Lecture 13

§1 Angular momentum

1. Angular momentum（角动量）

1° 线动量：$\vec{p}=m\vec{v}$

角动量：$\vec{l}=\vec{r}\times\vec{p}=m\cdot(\vec{r}\times\vec{v})$

2° SI Unit：$kg\cdot m^2/s$（$J\cdot s$）

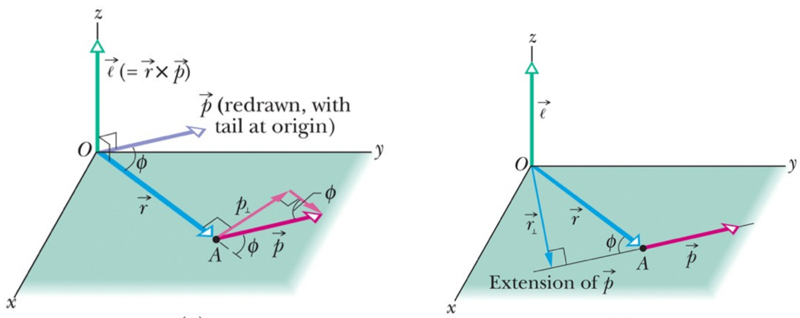
3° 方向：右手定则

4° 数值：$l=rmv\sin\phi=r_\perp mv=v_\perp\cdot mr$

ϕ为 $\vec{r}$ 与 $\vec{v}$ 间较小的夹角

5° 注意：① 只有在对于一个特定的点时，角动量才有意义

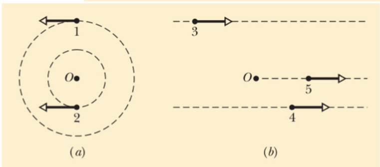
② 角动量永远垂直于矢径与线动量构成的平面



(a) (b)

例：

In part *a* of the figure, particles 1 and 2 move around point *O* in circles with radii 2 m and 4 m. In part *b*, particles 3 and 4 travel along straight lines at perpendicular distances of 4 m and 2 m from point *O*. Particle 5 moves directly away from *O*. All five particles have the same mass and the same constant speed. (a) Rank the particles according to the magnitudes of their angular momentum about point *O*, greatest first. (b) Which particles have negative angular momentum about point *O*?



Answer: (a) 1 & 3, 2 & 4, 5

(b) 2 and 3 (assuming counterclockwise is positive)

例：

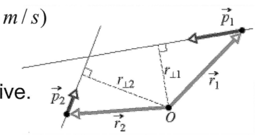
Problem

- Figure 12-12 shows an overhead view of two particles moving along horizontal paths. Particle 1, with momentum magnitude p_1 = 5.0 kg·m/s, has position vector $\vec{r}_1$ and will pass 2.0 m from point *O*. Particle 2, with momentum magnitude p_2 = 2.0 kg·m/s, has position vector $\vec{r}_2$ and will pass 4.0 m from point *O*.

  What is the net angular momentum $\vec{L}$ about point *O* of the two-particle system?

SOLUTION:

$$l_1=r_{\perp1}p_1=(2.0\,m)(5.0\,kg\cdot m/s)$$
$$=10\,\mathrm{kg\cdot m^2/s}$$



The RHR indicates that l_1 is positive.

$$l_1=+10\,kg\cdot m^2/s$$

$$l_2=r_{\perp2}p_2=(4.0\,m)(2.0\,kg\cdot m/s)$$
$$=8.0\,\mathrm{kg\cdot m^2/s}$$

The RHR indicates that l_2 is negative.

$$l_2=-8.0\,kg\cdot m^2/s$$
$$L=l_1+l_2=+10\,kg\cdot m^2/s+(-8.0\,kg\cdot m^2/s)$$
$$=+2.0\,\mathrm{kg\cdot m^2/s}$$

2、Newton's Second Law in angular form

对于 single particles:

- 平动:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

- Angular form:

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

- 所有力矩的向量和就等于该点角动量的变化率
- $\vec{\tau}_{net}$ 与 $d\vec{l}$ 必须定义相对于同一点.

证明:

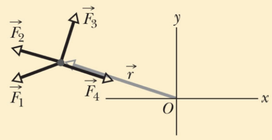
- $\vec{l} = m(\vec{r}\times\vec{v})$
- $\frac{d\vec{l}}{dt} = m(\vec{r}\times\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\times\vec{v})$

$$= m(\vec{r}\times\vec{a} + \vec{v}\times\vec{v})$$
$$= m(\vec{r}\times\vec{a})$$
$$= \vec{r}\times m\vec{a}$$
$$= \vec{r}\times\vec{F}_{net}$$
$$= \vec{\tau}_{net}$$

例:

Checkpoint 5

The figure shows the position vector $\vec{r}$ of a particle at a certain instant, and four choices for the direction of a force that is to accelerate the particle. All four choices lie in the xy plane. (a) Rank the choices according to the magnitude of the time rate of change ($d\vec{\ell}/dt$) they produce in the angular momentum of the particle about point O, greatest first. (b) Which choice results in a negative rate of change about O?



Answer: (a) F_3, F_1, F_2 & F_4 (b) F_3
(assuming counterclockwise is positive)

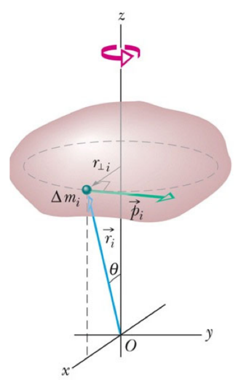
3、Angular momentum of a system of particles

1° $\vec{L} = \sum_{i=1}^{n} \vec{l}_i$

2° $\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$

4° 绕定轴转动的刚体的角动量

$$L = I\omega$$



- The magnitude of the angular momentum $\vec{l}_i$ with respect to O, is:

$$l_i = (r_i)(p_i)(sin90°) = \Delta m_i v_i r_i$$

- The z Component

$$l_{iz} = l_i sin\theta = \Delta m_i v_i r_i sin\theta = \Delta m_i v_i r_{\perp,i}$$

- For the continuous rigid body

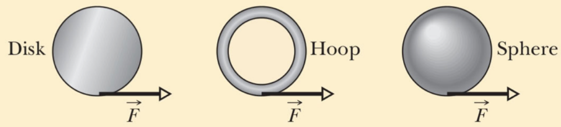
$$L_z = \sum_{i=1}^{n} l_{iz} = \sum_{i=1}^{n} \Delta m_i v_i r_{\perp,i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp,i}) r_{\perp,i} = \omega \sum_{i=1}^{n} \Delta m_i r_{\perp,i}^2$$
$$= I\omega$$

$L = I\omega$ (rigid body; Fixed axis)

例:

Checkpoint 6

In the figure, a disk, a hoop, and a solid sphere are made to spin about fixed central axes (like a top) by means of strings wrapped around them, with the strings producing the same constant tangential force $\vec{F}$ on all three objects. The three objects have the same mass and radius, and they are initially stationary. Rank the objects according to (a) their angular momentum about their central axes and (b) their angular speed, greatest first, when the strings have been pulled for a certain time t.



Answer: (a) All angular momenta will be the same, because the torque is the same in each case (b) sphere, disk, hoop

§2 Conservation of angular momentum

1. Law of conservation of angular momentum

1° 若作用于一个系统的合外力矩为0，则系统角动量保持不变

$\vec{L}_i = \vec{L}_f$ (closed, isolate system)

$I_i \omega_i = I_f \omega_f$

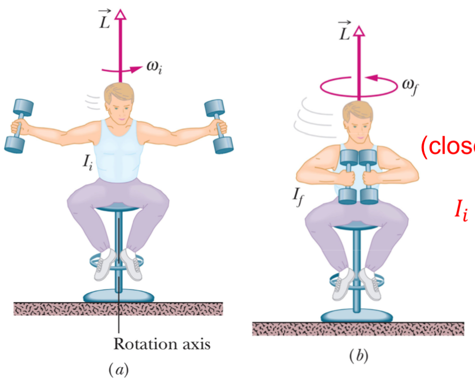
2° 若作用于系统特定轴上的合外力矩分量为0，则系统沿轴方向角动量保持不变

$\tau_{z,net} = \frac{dL_z}{dt} = 0$

L_z = constant

2. 应用

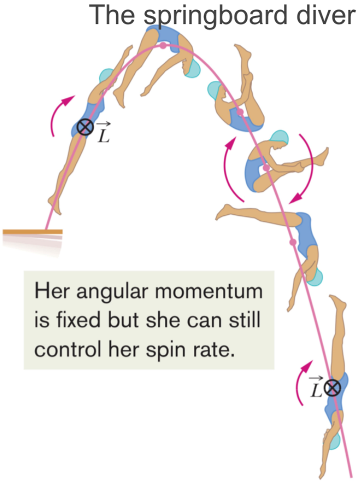
The spinning Volunteer



$\vec{L}_i = \vec{L}_f$
(closed, isolate system)
$I_i \omega_i = I_f \omega_f$
$I_i > I_f \rightarrow \omega_i < \omega_f$

15

The springboard diver



$\vec{L}_i = \vec{L}_f$
(closed, isolate system)

Which position has the largest and smallest angular speed?

- The position with smallest I → large angular speed
- Position six has a large I, therefore the diver's angular speed is small.

例:

Checkpoint 7

A rhinoceros beetle rides the rim of a small disk that rotates like a merry-go-round. If the beetle crawls toward the center of the disk, do the following (each relative to the central axis) increase, decrease, or remain the same for the beetle–disk system: (a) rotational inertia, (b) angular momentum, and (c) angular speed?

Answer: (a) decreases (b) remains the same (c) increases

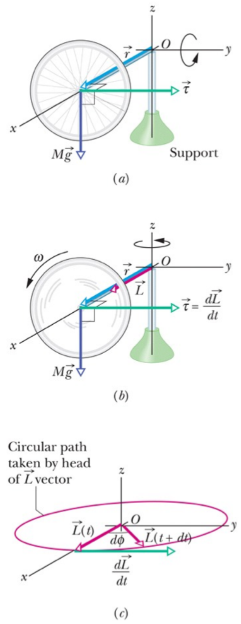
3、平动与滚动

Table 11-1 More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \frac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

[b]For systems of particles, including rigid bodies.
[c]For a rigid body about a fixed axis, with L being the component along that axis.
[d]For a closed, isolated system.

4. Precession (进动) of a gyroscope (陀螺仪)



- No spinning: The torque created by the gravitational force

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\tau = Mgr\sin 90° = Mgr$$

- Spinning: Precession, rotate horizontally about O

$$L = I\omega$$

$$dL = L\, d\emptyset$$

$$\tau = \frac{dL}{dt} = Mgr$$

$$dL = Mgr dt$$

$$d\emptyset = \frac{dL}{L} = \frac{Mgrdt}{I\omega}$$

Procession rate: $$\Omega = \frac{d\emptyset}{dt} = \frac{Mgr}{I\omega}$$

- 与 mass 无关
- 当转速足够快时, $\omega \gg \Omega$, gyroscope 将不会水平旋转
- 当 gyroscope 与水平面呈一定角度时, 结论同样适用 (垂直时除外)

Summary

- Rolling Smoothly

$$v_{com} = \frac{s}{\Delta t} = \frac{ds}{dt} = \frac{d\theta}{dt}R = \omega R$$

$$K = \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv_{com}^2$$

$$a_{com} = \frac{dv_{com}}{dt} = \frac{d\omega}{dt}R = \alpha R$$

$$a_{com,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2}$$

Summary

- Torque as a Vector

$$\vec{\tau} = \vec{r} \times \vec{F} \;\&\; \tau = rF\sin\emptyset$$

- Angular Momentum as a particle/ of a System of Particles

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

$$\vec{L} = \sum_{i=1}^{n} \vec{l}_i$$

- Newton's Second Law in Angular Form

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

Summary

- Angular Momentum of a Rigid Body

$$L = I\omega \text{ (rigid body; Fixed axis)}$$

- Conservation of Angular Momentum

$$\vec{L}_i = \vec{L}_f \text{ (closed, isolate system)}$$

$$I_i\omega_i = I_f\omega_f$$

- Precession of a Gyroscope

$$\Omega = \frac{d\emptyset}{dt} = \frac{Mgr}{I\omega}$$