Lecture 8

§1 Probability review

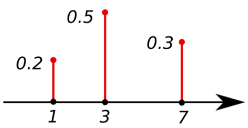
1. Terminologies

1° 基础概念

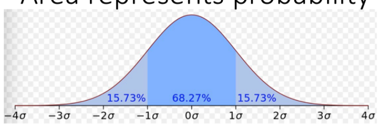
- **Random Experiment**: a repeatable procedure
- **Sample space**: set of all possible outcomes Ω.
- **Event**: a subset of the sample space.
- **Probability function**, P(ω): gives the probability for each outcome ω ∈ Ω
  - Probability is between 0 and 1
  - Total probability of all possible outcomes is 1.

2° 随机变量

- Discrete:
  - ✓ Probability mass function.
  - ✓ f(ω): gives the probability for each outcome ω ∈ Ω

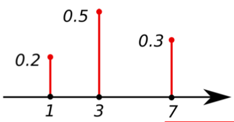


- Continuous
  - ✓ Probability density function.
  - ✓ f(ω): gives the probability density for each outcome ω ∈ Ω
  - ✓ Area represents probability



3° 期望与方差

- Discrete:
  - ✓ Probability mass function.



- Continuous
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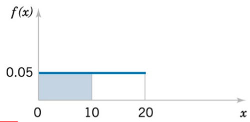


Summation ↔ Integration

- Mean

  $E[X]=\sum x\, f(x)$

- Variance

  $\mathrm{Var}[X]=\sum (x-\mathrm{E}[X])^2 f(x)$

- Mean

  $E[X]=\int_{-\infty}^{\infty} x f(x)dx$

- Variance

  $\mathrm{Var}[X]=\int_{-\infty}^{\infty} (x-E(X))^2 f(x)dx$

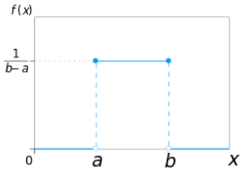
4° 概率分布(离散型)

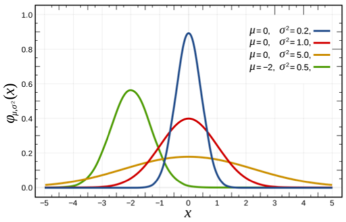
	p.m.f	Mean	Variance
Bernoulli; Ber(p)	$P(X=1)=p$ $P(X=0)=1-p$	p	$p(1-p)$
Binomial; Bin(N,p)	$\Pr(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$	Np	$Np(1-p)$
Geometric; Geo(p)	$\Pr(X=k)=(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson; Poi(λ)	$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}, k=0,1,\cdots$	λ	λ

Bernoulli distribution	Binomial distribution	Geometric Distribution
X: Whether the machine breaks down on a spefic day.	X: the number of breadowns during the first N days.	X: the first day the machine breaks down

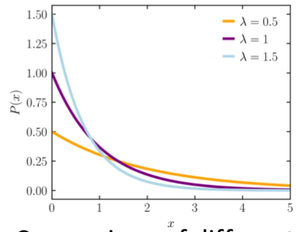
5° 概率分布(连续型)

	p.d.f	Mean	Variance
Uniform; Unif[a,b]	$f(x)=\dfrac{1}{b-a}, a\le x\le b$	$\dfrac{a+b}{2}$	$\dfrac{1}{12}(b-a)^2$
Normal; $N(\mu,\sigma^2)$	$f(x;\mu,\sigma)=\dfrac{1}{\sigma\sqrt{2\pi}}\exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right).$	μ	σ^2
Exponential; Exp(λ)	$f(x;\lambda)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0,\\ 0 & , x<0.\end{cases}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$





Comparison of different normal



Comparison of different exp

2. Useful formulas

1° Linearity

Formula 1: Linearity

$$E[X+Y]= E[X]+E[Y]$$

$$\begin{aligned}E[X+Y]&=\sum_x\sum_y(x+y)f(x,y)\\&=\sum_x\sum_y xf(x,y)+\sum_x\sum_y yf(x,y)\\&=\sum_x x\sum_y f(x,y)+\sum_y y\sum_x f(x,y)\\&=\sum_x xf_X(x)+\sum_y yf_Y(y)\\&=E[X]+E[Y]\end{aligned}$$

Formula 1: Linearity

Some useful variants:

$E[X+a]= E[X]+a$ (a is a constant)

$E[\Sigma_i X_i]= \Sigma_i E[X_i]$ (For >2 random variables)

$E[\Sigma_i c_iX_i]= \Sigma_i c_iE[X_i]$ (c_i are constants)

Note $E[cX]= cE[X]$

2° 期望

Formula 2

$$E[g(X)]= \Sigma_x g(x)P(X=x) = \Sigma_x g(x)f(x)$$

Some useful variants:

$$E[g(X)+h(X)]=E[g(X)]+ E[h(X)]$$

Proof: $$\begin{aligned}E[g(X)+h(X)]&= \Sigma_x \big(g(x)+h(x)\big)P(X=x)\\&=\Sigma_x g(x)f(x)+\Sigma_x h(x)f(x)\\&=E[g(X)]+ E[h(X)]\end{aligned}$$

3° 方差

Formula 3: Variance

$$\text{Var}[X] = \text{E}[(X - E[X])^2]$$

$$= \text{E}[X^2 - 2X \times E[X] + (E[X])^2]$$

$$= \text{E}[X^2] - 2E[X] \times E[X] + (E[X])^2]$$

$$= \text{E}[X^2] - (E[X])^2]$$

4° 方差与协方差

Formula 4: variance and covariance

$$\text{Var}[X] = \text{E}[(X - E[X])^2] = \text{E}[(X - E[X]) \times (X - E[X])] = \text{Cov}(X, X)$$

A useful result:

$$\text{Var}[X+Y] = \text{Cov}(X+Y, X+Y) = \text{E}[(X+Y-E[X]-E[Y]) \times (X+Y-E[X]-E[Y])]$$

$$= \text{E}[(X-E[X])^2] + \text{E}[(Y-E[Y])^2] + 2\,\text{E}[(X-E[X]) \times (Y-E[Y])]$$

$$= \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X, Y)$$

>2 RVs: $\text{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

3、辨析

1° 互斥、独立与不相关

Disjoint≠independent≠uncorrelated

- Disjoint (mutually exclusive): $A \cap B = \emptyset$
- Independent: P(AB)=P(A)P(B)

(for random variables X and Y, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for any x,y.)

- Uncorrelated: E[XY]=E[X]E[Y]

Disjoint $\underset{\not\longleftarrow}{\not\longrightarrow}$ independent $\underset{\not\longleftarrow}{\longrightarrow}$ uncorrelated

2° 0概率事件

Correction

- A and B are joint events if $A \cap B \neq \emptyset$.
- It is not always true that $P(A\ or\ B) < P(A) + P(B)$ or $P(A \cap B) > 0$!
- For example, $X \sim Unif[0,2]$. $A = \{0 \le X \le 1\}$, $B = \{1 \le X \le 2\}$. $A \cap B = \{X = 1\} \neq \emptyset$, but $P(A \cap B) = 0$.
- Note that P(A)=0 does not imply event A is impossible!

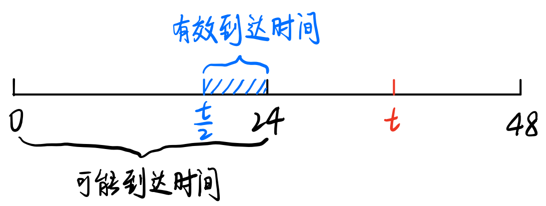
§2 Exercises

1.
Exercise #1

- Suppose your friend is to arrive sometime on Monday, with a uniform distribution over the full 24 hour day.
- Whatever time they arrive dictates how long they will stay at your place, e.g.:
  - If your friend arrives exactly at midnight (00:00) on Monday, they won't stay at all, and simply say "hello" while passing by.
  - If they arrive at 1:30am (01:30) they will stay for an hour and a half, until 3:00am (03:00).
  - If they arrive at 2:00pm (14:00) they will stay for 14 hours, until 4:00am the next day, Tuesday.

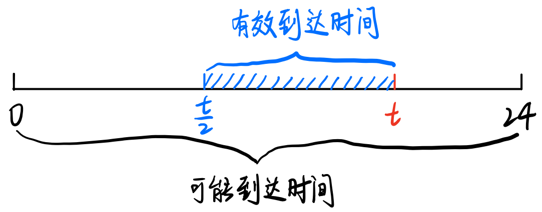
- If t is a time between the beginning of Monday and the end of Tuesday, what is the probability your friend is at your house at time t?

① $24 \le t \le 48$ 时



$$f(t) = \frac{24-\frac{t}{2}}{24} = 1-\frac{t}{48}$$

② $0 \le t \le 24$ 时

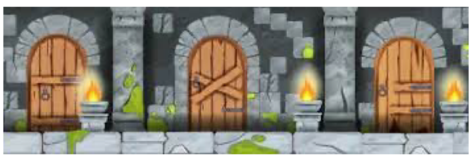


$$f(t) = \frac{\frac{t}{2}}{24} = \frac{t}{48}$$

2.
Exercise #2

A miner is trapped in a mine containing three doors.
- The first door leads to a tunnel that takes him to safety after one hour of travel.
- The second door leads to a tunnel that returns him to the mine after two hours of travel.
- The third door leads to a tunnel that returns him to his mine after three hours.

Assuming that the miner is at all times equally likely to choose any one of the doors, based on simulation, what is the expected length of time until the miner reaches safely?



Let X be the time it takes to safety.

Door A Door B Door C

1 hour -> safety 2 hours -> back to mine 3 hours -> back to mine

	First door chosen	Result
Case 1: w.p. 1/3	A	**Safety: X=1**
Case 2: w.p. 1/3	B	Return to mine: X = 2+ a new X (restart)
Case 3: w.p. 1/3	C	Return to mine: X = 3 + a new X (restart)

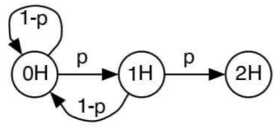
$E[X] = \frac{1}{3}\times 1 + \frac{1}{3}\times(2+E[X]) + \frac{1}{3}\times(3+E[X])$ ➡ E[X]=6 hours

3.

Exercise #3: A question for quantitative analyst

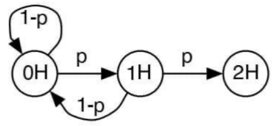
- Toss a coin continuously.
- What's the expected steps you got two consecutive heads?

0H: Currently 0 consecutive heads;
1H: Currently 1 consecutive heads;
2H: Currently 2 consecutive heads.



If the current is 0H, then with probability p, the next state is 1H; with probability 1-p remains at 0H.

If the current is 1H, then with probability p, the next state is 2H; with probability 1-p, returns to 0H.



Suppose that after D_S times on average: change from state S to 2H

$$D_{0H} = p\,(1 + D_{1H}) + (1-p)(1 + D_{0H})$$
$$D_{1H} = p\; + (1-p)(1 + D_{0H})$$

The value of D_{0H}?

D_{0H}：当前连续0次head，平均需要D_{0H}次能连续两次head.
D_{1H}：当前连续1次head，平均需要D_{1H}次能连续两次head.

$$\begin{cases} D_{0H} = p(1+D_{1H}) + (1-p)(1+D_{0H}) \\ D_{1H} = p(1) + (1-p)(1+D_{0H}) \end{cases}$$

$$\Rightarrow D_{0H} = \frac{1+p}{p^2}$$

4.

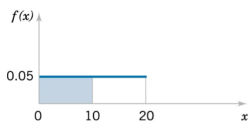
Exercise #4*: CDF

- For any random variable X and a function g
- g(X) is also a random variable!
- What is the distribution of the random variable g(X)?

A special case: recall the CDF

- CDF: $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du$

✓ $0 \le F(x) \le 1$
✓ If $x \le y$, then $F(x) \le F(y)$

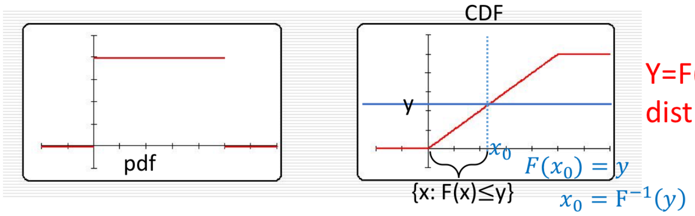


Exercise #4 *: CDF

A special case: CDF

- For any random variable X and its CDF function F(x)
- Y=F(X) is also a random variable! What is its distribution?

$$P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = F\big(F^{-1}(y)\big) = y$$



Y=F(X) is a uniform distribution on [0,1]

Exercise #4 *: CDF

- Recall that $F(x) = P(X \leq x)$ is increasing in x.
- Suppose that X = h(U) and h=F^{-1}(inverse of CDF) is an increasing function.

U ~ Uniform [0,1]

What do we have?

$P(X \leq x)$ = $P(h(U) \leq x)$ = $P(U \leq h^{-1}(x))$= $P(U \leq F(x))$ = $F(x)$

X = h(U) follows the distribution F