Lecture 4

§1 Alternating series approximation

Suppose for a series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, we have

- $u_n>0$, $\forall n\geq 1$
- $\{u_n\}$ is nonincreasing
- $u_n \longrightarrow 0$ as $n\to\infty$

Then $\sum(-1)^{n+1}u_n = L$ for some $L\in\mathbb{R}$.

If we want to approximate L using the sum of the first K terms (S_k), then

- L is between S_k and S_{k+1}
- $\underbrace{|L-S_k|}_{\text{Error}} < \underbrace{u_{k+1}}_{\text{first unused term, in absolute value}}$
- $L-S_k$ has same sign as first unused term

e.g. Find an approximated value of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}$ with error less than 0.001.

Sol: · $u_n=\frac{1}{(n-1)!}$ is positive, nonincreasing.

For $n\geq 2$, $0<\frac{1}{(n-1)!}\leq\frac{1}{n-1}\to 0$, so $\lim_{n\to\infty}u_n=0$.

Hence series converge (by alternating series test)

· When $K=8$, $\frac{1}{(K-1)!}=\frac{1}{7!}=\frac{1}{5040}<\frac{1}{5000}=0.0002$.

· Take $S_7=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\cdots\cdots+\frac{1}{6!}=0.36805\cdots\cdots$ as an approximated value.

Fact: exact value is $0.367879\cdots\cdots=e^{-1}$

§2 Conditional Convergence (条件收敛)

1、定义

> Def: A series $\sum a_n$ is said to be convergent conditionally if $\sum a_n$ converges but $\sum|a_n|$ diverges.

e.g.

$$\sum_n(-1)^{n+1}\frac{1}{n^p}\begin{cases}\text{converges absolutly}, & \text{if } p>1\\ \text{converges conditionally}, & \text{if } 0<p\leq 1\\ \text{diverges}, & \text{if } p\leq 0\end{cases}$$

2、无穷项的 re-arrangements

- 级数 $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\cdots\cdots$ 条件收敛

  令 $L:=\sum_{n=3}^{\infty}(-1)^{n+1}\frac{1}{n}=\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\cdots$

  因为 $\frac{1}{3}-\frac{1}{4}$, $\frac{1}{5}-\frac{1}{6}$, $\cdots\cdots$, $\frac{1}{2k-1}-\frac{1}{2k}$, $\cdots\cdots$ 为正数，$L>0$.

- 按下式 rearraging $\sum_{n=3}^{\infty}(-1)^{n+1}\frac{1}{n}$ 的项

  $\frac{1}{3}-\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{8}-\frac{1}{10}+\frac{1}{7}-\frac{1}{12}-\frac{1}{14}+\cdots\cdots$

因为 $\frac{1}{3}-\frac{1}{4}-\frac{1}{6}<0$，$\frac{1}{5}-\frac{1}{8}-\frac{1}{10}<0$，$\frac{1}{2k+1}-\frac{1}{4k}-\frac{1}{4k+2}<0$，

所以 rearranged 级数 不收敛于 L，因为 $L>0$.

Fact：一个级数条件收敛，改变其部分项的顺序可能会改变它的值.

通过改变其部分项的顺序，它可以收敛于任意数.

Fact：一个级数绝对收敛，改变其部分项的顺序不会改变它的值.

§3 Power Series (幂级数)

1、定义

Definition

A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots.$$

$\{c_n\}$ sequence of real numbers

The number a above is called the center of the power series. In particular, a power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots.$$

注：幂级数可被看作一个有无穷阶的多项式，但它并不对于 $\forall x\in\mathbb{R}$ 都收敛.

e.g. Consider $f(x) := 1+x+x^2+x^3+\cdots\cdots = \sum_{n=0}^{\infty} x^n$

By geometric series,

$$f(x)=\begin{cases}\frac{1}{1-x}, & \text{if } |x|<1;\\ \text{D.N.E}, & \text{if } |x|\geq 1.\end{cases}$$

Whether $f(x)$ is defined depends on the value of x.

注：在 power series notation 中，$0^0:=1$

如 $f(x):=\sum_{n=0}^{\infty}x^n$，$f(0)=\sum_{n=0}^{\infty}0^n=0^0+0^1+0^2+\cdots\cdots=1$

任何幂级数 $\sum_{n=0}^{\infty}c_n(x-a)^n$ 在它的中心 a 处必收敛，因为 $\sum_{n=0}^{\infty}c_n(a-a)^n=c_0$

2、性质（阿贝尔定理）

Theorem If $\sum_{n=0}^{\infty}c_n(x-a)^n$ converges at $x=x_0$, then it converges absolutely for all x with $|x-a|<|x_0-a|$. If it diverges at $x=x_0$, then it diverges for all x with $|x-a|>|x_0-a|$.

证明：· Suppose series converges at $x=x_0$.

May assume $x_0\neq a$, so $|x_0-a|>0$.

Since $\lim_{n\to\infty}c_n(x_0-a)^n=0$, $\exists N$ s.t. $\forall n\geq N$

$|c_n(x_0-a)^n|<1 \Rightarrow |c_n|<\frac{1}{|x_0-a|^n}$

For any x with $|x-a|<|x_0-a|$, we have

$0\leq |c_n|\cdot|x-a|^n < \frac{|x-a|^n}{|x_0-a|^n} = \left|\frac{x-a}{x_0-a}\right|^n$

Since $\left|\frac{x-a}{x_0-a}\right| < 1$, $\sum_{n=0}^{\infty}\left|\frac{x-a}{x_0-a}\right|^n$ converges.

So $\sum_{n=0}^{\infty}|c_n||x-a|^n$ converges absolutely.

- Suppose $\sum c_n(x_0-a)^n$ diverges.

If $\exists x$ s.t. $|x-a| > |x_0-a|$ but $\sum c_n(x-a)^n$ converges,

then by the first part, $\sum c_n|x_0-a|^n$ would converge (absolutely)

This is a contradiction, so no such x can exist.

i.e. if $|x-a| > |x_0-a|$, then $\sum c_n(x-a)^n$ diverges.

3. Radius of convergence (收敛半径)

> **Theorem (Existence of the Radius of Convergence)**
>
> *For any power series $\sum c_n(x-a)^n$, one of the following three statements holds:*
>
> (i) *There exists a positive real number R such that the series converges absolutely for all x with $|x-a| < R$ but diverges for all x with $|x-a| > R$. The series may or may not converge for x with $|x-a| = R$.*
>
> (ii) *The series converges absolutely for all x ($R = \infty$).*
>
> (iii) *The series converges at $x = a$ and diverges elsewhere ($R = 0$).*

注：端点处 $(a \pm R)$ 的敛散性要分别判断

e.g. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $x \in (-1, 1]$

e.g. $\sum x^n$ converges for $x \in (-1, 1)$

4. 收敛半径/收敛域的求解

使用 Ratio test / Root test.

例：For what values of x do the following power series converges?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{n+1}\cdot\frac{n}{|x|^n} = |x|\frac{n}{n+1} \to |x| \text{ as } |n| \to \infty$$

By Ratio test, if $|x| < 1$, series converges (absolutely);
if $|x| > 1$, series diverges;
if $x = 1$, series converges;
if $x = -1$, series diverges.

(b) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty$$

By Ratio test, series always converges.

(c) $\sum_{n=0}^{\infty} n!\,x^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n} = (n+1)|x| \to \begin{cases} 0, & \text{if } x = 0 \\ \infty, & \text{if } x \neq 0 \end{cases}$$

By Ratio test, series only converges for $x = 0$

(d) 求 $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}x^{2n}$ 的收敛半径

$$\lim_{n\to\infty}\frac{\frac{(2(n+1))!}{((n+1)!)^2}x^{2n+2}}{\frac{(2n)!}{(n!)^2}x^{2n}} = 4|x|^2<1$$

$|x|<\frac{1}{2}$，$R=\frac{1}{2}$

(e) 求 $\sum_{n=1}^{\infty}\frac{(x-1)^n}{2^n\cdot n}$ 的收敛域

令 $t=x-1$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{2}=\rho$，$R=\frac{1}{\rho}=2$

$t=-2$，$\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n}\cdot\frac{1}{n}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ 收敛

$t=2$，$\sum_{n=1}^{\infty}\frac{2^n}{2^n}\cdot\frac{1}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$ 发散

$t\in[-2,2)$ 收敛

$x\in[-1,3)$ 收敛