Lecture 3: Elementary Probility Theory

§1 Definition of probability

1. 概率 (probability)

1° 给出 model，预测 outcome.

2° quantify (量化) random experiment 的结果的 likehood (可能性)，或 chance.

3° 重复同一实验无限次，给定结果对应的 observation 的占比 接近 该结果的概率.

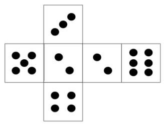
- Random Experiment:

- Consider one possible outcome： ω
- The outcome ω happens with probability P(ω)
- If we repeat such experiment **N** times
- We observe $\mathbf{N_1}$ observations that the outcome is ω.
- Then if N goes to infinity, $\mathbf{N_1/N}$ will approach P(ω).

Example 1

- Experiment: roll a fair dice once
- All possible outcomes: 1,2,3,4,5,6.
- What is the probability that the outcome is i, p_i. i=1,2,3,4,5,6
- You roll N times, you observe N_i times where the outcome is i. When N goes to infinity, N_i/N will be p_i.
- As $N_i/N=1/6$, $p_i=1/6$

Example 2



- Experiment: roll an unfair dice
- What is the probability that the outcome is 2.
- You roll the dice N times, you observe N_2 times where the outcome is 2. When N goes to infinity, N_2/N will be p.
- As $N_2/N=2/6$, p =1/3.

§2 5 probability 相关的 terminologies

1. Random experiment (随机试验)

1° 一个可导致多种 outcomes 的试验.

2° a repeatable procedure.

2. Sample space (样本空间)

1° 是所有可能 outcomes 的集合 (set)

2° 分为离散与连续

discrete (离散)：包含 finite 或 countable infinite 个结果. e.g. {H, T}, {1, 2, 3, 4 ……}, …

continuous (连续)：包含实数区间 (有界或无界). e.g. $[1,2]$, $(3,+\infty)$, …

Sample space

- Discrete or continuous: countable (listable) or not?
- Which of the following are continuous random variables?
  1. The sum of numbers on a pair of two dice.
  2. The possible sets of outcomes from flipping ten coins.
  3. The possible sets of outcomes from flipping (countably) infinite coins.
  4. The possible values of the temperature outside on any given day.
  5. The possible times that a person arrives at a restaurant.

3° 分为有序与无序

order-mattered (有序)：(A, B) 与 (B, A) 不同 e.g. {(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)}

not order-mattered (无序)：(A, B) 与 (B, A) 相同 e.g. {(1,2), (1,3), (2,3)}

4° 分为放回与不放回

select two items without replacement (不放回): {ab, ac, ba, bc, ca, cb}

select two items with replacement (有放回): {aa, ab, ac, ba, bb, bc, ca, cb, cc}

3. Events (事件)

1° Events 是可用 words / notation 描述的 sets，包含一个或多个 outcomes

- Experiment: toss a coin 2 times.
- Event -- You get 1 or more heads
  = { HH, HT, TH}

2° Events 遵从 set operations

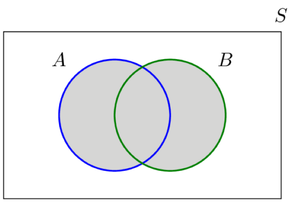
4. Set operations (集合运算)

1° Union (并集) (或)

union = A or B = $A \cup B$

Set operations - union

- The **union** of two events A and B is the event that consists of all outcomes that are contained in either A or B.
- We denote the union as (A or B) in words, and $A \cup B$ in notation



Venn diagrams

2° Intersection (交集) (和)

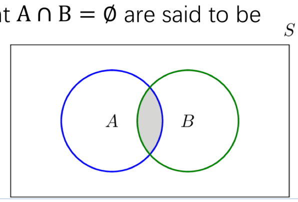
intersection = A and B = $A \cap B$

若 events A与B满足 $A \cap B = \varnothing$，则A与B mutually exclusive (互斥的)

Set operations - intersection

- The **intersecion** of two events A and B is the event that consists of all outcomes that are contained in both A and B.
- We denote the intersection as (A and B) in words, and $A \cap B$ in notation
- Two events A and B, such that $A \cap B = \emptyset$ are said to be **mutually exclusive**.

Note: The intersection can be an empty set.

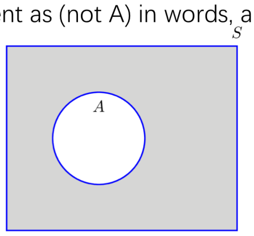


3° Complement (补集) (对立事件)

complement = not A = A' = A^c

Set operations - complement

- The **complement** of an events A in a sample space S is the set of outcomes in the sample space that are not in the event A.
- We denote the complement as (not A) in words, and A' or A^c in notation



Note: The complement can also be an empty set, it can also be the whole sample space.

Set operations – examples

- Toss a coin 2 times: sample space: {TT, HT, TH, HH}
- Event A: there is at least one Head: A = {HH, HT, TH}
- Event B: there is at least one Tail: B = {TT, HT, TH}
- Question:
  - What is the event A or B ($A \cup B$)? {TT, HT, TH, HH}
  - What is the event A and B ($A \cap B$)? {HT, TH}
  - What is the complement of event A? {TT}
  - What is the intersection of event B and the complement of event A? {TT}

4° Some useful properties

(1) $(E')' = E$

(2) $A \cap B = B \cap A$ and $A \cup B = B \cup A$

(3) $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

- The complement of the complement of an event is itself

$$(E')' = E$$

- The intersection and union does not depend on the order

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A$$

- The complement of the union (intersection) is the intersection (union) of the complement

$$(A \cup B)' = A' \cap B' \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

5. Probability function (概率函数) $P(\omega)$

- $P(\omega)$ gives the probability for each outcome $\omega \in \Omega$

Example:

Whenever a sample space consist of N possible outcomes that are *equally likely*, the probability of each outcome is 1/N

- 在定义了每个 outcome 的 probability 后，我们可以 assign probabilities to 由多个 outcomes 组成的 events
- 对于 discrete sample space，事件 E 的可能性被表示为 P(E)，P(E) 等于 E 中的 outcomes 的概率和

Example

Head and Tail are equally likely

- Experiment: toss a fair coin, report heads or tails.
- Sample space: Ω = {H, T}.
- Probability function: P(H) = .5, P(T) = .5, P(Ω) = 1.

Event	H	T	Ω: H or T
Probability	0.5	0.5	1

Tables can really help in complicated examples !

Example

A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$. Then,

$$P(A) = 0.1 + 0.3 = 0.4$$
$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$
$$P(C) = 0.1$$

Example

1,2,3,4,5,6 are equally likely

- Experiment: roll a fair dice, report the number
- Sample space: $\Omega = \{1,2,3,4,5,6\}$.
- Probability function: $P(i) = 1/6$, $i=1,2,3,4,5,6$.

- Event A: the number is > 3: {4,5,6}

- ➡ $P(A) = P(\{4\})+P(\{5\})+P(\{6\}) = 3/6=1/2$

Axioms of probability (概率公理)

1° $P(S)=1$, S is the sample space
total probability of all possible outcomes is 1

2° $0 \le P(A) \le 1$ for any event A
probability is between 0 and 1

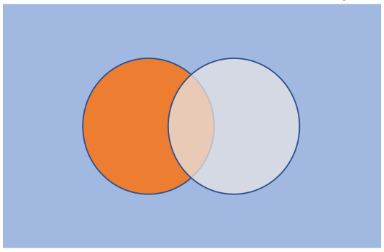
3° 对于 mutually exclusive ($A \cap B = \varnothing$) 的 events A 和 B，有
$P(A \cup B) = P(A) + P(B)$

利用以上三条公理，可以证得：

- $P(\varnothing) = 0$
- $P(E') = 1 - P(E)$

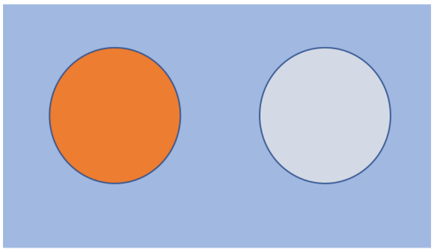
§3 Events 间的关系 (Joint and disjoint)

1. joint (相容的)
若 A 发生，B 也有可能发生
$P(A \text{ or } B) < P(A) + P(B)$



例：A: Infected with COVID-19
B: Nucleic acid test is negative

2. disjoint (互斥的)
等同于 mutually exclusive
($A \cap B = \varnothing$) 若 A 发生，B 不可能发生
$P(A \text{ or } B) = P(A) + P(B)$



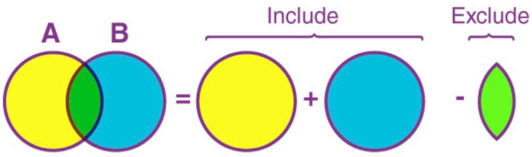
例: Experiment: toss a coin 3 times.
A: “exactly 2 heads”
B: “exactly 2 tails

3. addition rules (加法法则)

1° 对于两个 events:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

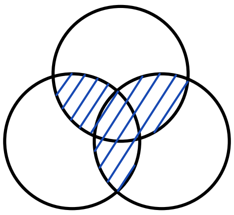
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



2° 对于多个 events:

$$P(A \cup B \cup C) = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



§4 Events 间的关系 (Independence (独立性))

1. independent (独立的)

A(B)事件的发生对B(A)的发生没有影响

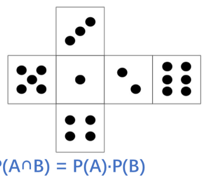
事件A与B是独立的 $\Longleftrightarrow$

$$\begin{cases} 1^\circ & P(A \text{ and } B) = P(A) \cdot P(B) \\ & P(A \cap B) = P(A) \cdot P(B) \\ 2^\circ & P(A \mid B) = P(A) \\ 3^\circ & P(B \mid A) = P(B) \end{cases}$$

Independence

- Two events: A and B
- Does knowing something about A tell us whether B happens (and vice versa)?
- **Independent**
  - A: the first number is 1;
  - B: the second number is 1

  P(A and B) = P(A)·P(B)



Two events A, and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$

2、dependent

A(B)事件的发生对B(A)的发生有影响

$P(A \text{ and } B) \neq P(A)\cdot P(B)$

Does knowing something about A tell us whether B happens (and vice versa)?

- Dependent
  - A: the first number is 1;
  - B: the sum of numbers is larger than 2

P(A) = 1/6
P(B) = 35/36
P(A and B) = 5/36
P(A and B) ≠ P(A)·P(B)

SAMPLE SPACE FOR A PAIR OF DICE

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

补：1、证明：若事件A与事件B发生的概率都不为0，那么独立和互斥有这样一层关系：
互斥不独立，独立不互斥

① 若A、B互斥，则$A\cap B=\varnothing$
那么$P(A\cap B)=0$
而$P(A)\cdot P(B)\neq 0$，
因此$P(A\cap B)\neq P(A)\cdot P(B)$，即A、B不独立

② 若A、B独立，则$P(A\cap B)=P(A)\cdot P(B)$
假设A、B互斥，则$A\cap B=\varnothing$
那么$P(A\cap B)=0$
而$P(A)\cdot P(B)\neq 0$
则$P(A\cap B)\neq P(A)\cdot P(B)$
矛盾
因此A、B独立，则A、B不互斥

2、零概率事件与不可能事件

零概率事件：$P(A)=0$
不可能事件：$A=\varnothing$
在离散情况下，两者等价.
在连续情况下，零概率事件可能发生
e.g. X服从在(0,1)上均匀分布
则事件$\{X=0.5\}$发生的概率为0
但事件$\{X=0.5\}$有可能发生

$P(A)=0 \underset{\Leftarrow}{\not\Rightarrow} A=\varnothing$

3、证明：零概率事件与任何事件独立

设 $P(A)=0$

对于任意事件 B，有 $A\cap B\subseteq A$

那么 $P(A\cap B)\leq P(A)$

则 $P(A\cap B)=0$

于是有 $P(A\cap B)=P(A)\cdot P(B)=0$

即 A、B 独立

4、证明：不可能事件与任意事件互斥

设 A 为不可能事件

则 $A=\varnothing$

对任何事件 B，有 $A\cap B=\varnothing$

则 A、B 互斥

5、综上：不可能事件与任何事件既独立又互斥

6、不独立可能不互斥，不互斥可能不独立