Lecture 22

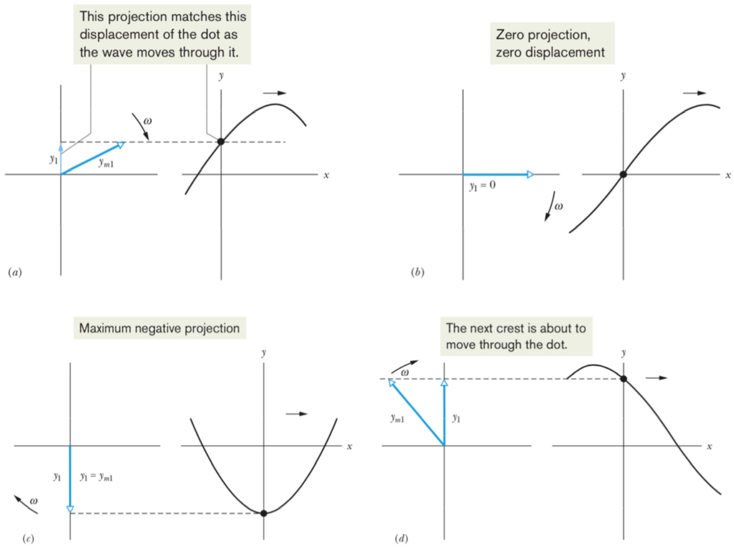
§1 Phasor

1. Phasor

- A **Phasor** is a vector that rotates around its tail, which is pivoted at the origin of a coordinate system.
- The angular speed of the phasor is equal to the angular frequency ω of the wave.

$$y(x,t) = y_m sin(kx - \omega t)$$

- The displacement amplitude y_m of all waves with the same k and ω may be viewed as a vector that rotates clockwise (- sign for t) in time. Because of the sine function, $y(x,t)$ is the vertical projection of the vector.



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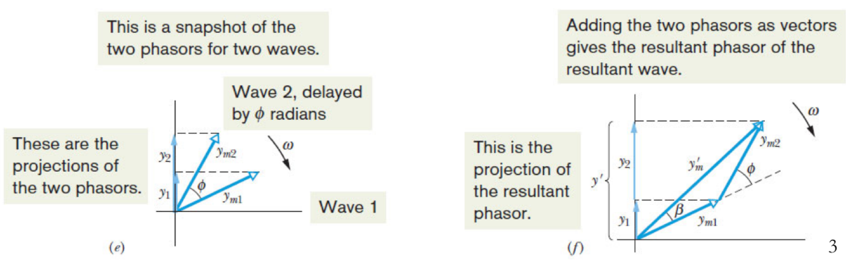
2. 波的叠加

- Phasors can be used to combine waves same k and ω even if their amplitudes are different.
- Wave 1: $y_1(x,t) = y_{m1} sin(kx - \omega t)$
- Wave 2: $y_2(x,t) = y_{m2} sin(kx - \omega t + \emptyset)$
- Thus, the resultant wave should be

$$y'(x,t) = y'_m sin(kx - \omega t + \beta)$$



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$$y_1 = y_{m1} \sin(kx - \omega t + \emptyset_1)$$

$$y_2 = y_{m2} \sin(kx - \omega t + \emptyset_2)$$

$$y_m' = \sqrt{y_{m1}^2 + y_{m2}^2 + 2 y_{m1} y_{m2} \cos(\emptyset_2 - \emptyset_1)}$$

$$\tan\emptyset' = \frac{y_{m1}\sin\emptyset_1 + y_{m2}\sin\emptyset_2}{y_{m1}\cos\emptyset_1 + y_{m2}\cos\emptyset_2}$$

$$y = y_m' \sin(kx - \omega t + \emptyset')$$

Additional Information

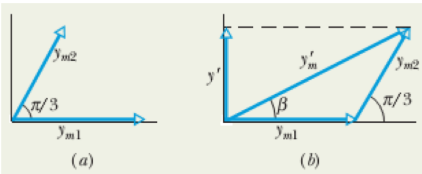
- $y_1(x,t) = y_{m1}\sin(kx - \omega t + \emptyset_1) = y_{m1}[\sin(kx - \omega t)\cos\emptyset_1 + \cos(kx - \omega t)\sin\emptyset_1]$
- $y_2(x,t) = y_{m2}\sin(kx - \omega t + \emptyset_2) = y_{m2}[\sin(kx - \omega t)\cos\emptyset_2 + \cos(kx - \omega t)\sin\emptyset_2]$
- $y'(x,t) = y_1 + y_2 = (y_{m1}\cos\emptyset_1 + y_{m2}\cos\emptyset_2)\sin(kx - \omega t) + (y_{m1}\sin\emptyset_1 + y_{m2}\sin\emptyset_2)\cos(kx - \omega t)$
- Let $A_1 = y_{m1}\cos\emptyset_1 + y_{m2}\cos\emptyset_2$ and $A_2 = y_{m1}\sin\emptyset_1 + y_{m2}\sin\emptyset_2$
- Set $y'_m = \sqrt{A_1^2 + A_2^2}$, $\cos\beta = \frac{A_1}{y'_m}$ and $\sin\beta = \frac{A_2}{y'_m}$
- Thus,

$$y'(x,t) = y'_m\cos\beta\,\sin(kx - \omega t) + y'_m\sin\beta\,\cos(kx - \omega t)$$
$$= y'_m\sin(kx - \omega t + \beta)$$

例：

Problem

Two sinusoidal wave $y_1(x,t)$ and $y_2(x,t)$ have the same wavelength and travel together in the same direction along a string. Their amplitudes are y_{m1}=4.0mm and y_{m2}=3.0mm, and their phase constants are 0 and π/3 rad, respectively. What are the amplitudes y_m' and phase constant β of the resultant wave?



Solution:

- Same string means same v and same wavelength means same k → k and ω are the same
- For the horizontal components

$$y'_{mh} = y_{m1}\cos 0 + y_{m2}\cos\pi/3$$
$$= 4.0\text{ mm} + (3.0\text{ mm})\cos\pi/3 = 5.50\text{ mm}.$$

- For the vertical components

$$y'_{mv} = y_{m1}\sin 0 + y_{m2}\sin\pi/3$$
$$= 0 + (3.0\text{ mm})\sin\pi/3 = 2.60\text{ mm}.$$

Thus, the resultant wave has an amplitude of

$$y'_m = \sqrt{(5.50\text{ mm})^2 + (2.60\text{ mm})^2}$$
$$= 6.1\text{ mm}$$

and a phase constant of

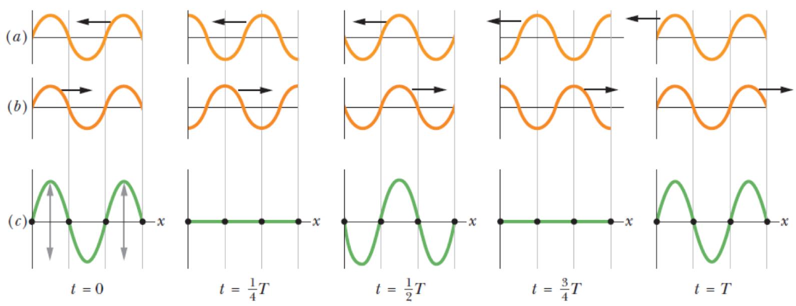
$$\beta = \tan^{-1}\frac{2.60\text{ mm}}{5.50\text{ mm}} = 0.44\text{ rad}.$$

§2 Standing wave

1. Standing wave (驻波)

- 若两个正弦波有相同的振幅.波长，且传播方向相反.则它们干涉会形成驻波.
- nodes (波节) never move

  antinodes (波腹) move the most.



2、表达式

Wave 1: $y_1(x,t) = y_m \sin(kx - \omega t)$

Wave 2: $y_2(x,t) = y_m \sin(kx + \omega t)$

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$
$$= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

$$y'(x,t) = 2y_m \sin(kx) \cos(\omega t)$$

- The resultant wave is no longer a traveling wave!!!

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{[2y_m \sin kx]}_{\substack{\text{Magnitude gives}\\ \text{amplitude}\\ \text{at position } x}} \underbrace{\cos \omega t}_{\substack{\text{Oscillating}\\ \text{term}}}$$

3、波节与波腹

在点 x 处: $2y_m \sin(kx)$

波节: $2y_m \sin(kx) = 0 \Rightarrow kx = \frac{2\pi}{\lambda}x = n\pi$

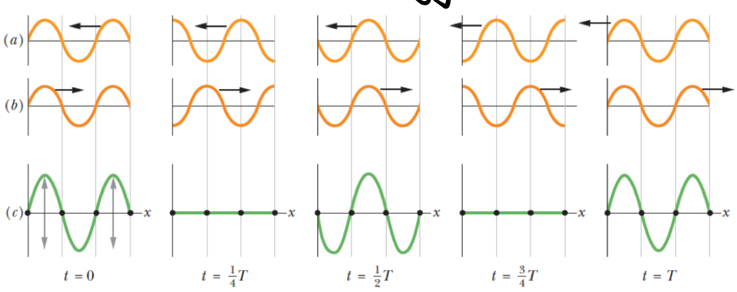
$x = n \cdot \frac{\lambda}{2} \quad (n = 0, 1, 2, \cdots\cdots)$

波腹: $2y_m \sin(kx) = \pm 2y_m \Rightarrow kx = \frac{2\pi}{\lambda}x = (n+\frac{1}{2})\pi$

$x = (n+\frac{1}{2}) \cdot \frac{\lambda}{2} \quad (n = 0, 1, 2, \cdots\cdots)$

4、能量

- When string is at maximum curvature
  - No K.E.
  - nodes have maximum energy (P.E.)
- When string is straight
  - Minimum P.E.
  - antinodes have maximum energy (K.E.)



- When the string is at maximum curvature, there is no K.E. At this moment, the nodes have maximum energy, in the form of P.E. (they stretch the most)

- When the string is straight, there is minimum P.E. At this moment, the antinodes have maximum energy, in the form of K.E. (largest speed)

- The mechanical energy flows back and forth from a node to an antinode as the string goes from maximum curvature to straight, and does not propagate along the string.

例: **Problem**

Checkpoint 5

Two waves with the same amplitude and wavelength interfere in three different situations to produce resultant waves with the following equations:

(1) $y'(x,t) = 4\sin(5x - 4t)$

(2) $y'(x,t) = 4\sin(5x)\cos(4t)$

(3) $y'(x,t) = 4\sin(5x + 4t)$

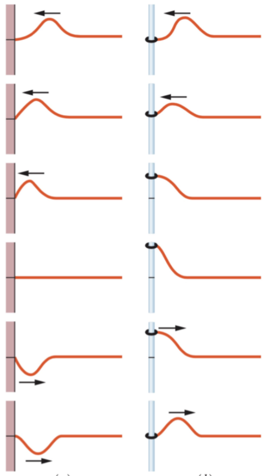
In which situation are the two combining waves traveling (a) toward positive x, (b) toward negative x, and (c) in opposite directions?

Answer: (a) 1; (b) 3; (c) 2

5. Reflection at a boundary

- Hard boundary: 位移永远为0，一定为波节
- Free boundary: 位移可以到达最大值，可以为波腹

Reflection at a Boundary



(a) (b)

- A wave reflects at the boundary and travels in the opposite direction to its incident direction.
- At a "Hard" boundary (a): the displacement is fixed to be zero. So the boundary is a node.
- At a "free" boundary(b): the displacement can be the maximum. So the boundary can be an antinode.

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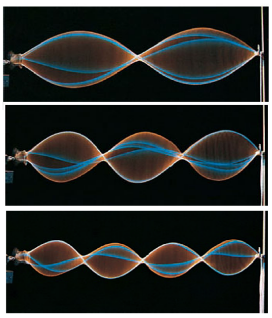
6. Standing wave and resonance

- 反射波与 incident wave (入射波) 会发生干涉
- 对于特定的频率，这种干涉会造成 standing wave pattern (or oscillation mode)，导致 resonance.
- The string is said to resonate at resonant frequencies.

Standing Wave and Resonance



Imperfect Standing Wave Patterns

- Two traveling wave reflects at hard boundaries at both ends.
- The reflected wave and the incident wave interfere with one another.
- For certain frequencies, such interference produces a standing wave pattern (or oscillation mode) with nodes and antinodes, causing resonance.
- The string is said to resonate at resonant frequencies.
- Not at resonant frequency, no standing wave pattern

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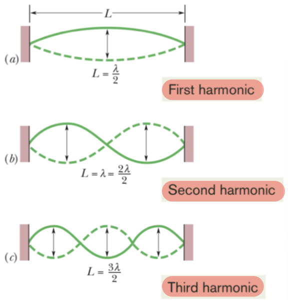
1° 两端均为 Hard boundaries.

$L = n \cdot \frac{\lambda}{2} \quad (n = 1, 2, 3 \cdots\cdots)$

resonance frequencies f:

$f = \frac{v}{\lambda} = \frac{nv}{2L} \quad (n = 1, 2, 3 \cdots\cdots)$

- For a standing Wave between two Hard boundaries, Each boundary is a node point.



- The distance L between two boundaries:

$$L = n\frac{\lambda}{2}$$

(for $n = 1, 2, 3, \ldots$)

- The resonance frequencies f:

$$f = \frac{v}{\lambda} = n\frac{v}{2L}$$

(for $n = 1, 2, 3, \ldots$)

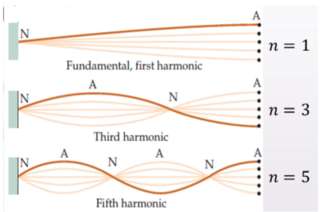
2° 一端 hard, 一端 free

$L = n \cdot \frac{\lambda}{4} \quad (n = 1, 3, 5 \cdots\cdots)$

resonance frequencies f:

$f = \frac{v}{\lambda} = \frac{nv}{4L} \quad (n = 1, 3, 5 \cdots\cdots)$

- Resonant Mode: one Hard and one Free boundaries
- The distance between two boundaries must be an odd integer of quarter wavelength



- The distance L between two boundaries:

$$L = n\frac{\lambda}{4}$$

(for $n = 1, 3, 5, \ldots$)

- The resonance frequencies f:

$$f = \frac{v}{\lambda} = n\frac{v}{4L}$$

(for $n = 1, 3, 5, \ldots$)

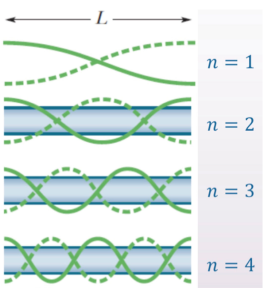
3° 两端 free

$L = n \cdot \frac{\lambda}{2} \quad (n = 1, 2, 3 \cdots\cdots)$

resonance frequencies f:

$f = \frac{v}{\lambda} = \frac{nv}{2L} \quad (n = 1, 2, 3 \cdots\cdots)$

- Resonant Mode: two Free boundaries, Each boundary is an antinode point.



- The distance L between two boundaries:

$$L = n\frac{\lambda}{2}$$

(for $n = 1, 2, 3, \ldots$)

- The resonance frequencies f:

$$f = \frac{v}{\lambda} = n\frac{v}{2L}$$

(for $n = 1, 2, 3, \ldots$)

- Resonance in 2D:

例:

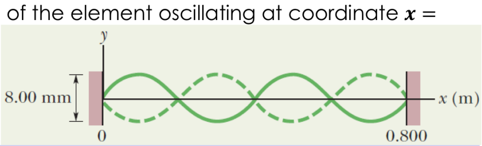
Problem

The figure shows a pattern of resonant oscillation of a string of mass $\boldsymbol{m = 2.500\ g}$ and length $\boldsymbol{L = 0.800\ m}$ and that is under tension $\boldsymbol{\tau = 325.0\ N}$.

a) What is the wavelength $\boldsymbol{\lambda}$ of the transverse waves producing the standing-wave pattern?

b) What is the frequency $\boldsymbol{f}$ of the transverse waves and of the oscillations of the moving string elements?

c) What is the maximum magnitude of the transverse velocity $\boldsymbol{u_m}$ of the element oscillating at coordinate $\boldsymbol{x = 0.180\ m}$?



Solution:

a)From the figure:

$$2\lambda = L \Rightarrow \lambda = \frac{L}{2} = \frac{0.800\text{ m}}{2} = 0.400\text{ m}$$

b)The Frequency is

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{\tau}{m/L}} = \sqrt{\frac{\tau L}{m}}$$

$$= \sqrt{\frac{(325\text{ N})(0.800\text{ m})}{2.50 \times 10^{-3}\text{ kg}}} = 322.49\text{ m/s}$$

$$f = \frac{v}{\lambda} = \frac{322.49\text{ m/s}}{0.400\text{ m}}$$

$$= 806.2\text{ Hz} \approx 806\text{ Hz}$$

Every piece of string element (so the whole string) oscillates vertically in simple harmonic motion with this frequency.

Solution:

- For the speed of the element:

$$y'(x,t) = 2y_m \sin(kx)\cos(\omega t)$$

$$u(x,t) = \frac{\partial y'}{\partial t} = \frac{\partial}{\partial t}[(2y_m \sin kx)\cos \omega t]$$

$$= [-2y_m\omega \sin kx]\sin \omega t.$$

$$u_m = |-2y_m\omega \sin kx|$$

- From the figure:

$$2y_m = 4.00\text{ mm}, x = 0.180\text{ m}, k = \frac{2\pi}{\lambda}, \text{and } \omega = 2\pi f$$

$$u_m = \left|-2(2.00 \times 10^{-3}\text{ m})(2\pi)(806.2\text{ Hz}) \times \sin\left(\frac{2\pi}{0.400\text{ m}}(0.180\text{ m})\right)\right|$$

$$= 6.26\text{ m/s}$$

- The time to reach its largest speed

$$\sin(\omega t) = \pm 1 \rightarrow t = \left(n + \tfrac{1}{2}\right)\tfrac{T}{2}$$

Summary

- Average Power:

$$P_{avg} = \frac{1}{2}\mu v \omega^2 y_m^2$$

- Superposition of Waves: Sum of the displacement
- Interference of Waves with same k and ω:
  - Same direction with same amplitude: **Traveling Sinusoidal Wave**

$$y' = \left[2y_m \cos\left(\frac{1}{2}\phi\right)\right]\sin\left(kx - \omega t + \frac{1}{2}\phi\right)$$

  - Same direction with different amplitude: **Traveling Sinusoidal Wave (using phasor)**

$$y'(x,t) = y'_m \sin(kx - \omega t + \beta)$$

  - Opposite direction with same amplitude: **Standing Wave**

$$y'(x,t) = 2y_m \sin(kx)\cos(\omega t)$$

Summary

- The Resonance Frequencies f:
  - Two Hard Boundaries:

$$f = \frac{v}{\lambda} = n\frac{v}{2L}\ (\text{for } n = 1, 2, 3, \ldots)$$

  - One Hard and one Free Boundaries:

$$f = \frac{v}{\lambda} = n\frac{v}{4L}\ (\text{for } n = 1, 3, 5, \ldots)$$

  - Two Free Boundaries:

$$f = \frac{v}{\lambda} = n\frac{v}{2L}\ (\text{for } n = 1, 2, 3, \ldots)$$