Lecture 16

对于给定 states i, j, 希望知道 transition probabilities $P_{ij}(t)$, $\forall t \geq 0$

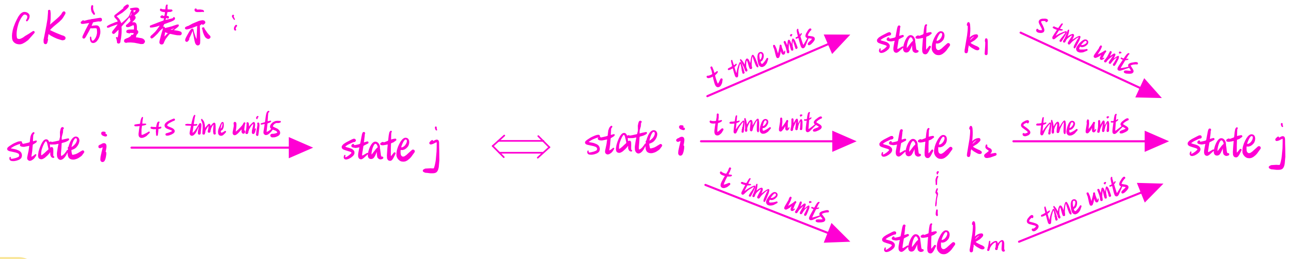
§1 Chapman-Kolmogorov equation

1. Theorem: Chapman-Kolmogorov equation (C-K 方程)

Chapman-Kolmogorov equation 为:

$$P_{ij}(t+s) = \sum_{k\in E} P_{ik}(t) P_{kj}(s), \quad t, s \geq 0, \; i, j \in E$$

注: CK 方程表示:



证明:

$$\begin{aligned}
P_{ij}(t+s) &= P\{X(t+s)=j \mid X(0)=i\} \\
&= \sum_{k\in E} P\{X(t+s)=j, X(t)=k \mid X(0)=i\} \\
&= \sum_{k\in E} P\{X(t+s)=j \mid X(t)=k, X(0)=i\} \cdot P\{X(t)=k \mid X(0)=i\} \\
&= \sum_{k\in E} P_{ik}(t) P_{kj}(s)
\end{aligned}$$

2. CK 方程的矩阵表示

根据 C-K 方程, 有

$$P(t+s) = P(t)\cdot P(s)$$

进一步可推出:

$$P(2t) = P(t+t) = P(t)^2$$

证明:

$$\begin{aligned}
P_{ij}(t+s) &= \sum_{k\in E} P_{ik}(t) P_{kj}(s) \\
&= [P_{i1}(t), P_{i2}(t), \cdots, P_{i|E|}(t)] \cdot [P_{1j}(s), P_{2j}(s), \cdots, P_{|E|j}(s)]^T \\
&= i\text{-th row of } P(t) \cdot j\text{-th column of } P(s)
\end{aligned}$$

$$\Rightarrow P(t+s) = P(t)\cdot P(s)$$

e.g.

- Suppose

$$P(0.1) = \begin{pmatrix} 0.7486327 & 0.1607327 & 0.0906346 \\ 0.0783127 & 0.8310527 & 0.0906346 \\ 0.0041073 & 0.0865273 & 0.9093654 \end{pmatrix}$$

- Compute

$$\begin{aligned}
&\mathbb{P}\{X(.4)=3, X(.2)=1, X(.1)=3 \mid X(0)=2\} \\
&= (P(0.1))^2_{1,3} P_{3,1}(0.1) P_{2,3}(.1) \\
&= (0.164840)(0.0041073)(0.0906346).
\end{aligned}$$

§2 Transition probability matrix 的求解

1. Transition probability matrix 的求解 (Generator matrix 的 motivation)

令 G 为 the generator matrix，则

$$P(s) = \sum_{k=0}^{\infty} \frac{s^k G^k}{k!} \quad \text{，其中 } G^0 = I$$

证明：

由 $P(t+s) = P(t)P(s)$，$\forall t, s \geq 0$ 可知，

$$\begin{aligned} P'(s) &= \lim_{t\to 0^+} \frac{P(s+t)-P(s)}{t} \\ &= \lim_{t\to 0^+} \frac{P(s)P(t)-P(s)}{t} \\ &= \lim_{t\to 0^+} \frac{(P(t)-I)}{t} P(s) \\ &= P'(0+) \cdot P(s) \quad , \quad s \geq 0 \qquad (P(0) = I) \end{aligned}$$

其中，$P'(0+)$ 存在，且 $P'(0+) = G$ (#)

解 $P'(s) = GP(s)$ (被称为 Kolmogorov backward equation) 可得：

$$\begin{aligned} P(s) &= e^{sG} \quad (\text{尚未定义 } e^G \text{ 如何计算}) \\ &= \sum_{k=0}^{\infty} \frac{s^k G^k}{k!} \end{aligned}$$

证明：(#) 式

W.T.S. $P_{ij}(0+) = G_{ij}$

① 当 $i \neq j$ 时，有

$$\begin{aligned} P_{ij}'(0+) &= \lim_{t\downarrow 0} \frac{P_{ij}(t) - P_{ij}(0)}{t} \\ &= \lim_{t\downarrow 0} \frac{P(X(t)=j \mid X(0)=i)}{t} \qquad (P_{ij}(0) = 0 \text{ ， } P_{ij}(t) = P(X(t)=j \mid X(0)=i)) \\ &\approx \lim_{t\downarrow 0} \frac{P(\text{one jump in } [0,t) \text{ from } i \text{ to } j)}{t} \qquad (t \to 0 \text{ 时，} [0,t) \text{ 内至多一次 jump}) \\ &= \lim_{t\downarrow 0} \frac{P(\text{one jump in } [0,t)) \cdot P(i \to j \mid \text{one jump})}{t} \\ &= \lim_{t\downarrow 0} \frac{P(Exp(\lambda_i) < t) \cdot J_{ij}}{t} \\ &= \lim_{t\downarrow 0} \frac{(1 - e^{-\lambda(i)t}) \cdot J_{ij}}{t} \\ &= \lim_{t\downarrow 0} \frac{\lambda(i) e^{-\lambda(i)t} \cdot J_{ij}}{1} \qquad (\text{L'Hôpital's rule}) \\ &= \lambda(i) J_{ij} \\ &= G_{ij} \end{aligned}$$

② 当 $i = j$ 时，注意到

$$\sum_j P_{ij}'(0) = \lim_{t\downarrow 0} \frac{\sum_j P_{ij}(t) - \sum_j P_{ij}(0)}{t} = \lim_{t\downarrow 0} \frac{0}{t} = 0$$

因此，$P_{ii}'(0) = -\sum_{j\neq i} P_{ij}'(0)$

$$= -\sum_{j\neq i} \lambda_{ij}$$

$$= G_{ii}$$

注：① Forward equation：$P'(t) = P(t)\,G$，其与 backward equation 的解相同.

② Backward equation 永远成立

Forward equation 在 state space 为 infinity 时可能不成立

③ 对于 multiple-states CTMC，使用 backward equation 求解 $P(t)$ 仍不容易

例1：机器在损坏前的工作时长服从 exponential distribution with mean $\frac{1}{\lambda}$ ($Exp(\lambda)$)

修理时长服从 exponential distribution with mean $\frac{1}{\mu}$ ($Exp(\mu)$)

若机器在 time 0 时处于 work condition，

求其在 time t 时处于 work condition 的概率

- 可作出 transition rate diagram

$$\begin{tikzcd} 0 \arrow[r, shift left, "\boxed{\lambda}"] & 1 \arrow[l, shift left, "\boxed{\mu}"] \end{tikzcd}$$

因此 generator matrix G 为

$$G = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

- 由 backward equation $P'(t) = G\,P(t)$ 可知：

$$P_{ij}'(t) = \sum_k G_{ik}\cdot P_{kj}(t)$$

代入 generator matrix，有

$$\begin{cases} P_{00}'(t) = G_{01}P_{10}(t) + G_{00}P_{00}(t) = \lambda P_{10}(t) - \lambda P_{00}(t) & (*) \\ P_{10}'(t) = G_{10}P_{00}(t) + G_{11}P_{10}(t) = \mu P_{00}(t) - \mu P_{10}(t) & (\#) \end{cases}$$

（由于 $P_{01}(t) = 1 - P_{00}(t)$，$P_{10}(t) = 1 - P_{11}(t)$，因此仅需列出两个方程）

- 对 (*) 两侧同乘 μ，对 (#) 两侧同乘 λ，随后相加，有：

$$\mu P_{00}'(t) + \lambda P_{10}'(t) = 0$$

$$\Rightarrow (\mu P_{00}(t) + \lambda P_{10}(t))' = 0$$

$$\Rightarrow \mu P_{00}(t) + \lambda P_{10}(t) = C$$

代入 $t=0$ 可得：

$$C = \mu P_{00}(0) + \lambda P_{10}(0)$$

$$= \mu\cdot 1 + \lambda\cdot 0$$

$$= \mu$$

因此，

$$P_{10}(t) = \frac{\mu}{\lambda}(1 - P_{00}(t))$$

- 代入 (*) 可知，

$$P_{00}'(t) = \lambda P_{10}(t) - \lambda P_{00}(t)$$
$$= \mu(1 - P_{00}(t)) - \lambda P_{00}(t)$$
$$= \mu - (\lambda+\mu) P_{00}(t) \quad (\Delta)$$

· 由 ODE 知识可知，

$$P_{00}(t) = C_1 + C_2 e^{-(\lambda+\mu)t}$$

由 (Δ) 可知，

$$-(\lambda+\mu) C_2 e^{-(\lambda+\mu)t} = \mu - (\lambda+\mu)[C_1 + C_2 e^{-(\lambda+\mu)t}]$$

$$\Rightarrow C_1 = \frac{\mu}{\lambda+\mu}$$

代入 $t=0$ 可得：

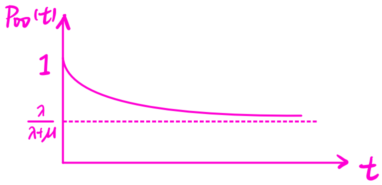
$$1 = \frac{\mu}{\lambda+\mu} + C_2$$

$$\Rightarrow C_2 = \frac{\lambda}{\lambda+\mu}$$

因此，

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

注：$P_{00}(t)$ 随时间 t 的曲线为：



其中 $\frac{\lambda}{\lambda+\mu}$ 为极限概率

e.g.

- Assume $X = \{X(t), t \geq 0\}$ is a CTMC on state space $E = \{1, 2, 3\}$ with generator

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

- Find

$$\mathbb{P}\{X(5.4) = 3, X(2.1) = 1 | X(0) = 2\} = P_{2,1}(2.7) P_{1,3}(3.3),$$

此处应为 2.1

- Use function expm($\cdot$) in *Python* or similar functions in other programming languages,

$$P(2.7) = e^{2.7G} = \begin{pmatrix} 0.12614 & 0.37612 & 0.49774 \\ 0.12612 & 0.37614 & 0.49774 \\ 0.12387 & 0.37387 & 0.50226 \end{pmatrix}, \quad P(3.3) = e^{3.3G}.$$