

Lecture 19

§1 Taylor's theorem

1. Definition: n -th order Taylor's polynomial

Goal: 用 polynomials locally near x_0 来近似 $f(x)$

- Use 1st order polynomial $p_1(x)$ (linearization of f at x_0)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0) \text{ as } x \rightarrow x_0 \text{ if } f \text{ differentiable at } x_0$$

- Use 2nd order polynomial $p_2(x)$

$$f(x) \approx p_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2$$

$$\text{Demand: } f(x_0) = p_2(x_0) \Rightarrow c_0 = f(x_0)$$

$$f'(x_0) = p_2'(x_0) \Rightarrow c_1 = f'(x_0)$$

$$f''(x_0) = p_2''(x_0) \Rightarrow c_2 = \frac{f''(x_0)}{2}$$

- Better approximation by higher-order polynomial $p_n(x)$?

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

这被称为 n -th order Taylor's polynomial of $f(x)$ at x_0

Question: What can we say about $\text{error}(x) = f(x) - p_{n-1}(x)$, $n \geq 1$

$$\text{特例: } n=1 \Rightarrow p_0(x) = f(x_0)$$

$$\text{error} = f(x) - f(x_0) \stackrel{\text{M.V.T.}}{=} f'(c)(x-x_0)$$

2. Theorem: Taylor's theorem (带拉格朗日型余项)

Let $n \geq 1$, suppose $f^{(n-1)}$ exists & continuous on $[a, b]$, and $f^{(n)}$ exists on (a, b) , $x_0 \in [a, b]$.

Then $f(x) - p_{n-1}(x) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n$, $\forall x \in [a, b]$, for some c between x and x_0 .

其中 $\frac{f^{(n)}(c)}{n!}(x-x_0)^n$ 被称为 Cauchy's remainder

注: 另一种表示形式为 $f(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$

证明:

- If $x = x_0$, done

- Suppose $x \neq x_0$, fix x, x_0 , Define $M = \frac{f(x) - p_{n-1}(x)}{(x-x_0)^n}$

W.T.S. $M = \frac{f^{(n)}(c)}{n!}$ for some c between x and x_0

W.L.D.G. Let $x_0 < x$

Define $g(t) = f(t) - p_{n-1}(t) - M(t-x_0)^n$

- $g^{(n-1)}(t)$ exists & continuous on closed interval $[x_0, x]$

$g^{(n)}(t)$ exists on (x_0, x)

- $g(x) = 0$

- $g(x_0) = 0 \stackrel{\text{M.V.T.}}{\Rightarrow}_{\text{on } [x_0, x]} g(x) - g(x_0) = g'(x_1)(x-x_0) \text{ for some } x_1 \in (x_0, x) \Rightarrow g'(x_1) = 0$

$$\begin{aligned}
 & \cdot g'(x_0) = 0 \xRightarrow{\text{M.V.T. on } [x_0, x_1]} g'(x_1) - g'(x_0) = g''(x_2)(x_1 - x_0) \text{ for some } x_2 \in (x_0, x_1) \Rightarrow g''(x_2) = 0 \\
 & \vdots \\
 & \cdot g^{(n-1)}(x_0) = 0 \xRightarrow{\text{M.V.T. on } [x_0, x_{n-1}]} \exists x_n \in (x_0, x_{n-1}) \text{ s.t. } g^{(n)}(x_n) = 0 \Rightarrow f^{(n)}(x_n) = 0 = M \cdot n! \\
 & \therefore M = \frac{f^{(n)}(c)}{n!}, \quad C = x_n \in (x_0, x_{n-1}) \subset (x_0, x)
 \end{aligned}$$

3. Theorem: Taylor's theorem (带佩亚诺型余项)

Suppose $f^{(n)}$ exists for some $n \geq 1$. Then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o(x-x_0)^n \text{ as } x \rightarrow x_0$$

其中 $o(x-x_0)^n$ 被称为 Peano's remainder

证明:

$$\text{W.T.S. } \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \dots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^n} = o(1) \text{ as } x \rightarrow x_0$$

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \dots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^n}$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow x_0} \frac{f'(x) - f'(x_0) - f''(x_0)(x-x_0) - \dots - \frac{f^{(n)}(x_0)}{(n-1)!}(x-x_0)^{n-1}}{n(x-x_0)^{n-1}}$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow x_0} \frac{f''(x) - f''(x_0) - f'''(x_0)(x-x_0) - \dots - \frac{f^{(n)}(x_0)}{(n-2)!}(x-x_0)^{n-2}}{n(n-1)(x-x_0)^{n-2}}$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{n(n-1)\dots \cdot 2(x-x_0)}$$

$$= \frac{1}{n(n-1)\dots \cdot 2} \lim_{x \rightarrow x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{x-x_0} \quad \text{linear approximation (Let } g = f^{(n-1)})$$

$$= \frac{1}{n(n-1)\dots \cdot 2} \lim_{x \rightarrow x_0} \frac{o(x-x_0)}{x-x_0}$$

$$= 0$$

$$\therefore \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \dots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^n} = o(1) \text{ as } x \rightarrow x_0$$

$$\text{e.g. } e^x = e^0 + (e^x)'|_{x=0}(x-0) + \dots + \frac{(e^x)^{(n-1)}|_{x=0}}{(n-1)!}(x-0)^{n-1} + \frac{(e^x)^{(n)}|_{x=c}}{n!}(x-0)^n$$

$$= 1 + x + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{e^c}{n!} x^n, \quad \forall x \in \mathbb{R}, n \geq 1, c \in (0, x)$$

$$\left| \frac{e^c x^n}{n!} \right| \leq \frac{e^{|x|} |x|^n}{n!} = e^{|x|} \frac{|x| \cdot |x| \dots |x|}{n \cdot (n-1) \dots 1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\therefore e^x = 1 + x + \dots + \frac{x^n}{n!} + \dots \quad \forall x \in \mathbb{R}$$

例 1: Find $\lim_{x \rightarrow \infty} [(x^3 - x^2 + \frac{x}{2}) e^{\frac{1}{x}} - x^3]$

$$\begin{aligned} & \lim_{x \rightarrow \infty} [(x^3 - x^2 + \frac{x}{2}) e^{\frac{1}{x}} - x^3] \\ \stackrel{y=\frac{1}{x}}{=} & \lim_{y \rightarrow 0} [(y^{-3} - y^{-2} + \frac{y^{-1}}{2}) e^y - y^{-3}] \\ = & \lim_{y \rightarrow 0} [(y^{-3} - y^{-2} + \frac{y^{-1}}{2})(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + o(y^3)) - y^{-3}] \\ = & \lim_{y \rightarrow 0} [y^{-3} - y^{-2} + \frac{y^{-1}}{2} + y^{-2} - y^{-1} + \frac{1}{2} + \frac{y^{-1}}{2} - \frac{1}{2} + \frac{y}{4} + (\frac{1}{3!} - \frac{y}{3!} + \frac{y}{3!2})(1 + o(1)) - y^{-3}] \\ = & \frac{1}{6} \end{aligned}$$

§2 关于 Taylor's theorem 的 facts

1. Theorem: second derivative test for local extreme

Suppose $f'(x_0)$ exists.

(i) Necessary conditions:

If $f(x_0)$ is local max, then $f''(x_0) \leq 0$

If $f(x_0)$ is local min, then $f''(x_0) \geq 0$

(ii) Sufficient conditions:

If $f'(x_0) = 0$, and $f''(x_0) < 0$, then $f(x_0)$ is local max

If $f'(x_0) = 0$, and $f''(x_0) > 0$, then $f(x_0)$ is local min

证明: (proof of (i))

证法一: (bad proof)

$\therefore f(x_0)$ is local max

$$\therefore \begin{cases} f'(x_0) = 0 \\ f(x_0) \geq f(x) \quad \forall x \approx x_0 \end{cases}$$

Take $x > x_0$, then $f(x_0) - f(x) \stackrel{\text{M.V.T.}}{=} f'(z)(x_0 - x)$, z between x and x_0

$$\text{Now, } f''(x_0) = \lim_{z \rightarrow x_0^+} \frac{f'(z) - f'(x_0)}{z - x_0} \leq 0$$

证法二: (better proof)

$$\text{By Peano, } f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + o((x - x_0)^2)$$

Argue by contradiction,

$$\text{If } f''(x_0) > 0, \text{ then } f(x) = f(x_0) + (\frac{f''(x_0)}{2!} + o(1))(x - x_0)^2$$

$$\therefore \frac{f''(x_0)}{2!} + o(1) > 0 \text{ if } x \text{ very close to } x_0$$

$\therefore f(x_0)$ is a local min (contradiction)

2. Theorem: Inverse function theorem

Let Ω be open in \mathbb{R}^n , $f: \Omega \rightarrow \mathbb{R}^n$ be C^1 -smooth.

Suppose $\exists a \in \Omega$ s.t. $(f'(a))_{n \times n}$ is non-singular

Then \exists open sets U & V in \mathbb{R}^n , s.t.

- $a \in U, f(a) \in V$
- f is one-to-one on U & $f(U) = V$
- Let g be the inverse of $f|_U$, then g is also C^1 -smooth on V ,
and $(g'(y))_{n \times n} = (f'(x))_{n \times n}^{-1}$

f 被称为 diffeomorphism from U to V

注: 对于 function of one variable $g(x) = f^{-1}(x)$, $g'(x) = \frac{1}{f'(x)}$