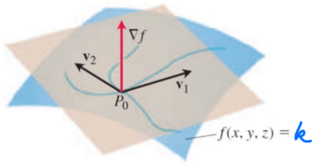
Lecture 16

§1 Tangent planes and linear approximation

- Let $\mathcal{S}$ be a surface in the domain of f described by $f(x, y, z) = k$, where f is differentiable.
- Let (x_0, y_0, z_0) be a point in $\mathcal{S}$.
- Take any curve $\mathcal{C}$ on $\mathcal{S}$ passing through (x_0, y_0, z_0).
- Just like what happened to level curves, $\nabla f(x_0, y_0, z_0)$ is perpendicular to any tangent vector of $\mathcal{C}$ at (x_0, y_0, z_0).



By considering different curves passing through the point, we have the picture above.

1. 定义

Def: Given any level surface S given by $f(x,y,z)=K$ of a differentiable function f, the tangent plane at a point $P_0 \in S$ to S is the plane through P_0 with normal $\nabla f(P_0)$, and the normal line of the surface at P_0 is the line through P_0 in the direction of $\nabla f(P_0)$.

2. 切平面方程

1° 等值面f在(x_0,y_0,z_0)处的切平面方程为

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$

2° 对于$f(x,y)$ [曲面$z=f(x,y)$]的图象，在$P_0=(x_0,y_0,f(x_0,y_0))$处的切平面为

$$z=f(x_0,y_0)+f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)$$

3. linear approximation

1° 切平面的方程 L:

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)$$

也被称作f在(x_0,y_0)处的 linearization.

2° The approximation:

$$f(x,y)\approx L(x,y)$$

指的是f在(x_0,y_0)处的 (standard) linear approximation 或 tangent plane approximation

4. linear approximation 的误差

选定一个起始点(x_0,y_0)，进行线性近似.

则近似的 error 为 $E(x,y):=f(x,y)-L(x,y)$

令：① $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ 在一个 open region S 上均连续

② 在某个以(x_0,y_0)为中心的长方形区域R内 $(R\subseteq S)$，存在M，使得 $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ $(=|f_{yx}|)$ are all bounded above by M

则：

$|E(x,y)| \le \frac{1}{2}M(|x-x_0|+|y-y_0|)^2$ for all $(x,y)\in R$.

§2 Differential

1. total differential (全微分)

对于f定义域内任一点 (x_0,y_0)：

- Δx 与 Δy independent
- $dx := \Delta x$, $dy := \Delta y$
- $\Delta z = \Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$ (change in f)

则 $dz = df = L(x_0+dx, y_0+dy) - L(x_0,y_0)$

则 $dz = f_x(x_0,y_0)\cdot dx + f_y(x_0,y_0)\cdot dy$ (change in L)

注：dz被称为 total differential

dz是关于 x_0, y_0, dx 与 dy 的一个函数

全微分可以用于估计 the sensitivity of the function value subject to a small change in values of the independent variables.

例：e.g. A right circular cylindrical storage has height $h_0=2.5m$ and base radius $r_0=0.5m$ (which you don't know yet). You try to measure the h and r of the storage. Which parameter should you be more careful with the measurement in order to avoid big measurement error in volume?

Sol: $V=\pi r^2 h$

$$dV = \frac{\partial V}{\partial r}\cdot dr + \frac{\partial V}{\partial h}\cdot dh$$

$$= 2\pi hr\cdot dr + \pi r^2 dh$$

$$dV|_{(r,h)=(0.5,2.5)} = \underbrace{2.5\pi\, dr}_{\text{more sensitive, be more careful}} + 0.25\pi\, dh$$

§3 Functions with three variables

For $w=f(x,y,z)$, differentiable, "base point" $P_0=(x_0,y_0,z_0)$

- Linearization of f at P_0:

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

- Error:

若所有的二阶导均在一个 open S上连续，且它们的绝对值在R (rectangular solid) $R\subseteq D$上的上界均为M，则

$|E(x,y,z)| \le \frac{1}{2}M(|x-x_0|+|y-y_0|+|z-z_0|)^2$ for all $(x,y,z)\in R$

- total differential:

$$dw = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

例: For $f(x,y,z) = x^2 - xy + 3\sin z$, consider the standard linear approximation at $P_0 = (2,1,0)$ on

$R = [1.99, 2.01] \times [0.98, 1.02] \times [-0.01, 0.01]$

$= \{(x,y,z): x \in [1.99, 2.01], y \in [0.98, 1.02], z = [-0.01, 0.01]\}$

- $f(P_0) = 2$, $f_x(P_0) = 3$, $f_y(P_0) = -2$, $f_z(P_0) = 3$, so

  $L(x,y,z) = 2 + 3(x-2) - 2(y-1) + 3z = 3x - 2y + 3z - 2$.

- Since $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = -3\sin z$, $f_{xy} = -1$, $f_{xz} = 0$, $f_{yz} = 0$,

  all their absolute values ≤ 2

- Hence, for all (x,y,z) in R,

  $|E(x,y,z)| \le \frac{1}{2}\cdot 2\cdot(|x-2| + |y-1| + |z-0|)^2$

  $\le (0.01 + 0.02 + 0.01)^2 = 0.0016$

§4 Optimization (local extrema)

1. Local extrema (极值)

Definition

Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, be a function, and let $(a_1, \ldots, a_n)$ be a point in D.

- We say that f has a local maximum at $(a_1, \ldots, a_n)$ if there exist an $r > 0$ such that

$$f(a_1, \ldots, a_n) \ge f(x_1, \ldots, x_n)$$

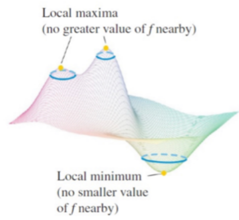
for every $(x_1, \ldots, x_n) \in D \cap B_r(a_1, \ldots, a_n)$.

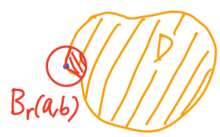
open ball with radius r (disk if $n=2$)

- We say that f has a local minimum at $(a_1, \ldots, a_n)$ if there exist an $r > 0$ such that

$$f(a_1, \ldots, a_n) \le f(x_1, \ldots, x_n)$$

for every $(x_1, \ldots, x_n) \in D \cap B_r(a_1, \ldots, a_n)$.





注: 1° local extrema 也被称为 relative extrema.

2° For $f(x,y) = x^2 + y^2$ defined on the closed disk $D = \{(x,y) : x^2 + y^2 \le 1\}$

f has local maxima at all (x,y) satisfying $x^2 + y^2 = 1$, the boundary of D

2. Theorem (First derivative test) (已知极值, 判断偏导情况)

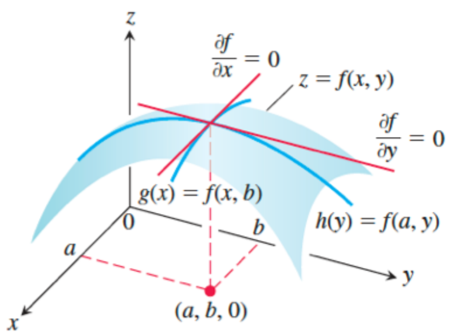
Although many theorems for extrema (i.e., maxima or minima) hold for n-variable functions, we will focus on these theorems for functions with 2 or 3 variables.

Theorem (First Derivative Test)

For a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, suppose that:

- *f has a local maximum or a local minimum at an interior point (a,b) of D, and;*
- *both $f_x(a,b)$ and $f_y(a,b)$ exist*

Then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. That is, $\nabla f(a,b) = \vec{0}$.



证明：

- Define $g(x) := f(x,b)$. Then $g(a)$ is a local extremum, where a is an interior point of the domain of g. By one-variable theory, $g'(a)$ D.N.E or $g'(a)=0$
- $g'(a) = \frac{\partial}{\partial x} f(a,b)$, which exists by assumption, so

  $f_x(a,b) = g'(a) = 0$
- Similarly, $f_y(a,b) = 0$

3. Critical point (可疑极值点)

> **Definition**
> An interior point (a, b) in the domain of a function f is called a critical point of f if either $\nabla f(a, b) = \vec{0}$ or at least one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.

注：若f在定义域的一个内点 P_0 处取 local extremum，则 P_0 为 critical point.

逆命题不成立

e.g. For $f(x,y) = y^2 x^2$, $(0,0)$ 为 critical point，但该点处并不取 local extremum.

4. Saddle point (鞍点)

> Def: If all partial derivatives of f are zero at an interior point (a,b) of the domain, 且 $f(a,b)$ 不是 local extremum, then (a,b) is called a saddle point of f, and $(a,b,f(a,b))$ is called a saddle point of the surface $z=f(x,y)$.

5. Theorem (Second derivative test) (已知偏导情况，判断极值)

> **Theorem (Second Derivative Test)**
> *Let f be a function whose second partial derivatives are all continuous on a open ~~ball~~ disk centered at (a, b). Suppose that $\nabla f(a, b) = \vec{0}$ (so (a, b) is a critical point of f). Let*
>
> $$H := H(a, b) := f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2.$$
>
> - *If $H > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).*
> - *If $H > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).*
> - *If $H < 0$, then f has no local extremum at (a, b); that is, (a, b) is a saddle point ~~ø~~ of f.*

注：若 $H=0$，则本方法失效 (inconclusive)，使用其他方法

例： **Example**

Find all local extrema of the function

$$f(x, y) := x^4 + y^4 - 4xy + 1$$

defined on $\mathbb{R}^2$.

- Find critical point :

$$\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{cases}$$

$\Rightarrow x(x^8 - 1) = 0$

$\Rightarrow x = 0$ or $x = \pm 1$

$\therefore (x, y) = (0,0), (1,1), (-1,-1)$ are all the critical points

- Use 2^{nd} der. test

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = f_{yx} = -4$

$H = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$

$H(0,0) = -16 < 0$

$H(1,1) = 144 - 16 > 0$, $f_{xx}(1,1) = 12 > 0 \Rightarrow$ local min

$H(-1,-1) = 144 - 16 > 0$, $f_{xx}(-1,-1) = 12 > 0 \Rightarrow$ local min

- No local max ; local min at $(1,1)$, $(-1,-1)$.