Lecture 7 (2021.9.30)

# §1 Linearization / Linear approximation

## 1. Definition of Linearization

If $f$ is differentiable at $x=a$, then the approximating function

$$L_a(x)/L(x) = f(a) + f'(a)(x-a)$$

is the linearization of $f$ at $a$. The approximation

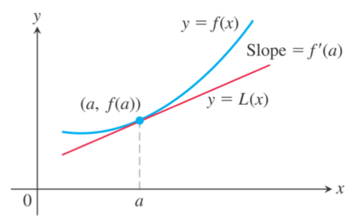
$$f(x) \approx L(x)$$

of $f$ by $L$ is the standard linear approximation of $f$ at $a$. The point $x=a$ is the center of the approximation.

If $f$ is differentiable at $a$, for $x$ "near" $a$,

$$\frac{f(x)-f(a)}{x-a} \approx f'(a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$




e.g. Approximate the numerical value of $\sqrt[3]{27.4}$ by considering linearization of $f(x) = \sqrt[3]{x}$ centered at $x=27$

$$f(x) = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$$

$$f'(27) = \frac{1}{3\times 9} = \frac{1}{27}$$

$$L_{27}(x) = f(27) + f'(27)(x-27)$$

$$= 3 + \frac{1}{27}(x-27)$$

$$= \frac{1}{27}x + 2$$

$$\sqrt[3]{27.4} \approx \frac{1}{27}\times 27.4 + 2 = 3.0148148\cdots \text{ (error} < 0.0001)$$

## 2. Approximate $(1+\varepsilon)^k$ for small $\varepsilon$

Set $f(x) = x^k$. Consider $L(x)$ with center $x=1$

$$f'(x) = kx^{k-1} \longrightarrow f'(1) = k$$

$$L(x) = L_1(x) = f(1) + f'(1)(x-1)$$

$$L_1(x) = 1 + k(x-1)$$

For $x = 1+\varepsilon$:

$$(1+\varepsilon)^k = f(1+\varepsilon) \approx L(1+\varepsilon) = 1 + k\varepsilon$$

e.g. $\sqrt{1.01} = 1.01^{\frac{1}{2}} \approx 1 + 0.01 \times \frac{1}{2} = 1.005$

e.g. $\sqrt[3]{1+5x^4} = (1+5x^4)^{\frac{1}{3}} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4$

# §2 Differentials

## 1. Definition

Let $y = f(x)$ be a differentiable function. The differential $dx$ is an indepedent variable. The differential $dy$ is

$$dy = f'(x)\,dx$$

1° $dx$ and $dy$ are both called differentials.

2° $dx$ is independent.

3° $dy$ depends on both $x$ and $dx$

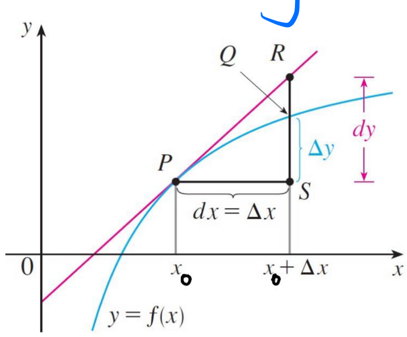
4° For $dx \neq 0$, we have $f'(x) = \frac{dy}{dx}$, compatible with previous notation.

e.g. For $y = f(x) = x^5 + 37x$,

when $x = x_0 = 1$ and $dx = 0.2$

$$dy = f'(x)\,dx = (5x^4 + 37)\big|_{x=1}(0.2) = 8.4$$

## 2. Geometric meaning



In this picture:

1° $\Delta x$ is an independent variable

2° $\Delta y = f(x_0+\Delta x) - f(x_0)$, dependent on $x_0$ and $x$.

3° $dx = \Delta x$

4° $dy = f'(x_0)dx$, dependent on $x_0$ and $dx$.

### 3. The relationship

Consider the linearization $L$ of $f$ centered at $x = x_0$. Then

$$L(x_0+dx) = f(x_0) + f'(x_0)(x_0+dx-x_0)$$

$$L(x_0+dx) = f(x_0) + f'(x_0)\,dx$$

$$dy = f'(x_0)dx$$

$$\Rightarrow \; dy = L(x_0+dx) - f(x_0)$$

Hence, standard linear approximation of $f(x_0+dx)$ centred at $x=x_0$ becomes $f(x_0+dx) \approx L(x_0+dx) = f(x) + dy$

### 4. Notation

If $y = f(x)$, can write $df$ to mean $dy$

## §3 Errors of approximation

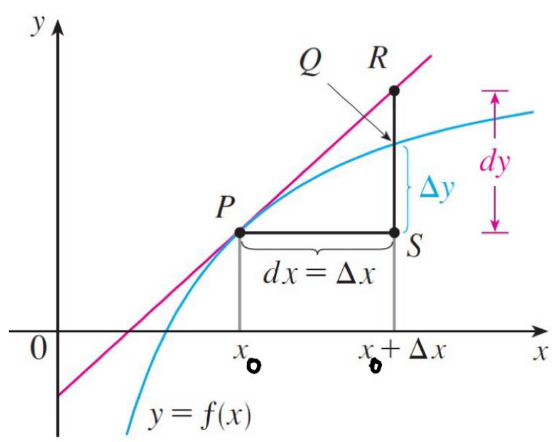
### 1. What's $f(x_0+\Delta x) - L(x_0+\Delta x)$

$$f(x_0+\Delta x) - L(x_0+\Delta x)$$

$$= f(x_0+\Delta x) - f(x_0) - f'(x_0)\cdot\Delta x$$

$$= \left(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - f'(x_0)\right)\cdot\Delta x$$

$$= \varepsilon\,\Delta x$$




### 2. About $\varepsilon$

Note that since

$$\varepsilon = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0)$$

then

$$\lim_{\Delta x \to 0} \varepsilon = f'(x_0) - f'(x_0) = 0$$

## 3. About error

$$\text{Error} = f(x_0+\Delta x) - L(x_0+\Delta x) = \varepsilon \Delta x$$

Since $\dfrac{f(x_0+\Delta x) - L(x_0+\Delta x)}{\Delta x} = \varepsilon$,

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - L(x_0+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \varepsilon = 0$$

So, error decreases faster than $\Delta x$ as $\Delta x \to 0$

Another way to write it is:

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x = dy + \varepsilon \Delta x$$

for some $\varepsilon$ with $\varepsilon \to 0$ as $\Delta x \to 0$.

($f$ is differentiable, $f$ changes from $x_0$ to $x_0 + \Delta x$)

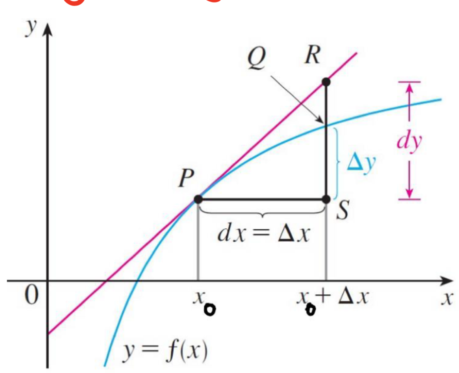
## 4. The above follows from:

1° $f(x_0+\Delta x) = f(x) + \Delta y$

2° $L(x_0+\Delta x) = f(x_0) + dy = f(x_0) + f'(x_0)\Delta x$

3° $f(x_0+\Delta x) = L(x_0+\Delta x) + \varepsilon \Delta x$

4° $\Delta y = dy + \varepsilon \Delta x$



## §4 Formal proof of the chain rule

Want to show $(g\circ f)'(x) = g'(f(x))\cdot f'(x_0)$

Let $y=f(x)$, $z=g(y)=g(f(x))$

Consider a change of $x$-value by $\Delta x$

Let $\Delta y$ and $\Delta z$ be corresponding changes in $y$- and $z$- values, respectively.

Then $\Delta y = f'(x_0)\,\Delta x + \varepsilon_1 \Delta x$ for some $\varepsilon_1$ with $\lim\limits_{\Delta x\to 0}\varepsilon_1 = 0$

Also $\Delta z = g'(y_0)\,\Delta y + \varepsilon_2 \Delta y$ for some $\varepsilon_2$ with $\lim\limits_{\Delta y\to 0}\varepsilon_2 = 0$

So, $\dfrac{\Delta y}{\Delta x} = f'(x_0)+\varepsilon_1$ $\qquad$ $\dfrac{\Delta z}{\Delta y} = g'(y_0)+\varepsilon_2$

Now $\dfrac{\Delta z}{\Delta x} = (g'(y_0)+\varepsilon_2)(f'(x_0)+\varepsilon_1)$

where $\varepsilon_1\to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$

Therefore,

$$
\begin{aligned}
(g\circ f)'(x) &= \left.\frac{dz}{dx}\right|_{x=x_0} = \lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}\\
&= (g'(y_0)+0)(f'(x_0)+0)\\
&= g'(f(x_0))\cdot f'(x_0)
\end{aligned}
$$

## §5 Sensitivity in change

The equation $df = f'(x_0)\,dx$ measures how sensitive the output $f$ is to a small change in $x$-value.

e.g. An object falls such that its position from the starting point is given by $S = 4.9t^2$.

Time measurement may have an error of $\pm 0.01\,s$

Error in distance may then be:

$$ds = S'(t)\cdot dt = 9.8t\cdot dt = \pm 0.098t$$

## §6 Extreme values of functions

### 1. Definitions of absolute extreme

Let $f$ be a function with domain $D$. Then $f$ has an absolute maximum value on $D$ at a point $c$ if

$$f(x) \le f(c) \quad \text{for all } x \text{ in } D$$

and an absolute minimum value on $D$ at a point $c$ if

$$f(x) \ge f(c) \quad \text{for all } x \text{ in } D$$

* "Absolute" maximum/minimum = "Global" maximum/minimum

e.g. $f: D \to \mathbb{R}$, $f(x) = x^2$

① $D = \mathbb{R}$: no absolute max

absolute min at $x=0$, $f(0)=0$

② $D = (0, 3]$: no absolute min

absolute max at $x=3$, $f(3)=9$

③ $D = [0, 3]$: absolute min at $x=0$, $f(0)=0$

absolute max at $x=3$, $f(3)=9$

### 2. The extreme value theorem

If $f$ is continuous on a closed interval $[a,b]$, then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a,b]$. That is, there are numbers $x_1$ and $x_2$ in $[a,b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other $x$ in $[a,b]$.

Fails in general if $[a,b]$ is replaced with other intervals such as open or half-open intervals.

Fails in general if $f$ is not continuous.

### 3. Definitions of local extreme

A function $f$ has a local maximum value at a point $c$ within its domain $D$ if $f(x) \le f(c)$ for all $x \in D$ lying in some open interval containing $c$.

A function $f$ has a local minimum value at a point $c$ within its domain $D$ if $f(x) \ge f(c)$ for all $x \in D$ lying in some open interval containing $c$.

e.i for $D \to R$.

the function $f$ has a local maximum at $c$ if there exists $a > 0$ such that

$$f(x) \le f(c) \text{ for all } x \in (c-a, c+a) \cap D$$

the function $f$ has a local minimum at $c$ if there exists $a > 0$ such that

$$f(x) \ge f(c) \text{ for all } x \in (c-a, c+a) \cap D$$

## 4. Terminology

Maxima = plural of maximum

Minima = plural of minimum

Extremum = a maximum or minimum

Extrema = plural of extremum

## 5. Definition of critical point

Let $f: D \to R$, and let $c$ be an interior point of $D$

Then $c$ is a critical point of $f$ if:

(i) $f'(c) = 0$ ; or

(ii) $f'(c)$ doesn't exist (in $R$)

e.g. Find the critical points of $f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

$x=0$ and $x \geq 1$

## 6. How can we find local extrema

1° The first derivative theorem for local extreme values.

If $f$ has a local maximum or minimum value at an interior point $c$ of its domain, and if $f'$ is defined at $c$, then

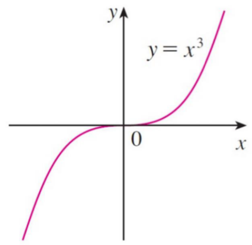
$$f'(c)=0$$

2° Alternative phrasing

Let $c$ be an interior point of $D$. If a function $f: D \to R$ has a local extremum at $c$, the $c$ is a critical point of $f$.

3° The converse of the theorem is not true

Counterexample:




## 7. How to find the absolute extrema of a Continuous Function $f$ on a finite closed interval

1° Evaluate $f$ at all critical points and endpoints.

2° Take the largest and smallest of these values.