Lecture 27

§1 Divergence

1. Divergence (散度)

Def: Given a vector field $\vec{F}$, the divergence of $\vec{F}$ is
$$\operatorname{div}(\vec{F}) := \nabla\cdot\vec{F}.$$

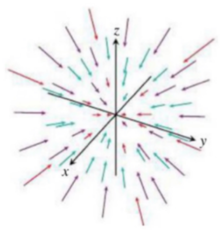
e.g. · If $\vec{F} = \langle M, N, P\rangle$, then $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N + \frac{\partial}{\partial z}P$

· If $\vec{F} = \langle M, N\rangle$, then $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N$

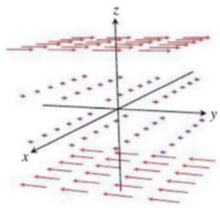
注: 直观上, $\operatorname{div}\vec{F}$ at P 描述了 P 处的 outward flux density

$$\operatorname{div}(\vec{F})(P)\begin{cases} >0 \Rightarrow \text{expanding (diverging) at } P \\ <0 \Rightarrow \text{compressing (shrinking) at } P \\ =0 \Rightarrow \text{neither}\end{cases}$$

例:

(a) 

$\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(b) 

$\mathbf{F}(x, y, z) = z\mathbf{j}$

(a) $\operatorname{div}\vec{F} = -1-1-1 = -3$ (Compressing)

(b) $\operatorname{div}\vec{F} = 0+0+0 = 0$

2. Theorem (旋度场的性质二)

Theorem

If $\mathbf{F}$ *is a vector field in* $\mathbb{R}^3$ *whose components have continuous second partial derivatives, then*

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) \equiv 0.$$

注: 换言之, 任意旋度场的散度为 0.

证明:

$$\operatorname{div}(\operatorname{curl}\vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$
$$= P_{xy} - N_{zx} + M_{zy} - P_{xy} + N_{xz} - M_{yz}$$
$$= 0$$

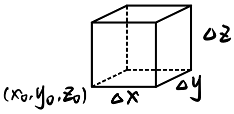
例: **Example**

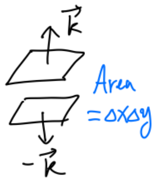
Show that $\mathbf{F}(x, y, z) := \langle xy, xyz, -y^2\rangle$ is not the curl field of any field.

$$\operatorname{div}(\vec{F}) = y + xz + 0 \not\equiv 0$$
$$\therefore \vec{F} \neq \operatorname{curl}\vec{G}, \ \forall \vec{G}$$

3. See $\operatorname{div}(\vec{F})$ as (outward) flux density

Consider the flux of $\vec{F}$ across a very tiny cube



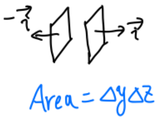


Flux 1

$$\approx \vec{F}(x_0, y_0, z_0+\Delta z)\cdot\vec{k}\,\Delta x\Delta y - \vec{F}(x_0,y_0,z_0)\cdot\vec{k}\,\Delta x\Delta y$$

$$= \frac{P(x_0,y_0,z_0+\Delta z) - P(x_0,y_0,z_0)}{\Delta z}\,\Delta x\Delta y\Delta z \quad \Delta V$$

$\hookrightarrow \frac{\partial P}{\partial z}(x_0,y_0,z_0)$ as $\Delta z \to 0$

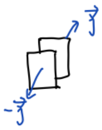


Flux 2

$$\approx \vec{F}(x_0+\Delta x, y_0, z_0)\cdot\vec{i}\,\Delta y\Delta z - \vec{F}(x_0,y_0,z_0)\cdot\vec{i}\,\Delta y\Delta z$$

$$= \frac{M(x_0+\Delta x, y_0, z_0) - M(x_0,y_0,z_0)}{\Delta x}\,\Delta x\Delta y\Delta z$$

$\hookrightarrow \frac{\partial M}{\partial x}(x_0,y_0,z_0)$ as $\Delta x \to 0$



Flux 3

$$\approx \vec{F}(x_0, y_0+\Delta y, z_0)\cdot\vec{j}\,\Delta x\Delta z - \vec{F}(x_0,y_0,z_0)\cdot\vec{j}\,\Delta x\Delta z$$

$$= \frac{N(x_0, y_0+\Delta y, z_0) - N(x_0,y_0,z_0)}{\Delta y}\,\Delta x\Delta y\Delta z$$

$\hookrightarrow \frac{\partial N}{\partial y}(x_0,y_0,z_0)$ as $\Delta y \to 0$

Hence, when cube is tiny, flux across cube

$$\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)\Delta V = \operatorname{div}(\vec{F})\,\Delta V,$$

so $\boxed{\operatorname{div}(\vec{F}) \approx \dfrac{\text{outward flux across cube}}{\text{volume of cube}}}$.

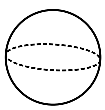
This cube has (x_0,y_0,z_0) as its corner. But when cube is "small" and $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}$ & $\frac{\partial P}{\partial z}$ are cts, then $\operatorname{div}(\vec{F})$ at corner $\approx \operatorname{div}(\vec{F})$ at center of cube.

§2 Divergence theorem / Gauss' theorem

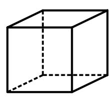
1. Closed surface (闭合曲面)

Roughly speaking, a closed surface is a surface that divides the space into two regions, "inside" (bounded) and "outside" (unbounded)

e.g. closed:

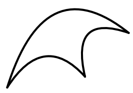


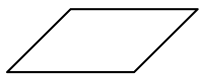
$x^2+y^2+z^2=a^2$



cubic shell

not closed:





e.g. $x^2+y^2+z^2 \le a^2$: closed bounded solid (closed ball $\bar{B}_a$).

$x^2+y^2+z^2 = a^2$: closed surface that is the boundary of $\bar{B}_a$.

e.g. Whole peach : closed bounded solid

peach skin : closed surface that is the boundary of the peach.

2. Divergence theorem / Gauss' theorem (高斯公式)

Theorem (Divergence Theorem) A.K.A. Gauss' theorem

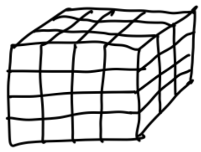
Let $\mathbf{F} = \langle M, N, P\rangle$ *be a vector fields whose components have continuous partial derivatives. If* E *is a bounded solid having* S *as its boundary, where* S *is a closed, piecewise smooth surface, oriented outward (i.e. its unit normals point out from* E*), then*

$$\underbrace{\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma}_{\text{Total flux across } S} = \iiint_E \underbrace{\operatorname{div}\mathbf{F}}_{\text{"Microscopic flux" at a point}}\,dV.$$

Total flux across S

"Microscopic flux" at a point.

$\approx$ flux across a tiny cube centered at the point



When summing microscopic flux, adjacent face get cancelled.

例: Find the flux of $\vec{F} = \langle z, y, x\rangle$ across the surface $x^2+y^2+z^2=1$

$$\begin{aligned}\iint_S \vec{F}\cdot\vec{n}\,d\sigma &= \iiint_B \operatorname{div}\vec{F}\,dV\\ &= \iiint_B 1\,dV\\ &= \operatorname{Vol}(B)\\ &= \tfrac{4}{3}\pi\end{aligned}$$

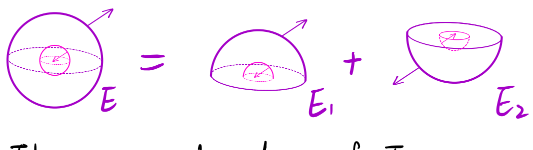
例: Find the flux of $\vec{F} = \langle x^2, 4xyz, ze^x\rangle$ across the boundary of the box $0\le x\le 3$, $0\le y\le 2$, $0\le z\le 1$

$$\begin{aligned}\iiint_S \vec{F}\cdot\vec{n}\,d\sigma &= \iiint_E \operatorname{div}\vec{F}\,dV\\ &= \iiint_E (2x+4xz+e^x)\,dV\\ &= \int_0^1\int_0^2\int_0^3 (2x+4xz+e^x)\,dx\,dy\,dz\\ &= 2\int_0^1 9+18z+e^3-1\,dz\\ &= 2(9+9+e^3-1)\\ &= 34+2e^3\end{aligned}$$

3. More general solids

若一个 solid 可以被 decomposed into 有限多个以 closed surface 为界的 solids，则高斯公式同样适用.

例: $E = b^2 \le x^2+y^2+z^2 \le a^2$ ($\bar{B}_a$, with B_b removed)



$$\begin{aligned}&\text{Flux across boundary of } E\\ =\ &\text{Flux across boundary of } E_1 + \text{Flux across boundary of } E_2\\ =\ &\iiint_{E_1}\operatorname{div}\vec{F}\,dV + \iiint_{E_2}\operatorname{div}\vec{F}\,dV\\ =\ &\iiint_E \operatorname{div}\vec{F}\,dV\end{aligned}$$

例: Consider $\vec{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$

(a) Show that the outward flux across any S_a is the same for all $a>0$, where S_a is the sphere given by $x^2+y^2+z^2=a^2$

(b) Find the value of such a flux

Sol: (a)

- Fix any a and b with $0<b<a$.
  Consider the solid E given by $b^2 \le x^2+y^2+z^2 \le a^2$
- Then $S_a \cup -S_b$ is the outward-oriented boundary of E, where $-S_b$ denotes S_b with normal pointing towards the origin.
- By the divergence theorem,
  $$\iint_{S_a \cup -S_b} \vec{F}\cdot\vec{n}\, d\sigma = \iiint_E \operatorname{div}(\vec{F})\, dV$$
- Let $\rho := \rho(x,y,z) = \sqrt{x^2+y^2+z^2}$. Then
  $$\frac{\partial M}{\partial x} = \frac{\rho^3 - x\cdot\frac{3}{2}\rho\cdot 2x}{\rho^6} = \frac{\rho^2-3x^2}{\rho^5}$$
  Similarly, $\frac{\partial N}{\partial y} = \frac{\rho^2-3y^2}{\rho^5}$ $\quad \frac{\partial P}{\partial z} = \frac{\rho^2-3z^2}{\rho^5}$
  $$\operatorname{div}(\vec{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{3\rho^2-3\rho^2}{\rho^5} = 0, \ \forall (x,y,z)\in\mathbb{R}^3 \,|\, \{0,0,0\}$$
- So $\iint_{S_a \cup -S_b} \vec{F}\cdot\vec{n}\, d\sigma = 0$
- But $\iint_{S_a \cup -S_b} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_a} \vec{F}\cdot\vec{n}\, d\sigma - \iint_{S_b} \vec{F}\cdot\vec{n}\, d\sigma$
  So $\iint_{S_a} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_b} \vec{F}\cdot\vec{n}\, d\sigma$
  Since this holds for all a and b with $b>a>0$, part (a) holds.

注: 事实上, 对于任一 outward flux by this $\vec{F}$ across any closed surface enclosing $(0,0,0)$, 这一结论均成立.

(b)

- It suffices to consider the flux be $\vec{F}$ across S_1, where S_1 is given by $x^2+y^2+z^2=1$
- The outward normal of S_1 is $\nabla g = \langle 2x, 2y, 2z \rangle$. Therefore, at any $(x_0,y_0,z_0)\in S_1$,
  $$\vec{n} = \frac{2\langle x_0, y_0, z_0\rangle}{\sqrt{4x_0^2+4y_0^2+4z_0^2}} = \langle x_0, y_0, z_0\rangle$$
- $\iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_1} \frac{1}{\sqrt{x^2+y^2+z^2}}\, d\sigma = \iint_{S_1} d\sigma = \text{Area}(S_1) = 4\pi$
- So the flux across any sphere (or closed surface) enclosed the origin in 4π

§3 Summary of "differentiation operation"

① $f(x) \xrightarrow{\frac{d}{dx}} f'(x)$

scalar (real-valued) $\quad$ scalar

Fundamental Thm of Calculus:

$$\int_{[a,b]} f'(x)\,dx := \int_a^b f'(x)\,dx = f(b) - f(a)$$

② $f(x,y,z) \xrightarrow{\text{grad}} \nabla f(x,y,z)$

Scalar $\quad$ vector

Fundamental Thm of Line Integrals:

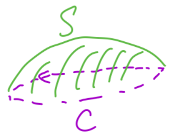
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

③ $\vec{F}(x,y,z) \xrightarrow{\text{curl}} \text{curl}\,\vec{F} = \nabla \times \vec{F}$

vector $\quad$ vector

Stokes' Thm (3D)

$$\iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$



Green's Thm (Circulation) (xy-plane)

$$\iint_R \text{curl}\,\vec{F} \cdot \vec{k}\, dA = \oint_C \vec{F} \cdot d\vec{r}$$

④ $\vec{F} \xrightarrow{\text{div}} \text{div}\,\vec{F} = \nabla \cdot \vec{F}$

vector $\quad$ scalar

Divergence Thm (3D)

$$\iiint_E \text{div}\,\vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

Green's Thm (Flux) (2D)

$$\iint_R \text{div}\,\vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, ds$$