Lecture 9

§1 Kinetic Energy of Rotation

1. Kinetic energy of rotation (转动动能)

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots\cdots$$

$$= \sum \frac{1}{2}m_iv_i^2$$

1° 将刚体视作速度不同的点的集合

2° m_i : 第i个点的 mass

v_i : 第i个点的 speed

3° 由 $v = \omega r$，且各点 ω 相同：

$$K = \sum \frac{1}{2}m_iv_i^2 = \frac{1}{2}\left(\sum m_ir_i^2\right)\cdot \omega^2$$

2. Rotational inertia (转动惯量)

1° Rotational inertia (or Moment of inertia) of the body with respect to the axis of rotation.

$$I = \sum m_ir_i^2$$

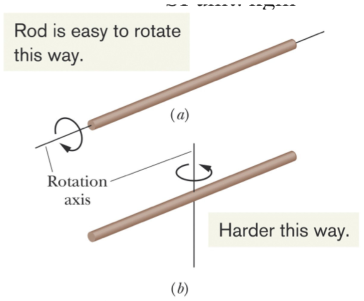
2° Kinetic energy

$$K = \frac{1}{2}I\omega^2$$

3° I 同时包含质量与分布，取决于旋转轴

4° SI unit : $kg \cdot m^2$

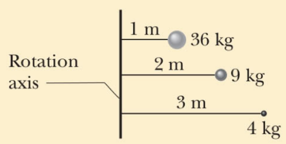
5° 决定了改变旋转状态的难度 (加速，减速，改变旋转轴)



- In the figure, because the mass is distributed much closer to the rotation axis in (a), the moment of inertia is much smaller for the case in (a) than in (b). In general, a smaller moment of inertia means easier rotation.
- Moment of Inertia depends on the rotational axis!

例: **Checkpoint 4**

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



Answer: They are all equal!

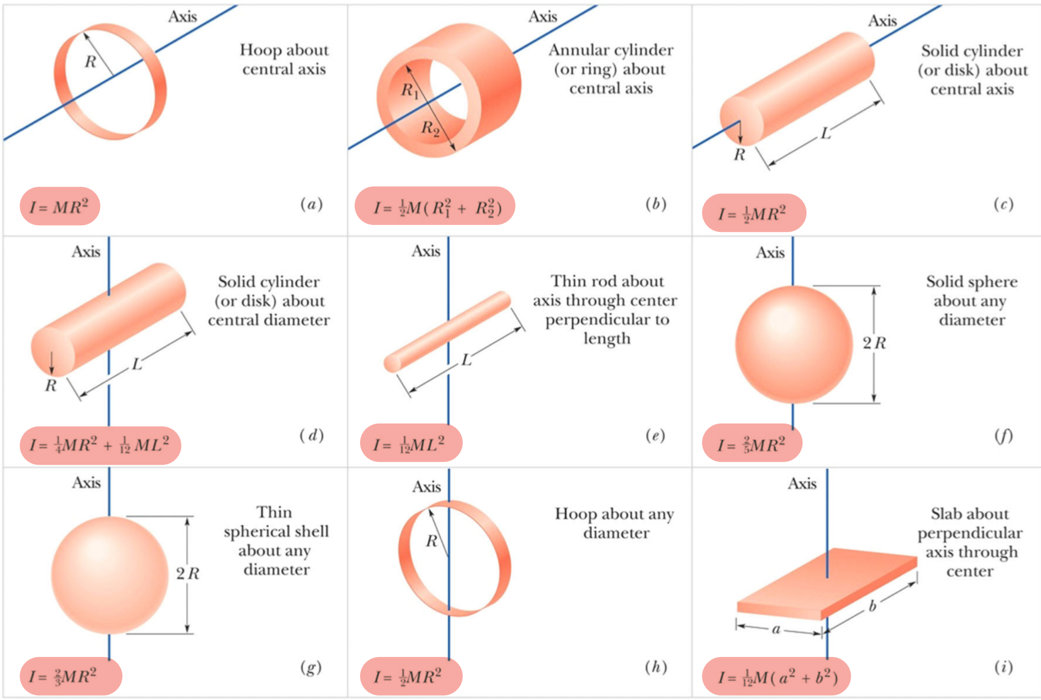
6° 对于 continuous rigid body,

$$I = \sum m_ir_i^2$$

$$I = \int r^2 dm$$

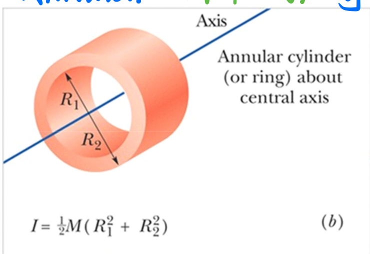
§2 常见的转动惯量

Table 10-2 Some Rotational Inertias



1° 类型一

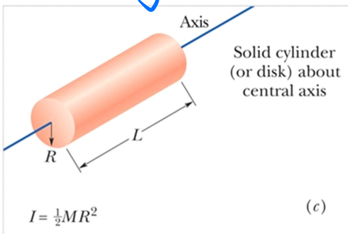
☆ (b) Annular (环状的) cylinder about it central axis



$$
\begin{aligned}
dM &= \rho \cdot dv \\
&= \rho \cdot 2\pi r l \, dr \\
&= 2\pi\rho l r \, dr
\end{aligned}
$$

$$
\begin{aligned}
I &= \int r^2 dM \\
&= \int_{R_1}^{R_2} 2\pi r^3 \rho l \, dr \\
&= \frac{1}{2}\pi r^4 \rho l \Big|_{R_1}^{R_2} \\
&= \frac{1}{2}\pi l \rho \cdot (R_2^2 - R_1^2)(R_2^2 + R_1^2) \\
&= \frac{1}{2}M(R_2^2 + R_1^2)
\end{aligned}
$$

(c) Solid cylinder (or ring) about central axis



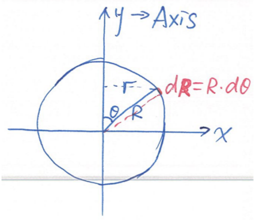
$$dM = \rho \cdot 2\pi r l \, dr$$

$$I = \int r^2 dM$$

$$= \int_0^R \rho \cdot 2\pi r^3 l \, dr$$
$$= \frac{1}{2}\rho\pi l R^4$$
$$= \frac{1}{2}MR^2$$

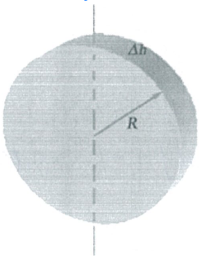
2° 类型二

(h) hoop about any diameter



$$dm = \frac{M}{2\pi R} \cdot R\,d\theta$$
$$I = 4\int_0^{\frac{\pi}{2}} R^2 \sin^2\theta \cdot \frac{M}{2\pi R} \cdot R\,d\theta$$
$$= 4 \cdot \frac{M}{2\pi} \cdot R^2 \int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$$
$$= 4 \cdot \frac{M}{2\pi} \cdot R^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$
$$= \frac{1}{2}MR^2$$

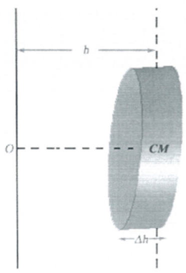
(d) Solid disk of width Δh



$$I_{Hoop} = \frac{1}{2}MR^2$$
$$dM = \frac{M}{\pi R^2} \cdot 2\pi r \cdot dr$$
$$dI_{Hoop} = \frac{1}{2} r^2 dM$$
$$= \frac{1}{2} r^2 \cdot \frac{M}{\pi R^2} \cdot 2\pi r \cdot dr$$
$$= \frac{M}{R^2} \cdot r^3\,dr$$
$$I = \int_0^R dI_{Hoop}$$
$$= \int_0^R \frac{M}{R^2} \cdot r^3\,dr$$
$$= \frac{1}{4}MR^2$$

变式

We will use the parallel axis theorem for finding the rotational inertia of a thin disk about an axis parallel to the vertical axis passing through its centre.



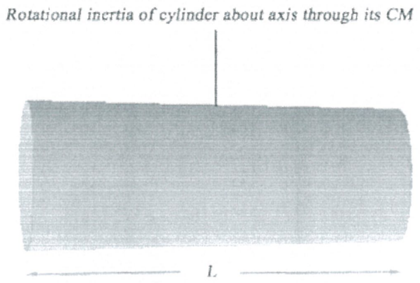
This gives

$$I_O = I_{CM} + Mh^2.$$

Using the expression of rotational inertia of a thin disk, we have

$$I_{O\text{ thin disk}} = \frac{\pi}{4}\rho R^4 \Delta h + \pi R^2 \rho \Delta h h^2.$$

(d2) Cylinder about axis through its CM



由 $I_{\text{0 thin disk}} = \frac{\pi}{4}\rho R^4 \Delta h + \pi R^2 \rho \Delta h h^2$：

$I = 2[\frac{\pi}{4}\rho R^4 \int_0^{\frac{L}{2}} dh + \rho R^2 \pi \int_0^{\frac{L}{2}} h^2 dh]$

$= \frac{1}{4}\rho R^4 \pi L + \frac{1}{12}\rho R^2 \pi L^3$

因为 $M = \pi R^2 L \rho$，

$I_{cyl} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$

$dM = \frac{M}{L} \cdot dh$

$dI = \frac{1}{4}dMR^2 + dMh^2$

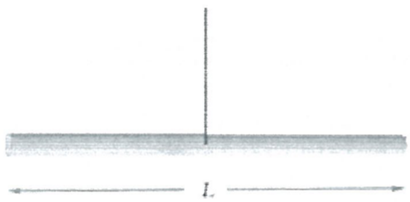
$= \frac{1}{4} \cdot \frac{M}{L} R^2 dh + \frac{M}{L} h^2 dh$

$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} dI$

$= \int_{-\frac{L}{2}}^{\frac{L}{2}} (\frac{1}{4}\frac{M}{L}R^2 + \frac{M}{L}h^2) \cdot dh$

$= \frac{1}{4}MR^2 + \frac{1}{12}ML^2$

❀ (e) Thin rod about an axis through its centre

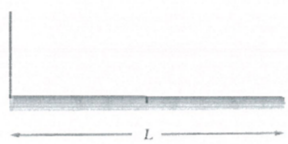


由 $I_{cyl} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$

取 $R = 0$：

$I_{\text{thin rod}}$: $\frac{1}{12}ML^2$

变式：



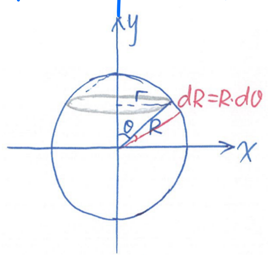
By applying parallel axis theorem and using the expression of rotational inertia of a thin rod about axis through its CM, we get

$$I_{thinrod-axis\text{ at end}} = \frac{1}{12} \times ML^2 + M(L/2)^2,$$

$$= \frac{1}{3} \times ML^2.$$

3° 类型三

❀ (g) Thin spherical shell about any diameter



$$dM = \frac{M}{4\pi R^2}\cdot 2\pi R\sin\theta\cdot R\,d\theta$$
$$= \frac{1}{2}M\sin\theta\,d\theta$$
$$I = 2\cdot\int_0^{\frac{\pi}{2}} R^2\sin^2\theta\cdot\frac{1}{2}M\sin\theta\,d\theta$$
$$= MR^2\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta$$
$$= \frac{2}{3}MR^2$$

(f) Rotational inertia of a solid sphere

由 $I_{\text{thin shell}} = \frac{2}{3}Mr^2$
$$= \frac{2}{3}4\pi r^2\cdot\rho\cdot dr\cdot r^2$$
$$= \frac{8}{3}\pi r^4\rho\,dr$$
$$I = \int_0^R \frac{8}{3}\pi\rho\cdot r^4\,dr$$
$$= \frac{8}{3}\pi\rho\cdot\frac{1}{5}R^5$$
$$= \frac{2}{5}MR^2$$

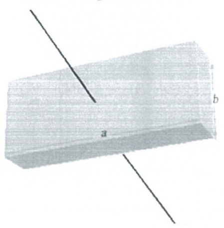
球壳的质量

$$dM = \frac{M}{\frac{4}{3}\pi R^3}\cdot 4\pi r^2\,dr$$
由 $I_{\text{thin shell}} = \frac{2}{3}Mr^2$
$$dI = \frac{2}{3}dM\cdot r^2 = 2\frac{M}{R^3}\cdot r^4\,dr$$
$$I = \int_0^R dI$$
$$= \int_0^R 2\frac{M}{R^3}r^4\,dr$$
$$= \frac{2}{5}MR^2$$

4° 类型四

(i) Rotational inertia of a thin slab (板)

Thin slab of length a and width b



$$\Delta M = \frac{M}{ab}\cdot dx\cdot dy$$
$$I = \int_{-\frac{b}{2}}^{\frac{b}{2}}dy\int_{-\frac{a}{2}}^{\frac{a}{2}}(x^2+y^2)\cdot\frac{M}{ab}\,dx$$
$$= \frac{M}{ab}\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{a^3}{12}+ay^2\,dy$$
$$= \frac{1}{12}M(a^2+b^2)$$

* 解题思路

1° 面(壳)状物体由环状物体推 $I_{shell} = \int_0^R dI_{hoop}$

体状物体由面状物体利用平行轴定理推，或利用壳状物体由 $I = \int_0^R dI_{shell}$ 推

2° 合理选用线密度，面密度，体密度

3° 先利用密度求 dM，再利用 $I = \int r^2 dM$ 求转动惯量

§3 The Parallel-Axis Theorem

1. The Parallel-Axis Theorem (平行轴定理)

1° 要求：

① 两条轴平行

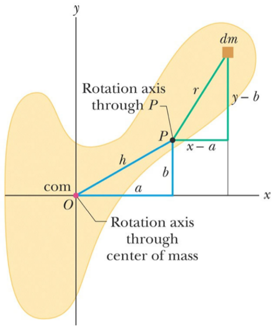
② 其中一条轴通过质心

2° 公式

$I = I_{com} + Mh^2$

其中 h 为两轴间的 perpendicular distance

We need to relate the rotational inertia around the axis at *P* to that around the axis at the com.



- If we know the moment of inertia for the center of mass axis (I_{com}), we can find the moment of inertia for a parallel axis with the parallel-axis theorem.
- Let h be the perpendicular distance between the axis that we need and the axis through the center of mass (remember these two axes must be parallel). Then the rotational inertia I about the required axis is

$$I = I_{com} + Mh^2$$

(Parallel-axis Theorem)

2. 证明

Proof of the Parallel-Axis Theorem

$$I = I_{com} + Mh^2$$

(Parallel-axis Theorem)

- For Center of Mass:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i = \frac{1}{M}\int x dm = 0$$

$$y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i = \frac{1}{M}\int y dm = 0$$

- For Point at (x, y) rotating at axis-P

$$I = \int r^2 dm = \int [(x-a)^2 + (y-b)^2] dm$$

$$= \int (x^2 + y^2) dm - 2a\int x dm - 2b\int y dm + \int (a^2 + b^2)\, dm$$

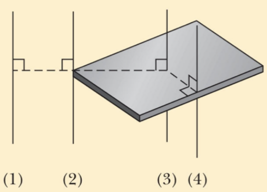
$$\int (x^2 + y^2) dm = \int R^2 dm = I_{com}$$

$$\int (a^2 + b^2)\, dm = Mh^2$$

例：

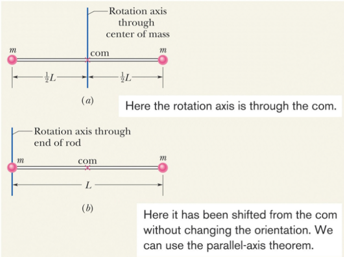
Checkpoint 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.



Answer: (1), (2), (4), (3)

例:



As shown in the left, a rigid body consisting of two particles of mass **m** connected by a rod of length **L** with negligible mass

(a) what is the rotational inertia about an axis through the center of mass, perpendicular to the rod as shown?

SOLUTION:

$$I = \sum m_i r_i^2 = (m)(\frac{1}{2}L)^2 + (m)(\frac{1}{2}L)^2$$

$$= \frac{1}{2}mL^2$$

(b) what is the rotational inertia of the body about an axis through the left end of the rod and parallel to the first axis?

SOLUTION:

$$I = m(0)^2 + mL^2 = mL^2$$

Or apply the Parallel-Axis Theorem:

$$I = I_{com} + Mh^2 = \tfrac{1}{2}mL^2 + (2m)(\tfrac{1}{2}L)^2$$

$$= mL^2$$

Summary

- Rotational Kinetic Energy

$$K = \frac{1}{2}I\omega^2$$

- Rotational Inertia

$$I = \sum m_i r_i^2 = \int r^2 dm$$

- The parallel-Axis Theorem

$$I = I_{com} + Mh^2$$