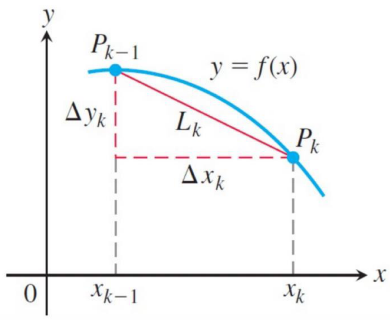
Lecture 16

§1 Arc Length

1. How to compute arc length

Consider a curve given by a continuous function $y=f(x)$ defined on the interval $[a,b]$, and let P be a partition of $[a,b]$.



If $y_k=f(x_k)$ and $\Delta y_k = y_k - y_{k-1}$, then the length of the curve between the points (x_{k-1}, y_{k-1}) and (x_k, y_k) is approximately

$$L=\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}=\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x_k \quad \text{when } \Delta x_k \text{ is small.}$$

The definition of arc length is obtain by taking the limit of $\sum_{k=1}^{n} L_k$ as $\|P\|\to 0$

2. Definition

Let f be a function such that f' is continuous on $[a,b]$. The length (or arc length) L of the curve $y=f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is defined as

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

e.g. Compute the length of the curve given by $y=x^{\frac{3}{2}}$, $0\le x\le 3$

$$L=\int_0^3 \sqrt{1+\left(\tfrac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$=\int_0^3 \sqrt{1+\tfrac{9}{4}x}\,dx$$

Let $u=1+\frac{9}{4}x$,

$$dx=\tfrac{4}{9}du$$

$u = \frac{31}{4}$ when $x = 3$

$u = 1$ when $x = 0$

$$L = \int_1^{\frac{31}{4}} \frac{4}{9}\sqrt{u}\, du$$

$$= \left[\frac{8}{27} u^{\frac{3}{2}} \right]_1^{\frac{31}{4}}$$

$$= \frac{8}{27}\left(\left(\frac{31}{4}\right)^{\frac{3}{2}} - 1 \right)$$

3. Note:

If the curve is given by $x = g(y)$, $c \le y \le d$, and g' is continuous, then the arc length can be computed by

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

e.g. Compute the length of the curve given by $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$, $0 \le x \le 2$

Try: $L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^2 \sqrt{1 + \left(\left(\frac{1}{2}\right)^{\frac{2}{3}} \frac{2}{3} (x)^{-\frac{1}{3}} \right)^2}\, dx$$

Hard to compute!

Try: $x = 2y^{\frac{3}{2}}$, $0 \le y \le 1$

$$L = \int_0^1 \sqrt{1 + (3y^{\frac{1}{2}})^2}\, dy$$

$$= \int_0^1 \sqrt{1 + 9y}\, dy$$

Let $u = 1 + 9y$

$dy = \frac{1}{9} du$

$u = 10$ when $y = 1$

$u = 1$ when $y = 0$

$$L = \int_1^{10} \frac{1}{9}\sqrt{u}\, du$$

$$= \left[\frac{2}{27} u^{\frac{3}{2}} \right]_1^{10}$$

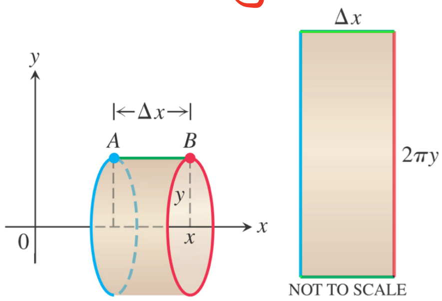
$$= \frac{2}{27}\left(10^{\frac{3}{2}} - 1 \right)$$

§2 Areas of Surfaces of Revolution

1. How to compute the areas of surfaces of revolution

1° Consider a cylindrical surface, generated by revolving a horizontal line around the x-axis:

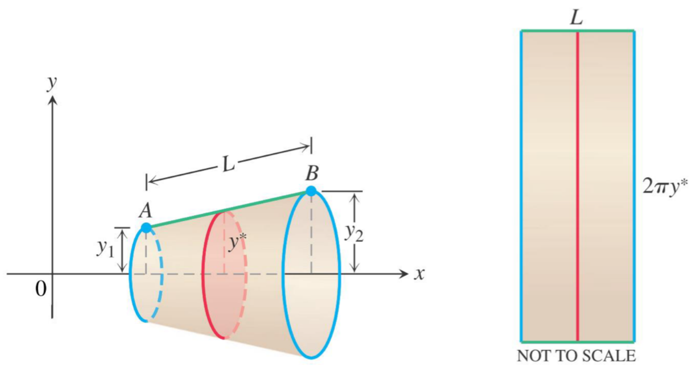
$$A = 2\pi y \Delta x$$



2° Consider a conical surface, generated by revolving a straight line around the x-axis:

$$A = 2\pi y^* L$$

$$y^* = \frac{y_1 + y_2}{2}$$



(a) (b)

3° For area of a surface of revolution about the x-axis in general ($y = f(x)$ for $a \le x \le b$):

Partition $[a,b]$ using $x_0, x_1 \cdots x_n$.

The k^{th} portion of the curve has length $\approx \sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$

The k^{th} portion of the surface has area

$$\approx \pi\left(f(x_{k-1})+f(x_k)\right)\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x_k$$

2. Definition

If the function $f(x) \geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$$

3. Note:

If $x=g(y) \geq 0$ is continuously differentiable on $[c,d]$, the area of the surface generated by revolving the graph of $x=g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y) \sqrt{1+(g'(y))^2}\, dy$$

e.g. The curve $y=x^{\frac{1}{3}}, 0 \leq x \leq 1$, is revolved about the y-axis to generate a surface, find its area

$x = y^3,\ 0 \leq y \leq 1$

$$S = \int_0^1 2\pi y^3 \sqrt{1+(3y^2)^2}\, dy$$

$$= 2\pi \int_0^1 y^3 \sqrt{1+(3y^2)^2}\, dy$$

Let $u = 1+9y^4$

$dy = \frac{1}{36y^3}\, du$

$u = 10$ when $y=1$

$u = 1$ when $y=0$

$$L = \int_1^{10} \frac{1}{36} \sqrt{u}\, du$$

$$= \left[\frac{1}{54} u^{\frac{3}{2}}\right]_1^{10}$$

$$= \frac{1}{54}\left(10^{\frac{3}{2}}-1\right)$$

§3 Work (功)

1. Definition

The work done by a variable force $F(x)$ in moving an object along the x-axis from $x=a$ to $x=b$ is

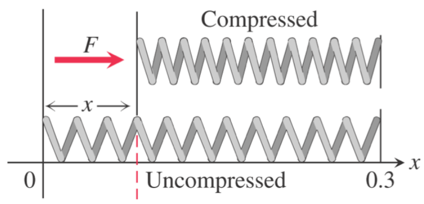
$$W = \int_a^b F(x)\,dx$$

e.g. **Hooke's Law** states that the force required to stretch or compress a spring is directly proportional to its distance x away from the natural position of the spring:

$$F(x) = kx$$

where k is the **spring constant** (units newtons per metre).

What is the work required to compress a spring from its natural length of 30 cm to a length of 20 cm?



$$\begin{aligned} W &= \int_0^{0.1} F(x)\,dx \\ &= \int_0^{0.1} kx\,dx \\ &= \left[\frac{k}{2}x^2\right]_0^{0.1} \\ &= 0.005k \end{aligned}$$

e.g.

EXAMPLE 5 The conical tank in Figure 6.39 is filled to within 2 m of the top with olive oil weighing 0.9 g/cm^3 or 8820 N/m^3. How much work does it take to pump the oil to the rim of the tank?

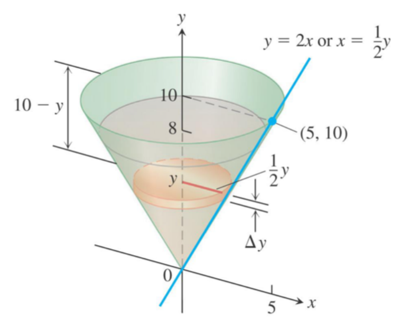


FIGURE 6.39 The olive oil and tank in Example 5.

$$\begin{aligned} W &= \int_0^8 \pi \left(\tfrac{1}{2}y\right)^2 \cdot 8820\,(10-y)\,dy \\ &= \int_0^8 2205\pi\,(10y^2 - y^3)\,dy \\ &= 2205\pi \int_0^8 10y^2 - y^3\,dy \\ &= 2205\pi \left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_0^8 \\ &= 2940\pi \times 8^3 \end{aligned}$$