STAT201B Lecture 5-6 Sufficiency

1 Sufficiency

Logic

在对 θ 进行 inference 的时候, 我们希望仅将 data 中相关的 information 分离出来, 即在不损失信息的情况下将 data 压缩成 $T(X)$, 这样的好处在于:

1. 提升 computational efficiency
2. 降低 storage requirements
3. 包含 irrelevant information 可能会增加 estimator 的 risk (见 Rao-Blackwell Theorem)
4. 提升数据的 scientific interpretability

1.1 Definition: Sufficient Statistic

Logic

关于 Sufficient Statistics 的更多论述, 见 STA3020 Lecture 4, 包括:

- Rank statistics 和 order statistics 的性质
- Sufficient statistics 的存在性
- Sufficient statistics 的一个充分条件:

$$T(x) = T(x') \implies \frac{f(x|\theta)}{f(x'|\theta)} \text{ is invariant over } \theta, \forall x, x' \in \Omega$$

- One-to-one mapping 保证 sufficiency

令

1. X 的分布来自于 $\mathcal{P} = \{P_\theta : \theta \in \Omega\}$ (一个与 θ 相关的分布族),
2. Statistic T 的 range 为 $\mathcal{T}$

则 statistic T 被称为 **sufficient**,

若 **对于任意 $t \in \mathcal{T}$, conditional distribution $P_\theta(X|T(X) = t)$ 与 θ independent**

Example

令 $X_i \overset{i.i.d.}{\sim} Ber(\theta), i = 1, \ldots, n$, 则 $T = \sum_{i=1}^n X_i$ 是 θ 的 sufficient statistic:

由于

$$\begin{aligned} P_\theta(X|T(X) = t) &= \frac{P_\theta(X_1, \ldots, X_n, T(X) = t)}{P_\theta(T(X) = t)} \\ &= \begin{cases} 0 & \text{if } t \neq \sum_{i=1}^n X_i \\ \frac{P_\theta(X_1 = x_1, \ldots, X_n = x_n)}{P_\theta(\sum_{i=1}^n x_i = t)} = \frac{\prod_{i=1}^n \theta^{x_i}(1-\theta)^{x_i}}{\binom{n}{t}\theta^t(1-\theta)^t} = \frac{\theta^t(1-\theta)^t}{\binom{n}{t}\theta^t(1-\theta)^t} = \frac{1}{\binom{n}{t}} & \text{if } t = \sum_{i=1}^n X_i \end{cases} \end{aligned}$$

与 θ independent, $\forall t \in \mathcal{T}$, T 为 θ 的 sufficient statistic

1.2 Theorem: Neyman Factorization Theorem

Logic

使用定义求解 sufficient statistics 较为繁琐, 可以使用 Neyman Factorization Theorem 快速求解

令 distribution family $\{P_\theta : \theta \in \Omega\}$ 有 joint mass / density $\{p(x;\theta) : \theta \in \Omega\}$

则

$$T \text{ is sufficient for } \theta \iff \exists \text{ functions } h \text{ and } g, \text{ such that } p(x;\theta) = h(x) \cdot g(T(x), \theta)$$

Proof (仅考虑 discrete case)

Proof. (of Theorem.1.10: Neyman-Fisher Factorization Theorem) We prove the FactorizationTheorem for discrete random variables as an illustration, the general proof follow the same line.

- Suppose T is sufficient. We want to prove the right hand side of (1.2). Let $t = T(x)$. The joint pmf. of X is

X=x 时的 realization

X=x 时一定有T=t

$$f(x|\theta) = \mathbb{P}_\theta(X = x) = \mathbb{P}_\theta(\{X = x\} \bigcap \{T = t\})$$
$$= \underbrace{\mathbb{P}_\theta(X = x|T = t)}_{\text{由 sufficiency: } \perp \theta}\mathbb{P}_\theta(T = t) =: h(x)g(t, \theta).$$

- Suppose the right hand side of (1.2) holds. We want to prove T is sufficient. Apparently, when $T(x) \neq t$, we have $\mathbb{P}_\theta(X = x|T = t) = 0$ by definition, which is invariant over θ. Meanwhile, when $T(x) = t$. Let $S = \{x' \in \mathcal{X}_n : T(x') = t\}$. Then

$$\mathbb{P}(X = x|T = t) = \frac{\mathbb{P}_\theta(\{X = x\} \bigcap \{T = t\})}{\mathbb{P}_\theta(T = t)} = \frac{\mathbb{P}_\theta(\{X = x\})}{\mathbb{P}_\theta(T = t)}$$
$$= \boxed{\frac{\mathbb{P}_\theta(\{X = x\})}{\{\sum_{x' \in S} \mathbb{P}_\theta(X = x')\}}} = \frac{g(t, \theta)h(x)}{\{\sum_{x' \in S} g(t, \theta)h(x')\}} = \frac{h(x)}{\{\sum_{x' \in S} h(x')\}}$$

which is invariant over θ, and conclude that T is sufficient by definition.

Example

令 $Y_i \overset{i.i.d.}{\sim} Uniform(0, \theta), i = 1, \dots, n$
证明: $T = Y_{(n)}$ 为 θ 的 sufficient statistic

$$\begin{aligned}
P_\theta(Y) &= \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) \mathbf{1}(0 < Y_i < \theta) \\
&= \left(\frac{1}{\theta}\right)^n \left[\prod_{i=1}^{n} \mathbf{1}(0 < Y_i < \theta)\right] \\
&= \left(\frac{1}{\theta}\right)^n \cdot \mathbf{1}(Y_{(1)} > 0) \cdot \mathbf{1}(Y_{(n)} < \theta) \\
&= \mathbf{1}(Y_{(1)} > 0) \cdot \left[\left(\frac{1}{\theta}\right)^n \cdot \mathbf{1}(Y_{(n)} < \theta)\right]
\end{aligned}$$

其中 $\mathbf{1}(Y_{(1)} > 0)$ 可以被视作 $h(y)$, $(\frac{1}{\theta})^n \cdot \mathbf{1}(Y_{(n)} < \theta)$ 可以被视作 $g(T(y), \theta)$

Example

例 1: $X_1, \cdots, X_n \overset{i.i.d}{\sim} \text{Poisson}(\lambda)$，求 λ 的 sufficient statistic

$$f(x|\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \cdot 1\{x_i \in N\} \right)$$

$$= \underbrace{\left(\prod_{i=1}^{n} \frac{1}{x_i!} \cdot 1\{x_i \in N\} \right)}_{h(x)} \cdot \underbrace{\left(e^{-n\lambda} \cdot \lambda^{n\bar{x}} \right)}_{g(t,\lambda)}$$

$t = \bar{x}$, i.e. $T(X) = \bar{X}$

☰ **Example**

例 2: $X_1, \cdots, X_n \overset{i.i.d}{\sim} \text{Bernoulli}(p)$，其中 $p \in (0,1)$，求 λ 的 sufficient statistic

$$f(x|p) = \prod_{i=1}^{n} \left(p^{x_i} (1-p)^{1-x_i} \cdot 1_{\{x_i \in \{0,1\}\}} \right)$$

$$= \underbrace{\left(\frac{p}{1-p}\right)^{n\bar{x}} \cdot (1-p)^n}_{g(t,\theta)} \cdot \underbrace{\left[\prod_{i=1}^{n} 1_{\{x_i \in \{0,1\}\}} \right]}_{h(x)}$$

$t = \bar{x}$, i.e. $T(X) = \bar{X}$

☰ **Example**

例 3: $X_1, \cdots, X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$，其中 $\mu \in R$，$\sigma^2 \in R^+$，求下述情况下 θ 的 sufficient statistic

① μ 未知，σ^2 已知，$\theta = \mu$

$$f(x|\theta) = \prod_{i=1}^{n} \left\{ (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \right\}$$

$$= \underbrace{\left[\frac{\exp\left(-\frac{\sum x_i^2}{2\sigma^2}\right)}{(2\pi\sigma^2)^{n/2}} \right]}_{h(x)} \cdot \underbrace{\exp\left\{ -\frac{n\mu^2 - 2n\mu\bar{x}}{2\sigma^2} \right\}}_{g(t,\theta)}$$

$t = \bar{x}$, i.e. $T(X) = \bar{X}$

② μ 已知，σ^2 未知，$\theta = \sigma^2$

$$f(x|\theta) = \prod_{i=1}^{n} \left\{ (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \right\}$$

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{n/2}} \cdot \exp\left\{ -\frac{n \cdot \left(\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}\right)}{2\sigma^2} \right\}}_{g(t,\theta)}$$

$t = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$, i.e. $T(X) = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$

③ μ，σ^2 未知，$\theta = (\mu, \sigma^2)$

$$f(x|\theta) = \prod_{i=1}^{n} \left\{ (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \right\}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \left[\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2 \right] \right\}$$

$$= \underbrace{(2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \left[(n-1)\cdot s^2 + n(\bar{x}-\mu)^2 \right] \right\}}_{g(t,\theta)} \quad \left(s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1} \right)$$

$T(X) = (S^2, \bar{X})$

1.3 Theorem: Rao-Blackwell Theorem

Logic

关于 loss, risk, admissibility, Rao-Blackwell Theorem 的更多描述, 见 STA3020 Lecture 5, 包括:

- Estimand, estimator, estimate 的区别
- Bias 和 unbiasedness 的定义
- Loss function 和 risk function 的定义
- Admissibility 的定义:
  - 一个 estimator δ 被称为 **inadmissible**, 若存在另一个 estimator δ' dominates δ, 即:

$$\exists \delta' \quad s.t. \quad \begin{cases} R(\theta,\delta') \leq R(\theta,\delta) & \text{for all } \theta \in \Theta \\ R(\theta,\delta') < R(\theta,\delta) & \text{for some } \theta \in \Theta \end{cases}$$

  - 一个 estimator δ 被称为 **admissible**, 若上述 estimator 不存在
- Strictly convex loss function 下 admissible estimator 的

令:

- X 为分布为 $P_\theta \in \mathcal{P} = \{P_\theta, \theta \in \mathbb{R}\}$ 的随机变量
- $\delta(X)$ 为 θ 的 (任意) 一个 estimator

若:

- $T(X)$ 为 θ 的一个 **sufficient** statistic
- Loss function $\mathcal{L}(\theta, \delta(X))$ 为关于 $\delta(X)$ 的 **strictly convex** function (如 L_2 loss)
- $\delta(X)$ 有 finite expectation 与 risk，即 $R(\theta, \delta(X)) = \mathbb{E}[\mathcal{L}(\theta, \delta(X))] < \infty$

则:

- 若定义 $\eta(t) = E_\theta[\delta(X)|T = t], \forall t$ (即 $\delta(X)$ 在 T 下的条件期望，是一个关于 T 的函数)
- 则 **estimator $\eta(T) = E_\theta[\delta(X)|T(X)]$** 满足:

$$R(\theta, \eta) < R(\theta, \delta)$$

  除非 $\delta(X) = \eta(T)$ with probability 1

Remark

1. 若 strictly convex 被替换为 convex, 则 $<$ 被替换为 $\leq$, 但是若去除 convexity assumption, 则定理不成立
2. 此处 T 被要求为 sufficient statistic, 这主要是为了**确保 $\eta(T)$ independent with θ** (因此可以被视作一个 estimator)
3. Rao-Blackwell theorem 的实际意义在于: 我们**可以通过 conditioning on sufficient statistics 来优化现有的 estimator**

Proof

由 Jensen's inequality, 有

$$\begin{aligned} \mathcal{L}(\theta, \eta(t)) &= \mathcal{L}(\theta, \mathbb{E}[\delta(X)|T(X) = t]) \\ &\leq \mathbb{E}[\mathcal{L}(\theta, \delta|T(X) = t)] \end{aligned}$$

当且仅当 $\theta = \eta(t)$ with probability 1 时取等;
对两侧取期望, 有:

$$R(\theta, \eta) = \mathbb{E}[\mathcal{L}(\theta, \eta)] \leq \mathbb{E}[\mathbb{E}[\mathcal{L}(\theta, \delta(X)|T)]] = R(\theta, \delta)$$

Example: 对 MSE 的优化

考虑 estimator δ 和 L2 loss $\mathcal{L}(\theta, \delta) = (\theta - \delta)^2$, 则经过 Rao-Blackwell 优化过的 estimator $\eta = \mathbb{E}_\theta[\delta|T]$ 满足:

$$
\begin{aligned}
R(\theta,\eta) &= \mathbb{E}_\theta[\mathcal{L}(\theta,\eta)] \\
&= \mathbb{E}_\theta[(\theta-\mathbb{E}_\theta[\delta|T])^2] \\
&= \mathbb{E}_\theta[(\mathbb{E}_\theta[\theta-\delta|T])^2] \quad (\text{conditional on } \theta, \text{可以将 } \theta \text{ 视作常数放入期望}) \\
&\le \mathbb{E}_\theta[\mathbb{E}_\theta[(\theta-\delta)^2|T]] \quad (\text{Jensen's inequality}) \\
&= \mathbb{E}_\theta[(\theta-\delta)^2] \\
&= R(\theta,\delta)
\end{aligned}
$$

1.4 Jensen's Inequality

令:

1. $(\Omega,\mathcal{F},\mathbb{P})$ 为一个 probability space
2. $X:\Omega\to R$ 为一个 integrable random variable (即 $\mathbb{E}[|X|]<\infty$)

若:

1. $\varphi:\mathbb{R}\to\mathbb{R}$ 为 **convex** function
2. $\varphi:\mathbb{R}\to\mathbb{R}$ 为 integrable

则:

$$\varphi(\mathbb{E}[X])\le\mathbb{E}[\varphi(X)]$$

Remark

若 φ 为 strictly convex, 则当且仅当 X is almost surely constant 时取等

Example

$(\mathbb{E}[X])^2\le\mathbb{E}[X^2]$

1.5 Minimal Sufficiency

Logic

关于 minimal sufficiency 的更多论述, 见 STA3020 Lecture 4, 包括:

- minimal sufficient 的定义
- Lehmann-Scheffé theorem
- Bahadur's theorem
- Exponential family 的 minimal sufficient statistic

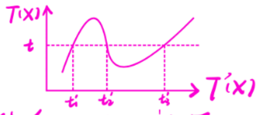
令 $T(X)$ 为一个 sufficient statistic
若对于**任意**其他 sufficient statistic $S(X)$, **$T(X)$ 为 a function of $S(X)$**, 即 $T=f(S)$ for some f
则 $T(X)$ 为 **minimal sufficient**

Remark

- $T=f(S)$ 表示了两件事:
  - 关于 S 的 knowledge implies 关于 T 的 knowledge
  - T 提供了 greater reduction of data, 除非 f 为 one-to-one

注: 换言之, $T'(x)=T'(y) \Rightarrow T(x)=T(y)$

$T(x)=T(y) \not\Rightarrow T'(x)=T'(y)$



也就是说, 不同的 $\underbrace{\text{values}}_{(t_i,\, t_i',\, t_i'')}$ of $T'(x)$ 所包含的信息, 可以被 $T(x)$ 的一个 $\underbrace{\text{value}}_{t}$ 所包含.

- Minimal sufficient statistic 仍然不是 unique 的 (通过 one-to-one mapping preserve)

1.6 Theorem: Lehmann-Scheffé theorem

令:

- $X_1, \dots, X_n \overset{i.i.d.}{\sim} f(\cdot|\theta)$, 其中 $\theta \in \Theta$
- $T = T(X)$ 为一个 statistics

若:

$$T(x) = T(x') \iff \frac{f(x|\theta)}{f(x'|\theta)} \text{ is invariant over } \theta, \forall x, x' \in \Omega$$

则 T 为 **minimal sufficient** for θ

Remark

换言之, 当且仅当 $T(X)$ 的 value 变化时, likelihood function 中关于 θ 的信息才会变

1.7 Definition: Exponential family

Logic

关于 exponential family 的更多论述, 见 STA3020 Lecture 3, 包括:

- Exponential family, parameter space, canonical form, curved exponential family 的定义
- Exponential family 的例子
- Exponential family 的性质

d-parameter exponential family:

一个 **d-parameter exponential family** 的 pmf/pdf 满足以下形式:

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{\theta}) &= h(\boldsymbol{x}) \cdot c(\boldsymbol{\theta}) \cdot exp\left[\sum_{i=1}^{d} \eta_i(\boldsymbol{\theta})\, T_i(\boldsymbol{x})\right] \\ &= h(\boldsymbol{x}) \cdot exp\left[\sum_{i=1}^{d} \eta_i(\boldsymbol{\theta}) T_i(\boldsymbol{x}) \ - A(\boldsymbol{\theta})\right] \end{aligned}$$

其中,

- $h(\boldsymbol{x}) \geq 0$
- $c(\boldsymbol{\theta}) \geq 0$
- $T_1(\boldsymbol{x}), \dots, T_d(\boldsymbol{x})$ 为关于 $\boldsymbol{x} = (x_1, \dots, x_n)$ 的 real value functions, 且不取决于 $\boldsymbol{\theta}$
- $\eta_1(\boldsymbol{\theta}), \dots, \eta_d(\boldsymbol{\theta})$ 为关于 $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)$ 的 real value functions, 且不取决于 $\boldsymbol{x}$

full rank exponential family:

若:

- $\boldsymbol{\eta}(\Theta) = \{\eta_1(\boldsymbol{\theta}), \dots, \eta_d(\boldsymbol{\theta})\}$ 在 $\mathbb{R}^d$ 中有 non-empty interior

- $T_1(\boldsymbol{x}), \ldots, T_d(\boldsymbol{x})$ 为 linearly independent

则该 exponential family 被称为 **full rank**

Remark

若 $\eta(\Theta)$ 仅包含 $\mathbb{R}^s$ $(s < d)$ 中的 open set, 则该 exponential family 被称为 curved exponential family with dimension s

Example

$\mathcal{N}(\mu, \mu), \mu > 0$ 构成一个 curved exponential family

1.8 Theorem: Exponential family 的 minimal sufficient statistic

若: $X = \{X_1, \ldots, X_n\}$ 的分布来自 **full rank exponential family**
则: $T = (T_1, \ldots, T_d)$ 为 minimal sufficient statistics

Example

问题:
令 $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$, 求 μ 和 σ^2 的 minimal sufficient statistic

解答:
对于 Normal random variables, 我们可以做以下变形:

$$\begin{aligned} f_{\mu,\sigma^2}(x_1, \ldots, x_n) &= \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n \prod_{i=1}^n exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n exp\left\{-\frac{\sum_{i=1}^n x_i^2 - 2\mu\sum_{i=1}^n x_i + \mu^2}{2\sigma^2} - n\, ln\sigma\right\} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2 - \frac{\mu}{2\sigma^2}\sum_{i=1}^n x_i + \frac{n\mu^2}{2\sigma^2} - n\, ln\sigma\right\} \end{aligned}$$

其中:

- $h(\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n$
- $\eta_1(\boldsymbol{\theta}) = -\frac{1}{2\sigma^2}$
- $T_1(\boldsymbol{x}) = \sum_{i=1}^n x_i^2$
- $\eta_2(\boldsymbol{\theta}) = -\frac{\mu}{2\sigma^2}$
- $T_2(\boldsymbol{x}) = \sum_{i=1}^n x_i$
- $A(\boldsymbol{\theta}) = \frac{n\mu^2}{2\sigma^2} - n\, ln\sigma$

因此 $(T_1(\boldsymbol{x}), T_2(\boldsymbol{x})) = (\sum_{i=1}^n x_i^2, \sum_{i=1}^n x_i)$ 为 minimal sufficient statistics