Lecture 2

§1 Probability, event, random variables

1、Definition: random experiment (随机实验), sample space (样本空间), event (事件)

- Random experiment: we describe a random experiment by its **procedure** and observations of its **outcomes**. For example, we toss a coin 2 times, and observe which side is up after each toss.
- Sample space: All possible outcomes of the random experiment form a sample space S. For the above coin toss example, we define

$$S = \{(Head, Head), (Head, Tail), (Tail, Head), (Tail, Tail)\}.$$

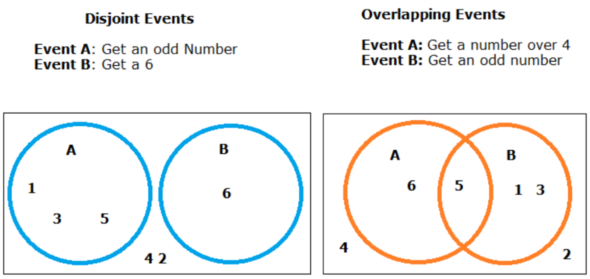
- Event: A **subset** of sample space S, denoted as A, can be called as an event in a random experiment, *i.e.*, $A \subset S$. In the above example, we define an event A as *at least one head up*, then it can be represented by

$$A = \{(Head, Head), (Head, Tail), (Tail, Head)\} \subset S.$$

2、概率公理

Assuming events $A \subset S$ and $B \subset S$, the probabilities of events related with and must satisfy,

- $P(A) \geq 0$
- $P(S) = 1$
- If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$; otherwise, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



3、Definition: random variables (随机变量)

- A random variable is a real valued function from the sample space S to a real space $\mathbb{R}$, as follows:

$$X : S \to \mathbb{R}$$

- Still take the 2-times coin toss as example, if we define the random variable as the number of tails, then we have

$$X((H,H)) = 0, X((H,T)) = 1, X((T,H)) = 1, X((T,T)) = 2.$$

  Then, the output space of X is denoted as $\{0, 1, 2\}$, also called state space $\mathcal{X}$.
- There are two types of random variables:
  - **Discrete**: $\mathcal{X}$ is discrete
  - **Continuous**: $\mathcal{X}$ is continuous

§2 Probability of discrete random variable

1、Definition: Probability of discrete random variable

- Probability of discrete random variable describes the chance of each state x in $\mathcal{X}$ for random variable X in a random experiment, denoted as

$$P(X = x), x \in \mathcal{X}.$$

2、Definition: joint, marginal probability

- Probability of a union of two events: Given two events A and B, we define the probability of A or B as follows:

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B), \\
&= P(A) + P(B) \text{ if } A \text{ and } B \text{ are } \textbf{mutually exclusive}.
\end{aligned} \tag{1}
$$

- Joint probabilities: The probability of the joint event A and B is defined as follows:

$$
P(A, B) = P(A \cap B) = P(A|B)P(B) = P(B|A)P(A), \tag{2}
$$

It is called the **product rule**.

- Marginal distribution: Given the above joint distribution, we can define the **marginal distribution** as follows:

$$
P(A) = \sum_b P(A, B) = \sum_b P(A|B = b)P(B = b), \tag{3}
$$

which sums over all possible states of B. It is called the **sum rule**.

3、Definition: conditional probability (条件概率), Bayes rule (贝叶斯法则)

- Conditional probability: Recalculating probability of event A after someone tells you that event B happened, as follows:

$$
P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{4}
$$

$$
P(A \cap B) = P(A|B)P(B) \tag{5}
$$

- Bayes Rule: Combining the definition of conditional probability with the product and sum rules yields Bayes rule, as follows:

$$
P(B|A) = \frac{P(A|B)P(B)}{P(A)}, \tag{6}
$$

$$
P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{P(X = x)P(Y = y|X = x)}{\sum_{x' \in \mathcal{X}} P(X = x')P(Y = y|X = x')} \tag{7}
$$

e.g.

- Suppose that you do a medical test for breast cancer, the test result could be *positive* or *negative*. We denote $x = 1$ as the event of positive test, while $x = 0$ as the event of negative test. We denote $y = 1$ as the event of having breast cancer, while $y = 0$ as the event of no breast cancer.
- Suppose that if one has breast cancer, the test will be positive with the probability 0.8, *i.e.*,

$$
P(x = 1|y = 1) = 0.8. \tag{8}
$$

- Then, if one gets a positive test result, what is the probability of having breast cancer? $P(y = 1|x = 1) = 0.8?$

- It is WRONG! It ignores the prior probability of having breast cancer.
- According to statistics, the average risk of a woman in the United States developing breast cancer sometime in her life is about 13%, *i.e.*,

$$
P(y = 1) = 0.13. \tag{9}
$$

- We also need to take into account the fact that the test may be a **false positive** or **false alarm**. Unfortunately, such false positives are quite likely (with current screening technology):

$$
P(x = 1|y = 0) = 0.1. \tag{10}
$$

- Combining all above probabilities using Bayes rule, we can compute

$$
\begin{aligned}
P(y = 1|x = 1) &= \frac{P(x = 1|y = 1)P(y = 1)}{P(x = 1|y = 1)P(y = 1) + P(x = 1|y = 0)P(y = 0)} \\
&= \frac{0.8 \times 0.13}{0.8 \times 0.13 + 0.1 \times 0.87} = 0.5445.
\end{aligned} \tag{11}
$$

It tells that if you test positive, you have have about a 54% chance of really having breast cancer!

4. Definition: independent random variables

- Independent: If X and Y are independent, denoted as $X \perp Y$, then the joint probability can be represented as the product of two marginals, *i.e.*,

$$X \perp Y \Longleftrightarrow P(X, Y) = P(X)P(Y). \tag{12}$$

- Given the above independence, we can use fewer parameters to define a joint probability. Suppose that X has 3 states, Y has 4 states, then we need $3 - 1 = 2$ and $4 - 1 = 3$ free parameters to define $P(X)$ and $P(Y)$, respectively.
- If without the independence, how many free parameters do we need to define the joint probability $P(X, Y)$? $(3 \times 4) - 1 = 11$.
- If given the independence, *i.e.*, $P(X, Y) = P(X)P(Y)$, how many free parameters do we need? $(3 - 1) + (4 - 1) = 5$.

5. Definition: expectation and variance of discrete random variables

- Expectation (or mean): $E(X) = \sum_{x \in \mathcal{X}} xP(X = x)$
- Expectation of a function: $E(f(X)) = \sum_{x \in \mathcal{X}} f(x)P(X = x)$
- Moments: expectation of power of X: $M_k = E(X^k)$
- Variance: Average (squared) fluctuation from the mean

$$\mathrm{Var}(X) = E((X - E(X))^2) = E(X^2) - E(X)^2 = M_2 - M_1^2. \tag{13}$$

- Standard deviation: Square root of variance, *i.e.*,

$$\mathrm{Std} = \sqrt{\mathrm{Var}(X)}. \tag{14}$$

§3 Probability of continuous random variable

1. Definition: continuous random variable

- A random variable X is continuous if its state space $\mathcal{X}$ is uncountable.
- In this case, $P(X = x) = 0$ for each x.
- If $p_X(x)$ is a probability density function (PDF) for X, then

$$P(a < X < b) = \int_a^b p(x)dx \tag{15}$$

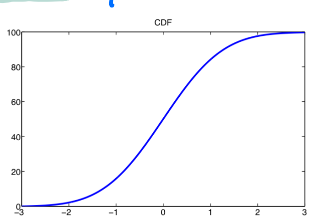
$$P(a < X < a + dx) \approx p(a) \cdot dx \tag{16}$$

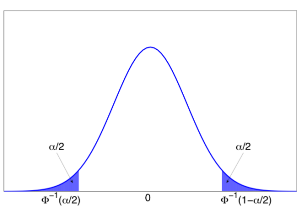
- The cumulative distribution function (CDF) is $F_X(x) = P(X < x)$. We have that $p_X(x) = F'(x)$, and $F(x) = \int_{-\infty}^{x} p(s)ds$.

2. Definition: marginal probability, conditional probability, independence

- $p_{X,Y}(x, y)$, joint probablity density function of X and Y
- $\int_x \int_y p(x, y)dxdy = 1$
- Marginal distribution: $p(x) = \int_{-\infty}^{\infty} p(x, y)dy$
- Conditional distribution: $p(x|y) = \frac{p(x,y)}{p(y)}$
- Note: $P(Y = y) = 0$! Formally, conditional probability in the continuous case can be derived using infinitesimal events.
- Independence: X and Y are independent if $p_{X,Y}(x, y) = p_X(x)p_Y(y)$

3. Definition: quantile





- Since the CDF $F(\cdot)$ is a monotonically increasing function, it has an inverse; let us denote this by $F^{-1}(\cdot)$.
- If $F(x)$ is the CDF of X, then $F^{-1}(\alpha)$ is the value of x_α such that $P(X \leq x_\alpha) = \alpha$; this is called the a quantile of F. The value $F^{-1}(0.5)$ is the median of the distribution, with half of the probability mass on the left, and half on the right. The values $F^{-1}(0.25)$ and $F^{-1}(0.75)$ are the **lower** and **upper quartiles**.

- We can also use the inverse CDF to compute tail area probabilities.
- For example, if Φ is the CDF of the Gaussian distribution $\mathcal{N}(0,1)$, then points to the left of $\Phi^{-1}(\alpha/2)$ contain $\alpha/2$ probability mass. By symmetry, points to the right of $\Phi^{-1}(1-\alpha/2)$ also contain $\alpha/2$ probability mass.
- Hence, the central interval $(\Phi^{-1}(\alpha/2), \Phi^{-1}(1-\alpha/2))$ contains $1-\alpha$ of the mass. If we set $\alpha = 0.05$, the central 95% interval is covered by the range

$$(\Phi^{-1}(0.025), \Phi^{-1}(0.975)) = (-1.96, 1.96). \tag{17}$$

For a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, the central 95% interval is $(\mu - 1.96\sigma, \mu + 1.96\sigma)$.

4. Definition: expectation and variance of continuous random variables

Similar to that of discrete random variables, only change the summation $\sum$ to the integral $\int$.

- Expectation (or mean): $\mu = E(X) = \int_{\mathcal{X}} x \cdot p(x)dx$
- Moments: expectation of power of X: $M_k = E(X^k) = \int_{\mathcal{X}} x^k \cdot p(x)dx$
- Variance: Average (squared) fluctuation from the mean

$$\mathrm{Var}(X) = E((X - E(X))^2) = E(X^2) - E(X)^2 = M_2 - M_1^2. \tag{18}$$

- Standard deviation: Square root of variance, *i.e.*,

$$\mathrm{Std} = \sqrt{\mathrm{Var}(X)}. \tag{19}$$

§4 Common distributions

1. Definition: bernoulli distribution (discrete)

- We firstly consider the probability of a binary random variable $x \in \{0, 1\}$. Suppose that you toss a coin, and $x = 1$ denotes the event of 'heads', while $x = 0$ indicates the event of 'tails'.
- The probability of $x = 1$ is described by a parameter μ,

$$p(x = 1|\mu) = \mu, \tag{20}$$

where $\mu \in [0, 1]$, and we can obtain that $p(x = 0|\mu) = 1 - \mu$.
- The probability distribution over x can therefore be written in the form

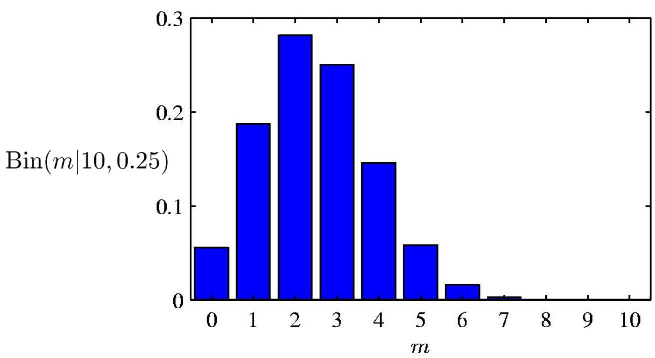
$$\mathrm{Bern}(x|\mu) = \mu^x(1-\mu)^{1-x}, \tag{21}$$

which is called Bernoulli distribution.
- Its mean and variance are

$$\mathbb{E}[x] = \sum_x x\mathrm{Bern}(x|\mu) = \mu, \tag{22}$$

$$\mathrm{var}[x] = \mathbb{E}[(x-\mu)^2] = \mu(1-\mu) \tag{23}$$

2. Definition: binomial distribution (discrete)



- Imagine that you toss the coin N times, and each tossing follows the Bernoulli distribution $p(x|\mu)$. We denote the variable m as the numbers of heads, then its distribution is formulated as follows:

$$\mathrm{Bin}(m|N,\mu) = \binom{N}{m} \mu^m (1-\mu)^{N-m}, \tag{24}$$

which is called Binomial distribution, where

$$\binom{N}{m} = \frac{N!}{(N-m)!m!}. \tag{25}$$

- Its mean and variance are

$$\mathbb{E}[m] = \sum_{m=0}^{N} m\mathrm{Bin}(m|N,\mu) = N\mu, \tag{26}$$

$$\mathrm{var}[m] = \mathbb{E}[(m-N\mu)^2] = N\mu(1-\mu). \tag{27}$$

3、Definition: Gaussian distribution (continuous)

- The Gaussian, also known as the **normal** distribution, is a widely used model for the distribution of **continuous** variables. In the case of a single variable x, the Gaussian distribution can be written in the form

$$\mathcal{N}(x|\mu,\sigma) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \tag{28}$$

where μ is the **mean** and σ^2 is the **variance**.

- For a D-dimensional vector $\boldsymbol{x}$, the multivariate Gaussian distribution takes the form

$$\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{D}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right), \tag{29}$$

where $\boldsymbol{\mu}$ is a D-dimensional **mean vector**, and $\boldsymbol{\Sigma}$ is a $D \times D$ **covariance matrix**, and $|\boldsymbol{\Sigma}|$ denotes the determinant of $\boldsymbol{\Sigma}$.