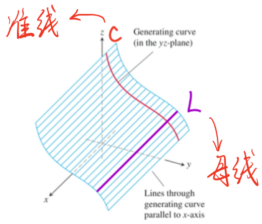
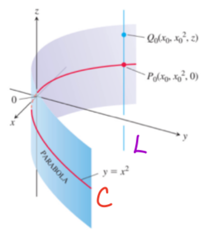
Lecture 11

§1 Cylinder and quadric surfaces

1. Definition of cylinder (柱面)

> Def: Let C be a plane curve lying in $\mathbb{R}^3$. If L is a line that passes through C and is not parallel to the plane in which C lies, and S is the union of all lines which are parallel to L and which pass through C, then S is called a cylinder or a cylindrical surface.





2. Definition of quadric surface (二次曲面)

> Def: A quadric surface is a the set of all points (x, y, z) in $\mathbb{R}^3$ that satisfy an equation $f(x,y,z)=0$, where $f(x,y,z)$ is a degree-two polynomial in x, y and z.

3. xy (xz/yz)-trace (截痕)

二面曲面 S 的 xy-trace 是 S 与平面 $z=z_0$ 的交线

xz/yz-trace 类似

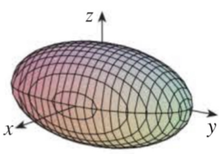
4. 六种基本二次曲面

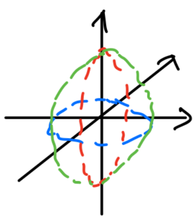
1° ellipsoid (椭球面)

① 方程: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

② 所有的 (nonempty) trace 均为 ellipses (椭圆)

③ 若 $a=b=c$, 这是一个 sphere



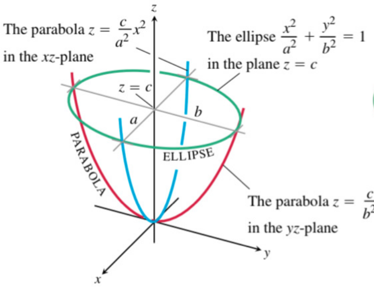


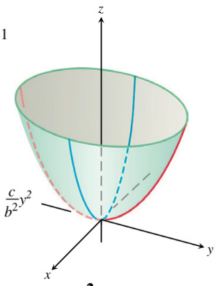
2° elliptic paraboloid (椭圆抛物面)

① 方程: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$

② 所有的 xy-traces 均为 ellipses

③ 所有的 xz与yz-traces 均为 parabolas（抛物线）





3° hyperbolic paraboloid（双曲抛物面）

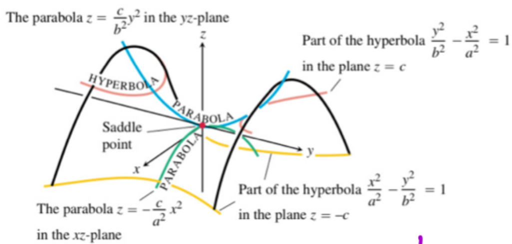
① 方程：$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$

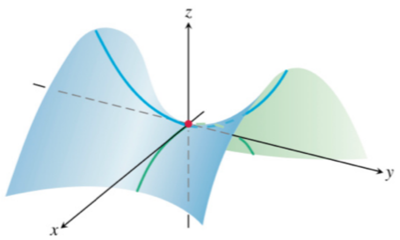
② 所有的 xy-traces 均为 hyperbolas（双曲线）

（除了平面 $z=0$：为两条线 $y = \pm\frac{b}{a}x$）

平面 $z=0$ 上下方的双曲线渐近线相同，实轴不同

③ 所有的 xz与yz-traces 为 parabolas





here $c>0$.

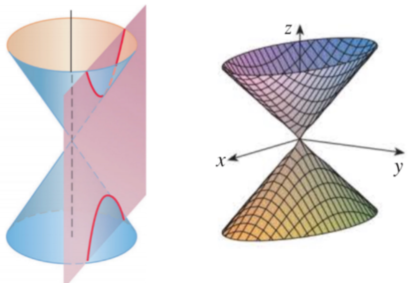
4° (elliptic) cone（椭圆锥面）

① 方程：$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

② 所有的 xy-traces 均为 ellipses

③ 所有的 xz与yz-traces 均为 hyperbolas

（除了平面 $x=0$ 与 $y=0$：为一对渐近线）

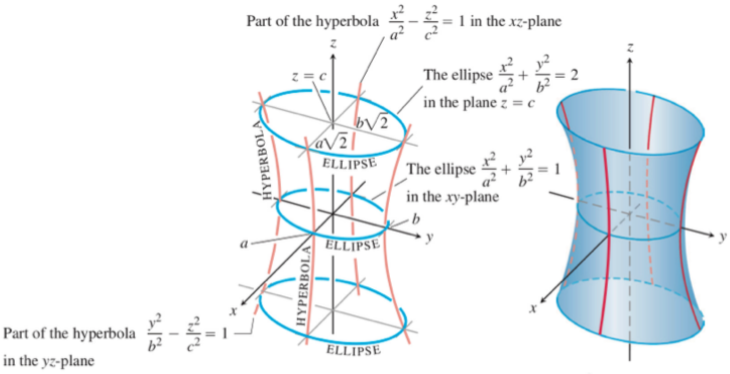


5° hyperboloid of one sheet（单叶双曲面）

① 方程：$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

② 所有的 xy-traces 均为 ellipses

③ 所有的 xz与yz-traces 均为 hyperbolas

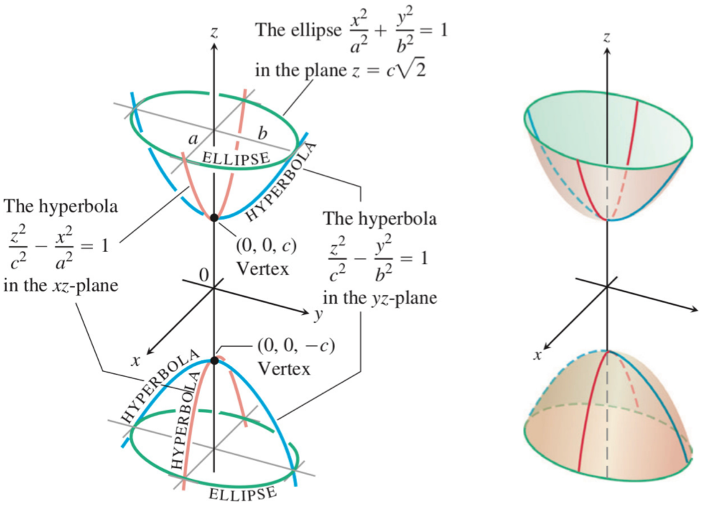


6° hyperboloid of two sheets (双叶双曲面)

① 方程: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

② 所有的 xy-traces 均为 ellipses

③ 所有的 xz 与 yz-traces 均为 hyperbolas



§2 Vector-valued functions (on intervals)

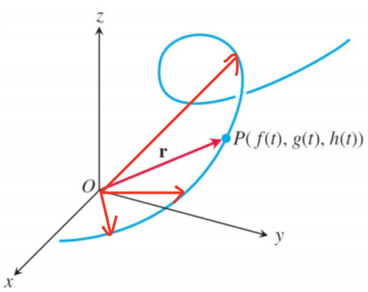
1. 定义

Definition

A vector-valued function or a vector function (on I) is a function $\vec{r}: I \to \mathbb{R}^n$, where I is an interval and $\mathbb{R}^n$ is viewed as a set of vectors.

注: 对于一个 vector function $\vec{r}: I \to R^n$, 我们有 $\vec{r}(t) = \langle r_1(t), r_2(t), \cdots\cdots, r_n(t)\rangle$

每个 r_i 均为一个由 I 到 R 的函数.



e.g. $\vec{r}(t) := \langle \cos(t), \sin(t), t \rangle, t \in \mathbb{R}$. 被称为 helix (螺旋线)

2. 极限的定义

Definition

Let $\vec{r}: I \to \mathbb{R}^n$ be a vector function. We say that $\vec{r}$ has a limit $\vec{L}$ as t approaches t_0, and write

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L},$$

if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $t \in I$, $0 < |t - t_0| < \delta$ implies $|\vec{r}(t) - \vec{L}| < \epsilon$.

3. Theorem: "Component-wise Limits"

Theorem ("Component-Wise Limits")

A vector function $\vec{r} = \langle r_1, r_2, \ldots r_n \rangle$ *satisfies*

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L} = \langle L_1, L_2, \ldots, L_n \rangle$$

if and only if $\lim_{t \to t_0} r_i(t) = L_i$ *for all* i.

e.g. If $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

$\lim_{t \to \frac{\pi}{4}} \vec{r}(t) = \langle \lim_{t \to \frac{\pi}{4}} \cos t, \lim_{t \to \frac{\pi}{4}} \sin t, \lim_{t \to \frac{\pi}{4}} t \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \rangle$

证明: $\lim_{x \to c} \vec{r}(x) = \vec{L} \Rightarrow \lim_{x \to c} r_i(x) = L_i, \forall i = 1, 2, \cdots, n$

- Let $\varepsilon > 0$. since $\lim_{x \to c} \vec{r}(x) = \vec{L}$,

  $\exists \delta > 0$, s.t. $\forall x \in (c-\delta, c+\delta) \setminus \{c\}$, $|\vec{r}(x) - \vec{L}| < \varepsilon$

  $\sqrt{(r_1(x)-L_1)^2 + (r_2(x)-L_2)^2 + \cdots + (r_n(x)-L_n)^2} < \varepsilon$

- Fix i, since $|r_i(x) - L_i| = \sqrt{(r_i(x)-L_i)^2} \leq \sqrt{(r_1(x)-L_1)^2 + (r_2(x)-L_2)^2 + \cdots + (r_n(x)-L_n)^2} < \varepsilon$

  so $\lim_{x \to c} r_i(x) = L_i$, $\forall i = 1, 2, \cdots, n$

证明: $\lim_{x \to c} r_i(x) = L_i, \forall i = 1, 2, \cdots, n \Rightarrow \lim_{x \to c} \vec{r}(x) = \vec{L}$

- Let $\varepsilon > 0$. since $\forall i = 1, 2, \cdots, n \Rightarrow \lim_{x \to c} \vec{r}(x) = \vec{L}$

  $\exists \delta > 0$, s.t. $\forall x \in (c-\delta_i, c+\delta_i) \setminus \{c\}$, $|r_i(x) - L_i| < \frac{\varepsilon}{\sqrt{n}}$

- Let $\delta := \min\{\delta_1, \delta_2, \cdots, \delta_n\}$

  Suppose $x \in (c-\delta, c+\delta) \setminus \{c\}$

  Then $x \in (c-\delta_i, c+\delta_i) \setminus \{c\}$ for $\forall i$. Now

  $|\vec{r}(x) - \vec{L}| = \sqrt{|r_1(x)-L_1|^2 + |r_2(x)-L_2|^2 + \cdots + |r_n(x)-L_n|^2}$

  $< \sqrt{\frac{\varepsilon^2}{n} \cdot n}$

  $= \varepsilon$

  so $\lim_{x \to c} \vec{r}(x) = \vec{L}$

§3 Continuity

1. 定义

Definition
A vector function $\vec{r}: I \to \mathbb{R}^n$ is said to be continuous at $t_0 \in I$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0),$$

and it is called a continuous function if it is continuous at all points in I.
and one-sided continuous at endpoints

2. Theorem

由 componentwise limits，有

$\vec{r} = \langle r_1, r_2, \cdots\cdots, r_n \rangle$ is continuous at $t_0 \iff r_i$ is continuous at t_0, $\forall i$

Example
The function $\vec{r}(t) := \langle \cos(t), \sin(t), \lfloor t \rfloor \rangle$ is discontinuous at every $t \in \mathbb{Z}$, since so is the third component.

§4 Derivatives

1. 定义

Definition Let $\vec{r}: I \to \mathbb{R}^n$ be a function, and let t be an interior point of I.

- ~~A vector~~ The function $\vec{r}$ is said to be differentiable at t if the limit

$$\vec{r}\,'(t) := \frac{d\vec{r}}{dt} := \lim_{h \to 0} \frac{\overbrace{\vec{r}(t+h) - \vec{r}(t)}^{\Delta \vec{r}}}{\underbrace{h}_{\Delta t}}$$

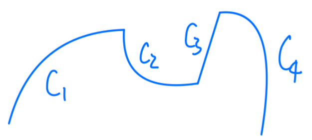
  exists.
- The vector $\vec{r}\,'(t)$ is called the derivative of $\vec{r}$ at t.
- If $\vec{r}\,'(t) \neq \vec{0}$, $\vec{r}\,'(t)$ is also called the tangent vector to the curve (given by $\vec{r}$) at the point $P = P(t)$, where $\overrightarrow{OP} = \vec{r}(t)$.
- The corresponding unit tangent vector is

$$\vec{T}(t) := \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

- The tangent line to the curve at a point $P(t)$ is the line through $P(t)$ in the direction of $\vec{r}\,'(t)$.

- $\vec{r}$ is said to be smooth if $\vec{r}\,'$ is continuous and never $\vec{0}$ on I.
- A curve in $\mathbb{R}^n$ is said to be smooth if it admits a smooth parametrization. $\vec{r}$

e.g.



← Not smooth, but piecewise smooth

← Each C_i is smooth

2. 物理意义

若 $\vec{r}(t)$ 为曲线上一点在 t 时刻 position vector，则

1° $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ 为 t 时刻的 direction of motion

2° $\vec{r}'(t)$ 为 t 时刻的 velocity vector

3° $|\vec{r}'(t)|$ 为 t 时刻的 speed

4° $\frac{d\vec{r}'}{dt} = \frac{d^2\vec{r}}{dt}$ 为 t 时刻的 acceleration

3. Theorem

The derivative of a vector function can be computed by taking derivatives component-wise.

> **Theorem**
> *If $\vec{r}(t) := \langle r_1(t), r_2(t), \dots, r_n(t)\rangle$ is a vector function, where each r_i is differentiable, then*
>
> $$\vec{r}'(t) = \langle r_1'(t), r_2'(t), \dots, r_n'(t)\rangle.$$

例：A curve is traced out by $\vec{r}(t) := \langle 1+t^3, te^{-t}, \sin(2t)\rangle$.
Find direction of motion and acceleration of the particle at the point $(1,0,0)$

Sol: $\vec{r}'(t) = \langle 3t^2, -te^{-t}+e^{-t}, 2\cos(2t)\rangle$

$\vec{r}(t) = \langle 1,0,0\rangle \Rightarrow t=0$

$\vec{T}(0) = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{1}{\sqrt{5}}\langle 0,1,2\rangle$

$\frac{d\vec{r}'(t)}{dt} = \langle 6t, te^{-t}-e^{-t}-e^{-t}, -4\sin(2t)\rangle$

$= \langle 6t, (t-2)e^{-t}, -4\sin(2t)\rangle$

$\frac{d\vec{r}'(t)}{dt}\Big|_{t=0} = \langle 0,-2,0\rangle$

4. Differentiation rules

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:*	$\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:*	$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
	$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:*	$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:*	$\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:*	$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:*	$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:*	$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Scalar function / real-valued function

证明 6：

Fact: $\lim_{t\to t_0}(\vec{a}(t)\times\vec{b}(t)) = [\lim_{t\to t_0}\vec{a}(t)]\times[\lim_{t\to t_0}\vec{b}(t)]$

利用 Fact：

$$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \lim_{h\to 0}\frac{\vec{u}(t+h)\times\vec{v}(t+h)-\vec{u}(t)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{u}(t+h)\times\vec{v}(t+h)-\vec{u}(t+h)\times\vec{v}(t)+\vec{u}(t+h)\times\vec{v}(t)-\vec{u}(t)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{u}(t+h)\times(\vec{v}(t+h)-\vec{v}(t))}{h} + \lim_{h\to 0}\frac{(\vec{u}(t+h)-\vec{u}(t))\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\vec{u}(t+h)\times\lim_{h\to 0}\frac{(\vec{v}(t+h)-\vec{v}(t))}{h} + \lim_{h\to 0}\frac{(\vec{u}(t+h)-\vec{u}(t))}{h}\times\lim_{h\to 0}\vec{v}(t)$$

$$= \vec{u}(t)\times\vec{v}'(t)+\vec{u}'(t)\times v(t)$$

5. 性质: vector functions of constant length

Suppose that a curve given by a differentiable function $\vec{r}$ lies on a sphere centered at the origin. Then at any t, the tangent vector $\vec{r}'(t)$ is perpendicular to the position vector $\vec{r}(t)$. This can be shown by algebraically proving the following statement: if $\vec{r}$ is a differentiable vector function of constant length, then for all t,

$$\vec{r}(t)\cdot\vec{r}'(t) = 0.$$

证明:

$$|\vec{r}(t)| = a$$
$$\Rightarrow \vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = a^2$$
$$\Rightarrow \vec{r}'(t)\cdot r(t) + r(t)\cdot r'(t) = 0$$
$$\Rightarrow \vec{r}'(t)\cdot r(t) = 0$$

* 该性质对于任意 $\vec{r}: I\to R^n$, $|\vec{r}(t)|=K$ 均成立

§5 Integrals

1. Definition

Definition

Let $\vec{r}: I\to\mathbb{R}^n$ be a vector function, with $\vec{r}=\langle r_1, r_2,\ldots, r_n\rangle$.

- A function $\vec{R}$ is called an antiderivative of $\vec{r}$ if $\vec{R}'(t)=\vec{r}(t)$ for all $t\in I$.
- The indefinite integral of $\vec{r}$, denoted by $\int\vec{r}(t)\,dt$, is defined to be the set of all antiderivatives of $\vec{r}$. We write

$$\int\vec{r}(t)\,dt = \vec{R}(t)+\vec{C},$$

where $\vec{C}$ represents any constant vector.

- The definite integral, denoted by $\int_a^b\vec{r}(t)\,dt$, is defined by

$$\int_a^b\vec{r}(t)\,dt := \left\langle\int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt,\ldots,\int_a^b r_n(t)\,dt\right\rangle.$$

2. FTC

If $\vec{R}$ is an antiderivative of $\vec{r}$ on $[a,b]$ and $\vec{r}$ is continuous, then

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$$

e.g. 若$\vec{r}(t)$为t时刻的位置向量，则

$\vec{r}\,'(t) = \vec{v}(t)$ (velocity)

$\vec{v}\,'(t) = \vec{a}(t)$ (acceleration)

$\int_a^b \vec{a}(t)\,dt = \vec{v}(b) - \vec{v}(a)$

$\int_a^b \vec{v}(t)\,dt = \vec{r}(b) - \vec{r}(a)$ (displacement)