Lecture 7

§1 关于级数的 basic facts（接上）

1. **Fact 9**（Root test）

Let $\alpha = \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}}$（也可换成 $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$），

(i) if $\alpha < 1$, then $\Sigma |a_n|$ converges

(ii) if $\alpha > 1$, then $\Sigma |a_n|$ diverges

(iii) if $\alpha = 1$, inconclusive（root test doesn't apply）

证明：

① proof of (i):

（思路：取 $\alpha < \beta < 1$，则 $|a_n|^{\frac{1}{n}} < \beta$，$\forall$ large n，则 $|a_n| < \beta^n$，由 comparison test 可证得）

Take $\beta \in (\alpha, 1)$

$\because \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = \alpha < \beta$

$\therefore \exists N$, s.t. $|a_n|^{\frac{1}{n}} < \beta$, $\forall n \geq N$

$\therefore |a_n| < \beta^n$, $\forall n \geq N$

$\because \Sigma \beta^n$ (geometric series) converges $(0 < \beta < 1)$

$\therefore$ By C.T. $\Sigma |a_n|$ converges.

② proof of (ii)

（思路：可以取 $\{|a_n|^{\frac{1}{n}}\}$ 的一个子序列，其极限大于 1，因此当 $n\to\infty$ 时，存在 $a_n \to \infty \neq 0$）

$\because \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = \alpha$

$\therefore \exists$ subseq $\{|a_{n_k}|^{\frac{1}{n_k}}\}_{k=1}^{\infty}$, s.t. $\lim_{k\to\infty} |a_{n_k}|^{\frac{1}{n_k}} = \alpha$

$\therefore \forall \varepsilon, \exists K$, s.t. $\alpha - \varepsilon < |a_{n_k}|^{\frac{1}{n_k}} < \alpha + \varepsilon$, if $k \geq K$

Take $\varepsilon = \frac{\alpha - 1}{2}$, then $\alpha - \varepsilon = \frac{\alpha + 1}{2} > 1$

$\therefore |a_{n_k}| > (\alpha - \varepsilon)^{n_k}$, $\forall k \geq K$

$\because \lim_{k\to\infty} (\alpha - \varepsilon)^{n_k} = \infty$

$\therefore \lim_{k\to\infty} a_{n_k} = \infty$

$\therefore \lim_{n\to\infty} a_n \neq 0$

$\therefore \Sigma a_n$ diverges

③ proof of (iii)

- $\Sigma \frac{1}{n}$ divergent

  $\alpha = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{e^{\ln n \cdot \frac{1}{n}}} = 1$

- $\Sigma \frac{1}{n^2}$ convergent

  $\alpha = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{2}{n}}} = 1$

2. Fact 10 (Ratio test)

(i) If $\overline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right|<1$, then $\sum|a_n|$ converges

(ii) If $\underline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right|>1$, then $\sum|a_n|$ diverges

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, inconclusive

证明:

① proof of (i)

(思路: 取 $\overline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right|<\alpha<1$, 则 $|a_{N+p}|<\alpha^p|a_N|$, 由比较审敛法证明)

Take $\overline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right|<\alpha<1$, then $\left|\frac{a_{n+1}}{a_n}\right|<\alpha$, $\forall n\geq$ some N

$\therefore |a_{n+1}|<\alpha|a_n|$

$\therefore |a_{N+1}|<\alpha|a_N|$

$|a_{N+2}|<\alpha|a_{N+1}|$

$\cdots\cdots$

$|a_{N+p}|<\alpha|a_{N+p-1}|$

$\therefore |a_{N+p}|<\alpha^p|a_N|$

$\therefore \sum_{p=1}^{\infty}|a_{N+p}|<\sum_{p=1}^{\infty}\alpha^p\cdot|a_N|$

$\therefore \sum_{n=N+1}^{\infty}|a_n|$ converges

$\therefore \sum|a_n|$ converges

② proof of (ii)

(思路: 若 $\underline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right|>1$, 则 n足够大时, $\left|\frac{a_{n+1}}{a_n}\right|>1$, 因此 $|a_n|\uparrow$, 因此$|a|$的极限不为0, 级数发散)

$\because \underline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right|>1$

$\therefore \left|\frac{a_{n+1}}{a_n}\right|>1$, $\forall n\geq$ some N

$\therefore |a_n|$ is increasing

$\therefore \lim_{n\to\infty}|a_n|=l$ exists (may be $+\infty$)

Then $l\neq0$, $\lim_{n\to\infty}a_n\neq0$

$\therefore$ By divergent test, $\sum a_n$ diverges

例1: 判断级数 $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\cdots$ 的敛散性

(先考虑 ratio test)

$$\frac{a_{n+1}}{a_n}=\begin{cases}\dfrac{\frac{1}{3^k}}{\frac{1}{2^k}}=\left(\frac{2}{3}\right)^k, & n=2k-1\\[2ex] \dfrac{\frac{1}{2^{k+1}}}{\frac{1}{3^k}}=\frac{1}{2}\left(\frac{3}{2}\right)^k, & n=2k\end{cases}$$

Then $\overline{\lim_{n\to\infty}}\frac{a_{n+1}}{a_n}=\infty$, $\underline{\lim_{n\to\infty}}\frac{a_{n+1}}{a_n}=0$

(Ratio test fails)

(再考虑 root test)

$$|a_n|^{\frac{1}{n}} = \begin{cases} \frac{1}{2^{\frac{k}{2k-1}}}, & n=2k-1 \\ \frac{1}{3^{\frac{k}{2k}}}, & n=2k \end{cases}$$

$\therefore E = \{x \mid \exists \text{ subseq } |a_{n_k}|^{\frac{1}{n_k}} \to x \text{ as } k\to\infty\}$

$= \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\}$

$\varlimsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \sup E = \frac{1}{\sqrt{2}} = \alpha < 1$

$\therefore \sum |a_n|$ converges

3. Theorem (根值审敛法较比值审敛法适用范围更广)

$$\varliminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \varliminf_{n\to\infty} |a_n|^{\frac{1}{n}} \le \varlimsup_{n\to\infty} |a_n|^{\frac{1}{n}} \le \varlimsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

证明:

(先证明 $\varliminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \varliminf_{n\to\infty} |a_n|^{\frac{1}{n}}$)

Let $\beta = \varliminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \ge 0$

Case 1: $\beta = 0$

then $\varliminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \le \varliminf_{n\to\infty} |a_n|^{\frac{1}{n}}$

Case 2: $\beta = \infty$

Case 3: $0 < \beta < \infty$

take $\gamma \in (0, \beta)$

$\because \varliminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \beta > \gamma$

$\therefore \exists N$, s.t. $\left|\frac{a_{n+1}}{a_n}\right| > \gamma$ if $n \ge N$

$\therefore |a_{N+p}| > \gamma^p |a_N|, \ \forall p \ge 1$

$\therefore |a_k| > \gamma^{k-N} |a_N|, \ \forall k \ge N+1$

$\therefore |a_k|^{\frac{1}{k}} > \gamma^{\frac{k-N}{k}} |a_N|^{\frac{1}{k}}, \ \forall k \ge N+1$

$\therefore \varliminf_{k\to\infty} |a_k|^{\frac{1}{k}} \ge \gamma, \ \forall \gamma \in (0, \beta)$

$\therefore \varliminf_{k\to\infty} |a_k|^{\frac{1}{k}} \ge \beta$

(再证明 $\varlimsup_{n\to\infty} |a_n|^{\frac{1}{n}} \le \varlimsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$)

Let $\alpha = \varlimsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

Case 1: $\alpha = \infty$

then $\varlimsup_{n\to\infty} |a_n|^{\frac{1}{n}} \le \varlimsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

Case 2: $0 \le \alpha < \infty$

take $\delta \in (\alpha, \infty)$

$\because \varlimsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \delta$

$\therefore \exists N$, s.t. $\left|\frac{a_{n+1}}{a_n}\right| < \delta$, if $n \ge N$

$\therefore |a_{N+p}| < \delta^p |a_N|, \ \forall p \ge 1$

$\therefore |a_k| < \delta^{k-N} |a_N|, \ \forall k \ge N+1$

$\therefore \quad |a_k|^{\frac{1}{k}} < \delta^{\frac{k-N}{k}} |a_N|^{\frac{1}{k}}, \ \forall k \geq N+1$

$\therefore \quad \overline{\lim_{n\to\infty}} |a_k|^{\frac{1}{k}} \leq \delta, \ \forall \delta > \alpha$

$\therefore \quad \overline{\lim_{n\to\infty}} |a_k|^{\frac{1}{k}} \leq \alpha$

Q.E.D.

Corollary:

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists (may be $+\infty$), then $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ also exists and $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$