0$

令 $u=x^2$，则 $1-\frac{u}{3!}+\frac{u^2}{5!}-\cdots\cdots=0$ roots：$u=\pi^2, (2\pi)^2, \cdots\cdots$

由 Lemma 可知：$-\frac{1}{3!}=-(\frac{1}{\pi^2}+\frac{1}{(2\pi)^2}+\cdots)$

$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$

注：这种解法"morally correct，technically wrong"！（因为引理针对的是有限项）

3、 18 世纪末：Driving force to rigorous Calculus

- George Berkeley 的 attack (1734)：发表名为"Discourse addresed to an infidel mathematician"的论文，称牛顿提出的"infinitely small"为"ghosts of departed quantities"。
- Teaching needs

- 18 世纪的 Calculus 变得 "decadent" (Lagrange)，它含有局限性.
  它无法回答诸如以下的问题:
- Suppose $\{a_n\}_{n=0}^{\infty}$ is bounded, $|q|<1$,
  prove the existence of $\lim_{n\to\infty}(a_0+a_1q+a_2q^2+\cdots+a_nq^n) = \sum_{n=0}^{\infty} a_nq^n$
- prove the existence of $\int_a^b f(x)\,dx$
- prove the Intermediate Value theorem
- A continuous function must be differentiable at somewhere? (Wrong)

§2 序列极限

1. **Definition**: convergent sequence (收敛序列)

   若序列 $\{a_n\}_{n=n_0}^{\infty}$ 满足 $\lim_{n\to\infty} a_n$ exists as a finite，则 $\{a_n\}_{n=n_0}^{\infty}$ 收敛

2. 序列极限 $\lim_{n\to\infty} a_n = l$ 定义的演化

   下述定义由 informal 到 rigorous:
   - a_n approaches l as n tends ∞
   - a_n is arbitrarily close to l, as close as desired, whenever n is large enough
   - $|a_n-l|$ is arbitrarily small, as small as desired, whenever n is large enough
   - $\forall$ (for any) $\varepsilon>0$, $|a_n-l|<\varepsilon$, whenever n is large enough

3. **Definition**: 序列极限 $\lim_{n\to\infty} a_n = l$

   $\forall\, \varepsilon>0$, $\exists\, N(\varepsilon)$ such that $|a_n-l|\le\varepsilon$, whenever (as long as) $n\ge N(\varepsilon)$

   这种情况我们记作 $\lim_{n\to\infty} a_n = l$，或 $a_n\to l$ as $n\to\infty$

   注: whenever = as long as = if = for

4. **Definition**: 序列极限 $\lim_{n\to\infty} a_n \ne l$

   定义1: $\exists$ (bad) $\varepsilon_0>0$ & $\exists$ (bad) subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, such that $|a_{n_k}-l|\ge\varepsilon_0$, $\forall\, k\ge1$

   定义2: $\exists$ (bad) $\varepsilon_0>0$ s.t. $\forall\, N$, $\exists$ (bad) $n(N)$ (positive integer) $\ge N$ s.t. $|a_{n(N)}-l|\ge\varepsilon_0$

   分析: 取 $N=1$，则存在 $n(1)=n_1$ (relabel 的过程)，s.t. $|a_{n_1}-l|\ge\varepsilon_0$

   再取 $N=n_1$，则存在 $n(n_1)=n_2\ge n_1$ (relabel)，s.t. $|a_{n_2}-l|\ge\varepsilon_0$

   再取 $N=n_2$，则存在 $n(n_2)=n_3\ge n_2$ (relabel)，s.t. $|a_{n_3}-l|\ge\varepsilon_0$

   ……

   因此，会存在一个 bad subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, s.t. $|a_{n_k}-l|\ge\varepsilon_0$, $\forall\, k\ge1$

5. **Definition**: infinite limits (无穷极限)

   我们说 a_n converges (diverges) to ∞，并记作 $\lim_{n\to\infty} a_n=\infty$，

   若 $\forall\, M>0$, $\exists\, N(M)$, s.t. $a_n>M$ as long as $n\ge N(M)$

6、利用 ε-N 语言证明数列极限

例1: Prove using rigorous definition: $\lim\limits_{n\to\infty}\sqrt{1-\frac{1}{n}}=1$

discussion: 问题转化为: $\forall \varepsilon>0$, want $\left|\sqrt{1-\frac{1}{n}}-1\right|<\varepsilon$, find $N(\varepsilon)$

$$\left|\sqrt{1-\frac{1}{n}}-1\right| = 1-\sqrt{1-\frac{1}{n}} = \frac{\left(1-\sqrt{1-\frac{1}{n}}\right)\left(1+\sqrt{1-\frac{1}{n}}\right)}{1+\sqrt{1-\frac{1}{n}}} = \frac{\frac{1}{n}}{1+\sqrt{1-\frac{1}{n}}} < \frac{1}{n} < \varepsilon \text{（放缩）}$$

因此, $N(\varepsilon)=\frac{1}{\varepsilon}$

proof: $\forall \varepsilon>0$, take $N(\varepsilon)=\frac{1}{\varepsilon}$, whenever $n\geqslant\frac{1}{\varepsilon}$,

we have $\varepsilon \geqslant \frac{1}{n} > \frac{\frac{1}{n}}{1+\sqrt{1-\frac{1}{n}}} = \left|1-\sqrt{1-\frac{1}{n}}\right|$

QED

例2: 证明: $\lim\limits_{n\to\infty}\frac{1}{n}=0$

discussion: 问题转化为, $\forall \varepsilon>0$, want $\left|\frac{1}{n}-0\right|<\varepsilon$, find $N(\varepsilon)$

$\left|\frac{1}{n}-0\right|=\frac{1}{n}<\varepsilon \Rightarrow n>\frac{1}{\varepsilon}$

因此, $N(\varepsilon)=\frac{2}{n}$

proof: $\forall \varepsilon>0$, take $N(\varepsilon)=\frac{2}{n}$, whenever $n\geqslant\frac{2}{\varepsilon}$,

we have $\left|\frac{1}{n}-0\right|=\frac{1}{n}\leq\frac{\varepsilon}{2}<\varepsilon$

QED

例3: 证明: $\lim\limits_{n\to\infty}\sqrt[n]{a}=1 \quad (a>1)$

discussion: $\left|\sqrt[n]{a}-1\right|=\sqrt[n]{a}-1<\varepsilon \Rightarrow n>\frac{1}{\log_a(1+\varepsilon)}$

因此, $N(\varepsilon)=\frac{2}{\log_a(1+\varepsilon)}$

proof: $\forall \varepsilon>0$, take $N(\varepsilon)=\frac{2}{\log_a(1+\varepsilon)}$, whenever $n\geqslant N(\varepsilon)$,

we have $\left|\sqrt[n]{a}-1\right| = \sqrt[n]{a}-1 \leq \sqrt{1+\varepsilon}-1 < \varepsilon$

QED