Lecture 23

# §1 Conservative field & 基本的定义、性质

## 1、Conservative field (保守场) 的定义

- A conservative force is a force with the property that the total work done in moving a particle between two points is independent of the path taken.
- Gravitational force is an example of a conservative force.

**Definition**

Let $\mathbf{F}$ be a vector field defined on an open region $D$. Then $\mathbf{F}$ is said to be conservative on $D$ if the following condition holds: for any two points $A$ and $B$ in $D$, if $C_1$ and $C_2$ are piecewise smooth curves in $D$ from $A$ to $B$, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Any such line integral, which depends only on the initial and terminal points (but not the path itself), is said to be path independent.

Consider the vector field $\mathbf{F}(x, y) := \langle -y, x \rangle$. One can check that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

if $C_1$ is the line segment from $(0,0)$ to $(1,1)$ and $C_2$ is the curve given by

$$\langle t, t^2 \rangle, \quad t \in [0,1].$$

Although both $C_1$ and $C_2$ are smooth curves from $(0,0)$ to $(1,1)$, the line integrals of $\mathbf{F}$ along them are different.

## 2、Theorem (Fundamental Theorem of Line Integrals) (FTLI)

**Theorem (Fundamental Theorem of Line Integrals)** (FTLI)

*Let $C$ be a piecewise smooth curve from $A$ to $B$, parametrized by $\mathbf{r}(t)$, with $t \in [a, b]$. Let $f$ be a real-valued function such that $\nabla f$ is continuous on a region $D$ containing $C$. Then*

Piecewise Smooth assumed

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

1° 证明：

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle\, dt$$

$$= \int_a^b f_x \cdot x'(t) + f_y \cdot y'(t) + f_z \cdot z'(t)\, dt$$

由 chain rule: $= \int_a^b \frac{d}{dt}\left(f(\vec{r}(t))\right) dt$

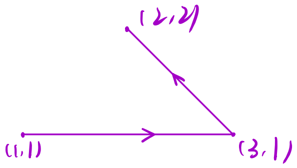
$$\overset{FTC}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A)$$

2° 推论

任一连续的 gradient field (梯度场) is conservative

例: Find the work done by force $\vec{F}=\langle x,y\rangle$ along $C$, where $C$ is




Sol: · $F=\nabla f$, where $f(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2$

· $\text{Work}=\int_C \vec{F}\cdot d\vec{r}$

$=\int_{(1,1)}^{(2,2)} \nabla f\cdot d\vec{r}$

$=f(2,2)-f(1,1)$

$=3$

3. potential function (势函数) 的定义

> Definition
> If $\mathbf{F}=\nabla f$, then $f$ is called a potential function for $\mathbf{F}$.

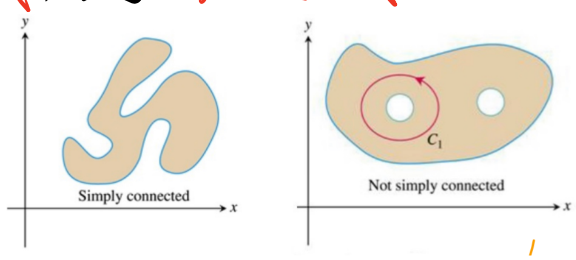
* $\vec{F}$ 被称为有势场

4. Connected region (连通区域) 的定义

> Definition
> A region $D$ is said to be connected if any two points in $D$ are connected by a curve lies entirely in $D$.

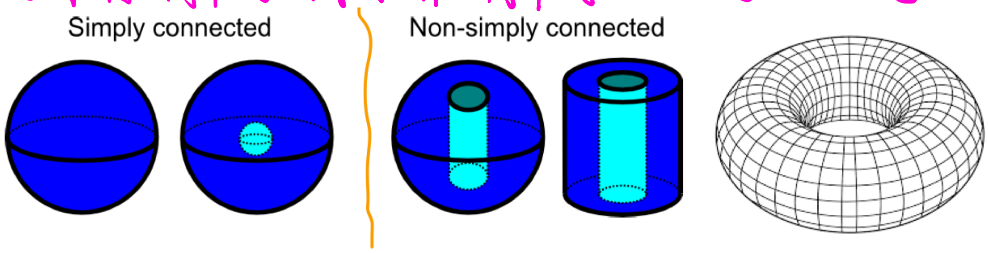
5. Simple connected region (单连通区域) 的定义

若平面区域 D 内的任意一条简单闭曲线的内部都包含于 D 之中, 则称 D 为单连通区域, 否则为多连通区域.




* 另一种理解思路: 若平面区域 D 内的任意一条简单闭曲线都可以在不脱离 D 的前提下 "shrank" 至任意一点, 则 D 为单连通区域.

如果将 "简单闭曲线" 换作 "简单闭壳面", 则该思路可以用于判断单连通空间 (3D)



## 5. (Exact) differential forms ((恰当)微分形式)

Def: · An expression $Mdx+Ndy+Pdz$ is a differential form.
· A differential form is exact on $D$ if it is the total differential $df$ for some real-valued (i.e. scalar) function $f$ (on $D$).

e.g. 考虑 $\vec{F}=\langle M,N,P\rangle$ (3D vector field). 则

· $\int_C \vec{F}\cdot d\vec{r}=\int_C Mdx+Ndy+Pdz$ (differential form)

若 $\vec{F}=\nabla f$ 为一个 gradient field. 则

· $\int_C Mdx+Ndy+Pdz=\int_C f_x dx+f_y dy+f_z dz$ (total differential)

## §2 Conservative field 的性质与条件(第二类曲线积分与路径无关的条件)

### 1. 性质&条件一：与 gradient field 的联系

**Theorem**
*Let* $\mathbf{F}$ *be a vector field whose components are continuous on an open connected region* $D$. *Then* $\mathbf{F}$ *is conservative on* $D$ *if and only if* $\mathbf{F}$ *is a gradient field (that is,* $\mathbf{F}=\nabla f$ *for some* $f$ *defined on* $D$*).*

设$\vec{F}$为定义在$D$上的一个光滑向量场，则$\vec{F}$在$D$内为保守场的充要条件为$\vec{F}$是有势场(梯度场)

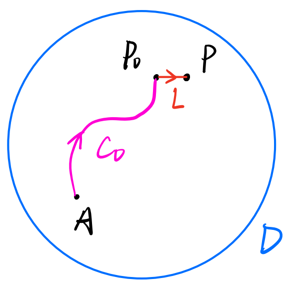
证明(二维)：

"$\Leftarrow$": 在 Fundamental Theorem of Line Integrals 处已证明.

"$\Rightarrow$":

· Assume that $\vec{F}$ is conservative.
Fix $A:=(a,b)\in D$.
Define $f(x,y):=\int_{(a,b)}^{(x,y)}\vec{F}\cdot d\vec{r}$

· Consider $P:=(x,y)$
Fix $P_0:=(x_0,y)$ in $D$ (say $x_0<x$)
令 $C_0$ 为一个分段连续的曲线，由$A$到$P_0$.
令 $L$ 为一个线段，由$P_0$到$P$.




· 则 $f(x,y)=\int_A^P \vec{F}\cdot d\vec{r}$

$$=\int_{C_0}\vec{F}\cdot d\vec{r}+\int_L \vec{F}\cdot d\vec{r}$$

$$=K+\int_{x_0}^{x}\vec{F}(\vec{r}(t))\cdot r'(t)\,dt$$

(令 $\int_{C_0}\vec{F}\cdot d\vec{r}$ 为 $K$, $L$: $\vec{r}(t)=\langle t,y\rangle$, $x_0\le t\le x$, $r'(t)=\langle 1,0\rangle$)

$$=K+\int_{x_0}^{x}M(t,y)\,dt$$

- 考虑将点P沿x轴方向移动一点：

$$\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x}\int_{x_0}^{x} M(t,y)\,dt$$
$$= M(x,y)$$

- 类似地，$\frac{\partial}{\partial y} f(x,y) = N(x,y)$
- 因此，$\nabla f = \vec{F}$，$\vec{F}$ 为一个 gradient field.

2. 性质&条件二：The loop property

- A given parametrization $\mathbf{r}(t)$, $a \le t \le b$, determines an orientation (or direction) of a curve $C$, with the positive direction corresponding to increasing values of $t$.
- In general, if $C$ is a curve with a given orientation, we use $-C$ to denote the same curve with the opposite orientation. Then

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}.$$

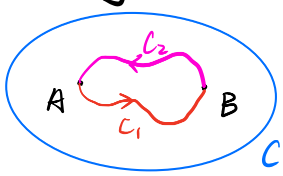
> **Theorem**
> *The following statements are equivalent for any vector field* $\mathbf{F}$.  Means: (a) $\Leftrightarrow$ (b); (a) if and only if (b).
> (a) *The field* $\mathbf{F}$ *is conservative on* $D$.
> (b) *For every closed curve* $C$ *in* $D$, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

在区域D内任意取定两点A, B. 曲线积分 $\int_{AB} Pdx + Qdy$ 在区域D内与路径无关的充要条件为：对于D内任意一条简单逐段光滑闭曲线C，沿C的曲线积分为0，即

$$\oint_{C^+} Pdx + Qdy = 0$$

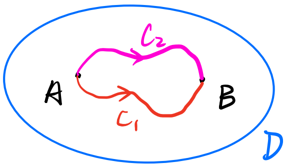
证明：

"$\Rightarrow$"：

- Assume $\vec{F}$ is conservative
  Fix any closed curve $C$.
  Fix any two distinct points A and B on C
- 


  Define $C_1$ & $C_2$ as above
  Then $C_1$ & $-C_2$ are curves from A to B
- $$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1 \cup -C_2} \vec{F}\cdot d\vec{r}$$
$$= \int_{C_1} + \int_{-C_2}$$
$$= \int_{C_1} - \int_{C_2} \quad (\text{path independent})$$
$$= 0$$
- Loop property is satisfied by $\vec{F}$

" $\Leftarrow$ ":

- Assume loop property.
  Fix any two distinct points A and B on D
- 


  Define $C_1$ & $C_2$ as above
  Then $C_1 \cup -C_2$ is a closed curve $C$
  Then $\oint_C \vec{F} \cdot d\vec{r} = 0$
- Since $\oint_C = \int_{C_1} + \int_{-C_2} = \int_{C_1} - \int_{C_2}$,
  $\int_{C_1} = \int_{C_2}$

例：**Example**

The vector field defined by

$$\mathbf{F}(x, y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

is not conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$, since its line integral along the unit circle is not equal to 0.

Let $C$ be $x^2+y^2=1$, $x=\cos t$, $y=\sin t$, $0 \le t \le 2\pi$

$$\oint_C \vec{F} \cdot d\vec{r} = \int M\,dx + N\,dy$$
$$= \int_0^{2\pi} -\sin t \cdot (-\sin t) + \cos t \cdot (\cos t)\, dt$$
$$= 2\pi \neq 0$$

$\therefore \vec{F}$ is not conservative.

## 3、性质&条件三：Component test

> **Theorem (Component Test for Conservative Fields)** (in $\mathbb{R}^2$)
>
> *Let $\mathbf{F}(x, y) := \langle M(x, y), N(x, y) \rangle$ be a vector field on an open simply connected domain $D$, such that $M$ and $N$ have continuous partial derivatives. Then $\mathbf{F}$ is conservative on $D$ if and only if*
>
> $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
>
> *on $D$.*

D是单连通区域，函数 $P(x,y)$ 与 $Q(x,y)$ 在D内有一阶连续偏导数，则在区域D内任意取定两点 A, B. 曲线积分 $\int_{AB} P\,dx + Q\,dy$ 在区域D内与路径无关的充要条件为

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

在D内处处成立

例:

**Example**

The vector field defined by $\mathbf{F}(x, y) := \langle -y, x\rangle$ is not conservative on $\mathbb{R}^2$, since it does not pass the component test.

$$\frac{\partial M}{\partial y} = -1 \ , \ \frac{\partial N}{\partial x} = 1$$

例:

**Problem**

Consider the vector field defined by

$$\mathbf{F}(x, y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

We have shown that $\mathbf{F}$ is not conservative. On the other hand, one can check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This seems to contradict the component test. What is wrong here?

$D = \mathbb{R}^2 \setminus \{0,0\}$ not simply connected

例:

**Example**

Consider the vector field defined by

$$\mathbf{F}(x, y) := \langle 3 + 2xy, x^2 - 3y^2 \rangle.$$

(a) Show that $\mathbf{F}$ is conservative on $\mathbb{R}^2$.

(b) Find a potential function for $\mathbf{F}$.

(c) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is parametrized by

$$\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle, \quad 0 \le t \le \pi.$$

(a) $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$

(b) Since $M = f_x = 3 + 2xy$,

$f(x,y) = 3x + x^2 y + K(y)$

Since $N = f_y = x^2 - 3y^2$,

$f(x,y) = 3x + x^2 y - y^3 + C$

(c) $\int_C \vec{F} \cdot d\vec{r} = f(0, -e^{\pi}) - f(0,1)$

$= e^{3\pi} + 1$

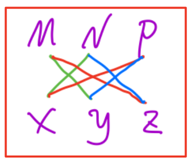
☆推广: Component test in 3D

## Component Test in 3D

M, N, P cts on a simply connected region

There is also a component test for vector fields $\mathbf{F} := \mathbf{F}(x, y, z)$ in $\mathbb{R}^3$. Under assumptions similar to those ~~on Page 25~~ for the 2D component test, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$




证明:

"⇒":

- Assume that $\vec{F}$ is conservative on $D$.
- Then $\exists f$ s.t. $\vec{F} = \nabla f = \langle f_x, f_y \rangle$ on $D$. Now

$$\frac{\partial M}{\partial y} = f_{xy} = f_{yx} = \frac{\partial N}{\partial x}$$

"$\Leftarrow$":

- Assume $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ on $D$
- 令 $C$ 为 $D$ 上的任意一条闭曲线，先假定 $C$ 为 simple
  因为 $D$ 为单连通区域，$C$ 围成的区域 $R$ 全部落在 $D$ 中
  由格林公式：
  $$\oint_C \vec{F}\cdot d\vec{r}=\iint_R \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\,dA=0$$
- 若 $C$ 不为 simple，可以 decompose it into simple closed curve.
- 由此，对 $\forall$ closed curve $C$ in $D$，$\oint_C \vec{F}\cdot d\vec{r}=0$
  由 loop property，$\vec{F}$ is conservative on $D$

4. 性质&条件四：Exact differential forms

$Mdx+Ndy+Pdz$ is exact if and only if $\vec{F}$ is conservative (with potential function $f$)

若函数 $P(x,y)$, $Q(x,y)$ 在单连通区域 $D$ 上有一阶连续偏导，则等式 $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ 在 $D$ 上恒成立的充分必要条件是 $Pdx+Qdy$ 恰是某个函数 $u(x,y)$ 的全微分，即有

$$du(x,y)=Pdx+Qdy$$

此时有：

$$\int_{\overset{\frown}{AB}} Pdx+Qdy=\int_A^B du=u(B)-u(A)$$

$u$ 被称作 $Pdx+Qdy$ 的原函数

5. 总结

We summarize some of the key results ~~of this week~~ about conservative fields as a theorem.

**Theorem**

*Let* **F** *be a vector field whose components are continuous on an open connected region D. Then the following conditions are equivalent.*

(a) *The field* **F** *is conservative on D.*

(b) *There exists a function f such that* $\mathbf{F}=\nabla f$ *on D.*

(c) *For every closed curve C in D,* $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$.

Simply Connected $D$

Under certain assumptions, whether (a) holds can be tested using the component test.