Lecture 15

§1 Even and Odd Functions

1. Definition

A function $f: D \to R$ is called

1° an even function (偶函数), if $f(x) = f(-x)$ for all $x \in D$.

2° an odd function (奇函数), if $f(x) = -f(-x)$ for all $x \in D$.

2. Theorem (Integrals of symmetric functions)

Let $f: [-a, a] \to R$ be an integrable function.

If f is an even function, the $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If f is an odd function, the $\int_{-a}^{a} f(x)\,dx = 0$

e.g. Show that $I = \int_{-\sqrt{2}}^{\sqrt{2}} (15x^4 - 4x^3 + 6x^2 + 7x)\,dx = 32\sqrt{2}$.

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} (15x^4 + 6x^2)\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} (-4x^3 + 7x)\,dx$$
$$= 2\int_{0}^{\sqrt{2}} (15x^4 + 6x^2)\,dx$$
$$= 2\left[3x^5 + 2x^3\right]_{0}^{\sqrt{2}}$$
$$= 32\sqrt{2}$$

e.g. Show that $I = \int_{-1}^{3} (x+1)^2 (x-3)^2\,dx = \frac{512}{15}$.

Function is symmetrical about $x = 1$.

Let $u = x - 1$. Then $du = dx$

$$I = \int_{-2}^{2} (u+2)^2 (u-2)^2\,du$$
$$= \int_{-2}^{2} (u^2 - 4)^2\,du$$
$$= \int_{-2}^{2} (u^4 - 8u^2 + 16)\,du$$
$$= \left[\frac{1}{5}u^5 - \frac{8}{3}u^3 + 16u\right]_{-2}^{2}$$
$$= \frac{512}{15}$$

Proof:

1° If f is even, then

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_0^a f(x)dx + \int_{-a}^{0} f(x)dx \\ &= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx \\ &= -\int_0^a f(-u)(-du) + \int_0^a f(x)dx \\ &= \int_0^a f(-u)du + \int_0^a f(x)dx \\ &= \int_0^a f(u)du + \int_0^a f(x)dx \\ &= 2\int_0^a f(u)du\end{aligned}$$

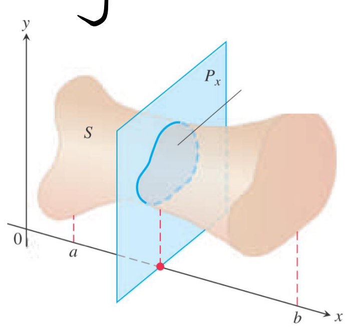
2° If f is odd, then

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \\ &= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx \\ &= -\int_0^a f(-u)(-du) + \int_0^a f(x)dx \\ &= \int_0^a f(-u)du + \int_0^a f(x)dx \\ &= -\int_0^a f(u)du + \int_0^a f(x)dx \\ &= 0\end{aligned}$$

§2 Volumes Using Cross-sections

1. Let S be a solid in the three dimensional ~~Euclidean~~ space (the xyz-space), lying between the planes $x = a$ and $x = b$. How do we compute the volume of the solid?

For $c \in [a,b]$, let $A(c)$ be the area of the cross-section obtained by intersecting S with the plane $x=c$



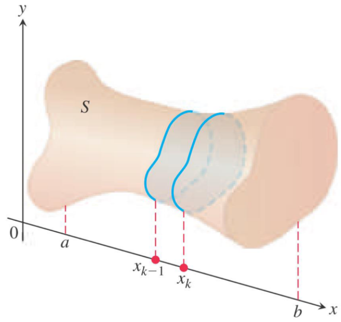
Considering a partition $P = \{x_0, x_1, x_2 \cdots\cdots x_n\}$ of $[a,b]$

When $\|P\|$ is small, the volume of the solid lying between $x = x_{k-1}$

and $x = x_k$ is approximately $A(x_k)\Delta x_k$.

When $\|P\|$ is small, the volume of S is approximately

$$\sum_{k=1}^{n} A(x_k)\cdot\Delta x_k$$



2. Definition

Let S be a solid that lies between the plane $x=a$ and $x=b$. The volume V of S is defined by

$$V = \int_a^b A(x)\,dx$$

provided that the cross-section area function $A(x)$ is integrable.

3. Cavalieri's principle

By definition of volume, the following two solids have the same volume:

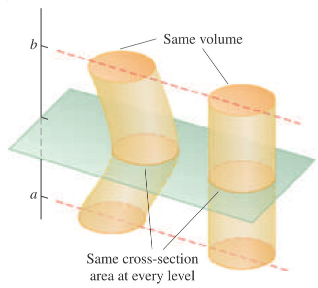
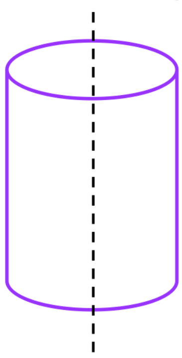


FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume, which can be illustrated with stacks of coins.

e.g. **EXAMPLE 2** A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.



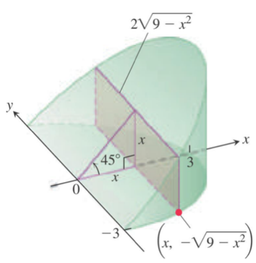


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

$A(x) = 2x\sqrt{9-x^2}, \quad 0 \le x \le 3$

$V = \int_0^3 2x\sqrt{9-x^2}\, dx$

Let $u = 9 - x^2$, then $du = -2x\, dx$

$V = \int_9^0 \sqrt{u}\, (-du)$

$= \int_0^9 \sqrt{u}\, du$

$= [\frac{2}{3} u^{\frac{3}{2}}]_0^9$

$= 18$

§3 Solids of Revolution (Using Cross-section)

1. Situation 1

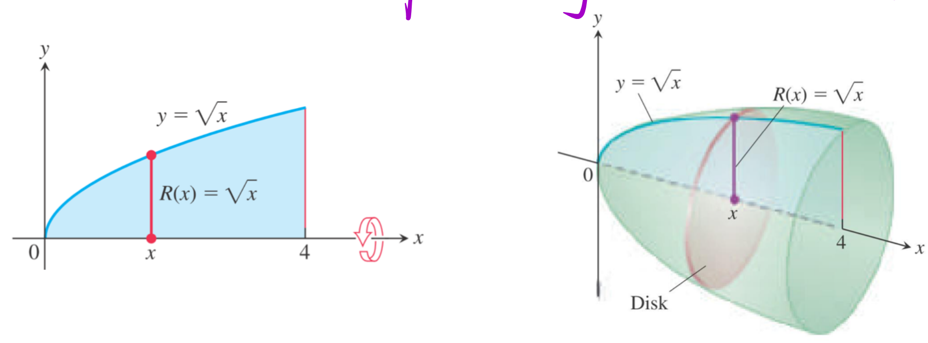
If the solid S is generated by rotating the region

$$\{(x,y): 0 \le y \le R(x), a \le x \le b\}$$

around the x-axis, then the cross-section of S are discs with radii $R(x)$. Consequently, $A(x) = \pi R(x)^2$, and

$$V = \int_a^b \pi R(x)^2 dx$$

e.g. What's the volume of the object shown below?



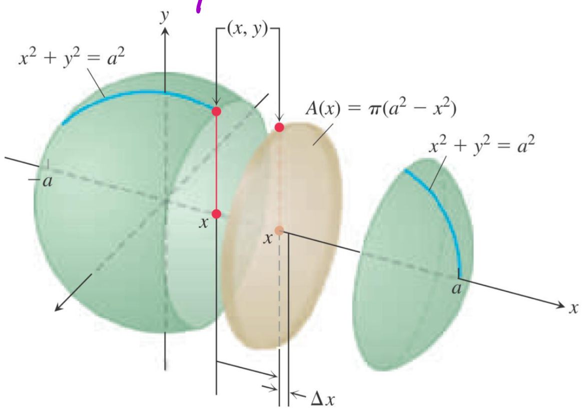
$V = \int_0^4 \pi (R(x))^2 dx \qquad R(x) = \sqrt{x}$

$V = \int_0^4 \pi x\, d(x)$

$= [\frac{1}{2}\pi x^2]_0^4$

$= 8\pi$

e.g. The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



$V = \int_{-a}^{a} \pi (R(x))^2 dx$ $\quad R(x) = \sqrt{a^2 - x^2}$

$$
\begin{aligned}
V &= \int_{-a}^{a} \pi (a^2 - x^2)\, dx \\
&= 2\pi \int_0^a a^2 - x^2\, dx \\
&= 2\pi \left[a^2 x - \tfrac{1}{3}x^3\right]_0^a \\
&= \tfrac{4}{3}\pi a^3
\end{aligned}
$$

2. Situation 2

If the solid S is generated by rotating the region

$$\{(x,y) : 0 \le r(x) \le y \le R(x),\ a \le x \le b\}$$

around the x-axis, then similarly,

$$V = \int_a^b \pi \left(R(x)^2 - r(x)^2\right) dx$$

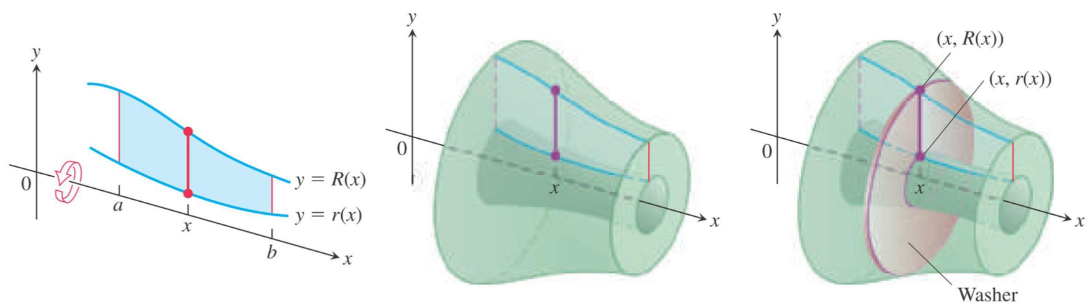
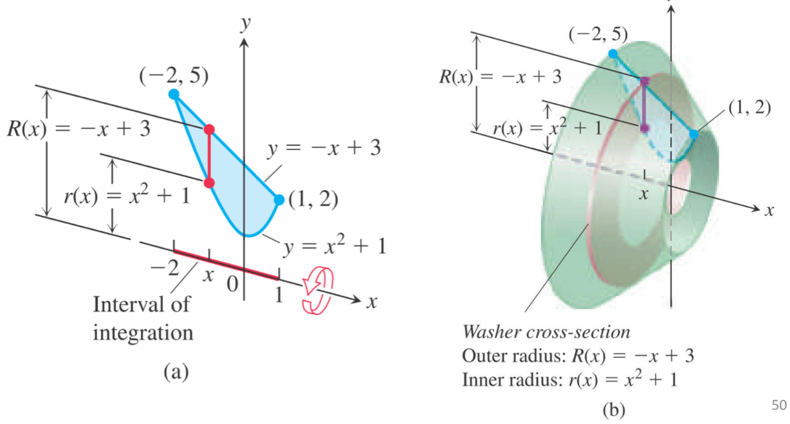


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\, dx$ leads to a slightly different formula.

e.g. The region is bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



Find the limits of integration in the x-domain, which are where $y=x^2+1$ and $y=-x+3$ intersect.

$$x=-2,\ y=5$$

$$x=1,\ y=2$$

$0\le x^2+1\le -x+3 \quad \forall x\in[-2,1]$

$V=\int_{-2}^{1}\pi\left(R(x)^2-r(x)^2\right)dx \qquad R=-x+3 \qquad r(x)=x^2+1$

$$V=\pi\int_{-2}^{1}(-x+3)^2-(x^2+1)^2\,dx$$

$$=\pi\int_{-2}^{1}-x^4-x^2-6x+8\,dx$$

$$=\pi\left[-\tfrac{1}{4}x^5-\tfrac{1}{3}x^3-3x^2+8x\right]_{-2}^{1}$$

$$=\frac{117}{5}\pi$$

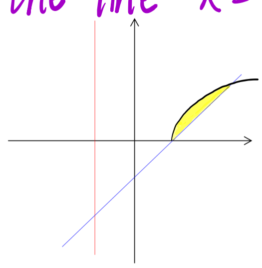
3. Situation 3

For revolution about the y-axis.

$$V=\int_c^d\pi\left(g_2(y)^2-g_1(y)^2\right)dy$$

where $0\le g_1(y)\le g_2(y) \quad \forall y\in[c,d]$

e.g. Find the volume of the solid obtained by rotating the region bounded by $y=2\sqrt{x-1}$ and $y=x-1$ about the line $x=-1$.



We can shift the plots to the right by 1 unit

i.e. add 1 to each function

$x = y+1 \longrightarrow x = y+2$

$x = \frac{y^2}{4}+1 \longrightarrow x = \frac{y^2}{4}+2$

Find the limits of integration in the y-domain, which are where $x=y+2$ and $x=\frac{y^2}{4}+2$ intersect.

$y=0$, $x=2$

or $y=4$, $x=6$

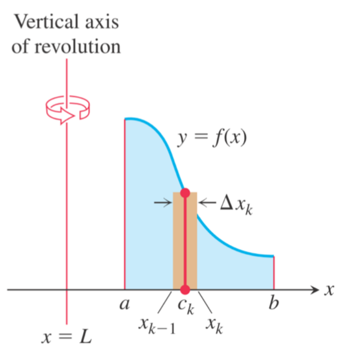
$V=\int_0^4 \pi ((g_2(y))^2-(g_1(y))^2)\,dy \quad g_2(y)=y+2 \quad g_1=\frac{y^2}{4}+2$

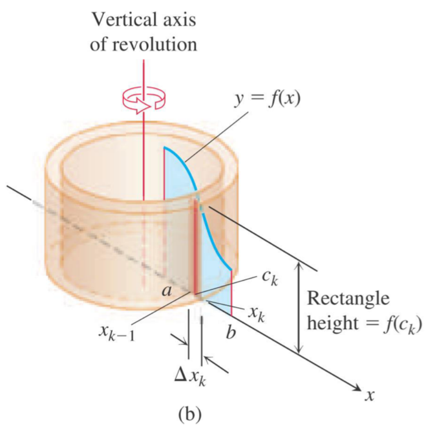
$$V=\pi\int_0^4 (y+2)^2-(\tfrac{y^2}{4}+2)^2\,dy$$
$$=\pi\int_0^4 \tfrac{y^4}{16}+2y^2+4y+8\,dy$$
$$=\pi\left[\tfrac{1}{80}y^5+\tfrac{2}{3}y^3+2y^2+8y\right]_0^4$$
$$=\tfrac{96}{5}\pi$$

§4 Solids of Revolution (Using Cylindrical Shells)

1. Cylindrical shells

Consider revolving the following region in blue about the y-axis to generate a solid. Its volume can be computed by adding the volumes of all the "cylindrical shells", one of which is displayed in orange in the figure.





(b)

Total volume of solid: $V \approx \sum V_k$

$$V \approx \sum_{k=1}^{n} 2\pi x_k \cdot h(x_k) \Delta x_k$$

2. Compute the volume using cylindrical shell

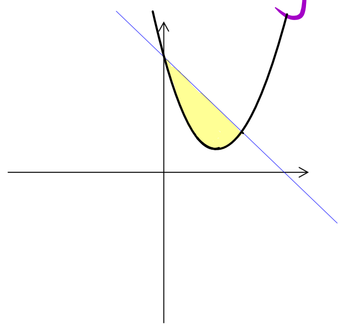
In general, given the solid S generated by revolving the region

$$\{(x,y) : g(x) \le y \le f(x),\ a < x < b\}$$

about the y-axis, let $h(x) = f(x) - g(x)$ be the height of the region at x. Then the volume V of S can be computed by

$$V = \int_a^b 2\pi x\, h(x)\, dx$$

e.g. A region is bounded by $y = (x-2)^2 + 1$ and $y = 5 - x$. Find the volume of the solid generated by revolving the region around the y-axis.



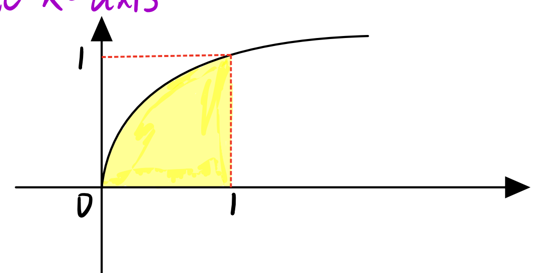
$$V = \int_0^3 2\pi x (5 - x - (x-2)^2 - 1)\, dx$$

$$= 2\pi \int_0^3 -x^3 + 3x^2\, dx$$

$$= 2\pi \left[-\tfrac{1}{4}x^4 + x^3\right]_0^3$$

$$= \frac{27}{2}\pi$$

e.g. Find the volume of the solid generated by revolving the region between $y = \sqrt{x}$ and $y = 0$, from $x = 0$ to $x = 1$, around the x-axis



Solution 1: $V = \pi \int_0^1 x \, dx$

$= \pi \left[\frac{1}{2} x^2 \right]_0^1$

$= \frac{\pi}{2}$

Solution 2: $V = \int_0^1 2\pi y (1 - y^2) \, dy$

$= 2\pi \int_0^1 (y - y^3) \, dy$

$= 2\pi \left[\frac{1}{2} y^2 - \frac{1}{4} y^4 \right]_0^1$

$= \frac{\pi}{2}$