

Lecture 21

§1 Implicit function theorem

Purpose: Give a system $f(x,y)=0 \Rightarrow \begin{cases} f_1(x_1, \dots, x_m, y_1, \dots, y_n)=0 \\ f_2(x_1, \dots, x_m, y_1, \dots, y_n)=0 \\ \vdots \\ f_n(x_1, \dots, x_m, y_1, \dots, y_n)=0 \end{cases}$

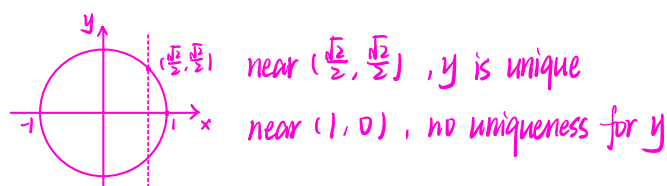
Want to solve (y_1, \dots, y_n) in terms of (x_1, \dots, x_m) from equations.

e.g.1 $x^2 + y^2 + 1 = 0$: Not possible to solve for y in terms of x

Moral of the story: need at least one point (a,b) s.t. $f(a,b)=0$

e.g.2 $x^2 + y^2 - 1 = 0$: $y^2 = 1 - x^2 \Rightarrow y = \pm \sqrt{1-x^2}$, $-1 \leq x \leq 1$

(lack of uniqueness. In order to determine y uniquely, need to be told near where we are supposed to solve for y)



e.g.3 $y + \sin y \ln x + x e^x - e - 1 = 0$. Find $\frac{dy}{dx} \Big|_{x=1, y=1}$

$$\frac{d}{dx}(y + \sin y \ln x + x e^x - e - 1) = 0$$

$$\Rightarrow \frac{dy}{dx} + \cos y \frac{dy}{dx} \ln x + \sin y \frac{1}{x} + (x+1)e^x = 0$$

More generally, $f(x,y)=0$. Assume $\exists (a,b) \in \mathbb{R}^2$ s.t. $f(a,b)=0$, want to find $\frac{dy}{dx} \Big|_{x=a, y=b}$

$$\frac{d}{dx} f(x,y) (= \frac{dz}{dx}) = 0 \xrightarrow{\text{chain rule}} \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx} = f_x(x,y) + f_y(x,y) \frac{dy}{dx} = 0$$

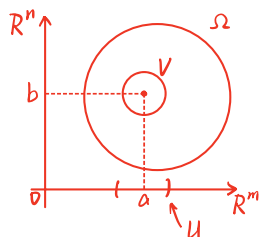
$$\Rightarrow \frac{dy}{dx} \Big|_{x=a, y=b} = \frac{-f_x(x,y)}{f_y(x,y)} \Big|_{x=a, y=b} \text{ provided } f_y(x,y) \neq 0$$

1. Theorem: Implicit function theorem

Let Ω be open in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, let $f: \Omega \subset \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ be C^1 -smooth.

- Write a generic point in \mathbb{R}^{m+n} as (x,y) , $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$
- Write f as $f(x,y)$, suppose $\exists (a,b) \in \Omega$ s.t. ① $f(a,b) = \vec{0}$

② $(D_y f(x,y) \Big|_{x=a, y=b})_{n \times n}$ is nonsingular



Then \exists open set U in \mathbb{R}^m , V in \mathbb{R}^{m+n} , s.t.

- $a \in U$, $(a,b) \in V$
- $\forall x \in U$, $\exists ! y \in \mathbb{R}^n$ s.t. $(x,y) \in V$, and $f(x,y) = \vec{0}$

Let $y=g(x)$, $x \in U$. $g(x)$ is called the function implicitly defined by $f(x,y)=0$

$g: U \rightarrow \mathbb{R}^n$ satisfies

- $g(a) = b$
- g is C^1 -smooth
- $f(x, g(x)) = 0, \forall x \in U$
- $(Dg(x))_{n \times m} = -(D_y f(x, y))_{n \times n}^{-1} (D_x f(x, y))_{n \times m}$

证明:

Going to use Inverse Function Theorem

Let $F(x, y) = \begin{bmatrix} x \\ f(x, y) \end{bmatrix}_{(m+n) \times 1}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\Rightarrow F: \Omega \in \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ is C^1 -smooth (by f is C^1 -smooth)

• $F(a, b) = \begin{bmatrix} a \\ 0 \end{bmatrix}$

$DF(a, b) = \begin{bmatrix} I_{m \times m} & 0_{m \times n} \\ D_x f(a, b)_{n \times m} & D_y f(a, b)_{n \times n} \end{bmatrix}_{(m+n) \times (m+n)}$

(Q: $DF(a, b)$ non-singular?)

$\therefore \det DF(a, b) = \det I_{m \times m} \cdot \det D_y f(a, b)_{n \times n} \neq 0$

$\therefore DF(a, b)$ non-singular

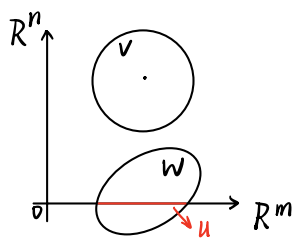
By Inverse function theorem,

\exists open sets of \mathbb{R}^{m+n} : $(a, b) \in V$, $\begin{pmatrix} a \\ 0 \end{pmatrix} \in W$ s.t.

$F: V \rightarrow W$ is 1-1 and onto. Let $G = (F|_V)^{-1}$.

Then G is C^1 -smooth on V

Let $U = \{x \in \mathbb{R}^m \mid (x, 0) \in W\} \Rightarrow a \in U$



Claim: U is open in \mathbb{R}^m

$\forall x_0 \in U$

$\therefore (x_0, 0) \in W$ and W open

$\therefore \text{nbhd } N_r(x_0, 0) \subset W$

$\Rightarrow \text{nbhd } N_r(x_0) \text{ (nbhd of } x_0 \text{ in } \mathbb{R}^m) \times \{0\} \subset W$

$\Rightarrow N_r(x_0) \subset U$

$\Rightarrow U$ is open in \mathbb{R}^m

Now $\forall x \in U$, $(x, 0) \in W \Rightarrow \exists ! (\tilde{x}, y) \in V$ s.t. $F(\tilde{x}, y) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$$\therefore F(\tilde{x}, y) = \begin{bmatrix} \tilde{x} \\ f(\tilde{x}, y) \end{bmatrix}$$

$$\therefore \begin{bmatrix} \tilde{x} \\ f(\tilde{x}, y) \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$\therefore \tilde{x} = x, f(x, y) = 0$$

Denote this y by $g(x)$, $x \in U$

$$\text{Then } f(a, b) = 0 \Rightarrow \begin{cases} f(x, g(x)) = 0 \\ g(a) = b \end{cases}$$

(Q: g C^1 -smooth on U ?)

$$\text{observe } F(x, g(x)) = \begin{bmatrix} x \\ f(x, g(x)) \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$\Rightarrow [g(x)] = F^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$\therefore F^{-1}$ is C^1 -smooth, $\forall x \in U$

$\therefore g$ is C^1 -smooth on U

(Q: $Dg(x) = ?$)

Observe $f(x, y) = 0$, $y = g(x)$, $x \in U$

$$\Rightarrow D_x(f(x, g(x))) \neq (D_x f)(x) = (D_x f)(x, g(x)) + D_y f(x, g(x))_{n \times n} \cdot (D_x g(x))_{n \times m}$$

$$\Rightarrow D_y f(x, g(x)) D_x g(x) = -D_x f(x, g(x))$$

$$\Rightarrow Dg(x) = -(D_y f(x, g(x)))^{-1} D_x f(x, g(x))$$

Q.E.D.

例1: $\begin{cases} e^{xu} \cos(yv) = u + \frac{\sqrt{2}}{2} \\ e^{xu} \sin(yv) = \frac{2\sqrt{2}v}{\pi} \end{cases}$ solve for (u, v) in terms of x & y near $(x, y, u, v) = (\underbrace{1}_a, \underbrace{1}_b, 0, \frac{\pi}{4})$

$$\text{Let } f(x, y, u, v) = \begin{bmatrix} e^{xu} \cos(yv) - u - \frac{\sqrt{2}}{2} \\ e^{xu} \sin(yv) - \frac{2\sqrt{2}v}{\pi} \end{bmatrix}$$

$$f(1, 1, 0, \frac{\pi}{4}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$D_{(u,v)} f = \begin{bmatrix} x e^{xu} \cos(yv) - 1 & -e^{xu} y \sin(yv) \\ x e^{xu} \sin(yv) & y e^{xu} \cos(yv) - \frac{2\sqrt{2}}{\pi} \end{bmatrix}$$

$$D_{(u,v)} f(1, 1, 0, \frac{\pi}{4}) = \begin{bmatrix} \frac{\sqrt{2}}{2} - 1 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{\pi} \end{bmatrix} \neq 0$$

By the Implicit function theorem, can solve for (u, v) in terms of $(x, y) \approx (1, 1)$