Lecture 8

§1 Power series

1. **Definition:** Power series (幂级数)

形如 $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ 的级数被称为 power series

注：在 power series 中，$0^0 = 1$

2. **Theorem:** 收敛半径的确定

令 $\alpha = \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}}$，令 $R = \frac{1}{\alpha}$（$R = \infty$ 若 $\alpha = 0$；$R = 0$ 若 $\alpha = +\infty$），则

① 对于 $x \in (-R, R)$，$\sum a_n x^n$ converges absolutely.

② 对于 $|x| > R$，$\sum a_n x^n$ diverges

③ 对于 $|x| = 0$，the theorem does not apply

例1: considering $\sum_{n=1}^{\infty} \frac{x^n}{n}$

$a_n = \frac{1}{n} \Rightarrow \alpha = \overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{n}} = 1 \Rightarrow R = \frac{1}{\alpha} = 1$

By theorem, $\sum \frac{x^n}{n}$ converges absolutely for $x \in (-1, 1)$, diverges for $|x| > 1$

- $x = 1 \Rightarrow \sum \frac{x^n}{n} = \sum \frac{1}{n}$ diverges
- $x = -1 \Rightarrow \sum \frac{x^n}{n} = \sum \frac{(-1)^n}{n}$ converges

Interval of convergence $= [-1, 1)$

证明:

Let $c_n = a_n x^n$, then $\sum a_n x^n = \sum c_n$

Use root test: $\overline{\lim_{n\to\infty}} |c_n|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} |x| = |x| \alpha$

By root test: if $|x| \alpha < 1$, i.e. $|x| < \frac{1}{\alpha}$, then $\sum c_n$ converges absolutely

if $|x| \alpha > 1$, i.e. $|x| > \frac{1}{\alpha}$, then $\sum c_n$ diverges

Q.E.D.

例2: Considering $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

（先考虑 theorem）

$a_n = \frac{1}{n!}$

$\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} \frac{1}{(n!)^{\frac{1}{n}}}$ (ugly)

（再考虑 ratio test）

Let $c_n = \frac{x^n}{n!}$

$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$

By ratio test, $\sum c_n$ converges absolutely, $\forall x \in \mathbb{R}$

§2 Summation by parts & Alternating series test

1. Technical issue: integration by parts

$$\int_a^b g(x)f(x)\,dx = \int_a^b g(x)\,dF(x)$$
$$= F(x)g(x)\Big|_a^b - \int_a^b F(x)\,dg(x)$$
$$= F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\,dx$$

其中 $F(x) = \int_a^x f(t)\,dt$, $F'(x) = f(x)$

2. Theorem: Summation by parts (分部求和)

Given two sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$,

Let $A_n = \sum_{k=0}^{n} a_k$, $\forall n \geq 0$ (define $A_{-1} = 0$). Then for $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = A_q b_q - A_{p-1} b_p - \sum_{n=p}^{q-1} A_n (b_{n+1} - b_n)$$

证明:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$
$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$
$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$
$$= A_q b_q + \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_p$$
$$= A_q b_q - A_{p-1} b_p - \sum_{n=p}^{q-1} A_n (b_{n+1} - b_n)$$

Q.E.D.

3. Theorem: 一个部分和序列有界(不一定收敛)的级数乘上一个递减趋向0的序列后仍收敛

Suppose

- the patial sums $\{A_n\}$ of $\sum_{n=0}^{\infty} a_n$ are bdd
- $b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots \to 0$ as $n \to \infty$

Then $\sum_{n=0}^{\infty} a_n b_n$ converges

证明:

$\forall\, 0 \leq p \leq q$, by summation by parts ,

$$\sum_{n=p}^{q} a_n b_n = A_q b_q - A_{p-1} b_p - \sum_{n=p}^{q-1} A_n (b_{n+1} - b_n)$$

$$\left|\sum_{n=p}^{q} a_n b_n\right| \leq M b_q + M b_p + \sum_{n=p}^{q-1} M\,|b_{n+1} - b_n|$$
$$= M(b_q + b_p) + M(b_p - b_q)$$
$$= 2M b_p$$

$\because \lim_{n\to\infty} b_n = 0$

$\therefore \forall \varepsilon > 0, \exists N > 0$, s.t. whenever $n \geq N$, $b_n \leq \varepsilon$

Now for each $p \geq N$, $\left|\sum_{n=p}^{q} a_n b_n\right| \leq 2M\varepsilon$

By Cauchy's criterion , $\sum a_n b_n$ converges

Q.E.D.

4. **Theorem**: Alternating series test (莱布尼茨定理)

Suppose $b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq \cdots \to 0$ as $n \to \infty$

Then $\sum (-1)^n b_n$ & $\sum (-1)^{n+1} b_n$ converges

证明:

Let $a_n = (-1)^n$, then

$A_n = a_0 + a_1 + \cdots + a_n = 0$ or 1

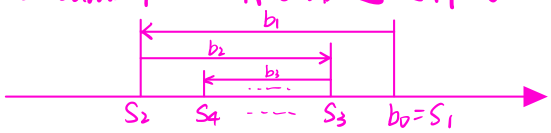
$\Rightarrow \{A_n\}$ bdd

$\because b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq \cdots \to 0$ as $n \to \infty$

$\therefore \sum (-1)^n b_n$ & $\sum (-1)^{n+1} b_n$ converges

Q.E.D.

注: 在 Calculus 中, 证明思路是这样的:



§3 Additions and multiplications of series

1. **Theorem**: 级数加法与数乘

Suppose $\sum_{n=n_0}^{\infty} a_n$ & $\sum_{n=n_0}^{\infty} b_n$ converge. Then

① $\sum_{n=n_0}^{\infty} (a_n \pm b_n)$ also converges, and $\sum_{n=n_0}^{\infty} (a_n \pm b_n) = \sum_{n=n_0}^{\infty} a_n \pm \sum_{n=n_0}^{\infty} b_n$

② $\sum_{n=n_0}^{\infty} (c a_n)$ also converges, and $\sum_{n=n_0}^{\infty} (c a_n) = c \sum_{n=n_0}^{\infty} a_n$

证明: 考虑 partial sum 即可

2. Multiplication of $\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n$

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n &= (a_0 + a_1 + a_2 + \cdots)(b_0 + b_1 + b_2 + \cdots) \\ &= \quad a_0 b_0 + a_0 b_1 + a_0 b_2 + \cdots \\ &\quad + a_1 b_0 + a_1 b_1 + a_1 b_2 + \cdots \\ &\quad + \cdots\end{aligned}$$

注: 我们没有必要研究 $\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n$ 的敛散性. No motivation!

3. Multiplication of $\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right)$

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) &= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n x^n, \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

4. **Definition**: Cauchy product (柯西乘积)

Given $\sum_{n=0}^{\infty} a_n$ & $\sum_{n=0}^{\infty} b_n$.

Define Cauchy product of these two series by $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $\forall n \geq 0$

注: ① 即使 $\sum a_n$ 与 $\sum b_n$ 均收敛, $\sum c_n$ 仍不一定收敛

反例：令 $\sum a_n=\sum b_n=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}=1-\frac{1}{\sqrt2}+\frac{1}{\sqrt3}-\frac{1}{\sqrt4}+\cdots$

则 $\sum c_n=1-\left(\frac{1}{\sqrt2}+\frac{1}{\sqrt2}\right)+\left(\frac{1}{\sqrt3}+\frac{1}{\sqrt2\cdot\sqrt2}+\frac{1}{\sqrt3}\right)-\left(\frac{1}{\sqrt4}+\frac{1}{\sqrt3\cdot\sqrt2}+\frac{1}{\sqrt2\sqrt3}+\frac{1}{\sqrt4}\right)+\cdots$

$c_n=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{(n-k+1)(k+1)}}$

由 $(n-k+1)(k+1)=\left(\frac{n}{2}+1\right)^2-\left(\frac{n}{2}-k\right)^2\leq\left(\frac{n}{2}+1\right)^2$,

$|c_n|\geq\sum_{k=0}^{n}\frac{2}{n+2}=\frac{2(n+1)}{n+2}$

因此 $|c_n|$ 极限不为0，$\sum c_n$ 发散

② 设 $\sum a_n(x-c)^n$ 的收敛半径为 R_a，$\sum b_n(x-c)^n$ 的收敛半径为 R_b.

令 $R:=\min\{R_a,R_b\}$, Then

$$\left(\sum_{n=0}^{\infty}a_n(x-c)^n\right)\cdot\left(\sum_{n=0}^{\infty}b_n(x-c)^n\right)=\sum_{n=0}^{\infty}c_n(x-c)^n$$

for all x with $|x-c|<R$, where

$$c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0=\sum_{k=0}^{n}a_k\cdot b_{n-k}$$

5. **Theorem**: $\sum_{n=0}^{\infty}a_n$ 与 $\sum_{n=0}^{\infty}b_n$ 的柯西乘积的值

If $\sum_{n=0}^{\infty}a_n=A$ converges abs. & $\sum_{n=0}^{\infty}b_n=B$ converges.

Then $\sum_{n=0}^{\infty}c_n$ converges and $\sum_{n=0}^{\infty}c_n=AB$

证明：

Let $A_n=a_0+a_1+\cdots+a_n$, $B_n=b_0+b_1+\cdots+b_n$. $C_n=c_0+c_1+\cdots+c_n$

Then
$$\begin{aligned}C_n&=a_0b_0+(a_0b_1+a_1b_0)+\cdots+(a_0b_n+a_1b_{n-1}+\cdots+a_nb_0)\\&=a_0B_n+a_1B_{n-1}+\cdots+a_nB_0\\&=a_0(B_n-B+B)+a_1(B_{n-1}-B+B)+\cdots+a_n(B_0-B+B)\\&=a_0(B_n-B)+a_1(B_{n-1}-B)+\cdots+a_n(B_0-B)+B(a_0+a_1+\cdots+a_n)\\&=a_0(B_n-B)+a_1(B_{n-1}-B)+\cdots+a_n(B_0-B)+BA_n\end{aligned}$$

W.T.S. $a_0(B_n-B)+a_1(B_{n-1}-B)+\cdots+a_n(B_0-B)\to0$ as $n\to\infty$

$\because$ $B_n\to B$ as $n\to\infty$

$\therefore$ $\forall\varepsilon>0$, $\exists N$, s.t. if $n\geq N$, then $|B_n-B|<\varepsilon$

$\therefore$
$$\begin{aligned}&|a_0(B_n-B)+a_1(B_{n-1}-B)+\cdots+a_n(B_0-B)|\\\leq&|a_0||B_n-B|+\cdots+|a_{n-N}||B_N-B|+\\&|a_{n-N+1}||B_{N-1}-B|+\cdots+|a_n||B_0-B|\\\leq&\varepsilon\cdot\sum_{k=0}^{\infty}|a_k|+|a_{n-N+1}||B_{N-1}-B|+\cdots+|a_n||B_0-B|\end{aligned}$$

$\because$ $\sum a_n$ converges absolutely

$\therefore$ $\lim_{n\to\infty}|a_n|=0$

$\sum_{k=0}^{\infty}|a_k|$ is a constant

$\therefore$
$$\begin{aligned}&\overline{\lim_{n\to\infty}}|a_0(B_n-B)+a_1(B_{n-1}-B)+\cdots+a_n(B_0-B)|\\\leq&\overline{\lim_{n\to\infty}}\varepsilon\cdot\sum_{k=0}^{\infty}|a_k|+|a_{n-N+1}||B_{N-1}-B|+\cdots+|a_n||B_0-B|\\=&\varepsilon\cdot\sum_{k=0}^{\infty}|a_k|+0+0+\cdots+0\end{aligned}$$
(Q.E.D.)