Lecture 8

§1 马氏链的应用

1. Gambler's ruin probability：赌徒破产问题

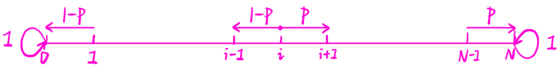
1° 问题描述与分析

- The Gambler's Ruin Probability (refer to Example 2.6). Let X_n denote the player's fortune at time n and thus $(X_n)_{n\geq 0}$ is a Markov chain with the one-step transition probability matrix given in Example 2.6.

- Suppose the initial "money" is i units ($i = 0, 1, 2, \ldots, N$), what is the probability that the gambler goes broke? (i.e. Ruin Probability; absorbing probability to state 0)

- Note that $\{1, 2, \ldots, N-1\}$ is a transient class, then the Markov chain will, after some finite amount of time, either reach state 0 or state N (because Markov chain visits any transient state finitely often). Hence "Ruin Probability" + "Reaching N Fortune Probability" = 1.

- The latter probability is easier to analyze!



2° 列式

- Let a_i ($i = 0, 1, \ldots, N$) denote the probability that, starting with i, the gambler's fortune will eventually reach N (before reaching 0, of course). Then we have

$$a_i = pa_{i+1} + qa_{i-1}, \quad i = 1, 2, \ldots, N-1.$$

First step method: $a_i = P(X_n \to n \mid X_0 = i) = P(X_n \to n, X_1 = i+1 \mid X_0 = i) + P(X_n \to n, X_1 = i-1 \mid X_0 = i)$

$$= P(X_n \to n \mid X_1 = i+1, X_0 = i) \cdot P(X_1 = i+1 \mid X_0 = i) + P(X_n \to n \mid X_1 = i-1, X_0 = i) \cdot P(X_1 = i-1 \mid X_0 = i)$$

- Idea: Conditioning on the outcome of the first play!

$$= P(X_n \to n \mid X_1 = i+1) \cdot P(X_1 = i+1 \mid X_0 = i) + P(X_n \to n \mid X_1 = i-1) \cdot P(X_1 = i-1 \mid X_0 = i) = pa_{i+1} + qa_{i-1}$$

$$i-1 \xleftarrow{q} i \xrightarrow{p} i+1$$

Also we have

$$a_0 = 0, \quad a_N = 1 \quad \text{(easy)}$$

- The above is called difference equation which can be easily solven by introducing the generating function. But this example is easy and can be solved directly.

3° 求解

① 方法一：特征值法

- 对于 difference equation $a_i = pa_{i+1} + qa_{i-1}$，有 characteristic equation：

$$\lambda = p\lambda^2 + q$$
$$\iff \lambda = p\lambda^2 + (1-p)$$
$$\iff (p\lambda - (1-p))(\lambda - 1) = 0$$
$$\iff \lambda_1 = \frac{1-p}{p} = \frac{q}{p}, \quad \lambda_2 = 1$$

- 情况1：$\lambda_1 \neq \lambda_2 \iff p \neq q \iff p \neq \frac{1}{2}$

由于 difference equation 为齐次的，解的形式为

$$a_i = c_1\left(\frac{q}{p}\right)^i + c_2(1)^i = c_1\left(\frac{q}{p}\right)^i + c_2$$

代入 $a_0=0$，$a_N=1$，有

$$\begin{cases} c_1+c_2=0 \\ c_1\left(\frac{q}{p}\right)^N + c_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{1}{\left(\frac{q}{p}\right)^N - 1} \\ c_2 = -\frac{1}{\left(\frac{q}{p}\right)^N - 1} \end{cases}$$

令 $\alpha = \frac{q}{p}$，则

$$a_i = \frac{1-\alpha^i}{1-\alpha^N}$$

- 情况2：$\lambda_1=\lambda_2 \Leftrightarrow p=q \Leftrightarrow p=\frac{1}{2}$

由于 difference equation 为齐次的，解的形式为

$$a_i = (c_1+c_2 i)(1)^i$$

代入 $a_0=0$，$a_N=1$，有

$$\begin{cases} c_1=0 \\ c_1+c_2 N = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_2 = \frac{1}{N} \end{cases}$$

则 $a_i = \frac{i}{N}$

② 方法二：利用数列知识

- $$a_i = pa_{i+1} + qa_{i-1}$$

$$\Leftrightarrow a_i = pa_{i+1} + (1-p)a_{i-1}$$

$$\Leftrightarrow p(a_{i+1}-a_i) = (1-p)(a_i - a_{i-1})$$

$$\Leftrightarrow a_{i+1}-a_i = \frac{q}{p}(a_i - a_{i-1}) = \cdots = \left(\frac{q}{p}\right)^i (a_1 - a_0)$$

因此，

$$\begin{aligned} a_k &= \sum_{i=0}^{k-1}(a_{i+1}-a_i) \\ &= \sum_{i=0}^{k-1}\left(\frac{q}{p}\right)^i (a_1-a_0), \quad k\geq 1 \end{aligned}$$

- 情况1：$\frac{q}{p}\neq 1 \Leftrightarrow p\neq q \Leftrightarrow p\neq\frac{1}{2}$

代入 $a_0=0$，令 $\alpha=\frac{q}{p}$，有

$$a_k = \frac{1-\alpha^k}{1-\alpha} a_1$$

代入 $a_N=1$，有

$$\frac{1-\alpha^N}{1-\alpha} a_1 = 1$$

$$\Rightarrow a_1 = \frac{1-\alpha}{1-\alpha^N}$$

$$\Rightarrow a_k = \frac{1-\alpha^k}{1-\alpha^N}$$

- 情况2：$\frac{q}{p}=1 \Leftrightarrow p=q \Leftrightarrow p=\frac{1}{2}$

$$a_{i+1}-a_i = a_i - a_{i-1} = \cdots = a_1 - a_0$$

代入 $a_0=0$，$a_N=1$，有

$$a_k=\frac{k}{N}$$

Finally we have, with $\alpha = q/p$,

$$a_i = \begin{cases} \frac{1-\alpha^i}{1-\alpha^N}, & p \neq \frac{1}{2}, \\ \\ \frac{i}{N}, & p = \frac{1}{2}, \end{cases}$$

with the ruin prbabilities given by: $b_i = 1 - a_i$.

4° 讨论：当 N 非常大时

Game with unbelievably high N:

Note that, as $N \to \infty$,

$$a_i \to \begin{cases} 1-\left(\frac{q}{p}\right)^i, & p > \frac{1}{2} \ (\alpha < 1), \\ 0, & p \leq \frac{1}{2} \ (\alpha \geq 1). \end{cases}$$

So the ruin probability $b_i = 1 - a_i$ tends to

$$\begin{cases} \alpha^i, & p > \frac{1}{2}, \\ 1, & p \leq \frac{1}{2}. \end{cases}$$

包括 fair game

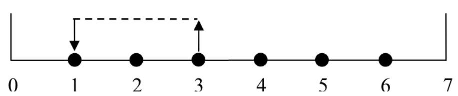
5° 讨论：Gambler's ruin probability 与 mean time calculation 的联系

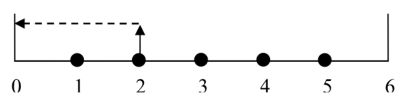
Calculate $f_{3,1}$ in Example 2.26 ($N = 7, p = 0.4$).

Solution: Note that $f_{3,1}$ is just the probability that a gambler starting with 3 reaches 1 before 7.(或0) That is, it is the probability that the gambler's fortune will go down 2 before going up 4;(或 going down 1) which is the probability that a gambler starting with 2 will go broke before reaching 6.

Idea:

$f_{3,1}$ = ruin probability starting from 2 （N 要减去 1）





Therefore

$$f_{3,1} = 1 - \underbrace{\frac{1-\left(\frac{0.6}{0.4}\right)^2}{1-\left(\frac{0.6}{0.4}\right)^6}}_{a_2} = 0.8797.$$

6、讨论：赌徒破产问题的一个等价表述：gambler v.s. banker

等价于初始财富 $i=a$，目标财富 $N=a+b$，求获胜概率 a_i

How about a gambler versus the banker in a gambling game ?

- We assume that the initial fortune of the gambler is a and that of the banker is b and $b \gg a$, and the probability that the gambler wins in each play is $p \leq \frac{1}{2}$.
  The total fortune is $N = a + b$.

- Even if the play is fair, i.e. $p = 0.5$, when $a = 100$ and $b = 1000$, we have

$$a_{100} = \frac{100}{100 + 1000} = 9.1\%. \quad \text{(attractive?)}$$

In the case of $p = 0.45$,

$$a_{100} = \frac{1 - \left(\frac{0.55}{0.45}\right)^{100}}{1 - \left(\frac{0.55}{0.45}\right)^{100+1000}} = 7.0766 \times 10^{-88}. \quad \text{(attractive???)}$$

(参考§1/4°的讨论)

- Then, the ruin probability of this gambler $\to 1$ when $b \to \infty$ (infinitely rich adversary).

2. Pagerank：页面排序问题

1° 问题描述与分析

- Suppose that you type "healthy food store" into a search engine. Each of the five web pages, A, B, C, D, E, contains the relevant information on the subject. Suppose that

  A has links to B and C,

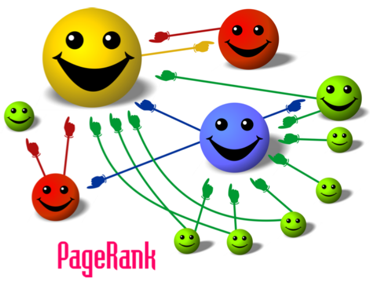
  B has links to A and D,

  C has link to D and E,

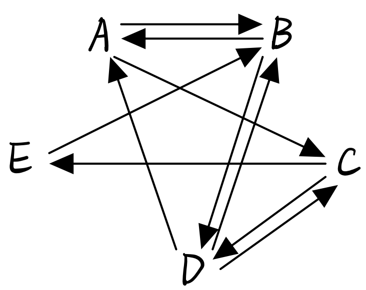
  D has link to A, B, and C,

  E has link to B. (根据访问概率/平均次数排序)

  Compute the "PageRank" of these five web pages.



2° 作出关系图



3° 求解 stationary probability / limiting probability (此处两者相等)

- $$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Using **Python** to compute P^{100}, one obtains the stationary distribution

$$0.215385 \quad 0.276923 \quad 0.184615 \quad 0.230769 \quad 0.092308.$$

Thus the PageRank of these five pages is

B, D, A, C, and E

with webpage B listed at the top.

3、Card shuffling : mixed time analysis

52张 cards (1, 2, ---, 52)

不停地洗牌，每个 state 均为 52张牌的 permutation，直到到达 stationary distribution

大致需要 $C\log_2 52 \approx 7$ 次洗牌