

# Lecture 18

## §1 关于 Differentiation 的 facts (接上)

### 1. Fact 11: 偏导连续 $\Rightarrow$ total differentiability

Suppose  $f: \text{open } D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  has continuous partial derivatives (i.e.  $\frac{\partial f_i}{\partial x_j}(x_1, \dots, x_n)$  exists & continuous on  $\Omega$ ,  $i=1, \dots, m, j=1, \dots, n$ )

Then  $\forall c \in \Omega$ ,  $f$  is (totally) differentiable at  $c$ .

证明:

$$\text{W.T.S. } f(c+h) = f(c) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(c) & \frac{\partial f_1}{\partial x_2}(c) & \dots & \frac{\partial f_1}{\partial x_n}(c) \\ \frac{\partial f_2}{\partial x_1}(c) & \frac{\partial f_2}{\partial x_2}(c) & \dots & \frac{\partial f_2}{\partial x_n}(c) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(c) & \frac{\partial f_m}{\partial x_2}(c) & \dots & \frac{\partial f_m}{\partial x_n}(c) \end{bmatrix}_{m \times n} h + o(h) \text{ as } h \rightarrow 0$$

Suffice to show:

$$f_i(c+h) = f_i(c) + \nabla f_i(c) \cdot h + o(h) \text{ as } h \rightarrow 0 \quad i=1, 2, \dots, m$$

To save notation, write  $f_i$  as  $f$

$$\begin{aligned} & f(c+h) - f(c) - \nabla f(c) \cdot h \\ &= f(c_1+h_1, c_2+h_2, \dots, c_n+h_n) - f(c_1, c_2, \dots, c_n) \\ & \quad - \frac{\partial f}{\partial x_1}(c_1, c_2, \dots, c_n) \cdot h_1 - \frac{\partial f}{\partial x_2}(c_1, c_2, \dots, c_n) \cdot h_2 - \dots - \frac{\partial f}{\partial x_n}(c_1, c_2, \dots, c_n) \cdot h_n \\ &= f(c_1+h_1, c_2+h_2, \dots, c_n+h_n) - f(c_1, c_2+h_2, \dots, c_n+h_n) \\ & \quad + f(c_1, c_2+h_2, \dots, c_n+h_n) - f(c_1, c_2, c_3+h_3, \dots, c_n+h_n) \\ & \quad + f(c_1, c_2, c_3+h_3, \dots, c_n+h_n) - f(c_1, c_2, c_3, c_4+h_4, \dots, c_n+h_n) \\ & \quad \dots \\ & \quad + f(c_1, c_2, \dots, c_{n-1}, c_n+h_n) - f(c_1, c_2, \dots, c_n) \\ & \quad - \frac{\partial f}{\partial x_1}(c_1, c_2, \dots, c_n) \cdot h_1 - \frac{\partial f}{\partial x_2}(c_1, c_2, \dots, c_n) \cdot h_2 - \dots - \frac{\partial f}{\partial x_n}(c_1, c_2, \dots, c_n) \cdot h_n \\ & \stackrel{\text{M.V.T.}}{=} \frac{\partial f}{\partial x_1}(\tilde{c}_1, c_2+h_2, \dots, c_n+h_n) \cdot h_1 \\ & \quad + \frac{\partial f}{\partial x_2}(c_1, \tilde{c}_2, c_3+h_3, \dots, c_n+h_n) \cdot h_2 \\ & \quad + \dots \\ & \quad + \frac{\partial f}{\partial x_n}(c_1, c_2, \dots, c_{n-1}, \tilde{c}_n) \cdot h_n \\ & \quad - \frac{\partial f}{\partial x_1}(c_1, c_2, \dots, c_n) \cdot h_1 - \frac{\partial f}{\partial x_2}(c_1, c_2, \dots, c_n) \cdot h_2 - \dots - \frac{\partial f}{\partial x_n}(c_1, c_2, \dots, c_n) \cdot h_n \\ & \quad (\tilde{c}_i \text{ between } c_i+h_i \text{ and } c_i) \\ &= \left[ \frac{\partial f}{\partial x_1}(\tilde{c}_1, c_2+h_2, \dots, c_n+h_n) - \frac{\partial f}{\partial x_1}(c_1, c_2, \dots, c_n) \right] \cdot h_1 \\ & \quad + \left[ \frac{\partial f}{\partial x_2}(c_1, \tilde{c}_2, c_3+h_3, \dots, c_n+h_n) - \frac{\partial f}{\partial x_2}(c_1, c_2, \dots, c_n) \right] \cdot h_2 \\ & \quad + \dots \\ & \quad + \left[ \frac{\partial f}{\partial x_n}(c_1, c_2, \dots, c_{n-1}, \tilde{c}_n) - \frac{\partial f}{\partial x_n}(c_1, c_2, \dots, c_n) \right] \cdot h_n \\ &= o(1) \cdot h_1 + o(1) \cdot h_2 + \dots + o(1) \cdot h_n \text{ as } h \rightarrow 0 \end{aligned}$$

$$\begin{aligned}
 & \left( \text{由于偏导连续, } \frac{\partial f}{\partial x_i}(c_1, c_2+h_2, \dots, c_n+h_n) - \frac{\partial f}{\partial x_i}(c_1, c_2, \dots, c_n) \rightarrow 0 \text{ as } h \rightarrow 0 \right) \\
 & = o(1)|h| + o(1)|h| + \dots + o(1)|h| \text{ as } h \rightarrow 0 \quad (\text{由于 } |h| \leq |h|) \\
 & = o(1)|h| \text{ as } |h| \rightarrow 0
 \end{aligned}$$

Therefore,  $f(c+h) - f(c) - \nabla f(c) \cdot h = o(|h|)$  as  $h \rightarrow 0$

$\therefore f$  total differentiable at  $c$

Q.E.D.

## 2. Definition: $C^k$ -smooth

- 我们称  $f: \text{open } D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  为  $C^1$ -smooth on  $D$ .  
若所有  $\frac{\partial f}{\partial x_i}$  存在且连续 on  $\Omega$
- 我们称  $f: \text{open } D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  为  $C^2$ -smooth on  $D$ .  
若  $f$  为  $C^1$ -smooth on  $D$ , 且所有 second partials  $\frac{\partial^2 f}{\partial x_i \partial x_j}$  存在且连续 on  $\Omega$
- 类似地, 可以定义  $C^k$ -smooth ( $k \geq 1$ ),  $C^\infty$ -smooth
- $C^1(D) = \text{real-valued functions which are } C^1\text{-smooth}$   
注:  $C^1(D)$  可理解为 vector space:  $f \in C^1(D), g \in C^1(D), c_1 f + c_2 g \in C^1(D)$

## 3. Lemma:

Let  $f(x, y)$  be real-valued & defined near  $(x_0, y_0) \in \mathbb{R}^2$  ( $f(x_0, y_0)$  may not be defined) &

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = A \text{ (finite \#)}.$$

Assume  $\forall y \approx y_0, y \neq y_0, \lim_{x \rightarrow x_0} f(x, y)$  exists as a finite, denoted as  $g(y)$ .

Then  $\lim_{y \rightarrow y_0} g(y)$  exists and equals to  $A$

$$\text{即 } \lim_{y \rightarrow y_0} \left( \lim_{x \rightarrow x_0} f(x, y) \right) = \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$$

证明:

$$\therefore \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = A$$

$\therefore \forall \varepsilon > 0, \exists \delta > 0$ , s.t. as long as  $0 < |(x, y) - (x_0, y_0)| < \delta, |f(x, y) - A| < \varepsilon$

Now fix  $y \approx y_0, y \neq y_0$ , s.t.  $|y - y_0| < \frac{\delta}{2}$

Then  $|f(x, y) - A| < \varepsilon$  for all  $x$  close to  $x_0, x \neq x_0$

Then sending  $x \rightarrow x_0$ ,

$$\lim_{x \rightarrow x_0} |f(x, y) - A| \leq \lim_{x \rightarrow x_0} \varepsilon$$

$$\Rightarrow |g(y) - A| \leq \varepsilon$$

$$\Rightarrow \lim_{y \rightarrow y_0} g(y) = A$$

Q.E.D.

## 4. Fact 12: Clairaut's theorem (克劳莱定理)

Let  $f: \text{open } D \subset \mathbb{R}^n \rightarrow \mathbb{R}, f \in C^1(D)$ , Suppose  $\frac{\partial f}{\partial x_j}(\frac{\partial f}{\partial x_i})$  exists and continuous on  $D$ .

Then  $\frac{\partial f}{\partial x_i}(\frac{\partial f}{\partial x_j})$  also exists and equals to  $\frac{\partial f}{\partial x_j}(\frac{\partial f}{\partial x_i})$  in  $D$

证明:

WLOG, assume  $f$  is a function of  $(x, y) \in D \subset \mathbb{R}^2$

Know:  $\forall c \in D, \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(c)$  exists  $= A$

$$\begin{aligned}\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(c) &= \frac{d}{dy}(\frac{\partial f}{\partial x}(c, y))|_{y=c_2} \\ &= \lim_{k \rightarrow 0} \frac{\frac{\partial f}{\partial x}(c, c_2+k) - \frac{\partial f}{\partial x}(c, c_2)}{k} = A\end{aligned}$$

W.T.S.  $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(c)$  exists  $= A$

$$\begin{aligned}\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(c) &= \frac{d}{dx}(\frac{\partial f}{\partial y}(x, c_2))|_{x=c_1} \\ &= \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial y}(c_1+h, c_2) - \frac{\partial f}{\partial y}(c_1, c_2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \lim_{k \rightarrow 0} \left( \frac{f(c_1+h, c_2+k) - f(c_1+h, c_2)}{k} - \frac{f(c_1, c_2+k) - f(c_1, c_2)}{k} \right) \right] \\ &= \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \frac{1}{hk} [f(c_1+h, c_2+k) - f(c_1+h, c_2) - f(c_1, c_2+k) + f(c_1, c_2)]\end{aligned}$$

Let  $F(h, k) = \frac{1}{hk} [f(c_1+h, c_2+k) - f(c_1+h, c_2) - f(c_1, c_2+k) + f(c_1, c_2)]$  defined for  $(h, k) \approx 0, h \neq 0, k \neq 0$   
 $F(h, k)$  satisfies:

- For each fixed  $h \approx 0, h \neq 0$

$$\lim_{k \rightarrow 0} F(h, k) = \frac{1}{h} \left[ \frac{\partial f}{\partial y}(c_1+h, c_2) - \frac{\partial f}{\partial y}(c_1, c_2) \right]$$

$$\begin{aligned}\lim_{\substack{(h,k) \rightarrow (0,0) \\ h \neq 0, k \neq 0}} F(h, k) &\stackrel{M.V.T.}{=} \lim_{\substack{(h,k) \rightarrow (0,0) \\ h \neq 0, k \neq 0}} \frac{1}{hk} \left[ \frac{\partial f}{\partial x}(\tilde{c}_1, c_2+k)h - \frac{\partial f}{\partial x}(\tilde{c}_1, c_2)h \right] \\ &= \lim_{\substack{(h,k) \rightarrow (0,0) \\ h \neq 0, k \neq 0}} \frac{1}{k} \left[ \frac{\partial f}{\partial x}(\tilde{c}_1, c_2+k) - \frac{\partial f}{\partial x}(\tilde{c}_1, c_2) \right] \\ &\stackrel{M.V.T.}{=} \lim_{\substack{(h,k) \rightarrow (0,0) \\ h \neq 0, k \neq 0}} \frac{1}{k} \frac{\partial f}{\partial y}(\frac{\partial}{\partial x})(\tilde{c}_1, \tilde{c}_2) \cdot k \\ &= \lim_{\substack{(h,k) \rightarrow (0,0) \\ h \neq 0, k \neq 0}} \frac{\partial f}{\partial y}(\frac{\partial}{\partial x})(\tilde{c}_1, \tilde{c}_2) \\ &= \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(c_1, c_2) = A\end{aligned}$$

$\therefore$  By lemma,  $\lim_{k \rightarrow 0} (\lim_{h \rightarrow 0} F(h, k))$  exists &  $= A$

$\therefore \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(c)$  exists &  $= A$

注: 有时  $\frac{\partial^2 f}{\partial x \partial y}(c) \neq \frac{\partial^2 f}{\partial y \partial x}(c)$

$$\text{e.g. } f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0)$$