Lecture 14

§1 关于连续性的 facts (接上)

1. **Fact 6:** 若f在紧集上连续, 则值域也为紧集

If $f: X \to Y$ is continuous & X is compact, then $f(X)$ is also compact.

证明:

仅需证明 $f(X)$ 是 sequentially compact 的

$\forall \{y_n\}_{n=1}^{\infty} \subset f(X) \Rightarrow \exists x_n \in X$ s.t. $f(x_n) = y_n$, $\forall n \geq 1$

$\because$ X is compact

$\therefore$ X is sequentially compact

$\therefore$ $\exists$ subseq $\{x_{n_k}\}_{k=1}^{\infty}$ s.t. $x_{n_k} \to$ some x_∞ as $k \to \infty$

$\because$ f is continuous on X

$\therefore$ $f(x_{n_k}) = y_{n_k} \to f(x_\infty) \in f(X)$ as $k \to \infty$

$\therefore$ $f(X)$ sequentially compact

Q.E.D.

2. **Fact 7:** Extreme value theorem

If $f: X \to R$ continuous & X is compact, then

$\exists p, q \in X$, s.t. $f(p) = \sup_{x\in X} f(x) = \max_{x \in X} f(x)$, $f(q) = \inf_{x \in X} f(x) = \min_{x\in X} f(x)$

注: 若 $f(x)$ 能 achieve $\sup_{x\in X} f(x)$, 则此时称 $\sup_{x\in X} f(x)$ 为 $\max_{x\in X} f(x)$

若 $f(x)$ 能 achieve $\inf_{x\in X} f(x)$, 则此时称 $\inf_{x\in X} f(x)$ 为 $\min_{x\in X} f(x)$

证明:

(利用HW中的定理: $E \subset R$, E bdd $\Rightarrow$ $\sup E \subset \bar{E}$)

$\because$ X compact

$\therefore$ $f(X)$ compact

$\therefore$ $f(X)$ closed & bdd

$\therefore$ $\overline{f(X)} = f(X)$

Suppose $M = \sup_{x\in X} f(x)$, $m = \inf_{x \in X} f(x)$, $-\infty < m \leq M < \infty$

$\therefore$ By ole HW, $m, M \in \overline{f(X)} = f(X)$

Q.E.D.

3. **Fact 8:** 单射+满射+函数连续+X为紧集 = 反函数连续

If $f: X \to Y$ is continuous & one-to-one & onto & X is compact, then $f^{-1}: Y \to X$ is also continuous (此处 f^{-1} 为反函数)

注: "one-to-one": if $x \neq y$, then $f(x) \neq f(y)$ (单射)

"onto": $f(X) = Y$ (满射)

证明：

$\forall y_0 \in Y$, W.T.S. f^{-1} is continuous at y_0

① If $y_0 \notin Y'$, then nothing to prove

② If $y_0 \in Y'$, W.T.S. $\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0)$

Argue by contradiction. Suppose not.

$\exists \varepsilon_0 > 0$, $\exists$ bad seq $\{y_n\}_{n=1}^{\infty} \subset Y$. $y_n \to y_0$ as $n \to \infty$. s.t. $\underbrace{d_X(f^{-1}(y_n), f^{-1}(y_0)) \geq \varepsilon_0}_{(*)}$ $\forall n \geq 1$

$\because$ f is onto $(f(X) = Y)$

$\therefore$ $\exists x_n \in X$ s.t. $f(x_n) = y_n$, $\forall n \geq 1$

$\because$ X is compact

$\therefore$ $\exists$ subseq $x_{n_k} \to$ some $x_\infty \in X$

$\because$ f is continuous

$\therefore$ $f(x_{n_k}) \to f(x_\infty)$ as $k \to \infty$

$\therefore$ $y_{n_k} \to f(x_\infty)$ as $k \to \infty$

$\therefore$ $f(x_\infty) \to y_0$

$x_\infty = f^{-1}(y_0)$

By $(*)$,

$d_X(x_{n_k}, x_\infty) \geq \varepsilon_0$, $\forall k \geq 1$ (contradiction)

Q.E.D.

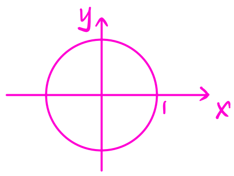
注：若 X 不为 compact，则结论可能不成立

取 $X = [0, 2\pi)$ (not compact), Y = unit circle on xy-plane, $f(\theta) = (\cos\theta, \sin\theta)$

① f continuous? Yes

② f one to one? Yes

③ f onto? Yes



$f^{-1}(1,0) = 0$

$f^{-1}(\cos(2\pi - \frac{1}{n}), \sin(2\pi - \frac{1}{n})) = 2\pi - \frac{1}{n} \to 2\pi$

But $(\cos(2\pi - \frac{1}{n}), \sin(2\pi - \frac{1}{n})) \to (1, 0)$ as $n \to \infty$

$\therefore$ f^{-1} not continuous at $(1, 0)$

Recall：我们称 f 在 $x_0 \in X$ 处连续，若 $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $d_Y(f(x), f(x_0)) < \varepsilon$ as long as $d_X(x, x_0) < \delta$

但此时 δ 的大小可能取决于 x_0 的选取（不同 x_0 处，$f(x_0)$ 的陡峭程度不同，同一个 ε，陡的地方 δ 要取更小的值）

4. Definition: Uniform continuity（一致连续）

令 $f: X \to Y$，我们称 f 为 uniformly continuous,

If $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), f(y)) < \varepsilon$ whenever $x, y \in X$, $d_X(x, y) < \delta$

(δ independent of locations of x, y!)

注：比较函数连续：$\forall x_0 \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $d(f(x), f(x_0)) < \varepsilon$, whenever $x \in X$, $d_X(x, x_0) < \delta$

5. Fact 9: 紧集上的连续函数一致连续

若 $f: X \to Y$ is continuous & X compact

则 f is uniformly continuous

证明:

Argue by contradiction.

Suppose not.

Then $\exists\, \varepsilon_0 > 0$ s.t. $\forall\, \delta > 0$, $\exists$ bad pair $x_\delta, y_\delta \in X$, s.t. $d_X(x_\delta, y_\delta) < \delta$, but $d_Y(f(x_\delta), f(y_\delta)) \geq \varepsilon_0$

Take $\delta = \frac{1}{n}$, $n = 1, 2, \cdots\cdots$

$$\Rightarrow \begin{array}{l} d_X(x_{\frac{1}{n}}, y_{\frac{1}{n}}) < \frac{1}{n}, \quad \forall\, n \geq 1 \quad (*) \\ d_Y(f(x_{\frac{1}{n}}), f(y_{\frac{1}{n}})) \geq \varepsilon_0, \quad \forall\, n \geq 1 \quad (\#) \end{array}$$

$\because$ X is compact

$\therefore$ $\exists$ subseq $\{x_{\frac{1}{n_k}}\}_{k=1}^{\infty} \to$ some x_∞ as $k \to \infty$

$\exists$ subseq $\{y_{\frac{1}{n_k}}\}_{k=1}^{\infty} \to$ some y_∞ as $k \to \infty$

By $(*)$, $d(x_{\frac{1}{n_k}}, y_{\frac{1}{n_k}}) < \frac{1}{n_k}$

$\therefore$ $d(x_\infty, y_\infty) \leq d(x_\infty, x_{\frac{1}{n_k}}) + d(x_{\frac{1}{n_k}}, y_{\frac{1}{n_k}}) + d(y_{\frac{1}{n_k}}, y_\infty) \to 0$ as $k \to \infty$

$\therefore$ $x_\infty = y_\infty$

$\because$ f continuous

$f(x_{\frac{1}{n_k}}) \to f(x_\infty)$

$f(y_{\frac{1}{n_k}}) \to f(y_\infty)$

$\therefore$ $f(x_\infty) = f(y_\infty)$

$\therefore$ $(\#)$ impossible (contradiction)

6. Fact 10: 连通集的映射仍构成连通集

Let $f: X \to Y$ & $E \subset X$ is connected.

Then $f(E)$ is connected.

证明:

Suppose Not. Then $\exists$ open O_1 & $O_2 \subset Y$, s.t.

- $f(E) = (f(E) \cap O_1) \cup (f(E) \cap O_2)$
- $f(E) \cap O_1 \cap O_2 = \emptyset$ $(*)$
- $f(E) \cap O_1 \neq \emptyset$, $f(E) \cap O_2 \neq \emptyset$

W.T.S. E disconnect.

即 $\exists$ open V_1 & $V_2 \subset X$, s.t.

- $E = (E \cap V_1) \cup (E \cap V_2)$
- $E \cap V_1 \cap V_2 = \emptyset$
- $E \cap V_1 \neq \emptyset$, $E \cap V_2 \neq \emptyset$

Let $V_1 = f^{-1}(O_1)$, $V_2 = f^{-1}(O_2)$

$\because$ f continuous

$\therefore$ V_1 & V_2 open in X

(先证 $E = (E \cap V_1) \cup (E \cap V_2)$)

observe $f(E) \subset O_1 \cup O_2 \Rightarrow E \subset f^{-1}(f(E)) \subset f^{-1}(O_1 \cup O_2) = f^{-1}(O_1) \cup f^{-1}(O_2) = V_1 \cup V_2$

$\therefore E \subset (V_1 \cap E) \cup (V_2 \cap E) \subset E$

$\therefore E = (V_1 \cap E) \cup (V_2 \cap E)$

(再证 $E \cap V_1 \neq \emptyset$, $E \cap V_2 \neq \emptyset$)

$\because f(E) \cap O_1 \neq \emptyset$

$\therefore \exists e \in E$ s.t. $f(e) \in O_1$

$\therefore e \in f^{-1}(O_1) = V_1$

$\therefore e \in E \cap V_1$

$\therefore E \cap V_1 \neq \emptyset$

Similarly, $E \cap V_2 \neq \emptyset$

(再证 $E \cap V_1 \cap V_2 = \emptyset$)

Suppose $E \cap V_1 \cap V_2 \neq \emptyset$, then $\exists e \in E, V_1, V_2$

$\therefore f(e) \in O_1, O_2, f(E)$ (contradict to (*))

$\therefore E \cap V_1 \cap V_2 = \emptyset$

$\therefore$ E disconnected (contradiction)

$\therefore f(E)$ connected

7. Fact 11: Intermediate value theorem (介值定理)

Let $f: [a,b] \to R$ be continuous. Let $m = \min\limits_{[a,b]} f$, $M = \max\limits_{[a,b]} f$.

If $m < M$, then $\forall c \in (m, M)$, $\exists d \in [a,b]$ s.t. $f(d) = c$

证明:

Recall $[a,b]$ connected

By Fact 10, $f([a,b])$ conneted.

Argue by contradiction. If $c \in (m, M)$ s.t. $c \notin f([a,b])$

Suppose $O_1 = (-\infty, c)$, $O_2 = (c, \infty)$, O_1, O_2 open

$\therefore f([a,b]) = (f[a,b] \cap (-\infty, c)) \cup (f[a,b] \cap (c, \infty))$

$= (f[a,b] \cap O_1) \cup (f[a,b] \cap O_2)$

observe: $f([a,b]) \cap O_1 \cap O_2 = \emptyset$

$m \in f([a,b]) \cap O_1 \Rightarrow f([a,b]) \cap O_1 \neq \emptyset$

$M \in f([a,b]) \cap O_2 \Rightarrow f([a,b]) \cap O_2 \neq \emptyset$

$\therefore f([a,b])$ disconnected. (contradiction)