Lecture 19

§1 Physical Pendulum

1. Physical Pendulum (复摆)

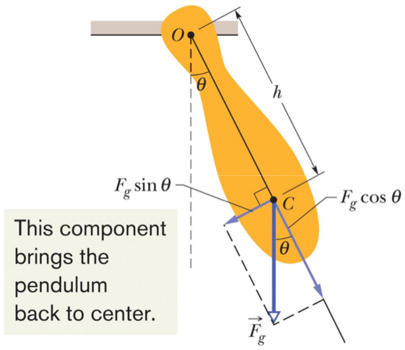
1° Complicated distribution of mass

2° 与单摆的唯一区别：h，COG 到 pivot point O 的距离

2. 周期

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{I}{mgL}} \Rightarrow T=2\pi\sqrt{\frac{I}{mgh}}$$

- 条件：小角度 swing
- I：绕 O 点的转动惯量
- T 取决于 I 与 h
- 若 O 位于 COM 处：$h=0$，无旋转



3. 用复摆测算 g

$$h=\frac{1}{2}L$$

$$I=I_c+md^2=\frac{1}{12}mL^2+m\left(\frac{L}{2}\right)^2=\frac{1}{3}mL^2$$

$$T=2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg\frac{1}{2}L}}=2\pi\sqrt{\frac{2L}{3g}}$$

$$g=\frac{8\pi^2L}{3T^2}$$

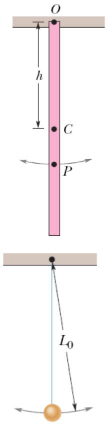
例：**Checkpoint 5**

Three physical pendulums, of masses m_0, $2m_0$, and $3m_0$, have the same shape and size and are suspended at the same point. Rank the masses according to the periods of the pendulums, greatest first.

Answer: All the same: mass does not affect the period of a pendulum (Independent of m)

$$T=2\pi\sqrt{\frac{I}{mgh}}$$

例: **Problem**



In the left figure, a meter stick (length $L = 1.0$ m) swings about a pivot point at one end, at distance $h = L/2$ from the stick's center of mass.

a) What is the period of oscillation T?

b) A simple pendulum whose length L_0 is chosen so that the periods of the two pendulums are equal. What is the distance L_0?

Because the mass in the meter stick is distributed across its length, the stick is not a simple pendulum → Physical Pendulum

4

Solution:

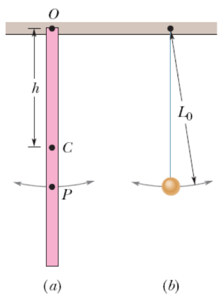
- The period is

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2+\frac{1}{4}L^2}{g\frac{1}{2}L}} = 2\pi\sqrt{\frac{2L}{3g}} = 1.64\ s$$

- The result is independent of m.
- If the simple pendulum has the same length as the physical pendulum, do they have the same period?

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- NO, the period of oscillation of the meter stick is different with that of a simple pendulum of the same length.



- (b) Simple Pendulum

$$T = 2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$L_0 = \frac{2}{3}L = 66.7\ cm$$

§2 简谐运动与匀速圆周运动

1. 关系

1° 匀速圆周运动：P'的角速度为 ω，初始角位置为 ϕ

2° 简谐运动可以视作匀速圆周运动在 x 轴上的 projection

2. Position

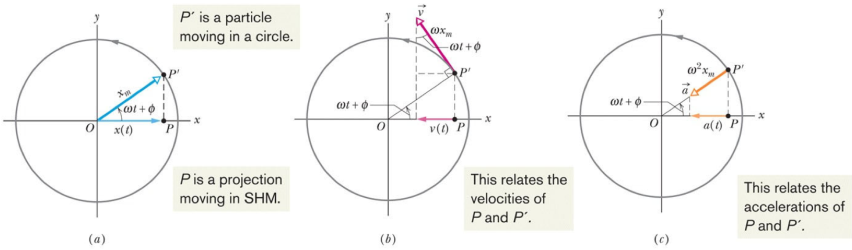
- $\theta(t) = \omega t + \phi$
- $x(t) = x_m \cos[\theta(t)] = x_m \cdot \cos(\omega t + \phi)$

3. Velocity

- $V = \omega R = \omega x_m$
- $V_x = V(t) = -\omega x_m \sin\theta = -\omega x_m \sin(\omega t + \phi)$

4、Acceleration

- $a = a_r = \frac{v^2}{R} = \omega^2 R = \omega^2 \cdot x_m$
- $a_x = a(t) = -\omega^2 x_m \cdot \cos\theta = -\omega^2 x_m \cos(\omega t + \varnothing)$



- Thus, whether we look at the displacement, the velocity, or the acceleration, the projection of uniform circular motion is indeed simple harmonic motion

10

§3 Damped oscillation (阻尼振动)

1、Damped simple harmonic motion

1° 若 oscillator 的运动被外力 reduce，则 oscillator 与它的运动 is said to be damped

2° 机械能转化为内能

3° Damping force (阻尼力) $\vec{F_d}$：

- 方向与 $\vec{v}$ 相反
- $F_d = -bv$

4° b: damping constant (阻尼系数)

取决于 vane 和 liquid 的特性

5° SI-unit：kg/s

- Newton's Second Law

$$F_{net} = F_d + F_{spring} = ma$$
$$-bv - kx = ma$$
$$-b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$
$$m\frac{dx^2}{dt^2} + b\frac{dx}{dt} + kx = 0$$

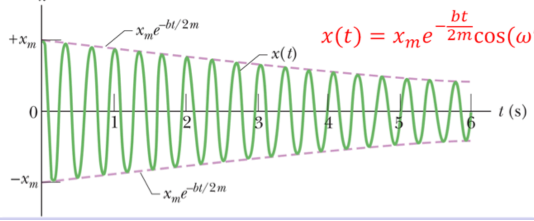
2、Damped oscillation

- 牛二：

$$F_{net} = F_d + F_{spring} = ma$$
$$-bv - kx = ma$$

- The solution is

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \varnothing)$$



- 解得

$$x(t) = x_m \cdot e^{-\frac{bt}{2m}} \cos(\omega' t + \varnothing)$$

- Angular frequency 满足

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

- 若 $b = 0$，$\omega' = \omega = \sqrt{\frac{k}{m}}$
- 若 $b \ll \sqrt{km}$，$\omega' \approx \omega = \sqrt{\frac{k}{m}}$

- Damped energy：随时间递减

若 damping 比较小，可以用 $x_m e^{-\frac{bt}{2m}}$ 代替 x_m

$$E(t) = \frac{1}{2} k x_m^2 e^{-\frac{bt}{m}}$$

- The Angular Frequency is

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

- If $b = 0$，$\omega' = \omega = \sqrt{\frac{k}{m}}$

$$x(t) = x_m \cos(\omega t + \varnothing)$$

- If $b \ll \sqrt{km}$，$\omega' \approx \omega = \sqrt{\frac{k}{m}}$
- Damped Energy: not a constant, decreases with time
- If the damping is small, we can replace x_m with $x_m e^{-\frac{bt}{2m}}$

$$E(t) = U(t) + K(t)$$
$$E(t) \approx \frac{1}{2} k x_m^2 e^{-\frac{bt}{m}}$$

例:

Problem

Checkpoint 6

Here are three sets of values for the spring constant, damping constant, and mass for the damped oscillator of Fig. 15-16. Rank the sets according to the time required for the mechanical energy to decrease to one-fourth of its initial value, greatest first.

Set 1	$2k_0$	b_0	m_0
Set 2	k_0	$6b_0$	$4m_0$
Set 3	$3k_0$	$3b_0$	m_0

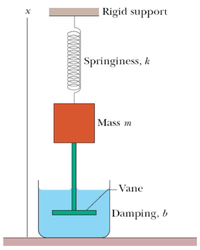
$$E(t) = e^{-\frac{bt}{m}}E_0$$
$$\frac{1}{4}E_0 = e^{-\frac{bt}{m}}E_0$$
$$t = \frac{m}{b}ln4$$

Answer: 1,2,3

例:

Problem

For the damped oscillator in the figure, ***m* = *250 g***, ***k* = 85 N/m**, and ***b* =*70 g/s***.



a) What is the period of the motion?

b) How long does it take for the amplitude of the damped oscillations to drop to **half** its initial value?

c) Approximately how long does it take for the mechanical energy to drop to **one-half** its initial value?

Solution:

(a) The period is related with angular frequency:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\frac{85}{0.25} - \frac{0.07^2}{4 \times 0.25^2}} \approx \sqrt{\frac{85}{0.25}}$$
$$= 18.44\, rad/s$$
$$T = \frac{2\pi}{\omega'} = 0.34\, s$$

- The period is approximately the same as the undamped oscillator

(b) The displacement function is

$$x(t) = x_m e^{-\frac{bt}{2m}}\cos(\omega' t + \emptyset)$$

- Initially, $t = 0\, s \rightarrow x(0) = x_m$
- After time t,

$$A(t) = x_m e^{-\frac{bt}{2m}} = \frac{1}{2}x_m$$
$$e^{-\frac{bt}{2m}} = \frac{1}{2}$$
$$t = \frac{2mln2}{b} = 5.0\, s$$

- About 15 cycles to reach half amplitude.

(c) For small damping,

$$E(t) \approx \frac{1}{2}kx_m^2 e^{-\frac{bt}{m}} = \frac{1}{2}E_0 = \frac{1}{2} \times \frac{1}{2}kx_m^2$$
$$e^{-\frac{bt}{m}} = \frac{1}{2}$$
$$t = \frac{mln2}{b} = 2.5\, s$$

- This is the half the time calculated in (b), about 7.5 cycles.

§4 Forced oscillation and resonance

1. Free oscillation (自由振动)

$\omega = \sqrt{\frac{k}{m}}$

2. Forced (or driven) oscillation (受迫振动) and resonance (共振)

- 受到一个周期性外力而振动
- 振子与 driving force 的角频率 ω_d 相同

  $x(t) = x_m \cos(\omega_d t + \phi)$
- 当 $\omega_d = \omega$ 时，displacement amplitude x_m (approximately) 与 velocity amplitude v_m 到达最大值 (resonance)

Additional Information-Forced Oscillation for Linear Oscillator

- Newton's 2nd Law and Hooke's Law

$$F_{net} = ma$$

- When the oscillator is driven with a sinusoidal Force

$$F(t) = F_0 \cos(\omega_d t)$$

- Then,

$$F(t) + (-kx) = ma$$
$$m\frac{d^2x}{dt^2} + kx = F_0 \cos(\omega_d t)$$

- The general solution is

$$x_g = x_c + x_d$$

- Where, x_c is the (complementary) solution to the equation $m\frac{d^2x}{dt^2} + kx = 0$
- Let's assume

$$x_c = x_0 cos\omega_0 t$$

20

- Where ω_0 is the natural angular frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- Let's assume

$$x_d = Acos\omega_d t$$
$$\frac{d^2x}{dt^2} = -A\omega_d^2 cos\omega_d t$$

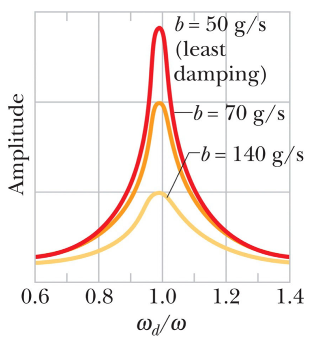
- Put it back into the differential equation

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos(\omega_d t)$$
$$m \times -A\omega_d^2 cos\omega_d t + k \times Acos\omega_d t = F_0 \cos(\omega_d t)$$
$$(k - m\omega_d^2)A = F_0$$
$$A = \frac{F_0}{k - m\omega_d^2} = \frac{F_0}{m(\omega_0^2 - \omega_d^2)}$$

- Thus,

$$x_g = x_c + x_d = x_0 cos\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega_d^2)} cos\omega_d t$$

Resonance



- If $\omega_d = \omega_0$, the driving frequency approaches that of the natural frequency of the object.
- then the object will oscillate with a very large amplitude
- Resonance can be generated

22

Resonance

On 12 April 1831, the Broughton suspension bridge (in England) collapsed, reportedly due to mechanical resonance induced by troops marching in step. As a result of the incident, the British Army issued an order that troops should "break step" when crossing a bridge.

Broughton-suspension-bridge

Forced oscillations at resonant frequency may result in rupture or collapse

Summary

- Frequency

$$1\ hertz = 1\ Hz = 1\ oscillation\ per\ second = 1\ s^{-1}$$

- Period

$$T = \frac{1}{f}$$

- Simple Harmonic Motion

$$x(t) = x_m \cos(\omega t + \emptyset)$$

$$\omega = \frac{2\pi}{\mathrm{T}} = 2\pi f$$

$$v(t) = -\omega x_m \sin(\omega t + \emptyset)$$

$$v_m = \omega x_m$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \emptyset) = -\omega^2 x(t)$$

$$a_m = \omega^2 x_m$$

Summary

- The Linear Simple Harmonic Oscillator

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Energy in SHM

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 cos^2(\omega t + \emptyset)$$

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2 sin^2(\omega t + \emptyset)$$

$$E = U + K = \frac{1}{2}kx_m^2$$

Summary

- Pendulum
  - Torsion Pendulum

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

  - Simple Pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

  - Physical Pendulum

$$T \approx 2\pi\sqrt{\frac{I}{mgh}}$$

Summary

- Simple Harmonic Motion and Uniform Circular Motion: SHM is the projection of UCM onto the diameter of the circle in which the circular motion occurs.
- Damped Harmonic Motion

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \emptyset)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$E(t) \approx \frac{1}{2}kx_m^2 e^{-\frac{bt}{m}}$$

Summary

Forced Oscillations and Resonance

- If an external driving force with angular frequency ω_d acts on an oscillating system with natural angular frequency ω, the system oscillates with angular frequency ω_d. The velocity amplitude v_m of the system is greatest when

$$\omega_d = \omega$$

- a condition called resonance, at which the amplitude x_m of the system is approximately greatest.