Lecture 17

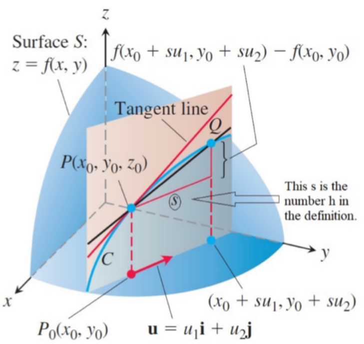
§1 关于 Differentiation 的 facts (接上)

1. **Definition:** directional derivative (方向导数)

$f: D\,(\text{open in } R^n) \to R^m$. Let $c \in D$, $\vec{u} \neq \vec{0} \in R^n$.

Then $\left.\dfrac{d f(c+t\vec{u})}{dt}\right|_{t=0}$ is called the directional derivative in direction of $\vec{u}$

Notations: $\dfrac{\partial f}{\partial u}(c)$, $D_u f(c)$



注: ① Meaning of $\frac{\partial f}{\partial u}(c)$: f 在 c 处沿 $\vec{u}$ 方向的变化率

② 在某些书中, 要求 $|\vec{u}|=1$, 但本课程不要求

③ 若 $\vec{u}=e_i=(0,\cdots,0,1,0,\cdots0)$ (第 i 项为 1), 则 $\frac{\partial f}{\partial e_i}=\frac{\partial f}{\partial x_i}$

2. **Fact 4:**

If f is differentiable at $c \in$ open $D \in R^n$ and $u \in R^n$. Then

$$\frac{\partial f}{\partial u}(c) = \left.\frac{df(c+t\vec{u})}{dt}\right|_{t=0}$$

$$= f'(g(0))\cdot g'(0) \quad (\text{令 } g(t)=c+t\vec{u})$$

$$= f'(c)\cdot u$$

注: Special case: 若 f 为 scale function (i.e. $m=1$), 则有

$$\frac{\partial f}{\partial u}(c) = \nabla f(c)\cdot u$$

3. **Definition:** local extreme

令 f 为 scale function. 我们称 $f(c)$ 为 local max (min) value,

若 $f(c) \geq (\leq) f(x)$, $\forall x \in N_r(c) \cap \mathrm{Dom}(f)$ for some $r>0$

4. **Fact 5:** First derivative test for local extremes

Suppose $c \in \mathrm{Dom}(f)$ & $f(c)$ is local extreme value & f is partially differentiable at c.

Then $\nabla f(c) = \vec{0}$

证明:

(考虑 special case: $\mathrm{Dom}(f) \subset R^1$, $f(c)$ is local max)

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$= \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} \quad \begin{matrix}\to -\\ \to +\end{matrix}$$

$$= \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \begin{matrix}\to -\\ \to -\end{matrix}$$

$$\because \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$$

$$\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$$

$$\therefore f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$$

(考虑 general case, $f(c)$ is local max)

Observe $\because f(c)$ is local max of $f(x_1, x_2, \cdots, x_n)$

$\therefore f(c)$ is also local max of $f(c_1, \cdots, c_{i-1}, x_i, c_{i+1}, \cdots, c_n)$

By special case:

$$\frac{\partial f}{\partial x_i}(c) = \left.\frac{d f(c_1, \cdots, c_{i-1}, x_i, c_{i+1}, \cdots, c_n)}{d x_i}\right|_{x_i = c_i} = 0 \quad , \quad i = 1, \cdots, n$$

$$\therefore \nabla f(c) = \vec{0}$$

注：若 $c \in \partial(\mathrm{Dom}(f))$，$\nabla f(c)$ 不一定为 $\vec{0}$

5. **Fact 6:** Rolle's Theorem (罗尔定理)

Suppose f is real valued function. f is continuous on $[a,b]$. f is differentiable on (a,b). $f(a)=f(b)$.

Then $\exists\, c \in (a,b)$, s.t. $f'(c)=0$

证明:

f always have global max or global min attained in (a,b), at c

Then by First derivative test, $f'(c)=0$

6. **Fact 7:** Mean value theorem (Laguange's version) (中值定理)

Assume that, except " $f(a)=f(b)$ ", $\frac{f(a)-f(b)}{a-b} = f'(c)$ for some c

7. **Fact 8:** Mean value theorem (Cauchy's version)

Suppose $f(x)$ and $g(x)$ are real valued, continuous on $[a,b]$, differentiable on (a,b).

Then $(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c)$ for some $c \in (a,b)$

(即 $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ for some $c \in (a,b)$)

注：若 $g(x)=x$，则可推出 Laguange's version

证明:

Let $h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$

Observe:
- h continuous on $[a,b]$
- $h(a) = f(b)g(a) - g(b)f(a)$
- $h(b) = f(b)g(a) - g(b)f(a)$

Then $h(a) = h(b)$

By Rolle's theorem, $\exists\, c \in (a,b)$, s.t. $h'(c) = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c) = 0$

Q.E.D.

8. Fact 9: 导数与增减性

Suppose f is real-valued & differentiable on (a,b)

(i) If $f'(x) \geq 0$ on (a,b), then f is increasing on (a,b)

(ii) If $f'(x) \leq 0$ on (a,b), then f is decreasing on (a,b)

(iii) If $f'(x) \equiv 0$ on (a,b), then $f \equiv$ constant

证明: 仅证明(i):

$\forall\, a < x_1 < x_2 < b$, by M.V.T.

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0$$

$\therefore$ f is increasing on (a,b)

9. Fact 10: MVT for vector-valued function

Suppose $f: D$ (open in $\mathbb{R}^n$) $\to \mathbb{R}^m$ is differentiable on D (i.e. $\forall c \in D$, f differentiable at c).

Assume D is convex & $\| Df(c)_{m\times n} \|$ is bounded on D (i.e. $\| Df(c)_{m\times n} \| \leq$ constant M, $\forall c \in D$)

Then $|f(b) - f(a)| \leq M |b-a|$

证明:

Fix arbitrary $z \in \mathbb{R}^n$, define $g(t) = z \cdot f(ta + (1-t)b)$, $t \in [0,1]$

By M.V.T. applied to $g(t)$ on $[0,1]$,

$$g(1) - g(0) = g'(c) \cdot (1-0), \quad c \in (0,1)$$

$$\therefore\ z \cdot f(a) - z \cdot f(b) = z \cdot f'(ca + (1-c)b)_{m\times n} (a-b)_{n\times 1}$$

$$\begin{aligned} \therefore\ |z \cdot (f(a) - f(b))| &\leq |z| \cdot |f'(ca + (1-c)b)_{m\times n} (a-b)_{n\times 1}| \\ &\leq |z| \cdot \| f'(ca + (1-c)b)_{m\times n} \| \cdot |(a-b)_{n\times 1}| \\ &\leq |z| \cdot M \cdot |a-b|, \quad \forall z \in \mathbb{R}^m \end{aligned}$$

Take $z = f(a) - f(b)$, then

$$|f(a) - f(b)|^2 \leq M |f(a) - f(b)| |a-b|$$

$$\therefore\ |f(a) - f(b)| \leq M |a-b|$$

Corollary:

If $Df = 0$ on D (convex), then take $M = 0 \Rightarrow f \equiv$ constant on D.

注: 事实上, 若 D 不为 convex, f 仍为 constant.