Lecture 11

§1 Newton's Method

1. Why we use Newton's method

Some functions are difficult or impossible to solve analytically. Newton's method (a.k.a Newton-Raphson method) is a numerical method for finding an approximate solution to a function.

E.g. Consider the special case where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad (a_n \neq 0)$$

is a polynomial with degree n; write $\deg(f) = n$.

- How to solve $f(x) = 0$?
- When $n \geq 5$ – general formulae **DO NOT EXIST** (famous result by Galois).

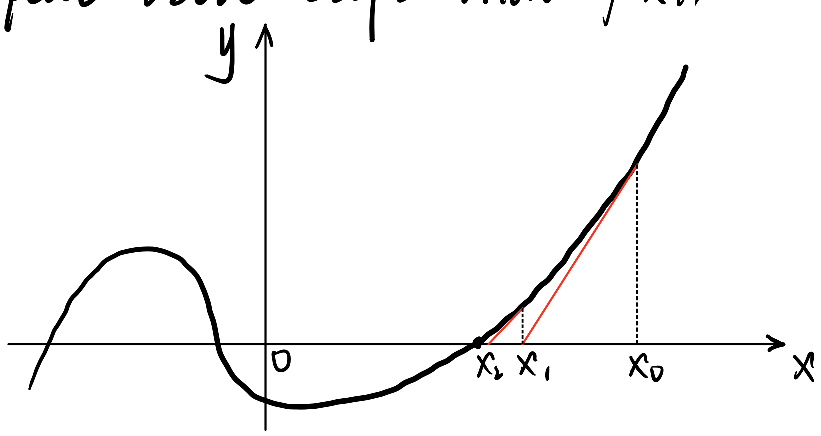
2. General procedure for Newton's method

1° Start with a point x_0 "near" a root.

2° Let L_i be the tangent line to $y=f(x)$ at $x=x_i$

3° If $f'(x_i) \neq 0$, L_i will intersect the x-axis at some point; call this point x_{i+1}.

4° Repeat above steps until $f(x_i) \approx 0$



Note: L_i is given by $y=f(x_i)+f'(x_i)(x-x_i)$, So $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, if $f'(x_i) \neq 0$

e.g. Consider $f(x) = x^3 - 2x - 5$

Since $f(2) = -1 < 0$ and $f(3) > 0$, there exists $r \in (2,3)$ such that $f(r) = 0$

Choose $x_0 = 2$.

$f'(x) = 3x^2 - 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2 - \frac{f(2)}{f'(2)}$$
$$= 2 + \frac{1}{10} = 2.1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.1 - \frac{f(2.1)}{f'(2.1)}$$
$$\approx 2.1 - 0.0054318$$
$$\approx 2.094568$$

$x_3 = 2.094551 \ldots\ldots$

$x_4 = 2.094551 \ldots\ldots$

Remark:

$x_0 = 2$ was an arbitrary choice

If we choose $x_0 = 3$, then

$x_1 = 2.36$

$x_2 = 2.127196 \ldots$

$x_3 = 2.095136 \ldots$

$x_4 = 2.094551 \ldots$

In both case, $x \to r$ as $n \to \infty$ (i.e. x_n converges to r)

Important:

Newton's method doesn't always work

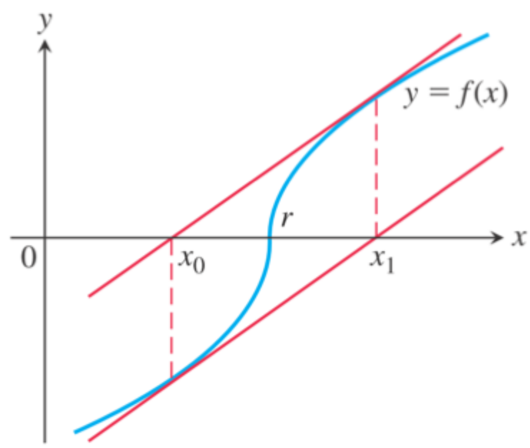
The sequence $x_1, x_2 \cdots\cdots$ may not converge (向一点汇合) to a root r, it may converge to a root we are not interested in (e.g. -ve root) or it or it may not converge at all.

e.g. $f(x) = \begin{cases} \sqrt{x-r} & \text{if } x \geq r \\ -\sqrt{r-x} & \text{if } x < r \end{cases}$

If we pick $x_0 = r-h$, then we find that

$x_1 = r+h \qquad x_2 = r-h \qquad x_3 = r+h$

x_n does not converge to any single number, and so does not converge to r.



One possible condition is:

Suppose f is continuous on $[a,b]$ and differentiable on (a,b) and for $f(r)=0$ where $r \in (a,b)$.

If $f'(r) \neq 0$, then there exists $\delta > 0$ such that, with any starting point, $x_0 \in (r-\delta, r+\delta)$, the sequence x_n converges to r.

§2 Antiderivatives and Indefinite integrals

1. Definition of antiderivative

If $F'(x)=f(x)$ for all x in an interval I,
then F is said to be an antiderivative of f on I.

2. By a corollary of the MVT. we have:

Theorem 4.7.8

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is:

$$F(x)+C$$

where C is an arbitrary constant.

3.

	Function	General antiderivative
1°	x^n	$\frac{1}{n+1}x^{n+1}+C$, $n\neq -1$
2°	$\sin kx$	$-\frac{1}{k}\cos kx+C$
3°	$\cos kx$	$\frac{1}{k}\sin kx+C$
4°	$\sec^2 kx$	$\frac{1}{k}\tan kx+C$

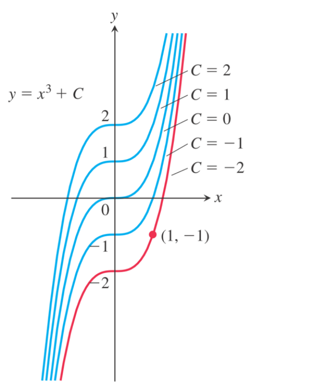


FIGURE 4.51 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

e.g. Find the antiderivative F of $f(x)=3\sqrt{x}+\sin 2x$ that satisfies $F(0)=1$.

$$f(x)=3x^{\frac{1}{2}}+\sin 2x=F'(x)$$

$$F(x)=3\cdot\left(\frac{2}{3}\right)x^{\frac{3}{2}}-\frac{1}{2}\cos 2x+C$$

$$F(0)=1 \Rightarrow 1=-\frac{1}{2}+C \Rightarrow C=\frac{3}{2}$$

$$F(x)=2x^{\frac{3}{2}}-\frac{1}{2}\cos 2x+\frac{3}{2}$$

4. Definition of indefinite integrals

The set of all antiderivative of f is call the indefinite integrals of f.

If F is one antiderivative of f, we write

$$\int f(x)\,dx = F(x) + C$$

5. Other definitions

1° $\int f(x)\,dx$ is called the indefinite intergral of f w.r.t. x.

2° $f(x)$ is called the integrand.

3° dx is called the variable of integration.

6. Remark:

$$\int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

e.g. Solve $\int (x^2+x)\,dx$

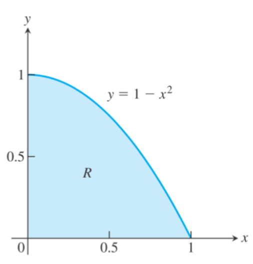
$$\begin{aligned}
&\int (x^2+x)\,dx \\
=\ &\int x^2\,dx + \int x\,dx \\
=\ &\tfrac{1}{3}x^3 + C_1 + \tfrac{1}{2}x^2 + C_2 \\
=\ &\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + C_1 + C_2 \\
=\ &\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + C
\end{aligned}$$

e.g. Solve $\int (x^2-2x+5)\,dx$

$$\begin{aligned}
&\int (x^2-2x+5) \\
=\ &\int x^2\,dx - 2\int x\,dx + 5\int x^0\,dx \\
=\ &\tfrac{1}{3}x^3 - x^2 + 5x + C
\end{aligned}$$

§3 Estimation with Finite Sums

Consider finding the area R under the graph of the function $y = 1 - x^2$, above the x-axis, between the vertical lines $x = 0$ and $x = 1$.



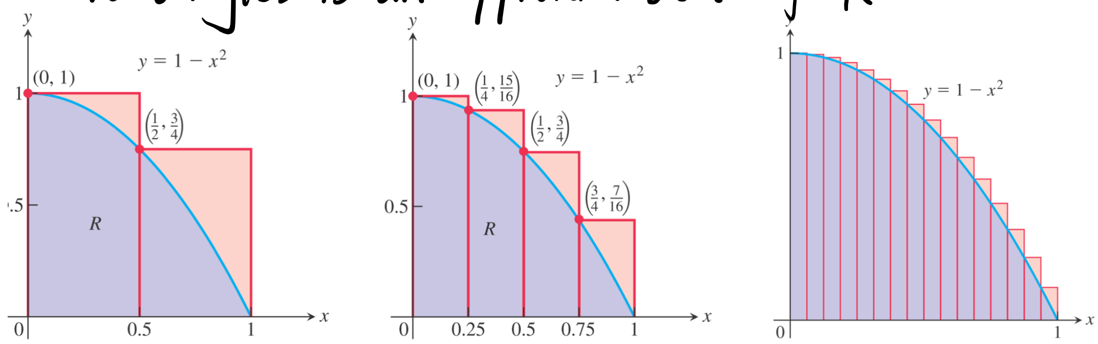
Note that $1 - x^2 \geq 0$ for all $x \in [0, 1]$.

1. Integrals: Approximating area by rectangles

We may approximate R by summing areas of rectangle:

1° Divide $[0,1]$ into subintervals with equal length, and construct rectangles using the function value of the left endpoint

2° The sum (Left endpoint sum) of the areas of these rectangles is an approximation of R



3° We devide $[0,1]$ into n subintervals of length $\Delta x = \frac{1}{n}$, corresponding to points $x_0, x_1, \cdots, x_n$ where

$$x_0 = 0, \quad x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \cdots\cdots, \quad x_n = \frac{n}{n} = 1$$

4° The approximated area is

$$\frac{1}{n}\left(f(c_1) + f(c_2) + \cdots\cdots + f(c_n)\right) = \frac{1}{n}\sum_{i=1}^{n} f(c_i)$$

where each $c_i \in [x_{i-1}, x_i]$ is chosen to be the left endpoint x_{i-1}

5° In this example, the function is decreasing, so $f(c_i) = f(x_{i-1})$ always gives the maximum f-value on $[x_{i-1}, x_i]$. The sum is called an upper sum.

6° One may also approximate by taking midpoints or right endpoints of the subintervals.

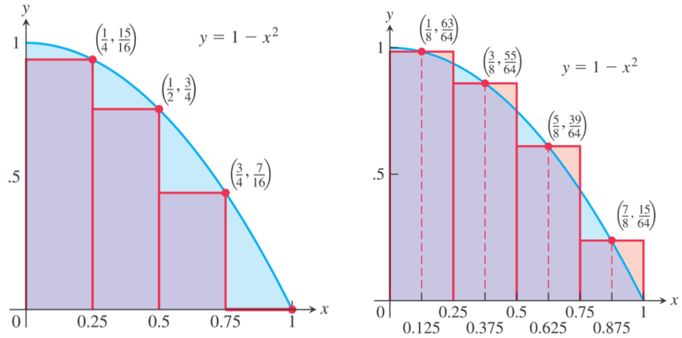


TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

§4. Riemann Sums

1. Definition of partition

A partition of the interval $[a,b]$ is a set

$$P = \{x_0, x_1, x_2, \cdots\cdots, x_{n-1}, x_n\}$$

such that

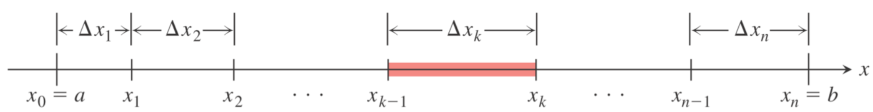
$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

2. Definition of Riemann sum

Given a function $f: [a,b] \to R$ with a partition P or $[a,b]$, a Riemann sum of f (w.r.t. P) is a sum of the form

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + \cdots\cdots + f(c_n)\Delta x_n$$

where $c_k \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$ for each $k \in \{1, \ldots, n\}$



3. Riemann sums

1° There are many Riemann sums for a function.

2° A Riemann sum depends on the partition P and the points c_k chosen from the subintervals.

3° The left-endpoint, midpoint and right-endpoint sums are all special cases of Riemann sum.