Lecture 13

§1 Functions of several variables

1. Functions of several variables (多元函数)

1° $f(x_1,\cdots\cdots,x_n)$ 表示一个含有 n 的实数变量的函数 (scalar function)

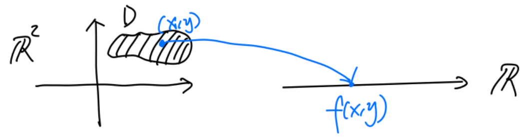
2° 另一种表示方法:

$$f(\vec{x})=f(x_1,\cdots\cdots,x_n)$$

$\vec{x}\in R^n$, $f: D\to R$, $D\subseteq R^n$, n 通常为 2 或 3.

向量表示仅仅用于强调存在多个变量.(考虑映射,可以把向量集想成点集)

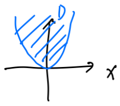
Vizualization: $\mathbb{R}^2$ $\mathbb{R}$



e.g. The area A of a rectangle depends on its length and width:

$$A=f(l,w)=lw,\quad l\geq 0,\ w\geq 0.$$

e.g. $f(x,y)=kx^{1/3}y^{2/3}$, $D=\mathbb{R}^2$.



$f(x,y)=\sqrt{y-x^2}$, $D=\{(x,y): y-x^2\geq 0\}=\{(x,y): y\geq x^2\}$.

2. 基本概念

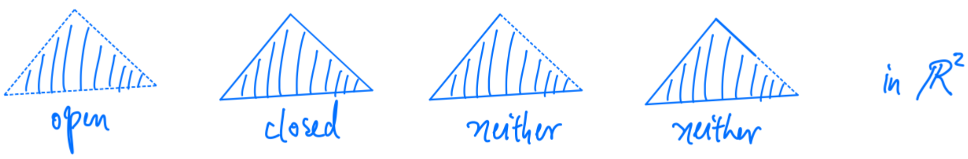
Definition

Let D be a subset of $\mathbb{R}^n$.

- A point P in $\mathbb{R}^n$ is called an interior point (内点) of D if some open ball centered at P completely lies in D.
- A point P in $\mathbb{R}^n$ is called a boundary point (边界点) of D if every open ball centered at P intersects both D and $\mathbb{R}^n\setminus D$.
- The set D is said to be open if every point in D is an interior point of D; it is said to be closed if it contains all of its boundary points.
- The set D is said to be bounded if it lies in some ball with finite radius; it is said to be unbounded otherwise.

in $\mathbb{R}^2$, "open disk"

disk in $\mathbb{R}^2$



- 邻域: 以某一点为圆心,指定半径的圆内的点的集合

$$U(P_0,\delta)=\{P \mid |PP_0|<\delta\}$$

去心 δ 邻域: $\mathring{U}(P_0,\delta)=\{P \mid 0<|PP_0|<\delta\}$

- 对于一个给定集合 $E\subset R^n$

内点 (interior point): 存在一个正数 r, 使 P 的 r 邻域整个包含于 E 中

$$U(P,r)\subset E$$

外点: 存在一个正数 r, 使 P 的 r 邻域与 E 不交

$$U(P,r)\cap E=\varnothing$$

边界点 (boundary point)：对于 $\forall$ 正数 r，$U(P,r)$ 中既有 E 中的点，又有非 E 中的点，

∂E 表示 E 的全体边界点的集合

聚点 (accumulation point)：对于 $\forall$ 正数 r，$\mathring{U}(P,r)$ 中总有 E 的点，

· 对于一个给定集合 $E\subset R^n$，

开集 (open)：所有点都为内点，

闭集 (closed)：包含所有的边界点

$\ast$ 直线为闭集，R^n 与 $\varnothing$ 既为开集又为闭集

有界集合 (bounded)：存在一个正数 ρ，使 E 包含于以原点为心，ρ 为半径的球内.

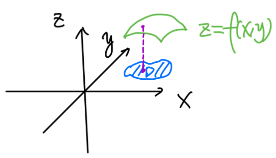
无界集合 (unbounded)：不存在一个正数 ρ，使 E 包含于以原点为心，ρ 为半径的球内.

3. 图像

1° n 变量函数的图像在 R^{n+1} 中

> **Def**: The graph of $f: D\to\mathbb{R}$, where $D\subseteq\mathbb{R}^n$, is the set
> $$\{(x_1,\dots,x_n,f(x_1,\dots,x_n)) : (x_1,\dots,x_n)\in D\}.$$

e.g. $n=2$; $z=f(x,y)$



Think of climbing a mountain:

(x,y) : 2D position on ground level

$f(x,y)$: mountain altitude at (x,y)

e.g. $n=3$; $w=f(x,y,z)$, graph is

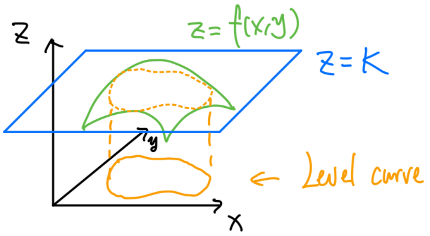
$$\{(x,y,z,w) : w=f(x,y,z),\ (x,y,z)\in D\}.$$

Can think of temperature : (x,y,z) : 3D position in space

$f(x,y,z)$: temparature at (x,y,z).

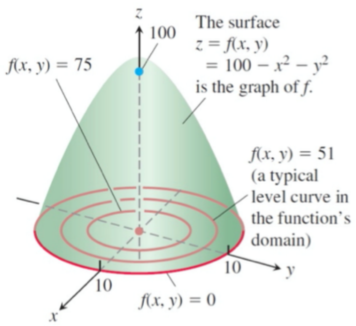
2° level curve (等值线) 与 level surface (等值面)

> **Def**: Given $f: D\to\mathbb{R}$, $D\subseteq\mathbb{R}^n$, and any constant $K\in\mathbb{R}$:
>
> · For $n=2$, the set $\{(x,y)\in D : f(x,y)=K\}$ is called a level curve of f.
>
> · For $n\geq 3$, the set $\{(x_1,\dots,x_n)\in D : f(x_1,\dots,x_n)=K\}$ is called a level surface of f.



注：函数 f 的 level curve / surface 在 f 的定义域内

例：作出 $f(x,y)=100-x^2-y^2$ 的 level curve



§2 Limits

1. 定义

Definition

Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$. We say that L is the limit of f as $\vec{x}$ approaches $\vec{x_0}$, and write

$\vec{x} \in \mathbb{R}^n$：
n real variables

$$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = L,$$

if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\vec{x} \in D$,

$$0 < |\vec{x} - \vec{x_0}| < \delta \text{ implies } |f(\vec{x}) - L| < \epsilon.$$

注：1° $0<|\vec{x}-\vec{x_0}|<\delta$ 等价于 $\vec{x}\in B_\delta(\vec{x_0})\setminus\{\vec{x_0}\}$.
即球心为 $\vec{x_0}$ 的终点，半径为 δ 的 open ball

2° 对于定义域为D的二元函数 f，

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

当对于 $\forall\varepsilon>0$，存在 $\delta>0$，使 $(x,y)\in D$，

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \Rightarrow |f(x,y)-L|<\varepsilon$$

2. 基本极限

Using the definition of limits, it can be shown that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0,\quad \lim_{(x,y)\to(x_0,y_0)} y = y_0, \text{ and } \lim_{(x,y)\to(x_0,y_0)} k = k,$$

where k is a constant.

- Similarly, $\lim_{(x,y,z)\to(x_0,y_0,z_0)} z = z_0$, etc.
- Note that $\lim_{(x,y)\to(x_0,y_0)} x$ means $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$, where $f(x,y)=x$. ← Graph is a cylindrical surface in $\mathbb{R}^3$.

证明：$\lim_{(x,y)\to(x_0,y_0)} x = x_0$

- 目标：对 $\forall\varepsilon>0$，$\exists\delta>0$，s.t. $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \Rightarrow |x-x_0|<\varepsilon$
- 令 $\delta:=\varepsilon$，Suppose $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$.
  则 $|x-x_0|=\sqrt{(x-x_0)^2}\le\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta=\varepsilon$（证毕）

3. 极限的性质

The following properties, although only stated for two-variable functions, hold for functions with finitely many variables.

THEOREM — Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$

if n even, assume $f(x,y)\geq 0$ for all (x,y) in some $B_\delta(x_0,y_0)\setminus\{(x_0,y_0)\}$

例：求 $\lim_{(x,y)\to(4,8)} 5x^2y^{\frac{1}{3}}$

$$\lim_{(x,y)\to(4,8)} 5x^2y^{\frac{1}{3}} = \left(\lim_{(x,y)\to(4,8)} 5\right)\cdot\left(\lim_{(x,y)\to(4,8)} x^2\right)\cdot\left(\lim_{(x,y)\to(4,8)} y^{\frac{1}{3}}\right)$$
$$= 5\cdot\left(\lim_{(x,y)\to(4,8)} x\right)^2\cdot\left(\lim_{(x,y)\to(4,8)} y\right)^{\frac{1}{3}}$$
$$= 5\cdot 4^2\cdot 8^{\frac{1}{3}}$$
$$= 160$$

例：求 $\lim_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}}\cdot x$$
$$= \lim_{(x,y)\to(0,0)} (\sqrt{x}+\sqrt{y})\cdot x$$
$$= 0$$

注：函数的定义域为 $D:\{(x,y): x\geq 0, y\geq 0, x\neq y\}$，且
- $(0,0)\notin D$
- $\forall \delta>0$, $B_\delta(0,0)\cap D\neq\emptyset$

4. Limits along paths (curves)

For a one-variable function, we know that $\lim_{x\to x_0} f(x) = L$ if and only if

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L.$$

An analogous statement for functions with multiple variables is the following.

Theorem
Let $f: D\to\mathbb{R}$ be a function, where $D\subseteq\mathbb{R}^n$. Then $\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = L$ if and only if

f converges to L as $\vec{x}$ approaches $\vec{x_0}$ along ~~any path~~ curve in D.

A curve in $\mathbb{R}^n$ is the range of a cts function $\vec{r}: I\to\mathbb{R}^n$.

注：1° 对于二元变量，若 $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$，且 $\vec{r}(t)$ 是 $\mathbb{R}^2$ 平面内任何满足 $\lim_{t\to t_0}\vec{r}(t) = \langle x_0, y_0\rangle$ 的曲线，则 $\lim_{t\to t_0} f(\vec{r}(t)) = L$.

2° 判断极限是否存在：

若f沿两条路径收敛于不同的数值，则 $\lim_{x\to x_0} f(x)$ 不存在

例：判断 $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$ 在(0,0)处极限是否存在

- On x-axis $(y=0, x\neq 0)$: $f(x,y)=1$
- On y-axis $(x=0, y\neq 0)$: $f(x,y)=-1$
- Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E

例：证明 $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ 不存在

- along the curve $y=kx$, $(x,y)\to(0,0)$ as $x\to 0$

  so $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(kx)}{x^2+(kx)^2} = \lim_{x\to 0} \frac{k^2x^2}{(1+k^2)x^2} = \frac{k}{1+k^2}$
- along $y=x$, limit $=\frac{1}{2}$; along $y=-x$, limit $=-\frac{1}{2}$
- so the limit D.N.E

例：证明 $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ 不存在

- along the curve $x=ky^2$, $(x,y)\to(0,0)$ as $y\to 0$

  so $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{ky^4}{(k^2+1)y^4} = \frac{k}{1+k^2} = \begin{cases} \frac{1}{2}, & \text{if } k=1 \\ -\frac{1}{2}, & \text{if } k=-1 \end{cases}$
- Hence limit D.N.E

5. Squeeze (Sandwich) theorem (夹逼准则)

> <u>Theorem</u>: If $g(x,y)\le f(x,y)\le h(x,y)$, $\forall (x,y)\in B_R(a,b)$ (for some $R>0$ fixed) and
>
> $$\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L,$$
>
> then
>
> $$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

例：求 $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

法一：

$0\le \left|\frac{3x^2y}{x^2+y^2}\right| \le 3|y| \to 0$ as $(x,y)\to(0,0)$

so $\lim_{(x,y)\to(0,0)} \left|\frac{3x^2y}{x^2+y^2}\right| = 0$

so $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

法二：(Polar coordinate)

$x=r\cos\theta$, $y=r\sin\theta$

$f(x) = \frac{3x^2y}{x^2+y^2} = \frac{3r^3\cos^2\theta\sin\theta}{r^2} = 3r\cdot(\cos^2\theta\sin\theta)$

Guess limit $=0$. Proof:

Let $\varepsilon>0$, (want $|f(x,y)-0| = |f(x,y)| < \varepsilon$)

Pick $\delta=\frac{\varepsilon}{3}$, suppose $|\langle x,y\rangle - \langle 0,0\rangle| = r < \delta = \frac{\varepsilon}{3}$

Then $|f(x,y)| \le 3|r||\cos^2\theta||\sin\theta| \le 3r < \varepsilon$

例: 证 $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ D.N.E

$f(x) = \frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$

so the limit D.N.E