Lecture 14

§1 Continuity

1. 定义

> **Definition**
> Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$, and let $\vec{x_0} \in D$.
> Then f is said to be continuous at $\vec{x_0}$ if
> $$\lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = f(\vec{x_0}).$$
> We say that f is continuous if f is continuous at $\vec{x_0}$ for every $\vec{x_0} \in D$.

注: 1° 对于二元函数:

$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = f(\vec{x_0})$ 等价于 $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ 等价于 $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$

2° 结合极限的性质,可知:

若干个连续函数的和、差、积、商、乘方在定义域内连续.

因此, polynomials, 如 $f(x,y,z) = 48z^2y - 61xyz + 13z - 7$, 为连续函数

Example

The function defined by

$$f(x,y) := \begin{cases} \frac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$.

2. Compositions (复合) of continuous functions

Just like functions with one variable, the composition of two continuous functions is continuous. In particular:

If $f : D \to \mathbb{R}$ is a multiple-variable continuous function, and g is a single-variable continuous function whose domain contains the range of f, then $g \circ f$ is a continuous function on D.

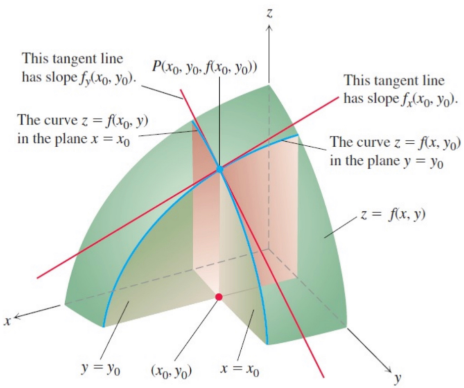
e.g. $f(x,y) = \cos\frac{xy}{x^2+1}$ 在 R^2 上连续; $g(x,y,z) = e^{x+y}\cos z$ 在 R^3 上连续

§2 Partial derivatives (偏导数)

1. Partial derivatives for two-variable functions

> **Definition**
> Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. The partial derivative of f at (x_0, y_0) with respect to x, denoted by $\frac{\partial}{\partial x} f(x_0, y_0)$, is defined by
>
> Also: $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$
> $$\frac{\partial}{\partial x} f(x_0,y_0) := \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h},$$
> and the partial derivative of f at (x_0, y_0) with respect to y, denoted by $\frac{\partial}{\partial y} f(x_0, y_0)$, is defined by
> $$\frac{\partial}{\partial y} f(x_0,y_0) := \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}.$$

$\frac{d}{dx} g(x_0)$, where $g(x) := g_{y_0}(x) := f(x, y_0)$



注: "Partial"的表达形式:

$\frac{\partial f}{\partial x}$, f_x, $D_x f$, $D_1 f$ ……

2、General partial derivatives

Definition

More generally, if $f : D \to \mathbb{R}$ is a function with $D \subseteq \mathbb{R}^n$, and

$$\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle \in D,$$

then the partial derivative of f at $\vec{a}$ with respect to the i-th coordinate (or i-th variable), denoted by $\frac{\partial}{\partial x_i} f(\vec{a})$, is defined to be the following limit:

$$\lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}.$$

注: 上述极限形式也可写作

$$f_i(\vec{a}) := \frac{\partial f}{\partial x_i}(\vec{a}) := \lim_{h \to 0} \frac{f(\vec{a} + h\vec{e_i}) - f(\vec{a})}{h}$$

$\vec{e_i} = \langle 0, 0, \cdots, 1, \cdots, 0 \rangle$ = ith component

3、偏导数的计算

To compute the partial derivative of f with respect to the i-th variable x_i, just treat all other variables as constants and differentiate f with respect to x_i.

例: (a) For the function defined by $f(x, y) := x^2 + 3xy + y - 1$, find

$$\frac{\partial}{\partial x} f(4, -5) \quad \text{and} \quad \frac{\partial}{\partial y} f(4, -5).$$

$f_x(x, y) = 2x + 3y \Rightarrow \frac{\partial}{\partial x} f(4, -5) = -7$

$f_y(x, y) = 3x + 1 \Rightarrow \frac{\partial}{\partial y} f(4, -5) = 13$

例: (b) Find the slope of tangent to the parabola, obtained by intersecting the surface $z = \underbrace{x^2 + y^2}_{f(x,y)}$ with the plane $x = 1$, at the point $(1, 2, 5)$.

Slope $= \frac{\partial}{\partial y} f(1, 2) = 2y \big|_{(x,y)=(1,2)} = 4$

4、高阶偏导

If f is a function with two variables, then f_x and f_y are also functions with two variables, so they can have partial derivatives. These are the second-order partial derivatives of f, whose notations are given as follows:

$$\frac{\partial^2 f}{\partial x^2} := f_{xx} := (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} := f_{xy} := (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} := f_{yx} := (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial y^2} := f_{yy} := (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

e.g. $\frac{\partial^3 f}{\partial x \partial y \partial z} = f_{zyx}$ ✓

These notations extend to functions with three or more variables.

例: Exercise

Compute the four second-order derivatives of the function f defined by

$$f(x, y) := x \cos y + ye^x.$$

Sol: $f_x = \cos y + ye^x$, $f_y = -x\sin y + e^x$

$f_{xx} = ye^x$

$f_{xy} = -\sin y + e^x$

$f_{yx} = -\sin y + e^x$

$f_{yy} = -x\cos y$

5. Clairaut's theorem (克劳莱定理)

14.3.2 Also called Clairaut's theorem

THEOREM ~~2~~—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives ~~f_x, f_y,~~ f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

注: 1° 并非对于所有二元函数, 混合偏导 $f_{xy}(a,b)$ 与 $f_{yx}(a,b)$ 均相等.

反例:

$$f(x,y) := \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

$f_{xy}(0,0) \neq f_{yx}(0,0)$

2° 根据定理, 要想利用其简化混合偏导的计算, 首先要检验两个混合偏导在 (a,b) 处的连续性, 这是很奇怪的 (无法起到简化计算的作用).

3° 事实上, 该定理的前提可放宽:

- f_{xy} is cts at (a,b)
- f_x, f_y, f_{xy} are defined near (a,b)

$\Big\}$ $f_{xy}(a,b) = f_{yx}(a,b)$

例: Example

Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + e^y(y^2+1)^{-1}$.

Sol: Since $f_{xy} = (f_x)_y = (y)_y = 1 \Rightarrow$ cts

So $f_{yx} = f_{xy} = 1$

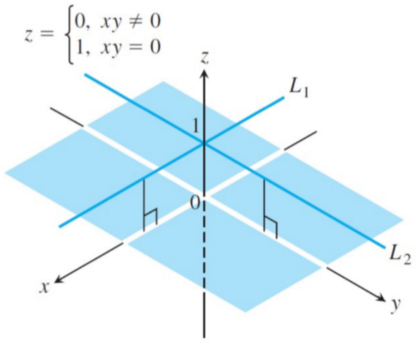
注: The Mixed Derivative Theorem extends to functions with more variables and higher order derivatives, as long as all these derivatives are defined in the open ball centered at the point and are cts at the point:

$$f_{xxyz} = f_{xyzx} = f_{zxxy} = \cdots$$

6. 偏导与连续

偏导存在 $\not\Rightarrow$ 连续

Even if all the partial derivatives of a function f exists at a point, it does not mean that the function is continuous at that point. One example is shown in the following figure.



Example

The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

has both partial derivatives at $(0,0)$. However, it is not continuous at $(0,0)$, since we have shown that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

§3 Differentiability (可微)

1. 单变量函数

对于单变量函数 $y=f(x)$，若 f 在 x_0 处 differentiable，则

$$\varepsilon := \varepsilon(\Delta x) = \frac{f(x_0+\Delta x) - (f(x_0) - f'(x_0)\Delta x)}{\Delta x} \to 0 \quad \text{as } \Delta x \to 0$$

换言之，存在 $\varepsilon := \varepsilon(\Delta x)$ 使得 $\Delta y = f(x_0+\Delta x) - f(x_0) = f'(x_0)\Delta x + \varepsilon \Delta x$，且 $\varepsilon \to 0$ as $\Delta x \to 0$.

2. 双变量函数

Def: A function $z=f(x,y)$ is differentiable at $(x_0,y_0)\in D$ if both f_x and f_y exist at (x_0,y_0), and

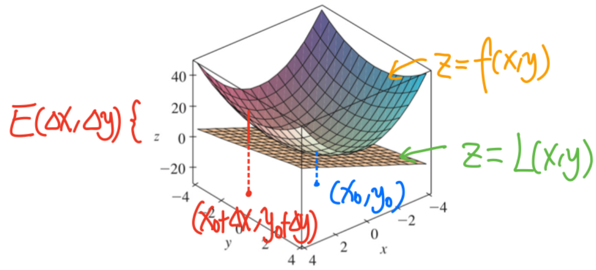
$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

satisfies

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

for some $\varepsilon_1 := \varepsilon_1(\Delta x, \Delta y)$ and $\varepsilon_2 := \varepsilon_2(\Delta x, \Delta y)$ such that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. We say f is differentiable on D if f is differentiable at all points in D.

注：1° 相较于单变量函数使用"tangent line"进行近似，双变量函数使用"tangent plane"进行近似.



显然，$L(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$.

令 Error term 为 $E(\Delta x,\Delta y)$，则

$$
\begin{aligned}
E(\Delta x,\Delta y) &:= f(x_0+\Delta x, y_0+\Delta y) - L(x_0+\Delta x, y_0+\Delta y) \\
&= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y \\
&= \Delta z - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y \\
&= \varepsilon_1\Delta x + \varepsilon_2\Delta y
\end{aligned}
$$

因此，为了起到较好的近似效果，定义中要求 ε_1 & $\varepsilon_2 \to 0$ as $(\Delta x,\Delta y)\to(0,0)$

2° 若 $E(\Delta x,\Delta y) := \varepsilon_1\Delta x + \varepsilon_2\Delta y$，则

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{E(\Delta x,\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0$$

例：$z=f(x,y)=xy$，show that f is differentiable on R^2

- Fix $(x_0,y_0)\in R^2$
- $$
\begin{aligned}
E(\Delta x,\Delta y) &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y \\
&= (x_0+\Delta x)\cdot(y_0+\Delta y) - x_0y_0 - y_0\Delta x - x_0\Delta y \\
&= \Delta x\Delta y \\
&= 0\cdot\Delta x + (\Delta y)\cdot\Delta x \to 0 \text{ as } (\Delta x,\Delta y)\to(0,0)
\end{aligned}
$$

  so f is differentiable at (x_0,y_0)

- Since $(x_0,y_0)\in D$ is arbitrary, f is differentiable on R^2

3、定理：偏导数 countinuous $\to$ differentiable

The following is a sufficient condition for a two-variable function f to be differentiable at a point.

> **Theorem**
> *Suppose that partial derivatives f_x and f_y exist in an open ~~ball~~ disk containing (x_0, y_0) and are continuous at (x_0, y_0). Then f is differentiable at (x_0, y_0).*

* 对于初等函数而言，只要偏导数存在，函数就可微

e.g. For $f(x,y)=xy$, since $f_x=y$ and $f_y=x$ are both continuous, we see that f is differentiable.

4. 定理: differentiable $\rightarrow$ countinuous

> **Theorem** If a two-variable function f is differentiable at (x_0,y_0), then it is continuous at (x_0,y_0).

证明: 若f在(x_0,y_0)可微, 则随着$(\Delta x,\Delta y)\to(0,0)$, $\varepsilon_1\Delta x+\varepsilon_2\Delta y\to 0$, 因此$\Delta z\to 0$.

根据定义, 也就是 $f(x_0+\Delta x, y_0+\Delta y)\to f(x_0,y_0)$.

因此, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0)$

因此f在(x_0,y_0)处连续.

5. 三(多)元函数可微

若f在(x_0,y_0,z_0)处可微, 且f_x, f_y, f_z在(x_0,y_0,z_0)处存在, 则

$$\Delta w = f_x(x_0,y_0,z_0)\cdot\Delta x + f_y(x_0,y_0,z_0)\cdot\Delta y + f_z(x_0,y_0,z_0)\cdot\Delta z + \varepsilon_1\Delta x + \varepsilon_2\Delta y + \varepsilon_3\Delta z$$

其中 $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 是 $\Delta x,\Delta y,\Delta z$ 的函数, 满足 $\varepsilon_i\to 0$ as $(\Delta x,\Delta y,\Delta z)\to(0,0,0)$, $\forall i\in\{1,2,3\}$.

上述定理也同样适用三(多)元函数.

注: 可微不一定偏导连续, 反例: $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{else}\\ 0 & (x,y)=0\end{cases}$