Lecture 4

§1 马氏链的重要性质

1. **Property**: Joint distribution

对于转移概率为 $\{P_{ij}\}$ 的 MC，有

$$P(X_n=i_n,\cdots,X_1=i_1,X_0=i_0)=P(X_0=i_0)\cdot P_{i_0i_1}\cdots\cdots P_{i_{n-1}i_n}$$

for $n\geq 0$ and $(i_0,i_1,\cdots,i_n)\in E^{n+1}$

证明：

由条件概率，有

$$\begin{aligned}P(X_n=i_n,\cdots,X_1=i_1,X_0=i_0)&=P(X_n=i_n\mid X_{n-1}=i_{n-1},\cdots X_0=i_0)\cdot P(X_{n-1}=i_{n-1},\cdots X_0=i_0)\\&=P_{i_{n-1}i_n}\cdot P(X_{n-1}=i_{n-1},\cdots X_0=i_0)\\&=P(X_0=i_0)\cdot P_{i_0i_1}\cdots\cdots P_{i_{n-1}i_n}\end{aligned}$$

2. **Property**: Conditional probability（马氏性的推广）

对任意 $n>k_1>k_2>\cdots>k_m\geq 0$，有 （$k_1\cdots k_m$ 不一定连续）

$$P(X_n=i_n\mid X_{k_1}=i_{k_1},X_{k_2}=i_{k_2},\cdots,X_{k_m}=i_{k_m})=P(X_n=i_n\mid X_{k_1}=i_{k_1})$$

Applications：

① 对任意 $0\leq k<n$，有

$$P(X_{n+1}=j\mid X_n=i,X_{n-1}=i_{n-1},\cdots,X_k=i_k)=P(X_{n+1}=j\mid X_n=i)=P_{ij}$$

$$P(X_{n+1}=j\mid X_n=i,X_k=i_k)=P(X_{n+1}=j\mid X_n=i)=P_{ij}$$

② 对任意 $0\leq k_m<\cdots<k_1<n$，有

$$P(X_{n+1}=j\mid X_n=i,X_{k_1}=i_{k_1},\cdots,X_{k_m}=i_{k_m})=P(X_{n+1}=j\mid X_n=i)=P_{ij}$$

③ 对 $n\geq 1$ 与 $n>k>0$，有

$$P(X_n=i_n\mid X_k=i_k,X_0=i_0)=P(X_n=i_n\mid X_k=i_k)$$

§2 Chapman-Kolmogorov equation（C-K 方程）

1. **Definition**: n-step transition probabilities $P_{ij}^{(n)}$（n步转移概率）

在n步 transition 后，process 由 state i 变为 state j 的概率

$$P_{ij}^{(n)}=P\{X_n=j\mid X_0=i\},\quad i,j\in E$$

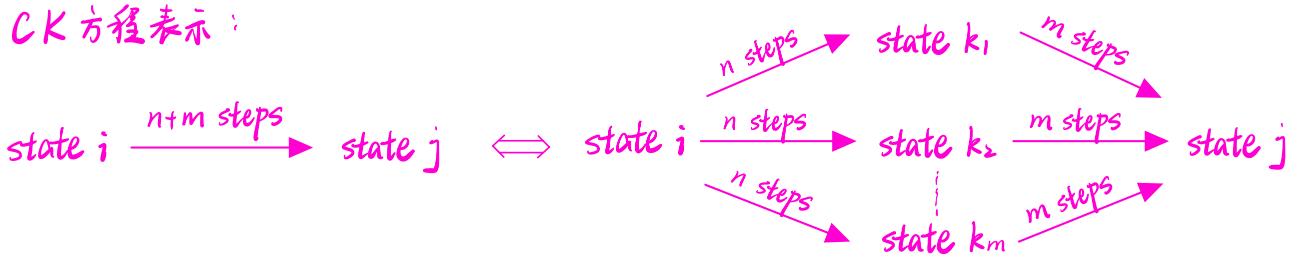
注：$P_{ij}^{(n)}\neq(P_{ij})^n$

2. Chapman-Kolmogorov equation（C-K 方程）

Chapman-Kolmogorov equation 为：

$$P_{ij}^{(n+m)}=\sum_{k\in E}P_{ik}^{(n)}P_{kj}^{(m)},\quad n,m\geq 0,\quad i,j\in E$$

注：CK 方程表示：



证明:

$$P_{ij}^{(n+m)} = P\{X_{n+m}=j \mid X_0=i\}$$
$$= \sum_{k\in E} P\{X_{n+m}=j,\ X_n=k \mid X_0=i\}$$
$$= \sum_{k\in E} P\{X_{n+m}=j \mid X_n=k, X_0=i\}\cdot P\{X_n=k \mid X_0=i\}$$
$$= \sum_{k\in E} P\{X_{n+m}=j \mid X_n=k\}\cdot P\{X_n=k \mid X_0=i\}$$
$$= \sum_{k\in E} P_{kj}^{(m)}\cdot P_{ik}^{(n)}$$

3. C-K方程的矩阵表示

令 $P^{(n)} = (P_{ij}^{(n)})$ 表示 n-step transition matrix (n步转移概率矩阵)

- 则根据C-K方程,有

$$P^{(n+m)} = P^{(n)}\cdot P^{(m)} \quad (*)$$

- 进一步可推出:

$$P^{(n)} = P^{(n-1)}\cdot P = P^n$$

注: 即n步转移概率矩阵 = 单步转移概率矩阵的n次幂

证明: $(*)$

$$P_{ij}^{(n+m)} = P_{ij}^{(n+m)}$$
$$= \sum_{k\in E} P_{ik}^{(n)}\cdot P_{kj}^{(m)}$$
$$= [P_{i1}^{(n)}, P_{i2}^{(n)}, \cdots, P_{iN}^{(n)}]\,[P_{1j}^{(m)}, P_{2j}^{(m)}, \cdots, P_{Nj}^{(m)}]^T$$
$$= \text{i-th row of } P^{(n)} \cdot \text{j-th column of } P^{(m)}$$
$$= (P^{(n)}\cdot P^{(m)})_{ij}$$

因此 $P^{(n+m)} = P^{(n)}\cdot P^{(m)}$

4. 关于 notation

① 单步转移概率为

$$P_{ij} = P\{X_{n+1}=j \mid X_n=i\}$$

② n步转移概率为

$$P_{ij}^{(n)} = P_{ij}^{(n)} = P_{ij}^{n} = P\{X_n=j \mid X_0=i\} \quad (\neq (P_{ij})^n = (P_{ij})^n)$$

注: ① 此处的 $P_{ij}^{(n)}$ 与 P_{ij}^{n} 分别表示矩阵 $P^{(n)}$ 与 P^n 的第 (i,j) 个 entry

② $P_{ij}^{1} = P_{ij} = P\{X_1=j \mid X_0=i\}$

③ $P_{ij}^{n} = (P\times P\times\cdots\times P)_{ij}$, 更好的 notation 为 $P_{ij}^{(n)}$

③ 对于 $n=0$, 一般认为 $P^0 = I$, the identity matrix, 即

$$P_{ij}^{0} = P\{X_0=j \mid X_0=i\} = 1_{\{i=j\}}$$

e.g. ▶ Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions.

Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

- If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the preceding is a two-state Markov chain whose transition probabilities are given by

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}.$$

- If $\alpha = 0.7$ and $\beta = 0.4$, then *calculate the probability that it will rain four days from today given that it is raining today.*

- Solution: recall the one-step transition probability matrix and we need to find $P^4 = P^2 P^2$:

$$P^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix},$$

$$P^4 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}\begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}.$$

e.g.

- Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it

(i) has rained for the past two days,	then it will rain tomorrow with probability	*0.7;*
(ii) rained today but not yesterday,		*0.5;*
(iii) rained yesterday but not today,		*0.4;*
(iv) has not rained in the past two days,		*0.2.*

- Cont'd. But if we introduce $X_n = (\text{weather on day } n, \text{ weather on day } n-1)$

State	Rains? Today	Yesterday
0	Y	Y
1	Y	N
2	N	Y
3	N	N

令 0表示 rain, 1表示 no rain,
则 $E = \{(0,0),(1,0),(0,1),(1,1)\}$

Then the preceding would then present a four-state Markov chain having a transition probability matrix

$$P = \begin{array}{c} \begin{matrix} (0,0) & (1,0) & (0,1) & (1,1) \end{matrix} \\ \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix} \end{array}. \quad \begin{matrix} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{matrix}$$

(How to derive e.g. $p_{33} = 0.8$?) $= P(X_{n+1} = (1,1) \mid X_n = (1,1)) = P(\text{tmr } 1 \mid \text{today } 1, \text{ yes } 1)$
$= 1 - 0.2 = 0.8$

Example 2.9. (refer to Example 2.4) Given that it rained on Monday and Tuesday, what is the probability that it will rain on Thursday.

Solution:

	X_t	next step	X_{t+1}	next step	X_{t+2}
	Mon Tue	$\longrightarrow$	Tue, Wed	$\longrightarrow$	Wed, Thu
rains?	Y, Y		Y, ?		Y/N, Y
state	0				0 or 1

$$P^2 = \begin{pmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{pmatrix}\begin{pmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{pmatrix} = \begin{array}{c} \begin{matrix} YY & NY & YN & NN \end{matrix} \\ \begin{pmatrix} .49 & .12 & .21 & .18 \\ .35 & .20 & .15 & .30 \\ .20 & .12 & .20 & .48 \\ .10 & .16 & .10 & .64 \end{pmatrix} \end{array} \begin{matrix} YY \\ NY \\ YN \\ NN \end{matrix}$$

Answer $= P^2_{00} + P^2_{01} = 0.49 + 0.12 = 0.61.$

$= P(X_{t+2} = (Y,Y) \mid X_t = (Y,Y)) + P(X_{t+2} = (N,Y) \mid X_t = (Y,Y))$

5. 状态 X_n 在 n 时刻的概率分布

对于 $n \geq 0$, 令 $\mu_n = X_n$ 的 pmf (写作 row vector) 表示状态 X_n 在 n 时刻的概率分布, 即

$\mu_n = (P\{X_n = 0\}, P\{X_n = 1\}, \cdots\cdots, P\{X_n = i\}, \cdots\cdots)$

· 因此，对 $\forall j \in E$，我们有

$$\mu_n(j) = P\{X_n=j\}$$
$$= \sum_{i\in E} P\{X_{n-1}=i, X_n=j\}$$
$$= \sum_{i\in E} P\{X_{n-1}=i\} \cdot P\{X_n=j \mid X_{n-1}=i\}$$
$$= \sum_{i\in E} \mu_{n-1}(i) \cdot p_{ij}$$
$$= \mu_{n-1} \cdot [p_{0j}, p_{1j}, \cdots, p_{Nj}]^T$$

· 写作矩阵形式，为

$$\mu_n = \mu_{n-1} \cdot P \quad , n \geq 1$$

· 进一步有

$$\mu_n = \mu_0 \cdot P^n \quad , n \geq 1$$

即初始状态分布向量 μ_0 与 n 步转移概率矩阵 P^n 的乘积

e.g.

- For instance, if $\alpha_0 = 0.4$ and $\alpha_1 = 0.6$, in Example 2.8, i.e. $\mu_0 = (0.4 \ \ 0.6)$.

  e.g. *Tomorrow it will rain with probability 0.4. What will be this probability 4 days after ?*

- Then the distribution of X_4, the weather 4 days after we begin keeping weather records is

$$\mu_4 = \mu_0 P^4 = (0.4 \ \ 0.6)\begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix} = (0.57 \ \ 0.43).$$

  In particular, the probability that it will rain equals 0.57.

补*：利用特征值分解求 n 步转移概率矩阵

例：$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$，求 P^n

① 求特征值和特征向量

· 特征多项式为

$$\begin{vmatrix} 0.7-\lambda & 0.3 \\ 0.4 & 0.6-\lambda \end{vmatrix} = \lambda^2 - 1.3\lambda + 0.3 = (\lambda-1)(\lambda-0.3) = 0$$

· 特征值为

$$\lambda_1 = 1 , \quad \lambda_2 = 0.3$$

· 当 $\lambda_1 = 1$ 时

$$\begin{bmatrix} -0.3 & 0.3 \\ 0.4 & -0.4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{特征向量为} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

当 $\lambda_2 = 0.3$ 时

$$\begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.3 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{特征向量为} \propto \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

② 特征值分解

· 将特征向量化为单位向量，组合可得到矩阵Q

$$Q=\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3}{5} \\ \frac{\sqrt{2}}{2} & \frac{4}{5} \end{bmatrix}$$

· 特征值分解

$$P=Q\Lambda Q^{-1}=\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3}{5} \\ \frac{\sqrt{2}}{2} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.3 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3}{5} \\ \frac{\sqrt{2}}{2} & \frac{4}{5} \end{bmatrix}^{-1}$$

③ 求n步转移概率矩阵

$$\begin{aligned}
P^n &= (Q\Lambda Q^{-1})(Q\Lambda Q^{-1})\cdots(Q\Lambda Q^{-1}) \\
&= Q\Lambda^n Q^{-1} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3}{5} \\ \frac{\sqrt{2}}{2} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 0.3^n \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3}{5} \\ \frac{\sqrt{2}}{2} & \frac{4}{5} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3}{5} \\ \frac{\sqrt{2}}{2} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 0.3^n \end{bmatrix}\begin{bmatrix} \frac{4}{7}\sqrt{2} & \frac{3}{7}\sqrt{2} \\ -\frac{5}{7} & \frac{5}{7} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{7}+\frac{3}{7}\times 0.3^n & \frac{3}{7}-\frac{3}{7}\times 0.3^n \\ \frac{4}{7}-\frac{4}{7}\times 0.3^n & \frac{3}{7}+\frac{4}{7}\times 0.3^n \end{bmatrix}
\end{aligned}$$