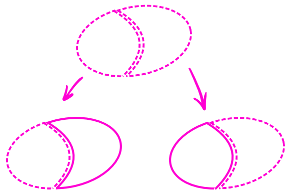
Lecture 12

§1 Connected set

1. **Definition:** separated (分离的)

Let $A, B \subset$ metric space X. We say A and B are separated if

$\bar{A} \cap B = \emptyset$ & $A \cap \bar{B} = \emptyset$ (In particular, $A \cap B = \emptyset$)



e.g. $X = \mathbb{R}$, $A = (0,1)$, $B = (1,2)$

2. **Definition:** Connected set (连通集)

$E \subset X$ is said to be connected if there doesn't exist nonempty A & $B \subset E$, s.t.

- $E = A \cup B$
- A and B are separated

注: 把E任意分成非空的两份, 这两份是separated

3. **Fact 1:** 连通集的等价条件

The following are equivalent:

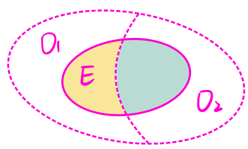
(a) E connected

(b) There doesn't exist open sets O_1 & $O_2 \in X$ s.t.

① $E = (E \cap O_1) \cup (E \cap O_2)$,

② $E \cap O_1$ & $E \cap O_2$ nonempty,

③ $E \cap O_1 \cap O_2 = \emptyset$



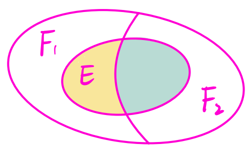
注: 若任意两个不与E互斥的开集, 它们无法既覆盖E, 又保持与E相交的部分互斥, 则E为连通集 (若保持互斥就一定无法覆盖E)

(c) There doesn't exist closed sets F_1 & $F_2 \in X$ s.t.

① $E = (E \cap F_1) \cup (E \cap F_2)$,

② $E \cap F_1$ & $E \cap F_2$ nonempty,

③ $E \cap F_1 \cap F_2 = \emptyset$



注: 若任意两个不与E互斥的闭集, 它们无法既覆盖E, 又保持与E相交的部分互斥, 则E为连通集 (若覆盖E就一定无法保持互斥)

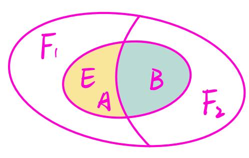
证明: (仅证明 (a) $\Longleftrightarrow$ (c))

① (a) $\Rightarrow$ (c)

Argue by contradiction.

Suppose $\exists$ closed F_1 & F_2, s.t. $E = (E \cap F_1) \cup (E \cap F_2)$, where $E \cap F_1$ & $E \cap F_2$ nonempty, $E \cap F_1 \cap F_2 = \emptyset$.

Call $E \cap F_1 = A$, $E \cap F_2 = B$, $E = A \cup B$



(W.T.S. $\exists$ 非空的 A, B, s.t. (1) $E = A \cup B$ (2) A, B separated,

即 $\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$, 由此证得 E不为连通集, 矛盾)

$\because$ $A \subset F_1$ and F_1 closed

$\therefore$ By ole results, $\bar{A} \subset F_1$. similarly, $\bar{B} \subset F_2$ (A的闭包是包含A的最小的闭集)

$\therefore$ $\bar{A} \cap B \subset F_1 \cap B = F_1 \cap (E \cap F_2) = \emptyset$

$\bar{B} \cap A \subset F_2 \cap A = F_2 \cap (E \cap F_1) = \emptyset$

$\therefore$ A & B are separated

$\therefore$ E is not connected

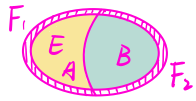
② (a) $\Leftarrow$ (c)

Argue by contradiction.

Suppose $\exists$ nonempty $A, B \subset E$, s.t. $E = A \cup B$, $\bar{A} \cap B = \emptyset = A \cap \bar{B}$

(W.T.S. $\exists$ closed F_1 & F_2, s.t. $E = (E \cap F_1) \cup (E \cap F_2)$, where $E \cap F_1$ & $E \cap F_2$ nonempty, $E \cap F_1 \cap F_2 = \emptyset$. 由此证得(c)的条件不满足，矛盾)

Take $F_1 = \bar{A}$ (closed), $F_2 = \bar{B}$ (closed)



· $F_1 \cap E = \bar{A} \cap E = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B)$

$= A \cup \emptyset = A$

· $F_2 \cap E = \bar{B} \cap E = \bar{B} \cap (A \cup B) = (\bar{B} \cap A) \cup (\bar{B} \cap B)$

$= \emptyset \cup B = B$

$\therefore$ $E = A \cup B = (F_1 \cap E) \cup (F_2 \cap E)$

$F_1 \cap E \cap F_2 = (F_1 \cap E) \cap (E \cap F_2) = A \cap B = \emptyset$

$\therefore$ (c) doesn't hold

4. Fact 2: 空集为连通集

Empty set $\emptyset$ is connected

5. Fact 3: 若X为连通集，则其既开又闭的子集仅有X与$\emptyset$

If X is connected, then any $E \subset X$ which is both open & closed must be either X or $\emptyset$

证明:

Argue by contradiction.

Suppose $\exists$ a nonempty set $E \neq X$, s.t. E is both open and closed

$\because$ E is closed

$\therefore$ $\bar{E} = E$

$\therefore$ $\bar{E} \cap E^c = E \cap E^c = \emptyset$

$\because$ E is open

$\therefore$ E^c is closed

$\therefore$ $\overline{E^c} = E^c$

$\therefore$ $\overline{E^c} \cap E = E^c \cap E = \emptyset$

$\because$ $X = E \cup E^c$

$\therefore$ z is not connected

6. **Fact 4:** R中的任意 interval 均为连通集

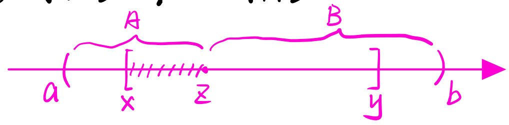
Any interval I in R is connected

证明:

Suppose $I=(a,b)$ (a,b may not be infinite)

Argue by contradiction. (证明若 I 不为连通集，会导致 I 中一点 z 不属于 I)

Suppose $\exists$ nonempty $A, B \subset X = R$, s.t. $I = A \cup B$, $\bar{A} \cap B = \emptyset = A \cap \bar{B}$



Pick $x \in A$, $y \in B$. WLOG, $a < x < y < b$

Let $z = \sup(A \cap [x,y]) \in I$

By HW5, Prob 5, $z = \sup(A \cap [x,y]) \in \overline{A \cap [x,y]} \subset \bar{A}$

$\because$ $\bar{A} \cap B = \emptyset$

$\therefore$ $z \notin B$

In particular, $z \neq y \Rightarrow x \leq z < y$

Case 1: $z \notin A$

$\therefore$ $z \notin A \cup B = I$ (contradiction)

Case 2: $z \in A$

$\because$ $\bar{B} \cap A = \emptyset$

$\therefore$ $z \notin \bar{B} = B \cup B'$

$\therefore$ $\exists z_1$ s.t. $z < z_1 < y$, $z_1 \notin B$ (z_1为z邻域中的一点, z邻域与B互斥)

$\because$ $x \leq z < z_1 < y$ & $z = \sup(A \cap [x,y])$

$\therefore$ $z_1 \notin A$

$\therefore$ $z_1 \notin A \cup B = I$ (contradiction)

7. **Fact 5:** R^n中的凸集必为连通集

Any convex subset E in R^n is connected

注: 称E为 convex 若 $\forall x,y \in E$, $\forall t \in [0,1]$. we have $tx + (1-t)y \in E$

证明:

Argue by contradiction

Suppose E is not connected. Then $\exists$ nonempty $A, B \subset R^n$, s.t. $A \cup B = E$, $\bar{A} \cap B = \emptyset = A \cap \bar{B}$

Take $a \in A$, $b \in B$.

Let $A_0 = \{t \in [0,1] \mid (1-t)a + tb \in A\}$, $B_0 = \{t \in [0,1] \mid (1-t)a + tb \in B\}$

$A_0 \cup B_0 = \{t \in [0,1] \mid (1-t)a + tb \in A \cup B\}$

$= \{t \in [0,1] \mid (1-t)a + tb \in E\}$

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