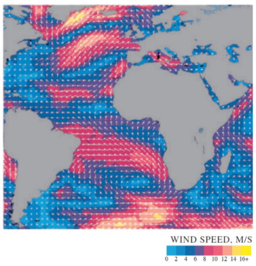
Lecture 22

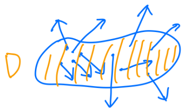
§1 Vector field

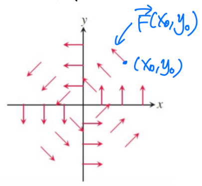
1. Vector field (向量场) 定义

令 $D \subseteq R^n$ 为一组点. R^n 空间内的一个 vector field 为一个函数 $\vec{F}: D \to R^n$, $\vec{F}$ 指定 $D \in R^n$ 中的每一个点为 R^n 中的一个向量



Def: Let $D \subseteq \mathbb{R}^n$ be a set of points. A vector field (in $\mathbb{R}^n$) is a function $\vec{F}: D \to \mathbb{R}^n$ that assigns each point in D a vector in $\mathbb{R}^n$.



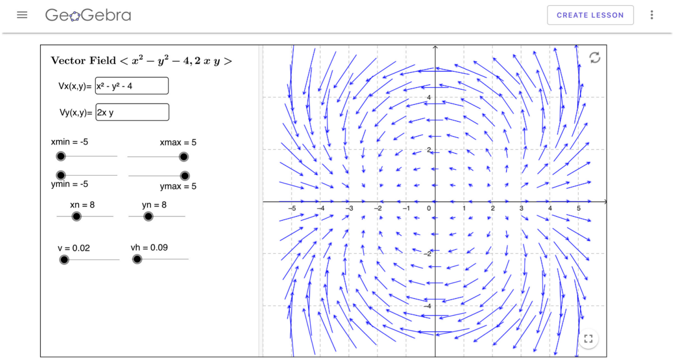


e.g. velocity fields, force fields, gradient field

2. 表达形式

R^2 中的向量场: $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$

R^3 中的向量场: $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$



§2 Line integrals of vector fields (第二类曲线积分)

1. Motivation: work & Definition

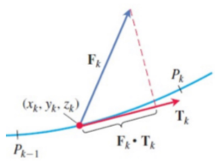
A basic formula in physics states that

$$\text{Work} = \text{Force} \cdot \text{Distance},$$

if the force is a constant. Suppose that a force field **F** in the space ~~drags~~ acts on a particle moving along a smooth curve C. To approximate the total work done by **F** in moving the particle, we may do the following.

- Partition C into $\{P_0, P_1, \ldots, P_n\}$.
- The work done by **F** in moving the particle from P_{k-1} to P_k is approximately $\mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k$, as shown in the following figure.
- We expect the total work done to be

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k.$$



We define the work done by a continuous force field **F** in moving a particle along C to be

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

This is an example of a line integral of a vector field.

Definition
Let **F** be a continuous vector field defined on a smooth curve C. The line integral of **F** along C is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

Note that $f(x,y,z) := \vec{F}(x,y,z)\cdot\vec{T}(x,y,z)$ is a real-valued function, so $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C f\,ds$.

talked about this in 16.1.

注：1° 若 $f(x,y,z) := \vec{F}(x,y,z)\cdot\vec{T}(x,y,z)$ 为一个实值函数，则 $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C f\,ds$，可以用第一类曲线积分解决

2° 若 C is parametrized by $\vec{r}(t)$, for $t\in[a,b]$，则

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\,|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

由此，$\int_C \vec{F}\cdot\vec{T}\,ds$ 可以写作 $\int_C \vec{F}\cdot d\vec{r}$

2、第二类曲线积分的计算

对于一个空间曲线 C：$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$，$d\vec{r} = \langle dx, dy, dz\rangle$

若 $\vec{F} = \langle M, N, P\rangle$，（$M, N, P$ 为关于 x, y, z 的函数），则

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

$$= \int_C M\,dx + \int_C N\,dy + \int_C P\,dz$$

☆ 注：1° 若曲线 L 由 $y=g(x)$ $(a\le x\le b)$ 给出，积分方向是由 $A(a,g(a))$ 到 $B(b,g(b))$. 又假定 $M(x,y)$, $N(x,y)$ 在 L 上连续，则有

$$\int_{\widehat{AB}} M(x,y)\,dx + N(x,y)\,dy = \int_a^b [M(x,g(x)) + N(x,g(x))\,g'(x)]\,dx$$

2° 若曲线 L 由 $x=h(y)$ $(c\le y\le d)$ 给出，积分方向是由 $C(c,g(c))$ 到 $D(d,g(d))$. 又假定 $M(x,y)$, $N(x,y)$ 在 L 上连续，则有

$$\int_{\widehat{AB}} M(x,y)\,dx + N(x,y)\,dy = \int_c^d [M(h(y),y)\,h'(y) + N(h(y),y)]\,dy$$

3° 若曲线 L 由参数方程给出

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} \quad \alpha\le t\le\beta$$

其中 $\varphi(t)$ 与 $\psi(t)$ 在 $[a,b]$ 上有连续的一阶导数. 当 t 单调地由 α 变为 β 时，曲线 L 上的点由 A 变到 B. 若 $M(x,y)$, $N(x,y)$ 在 L 上连续，则有：

$$\int_{\widehat{AB}} M(x,y)\,dx + N(x,y)\,dy = \int_\alpha^\beta [M(\varphi(t),\psi(t))\cdot\varphi'(t) + N(\varphi(t),\psi(t))\cdot\psi'(t)]\,dt$$

4° 若L为一空间曲线，由参数方程给出

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \quad (\alpha\le t\le\beta) \\ z=\chi(t) \end{cases}$$

其中$\varphi(t),\psi(t),\chi(t)$在$[\alpha,\beta]$上有连续的导数. 若$M(x,y,z)$, $N(x,y,z)$, $P(x,y,z)$在L上连续，则有：

$$\int_{\widehat{AB}} M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$$
$$= \int_\alpha^\beta [M(\varphi(t),\psi(t),\chi(t))\cdot\varphi'(t) + N(\varphi(t),\psi(t),\chi(t))\cdot\psi'(t) + P(\varphi(t),\psi(t),\chi(t))\cdot\chi'(t)]\,dt$$

例：Example

Find the work done by the force field $\mathbf{F}=\langle x,y,z\rangle$ in moving a particle along the curve C parametrized by

$$\mathbf{r}(t)=\langle\cos(\pi t),t^2,\sin(\pi t)\rangle,\quad t\in[0,1].$$

Sol: $\vec{r}(t)=\langle\cos(\pi t),t^2,\sin(\pi t)\rangle$

$\vec{F}(t)=\langle x,y,z\rangle$
$=\langle\cos(\pi t),t^2,\sin(\pi t)\rangle$

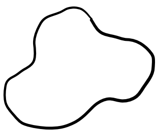
$$\text{Work}=\int_C \vec{F}\cdot d\vec{r}$$
$$=\int_0^1 \cos(\pi t)\cdot(-\pi\sin(\pi t))+t^2\cdot 2t+\sin(\pi t)\cdot(\pi\cos\pi t)\,dt$$
$$=\int_0^1 2t^3\,dt$$
$$=\frac{1}{2}$$

3、Flows (流量) and circulation (环量)

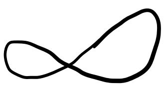
1° $\vec{F}$沿着C的第二类曲线积分$\int_C\vec{F}\cdot\vec{T}ds$也被称作flow integral (流量积分).

2° 一个曲线$\vec{r}(t)$, $a\le t\le b$被称作closed若$\vec{r}(a)=\vec{r}(b)$.
另外，若$\vec{r}$在$[a,b)$上是one-to-one的，则被称作simple closed curve (简单闭曲线)
(可以理解为doesn't cross itself, except possibly at endpoints)

e.g.



simple closed curve



not simple closed curve

3° 若C为closed，可以将$\int_C\vec{F}\cdot\vec{T}ds$写作$\oint_C\vec{F}\cdot\vec{T}ds$，这个积分经常被称作circulation
对于flows和circulations，曲线C的oritation (direction)是有影响的. 改变C的方向会改变$\vec{T}$的方向.
可以用$\oint_C$来表示C的方向 (counterclockwise)

§3. Flux (for vector fields in $\mathbb{R}^2$)

1、Flux（通量）的定义

1° 考虑水流以恒定速度 $\vec{v}$ 垂直穿过长度为 l 的线段 L，则 $|\vec{v}|l$ 测量了单位时间穿过 L 的水量

2° 若 $\vec{v}$ 不与 L 垂直，则需要点乘 L 的单位法向量 $\vec{n}$.
单位时间穿过 L 的水量 $= (\vec{v}\cdot\vec{n})\,l$ m^2/s

3° 若将 L 替换为简单闭曲线 C，考虑 flux：

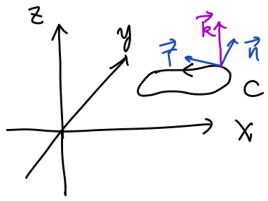
> Def: If C is a simple closed curve and $\vec{F} = \langle M(x,y), N(x,y)\rangle$ is a vector field in $\mathbb{R}^2$, and $\vec{n}$ is the outward normal C, then the (outward) flux of $\vec{F}$ across C is the integral
> $$\int_C \vec{F}\cdot\vec{n}\,ds.$$

If $\vec{F}$ is the velocity field of some flowing fluid, then the (outward) flux measures the total amount (e.g., m^2) of fluid flowing out of C per unit time (e.g., sec).

2、Flux 的计算

令 C is oriented counterclockwise，由右手定则可知，

$$\begin{aligned}\vec{n} &= \vec{T}\times\vec{k}\\ &= \frac{d\vec{r}}{ds}\times\vec{k}\\ &= \left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0\right\rangle \times \langle 0,0,1\rangle\\ &= \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}\end{aligned}$$



由此，$\vec{n}\,ds = dy\,\vec{i} - dx\,\vec{j}$，

✿ $\int_C \vec{F}\cdot\vec{n}\,ds = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C (M\,dy - N\,dx)$

例: Find the flux of $\vec{F}\langle x-y, x\rangle$ across the circle $C: x^2+y^2=1$

Sol: C: $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$, counterclockwise, does not repeat except $\vec{r}(0)=\vec{r}(2\pi)$.

$$dy = \cos t\,dt, \quad dx = -\sin t\,dt$$

$$\begin{aligned}\Rightarrow \text{flux} &= \oint_C \vec{F}\cdot\vec{n}\,ds\\ &= \oint_C M\,dy - N\,dx\\ &= \int_0^{2\pi} (\cos t - \sin t)\cdot\cos t\,dt - \cos t\cdot(-\sin t)\,dt\\ &= \int_0^{2\pi} \cos^2 t\,dt\\ &= \pi\end{aligned}$$