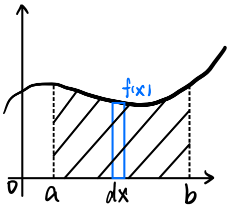
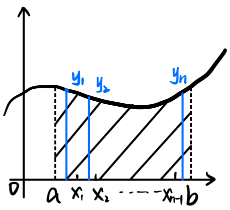
Lecture 22

Recall:

1. $f(x) \geq 0$ on $[a,b]$



Area $= \int_a^b f(x)\,dx$

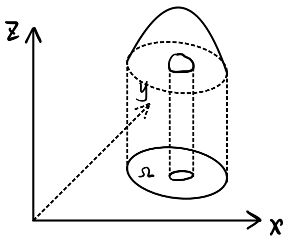


partition $P = \{\overset{a}{\uparrow}{x_0}, x_1, \cdots, \overset{b}{\uparrow}{x_n}\}$

$\|P\|$: norm of P = longest subinterval

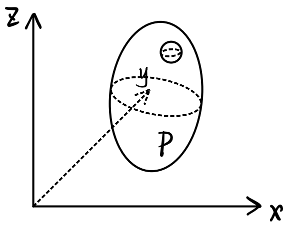
Area $= \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(y_i)\,\Delta x_i$ (Riemann sum)

2. $z = f(x,y) \geq 0$, $(x,y) \in \Omega$



Volume $V = \iint_\Omega f(x,y)\,dA$

3. density function $f(x,y,z) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \approx \frac{\Delta m}{\Delta V}$



$\Delta m = f(x,y,z)\,\Delta V$

total mass $= \iiint_P f(x,y,z)\,dV$

Goal:

Q1: Given set $S \subset R^n$, how to define rigorously $\int_S f(x)\,dx$

Q2: Necessary & sufficient condition on $f(x)$ so that $\int_S f(x)\,dx$ exists.

Q3: Generalization of Fubini's theorem ($\iint_{[a,b]\times[c,d]} = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$)

§1 Integration theory

1. Jargons

1° Closed rectangle in R^n

$Q = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n]$

$= \{(x_1,x_2,\cdots,x_n) \in R^n \mid a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2,\ \cdots,\ a_n \leq x_n \leq b_n\}$

Volume of $Q \overset{\text{def}}{=} (b_1-a_1)(b_2-a_2)\cdots(b_n-a_n)$

2° Partition

Let P_1 be a partition of $[a_1, b_1]$.

Let P_2 be a partition of $[a_2, b_2]$.

$\vdots$

Let P_n be a partition of $[a_n, b_n]$.

则称 n-tuple $(P_1, P_2, \cdots, P_n)$ 为一个 partition of Q

若 I_j 为一个 subinterval determined by P_j, $j=1,\cdots,n$, 则 the rectangle $R \overset{def}{=} I_1 \times I_2 \times \cdots \times I_n$ 被称为一个 subrectangle determined by P. 记作 $R \in P$. $|R|$ 表示 rectangle 的 volume

3° norm of $\|P\|$ = 所有 subrectangle $\in P$ 的最大的边

2. Definition: Riemann integrable

令 $f(x)$ 为 real-valued & bdd on Q, 则称 f 为 Riemann integrable on Q,

若 $\forall \varepsilon>0$, $\exists \delta>0$, s.t. as long as $\|P\|<\delta$, we have $\left|\sum_{R\in P} f(x_R)|R| - A\right| < \varepsilon$ for some constant A for any $x_R \in R$.

其中 $\sum_{R\in P} f(x_R)|R|$ 被称为 Riemann sum.

在这种情况下, 记 $\lim_{\|P\|\to 0} \sum_{R\in P} f(x_R)|R| = A$, 其中 A 被称作 integral of f on Q, 记作 $\int_Q f(x)\,dx$

例: $Q=[0,1]\times[0,1111]$

Dirichlet function $D = \begin{cases} 1, & \text{if } (x,y) \text{ irrational} \\ 0, & \text{if } (x,y) \text{ rational} \end{cases}$

判断 D 是否 Riemann integrable

$$\sum_{R\in P} D(x_R, y_R)|R| = \begin{cases} \sum_{R\in P}|R| = 1111, & \text{if always take } (x_R, y_R) \text{ irrational} \\ 0, & \text{if always take } (x_R, y_R) \text{ rational} \end{cases}$$

$\Rightarrow$ "A" does not exist

$\Rightarrow$ $D(x,y)$ not Riemann integrable on Q

Q: When is f R-integrable on Q?

Agony: 不知道 A 的值

Discussion: $\left|\sum_{R\in P} f(x_R)|R| - A\right| < \varepsilon$

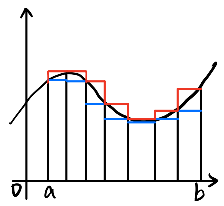
$\Longleftrightarrow A-\varepsilon < \sum_{R\in P} f(x_R)|R| < A+\varepsilon$

$\Rightarrow \begin{cases} A-\varepsilon \le \sum_{R\in P} \inf_{x\in R} f(x)|R| & \text{记 } \inf_{x\in R} f(x) = m_R(f) \\ \sum_{R\in P} \sup_{x\in R} f(x)|R| \le A+\varepsilon & \text{记 } \sup_{x\in R} f(x) = M_R(f) \end{cases}$

$\Rightarrow A-\varepsilon \le \sum_{R\in P} m_R(f)|R| \le \sum_{R\in P} M_R(f)\cdot|R| \le A+\varepsilon$

Define $L(f;P) = \sum_{R\in P} m_R(f)|R|$ lower sum determined by P

$U(f;P) = \sum_{R\in P} M_R(f)\cdot|R|$ upper sum determined by P



由此可推出一些 Facts

3. Fact 1

f is R-integrable on Q

$\Longleftrightarrow$ $\forall \varepsilon>0$, $\exists \delta>0$, s.t. $0 \leq U(f;P)-L(f;P)<\varepsilon$, whenever $\|P\|<\delta$

证明:

(见作业)

4. Fact 2: (more user-friendly)

f is R-integrable on Q

$\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists$ partition P' s.t. $(0 \le)\ U(f;P') - L(f;P') < \varepsilon$ (※)

证明:

① proof of "$\Rightarrow$":

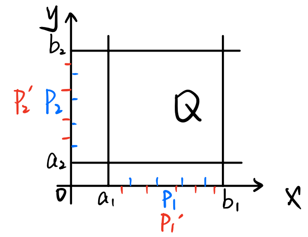
trival by Fact 1.

② proof of "$\Leftarrow$":

(W.T.S. Fact 1 holds)

Define the 3-rd partition $P'' = P \cup P' = (P_1 \cup P_1', P_2 \cup P_2', \cdots, P_n \cup P_n')$

P'' 被称为 common refinement of P & P'.



Observe: can obtain P'' from P as follows

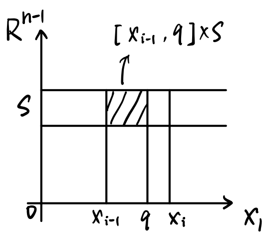
$$P \xrightarrow{\text{add 1 pt}} P^1 \xrightarrow{\text{add 1 pt}} P^2 \xrightarrow{\text{add 1 pt}} \cdots\cdots \xrightarrow{\text{add 1 pt}} P^k = P''$$

Suppose $x_{i-1} \le q \le x_i$ for some i,

WLOG, assume $P^1 = (P_1 \cup \{q\}, P_2, \cdots, P_n)$

$$0 \le L(f;P^1) - L(f;P) = \sum_{S \in (P_2,\cdots,P_n)} \{ m_{[x_{i-1},q]\times S}(f) \cdot |[x_{i-1},q]\times S| \} + m_{[q,x_i]\times S}(f) \cdot |[q,x_i]\times S| - \sum_{S \in (P_2,\cdots,P_n)} \{ m_{[x_{i-1},x_i]\times S}(f) \cdot |[x_{i-1},x_i]\times S| \}$$

(前两项 > 最后一项)



$\because$ $|f| \le$ constant M on Q

$$\therefore\ L(f;P^1) - L(f;P) \le 2M \sum_{S \in (P_2,\cdots,P_n)} |S| (x_i - x_{i-1})$$

$$\le 2M \|P\| \sum_{S \in (P_2,\cdots,P_n)} |S| \quad (x_i - x_{i-1} \le \|P\|)$$

$$= 2M \|P\| (b_2 - a_2) \cdots (b_n - a_n)$$

$$\le 2M \|P\| \cdot (\text{width of } Q)^{n-1}$$

$\therefore$ $0 \le L(f;P^1) - L(f;P) \le 2M \|P\| \cdot (\text{width of } Q)^{n-1}$

Similarly, $0 \le L(f;P^2) - L(f;P^1) \le 2M \|P\| \cdot (\text{width of } Q)^{n-1}$

$\cdots\cdots$

$0 \le L(f;P^k) - L(f;P^{k-1}) \le 2M \|P\| \cdot (\text{width of } Q)^{n-1}$

$$\therefore \quad 0 \le L(f;P'') - L(f;P) \le 2Mk\|P\| \cdot (\text{width of } Q)^{n-1} \quad (*)$$

Similarly, $0 \ge U(f;P'') - U(f;P) \ge -2Mk\|P\| \cdot (\text{width of } Q)^{n-1} \quad (\#)$

$(*) - (\#)$ 得:

$$0 \le L(f;P'') - L(f;P) - U(f;P'') + U(f;P) \le 4Mk\|P\| \cdot (\text{width of } Q)^{n-1}$$

$$\begin{aligned} \Rightarrow \quad U(f;P) - L(f;P) &\le 4Mk\|P\| \cdot (\text{width of } Q)^{n-1} + U(f;P'') - L(f;P'') \\ &< 4Mk\|P\| \cdot (\text{width of } Q)^{n-1} + \varepsilon \quad (\text{由}(☆)) \\ &< 2\varepsilon \end{aligned}$$

if $\|P\| < \delta$, which is taken s.t. $4Mk\|P\| \cdot (\text{width of } Q)^{n-1} < \varepsilon$

注: 在refine partition之后,

$U(f,P) \ge U(f,P \cup P')$ $\quad$ $L(f,P) \le L(f,P \cup P')$