

# Notes on rigidity and pinnings of algebraic groups

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August 7, 2023

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devoted to functions like the Weierstrass function which are continuous but differentiable at no points or just a finite set of points.

**Exercise 1.2.1.** Give an example of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  so that the first and second derivatives  $f', f''$  exist and are continuous, but  $f''$  is not differentiable (at at least one point). In other words,  $f$  is twice differentiable but not three times differentiable.

In parallel to the discussion above, morphisms of algebraic groups are much more rigid than abstract homomorphisms. So we do not, in general, expect an abstract homomorphism to be regular. However, sometimes we can prove that an abstract homomorphism is “almost” regular, in a particular sense. These are called “rigidity theorems.”

We want to prove rigidity theorems of the following form: for a particular fixed algebraic group  $G(k)$ , any abstract homomorphism  $\rho : G(k) \rightarrow \mathrm{GL}_m(K)$  must be “almost regular.” In this context, “almost regular” means a factorization of  $\rho$ , a way of writing  $\rho$  as a composition of two maps, where one of those is regular. The usual goal is to factor  $\rho$  into the form  $\rho = \sigma \circ F$  where  $\sigma$  is regular.

$$\begin{array}{ccc} G(k) & \xrightarrow{\rho} & \mathrm{GL}_m(K) \\ & \searrow F & \nearrow \sigma \\ & G(A) & \end{array}$$

To emphasize the rigidity of  $\sigma$ , the diagram above uses squiggly arrows for abstract group homomorphisms, and a straight arrow for the more rigid regular group homomorphism. Philosophically speaking, we think of  $\sigma$  as the “regular part” of  $\rho$ .

### 1.3 What is a pinning?

Algebraic groups have a lot of structure. We can “pin” them to understand and see the structure better. The word “pinning” here comes from the metaphor of pinning the wings of a butterfly specimen, as you might see in a museum. A pinning of an algebraic group  $G(k)$  is a collection of group homomorphisms

$$X_\alpha : V_\alpha(k) \rightarrow G(k)$$

where  $\alpha$  ranges over the set of “roots”  $\Phi$  of  $G$ , and  $V_\alpha(k)$  is just a vector space over  $k$  (often the dimension of  $V_\alpha(k)$  is just one, so in that case  $V_\alpha(k)$  is essentially just  $k$  with the group operation of addition). These pinning maps  $X_\alpha$  should satisfy a variety of properties.

(P1) They are group homomorphisms<sup>2</sup>, meaning that

$$X_\alpha(v + w) = X_\alpha(v) \cdot X_\alpha(w)$$

for all  $v, w \in V_\alpha(k)$ .

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<sup>2</sup>Usually they are group homomorphisms. In some more complicated situations, we get a slightly different version of this equation with an additional factor on one side, so in those situations they aren’t quite homomorphisms.

- (P2) The group  $G(k)$  is generated by the images  $X_\alpha(v)$ , where  $\alpha$  ranges over  $\Phi$  and  $v$  ranges over  $V_\alpha(k)$ .
- (P3)  $G(k)$  contains a subgroup  $T(k)$  called a “maximal  $k$ -split torus” and for any  $t \in T(k)$ , conjugating  $X_\alpha(v)$  by  $t$  corresponds to scalar multiplication of the input vector  $v$  by a scalar called  $\alpha(t)$ .

$$t \cdot X_\alpha(v) \cdot t^{-1} = X_\alpha(\alpha(t)v)$$

- (P4) There is a “Weyl group”  $W$  and for any root  $\alpha \in \Phi$ , there is a corresponding Weyl group element  $w_\alpha$ . Also, for any root  $\beta \in \Phi$ , conjugating  $X_\beta(v)$  by  $w_\alpha$  corresponds to changing the root  $\beta$  by some kind of reflection process to get a new root  $\sigma_\alpha(\beta)$ , and doing some invertible linear transformation  $\phi$  to  $v$ .

$$w_\alpha \cdot X_\beta(v) \cdot w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}(\phi(v))$$

Ideally,  $\phi$  is not very complicated, something like  $\phi(v) = -v$ .

- (P5) Given two roots  $\alpha$  and  $\beta$  with  $\alpha + \beta \neq 0$ , the commutator

$$[X_\alpha(v), X_\beta(w)] = \prod_{\gamma} X_\gamma(N(u, v))$$

is some product of other pinning map outputs  $X_\gamma$  and the inputs of those are determined by some function  $N$  (a polynomial function of  $u$  and  $v$ ). This product is determined by the root system  $\Phi$ . Often  $N(u, v)$  is something like

$$N(u, v) = \pm uv$$

or

$$N(u, v) = c_1 u_1 v_1 + c_2 u_2 v_2 + \cdots + c_n u_n v_n$$

where  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$  and the  $c_i$  are coefficients coming from the definition of the group  $G$ .

Next up are some exercises to get a better hands on understanding of some aspects of the pinning definition. For all of the exercises below, fix a field  $k$ .

**Exercise 1.3.1.** Let  $G$  be a group of matrices, and let  $X : k \rightarrow G$  be a group homomorphism from  $(k, +)$  to  $G$ . That is,

$$X(u + v) = X(u) \cdot X(v)$$

where  $+$  on the left means addition in  $k$ , and  $\cdot$  on the right means matrix multiplication. Show that  $X(0)$  is the identity matrix. (Combined with property (P1), what does this say about pinning maps?)

**Exercise 1.3.2.** Show that any matrix of the form

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$$

with  $a, b, c \in k$  can be written as a product of matrices of the form

$$\begin{pmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & v \\ & 1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & 1 & w \\ & & 1 \end{pmatrix}$$

Be careful: It's not just as simple as the product

$$\begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$$

(This exercise is illustrating how  $\mathrm{SL}_3(k)$  is generated by the images of its pinning maps, a la property (P2). We will study this pinning in detail in section 2.)

**Exercise 1.3.3.** Let  $t_1, t_2, t_3 \in k^\times$  be nonzero elements of  $k$ . Let  $t = \mathrm{diag}(t_1, t_2, t_3)$  be an arbitrary  $3 \times 3$  diagonal matrix with entries in  $k^\times$ . Let

$$X = X_{\alpha_{12}}(u) = \begin{pmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Verify by direct calculation that

$$t \cdot X \cdot t^{-1} = X_{\alpha_{12}}(t_1 t_2^{-1} u) = \begin{pmatrix} 1 & t_1 t_2^{-1} u & \\ & 1 & \\ & & 1 \end{pmatrix}$$

(This is just one example of (P3) for  $\mathrm{SL}_3(k)$ .)

**Exercise 1.3.4.** Let  $X$  and  $Y$  be the matrices below.

$$X = X_{\alpha_{12}}(u) = \begin{pmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{pmatrix} \quad Y = X_{\alpha_{23}}(v) = \begin{pmatrix} 1 & & v \\ & 1 & \\ & & 1 \end{pmatrix}$$

Verify that the commutator  $[X, Y]$  has the form

$$X_{\alpha_{13}}(N(u, v)) = \begin{pmatrix} 1 & & N(u, v) \\ & 1 & \\ & & 1 \end{pmatrix}$$

and determine a formula for  $N(u, v)$ , thinking of  $N$  as a function  $N : k \times k \rightarrow k$ . (This is an example of (P4) for  $\mathrm{SL}_3(k)$ .)

## 2 Special linear groups

Fix a field  $k$ . Whenever convenient, we can assume that the characteristic of  $k$  is zero. This assumption is mostly not necessary, but it simplifies some things. The group of units of  $k$  is denoted  $k^\times$  (in other words,  $k^\times$  is the set of nonzero elements in  $k$ ).

**Definition 2.0.1.** Fix a field  $k$  and a positive integer  $n$ .  $M_n(k)$  is the set of  $n \times n$  matrices with entries in  $k$ . For  $x \in M_n(k)$ , the **determinant** of  $x$  is an element of  $k$ . For  $2 \times 2$  matrices, the usual formula applies.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

For  $3 \times 3$  matrices, we can use the  $2 \times 2$  formula and a cofactor expansion to calculate the determinant.

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

This process generalizes to larger  $n \times n$  matrices. To calculate the determinant, we expand along the top row. We should not actually have to do this by hand at any point. The point being made here is just that determinant of an  $n \times n$  matrix is defined over any field  $k$ , because calculating a determinant only involves addition and multiplication operations in the field  $k$ .

**Definition 2.0.2.** Fix a positive integer  $n$ . The **special linear group** with coefficients in  $k$  is the group of  $n \times n$  matrices with determinant 1.

$$\mathrm{SL}_n(k) = \{X \in M_n(k) : \det X = 1\}$$

$\mathrm{SL}_n(k)$  is an algebraic group under matrix multiplication. The condition  $\det(X) = 1$  may not initially appear to be a polynomial equation, but if you expand out the determinant, it is just a big polynomial in terms of the coefficients of the matrix  $X$ .

### 2.1 Torus subgroup and character group

**Definition 2.1.1.** The **diagonal subgroup** or **torus subgroup** in  $\mathrm{SL}_n(k)$  is the subgroup of diagonal matrices. We will call it  $T(k)$ .

$$T(k) = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & \ddots \\ & & & t_n \end{pmatrix} : t_i \in k^\times, \det t = t_1 t_2 \cdots t_n = 1 \right\}$$

Notice that the equation  $t_1 t_2 \cdots t_n = 1$  means that none of the entries  $t_i$  can be zero for an element of  $T(k)$ . For an element of  $T(k)$ , the inverse is the matrix whose diagonal entries are the term-by-term inverses.

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}^{-1} = \begin{pmatrix} t_1^{-1} & & \\ & \ddots & \\ & & t_n^{-1} \end{pmatrix}$$

**Definition 2.1.2.** Let  $T(k)$  be the diagonal subgroup of  $\mathrm{SL}_n(k)$  as above. For  $i = 1, \dots, n$ , define a map  $\alpha_i : T(k) \rightarrow k^\times$  by

$$\alpha_i \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} = t_i$$

Since the determinant of an element of  $T(k)$  is one, none of the individual diagonal entries  $t_i$  can be zero, so  $t_i \in k^\times$ . More importantly,  $\alpha_i$  is a group homomorphism, meaning  $\alpha_i(ts) = \alpha_i(t)\alpha_i(s)$  for any  $s, t \in T(k)$ .

**Definition 2.1.3.** The **character group** of  $T(k)$  is the set of group homomorphisms from  $T(k)$  to  $k^\times$ . It is called  $X(T(k))$  or just  $X(T)$ .

$$X(T) = \mathrm{Hom}(T(k), k^\times) = \{ \alpha : T(k) \rightarrow k^\times \mid \alpha \text{ is a group homomorphism} \}$$

Group homomorphisms  $T(k) \rightarrow k^\times$  are called **characters** (of  $T$ ). The **sum** of two characters  $\alpha$  and  $\beta$  is defined to be

$$\alpha + \beta : T(k) \rightarrow k^\times \quad (\alpha + \beta)(t) = \alpha(t)\beta(t)$$

where the multiplication on the right side is just multiplication in  $k$ . The set of characters  $X(T)$  is an abelian group under this character sum operation (you checked this on the application last fall). In particular, the identity of  $X(T)$  is the constant function

$$0_{X(T)} : T(k) \rightarrow k^\times \quad t \mapsto 1$$

We call this function the **zero character**, and usually just denote it by 0. (You can check that  $\alpha + 0_{X(T)} = \alpha$  for any  $\alpha \in X(T)$ .)

**Definition 2.1.4.** Let  $T(k)$  and  $\alpha_i$  be as above. For  $1 \leq i, j \leq n$  and  $i \neq j$ , define  $\alpha_{ij} = \alpha_i - \alpha_j$ . In other words,  $\alpha_{ij}$  is a group homomorphism  $T(k) \rightarrow k^\times$  (that is,  $\alpha_{ij}$  is a character of  $T$ ), and it is described by

$$\alpha_{ij}(t) = \alpha_i(t)\alpha_j(t)^{-1} \quad \alpha_{ij} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = t_i t_j^{-1}$$

**Remark 2.1.5.** In  $T(k)$ , we have the equation  $\det(t) = 1$ . Since the determinant of a diagonal matrix is the product of the diagonal entries, we get the equation

$$1 = \det(t) = t_1 \cdots t_n = \alpha_1(t) \cdots \alpha_n(t)$$

That is,

$$\alpha_1(t) \cdots \alpha_n(t) = 1 = 0_{X(T)}(t)$$



where  $0_{X(T)} \in X(T)$  is the zero character. In other words, the sum

$$\alpha_1 + \cdots + \alpha_n \in X(T)$$

is the zero character. Translating this into an equation in  $X(T)$ , we get the following.

$$\alpha_1 + \cdots + \alpha_n = 0_{X(T)} = 0$$

We emphasize: this is not an equation in  $k$ , it is an equation in the abelian group  $X(T)$ .

**Definition 2.1.6.** For  $n \geq 1$ , define  $\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  ( $n$  copies of  $\mathbb{Z}$ ).

**Lemma 2.1.7.** Let  $\mathbb{Z}\vec{1}$  be the set of integer multiples of the vector  $\vec{1} = (1, \dots, 1)$ .

$$\mathbb{Z}\vec{1} = \{m(1, \dots, 1) = (m, \dots, m) : m \in \mathbb{Z}\}$$

Then  $X(T) \cong \mathbb{Z}^n / \mathbb{Z}\vec{1}$  (isomorphism of abelian groups)

*Proof.* Let  $e_i \in \mathbb{Z}^n$  be the  $n$ -tuple  $(0, \dots, 0, 1, 0, \dots, 0)$  which is all zeros except for a one in the  $i$ th position. Then consider the map

$$\phi : \mathbb{Z}^n \rightarrow X(T) \quad \phi \left( \sum_{i=1}^n m_i e_i \right) = \sum_{i=1}^n m_i \alpha_i$$

where the  $m_i$  are integers. Then  $\phi$  is a group homomorphism,  $\ker \phi = \mathbb{Z}\vec{1}$ , and  $\phi$  is onto, so by the first isomorphism theorem,  $\mathbb{Z}^n / \ker \phi = \mathbb{Z}^n / \mathbb{Z}\vec{1} \cong X(T)$ .  $\square$

The following elements correspond under the above isomorphism.

$$\begin{aligned} e_i &\longleftrightarrow \alpha_i \\ e_i - e_j &\longleftrightarrow \alpha_{ij} = \alpha_i - \alpha_j \\ (1, -1, 0, 0, \dots, 0) &\longleftrightarrow \alpha_{12} = \alpha_1 - \alpha_2 \\ (0, 1, -1, 0, \dots, 0) &\longleftrightarrow \alpha_{23} = \alpha_2 - \alpha_3 \\ (0, 0, 1, -1, \dots, 0) &\longleftrightarrow \alpha_{34} = \alpha_3 - \alpha_4 \end{aligned}$$

This is somewhat misleading – more literally, on the left side of this correspondence I mean that the coset  $\bar{e}_i = e_i + \mathbb{Z}\vec{1}$  corresponds to the character  $\alpha_i$ , and so on. This becomes apparent when we consider the corresponding elements

$$\vec{1} = (1, \dots, 1) \longleftrightarrow \alpha_1 + \cdots + \alpha_n$$

because  $\alpha_1 + \cdots + \alpha_n = 0$  in  $X(T)$ , but  $\vec{1}$  is not the zero element in  $\mathbb{Z}^n$ . However,  $\vec{1}$  does represent the zero coset in the quotient  $\mathbb{Z}^n / \mathbb{Z}\vec{1}$ ; that is,  $\vec{1} + \mathbb{Z}\vec{1} = \mathbb{Z}\vec{1} = 0$  in  $\mathbb{Z}^n / \mathbb{Z}\vec{1}$ .

## 2.2 Pinning the special linear group

Let  $k$  be a field of characteristic zero, and consider the special linear group  $\mathrm{SL}_n(k)$ .

**Definition 2.2.1.** Fix  $i, j$  with  $1 \leq i, j \leq n$  and  $i \neq j$ . For  $x \in k$ , let  $E_{ij}(x)$  be the matrix with  $x$  in the  $ij$  entry and 0 elsewhere. Let  $e_{ij}(x) = I + E_{ij}(x)$  be the matrix with  $x$  in the  $ij$  entry, 1's on the diagonal, and 0 elsewhere. For example, in  $\mathrm{SL}_2(k)$  we have

$$E_{12}(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \quad e_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

The determinant of  $e_{ij}(x)$  is one, so  $e_{ij}(x) \in \mathrm{SL}_n(k)$ . Now, consider  $e_{ij}$  as a group homomorphism from  $k$  (viewed as an abelian group under addition) to  $\mathrm{SL}_n(k)$ .

$$e_{ij} : k \rightarrow \mathrm{SL}_n(k) \quad x \mapsto e_{ij}(x)$$

This being a group homomorphism means that

$$e_{ij}(x + y) = e_{ij}(x) \cdot e_{ij}(y)$$

Since  $e_{ij}$  is a homomorphism, it must take the identity to the identity.

$$e_{ij}(0) = I = 1 \in \mathrm{SL}_n(k)$$

Also notice that  $\ker(e_{ij}) = \{0\}$ , so  $e_{ij}$  is injective.

**Definition 2.2.2.** Instead of  $e_{ij}$ , we will use the notation  $X_{\alpha_{ij}}$  for the same thing. That is,  $X_{\alpha_{ij}}(u) = e_{ij}(u)$  for  $u \in k$ .

**Remark 2.2.3.** It is useful to remember that  $X_{\alpha_{ij}}(u)^{-1} = X_{\alpha_{ij}}(-u)$ . This is a consequence of the fact that  $X_{\alpha_{ij}}$  is a group homomorphism.

**Definition 2.2.4.** The **root system** of  $\mathrm{SL}_n(k)$  is the set of characters  $\alpha_{ij}$  for  $i \neq j$ . We denote it by  $\Phi(\mathrm{SL}_n(k), T(k))$  or  $\Phi(\mathrm{SL}_n, T)$  just  $\Phi$ .

$$\Phi = \{\alpha_{ij} : 1 \leq i, j \leq n, i \neq j\}$$

Elements of  $\Phi$  are called **roots**. It is important not to forget that elements of  $\Phi$  are characters, i.e. functions from  $T(k)$  to  $k^\times$ . Specifically,  $\alpha_{ij}$  is the function

$$\alpha_{ij} : T(k) \rightarrow k^\times \quad \alpha_{ij}(t) = \alpha_i(t)\alpha_j(t)^{-1} = t_i t_j^{-1}$$

where  $t = \mathrm{diag}(t_1, \dots, t_n)$ .

### 2.2.1 Conjugation by torus

$T(k)$  acts on  $\mathrm{SL}_n(k)$  by conjugation. That is, there is a group action

$$T(k) \times \mathrm{SL}_n(k) \rightarrow \mathrm{SL}_n(k) \quad t * x = txt^{-1}$$

**Exercise 2.2.5.** Verify that this is a group action, by showing that

- For any  $t \in T(k)$  and  $x \in \mathrm{SL}_n(k)$ , the conjugation  $t * x = txt^{-1}$  is in  $\mathrm{SL}_n(k)$ .
- If  $I$  is the identity matrix and  $x \in \mathrm{SL}_n(k)$ , then  $I * x = IxI^{-1} = x$ .
- For any two elements  $s, t \in T(k)$ , we have  $(ts) * x = t * (s * x)$ .

**Lemma 2.2.6.** Let  $1 \leq i, j \leq n$  and  $i \neq j$ . Let  $t \in T(k)$  be a diagonal matrix, and let  $u \in k$ . Then

$$t * X_{\alpha_{ij}}(u) = t \cdot X_{\alpha_{ij}}(u) \cdot t^{-1} = X_{\alpha_{ij}}(\alpha_{ij}(t)u)$$

In other words, for  $\alpha \in \Phi(\mathrm{SL}_n(k), T(k))$ , we have

$$t * X_{\alpha}(u) = t \cdot X_{\alpha}(u) \cdot t^{-1} = X_{\alpha}(\alpha(t)u)$$

**Exercise 2.2.7.** In the group  $\mathrm{SL}_2(k)$ , confirm by direct matrix calculation that for any  $t = \mathrm{diag}(t_1, t_2)$  and  $u \in k$ , that we have

$$t \cdot X_{\alpha_{12}}(u) \cdot t^{-1} = X_{\alpha_{12}}(\alpha_{12}(t)u) = X_{\alpha_{12}}(t_1 t_2^{-1}u)$$

That is,

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & t_1 t_2^{-1}u \\ & 1 \end{pmatrix}$$

Then convince yourself that nothing really changes when you do the “same” calculation in  $\mathrm{SL}_n(k)$ , replacing  $t$  with  $\mathrm{diag}(t_1, t_2, \dots, t_n)$ .

## 2.2.2 Conjugation by Weyl group

**Definition 2.2.8.** Let  $\alpha \in \Phi(\mathrm{SL}_n(k), T(k))$  be a root, so  $\alpha = \alpha_{ij}$  for some  $i \neq j$ . Let  $u \in k$ , and assume  $u \neq 0$ . Then define

$$w_{\alpha}(u) = X_{\alpha}(u) \cdot X_{-\alpha}(u^{-1}) \cdot X_{\alpha}(u)$$

Here,  $-\alpha$  means the negative of  $\alpha$  in the abelian group  $X(T)$ . Concretely, if  $\alpha = \alpha_{ij} = \alpha_i - \alpha_j$ , then  $-\alpha = \alpha_{ji} = \alpha_j - \alpha_i$ . The elements  $w_{\alpha}(u)$  are called **Weyl group elements**. (There is a larger “Weyl group” including the  $w_{\alpha}(u)$ , but we omit the details for now.)

**Example 2.2.9.** In the group  $\mathrm{SL}_3(k)$ , with  $\alpha = \alpha_{12} = \alpha_1 - \alpha_2$ , where  $X_{\alpha}(u) = e_{12}(u)$ , that we get

$$w_{\alpha}(1) = w_{\alpha_{12}}(1) = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

**Exercise 2.2.10.** Work out  $w_{\alpha_{12}}(u)$  in more generality (still in  $\mathrm{SL}_3(k)$ ), by doing concrete matrix multiplication with  $e_{12}(u)$  matrices.

**Definition 2.2.11.** A **monomial matrix** is a matrix with exactly one nonzero entry in each row and each column. A **permutation matrix** is a monomial matrix where all nonzero entries are 1. A **signed permutation matrix** is a monomial matrix where all nonzero entries are 1 or  $-1$ .

**Remark 2.2.12.** In general, for  $\alpha \in \Phi(\mathrm{SL}_n(k), T(k))$ , the Weyl group elements  $w_\alpha(u)$  are monomial matrices, and  $w_\alpha(1)$  is a signed permutation matrix.

**Exercise 2.2.13.** Show that for  $\alpha \in \Phi(\mathrm{SL}_n(k), T(k))$  and  $u \in k^\times$ ,  $w_\alpha(u)^{-1} = w_\alpha(-u)$ .

**Definition 2.2.14.** Consider the space  $\mathbb{R}^n$ , and denote the dot product of two vectors  $v, w \in \mathbb{R}^n$  by  $(v, w)$ . Given  $v \in \mathbb{R}^n$ , the **hyperplane perpendicular to  $v$**  is the set of all vectors perpendicular to  $v$ .

$$P_v = \{w \in \mathbb{R}^n : (v, w) = 0\}$$

The **reflection across  $P_v$**  is the (linear) map

$$\sigma_v : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \sigma_v(w) = w - \frac{2(v, w)}{(v, v)}v$$

Geometrically, this is actually a mirror reflection across  $P_v$ . Now consider  $\Phi = \Phi(\mathrm{SL}_n(k), T(k))$ . Identify  $\Phi$  with a subset of  $\mathbb{R}^n$  by identifying  $\alpha_i$  with the standard basis vector  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$  with the 1 in the  $i$ th position. Then  $\alpha_{ij} = \alpha_i - \alpha_j$  is identified with

$$\alpha_{ij} \leftrightarrow (0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$$

with the 1 in the  $i$ th position and  $-1$  in the  $j$ th position. Under this identification, the dot product operation on  $\mathbb{R}^n$  becomes

$$(\alpha_i, \alpha_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and then

$$\begin{aligned} (\alpha_{ij}, \alpha_{kl}) &= (\alpha_i - \alpha_j, \alpha_k - \alpha_l) \\ &= (\alpha_i, \alpha_k) - (\alpha_i, \alpha_l) - (\alpha_j, \alpha_k) + (\alpha_j, \alpha_l) \\ &= \delta_{ik} - \delta_{il} - \delta_{jk} + \delta_{jl} \\ &\in \{-2, -1, 0, 1, 2\} \end{aligned}$$

Now, for  $\alpha \in \Phi$ , consider  $\alpha$  as an element of  $\mathbb{R}^n$  under the identification above. Then we have a hyperplane  $P_\alpha$  of vectors perpendicular to  $\alpha$ , and we can again use the reflection  $\sigma_\alpha$  across  $P_\alpha$ .

$$\sigma_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \sigma_\alpha(w) = w - \frac{2(\alpha, w)}{(\alpha, \alpha)}\alpha$$

**Lemma 2.2.15.** Let  $\alpha \in \Phi = \Phi(\mathrm{SL}_n(k), T(k))$ , and let  $\sigma_\alpha$  be the reflection above. If  $\beta \in \Phi$ , then  $\sigma_\alpha(\beta) \in \Phi$ . In other words,  $\sigma_\alpha$  can be thought of as a map  $\Phi \rightarrow \Phi$ .

$$\sigma_\alpha : \Phi \rightarrow \Phi \quad \sigma_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

**Example 2.2.16.** When  $n = 3$ , we have

$$\Phi = \Phi(\mathrm{SL}_n, T) = \{\alpha_{12}, \alpha_{21}, \alpha_{13}, \alpha_{31}, \alpha_{23}, \alpha_{32}\}$$

We identify  $\Phi$  with a subset of  $\mathbb{R}^3$ , the set of vectors

$$\tilde{\Phi} = \{(1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1)\}$$

These vectors are all perpendicular to the vector  $(1, 1, 1)$ . In other words,  $\tilde{\Phi}$  sits inside a 2D plane, so we can visualize  $\tilde{\Phi}$  in two dimensions. When we do, it looks like six vectors pointing at the six vertices of a regular hexagon.

Try to convince yourself that if you take any two of those vectors  $v$  and  $w$ , take  $P_v$  to be the line perpendicular to  $v$ , and reflect  $w$  across  $v$ , then that reflection  $\sigma_v(w)$  is another one of the six vectors in  $\Phi$ .

**Proposition 2.2.17.** *Let  $\alpha, \beta \in \Phi(\mathrm{SL}_n(k), T(k))$  be roots. Then*

$$w_\alpha(1) \cdot X_\beta(u) \cdot w_\alpha(1)^{-1} = X_{\sigma_\alpha(\beta)}(\pm u)$$

*Proof.* We're going to set up a computer to check these calculations explicitly. Also, we would like a more precise statement which tells us when you get a minus sign.  $\square$

### 2.2.3 Commutator formula

**Definition 2.2.18.** Let  $G$  be any group, and let  $x, y \in G$ . The **commutator** of  $x$  and  $y$  is

$$[x, y] = xyx^{-1}y^{-1}$$

**Exercise 2.2.19.** Show that  $x$  and  $y$  commute (i.e.  $xy = yx$ ) if and only if  $[x, y] = 1$  (here 1 is the identity element of the group  $G$ ).

**Exercise 2.2.20.** Let  $u, v \in k$ . In the group  $\mathrm{SL}_n(k)$ , compute the commutator

$$[X_{\alpha_{12}}(u), X_{\alpha_{23}}(v)]$$

**Proposition 2.2.21.** *Let  $\Phi = \Phi(\mathrm{SL}_n(k), T(k))$ , and let  $\alpha, \beta \in \Phi$  so that  $\alpha \neq \pm\beta$ . Let  $u, v \in k$ . Then*

$$[X_\alpha(u), X_\beta(v)] = \begin{cases} I & \alpha + \beta \notin \Phi \\ X_{\alpha+\beta}(\pm uv) & \alpha + \beta \in \Phi \end{cases}$$

*That is,  $X_\alpha(u)$  and  $X_\beta(v)$  commute if and only if  $\alpha + \beta \notin \Phi$  (or if  $uv = 0$ ).*

**Remark 2.2.22.** The signs in the previous proposition can be described more precisely, as follows. Let  $c_{\alpha\beta}$  be the sign, so

$$[X_\alpha(u), X_\beta(v)] = \begin{cases} I & \alpha + \beta \notin \Phi \\ X_{\alpha+\beta}(c_{\alpha\beta}uv) & \alpha + \beta \in \Phi \end{cases}$$

If  $\alpha, \beta \in \Phi$  and  $\alpha + \beta \in \Phi$ , then it must be that  $\alpha = \alpha_{ij}$  and  $\beta = \alpha_{k\ell}$  for some indices  $i, j, k, \ell$  where the set  $\{i, j, k, \ell\}$  contains exactly three distinct elements. There are four possibilities.

Case	$c_{\alpha\beta}$
$i = k$	1
$i = \ell$	-1
$j = k$	1
$j = \ell$	-1

You can check cases of this by hand by just calculating a few commutators, such as

$$\left[ X_{\alpha_{12}}(u), X_{\alpha_{31}}(v) \right] = X_{\alpha_{32}}(-uv)$$

which falls in the case  $i = \ell$ .

### 3 Background material

This section includes a large amount of technical background for algebraic groups. In section 3.1, we discuss bilinear, hermitian and skew-hermitian, and quadratic forms, and in particular their Witt indices. In section 3.2, we finally give a technical definition of algebraic groups, and in particular the special orthogonal and special unitary groups of interest for this project. In section 3.3, we discuss Lie algebras and in particular the Lie algebras associated with algebraic groups, as well as roots and root spaces inside Lie algebras. In section 3.4, we discuss abstract root systems. In section 3.5, we give a more precise definition of pinning.

Finally, in section 3.6, we discuss the paper [KS09], which applies the idea of pinnings to prove some rigidity statements about the special linear group (technically, they study the elementary linear group but it is basically the same most of the time). This paper is the beginning of the idea of using pinning structures to prove rigidity theorems.

#### 3.1 Bilinear, hermitian, and quadratic forms

This section gives an overview of bilinear, quadratic, and hermitian forms and their associated matrices. Our objective is just to study them enough to understand what the Witt index of a form is. An excellent alternate reference on bilinear and quadratic forms is the following notes by Keith Conrad <https://kconrad.math.uconn.edu/blurbs/linmultialg/bilinearform.pdf>.

##### 3.1.1 Bilinear forms

Before discussing bilinear forms in general, we look at the classical example of a bilinear form: dot product on  $\mathbb{R}^n$ .

**Example 3.1.1.** Let  $n \geq 1$ , and consider the vector space  $V = \mathbb{R}^n$  (as a vector space of dimension  $n$  over  $\mathbb{R}$ , viewed as column vectors). The usual dot product operation is a map

$$\begin{aligned} b : V \times V &\rightarrow \mathbb{R} \\ b(x, y) &= x \cdot y \\ &= b\left((x_1, \dots, x_n), (y_1, \dots, y_n)\right) \\ &= (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) \\ &= x_1 y_1 + \dots + x_n y_n \\ &= \sum_{i=1}^n x_i y_i \end{aligned}$$

The map  $b$  has the following properties: for any  $x, y, z \in V$ , and any  $s, t \in \mathbb{R}$ ,

$$\begin{aligned} b(x, y + z) &= b(x, y) + b(x, z) \\ b(x + y, z) &= b(x, z) + b(y, z) \\ b(sx, ty) &= (st)b(x, y) \end{aligned}$$

These properties are summarized by saying that  $b$  is a **bilinear map**, or by saying that  $b$  is **linear in each entry**. Another way to think about  $b$  is the following. Given  $x, y \in \mathbb{R}^n$ , view them as column vectors, and then

$$b(x, y) = x^T \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} y$$

where  $x^T$  means the transpose (so turning  $x$  from a column vector into a row vector).

A bilinear form (on a vector space over a field  $k$ ) is a generalization of these properties of the dot product.

**Definition 3.1.2.** Fix a field  $k$ . A **vector space over  $k$**  is an abelian group  $V$  (where we write the operation as  $+$ ) together with a map

$$k \times V \rightarrow V$$

with the properties that for any  $a, b \in k$  and  $u, v \in V$  we have

$$0v = 0$$

$$1v = v$$

$$a(bv) = (ab)v$$

$$a(v + w) = av + aw$$

$$(a + b)v = av + bv$$

**Example 3.1.3.** Fix a positive integer  $n$ . Then  $V = k^n$  is a vector space over  $k$ , where vector addition is component-wise,

$$(v_1, v_2, \dots, v_n) + (u_1, u_2, \dots, u_n) = (v_1 + u_1, v_2 + u_2, \dots, v_n + u_n)$$

and scalar multiplication is also component-wise.

$$a(v_1, \dots, v_n) = (av_1, \dots, av_n)$$

**Definition 3.1.4.** The **dimension** of a vector space  $V$  over  $k$  is the size of any basis (a linearly independent spanning set). We will only encounter finite dimensional vector spaces. It is a theorem that the size of any two bases are the same, so the dimension is well-defined.

The dimension of  $k^n$  is  $n$ . Essentially, any two vector spaces of the same dimension are “the same,” meaning that they are isomorphic as vector spaces, so every finite dimensional vector space over  $k$  is isomorphic to  $k^n$  for some  $n$ .

**Definition 3.1.5.** Let  $k$  be a field, and let  $V$  be a vector space of (finite) dimension  $n$  over  $k$  (so essentially you can think of  $V$  as  $k^n$ ). A **bilinear form** on  $V$  is a map

$$b : V \times V \rightarrow k$$

such that for any  $u, v, w \in V$  and  $\lambda, \mu \in k$ , we have

$$b(u, v + w) = b(u, v) + b(u, w)$$

$$b(u + v, w) = b(u, w) + b(v, w)$$

$$b(\lambda v, \mu w) = \lambda \mu b(v, w)$$



**Definition 3.1.6.** A bilinear form is **symmetric** if  $b(v, w) = b(w, v)$  for all  $v, w \in V$ .

**Example 3.1.7.** The dot product on  $\mathbb{R}^n$  is a symmetric bilinear form.

**Definition 3.1.8.** Let  $b : V \times V \rightarrow k$  be a bilinear form, and fix an ordered basis  $\beta = \{e_1, \dots, e_n\}$  of  $V$ . The **matrix of  $b$  with respect to the basis  $\beta$**  is the matrix  $B = \text{Mat}_\beta(b)$  whose  $ij$ th entry is  $b(e_i, e_j)$ .

$$B = \text{Mat}_\beta(b) = \left( b(e_i, e_j) \right)$$

**Definition 3.1.9.** The **Kronecker delta function** is defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

**Example 3.1.10.** Let  $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  be the usual dot product, and let  $e_i \in \mathbb{R}^n$  be the usual basis vector with 1 in the  $i$ th entry and zeros elsewhere. Then  $\beta = \{e_1, \dots, e_n\}$  is called the **standard basis** of  $\mathbb{R}^n$ . Note that

$$b(e_i, e_j) = \delta_{ij}$$

and the matrix of  $b$  with respect to  $\beta$  is

$$B = \text{Mat}_\beta(b) = \left( b(e_i, e_j) \right) = (\delta_{ij})$$

which is just the identity matrix.

**Definition 3.1.11.** Let  $B \in M_n(k)$  be an  $n \times n$  matrix. Let  $V$  be a vector space over  $k$  of dimension  $n$ , and fix an ordered basis  $\beta = \{e_1, \dots, e_n\}$  of  $V$ . For any  $v \in V$ , we can write it uniquely in terms of this basis.

$$v = \sum_{i=1}^n v_i e_i$$

Let  $[v]_\beta$  be the column vector  $(v_1, \dots, v_n) \in k^n$ . In particular,  $[e_i]_\beta$  is the vector with zeros except for a 1 in the  $i$ th entry. The **bilinear form associated to  $B$  (with respect to the basis  $\beta$ )** is the bilinear form  $b_{\beta, B} : V \times V \rightarrow k$  defined by

$$b_{\beta, B}(v, w) = [v]_\beta^T B [w]_\beta$$

Notice that taking  $v = e_i$  and  $w = e_j$ , that  $b_{\beta, B}(e_i, e_j)$  is the  $ij$ th entry of the matrix  $B$ .

$$b_{\beta, B}(e_i, e_j) = [e_i]_\beta^T B [e_j]_\beta = B_{ij}$$

**Lemma 3.1.12.** Let  $k$  be a field, let  $V$  be a vector space of dimension  $n$  over  $k$ , and fix an ordered basis  $\beta$  of  $V$ . Then there is a one-to-one correspondence between bilinear forms and elements of  $M_n(k)$ .

$$\begin{aligned} \{\text{bilinear forms}\} &\leftrightarrow M_n(k) \\ b &\leftrightarrow \text{Mat}_\beta(b) \\ b_{\beta, B} &\leftrightarrow B \end{aligned}$$

**Exercise 3.1.13.** Show that the matrix of a symmetric bilinear form is a symmetric matrix (satisfies  $B^T = B$ ).

**Definition 3.1.14.** A matrix  $B \in M_n(k)$  is **nondegenerate** if it has nonzero determinant (aka is an element of  $GL_n(k)$ ). A bilinear form  $b$  is **nondegenerate** if the associated matrix is nondegenerate.

**Definition 3.1.15.** Let  $b, b'$  be bilinear forms on a  $k$ -vector space  $V$  with respective associated matrices  $B$  and  $B'$ . Then  $b, b'$  are **equivalent** if there exists an invertible matrix  $P$  such that

$$B' = P^T B P$$

**Exercise 3.1.16.** Prove that equivalence of bilinear forms is an equivalence relation.

**Definition 3.1.17.** Let  $b$  be a bilinear form on  $V$  and fix a basis of  $V$ . A bilinear form is **diagonal** if its associated matrix  $B$  is diagonal. A bilinear form is **diagonalizable** if it is equivalent to a diagonal bilinear form.

**Theorem 3.1.18.** *As long as the characteristic is not 2, every nondegenerate symmetric bilinear form is diagonalizable. That is, there always exists a basis in which the matrix of  $b$  is a diagonal matrix.*

*Proof.* See <https://kconrad.math.uconn.edu/blurbs/linmultialg/bilinearform.pdf>. □

### 3.1.2 Hermitian forms

**Example 3.1.19.** Let  $n$  be a positive integer. We view  $\mathbb{C}^n$  as a vector space over  $\mathbb{C}$ , thinking of elements as column vectors  $(z_1, \dots, z_n)$  with  $z_i \in \mathbb{C}$ . The **complex dot product** is the following operation.

$$h : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$$

$$h(w, z) = h(w_1, \dots, w_n), (z_1, \dots, z_n) = w_1 \bar{z}_1 + \dots + w_n \bar{z}_n = \sum_{i=1}^n w_i \bar{z}_i$$

**Exercise 3.1.20.** Verify that the complex dot product  $h$  has the following properties. For any  $u, v, w \in \mathbb{C}^n$  and  $\lambda, \mu \in \mathbb{C}$ ,

$$h(u, v + w) = h(u, v) + h(u, w)$$

$$h(u + v, w) = h(u, w) + h(v, w)$$

$$h(\lambda u, v) = \lambda h(u, v)$$

$$h(u, \mu v) = \bar{\mu} h(u, v)$$

$$h(u, v) = \overline{h(v, u)}$$

In other words,  $h$  is almost a symmetric bilinear form, except for two differences. First, when you pull a scalar out of the second entry, you have to take the complex conjugate. Second,  $h$  is not symmetric, but “conjugate-symmetric.”

**Definition 3.1.21.** Let  $V$  be a vector space over  $\mathbb{C}$ . A **hermitian form** or **sesquilinear form** on  $V$  is a map

$$h : V \times V \rightarrow \mathbb{C}$$

such that for any  $u, v, w \in V$  and  $\lambda, \mu \in \mathbb{C}$ ,

$$h(u, v + w) = h(u, v) + h(u, w)$$

$$h(u + v, w) = h(u, w) + h(v, w)$$

$$h(\lambda u, v) = \lambda h(u, v)$$

$$h(u, \mu v) = \bar{\mu} h(u, v)$$

$$h(u, v) = \overline{h(v, u)}$$

The second-to-last equation is often phrased by saying that “ $h$  is conjugate-linear in the second entry.” The last equation can be thought of as saying that  $h$  is “conjugate-symmetric.”

**Example 3.1.22.** Complex dot product is a  $\sigma$ -hermitian form on  $\mathbb{C}^n$ , where  $\sigma$  is traditional complex conjugation.

**Definition 3.1.23.** A **skew-hermitian form** or **anti-hermitian form** is nearly the same as a hermitian form, except that  $h(u, v) = \overline{h(v, u)}$  is replaced by the condition  $h(u, v) = -\overline{h(v, u)}$ .

In order to generalize hermitian forms to arbitrary fields, we need to discuss quadratic field extensions.

**Definition 3.1.24.** A **field extension**  $L/k$  is a pair of fields  $L$  and  $k$  where  $k$  is a subfield of  $L$ .

**Example 3.1.25.** The following are examples of field extensions.

$$\mathbb{C}/\mathbb{R} \quad \mathbb{R}/\mathbb{Q} \quad \mathbb{Q}(\sqrt{2})/\mathbb{Q} \quad \mathbb{Q}(\sqrt{3})/\mathbb{Q}$$

**Definition 3.1.26.** Let  $L/k$  be a field extension. We can view  $L$  as a vector space over  $k$ . The **degree** of the field extension is the dimension of  $L$  as a  $k$ -vector space.

**Example 3.1.27.** The degree of the field extension  $\mathbb{C}/\mathbb{R}$  is 2, because the dimension is 2. A basis is  $\{1, i\}$ .

**Definition 3.1.28.** Let  $k$  be a field. Let  $d \in k$  be a nonzero element, such that there is no element  $a \in k$  so that  $a^2 = d$  (that is,  $d$  does not already have a “square root” in  $k$ ). Then we can define a new field by

$$k(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in k\}$$

Addition in  $k(\sqrt{d})$  is described by

$$(a_1 + b_1\sqrt{d}) + (a_2 + b_2\sqrt{d}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{d}$$

and multiplication is

$$(a_1 + b_1\sqrt{d})(a_2 + b_2\sqrt{d}) = (a_1a_2 + b_1b_2d) + (a_1b_2 + a_2b_1)\sqrt{d}$$

One way to think about  $k(\sqrt{d})$  is that we are taking  $k$  and adding in solutions to the quadratic equation  $x^2 - d = 0$ . The way to make this precise is to quotient the polynomial ring  $k[x]$  by the principal ideal generated by  $x^2 - d$ .

$$k(\sqrt{d}) \cong k[x]/\langle x^2 - d \rangle$$

**Example 3.1.29.**  $\mathbb{C} = \mathbb{R}(\sqrt{-1})$

**Example 3.1.30.** For any integer  $d$  which is not the square of another integer, we can construct the field  $\mathbb{Q}(\sqrt{d})$  and get a field which is strictly bigger than  $\mathbb{Q}$ . If  $d > 0$ , then we can think of  $\mathbb{Q}(\sqrt{2})$  as a subfield of  $\mathbb{R}$ , but if  $d < 0$  then we can just think of  $\mathbb{Q}(\sqrt{d})$  as a subfield of  $\mathbb{C}$ .

**Exercise 3.1.31.** Show that  $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{18})$ .

**Lemma 3.1.32.** For any field  $k$  and any  $d \in k$  such that  $d \neq a^2$  for any  $a \in k$ , the field extension  $k(\sqrt{d})/k$  has degree 2.

*Proof.* The set  $\{1, \sqrt{d}\}$  is a basis, because every element of  $k(\sqrt{d})$  can be uniquely written in the form  $a + b\sqrt{d}$  where  $a, b \in k$ .  $\square$

**Definition 3.1.33.** A **quadratic field extension** is a field extension of the form  $k(\sqrt{d})/k$  for some field  $k$ .

**Lemma 3.1.34.** If  $k$  is a field and  $\text{char}(k) \neq 2$ , then every field extension  $L/k$  of degree 2 is a quadratic extension.

Since we will generally assume that  $\text{char } k = 0$  or at least  $\text{char } k \neq 2$ , for us “quadratic extension” and “extension of degree 2” are effectively synonymous.

**Definition 3.1.35.** For a matrix  $X$  with entries in  $L$ , the **conjugate matrix**, denoted  $\overline{X}$ , is the matrix formed by applying conjugation to each individual entry of  $X$ . The transpose matrix is denoted  $X^t$  or  $X^T$ . The **conjugate transpose** matrix  $\overline{X^T} = \overline{X}^T$  is denoted  $X^*$  for short. (Other common notations include  $X^\dagger$  and  $X^H$ .)

**Definition 3.1.36.** Let  $L$  be a field. An **involution** on  $L$  is a field automorphism  $\sigma : L \rightarrow L$  such that  $\sigma^2 = \text{Id}_L$  (that is,  $\sigma(\sigma(x)) = x$  for every  $x \in L$ ). Being a field automorphism is the same as being a ring isomorphism from  $L$  to itself. In other words,  $\sigma$  is a bijection and

$$\sigma(x + y) = \sigma(x) + \sigma(y) \quad \sigma(xy) = \sigma(x)\sigma(y)$$

for all  $x, y \in L$ . Usually when we have an involution on a field, we don’t write it as  $\sigma(x)$ , instead we just write  $\overline{x}$  (because the traditional example is complex conjugation on  $\mathbb{C}$ ).

**Exercise 3.1.37.** Let  $L/k$  be a quadratic field extension, so  $L = k(\sqrt{d})$  for some non-square  $d \in k$ . Verify that

$$L \rightarrow L \quad a + b\sqrt{d} \mapsto \overline{a + b\sqrt{d}} = a - b\sqrt{d} \quad a, b \in k$$

is an involution on  $L$ , and that  $\bar{x} = x$  for any  $x \in k$ .

The next definition generalizes hermitian forms to vector spaces over fields other than  $\mathbb{C}$ .

**Definition 3.1.38.** Let  $L$  be a field with an involution  $\sigma$ , which we write as  $\sigma(x) = \bar{x}$ . Let  $V$  be a vector space over  $L$ . A  **$\sigma$ -hermitian form** or just **hermitian form** on  $V$  is a map

$$h : V \times V \rightarrow L$$

such that for any  $u, v, w \in V$  and  $\lambda, \mu \in L$ ,

$$\begin{aligned} h(u, v + w) &= h(u, v) + h(u, w) \\ h(u + v, w) &= h(u, w) + h(v, w) \\ h(\lambda u, v) &= \lambda h(u, v) \\ h(u, \mu v) &= \bar{\mu} h(u, v) \\ h(u, v) &= \overline{h(v, u)} \end{aligned}$$

Essentially, a hermitian form is a symmetric bilinear form except for two changes: bringing out scalars from the second entry requires conjugation, and instead of being symmetric it is “conjugate-symmetric.”

**Definition 3.1.39.** As in the complex case, a **skew- $\sigma$ -hermitian form** is the same as a  $\sigma$ -hermitian form, except that instead of  $h(u, v) = \overline{h(v, u)}$  we have  $h(u, v) = -\overline{h(v, u)}$ .

**Definition 3.1.40.** Let  $h$  be a hermitian or skew-hermitian form on an  $n$ -dimensional  $L$ -vector space  $V$ . Fix a basis  $\beta = \{v_1, \dots, v_n\}$  of  $V$ . The **matrix of  $h$  with respect to the basis  $\beta$**  is a matrix  $H = \text{Mat}_\beta(h)$  whose  $(i, j)$  entry is  $h(v_i, v_j)$ .

$$H = (h(v_i, v_j))$$

Conversely, given any hermitian matrix  $H$  there is an associated hermitian form  $h$  defined by

$$h(v_i, v_j) = H_{ij}$$

For a hermitian form, the equation  $h(u, v) = \overline{h(v, u)}$  is equivalent to the condition  $H = \overline{H}^T$ , i.e.  $H$  is a hermitian matrix. For a skew-hermitian form,  $h(u, v) = -\overline{h(v, u)}$  is equivalent to  $H = -\overline{H}^T$ , i.e.  $H$  is a skew-hermitian matrix. We will denote the conjugate transpose of a matrix by  $H^*$ .

**Lemma 3.1.41.** *There is a one-to-one correspondence between  $\sigma$ -hermitian forms and  $\sigma$ -hermitian matrices. (This is also true if you add the modifier “skew” to both.)*

**Definition 3.1.42.** A bilinear or hermitian form is **nondegenerate** if its associated matrix is invertible.

### 3.1.3 Quadratic forms

**Definition 3.1.43.** Let  $V$  be an  $n$ -dimensional  $k$ -vector space. A **quadratic form** on  $V$  is a map  $q : V \rightarrow k$  described by a quadratic polynomial. That is,  $q$  is of the form

$$q(x_1, \dots, x_n) = \sum_{i,j} a_{ij} x_i x_j$$

with  $a_{ij} \in k$ .

**Example 3.1.44.** Let  $k = \mathbb{R}$  and  $V = \mathbb{R}^3$ . Then the following are quadratic forms on  $V$ .

$$q_1(x, y, z) = x^2 + y^2 + z^2 \quad q_2(x, y, z) = 3x^2 - y^2 + xy \quad q_3(x, y, z) = xy + 3xz$$

**Remark 3.1.45.** A quadratic form  $v$  always satisfies

$$q(\lambda v) = \lambda^2 q(v)$$

for all  $\lambda \in k$  and  $v \in V$ . Slightly more subtly, given any quadratic form  $v$ , the map

$$b_q : V \times V \rightarrow k \quad b_q(u, v) = \frac{1}{2} (q(u + v) - q(u) - q(v))$$

is a bilinear form on  $V$ .

**Example 3.1.46.** Let  $q \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$  on  $\mathbb{R}^2$ . Then the associated bilinear form  $b_q$  is

$$\begin{aligned} b_q \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) &= \frac{1}{2} \left( q \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} - q \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - q \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \\ &= \frac{1}{2} \left( (x_1 + x_2)^2 + (y_1 + y_2)^2 - (x_1^2 + y_1^2) - (x_2^2 + y_2^2) \right) \\ &= \frac{1}{2} (2x_1x_2 + 2y_1y_2) \\ &= x_1x_2 + y_1y_2 \end{aligned}$$

which is the usual dot product operation on  $\mathbb{R}^2$ . If we use the standard basis  $\{e_1, e_2\}$  on  $\mathbb{R}^2$ , the matrix of this bilinear form is the identity matrix.

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Example 3.1.47.** Let  $q \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2$  on  $\mathbb{R}^2$ . The associated bilinear form is

$$\begin{aligned} b_q \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) &= \frac{1}{2} \left( q \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} - q \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - q \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \\ &= \frac{1}{2} \left( (x_1 + x_2)^2 - (y_1 + y_2)^2 - (x_1^2 - y_1^2) - (x_2^2 - y_2^2) \right) \\ &= \frac{1}{2} (2x_1x_2 - 2y_1y_2) \\ &= x_1x_2 - y_1y_2 \end{aligned}$$

If we use the standard basis  $\{e_1, e_2\}$  on  $\mathbb{R}^2$ , the matrix of this bilinear form is

$$B = \begin{pmatrix} b(e_1, e_1) & b(e_1, e_2) \\ b(e_2, e_1) & b(e_2, e_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**Lemma 3.1.48.** *When  $\text{char } k \neq 2$ , there is a one-to-one correspondence between symmetric bilinear forms and quadratic forms. Given a quadratic form  $q$ , the associated bilinear form is described by*

$$b_q(u, v) = \frac{1}{2} \left( q(u + v) - q(u) - q(v) \right)$$

*Given a symmetric bilinear form  $b$ , the associated quadratic form is*

$$q_b(v) = b(v, v)$$

**Remark 3.1.49.** If you want to associate a matrix to a quadratic form, just turn it into the equivalent bilinear form first and use that matrix.

**Theorem 3.1.50.** *Any quadratic form on a  $k$ -vector space  $V$  of dimension  $n$  is equivalent to a quadratic form that looks like*

$$q(x_1, \dots, x_n) = a_1 x_1^2 + \dots + a_n x_n^2$$

*for some  $a_1, \dots, a_n \in k$ .*

**Definition 3.1.51.** A **quadratic space** is a pair  $(V, q)$  where  $V$  is a vector space and  $q$  is a quadratic form on  $V$ .

**Example 3.1.52.** Let  $q(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$ . Then  $(\mathbb{R}^n, q)$  is a quadratic space. (This is the quadratic form associated to the traditional dot product on  $\mathbb{R}^n$ .)

**Definition 3.1.53.** Let  $(V, q)$  be a quadratic space. A **quadratic subspace** or just **subspace** of  $(V, q)$  is just a vector subspace with the restriction of  $q$  to that subspace.

### 3.1.4 Isotropy and Witt index

**Definition 3.1.54.** Let  $b$  be a symmetric bilinear form or hermitian form or skew-hermitian form on a vector space  $V$ . A vector  $v \in V$  is **isotropic** (with respect to  $b$ ) if  $v \neq 0$  and  $b(v, v) = 0$ . The form  $b$  is **isotropic** if there exists at least one isotropic vector. (Note that if  $v$  is isotropic, then any nonzero scalar multiple of  $v$  is also isotropic, so if there is any isotropic vector then there are many.) The form  $b$  is **anisotropic** if there are no isotropic vectors. The form  $b$  is **totally isotropic** if all nonzero vectors are isotropic.

Let  $q$  be a quadratic form on a vector space  $V$ . A vector  $v \in V$  is **isotropic** (with respect to  $q$ ) if  $v \neq 0$  and  $q(v) = 0$ . As with bilinear and hermitian forms, the form  $q$  is called **isotropic** if there exists at least one isotropic vector, **anisotropic** if there are no isotropic vectors, and **totally isotropic** if all nonzero vectors are isotropic.

**Example 3.1.55.** The traditional dot product on  $\mathbb{R}^n$  is an anisotropic, nondegenerate, symmetric bilinear form.

**Example 3.1.56.** The traditional complex dot product on  $\mathbb{C}^n$  is an anisotropic, nondegenerate, hermitian form.

**Example 3.1.57.** Return to the quadratic form  $q(x, y) = x^2 - y^2$  on  $\mathbb{R}^2$ , with associated bilinear form  $b((x_1, y_1), (x_2, y_2)) = x_1x_2 - y_1y_2$ . This  $q$  is isotropic, because  $(1, 1)$  is an isotropic vector. However, it is not totally isotropic because  $q(v)$  isn't always zero.

**Example 3.1.58.** Let  $h$  be the skew-hermitian form represented by the matrix

$$H = \begin{pmatrix} & -1 & \\ 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}$$

Then  $h$  is isotropic, but not totally isotropic.

**Lemma 3.1.59.** Let  $b$  be a symmetric bilinear form on a  $k$ -vector space  $V$  where  $\text{char } k \neq 2$ . Then  $b$  is totally isotropic if and only if  $b(v, w) = 0$  for all  $v, w \in V$ .

*Proof.* Clearly if  $b(v, w) = 0$  for all  $v, w$  then  $b$  is totally isotropic, so we just need to prove the other direction. Suppose  $b$  is totally isotropic, that is,  $b(v, v) = 0$  for all  $v \in V$ . Let  $v, w \in V$ , and then

$$0 = b(v + w, v + w) = b(v, v) + b(v, w) + b(w, v) + b(w, w) = b(v, w) + b(w, v) = 2b(v, w)$$

using the fact that  $b$  is symmetric. Then since we are not in characteristic 2, the equation above implies that  $b(v, w) = 0$ .  $\square$

**Definition 3.1.60.** Let  $b$  be a nondegenerate symmetric bilinear form, nondegenerate hermitian form, or nondegenerate skew-hermitian form on a vector space  $V$ . The **Witt index** or **isotropy index** of  $b$  is the maximum dimension of a totally isotropic subspace. Traditionally we use  $q$  for the Witt index.

**Remark 3.1.61.** The Witt index of an anisotropic form is zero.

**Example 3.1.62.** Return to the quadratic form  $q(x, y) = x^2 - y^2$  on  $\mathbb{R}^2$ , with associated bilinear form  $b((x_1, y_1), (x_2, y_2)) = x_1x_2 - y_1y_2$ . As previously noted,  $q$  is isotropic, because  $(1, 1)$  is an isotropic vector, but it is not totally isotropic. So the maximum dimension of a totally isotropic subspace is 1.

**Lemma 3.1.63.** The Witt index of a form on a vector space  $V$  is at most half the dimension of  $V$ .

**Definition 3.1.64.** A vector space with a form is **split** or **metabolic** if the Witt index is exactly half the dimension.

**Example 3.1.65.** Let  $k$  be an arbitrary field, of characteristic  $\neq 2$ . The quadratic space  $(k^2, x^2 - y^2)$  is split, because it has dimension 2 and Witt index 1. This particular quadratic space is called the **hyperbolic plane** (over  $k$ ).



**Exercise 3.1.66.** Let  $h$  be the skew-hermitian form represented by the matrix

$$H = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

Show that  $h$  has Witt index 2, i.e.  $h$  is split.

**Theorem 3.1.67.** *Assuming  $\text{char } k \neq 2$ , every split quadratic space is a direct sum of copies of the hyperbolic plane.*

**Exercise 3.1.68.** Let  $h$  be the hermitian form represented by the matrix

$$H = \begin{pmatrix} & I_q & \\ \varepsilon I_q & & \\ & & C \end{pmatrix}$$

where  $I_q$  is the  $q \times q$  identity matrix and  $\varepsilon = \pm 1$  and  $C$  is an invertible diagonal matrix. Show that the Witt index of  $h$  is  $q$ . Similarly, if  $b$  is the symmetric bilinear form represented by

$$B = \begin{pmatrix} & I_q & \\ I_q & & \\ & & C \end{pmatrix}$$

then show the Witt index of  $b$  is  $q$ .

**Theorem 3.1.69** (Witt decomposition). *Let  $(V, q)$  be a quadratic space over a field  $k$ . Then  $V$  can be written (essentially uniquely) in the form*

$$V \cong V_t \oplus V_a \oplus V_h$$

where  $V_t$  is totally isotropic ( $q|_{V_t} = 0$ ),  $V_a$  is anisotropic, and  $V_h$  is split. We can also emphasize the forms more by writing this as

$$(V, q) \cong (V_t, 0) \oplus (V_a, q_a) \oplus (V_h, q_h)$$

Also, given this decomposition, the Witt index is  $\frac{1}{2} \dim V_h$ , the number of hyperbolic planes comprising  $V_h$ .

## 3.2 Algebraic groups

**Definition 3.2.1.** An **algebraic variety** is the set of solutions to a polynomial equation (or list of polynomial equations). More precisely, we need to fix a field  $k$  and talk about an **algebraic variety over  $k$** , which is the set of solutions in  $k$  to a polynomial equation.

**Example 3.2.2.** A circle is an algebraic variety because it is the set of point  $(x, y)$  which are solutions to  $x^2 + y^2 = 1$ . More precisely, the usual circle is a variety over the field  $\mathbb{R}$ . If we take a bigger field, such as  $\mathbb{C}$ , we might get more solutions like  $(i, \sqrt{2})$ , or if we take a smaller field like  $\mathbb{Q}$  we will get fewer solutions (just the points on the circle with rational coordinates).

**Example 3.2.3.** For any polynomial function  $f(x)$ , the curve  $y = f(x)$  is an algebraic variety.

**Definition 3.2.4.** Let  $V_1, V_2$  be algebraic varieties. A **regular map** or **morphism of algebraic varieties** is, very roughly, a map  $V_1 \rightarrow V_2$  which can be described (locally) in terms of polynomials. (You can approximate regular by thinking of “smooth” or “infinitely differentiable” or “essentially a polynomial function.”)

**Definition 3.2.5.** An **algebraic group** is a set which has both a group structure and the structure of an algebraic variety (over a base field  $k$ ), and the group operations are regular maps. That is, the multiplication map

$$m : G \times G \rightarrow G \quad m(x, y) = xy$$

and the inversion map

$$i : G \rightarrow G \quad i(x) = x^{-1}$$

are regular maps.

**Example 3.2.6.** The special linear group is an algebraic group.

$$\mathrm{SL}_n(k) = \{X \in \mathrm{M}_n(k) : \det(X) = 1\}$$

**Example 3.2.7.** Let  $b$  be a nondegenerate symmetric bilinear form with Witt index  $q$  and matrix  $B$  (in a fixed basis). The **special orthogonal group** of  $b$  is

$$\mathrm{SO}_{n,q}(k, b) = \{X \in \mathrm{SL}_n(k) : X^T B X = B\}$$

This is a subgroup of  $\mathrm{SL}_n(k)$ , and is itself an algebraic group.

**Example 3.2.8.** Let  $h$  be a nondegenerate hermitian or skew-hermitian form with Witt index  $q$ , and with matrix  $H$  (in a fixed basis). The **special unitary group** of  $h$  is

$$\mathrm{SU}_{n,q}(L, h) = \{X \in \mathrm{SL}_n(L) : X^* H X = H\}$$

where  $X^* = \overline{X}^T$ . The special unitary group is a subgroup of  $\mathrm{SL}_n(L)$ , and is itself an algebraic group.

**Definition 3.2.9.** Let  $G(k), G'(k)$  be algebraic groups over a field  $k$ . A **morphism of algebraic groups** is a group homomorphism  $G(k) \rightarrow G'(k)$  which is also a regular map.

**Definition 3.2.10.** Let  $G(k), G'(k)$  be algebraic groups over a field  $k$ . An **abstract homomorphism** is a group homomorphism  $G(k) \rightarrow G'(k)$  (but not necessarily regular).

### 3.3 Lie algebras

For this whole section, let  $k$  be a field and assume  $\text{char } k \neq 2$ .

**Definition 3.3.1.** An **algebra** over  $k$  or  **$k$ -algebra** is a  $k$ -vector space  $A$  equipped with an additional multiplication map  $A \times A \rightarrow A$ . First, since  $A$  is a vector space, it has an addition operation which makes it an abelian group.

$$A \times A \rightarrow A \quad (a, b) \mapsto a + b$$

$A$  has a zero element and additive inverses for every element, and addition is commutative.

$$a + 0 = 0 + a = a \quad a + (-a) = (-a) + a = 0 \quad a + b = b + a$$

for every  $a, b \in A$ . Also,  $A$  has a scalar multiplication map.

$$k \times A \rightarrow A$$

This scalar multiplication map distributes over addition, meaning

$$\lambda(a + b) = \lambda a + \lambda b$$

for all  $\lambda \in k$  and  $a, b \in A$ .  $A$  also has a multiplication map, which distributes over addition.

$$A \times A \rightarrow A \quad (a, b) \mapsto ab$$

Distributing over addition means that

$$a(b + c) = ab + ac \quad (a + b)c = ac + bc$$

for every  $a, b, c \in A$ . Note that we do NOT assume that multiplication is associative, that is, we do not assume that  $(ab)c = a(bc)$ . If you have this property, then  $A$  is called an **associative algebra**, and in this case  $A$  is a ring under its addition and multiplication operations. But in general,  $A$  may not be associative.

Note that we do NOT assume that  $A$  has a unity, or that multiplication is commutative.  $A$  also has the property that multiplication is compatible with scalars, in the sense that

$$(\lambda a)(\mu b) = (\lambda \mu)ab$$

for every  $\lambda, \mu \in k$  and  $a, b \in A$ .

**Definition 3.3.2.** A **Lie algebra** over  $k$  is an algebra  $L$  over  $k$  with two additional properties. First, traditionally in a Lie algebra the operation is denoted with brackets, like so.

$$[, ] : L \times L \rightarrow L$$

so the multiplication of two elements  $x, y$  is written  $[x, y]$  instead of  $xy$ . The two additional equations that make  $L$  into a Lie algebra are

$$[x, x] = 0 \quad \forall x \in L$$

and

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

This second one is called the **Jacobi identity**.

**Exercise 3.3.3.** Show that in any Lie algebra,  $[x, y] = -[y, x]$  by applying the fact that  $[x + y, x + y] = 0$  and using the distributive property.

**Exercise 3.3.4.** Show that if  $L$  is an algebra (not necessarily a Lie algebra) in which  $[x, y] = -[y, x]$  and the characteristic of  $k$  is not 2, then  $[x, x] = 0$ .

**Example 3.3.5.**  $\mathbb{R}^3$  with the cross product operation is Lie algebra. Verifying the Jacobi identity is tedious, I do not recommend.

**Example 3.3.6.** Fix  $n \geq 1$  and let  $M_n(k)$  be the set of  $n \times n$  matrices with entries in  $k$ . Then  $M_n(k)$  is a Lie algebra under the operation

$$[A, B] = AB - BA$$

This is the most important example of a Lie algebra.

### 3.3.1 Lie algebras associated to algebraic groups

Every algebraic group has an associated Lie algebra. This is somewhat complicated to describe, so rather than talk about the general definition, I will just describe the specific Lie algebra associated with each of the groups we are dealing with. All of these are matrix Lie algebras that are subalgebras of  $M_n(k)$ , meaning they have the same bracket operation but are just a particular subset.

**Definition 3.3.7.** Let  $n \geq 1$  and consider the special linear group  $SL_n(k)$ . The associated Lie algebra to this is

$$\mathfrak{sl}_n(k) = \{x \in M_n(k) : \text{tr}(x) = 0\}$$

The bracket operation is the same as on  $M_n(k)$ . That is,

$$[x, y] = xy - yx$$

where  $xy$  refers to usual matrix multiplication.

**Exercise 3.3.8.** Show that if  $\text{tr}(x) = \text{tr}(y) = 0$ , then  $\text{tr}[x, y] = 0$ . In other words,  $\mathfrak{sl}_n(k)$  is actually closed under the bracket operation.

**Definition 3.3.9.** Let  $b$  be a nondegenerate symmetric bilinear form with Witt index  $q$  and matrix  $B$ , and let  $SO_n(k, B)$  be the associated special orthogonal group. The **special orthogonal Lie algebra** associated to  $B$  is

$$\mathfrak{so}_{n,q}(k, b) = \{X \in M_n(k) : X^T B + BX = 0\}$$

again with the usual matrix bracket on  $M_n(k)$ .

**Definition 3.3.10.** Let  $h$  be a nondegenerate hermitian or skew-hermitian form with Witt index  $q$  and matrix  $H$ , and let  $SU_n(L, H)$  be the associated special unitary group. The **special unitary Lie algebra** associated to  $H$  is

$$\mathfrak{su}_{n,q}(L, h) = \{X \in M_n(L) : X^* H + HX = 0\}$$

with the usual matrix bracket on  $M_n(L)$ .

### 3.3.2 Roots and root spaces

**Definition 3.3.11.** Let  $G(k)$  be an algebraic group that is one of  $\mathrm{SL}_n(k)$ ,  $\mathrm{SO}_n(k, B)$ , or  $\mathrm{SU}_n(L, H)$ , and let  $\mathfrak{g}(k)$  be the associated Lie algebra (either  $\mathfrak{sl}_n(k)$ ,  $\mathfrak{so}_n(k, b)$ , or  $\mathfrak{su}_n(L, h)$ ). Let  $S(k)$  be the torus subgroup of  $G(k)$  (the full subgroup of  $\mathrm{SL}_n(k)$ , or the subgroup we previously specified for  $\mathrm{SO}_{n,q}(k, b)$  or  $\mathrm{SU}_{n,q}(L, h)$ ). Then  $S(k)$  acts on  $\mathfrak{g}(k)$  by conjugation. That is, the map

$$S(k) \times \mathfrak{g}(k) \rightarrow \mathfrak{g}(k) \quad (s, x) \mapsto sxs^{-1}$$

is a well-defined group action. (In the expression  $sxs^{-1}$ , we are just doing matrix multiplication.)

**Exercise 3.3.12.** Verify this is well-defined in the case of  $\mathrm{SL}_n(k)$ . That is, show that if  $s \in \mathrm{SL}_n(k)$  is a diagonal matrix, and  $X \in \mathfrak{sl}_n(k)$  is a matrix with trace zero, then  $sXs^{-1}$  has trace zero.

**Definition 3.3.13.** Let  $\alpha \in X^*(S)$  be a character (not necessarily a root). That is,  $\alpha$  is a group homomorphism  $S(k) \rightarrow k^\times$ . Define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}(k) : sXs^{-1} = \alpha(s)X, \forall s \in S(k)\}$$

The expression  $\alpha(s)x$  can be confusing. What it means is we are plugging in  $s$  into  $\alpha$ , since  $\alpha$  is a function with domain  $S(k)$ , so  $\alpha(s)$  is just a scalar, an element of  $k^\times$ . Then thinking of  $x \in \mathfrak{g}(k)$  as a matrix, we multiply each individual entry of the matrix  $x$  by the scalar  $\alpha(s)$ .

Conceptually,  $\mathfrak{g}_\alpha$  is kind of like the set of “simultaneous eigenvectors” for all of  $S(k)$ , if we think of each  $s \in S(k)$  as a linear map from  $\mathfrak{g}(k)$  to itself (which is possible using the group action). If this is confusing, just focus on the definition above.

**Exercise 3.3.14.** Show that  $\mathfrak{g}_\alpha$  is a vector subspace of  $\mathfrak{g}(k)$ .

**Example 3.3.15.** For most characters, the set  $\mathfrak{g}_\alpha$  is just the zero subspace. To illustrate, we consider the character  $\alpha_1$  for the group  $\mathrm{SL}_2(k)$  with its full diagonal subgroup  $S(k)$ . Recall that

$$S(k) = \left\{ \begin{pmatrix} s_1 & \\ & s_1^{-1} \end{pmatrix} : s_1 \in k^\times \right\}$$

and  $\alpha_1$  is the map

$$\alpha_1 : S(k) \rightarrow k^\times \quad \alpha_1 \left( \begin{pmatrix} s_1 & \\ & s_1^{-1} \end{pmatrix} \right) = s_1$$

In this case, the Lie algebra  $\mathfrak{g}(k)$  is the set of  $2 \times 2$  matrices with entries in  $k$  and trace zero, so concretely

$$\mathfrak{g}(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in k \right\}$$

We will try to determine  $\mathfrak{g}_{\alpha_1}$ . Let

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$$

If  $x \in \mathfrak{g}(k)$  belongs to  $\mathfrak{g}_{\alpha_1}$ , then  $sxs^{-1} = s_1x$  for every  $s \in S(k)$ . This means that

$$\begin{aligned} \begin{pmatrix} s_1 & \\ & s_1^{-1} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \begin{pmatrix} s_1 & \\ & s_1^{-1} \end{pmatrix}^{-1} &= s_1 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \\ \begin{pmatrix} s_1x_{11} & s_1x_{12} \\ s_1^{-1}x_{21} & -s_1^{-1}x_{11} \end{pmatrix} \begin{pmatrix} s_1^{-1} & \\ & s_1 \end{pmatrix} &= \begin{pmatrix} s_1x_{11} & s_1x_{12} \\ s_1x_{21} & -s_1x_{11} \end{pmatrix} \\ \begin{pmatrix} x_{11} & s_1^2x_{12} \\ s_1^{-2}x_{21} & -x_{11} \end{pmatrix} &= \begin{pmatrix} s_1x_{11} & s_1x_{12} \\ s_1x_{21} & -s_1x_{11} \end{pmatrix} \end{aligned}$$

Remember that for  $x$  to be in  $\mathfrak{g}_{\alpha}$ , this has to be true for every  $s$ . So it has to be true for every  $s_1 \in k^\times$ . Comparing the  $(1,1)$  entries, this implies that

$$x_{11} = s_1x_{11}$$

If this is true for every  $s_1$ , then it must be that  $x_{11} = 0$ . Similarly, comparing the  $(1,2)$  entries we get

$$s_1^2x_{12} = s_1x_{12}$$

which again is only possibly true for every  $s_1$  if  $x_{12} = 0$ . And a similar argument shows that  $x_{21} = 0$  as well. So the only possible matrix  $x$  belonging to  $\mathfrak{g}_{\alpha_1}$  is the zero matrix.

**Example 3.3.16.** In contrast with the previous example,  $\mathfrak{g}_{\alpha}$  is not always zero. Take the character  $2\alpha_1$  in the same scenario as above.

$$2\alpha_1 : S(k) \rightarrow k^\times \quad 2\alpha_1 \begin{pmatrix} s_1 & \\ & s_1^{-1} \end{pmatrix} = s_1^2$$

Then we repeat the calculation. Suppose  $x \in \mathfrak{sl}_n(k)$  belongs to  $\mathfrak{g}_{2\alpha_1}$ . Then reusing the left-hand side of the calculation above, we conclude that

$$sxs^{-1} = \begin{pmatrix} x_{11} & s_1^2x_{12} \\ s_1^{-2}x_{21} & -x_{11} \end{pmatrix} = s_1^2x = \begin{pmatrix} s_1^2x_{11} & s_1^2x_{12} \\ s_1^2x_{21} & -s_1^2x_{11} \end{pmatrix}$$

Again by comparing the  $(1,1)$  entries, we can conclude that  $x_{11}$  must be zero, since  $x_{11} = s_1^2x_{11}$  can only be true for every  $s_1 \in k^\times$  if  $x_{11} = 0$ . And we can do a similar argument to conclude that  $x_{21} = 0$ . However, the  $(1,2)$  entry does not work out quite the same way, because it gives

$$s_1^2x_{12} = s_1^2x_{12}$$

which is just always true, regardless of  $s_1 \in k^\times$  and  $x_{12} \in k$ . As a result, we conclude that

$$\mathfrak{g}_{2\alpha_1} = \left\{ \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} : u \in k \right\}$$

which is a subspace of  $\mathfrak{sl}_2(k)$  of dimension 1.

**Definition 3.3.17.** A **root of  $\mathfrak{g}$  with respect to  $S$**  is a character  $\alpha$  so that the subspace  $\mathfrak{g}_{\alpha}$  is nonzero.

**Exercise 3.3.18.** Show that  $-2\alpha_1$  is also a root for  $\mathfrak{sl}_2(k)$  with root space

$$\mathfrak{g}_{-2\alpha_1} = \left\{ \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} : u \in k \right\}$$

**Definition 3.3.19.** Let  $X$  be an  $n \times n$  matrix. The **matrix exponential** of  $X$  is

$$e^X = \exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!} = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$$

In general, we might need to worry about whether this infinite series converges. However, for all of the matrices that we'll deal with, this series terminates after just 2-3 nonzero terms.

**Example 3.3.20.** Let  $X$  be an element of the root space  $\mathfrak{g}_{2\alpha_1}$  for  $\mathfrak{sl}_2(k)$ , so

$$X = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$$

for some  $u \in k$ . Then  $X^2 = 0$ , so

$$\exp(X) = 1 + X = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

**Definition 3.3.21.** Let  $G(k)$  be a special linear, special orthogonal, or special unitary group, with Lie algebra  $\mathfrak{g}(k)$  and torus  $S(k)$ . Let  $\alpha$  be a root, with nonzero root space  $\mathfrak{g}_\alpha$ . The **root subgroup** associated to  $\alpha$  is

$$U_\alpha = \exp(\mathfrak{g}_\alpha) = \{\exp(x) : x \in \mathfrak{g}_\alpha\}$$

**Example 3.3.22.** The root subgroup associated to  $2\alpha_1$  for  $\mathrm{SL}_2(k)$  is

$$U_{2\alpha_1} = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in k \right\}$$

#### Determining pinning maps.

Let  $G(k)$  be a special linear, special orthogonal, or special unitary group with Lie algebra  $\mathfrak{g}(k)$  and torus  $S(k)$ . Let  $\alpha$  be a root, with nonzero root space  $\mathfrak{g}_\alpha$ . The associated pinning map  $X_\alpha$  is the exponential map from  $\mathfrak{g}_\alpha$  to  $G(k)$ .

**Example 3.3.23.** Let  $G = \mathrm{SL}_2(k)$  and  $\alpha = 2\alpha_1$ . The pinning map  $X_{2\alpha_1}$  is just the exponential map

$$\exp : \mathfrak{g}_{2\alpha_1} \rightarrow U_{2\alpha_1}$$

(followed by inclusion into  $\mathrm{SL}_2(k)$ ).

**Remark 3.3.24.** In general, we don't really want to do the whole calculation we did in Example 3.3.15 for a bunch of potential roots, so it is usually faster to go about things as follows. Let  $X$  be an arbitrary element of the Lie algebra  $\mathfrak{g}(k)$ , and let  $s$  be an arbitrary element of the torus subgroup  $S(k)$ . Then compute  $sXs^{-1}$ , and look at the various coefficients that appear. Are any of these equal to  $\alpha(s)$  for some character  $\alpha$ ? Those are usually the roots.

### 3.4 Root systems

We start with a very basic intuition: a root system is a “very symmetrical” finite set of vectors in  $\mathbb{R}^n$ .

**Definition 3.4.1.** Fix a positive integer  $n$ , and let  $E = \mathbb{R}^n$ . Let  $(-, -)$  denote the usual dot product on  $E$ . Let  $\alpha, \beta \in E$ . We say  $\alpha$  and  $\beta$  are **orthogonal** if  $(\alpha, \beta) = 0$ . Also define

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

Note that  $\langle -, - \rangle$  is linear in the first component, but not in the second. Also note that  $\langle \alpha, \beta \rangle = 0 \iff (\alpha, \beta) = 0$  (assuming  $\beta \neq 0$ ).

**Definition 3.4.2.** Let  $E$  be as above, and let  $\alpha \in E$ . The **hyperplane orthogonal to  $\alpha$**  is the set of all vectors orthogonal to  $\alpha$ , and is called  $P_\alpha$ .

$$P_\alpha = \{\beta \in E : (\alpha, \beta) = 0\}$$

**Definition 3.4.3.** For  $\alpha \in E$ , let  $\sigma_\alpha : E \rightarrow E$  be the reflection across  $P_\alpha$ . In terms of a formula,

$$\sigma_\alpha : E \rightarrow E \quad \sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$$

This is a reflection in a geometric sense, and also in the sense that  $\sigma_\alpha^2 = \text{Id}_E$ .

**Definition 3.4.4.** Let  $E = \mathbb{R}^n$ . A **root system** in  $E$  is a finite set  $\Phi \subset E$  such that

(R1)  $0 \notin \Phi$  and the span of  $\Phi$  is  $E$

(R2) For  $\alpha \in \Phi$ , the only multiples of  $\alpha$  that are in  $\Phi$  are  $\pm\alpha$  and  $\pm 2\alpha$ .

(R3) For  $\alpha \in \Phi$ ,  $\sigma_\alpha(\Phi) = \Phi$ .

(R4) For  $\alpha, \beta \in \Phi$ ,  $\langle \alpha, \beta \rangle \in \mathbb{Z}$ .

Technically speaking, (R2) is redundant because one can prove that (R2) is a consequence of (R1), (R3), and (R4). However, we just include it in the definition. The root system  $\Phi$  is called **reduced** if (R2) can be replaced by the stronger statement

(R2') For  $\alpha \in \Phi$ , the only multiples of  $\alpha$  that are in  $\Phi$  are  $\pm\alpha$ .

An element of a root system is called a **root**.

**Remark 3.4.5.** There is a complete classification of root systems via something called “Dynkin diagrams.” The details of the proof of this classification are not important for us, and really we only need to care about type A, B, C, and BC root systems. There are also type E, F, G root systems.



**Example 3.4.6.** We describe the  $A_{n-1}$  root system. Let  $E = \mathbb{R}^n$ , and let  $\alpha_i = (0, \dots, 1, \dots, 0)$  be the vector with 1 in the  $i$ th position and 0 elsewhere. Let  $\alpha_{ij} = \alpha_i - \alpha_j$ . Then set

$$\Phi = \{\alpha_{ij} : 1 \leq i, j \leq n, i \neq j\}$$

Then  $\Phi$  is a reduced root system, called the type  $A_{n-1}$  root system. Note that  $\Phi$  does not actually span all of  $E$  – the span of  $\Phi$  is actually the subspace orthogonal to the vector  $(1, 1, 1, \dots, 1)$ . So technically speaking,  $\Phi$  is a root system not in  $E$ , but rather in this  $(n-1)$ -dimensional subspace of  $E$ .

**Definition 3.4.7.** The **rank** of  $\Phi$  is the dimension of the space spanned by  $\Phi$ .

**Example 3.4.8.** The rank of  $A_{n-1}$  is  $n-1$ . In general, for any root system the subscript denotes the rank, so for example  $D_4$  is the type  $D$  root system of rank 4.

**Example 3.4.9.** We describe the  $B_n, C_n$ , and  $BC_n$  root systems. Let  $E = \mathbb{R}^n$ , and  $\alpha_i = (0, \dots, 1, \dots, 0)$  as before. Then

$$\begin{aligned} B_n &= \{\pm\alpha_i : 1 \leq i \leq n\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq n, i \neq j\} \\ C_n &= \{\pm 2\alpha_i : 1 \leq i \leq n\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq n, i \neq j\} \end{aligned}$$

Note that the second part is identical, the only difference is the 2 in the first set of roots. Both  $B_n$  and  $C_n$  are reduced root systems (in  $E$ ). The  $BC_n$  root system is simply the union of the two.

$$BC_n = \{\pm\alpha_i, \pm 2\alpha_i : 1 \leq i \leq n\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq n, i \neq j\}$$

$BC_n$  is not reduced. In fact, the root systems  $BC_n$  are the only non-reduced root systems (we omit the proof).

**Definition 3.4.10.** A root system is **reducible** if it can be written as a disjoint union of two subsets which are orthogonal to each other. A root system is **irreducible** if it is not reducible.

**Remark 3.4.11.** All of the example root systems above ( $A_n, B_n, C_n, BC_n$ ) are irreducible.

**Remark 3.4.12.** Despite the unfortunate similarity of terminology, there is no relationship between a root system being “reduced/non-reduced” and “reducible/irreducible.” For example, it is possible to have a root system which is non-reduced and irreducible ( $BC_n$  is the key example).

**Lemma 3.4.13.** Any root system  $\Phi$  can be written uniquely as a disjoint union of irreducible root systems. These are called the **irreducible components** of  $\Phi$ .

Because of the lemma, it is sufficient to study irreducible root systems.

**Definition 3.4.14.** Let  $\Phi$  be a root system of rank  $n$  in  $E = \mathbb{R}^n$ . For any  $\alpha \in \Phi$ , we can consider the reflection  $\sigma_\alpha$  as an invertible linear map  $E \rightarrow E$ . By (R3), if we restrict  $\sigma_\alpha$  to  $\Phi$ , the output is  $\Phi$ . So we can think of  $\sigma_\alpha$  as a map  $\Phi \rightarrow \Phi$ . The **Weyl group** of  $\Phi$ , denoted  $W(\Phi)$ , is the subgroup of invertible linear maps  $E \rightarrow E$  generated by all of the  $\sigma_\alpha$ . Since all the generators take  $\Phi$  to itself, every element of  $W(\Phi)$  sends  $\Phi$  to itself.

**Remark 3.4.15.** In general, there are elements of  $W(\Phi)$  which are not  $\sigma_\alpha$  for any  $\alpha$ .

Despite the fact that  $W(\Phi)$  is generated by elements of order 2, not every element of  $W(\Phi)$  has to have order 2. However, since  $\Phi$  is a finite set and  $W(\Phi)$  is a subgroup of the permutation group of  $\Phi$ ,  $W(\Phi)$  is a finite group. Usually,  $W(\Phi)$  is not the full permutation group of  $\Phi$ , since  $\sigma_\alpha(\beta)$  must have the same length as  $\beta$ . So  $W(\Phi)$  must permute the set of roots of the same length. For example, in  $\mathbb{C}_n$ , this means that  $W(\Phi)$  can only send roots of the form  $\pm\alpha_i$  to other roots of this form, never to a root such as  $\alpha_1 + \alpha_2$ .

Also, if  $\Phi$  is reducible, then  $W(\Phi)$  must only send roots in one irreducible component within that irreducible component. However, within that restriction,  $W(\Phi)$  “acts transitively,” meaning that for any two roots  $\alpha, \beta$  of the same length in the same irreducible component, there exists a  $w \in W(\Phi)$  such that  $w(\alpha) = \beta$ . Since most of the time we will deal with irreducible root systems, the main restriction here is that  $\alpha, \beta$  have the same length.

### 3.5 Pinnings

**Definition 3.5.1.** Let  $k$  be a field, and let  $G = G(k)$  be an algebraic group, with a maximal  $k$ -split torus subgroup  $S = S(k)^3$ . Let  $\Phi = \Phi(G, S)$  be the root system of  $G$  with respect to  $S$ . A **pinning** of  $G$  (with respect to  $S$ ) is a set of maps

$$X_\alpha : V_\alpha(k) \rightarrow G(k)$$

for each  $\alpha \in \Phi$ , such that:

1. Each  $V_\alpha(k)$  is a vector space over  $k$  (often dimension 1, sometimes 2, and occasionally more than 2).
2.  $X_\alpha$  is a group homomorphism, meaning that for each  $u, v \in V_\alpha(k)$ ,

$$X_\alpha(u + v) = X_\alpha(u) \cdot X_\alpha(v)$$

3. (Torus conjugation formula) For every  $s \in S(k)$ ,

$$s \cdot X_\alpha(u) \cdot s^{-1} = X_\alpha(\alpha(s)u)$$

4. (Commutator formula) For every  $\alpha, \beta \in \Phi$  with  $\alpha \neq c\beta$  for any scalar  $c$ , there are polynomial maps

$$N_{ij}^{\alpha\beta} : V_\alpha(k) \times V_\beta(k) \rightarrow V_{\alpha+\beta}(k)$$

such that for any  $u \in V_\alpha(k)$  and  $v \in V_\beta(k)$ , we have

$$\left[ X_\alpha(u), X_\beta(v) \right] = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi}} X_{i\alpha + j\beta} \left( N_{ij}^{\alpha\beta}(u, v) \right)$$

---

<sup>3</sup>You don't need to know exactly what a maximal  $k$ -split torus is, just imagine  $S$  as some generalization of the diagonal subgroup of  $\mathrm{SL}_n(k)$ .

**Definition 3.5.2.** A **Weyl group element** is an element  $w \in G$  which normalizes the torus  $S$ . In other words,  $w \in N_G(S)$ , which is to say,

$$wSw^{-1} = S$$

or equivalently

$$ws w^{-1} \in S \quad \forall s \in S$$

A **pinning compatible with the Weyl group** is a pinning with an additional property:

5. (Weyl group conjugation formula) For every  $\alpha \in \Phi$ , there is a Weyl group element  $w_\alpha \in G(k)$  which belongs to the subgroup generated by  $X_\alpha(V_\alpha(k))$  and  $X_{-\alpha}(V_{-\alpha}(k))$  and such that for every  $\beta \in \Phi$  and  $v \in V_\beta(k)$ ,

$$w_\alpha \cdot X_\beta(v) \cdot w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}(\varphi_{\alpha\beta}(v))$$

where  $\varphi_{\alpha\beta} : V_\beta \rightarrow V_{\sigma_\alpha(\beta)}$  is some invertible linear function.

**Remark 3.5.3.** By general theory that we are not going into, under appropriate assumptions about the group  $G$ ,  $G$  always contains a maximal  $k$ -split torus  $S$ , and there is always a root system  $\Phi(G, S)$ , and a pinning of  $G$  always exists.

I'm not 100% confident, but I believe that a pinning compatible with the Weyl group should always exist as well. For the purposes of this project, we will assume these exist at least for the groups we are working with, and try to find them, write them down as explicitly as possible, and set up a computer to check the various equations involved.

### 3.6 *Nonlinearity of matrix groups*

In this section, we give some notes intended to accompany and explain the short 2009 paper *Nonlinearity of matrix groups* by Martin Kassabov and Mark Sapir, a paper which utilizes a pinning of  $\mathrm{SL}_n(k)$  to prove a theorem about rigidity. In particular we focus on the ideas of Theorem 3 as they apply to rigidity statements for abstract homomorphisms of algebraic groups.

**Definition 3.6.1.** A **unital ring** is a set  $R$  with two operations  $+$  and  $\times$ , and these operations satisfy:

1.  $(R, +)$  is an abelian group with identity element 0. That is,  $+$  is associative, commutative, and every element  $a$  has a unique additive inverse  $-a$ .

$$a + (b + c) = (a + b) + c$$

$$a + b = b + a$$

$$a + 0 = a$$

$$a + (-a) = 0$$

2.  $(R, \times)$  is not a group, but is associative and has an identity element 1. Note that we do not assume  $\times$  is commutative.

$$a(bc) = (ab)c \quad a1 = 1a = a$$

3. Addition and multiplication play well together via the distributive property.

$$a(b + c) = ab + ac \quad (a + b)c = ac + bc$$

Let  $R$  be a unital ring, and let  $K$  be a field of characteristic zero. Let  $n$  be a positive integer.

**Definition 3.6.2.** The **general linear group** of  $R$  is  $\mathrm{GL}_n(R)$ , the group of  $n \times n$  invertible matrices with entries in  $R$ . Equivalently,  $\mathrm{GL}_n(R)$  is the set of  $n \times n$  matrices whose determinant is a unit (multiplicatively invertible) element of  $R$ .  $\mathrm{SL}_n(R)$  is the subgroup of  $\mathrm{GL}_n(R)$  of matrices of determinant one.

**Definition 3.6.3.** For  $1 \leq i, j \leq n$  and  $i \neq j$ , and  $r \in R$ , let  $x_{ij}(r)$  be the matrix with 1's on the diagonal and  $r$  in the  $(i, j)$  off-diagonal entry. Note that  $\det x_{ij}(r) = 1$ , so  $x_{ij}(r) \in \mathrm{SL}_n(R)$ .

In other places in this document,  $x_{ij}(r)$  is called  $X_{\alpha_{ij}}(r)$ . We have previously observed that  $X_{\alpha_{ij}} : R \rightarrow \mathrm{SL}_n(R)$  is a group homomorphism, which means in this context that for any  $r_1, r_2 \in R$ , we have

$$x_{ij}(r_1 + r_2) = x_{ij}(r_1) \cdot x_{ij}(r_2)$$

**Definition 3.6.4.** Let  $G$  be a group, and let  $S \subset G$  be a subset. The **subgroup generated by  $S$**  is the subset of  $G$  consisting of all elements that can be written as a product of elements in  $S$  and their inverses. Equivalently, it is the smallest subgroup of  $G$  containing  $S$ .

**Definition 3.6.5.** Let  $S$  be the following subset of  $\mathrm{GL}_n(R)$ .

$$S = \{x_{ij}(r) : 1 \leq i, j \leq n \text{ and } i \neq j \text{ and } r \in R\}$$

We define  $\mathrm{EL}_n(R)$  to be the subgroup of  $\mathrm{GL}_n(R)$  generated by  $S$ .

**Lemma 3.6.6.**  $\mathrm{EL}_n(R) \subset \mathrm{SL}_n(R)$

*Proof.* Since every  $x_{ij}(r)$  has determinant one, any product of them (or their inverses) will also have determinant one.  $\square$

**Lemma 3.6.7.** If  $R$  is a field, then  $\mathrm{EL}_n(R) = \mathrm{SL}_n(R)$ .

*Proof.* If  $R$  is a field, then Gaussian elimination works. What does Gaussian elimination do? It uses elementary row operations to rewrite a square matrix as the identity matrix. But each time you do an elementary row operation, that's like multiplying by some  $x_{ij}(r)$  or a permutation matrix.  $\square$

## The subgroups $A$ and $U$

Let  $n$  and  $k$  be positive integers, and assume  $k \geq 3$ . For all of these calculations, it will not hurt to assume  $k = 3$ . Let

$$\pi : \mathrm{EL}_k(R) \rightarrow \mathrm{GL}_n(K)$$

be a group homomorphism. The paper refers to  $\pi$  as a “linear representation,” but this is just another way of saying group homomorphism into some general linear group. In particular,  $\pi$  is what I call an “abstract homomorphism,” because we do not assume it is algebraic at all, even though  $\mathrm{EL}_k(R)$  and  $\mathrm{GL}_n(K)$  can be thought of as algebraic groups.

**Definition 3.6.8.** This notation is not in the paper, but I think it helps clarify things. Let

$$A = \{x_{13}(r) : r \in R\}$$

That is,  $A$  is the subset of  $\text{EL}_k(R)$  of matrices of the form

$$\begin{pmatrix} 1 & 0 & r & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix}$$

In other words,  $A$  is the image of the map  $x_{ij} : R \rightarrow \text{EL}_k(R)$ .

$$A = x_{ij}(R)$$

Since  $x_{ij}$  is a group homomorphism and  $R$  is a group (under addition), the image must be a subgroup. That is,  $A$  is a subgroup of  $\text{EL}_k(R)$ . The group operation is matrix multiplication.

$$\begin{pmatrix} 1 & 0 & r_1 & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix} \begin{pmatrix} 1 & 0 & r_2 & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix} = \begin{pmatrix} 1 & 0 & r_1 + r_2 & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix}$$

Even though matrix multiplication is usually not commutative, in this context the matrix multiplication within  $A$  is commutative, because addition in  $R$  is commutative.

$$\begin{pmatrix} 1 & 0 & r_1 + r_2 & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix} = \begin{pmatrix} 1 & 0 & r_2 + r_1 & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix}$$

In fact, if we just view  $(R, +)$  as an abelian group, we can think of  $x_{13}$  as a group isomorphism between  $(R, +)$  and  $(A, \times)$ .

$$x_{13} : R \rightarrow A \quad x_{13}(r) = \begin{pmatrix} 1 & 0 & r & & \\ & 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix}$$

The inverse map is just sending that matrix to  $x_{13}(r)$ .

**Definition 3.6.9.** Let  $A$  be as above, and recall that we have a group homomorphism  $\pi : \text{EL}_k(R) \rightarrow \text{GL}_n(K)$ . Let

$$U = \pi(A) = \{\pi(a) : a \in A\} = \{\pi(x_{13}(r)) : r \in R\}$$

So we have  $A$  sitting inside  $\text{EL}_k(R)$ , and  $U$  sitting inside  $\text{GL}_n(R)$ . The diagram below depicts the situation. The upward arrows are just the inclusion maps, and the diagram commutes, meaning following a path of arrows along one side of the square is the same as going the other way.

$$\begin{array}{ccc}
\mathrm{EL}_k(R) & \xrightarrow{\pi} & \mathrm{GL}_n(R) \\
\uparrow & & \uparrow \\
A & \xrightarrow{\pi|_A} & U
\end{array}$$

Also note that  $\pi|_A(A) = U$ , or in other words,  $\pi|_A$  is a surjective map from  $A$  to  $U$ .

**Remark 3.6.10.**  $R$  is a ring, so  $(R, +)$  is an abelian group. Then  $x_{13} : R \rightarrow A$  is a group isomorphism, so  $A$  is an abelian group. Then  $\pi : A \rightarrow U$  is a surjective group homomorphism, so  $U$  is also an abelian group.

**Definition 3.6.11.** Let  $V$  be the “Zariski closure” of  $U$  inside  $\mathrm{GL}_n(R)$ . This is not something we need to understand in a very thorough way. Basically,  $V$  is still a subset of  $\mathrm{GL}_n(R)$ , and it should be “just a little bit bigger” than  $U$ . As an analogy, if  $U$  was an open interval on the real line, then  $V$  would just be the closed version of that interval, just adding in the endpoints. Or as a second analogy, if  $U$  was an open disk in the plane, not including its boundary, then  $V$  would be the closed disk, just  $U$  plus the boundary.

We defined  $V$ , but we will mostly ignore it and focus on  $U$ .

### Theorem 3 - multiplication in $U$

The big idea of Theorem 3 is to make  $U$  into a ring by adding a second operation. In order to make an abelian group into a unital ring, we need to:

- Define a second operation
- Verify that the second operation is associative and has a unity element
- Verify that the distributive property holds

In general, there is not a particularly “natural” way to make an abelian group into a ring. Even if you find a way, it may not be the only way. Technically, Theorem 3 is about making  $V$  into a ring, but you can mostly ignore this and pretend that it’s happening to  $U$ .

**Definition 3.6.12.** Next we describe the Weyl group elements of  $\mathrm{SL}_3(R)$  used in the proof of Theorem 3. Set

$$w_{12} = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \quad w_{23} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

The paper does not call these “Weyl group elements,” but they are elements of the Weyl group of  $\mathrm{SL}_3(R)$ .

**Lemma 3.6.13.** For any  $r \in R$ ,

$$w_{12}x_{13}(r)w_{12}^{-1} = x_{23}(r) \quad w_{23}x_{13}(r)w_{23}^{-1} = x_{12}(r)$$

**Remark 3.6.14.** I will not discuss the proof; you can just do some matrix calculations to verify it yourself. However, I want to make a note here on how these fit into the theory of root systems and our general Weyl group reflection formula. In the notation of this document, this first equation is

$$w_{\alpha_{12}}(-1) \cdot X_{\alpha_{13}}(r) \cdot w_{\alpha_{12}}(-1)^{-1} = X_{\alpha_{23}}(r)$$

Hopefully you remember that in the general version of the formula, the right hand side subscript on  $X$  is described in terms of a reflection. That general version is

$$w_{\alpha}(u) \cdot X_{\beta}(r) \cdot w_{\alpha}(u)^{-1} = X_{\sigma_{\alpha}(\beta)}(\pm r)$$

The reason  $\alpha_{23}$  shows up on the right side of the formula above is that  $\alpha_{23} = \sigma_{\alpha_{12}}(\alpha_{13})$ .

**Remark 3.6.15.** We have talked about the general commutator formula in  $\text{SL}_n$ . In the context of Theorem 3, the only version of this that we need is

$$[x_{12}(r_1), x_{23}(r_2)] = x_{13}(r_1 r_2)$$

Also note that in general, group homomorphisms can “go inside” commutators, as follows. If  $\pi : G \rightarrow H$  is a group homomorphism and  $x, y \in G$ , then

$$\pi([x, y]) = \pi(xyx^{-1}y^{-1}) = \pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} = [\pi(x), \pi(y)]$$

The version of this to remember is

$$\pi([x, y]) = [\pi(x), \pi(y)]$$

**Definition of multiplication in  $U$ .** Finally, we are ready to define the multiplication operation in  $U$ , which is the big accomplishment of Theorem 3. Our goal is to define a second operation  $U \times U \rightarrow U$  (and then extend this to  $V$ ) to make  $U$  into a ring, and furthermore to do this in a way which makes the map

$$\rho = \pi \circ x_{13} : R \rightarrow \text{GL}_n(K)$$

into a ring homomorphism. That is, we want to have

$$\begin{aligned} \rho \left( r_1 \underset{R}{+} r_2 \right) &= \rho(r_1) \underset{U}{+} \rho(r_2) \\ \rho \left( r_1 \underset{R}{\times} r_2 \right) &= \rho(r_1) \underset{U}{\times} \rho(r_2) \end{aligned}$$

On the left side, the operations  $\underset{R}{+}$  and  $\underset{R}{\times}$  are in the ring  $R$ . On the right side,  $\underset{U}{+}$  actually means matrix multiplication and  $\underset{U}{\times}$  means some other weird operation, which we haven't defined yet. The way to define it is

$$U \times U \rightarrow \text{GL}_n(K) \quad u_1 \times u_2 = \left[ \pi(w_{23}) \cdot u_1 \cdot \pi(w_{23})^{-1}, \pi(w_{12}) \cdot u_2 \cdot \pi(w_{12})^{-1} \right]$$

All of the dots here are matrix multiplication inside  $\text{GL}_m(K)$ . Note that in Kassabov and Sapir, they define it without some of these  $\pi$ 's. Their version looks like

$$u_1 \times u_2 = \left[ w_{23} \cdot u_1 \cdot w_{23}^{-1}, w_{12} \cdot u_2 \cdot w_{12}^{-1} \right]$$

However, I think they meant the version with  $\pi$  instead, because otherwise it doesn't work out.

We want this weird multiplication to be a map into  $U$ , not just a map into  $\text{GL}_n(K)$ , so let's confirm that. Let  $u_1, u_2 \in U$ . Then we can always write them in the form  $u_1 = \rho(r_1) = \pi(x_{13}(r_1))$  and  $u_2 = \rho(r_2) = \pi(x_{13}(r_2))$  for some  $r_1, r_2 \in R$ .

$$\begin{aligned} \rho(r_1) \times \rho(r_2) &= \left[ \pi(w_{23}) \cdot \rho(r_1) \cdot \pi(w_{23})^{-1}, \pi(w_{12}) \cdot \rho(r_2) \cdot \pi(w_{12})^{-1} \right] \\ &\text{rewrite } \rho(r_1) \text{ as } \pi(x_{13}(r_1)) \\ &= \left[ \pi(w_{23}) \cdot \pi(x_{13}(r_1)) \cdot \pi(w_{23})^{-1}, \pi(w_{12}) \cdot \pi(x_{13}(r_2)) \cdot \pi(w_{12})^{-1} \right] \\ &\text{use the fact that } \pi \text{ is a group homomorphism to "factor it out"} \\ &= \left[ \pi(w_{23} \cdot x_{13}(r_1) \cdot w_{23}^{-1}), \pi(w_{12} \cdot x_{13}(r_2) \cdot w_{12}^{-1}) \right] \\ &\text{move } \pi \text{ outside the commutator using the fact that it is a group homomorphism} \\ &= \pi \left( [w_{23} \cdot x_{13}(r_1) \cdot w_{23}^{-1}, w_{12} \cdot x_{13}(r_2) \cdot w_{12}^{-1}] \right) \\ &\text{now use Weyl group conjugation formulas} \\ &= \pi \left( [x_{12}(r_1), x_{23}(r_2)] \right) \\ &\text{now use commutator formula} \\ &= \pi \left( x_{13}(r_1 r_2) \right) \\ &= \rho(r_1 r_2) \end{aligned}$$

Ok, what was the point of all of this calculating? We defined a map  $\times : U \times U \rightarrow \text{GL}_n(K)$  by this weird conjugation formula, and what we just showed is that for any  $r_1, r_2 \in R$ , we have

$$\rho(r_1 r_2) = \rho(r_1) \times \rho(r_2)$$

Since  $\rho$  is a surjective map from  $R$  to  $U$ , every element of  $U$  can be written in the form  $\rho(r)$  for some  $r \in R$ . Thus, for any  $u_1, u_2 \in U$  we can write them as

$$u_1 = \rho(r_1) \quad u_2 = \rho(r_2)$$

and then

$$u_1 \times u_2 = \rho(r_1) \times \rho(r_2) = \rho(r_1 r_2)$$

In particular, the product  $u_1 \times u_2$  is in  $U$ . So the map we defined as a map  $U \times U \rightarrow \text{GL}_n(K)$  is actually a map  $U \times U \rightarrow U$ .

At this point, there is a bit of technical work to show that the two operations we have defined actually satisfy the axioms for a ring. This involves some details beyond the scope of our project, but it's not actually super interesting.



**Summary.** In their paper *Nonlinearity of matrix groups*, particularly in Definition 2 and Theorem 3, Kassabov and Sapir showed how using some of the formulas involved in a pinning of a matrix group can be useful to prove something about rigidity. In particular, they used pinning formulas related to the group  $\mathrm{SL}_n$ , and used the formulas to assist in defining a weird kind of multiplication operation on the set of matrices of the form

$$\begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix}$$

It's a few steps more from there to proving something about rigidity, but that's beyond our current scope.

## 4 Pinnings in Matlab

Matlab code is primarily built up through “scripts.” A script is a single file in the file directory. In a script, you can define as many functions as you want. However, each script has a “main” function, which is by definition the function whose name is the same as the name of the file. **Only the main function in a script is visible to the “outside world.”** In other words, only the main function in a script is callable outside of that script. So if you write a function, and want that function to be usable by other scripts or usable in command line, then you need that function to be the main function of its own file. Put another way, you should only define a function within a script that isn’t the main function if you only ever need that function within that script.

As much as possible, we will build our code using test-driven development. That is, before even writing functions, we should write automated tests which we expect our functions to pass. This way, we can build up code and be confident at every stage that the code we are writing meets our goals and specifications (or at least meets the ones we have been able to get to pass so far).

### 4.1 The symbolic toolbox

The symbolic toolbox is an add-in for Matlab which allows us to do calculations with symbolic variables. For example, try out the sample code below in Matlab’s command line environment, or copy and paste it into a script and run the script.

```
syms u
syms v
X = [[1,u];[0,1]]
Y = [[1,v];[0,1]]
X*Y
```

The code above is the Matlab version of doing the following calculation by hand.

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & uv \\ 0 & 1 \end{pmatrix}$$

You can also create a whole matrix of symbolic variables at once. For example, “`sym('x',[3,3])`” creates a  $3 \times 3$  matrix with entries `x1_1`, `x1_2`,  $\dots$  `x3_3`.

### 4.2 Overall coding goals

Let  $k$  be a field of characteristic  $\neq 2$ , and let  $G$  be a special linear, special unitary, or special orthogonal group over  $k$ . Let  $S$  be a maximal  $k$ -split torus subgroup in  $G$ , and let  $\Phi = \Phi(G, S)$  be the root system of  $G$  with respect to  $S$ .

For each  $\Phi$  and  $G$  as above, we want to encode  $\Phi$  as an instance of the `RootSystem` class, and encode  $G$  as an instance of the `PinnedGroup` class, and then use the `PinnedGroup.RunTests()` method to verify all of the properties of a pinning. The main building blocks required to build a `PinnedGroup` are

- Root space maps, and accompanying the root space dimension map
- Root subgroup maps (just the exponentiated version of root space maps)
- Weyl group maps
- Commutator coefficient maps
- Weyl group conjugation coefficient maps

## 4.3 Coding specifications

Here we describe what Matlab tools we have built or want to build to analyze and verify pinnings of algebraic groups.

### 4.3.1 General purpose functions

Functions which we will be able to reuse for different groups.

**Name:** SymbolicIsEqual

**Inputs:** Two matrices  $A$  and  $B$

**Outputs:** Boolean

**Description:** Return true if  $A$  and  $B$  are equal (as symbolic matrices). The built-in Matlab function `isequal(A,B)` will say that  $A$  and  $B$  are not equal if the corresponding entries of  $A$  and  $B$  are the same but different “types,” for example according to `isequal` the numerical (floating point) value 1 is different from 1 as a “symbolic expression.”

**Name:** SymbolicIsDiag

**Inputs:** Matrix  $A$

**Outputs:** Boolean

**Description:** Return true if  $A$  is a diagonal matrix, false otherwise. There is a built-in Matlab function `isdiag` which also does this, but sometimes (I can’t figure out exactly why) the built-in function doesn’t behave as expected when given matrices with symbolic variable entries.

**Name:** Commutator

**Inputs:** Two invertible matrices  $A, B$

**Outputs:** The commutator matrix  $[A, B] = ABA^{-1}B^{-1}$

**Description:** Should maybe validate that  $A$  and  $B$  are invertible.

### 4.3.2 NForm class

NForm is short for nondegenerate isotropic form. This class is a data structure which encodes a nondegenerate isotropic form, which may be either symmetric bilinear, hermitian, or skew-hermitian.

#### Properties

Name	What it is
Dimension	Dimension of the underlying vector space
Index	Witt index
NameString	String describing the type
Epsilon	1 for hermitian or symmetric bilinear, $-1$ for skew-hermitian
Matrix	Matrix of the bilinear form in the standard basis
AnisotropicPartVector	Vector encoding the diagonal entries of the matrix of the anisotropic part of the form in the Witt decomposition
AnisotropicMatrix	Matrix of the anisotropic part from the Witt decomposition
PrimitiveElement	Primitive element of the associated quadratic field extension, only relevant for (skew-)hermitian forms

#### Methods

**Name:** NForm (constructor)

**Inputs:** Integer  $n$ , integer  $q$ , sign  $\varepsilon$ , vector  $\text{vec}_C$ , symbolic variable PrimitiveElement, string NameString

**Outputs:** NForm object

**Description:** Constructs an NForm object with the prescribed parameters. More abstractly, the output NForm object is the form with matrix

$$\begin{pmatrix} & I_q \\ \varepsilon I_q & \\ & C \end{pmatrix}$$

where  $C = \text{diag}(\text{vec}_C)$ . This is a nondegenerate isotropic form with Witt index  $q$ . If  $\varepsilon = 1$ , it is symmetric bilinear or hermitian depending on  $C$ , and if  $\varepsilon = -1$  it is skew-hermitian (provided  $C$  meets a certain condition).

**Name:** IsSymmetric

**Inputs:** NForm object

**Outputs:** Boolean

**Description:** Returns true if the form is symmetric.

**Name:** IsHermitian  
**Inputs:** NIForm object  
**Outputs:** Boolean  
**Description:** Returns true if the form is hermitian.

**Name:** IsSkewHermitian  
**Inputs:** NIForm object  
**Outputs:** Boolean  
**Description:** Returns true if the form is skew-hermitian.

**Name:** evaluate  
**Inputs:** NIForm object  
**Outputs:** Scalar  
**Description:** Evaluate the form on two vectors. That is, if the form has matrix  $M$  and the input vectors are  $u$  and  $v$ , output  $uMv^T$ .

**Name:** evaluateAnisotropicPart  
**Inputs:** NIForm object  
**Outputs:** Scalar  
**Description:** Evaluate the anisotropic form on two vectors. That is, if the anisotropic part is  $C$  and the input vectors are  $u$  and  $v$ , output  $uCv^T$ .

### 4.3.3 RootSystem class

The RootSystem class is a data structure which encodes an abstract root system. The main substructure is the cell array RootList, which lists all the roots as vectors of the same length. The key useful methods are RootLinearCombos to compute the set of all positive integral linear combinations of two given roots, and the static method ReflectRoot for computing the reflection of one root across the hyperplane perpendicular to a second root.

#### Properties

Name	What it is
Type	String encoding root system type: A, B, C, BC, D, E, F, or G
Rank	Positive integer (dimension of the span of roots)
VectorLength	Length of vectors in RootList
RootList	Cell array containing vectors represnting individual roots

#### Methods

**Name:** RootSystem (constructor)

**Inputs:** Type (string), Rank (positive integer), VectorLength (positive integer, optional)

**Outputs:** An instance of the RootSystem class

**Description:** Given type, rank, and vector length, the constructor builds a root system of that type and rank, using vectors of the specified length. Depending on the type, the vector length needs to be at least a certain value relative to the rank. If no VectorLength is provided, then a default minimum value is used.

Type	$A_q$	$B_q$	$C_q$	$BC_q$	$D_q$	$E_q$	$F_4$	$G_2$
Default vector length	$q + 1$	$q$	$q$	$q$	$q$	8	4	3

**Name:** IsRoot

**Inputs:** A vector  $\alpha$

**Outputs:** Boolean

**Description:** Return true if  $\alpha$  is in RootList

**Name:** LinearCombos

**Inputs:** Two roots  $\alpha, \beta$

**Outputs:** Cell array containing a list of triples  $\{i\alpha + j\beta, i, j\}$  (in Matlab this is ordered, not an unordered set).

**Description:** Compute the set of positive integral linear combinations of  $\alpha, \beta$  that are roots.

$$(\alpha, \beta) = \{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}_{\geq 1}\}$$

Since it can be important for other computations which use this method to know what  $i$  and  $j$  are used, the output is not just a list of roots. Rather, the output is a list of triples of the form  $\{i\alpha + j\beta, i, j\}$  where  $i\alpha + j\beta$  is one of the integral linear combinations and  $i$  and  $j$  are the integer coefficients used to obtain that new root. For example, if  $\alpha = [2, 0, 0, 0]$  and  $\beta = [-1, 1, 0, 0]$  then

$$\text{LinearCombos}(\alpha, \beta) = \left\{ \{[1, 1, 0, 0], 1, 1\}, \{[0, 2, 0, 0], 1, 2\} \right\}$$

The output list is a cell array, and within it each triple is a cell array, so if you want to extract the first triple, you would use the syntax

$$\text{LinearCombos}(\alpha, \beta) \{1\}$$

which would output the triple  $\{[1, 1, 0, 0], 1, 1\}$  and then to extract the root from that first triple use

$$\text{LinearCombos}(\alpha, \beta) \{1\} \{1\}$$

**Name:** VerifyProperties

**Inputs:** None

**Outputs:** Text log

**Description:** Run tests to verify root system axioms (R1)-(R4). Note that this does not check that the root system is reduced, because we wish to include systems of type BC.

### Static methods

**Name:** ReflectRoot

**Inputs:** Two roots (vectors)  $\alpha, \beta$  of the same length

**Outputs:** The reflection of  $\beta$  across the hyperplane perpendicular to  $\alpha$

**Description:** The reflection is described by the formula

$$\sigma_{\alpha}(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

where here  $(\alpha, \beta)$  and  $(\alpha, \alpha)$  refer to the standard dot product.

**Name:** IsProportionalRoot

**Inputs:** Two roots  $\alpha, \beta$

**Outputs:** Boolean

**Description:** Return true if  $\alpha = c\beta$  for some scalar  $c$ .

**Name:** BuildTypeA

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $A_r$  root system with rank  $r$ , representing roots as vectors of specified length. Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$A_r = \{\alpha_i - \alpha_j : 1 \leq i, j \leq r \text{ and } i \neq j\}$$

If VectorLength exceeds  $r + 1$ , then we simply pad the end of every vector with enough zeros.

**Name:** BuildTypeB

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $B_r$  root system with specified rank  $r$ , representing

roots as vectors of specified length. Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$B_r = \{\pm\alpha_i : 1 \leq i \leq r\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq r \text{ and } i \neq j\}$$

If VectorLength exceeds  $r$ , then we simply pad the end of every vector with enough zeros.

**Name:** BuildTypeC

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $C_r$  root system with specified rank  $r$ , representing roots as vectors of specified length. Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$C_r = \{\pm 2\alpha_i : 1 \leq i \leq r\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq r \text{ and } i \neq j\}$$

If VectorLength exceeds  $r$ , then we simply pad the end of every vector with enough zeros.

**Name:** BuildTypeBC

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $BC_r$  (non-reduced) root system with specified rank  $r$ , representing roots as vectors of specified length. This is the union of the type  $B_r$  and  $C_r$  systems. Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$BC_r = \{\pm\alpha_i, \pm 2\alpha_i : 1 \leq i \leq r\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq r \text{ and } i \neq j\}$$

If VectorLength exceeds  $r$ , then we simply pad the end of every vector with enough zeros.

**Name:** BuildTypeD

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $D_r$  root system with specified rank  $r$ , representing roots as vectors of specified length. Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$D_r = \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq r \text{ and } i \neq j\}$$

If VectorLength exceeds  $r$ , then we simply pad the end of every vector with enough zeros.



**Name:** BuildTypeE

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $E_r$  root system, noting that we must have  $6 \leq r \leq 8$ . This is complicated to describe, so we do not describe it in detail here.

**Name:** BuildTypeF

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $F_4$  root system. (The rank must be 4.) Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$\begin{aligned} F_4 = & \{\pm 2\alpha_i : 1 \leq i \leq 4\} \\ & \cup \{\pm \alpha_i \pm \alpha_j : 1 \leq i, j \leq 4 \text{ and } i \neq j\} \\ & \cup \{\pm \alpha_i \pm \alpha_j \pm \alpha_k \pm \alpha_\ell : 1 \leq i, j, k, \ell \leq 4 \text{ and } i, j, k, \ell \text{ are all distinct}\} \end{aligned}$$

If VectorLength exceeds 4, then we simply pad the end of every vector with enough zeros.

**Name:** BuildTypeG

**Inputs:** Rank (positive integer), VectorLength (positive integer)

**Outputs:** Cell array

**Description:** Construct the type  $G_2$  root system. (The rank must be 2.) Using the notation  $\alpha_i = [0, \dots, 1, \dots, 0]$  with 1 in the  $i$ th position,

$$\begin{aligned} G_2 = & \{\pm(\alpha_i - \alpha_j) : 1 \leq i, j \leq 3 \text{ and } i \neq j\} \\ & \cup \{\pm(2\alpha_i - \alpha_j - \alpha_k) : 1 \leq i, j, k \leq 3 \text{ and } i, j, k \text{ are all distinct}\} \end{aligned}$$

If VectorLength exceeds 3, then we simply pad the end of every vector with enough zeros.

#### 4.3.4 PinnedGroup class

A class encoding all of the data of an algebraic group with a pinning. To run tests to verify a pinning, create an instance of PinnedGroup with all the required data, and use the RunTests() method.

### Properties

Name	What it is
NameString	String describing the group, e.g. “special linear group”
MatrixSize	Positive integer
Root_System	An instance of the RootSystem class
RootList	Cell array, substructure of Root_System
RootSystemRank	Positive integer, the rank of Root_System
FormMatrix	Matrix, only used for special orthogonal/unitary groups
RootSpaceDimension	Function handle
RootSpaceMap	Function handle for $\text{Lie}X_\alpha$
RootSubgroupMap	Function handle for $X_\alpha$
WeylGroupMap	Function handle for $W_\alpha$
GenerictorusElementMap	Function handle
IsGroupElement	Function handle, tests if an input matrix is in $G(k)$
IsTorusElement	Function handle, tests if an input matrix is in $S(k)$
IsLieAlgebraElement	Function handle, tests if an input matrix is in $\mathfrak{g}(k)$
CommutatorCoefficientMap	Function handle, outputs $N_{ij}^{\alpha\beta}(u, v)$
WeylGroupCoefficientMap	Function handle

All of the properties above starting with RootSpaceDimension are function handles. To get an idea of how this works, take a look at the RunSLTests script and see where this is used. Below we specify what the format of inputs for each of these functions needs to be in order for PinnedGroup.RunTests() to work properly.

Below are specifications for the RunTests script, which we use to test all of our code. You don’t need to know absolutely every detail of what’s going on in this script, but you do need to be aware of what kinds of inputs it requires, and more importantly what formats and inputs the inputs require.

Function	Inputs	Type
RootSpaceDimension	MatrixSize Root_System Root $\alpha$	Integer RootSystem Vector of length MatrixSize
RootSpaceMap	MatrixSize Root_System FormMatrix Root $\alpha$ Vector $u$	Integer RootSystem Matrix Vector of length MatrixSize Vector of length RootSpaceDimension( $\alpha$ )
RootSubgroupMap	MatrixSize Root_System FormMatrix Root $\alpha$ Vector $u$	Integer RootSystem Matrix Vector of length MatrixSize Vector of length RootSpaceDimension( $\alpha$ )
WeylGroupMap	MatrixSize Root_System FormMatrix Root $\alpha$ Vector $u$	Integer RootSystem Matrix Vector of length MatrixSize Vector of length RootSpaceDimension( $\alpha$ )
GenericTorus ElementMap	MatrixSize RootSystemRank Diagonal values	Integer Integer Vector of length RootSystemRank
IsGroupElement	MatrixSize MatrixToTest FormMatrix	Integer Matrix Matrix
IsTorusElement	MatrixSize MatrixToTest FormMatrix	Integer Matrix Matrix
IsLieAlgebraElement	MatrixSize MatrixToTest FormMatrix	Integer Matrix Matrix
Commutator CoefficientMap	MatrixSize Root_System Root $\alpha$ Root $\beta$ $p$ $q$ $u$ $v$	Integer RootSystem Vector of length MatrixSize Vector of length MatrixSize Integer Integer Vector of length RootSpaceDimension( $\alpha$ ) Vector of length RootSpaceDimension( $\beta$ )
WeylGroup CoefficientMap	MatrixSize Root_System FormMatrix Root $\alpha$ Root $\beta$ $u$	Integer RootSystem Matrix Vector of length MatrixSize Vector of length MatrixSize Vector of length RootSpaceDimension( $\alpha$ )

Function	Output type
RootSpaceDimension	Integer
RootSpaceMap	Matrix of size MatrixSize
RootSubgroupMap	Matrix of size MatrixSize
WeylGroupMap	Matrix of size MatrixSize
GenericTorusElementMap	Matrix of size MatrixSize
IsGroupElement	Boolean
IsTorusElement	Boolean
IsLieAlgebraElement	Boolean
CommutatorCoefficientMap	Vector
WeylGroupCoefficientMap	Vector

### Methods

**Name:** PinnedGroup (constructor)

**Inputs:** NameString, MatrixSize, Root\_System, FormMatrix, RootSpaceDimension, RootSpaceMap, RootSubgroupMap, WeylGroupMap, GenericTorusElementMap, IsGroupElement, IsTorusElement, IsLieAlgebraElement, CommutatorCoefficientMap, WeylGroupCoefficientMap

**Outputs:** A new instance of the PinnedGroup class built from the input data

**Description:** The constructor doesn't do much of anything other than create a new instance of the class, and assign its properties to the various inputs.

**Name:** RunTests

**Inputs:** None

**Outputs:** Text log

**Description:** Do symbolic matrix calculations to verify that the data above satisfies all of the properties of a pinning of an algebraic group. Each test is outsourced to another method.

- (TestBasics) Check that  $\text{Lie}X_\alpha$  always outputs something in  $\mathfrak{g}(k)$
- (TestBasics) Check that  $X_\alpha(u)$  always outputs something in  $G(k)$
- (TestRootSpaceMapsAreHomomorphisms) Check that for  $\alpha \in \Phi$ ,

$$\text{Lie}X_\alpha(u + v) = \text{Lie}X_\alpha(u) + \text{Lie}X_\alpha(v)$$

- (TestRootSubgroupMapsAreAlmostHomomorphisms) Check that for  $\alpha \in \Phi$ ,

$$X_\alpha(u) \cdot X_\alpha(v) = X_\alpha(u + v) \prod_{i \geq 2} X_{i\alpha} \left( q_\alpha^i(u, v) \right)$$

Note that the product over  $i \geq 2$  is trivial (the identity) for all root systems except for type BC.

- (TestTorusConjugationFormula) Check that for  $s \in S(k)$  and  $\alpha \in \Phi$ ,

$$s \cdot X_\alpha(u) \cdot s^{-1} = X_\alpha(\alpha(s)u)$$

- (TestCommutatorFormula) Check the commutator formula: for  $\alpha, \beta \in \Phi$  a pair of non-proportional roots,

$$\left[ X_\alpha(u), X_\beta(v) \right] = \prod_{i,j \geq 1} X_{i\alpha+j\beta} \left( N_{ij}^{\alpha\beta}(u, v) \right)$$

- (TestWeylGroupElements) Check that for  $\alpha \in \Phi$  and  $s \in S(k)$ ,

$$w_\alpha(u) \cdot s \cdot w_\alpha(u)^{-1} \in S(k)$$

- (TestWeylGroupConjugationFormula) Check the Weyl group conjugation formula: for  $\alpha, \beta \in \Phi$ ,

$$w_\alpha(u) \cdot X_\beta(v) \cdot w_\alpha(u)^{-1} = X_{\sigma_\alpha(\beta)}(\varphi u)$$

where  $\varphi u$  is something like  $\pm u$  or  $\pm \bar{u}$ .

## Static methods

**Name:** CharacterEval

**Inputs:** An integer  $n = \text{MatrixSize}$ , and a character  $\alpha$  ( $n \times 1$  integer vector) and a diagonal matrix  $t = \text{diag}(t_1, \dots, t_n)$

**Outputs:** A scalar (possibly a number, or symbolic variable)

**Description:** Given a  $n \times 1$  integer vector  $\alpha = [v_1, \dots, v_n]$ , interpret it as the character  $v_1\alpha_1 + \dots + v_n\alpha_n$ , then evaluate that character at the diagonal matrix  $t$ .

$$(v_1\alpha_1 + \dots + v_n\alpha_n) \text{diag}(t_1, \dots, t_n) = \prod_{i=1}^n t_i^{v_i} = t_1^{v_1} \dots t_n^{v_n}$$

## 4.4 Example – $\text{SL}_n$ as a PinnedGroup object

In this section we describe the functions behind creating an instance of the PinnedGroup class modeling the special linear group. Refer to the RunSLTests.m file for details on how exactly to create the PinnedGroup class instance and run tests.

**Name:** LieX\_SL

**Inputs:** An integer  $n$ , an instance of the  $A_{n-1}$  root system using the RootSystem class, a placeholder input for FormMatrix (just input the identity or zero matrix, it doesn't matter), a root  $\alpha$  from the  $A_{n-1}$  root system ( $n \times 1$  vector), and a symbolic variable  $u$

**Outputs:** An element of the root space associated with  $\alpha$  (zeros everywhere except a  $u$  in the  $ij$ th entry)

**Description:** For example,

$$\text{LieX\_SL}(3, A_2, \sim, \alpha_{12}, u) = \begin{pmatrix} 0 & u & \\ & 0 & \\ & & 0 \end{pmatrix}$$

More generally,  $\text{LieX\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u)$  is the  $n \times n$  matrix with zeros everywhere except for  $u$  in the  $ij$ th entry (this matrix is often denoted  $E_{ij}(u)$ ).

**Name:** X\_SL

**Inputs:** An integer  $n$ , an instance of the  $A_{n-1}$  root system using the RootSystem class, a placeholder input for FormMatrix (just input the identity or zero matrix, it doesn't matter), a root  $\alpha$  from the  $A_{n-1}$  root system ( $n \times 1$  vector), and a symbolic variable  $u$

**Outputs:** An element of the root subgroup associated with  $\alpha$

**Description:** This is just the matrix exponential of  $\text{LieX\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u)$ .

$$\text{X\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u) = \exp \left( \text{LieX\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u) \right)$$

In general, the matrix exponential is

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots$$

However, the matrices  $\text{LieX\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u)$  all satisfy  $A^2 = 0$ , so the matrix exponential is just  $I + A$ . Conveniently, this means we don't have to worry about convergence questions regarding the infinite sum above.

$$\text{X\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u) = I + \text{LieX\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u)$$

For example,

$$\text{X\_SL}(n, A_{n-1}, \sim, \alpha_{12}, u) = \begin{pmatrix} 1 & u & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Another way to say this is that  $\text{X\_SL}(n, A_{n-1}, \sim, \alpha_{ij}, u) = e_{ij}(u)$ .

**Name:** W\_SL

**Inputs:** An integer  $n$ , an instance of the  $A_{n-1}$  root system using the RootSystem class, a placeholder input for FormMatrix (just input the identity or zero matrix, it doesn't matter), a root  $\alpha$  from the  $A_{n-1}$  root system ( $n \times 1$  vector), and a symbolic variable  $u$

**Outputs:** The matrix  $w_\alpha(u)$

**Description:** The formula for  $w_\alpha(u)$  is

$$w_\alpha(u) = X_\alpha(u) \cdot X_{-\alpha}(-u^{-1}) \cdot X_\alpha(u)$$

**Name:** CommutatorCoefficientSL

**Inputs:** An integer  $n$ , an instance of the  $A_{n-1}$  root system using the RootSystem class, two roots  $\alpha, \beta \in \Phi(\text{SL}_n, T)$ , two positive integers  $p$  and  $q$ , and two scalars  $u$  and  $v$

**Outputs:** A scalar (should always be  $\pm uv$ )

**Description:** Note that for  $\text{SL}_n$ , we always have  $p = q = 1$ . (We still need these as inputs to the function to comply with PinnedGroup.)

If  $\alpha + \beta$  is not a root, then output zero. If  $\alpha + \beta$  is a root, output the coefficient  $N_{pq}^{\alpha\beta}(u, v)$  from the formula

$$\left[ X_\alpha(u), X_\beta(v) \right] = X_{\alpha+\beta} \left( N_{11}^{\alpha\beta}(u, v) \right)$$

This coefficient  $N(u, v)$  should always be  $\pm uv$ . Concretely, if  $\alpha + \beta$  is a root, then they must be of the form  $\alpha = \alpha_{ij}$  and  $\beta = \alpha_{k\ell}$  where there are exactly three distinct indices in the set  $\{i, j, k, \ell\}$ , and

$$N_{11}^{\alpha_{ij}, \alpha_{k\ell}}(u, v) = \begin{cases} uv & i = k \text{ or } j = k \\ -uv & i = \ell \text{ or } j = \ell \end{cases}$$

**Name:** WeylGroupConjugationCoefficientSL

**Inputs:** An integer  $n$ , an instance of the  $A_{n-1}$  root system using the RootSystem class, two roots  $\alpha, \beta$ , and a scalar (often symbolic variable)  $u$

**Outputs:** A scalar (either  $u$  or  $-u$ )

**Description:** This should output  $\varphi_{\alpha\beta}(u) = \pm u$  appearing on the right hand side of the equation

$$w_\alpha(1) \cdot X_\beta(u) \cdot w_\alpha(1)^{-1} = X_{\sigma_\alpha(\beta)}(\varphi_{\alpha\beta}(u))$$

More concretely, if  $\alpha = \alpha_{ij}$  and  $\beta = \alpha_{k\ell}$ , then

$$\varphi_{\alpha\beta}(u) = \begin{cases} -u & i = k \text{ or } i = \ell \\ u & \text{else} \end{cases}$$

## 5 Special orthogonal groups

In this section I describe special orthogonal groups associated to bilinear forms.

**Definition 5.0.1.** Let  $b$  be a nondegenerate symmetric bilinear form on a  $k$ -vector space  $V$  of dimension  $n$ , with matrix  $B$  (with respect to a fixed basis of  $V$ ). The **orthogonal group** of  $B$  is the group of  $n \times n$  invertible matrices  $X$  with entries in  $k$  satisfying  $X^T B X = B$ .

$$O_n(k, b) = \{X \in GL_n(k) : X^T B X = B\}$$

The **special orthogonal group** of  $B$  is the subgroup of  $O_n(B, k)$  of matrices with determinant equal to one.

$$SO_n(k, b) = \{X \in SL_n(k) : X^T B X = B\}$$

In the future, we will also want to specify the Witt index of  $b$  in the notation, so let  $q$  be the Witt index of  $b$  (or  $B$  equivalently). Then we denote the special orthogonal group by

$$SO_{n,q}(k, b)$$

When it is clear from context, we can also just write this as  $SO(k, b)$  or even just  $SO(k)$ .

**Exercise 5.0.2.** Verify that  $O_{n,q}(k, b)$  is a group under matrix multiplication and that  $SO_{n,q}(k, b)$  is a subgroup.

**Exercise 5.0.3.** Show that if  $b$  and  $b'$  are equivalent bilinear forms, then the associated orthogonal groups  $O_{n,q}(k, b)$  and  $O_{n,q}(k, b')$  are isomorphic. Hint: if  $B' = P^T B P$ , then

$$X^T B' X = B' \iff X^T (P^T B P) X = P^T B P \iff ((P^T)^{-1} X^T P^T) B (P X P^{-1}) = B$$

and it is relevant that  $(P^T)^{-1} = (P^{-1})^T$ .

**Exercise 5.0.4.** Verify that the isomorphism you found in the previous exercise also gives an isomorphism between  $SO_{n,q}(k, b)$  and  $SO_{n,q}(k, b')$ .

**Remark 5.0.5.** The main takeaway from the exercise is: changing a bilinear form to an equivalent bilinear form doesn't change the group (up to isomorphism), so if we just want to study the group, then we are free to use whichever equivalent bilinear form we like. Because of Theorem 3.1.18, we can always choose a basis so that our nondegenerate symmetric bilinear form is diagonal, which makes it have an associated diagonal matrix. Then because of Exercise 5.0.3, changing our basis in this way doesn't affect the group. So any orthogonal or special orthogonal group can be described using an invertible diagonal matrix (every diagonal matrix is symmetric, so now symmetric becomes redundant). However, it is not always best to assume that  $B$  is completely diagonalized.



## 5.1 Split, quasisplit, and neither

Let  $k$  be an arbitrary field of characteristic  $\neq 2$ . If it helps, you can usually pretend that  $k = \mathbb{R}$  or  $k = \mathbb{Q}$  without running into any problems.

Let  $V$  be a vector space over  $k$  of dimension  $n$ , and let  $b$  be a nondegenerate symmetric isotropic bilinear form on  $V$ . Fix a basis of  $V$ , and let  $B$  be the matrix of  $b$  in that basis. Then  $B$  is a symmetric, invertible matrix. We could use Theorem 3.1.18 to assume that  $B$  is a diagonal matrix, but we don't actually want to do this.

Let  $Q$  be the quadratic form associated to  $b$ , and let  $q$  be the Witt index of  $Q$ . Since  $b$  is isotropic,  $q \geq 1$ . Note that  $q \leq \frac{n}{2}$ . By the Witt decomposition theorem, we can write  $V$  as

$$V = V_t \oplus V_o \oplus V_h$$

where  $V_t$  is totally isotropic (meaning  $b(v, w) = 0$  for all  $v, w \in V_t$ ) and  $V_o$  is anisotropic and  $V_h$  is a direct sum of  $q$  hyperbolic planes (so  $\dim V_h = 2q$ ). Since  $b$  is nondegenerate, the  $V_t$  part must be zero, so actually

$$V = V_o \oplus V_h$$

We can write out  $V_h$  as a direct sum of 2-dimensional spaces, so that

$$V = V_o \oplus V_h = V_a \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_q$$

where each  $H_i$  has dimension 2. The value of  $q$  determines the following: <sup>4</sup>

- If  $q$  is as big as possible, which is to say,  $q = \frac{n}{2}$  or  $q = \frac{n-1}{2}$  then the special orthogonal group  $\mathrm{SO}(k, b)$  is called “ $k$ -split” which for our purposes means it's already understood.
- If  $q$  is almost as big as possible, which it to say,  $q = \frac{n}{2} - 1$  (or equivalently  $2q + 2 = n$  so notably  $n$  must be even) then  $\mathrm{SO}(k, b)$  is quasisplit.
- In the remaining case,  $n > 2q + 2$  and  $\mathrm{SO}(k, b)$  is neither split nor quasisplit.

We will always work in the case  $q \geq 1$ . The split case is already reasonably well understood, so we will ignore that case and assume  $n \geq 2q + 2$ . Using the Witt decomposition, we can choose a basis of  $V$  so that in that basis the bilinear form has the matrix

$$B = \begin{pmatrix} & I_q & \\ I_q & & \\ & & C \end{pmatrix}$$

where  $I_q$  is the  $q \times q$  identity matrix and  $C$  is an  $(n - 2q) \times (n - 2q)$  invertible diagonal matrix. When working on examples, the following list of minimal cases can be helpful. The small quasisplit cases are  $(n, q) = (4, 1), (6, 2), (8, 3)$  etc. and the small non-quasisplit cases are  $(n, q) = (5, 1), (6, 2), (7, 2), (8, 3)$  etc.

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<sup>4</sup>See section 34.6 (page 222) of Humphrey's book on linear algebraic groups, or page 256 of Borel.

**Exercise 5.1.1.** Let  $B$  be the matrix above for the minimal quasisplit case  $n = 4, q = 1$ .

$$B = \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & c_1 & \\ & & & c_2 \end{pmatrix}$$

Assume that the fixed basis of  $V = k^n$  is  $\{e_1, \dots, e_4\}$  where  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  and so on. Show that

- The associated bilinear form  $b_C$  for the matrix  $C$  is

$$b_C \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = c_1 x_1 y_1 + c_2 x_2 y_2$$

- The associated bilinear form  $b_B$  for the matrix  $B$  is

$$b_B \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \right) = x_2 y_1 + x_1 y_2 + c_1 x_3 y_3 + c_2 x_4 y_4$$

- The associated quadratic form for the bilinear form  $b_C$  is

$$q_C \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = c_1 x_1^2 + c_2 x_2^2$$

- The associated quadratic form for the bilinear form  $b_B$  is

$$q_B \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = 2x_1 x_2 + c_1 x_3^2 + c_2 x_4^2$$

**Exercise 5.1.2.** Generalize the previous descriptions of the bilinear and quadratic forms of  $B$  and  $C$  to the general case where  $B$  has the form

$$B = \begin{pmatrix} & & I_q & \\ & I_q & & \\ & & & C \end{pmatrix}$$

where  $C = \text{diag}(c_1, \dots, c_{n-2q})$  and  $q \geq 1$  and  $n > 2q$ .

## 5.2 Special orthogonal Lie algebra

**Definition 5.2.1.** Let  $B$  be the  $(n \times n)$  matrix of a nondegenerate symmetric bilinear form with Witt index  $q \geq 1$ , and let  $\mathrm{SO}_{n,q}(k, b)$  be the associated special orthogonal group. The **special orthogonal Lie algebra** associated to  $B$  is

$$\mathfrak{so}_{n,q}(k, b) = \{X \in M_n(k) : X^T B + BX = 0\}$$

**Exercise 5.2.2.** Recall that we can write  $B$  in the form

$$B = \begin{pmatrix} & I_q & \\ I_q & & \\ & & C \end{pmatrix}$$

where  $C$  is a  $(n-2q) \times (n-2q)$  invertible matrix. Using this form, show that if  $X \in \mathfrak{so}_{n,q}(k, b)$ , then  $X$  has the block form

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

where the blocks have the same size as the corresponding blocks of  $B$ , and these blocks satisfy the matrix equations

$$\begin{aligned} X_{21}^T + X_{21} &= 0 \\ X_{11}^T + X_{22} &= 0 \\ X_{31}^T C + X_{23} &= 0 \\ X_{22}^T + X_{11} &= 0 \\ X_{12}^T + X_{12} &= 0 \\ X_{32}^T C + X_{13} &= 0 \\ X_{23}^T + C X_{31} &= 0 \\ X_{13}^T + C X_{32} &= 0 \\ X_{33}^T C + C X_{33} &= 0 \end{aligned}$$

Alternately, we can remove the redundant equations and do some manipulation to rewrite this as

$$\begin{aligned} X_{21} &= -X_{21}^T \\ X_{12} &= -X_{12}^T \\ X_{22} &= -X_{11}^T \\ X_{23} &= -X_{31}^T C \\ X_{13} &= -X_{32}^T C \\ X_{33} &= -C^{-1} X_{33}^T C \end{aligned}$$

so  $X$  has the form

$$X = \begin{pmatrix} X_{11} & X_{12} & -X_{32}^T C \\ X_{21} & -X_{11}^T & -X_{31}^T C \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

where  $X_{11}, X_{31}, X_{32}$  are arbitrary and  $X_{21}, X_{12}, X_{33}$  are subject to the previous constraint equations.

### 5.3 Torus subgroup, character group, and root system

Let  $q \geq 1$  and  $n > 2q$ , and let  $B$  be a the matrix of a nondegenerate symmetric bilinear form on  $k^n$  with Witt index  $q$ . The assumption  $n > 2q$  ensures that  $C$  is not an empty block.

$$B = \begin{pmatrix} & I_q \\ I_q & \\ & & C \end{pmatrix}$$

$C$  is an  $(n - 2q) \times (n - 2q)$  diagonal matrix with entries  $c_1, \dots, c_{n-2q} \in k^\times$ . Then consider the associated special orthogonal group  $\mathrm{SO}(k, b)$ .

$$\mathrm{SO}_{n,q}(k, b) = \mathrm{SO}_n(k, b) = \{X \in \mathrm{SL}_n(k) : X^T B X = B\}$$

**Exercise 5.3.1.** Suppose  $s \in \mathrm{SO}_{n,q}(k, b)$  is a diagonal matrix. Show that  $s$  has the form

$$s = \begin{pmatrix} s_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & s_q & & & & & & \\ & & & s_1^{-1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & s_q^{-1} & & & \\ & & & & & & \pm 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & \pm 1 \end{pmatrix}$$

for some  $s_1, \dots, s_q \in k^\times$ . (Each  $\pm 1$  in the middle block has an independent sign.)

**Definition 5.3.2.** The **diagonal torus** or **torus subgroup** of  $\mathrm{SO}_{n,q}(k, b)$  is the following subgroup of diagonal matrices.

$$S(k) = \left\{ \begin{pmatrix} s_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & s_q & & & & & & \\ & & & s_1^{-1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & s_q^{-1} & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} : s_1, \dots, s_q \in k^\times \right\}$$

In other words,  $S(k)$  is a subgroup of  $\mathrm{SO}_{n,q}(k, b)$ , but it does not consist of all diagonal matrices in  $\mathrm{SO}(k, b)$ . Rather, it only consists of those for which the bottom right diagonal block is the identity matrix.

**Definition 5.3.3.** For  $i = 1, \dots, n$ , let  $\alpha_i : S(k) \rightarrow k^\times$  be the map which picks off the  $i$ th diagonal entry of a diagonal matrix  $s \in S(k)$ . (The notation in Borel [Bor91] section 23.9 uses  $y_i$  instead of  $\alpha_i$ , but means exactly the same thing.) You can verify that  $\alpha_i$  is a group homomorphism.

**Definition 5.3.4.** The **character group** of  $S(k)$  is the set of group homomorphisms from  $S(k)$  to  $k^\times$ , and is denoted  $X^*(S)$  or  $\text{Hom}(S(k), k^\times)$ .  $X^*(S)$  is an abelian group, and we call the operation “character addition.” For example,  $\alpha_1$  and  $\alpha_2$  are characters and  $\alpha_1 + \alpha_2$  is another character described by the formula

$$(\alpha_1 + \alpha_2)(s) = \alpha_1(s)\alpha_2(s)$$

or in more concrete matrix terms,

$$(\alpha_1 + \alpha_2) \left( \begin{pmatrix} s_1 & & & & & \\ & \ddots & & & & \\ & & s_q & & & \\ & & & s_1^{-1} & & \\ & & & & \ddots & \\ & & & & & s_q^{-1} \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \right) = s_1 s_2$$

**Remark 5.3.5.** Assume  $n > 2q$ . Later, we will do a calculation to work out that the root system of  $\text{SO}_{n,q}(k, b)$  is

$$\Phi = \Phi(\text{SO}_{n,q}(k, b), S(k)) = \{\pm\alpha_i : 1 \leq i \leq q\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq q, i \neq j\}$$

This is the type  $\mathbf{B}_q$  root system. (Note that if  $n = 2q$ , the root system is type  $\mathbf{D}_q$ , but we do not deal with this case.) Elements of  $\Phi$  are also elements of the character group  $X^*(S)$ , but we call them roots.

**Exercise 5.3.6.** Show that if  $q \geq 2$ , then  $\Phi$  as above contains  $2q^2$  roots.

**Example 5.3.7.** Our minimal case of interest is  $(n, q) = (4, 1)$ , so the root system is  $\mathbf{B}_1$ . The  $\mathbf{B}_1$  and  $\mathbf{A}_1$  root systems are the same (they are isomorphic as root systems, whatever that means).  $\mathbf{B}_1$  has just 2 elements.

$$\Phi = \{\pm\alpha_1\}$$

The next smallest quasisplit example is  $(n, q) = (6, 2)$ , where the root system is  $\mathbf{B}_2$ . Incidentally,  $\mathbf{B}_2$  and  $\mathbf{C}_2$  are isomorphic (as root systems).  $\mathbf{B}_2$  has 8 elements.

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_1 \pm \alpha_2\}$$

And finally the next smallest is  $(n, q) = (8, 3)$ , where the root system is  $\mathbf{B}_3$ , which has 18 elements. (Note that  $\mathbf{B}_3$  is not isomorphic to  $\mathbf{C}_3$ .)

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_1 \pm \alpha_2, \pm\alpha_1 \pm \alpha_3, \pm\alpha_2 \pm \alpha_3\}$$

## 5.4 Pinning special orthogonal groups

In this section we describe pinning of special orthogonal groups. That is, given the special orthogonal group  $\mathrm{SO}_{n,q}(k, b)$ , with root system  $\Phi$ , we want to have a map  $X_\alpha$  for every root  $\alpha \in \Phi$  with all the various pinning properties.

### 5.4.1 $\mathrm{SO}_{4,1}(k, b)$ minimal quasisplit case

The smallest possible case of interest to study is the case where  $V$  is a  $k$ -dimensional vector space of dimension 4, and  $b$  is a bilinear form on  $V$  with Witt index 1. Then the matrix  $B$  of the bilinear form can be written as

$$B = \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & & C \\ 1 & & & \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & c_1 & \\ 1 & & & c_2 \end{pmatrix}$$

In this case, the equations from Exercise 5.2.2 imply that an arbitrary element of  $\mathfrak{so}_{4,1}(k, b)$  has the form

$$X = \begin{pmatrix} x_{11} & 0 & -c_1 x_{32} & -c_2 x_{42} \\ 0 & -x_{11} & -c_1 x_{31} & -c_2 x_{41} \\ x_{31} & x_{32} & 0 & x_{34} \\ x_{41} & x_{42} & -\frac{c_1}{c_2} x_{34} & 0 \end{pmatrix}$$

where  $x_{11}, x_{31}, x_{41}, x_{32}, x_{42}, x_{34}$  are arbitrary elements of  $k$ .

**Example 5.4.1.** Next we work out the roots, root spaces, and root subgroups for  $\mathrm{SO}_{4,1}(k, b)$ . Let  $X$  be an arbitrary element of  $\mathfrak{so}_{4,1}(k, b)$  as above. Let  $s \in S(k)$  be an arbitrary element of the torus subgroup.

$$s = \begin{pmatrix} s_1 & & & \\ & s_1^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

We then calculate the conjugation  $sXs^{-1}$  in general.

$$\begin{aligned} sXs^{-1} &= \begin{pmatrix} s_1 & & & \\ & s_1^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_{11} & 0 & -c_1 x_{32} & -c_2 x_{42} \\ 0 & -x_{11} & -c_1 x_{31} & -c_2 x_{41} \\ x_{31} & x_{32} & 0 & x_{34} \\ x_{41} & x_{42} & -\frac{c_1}{c_2} x_{34} & 0 \end{pmatrix} \begin{pmatrix} s_1 & & & \\ & s_1^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} s_1 x_{11} & 0 & -s_1 c_1 x_{32} & -s_1 c_2 x_{42} \\ 0 & -s_1^{-1} x_{11} & -s_1^{-1} c_1 x_{31} & -s_1^{-1} c_2 x_{41} \\ x_{31} & x_{32} & 0 & x_{34} \\ x_{41} & x_{42} & -\frac{c_1}{c_2} x_{34} & 0 \end{pmatrix} \begin{pmatrix} s_1^{-1} & & & \\ & s_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\ &= \begin{pmatrix} x_{11} & 0 & -s_1 c_1 x_{32} & -s_1 c_2 x_{42} \\ 0 & -x_{11} & -s_1^{-1} c_1 x_{31} & -s_1^{-1} c_2 x_{41} \\ s_1^{-1} x_{31} & s_1 x_{32} & 0 & x_{34} \\ s_1^{-1} x_{41} & s_1 x_{42} & -\frac{c_1}{c_2} x_{34} & 0 \end{pmatrix} \end{aligned}$$

If this is going to be equal to  $\alpha(s)X$  for some  $\alpha$ , with  $X$  being nonzero, this is only going to be possible if  $\alpha(s) = s_1$  or  $\alpha(s) = s_1^{-1}$ . That is, the roots are  $\alpha_1$  and  $-\alpha_1$ , and

$$\mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & -c_1 x_{32} & -c_2 x_{42} \\ & 0 & \\ x_{32} & 0 & \\ x_{42} & & 0 \end{pmatrix} : x_{32}, x_{42} \in k \right\}$$

$$\mathfrak{g}_{-\alpha_1} = \left\{ \begin{pmatrix} 0 & & & \\ & 0 & -c_1 x_{31} & -c_2 x_{41} \\ x_{31} & 0 & & \\ x_{41} & & 0 & \end{pmatrix} : x_{31}, x_{41} \in k \right\}$$

so the  $\text{LieX}_\alpha$  maps are

$$\text{LieX}_{\alpha_1} : k^2 \rightarrow \text{SO}_{4,1}(k, b) \quad \text{LieX}_{\alpha_1}(x_{32}, x_{42}) = \begin{pmatrix} 0 & -c_1 x_{32} & -c_2 x_{42} \\ & 0 & \\ x_{32} & 0 & \\ x_{42} & & 0 \end{pmatrix}$$

$$\text{LieX}_{-\alpha_1} : k^2 \rightarrow \text{SO}_{4,1}(k, b) \quad \text{LieX}_{-\alpha_1}(x_{31}, x_{41}) = \begin{pmatrix} 0 & & & \\ & 0 & -c_1 x_{31} & -c_2 x_{41} \\ x_{31} & 0 & & \\ x_{41} & & 0 & \end{pmatrix}$$

**Definition 5.4.2.** We develop some notation to make referring to these rows in particular blocks easier. Given a row vector  $v = (v_1, \dots, v_m) \in k^m$ , let  $E_{i, \ell \leq j \leq \ell+m}(v)$  denote the matrix with the vector  $v$  in the entries  $E_{i\ell}, \dots, E_{i, \ell+m}$ . Similarly,  $E_{\ell \leq i \leq \ell+m, j}(v^T)$  is the matrix with the column vector  $v^T$  in the entries  $E_{\ell j}, \dots, E_{\ell+m, j}$ . Using this notation,

$$\text{LieX}_{\alpha_1}(v_1, v_2) = \text{LieX}_{\alpha_1}(v) = E_{3 \leq i \leq 4, 2}(v^T) - E_{1, 3 \leq 4 \leq j}(vC)$$

$$\text{LieX}_{-\alpha_1}(v_1, v_2) = \text{LieX}_{-\alpha_1}(v) = E_{3 \leq i \leq 4, 1}(v^T) - E_{2, 3 \leq j \leq 4}(vC)$$

There are two things going on with  $\text{SO}_{4,1}$  that did not happen in the  $\text{SL}_2(k)$  case. First,  $\mathfrak{g}_{\pm\alpha_1}$  is two-dimensional, not one-dimensional. Second, elements of  $\mathfrak{g}_{\alpha_1}$  and  $\mathfrak{g}_{\alpha_2}$  do not square to zero.

$$\begin{pmatrix} 0 & -c_1 x_{32} & -c_2 x_{42} \\ & 0 & \\ x_{32} & 0 & \\ x_{42} & & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -(c_1 x_{32}^2 + x_2 x_{42}^2) & \\ & 0 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Q_C(x_{32}, x_{42}) & \\ & 0 & \\ & & 0 \end{pmatrix}$$

Interestingly, the thing that shows up is actually the output of the quadratic form  $Q_C$  associated to  $C$  applied to the vector  $(x_{32}, x_{42})$ . For this reason, it is probably good for us

to understand more about quadratic forms. Putting this aside for now, we see that

$$\begin{aligned}
U_{\alpha_1} = \exp(\mathfrak{g}_{\alpha_1}) &= \left\{ \begin{pmatrix} 0 & -\frac{1}{2}(c_1x_{32}^2 + c_2x_{42}^2) & -c_1x_{32} & -c_2x_{42} \\ & 0 & & \\ & x_{32} & 0 & \\ & x_{42} & & 0 \end{pmatrix} : x_{32}, x_{42} \in k \right\} \\
&= \left\{ \begin{pmatrix} 0 & -\frac{1}{2}Q_C(x_{32}, x_{42}) & -c_1x_{32} & -c_2x_{42} \\ & 0 & & \\ & x_{32} & 0 & \\ & x_{42} & & 0 \end{pmatrix} \right\} \\
U_{-\alpha_1} = \exp(\mathfrak{g}_{-\alpha_1}) &= \left\{ \begin{pmatrix} 0 & & & \\ -\frac{1}{2}(c_1x_{31}^2 + c_2x_{41}^2) & 0 & -c_1x_{31} & -c_2x_{41} \\ & x_{31} & 0 & \\ & x_{41} & & 0 \end{pmatrix} : x_{31}, x_{41} \in k \right\} \\
&= \left\{ \begin{pmatrix} 0 & & & \\ -\frac{1}{2}Q_C(x_{31}, x_{41}) & 0 & -c_1x_{31} & -c_2x_{41} \\ & x_{31} & 0 & \\ & x_{41} & & 0 \end{pmatrix} \right\}
\end{aligned}$$

Thus the pinning maps for  $\mathrm{SO}_{4,1}(k, b)$  are

$$\begin{aligned}
X_{\alpha_1} : k^2 \rightarrow \mathrm{SO}_{4,1}(k, b) \quad X_{\alpha_1}(x_{32}, x_{42}) &= \begin{pmatrix} 0 & -\frac{1}{2}(c_1x_{32}^2 + c_2x_{42}^2) & -c_1x_{32} & -c_2x_{42} \\ & 0 & & \\ & x_{32} & 0 & \\ & x_{42} & & 0 \end{pmatrix} \\
X_{-\alpha_1} : k^2 \rightarrow \mathrm{SO}_{4,1}(k, b) \quad X_{-\alpha_1}(x_{31}, x_{41}) &= \begin{pmatrix} 0 & & & \\ -\frac{1}{2}(c_1x_{31}^2 + c_2x_{41}^2) & 0 & -c_1x_{31} & -c_2x_{41} \\ & x_{31} & 0 & \\ & x_{41} & & 0 \end{pmatrix}
\end{aligned}$$

**Definition 5.4.3** (Weyl group elements). Let  $u = (u_1, u_2) \in k^2$  with  $Q_C(u) \neq 0$  and let  $\alpha \in \Phi$ . The **Weyl group element** associated to  $\alpha$  and  $u$  is

$$w_\alpha(u) = X_\alpha(u) \cdot X_{-\alpha}\left(\frac{2u}{Q_C(u)}\right) \cdot X_\alpha(u)$$

By direct computation, combined with some theoretical considerations, it is possible to verify that

- $w_\alpha(u)^2 = \mathrm{Id}$  for any  $\alpha, u$ .
- For any  $s \in S(k)$ ,  $w_\alpha(u) \cdot s \cdot w_\alpha(u) \in S(k)$ .



- For any  $\alpha, \beta \in \Phi$  and  $u, v \in k^2$ ,<sup>5</sup>

$$w_\alpha(u) \cdot X_\beta(v) \cdot w_\alpha(u)^{-1} = X_{\sigma_\alpha(\beta)}(\varphi(v))$$

where

$$\varphi_{\alpha\beta}(v) = \textcolor{red}{INCOMPLETE}$$

We have done nothing with commutator coefficients  $N_{ij}^{\alpha\beta}(u, v)$  for  $\text{SO}_{4,1}$ , because there are no nontrivial commutator formulas for  $\text{SO}_{4,1}$ . There are no roots that add up to another root, so the commutator formula is vacuous for this group.

#### 5.4.2 $\text{SO}_{6,2}(k, b)$ second smallest quasisplit case

Next we expand slightly to the case  $(n, q) = (6, 2)$ . Now  $B$  has the form

$$B = \begin{pmatrix} & & 1 & & & \\ & & & 1 & & \\ 1 & & & & & \\ & 1 & & & & \\ & & & & c_1 & \\ & & & & & c_2 \end{pmatrix}$$

and the root system is type  $B_2$ .

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_1 \pm \alpha_2\}$$

**Exercise 5.4.4.** Using your general calculations from Exercise 5.2.2, write down the general form of an element of the special orthogonal Lie algebra  $\mathfrak{so}_{6,2}(k, b)$ .

**Exercise 5.4.5.** Mimic the calculation of Example 5.4.1 to determine the root spaces for the Lie algebra  $\mathfrak{so}_{6,2}(k, b)$ . That is, take  $s \in S(k)$  an arbitrary element of the diagonal torus subgroup, and  $X \in \mathfrak{so}_{6,2}(k, b)$  an arbitrary element of the Lie algebra, and calculate  $sXs^{-1}$ . Then compare the result of that calculation with  $\alpha(s)X$  to determine which characters  $\alpha$  have nonzero root space  $\mathfrak{g}_\alpha$ . Confirm that the roots are

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_1 \pm \alpha_2\}$$

(the  $B_2$  root system). Write out general forms for elements of the root spaces, and use these to determine what the maps  $\text{Lie}X_\alpha$  and  $X_\alpha$  look like.

**Remark 5.4.6.** In the previous exercise, you should find that the dimensions of the root spaces are

Root $\alpha$	Dimension of $\mathfrak{g}_\alpha$
$\pm\alpha_i \pm \alpha_j$	1
$\pm\alpha_i$	$n - 2q = 2$

<sup>5</sup>Recall that  $\sigma_\alpha(\beta)$  is the reflection of  $\beta$  across the hyperplane perpendicular to  $\alpha$ , for which we have the formula

$$\sigma_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

### 5.4.3 $\mathrm{SO}_{n,q}(k, b)$ with $n = 2q + 2$ , general quasisplit case

Now we consider the general situation of a “quasisplit” special orthogonal group, where  $n = 2q + 2$ . Note that this includes the cases  $(n, q) = (4, 1)$  and  $(n, q) = (6, 2)$  that we studied above. Let  $q \geq 1$  and let  $n = 2q + 2$ , and let  $b$  be a symmetric nondegenerate bilinear form on  $k^n$  with Witt index  $q$ , and choose a basis of  $k^n$  so that the matrix of  $b$  has the form

$$B = \begin{pmatrix} & I_q & & \\ I_q & & & \\ & & c_1 & \\ & & & c_2 \end{pmatrix}$$

where  $I_q$  is the  $(q \times q)$  identity matrix, and  $c_1, c_2 \in k^\times$ .

**Exercise 5.4.7.** Generalize exercise 5.4.4 to the case  $n = 2q + 2$ . That is, reuse the calculation from Exercise 5.2.2 to write down the general form of an element of the special orthogonal Lie algebra  $\mathfrak{so}_{2q+2,q}(k, b)$ .

**Exercise 5.4.8.** Generalize exercise 5.4.5 to the case  $n = 2q + 2$ . That is, take  $s \in S(k)$  an arbitrary element of the diagonal torus subgroup, and  $X \in \mathfrak{so}_{2q+2,q}(k, b)$  an arbitrary element of the Lie algebra, calculate  $sXs^{-1}$ , and compare the result of that calculation with  $\alpha(s)X$  to determine which characters  $\alpha$  have nonzero root space  $\mathfrak{g}_\alpha$ . Confirm that the set of roots is the  $B_q$  root system as described below.

$$\Phi = \Phi\left(\mathrm{SO}_{2q+2,q}(k, b), S(k)\right) = \{\pm\alpha_i : 1 \leq i \leq q\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq q, i \neq j\}$$

Write out general forms for elements of the root spaces, and use these to determine what the maps  $\mathrm{Lie}X_\alpha$  and  $X_\alpha$  look like.

**Remark 5.4.9.** In the previous exercise, you should find that the dimensions of the root spaces are

Root $\alpha$	Dimension of $\mathfrak{g}_\alpha$
$\pm\alpha_i \pm \alpha_j$	1
$\pm\alpha_i$	$n - 2q = 2$

### 5.4.4 $\mathrm{SO}_{n,q}(k, b)$ with $1 \leq q < \lfloor \frac{n}{2} \rfloor$ , general non-split case

Let  $n \geq 4$  and let  $1 \leq q < \lfloor \frac{n}{2} \rfloor$ . Note that this includes the quasisplit case  $n = 2q + 2$ , but excludes the split cases  $n = 2q$  and  $n = 2q + 1$ . Let  $b$  be a symmetric nondegenerate bilinear form on  $k^n$  with Witt index  $q$ , and choose a basis of  $k^n$  so that the matrix of  $b$  is

$$B = \begin{pmatrix} & I_q & & \\ I_q & & & \\ & & C & \\ & & & \end{pmatrix}$$

where  $I_q$  is the  $(q \times q)$  identity matrix and  $C = \mathrm{diag}(c_1, c_2, \dots, c_{n-2q})$  is an  $(n - 2q) \times (n - 2q)$  invertible diagonal matrix with entries in  $k^\times$ . There is not much to say about generalizing

Exercise 5.4.7 to the general non-split case, since there isn't much more to do than just copy down the general result of the calculations from Exercise 5.2.2.

We generalize Exercise 5.4.8 to the general non-split case, though we do not write out all the details. We start with  $s \in S(k)$  and  $X \in \mathfrak{so}_{n,q}(k, b)$  and compute  $sXs^{-1}$  to determine the roots and root spaces for  $\mathrm{SO}_{n,q}(k, b)$ , and what the maps  $\mathrm{Lie}X_\alpha$  and  $X_\alpha$  look like. The dimensions of the root spaces are

Root $\alpha$	Dimension of $\mathfrak{g}_\alpha$
$\pm\alpha_i \pm \alpha_j$	1
$\pm\alpha_i$	$n - 2q$

Consider the long roots. As indicated in the table above, the root spaces and root subgroups are one-dimensional. There are two types of long roots: those of the form  $\pm(\alpha_i - \alpha_j)$ , and those of the form  $\pm(\alpha_i + \alpha_j)$

$$\begin{aligned}
\mathrm{Lie}X_{\alpha_i - \alpha_j} : k &\rightarrow \mathfrak{so}_{n,q} & \mathrm{Lie}X_{\alpha_i - \alpha_j}(u) &= E_{ij}(v) - E_{q+j, q+i}(v) \\
\mathrm{Lie}X_{\alpha_i + \alpha_j} : k &\rightarrow \mathfrak{so}_{n,q} & \mathrm{Lie}X_{\alpha_i + \alpha_j}(u) &= E_{i', j'+q}(v) - E_{j', i'+q}(v) \\
\mathrm{Lie}X_{-(\alpha_i + \alpha_j)} : k &\rightarrow \mathfrak{so}_{n,q} & \mathrm{Lie}X_{-(\alpha_i + \alpha_j)}(u) &= E_{i'+q, j'}(v) - E_{j'+q, i'}(v)
\end{aligned}$$

where  $i' = \min(i, j)$  and  $j' = \max(i, j)$ . For short roots, the root spaces and root subgroups have dimension  $n - 2q$ . Let  $v = (v_1, \dots, v_{n-2q}) \in k^{n-2q}$ .

$$\begin{aligned}
\mathrm{Lie}X_{\alpha_i} : k^{n-2q} &\rightarrow \mathfrak{so}_{n,q} & \mathrm{Lie}X_{\alpha_i}(v) &= \sum_{m=1}^{n-2q} \left( E_{2q+m, q+i}(v_m) - E_{i, 2q+m}(c_m v_m) \right) \\
\mathrm{Lie}X_{-\alpha_i} : k^{n-2q} &\rightarrow \mathfrak{so}_{n,q} & \mathrm{Lie}X_{-\alpha_i}(v) &= \sum_{m=1}^{n-2q} \left( E_{2q+m, i}(v_m) - E_{q+i, 2q+m}(c_m v_m) \right)
\end{aligned}$$

Below we give some concrete examples. First, the minimal quasisplit case is  $(n, q) = (4, 1)$  which has just two short roots. The maps are

$$\mathrm{Lie}X_{\alpha_1}(v) = \begin{pmatrix} 0 & -c_1 v_1 & -c_2 v_2 \\ & 0 & \\ v_1 & 0 & \\ v_2 & & 0 \end{pmatrix} \quad \mathrm{Lie}X_{-\alpha_1}(v) = \begin{pmatrix} 0 & & & \\ & 0 & -c_1 v_1 & -c_2 v_2 \\ v_1 & & 0 & \\ v_2 & & & 0 \end{pmatrix}$$

The minimal non-quasisplit example is  $(n, q) = (5, 1)$ , which looks essentially the same as  $(4, 1)$ . The two maps for this case are depicted below.

$$\mathrm{Lie}X_{\alpha_1}(v) = \begin{pmatrix} 0 & -c_1 v_1 & -c_2 v_2 & -c_3 v_3 \\ & 0 & & \\ v_1 & 0 & & \\ v_2 & & 0 & \\ v_3 & & & 0 \end{pmatrix} \quad \mathrm{Lie}X_{-\alpha_1}(v) = \begin{pmatrix} 0 & & & & \\ & 0 & -c_1 v_1 & -c_2 v_2 & -c_3 v_3 \\ v_1 & & 0 & & \\ v_2 & & & 0 & \\ v_3 & & & & 0 \end{pmatrix}$$

The next smallest quasisplit example is  $(n, q) = (6, 2)$ , but this looks very similar to the next non-quasisplit example of  $(n, q) = (7, 2)$ , which we depict below.

$$\begin{aligned} \text{Lie}X_{\alpha_1-\alpha_2}(v) &= \begin{pmatrix} 0 & v & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & -v & 0 & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} & \text{Lie}X_{\alpha_2-\alpha_1}(v) &= \begin{pmatrix} 0 & & & & & \\ v & 0 & & & & \\ & & 0 & -v & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} \\ \text{Lie}X_{\alpha_1+\alpha_2}(v) &= \begin{pmatrix} 0 & & & v & & \\ & 0 & -v & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} & \text{Lie}X_{-\alpha_1-\alpha_2}(v) &= \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & v & 0 & & \\ -v & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
\text{LieX}_{\alpha_1}(v) &= \begin{pmatrix} 0 & & & -c_1v_1 & -c_2v_2 & -c_3v_3 \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & v_1 & & 0 & & \\ & v_2 & & & 0 & \\ & v_3 & & & & 0 \end{pmatrix} \\
\text{LieX}_{-\alpha_1}(v) &= \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & -c_1v_1 & -c_2v_2 & -c_3v_3 \\ & & & 0 & & \\ v_1 & & & 0 & & \\ v_2 & & & & 0 & \\ v_3 & & & & & 0 \end{pmatrix} \\
\text{LieX}_{\alpha_2}(v) &= \begin{pmatrix} 0 & & & -c_1v_1 & -c_2v_2 & -c_3v_3 \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & v_1 & & 0 & & \\ & v_2 & & & 0 & \\ & v_3 & & & & 0 \end{pmatrix} \\
\text{LieX}_{-\alpha_2}(v) &= \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & -c_1v_1 & -c_2v_2 & -c_3v_3 \\ v_1 & & & 0 & & \\ v_2 & & & & 0 & \\ v_3 & & & & & 0 \end{pmatrix}
\end{aligned}$$

## 5.5 Coding specifications

Matlab functions describing pinnings of special orthogonal groups.

**Name:** LieX\_SO

**Inputs:** An integer  $n$ , an instance of the  $B_q$  root system using the RootSystem class, the matrix  $B$ , a root  $\alpha$  from the  $B_q$  root system ( $n \times 1$  vector), and a symbolic vector  $u$

**Outputs:**  $n \times n$  matrix

**Description:** Output an element of the  $\mathfrak{g}_\alpha$  root system using the vector  $u$  as coefficients. For example, in the  $(n, q) = (4, 1)$  case with  $\alpha = \alpha_1$ ,

$$\text{LieX\_SO}(4, B_1, B, \alpha_1, v) = \begin{pmatrix} 0 & & -c_1x_{32} & -c_2x_{42} \\ & 0 & & \\ x_{32} & & 0 & \\ & x_{42} & & 0 \end{pmatrix}$$

**Name:** X\_SO

**Inputs:** An integer  $n$ , an instance of the  $B_q$  root system using the RootSystem class, the matrix  $B$ , a root  $\alpha$  from the  $B_q$  root system ( $n \times 1$  vector), and a symbolic vector  $u$

**Outputs:**  $n \times n$  matrix

**Description:** Matrix exponential of LieX\_SO of same inputs.

**Name:** W\_SO

**Inputs:** An integer  $n$ , an instance of the  $B_q$  root system using the RootSystem class, the matrix  $B$ , a root  $\alpha$  from the  $B_q$  root system ( $n \times 1$  vector), and a symbolic vector  $u$

**Outputs:**  $n \times n$  matrix

**Description:**

$$w_\alpha(u) = \begin{cases} X_\alpha(u) \cdot X_{-\alpha}(u) \cdot X_\alpha(u) & \alpha = \pm(\alpha_i + \alpha_j) \\ X_\alpha(u) \cdot X_{-\alpha}(-u^{-1}) \cdot X_\alpha(u) & \alpha = \pm(\alpha_i - \alpha_j) \\ X_\alpha(u) \cdot X_{-\alpha}\left(\frac{2}{\text{tr}(b)}u\right) \cdot X_\alpha(u) & \alpha = \pm\alpha_i \end{cases}$$

**Name:** CommutatorCoefficientSO

**Inputs:** An integer  $n$ , an instance of the  $B_q$  root system using the RootSystem class, two roots  $\alpha, \beta$  from the  $B_q$  root system ( $n \times 1$  vector), two positive integers  $i, j$ , and two vectors  $u, v$

**Outputs:** The vector  $N_{ij}^{\alpha\beta}(u, v)$

**Description:**  $N_{ij}^{\alpha\beta}(u, v)$  is determined by the formula

$$\left[ X_\alpha(u), X_\beta(v) \right] = \prod_{i\alpha+j\beta \in B_q} X_{i\alpha+j\beta} \left( N_{ij}^{\alpha\beta}(u, v) \right)$$

where  $\alpha \neq \pm\beta$ . There are three cases: the sum of two long roots is a long root, the sum of two short roots is a long root, and the sum of two roots of different lengths results in two positive integer linear combinations.

1. If  $\alpha, \beta \in \Phi$  are long and  $\alpha + \beta \in \Phi$  (which must also be long), then all relevant root subgroups have dimension one and  $N_{11}^\alpha(u, v) = \pm uv$ .

$$[X_\alpha(u), X_\beta(v)] = X_{\alpha+\beta}(\pm uv) \quad u, v \in k$$

2. If  $\alpha, \beta \in \Phi$  are short and  $\alpha + \beta \in \Phi$  (which must be long), then

$$N_{11}^{\alpha\beta}(u, v) = \pm b_a(u, v) = \pm \sum_{i=1}^{n-2q} c_i u_i v_i \quad u, v \in k^{n-2q}$$

$$[X_\alpha(u), X_\beta(v)] = X_{\alpha+\beta}(\pm b_a(u, v)) \quad u, v \in k^{n-2q}$$

3. If  $\alpha, \beta \in \Phi$  are different lengths and  $\alpha + \beta \in \Phi$ , then  $(\alpha, \beta)$  has two elements. Without loss of generality, assume  $\alpha$  is short and  $\beta$  is long; then  $(\alpha, \beta) = \{\alpha + \beta, 2\alpha + \beta\}$  and  $N_{11}^{\alpha\beta}(u, v) = \pm uv$  and  $N_{21}^{\alpha\beta}(u, v) = \pm \frac{1}{2}vb_a(u, u)$ .

$$[X_\alpha(u), X_\beta(v)] = X_{\alpha+\beta}(\pm uv) \cdot X_{2\alpha+\beta}\left(\pm \frac{1}{2}vb_a(u, u)\right) \quad u \in k^{n-2q}, v \in k$$

**Name:** WeylGroupCoefficientSO

**Inputs:** An integer  $n$ , an instance of the  $B_q$  root system using the RootSystem class, the matrix  $B$ , two roots  $\alpha, \beta$ , and a vector  $v$

**Outputs:** The vector  $\varphi_{\alpha\beta}(v)$

**Description:**  $\varphi_{\alpha\beta}(v)$  is determined by the formula

$$w_\alpha(\vec{1}) \cdot X_\beta(v) \cdot w_\alpha(\vec{1})^{-1} = X_{\sigma_\alpha(\beta)}(\varphi_{\alpha\beta}(v)) \quad (1)$$

In the case  $(n, q) = (4, 1)$ , these are described explicitly via

$$\varphi_{\alpha\beta}(v) = \textcolor{red}{INCOMPLETE}$$

## 6 Special unitary groups

In this section, we study pinnings of special unitary groups. For the duration of this section, fix a quadratic field extension  $L/k = k(\sqrt{d})/k$  and let  $\sigma(x) = \bar{x}$  be the standard involution  $\overline{a + b\sqrt{d}} = a - b\sqrt{d}$  (for  $a, b \in k$ ). Also fix  $n$ , and fix a hermitian or skew-hermitian form  $h$  on  $V = L^n$ . Fix a basis of  $V$ , and let  $H$  be the matrix of  $h$  in the basis  $\beta$ . Let  $q$  be the Witt index of  $h$ . Via a “hermitian version” of the Witt decomposition theorem, it is possible to choose a basis of  $V$  so that the matrix  $H$  has the form

$$H = \begin{pmatrix} & I_q & \\ \varepsilon I_q & & \\ & & C \end{pmatrix}$$

where  $\varepsilon = \pm 1$  is 1 if  $h$  is hermitian and  $-1$  if  $h$  is skew-hermitian, and  $C = \text{diag}(c_1, \dots, c_{n-2q})$  is an invertible diagonal matrix corresponding to the anisotropic part of  $h$ . Note that when  $h$  is hermitian, the entries of  $C$  must lie in  $k$ , while if  $h$  is skew-hermitian the entries of  $C$  must be “purely imaginary,” i.e. of the form  $a\sqrt{d}$  for some  $a \in k$ . This can be described in both cases by saying that

$$\overline{C} = \varepsilon C$$

**Remark 6.0.1.** It is possible to assume  $\varepsilon = 1$  or  $\varepsilon = -1$  without any loss of generality. However, we will write things out for both cases as much as possible.

**Definition 6.0.2.** The **special unitary group** associated to  $h$  is

$$\text{SU}_{n,q}(L, h) = \{X \in \text{SL}_n(L) : X^* H X = H\}$$

Conceptually, we can think of  $\text{SU}_n(L, h)$  as linear transformations  $V \rightarrow V$  which preserve  $h$ . But we will never make serious use of this point of view; rather we will focus on the concrete matrix equation  $X^* H X = H$ .

**Remark 6.0.3.** If  $n = 2q$ , then  $\text{SU}_{n,q}$  is quasisplit, but if  $n > 2q$  then  $\text{SU}_{n,q}$  is not quasisplit (and hence also not split). The case  $n = 2q$  was previously studied in [RR22], although in that paper we had  $H$  in a different form than our general form in these notes. In particular, in that paper we put  $H$  in the block diagonal form below.

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

### 6.1 Features of special unitary groups

#### 6.1.1 $\text{SU}_{4,2}(L, h)$ minimal quasisplit case

As above, let  $L/k$  be a quadratic field extension with conjugation  $\sigma(x) = \bar{x}$ . Let  $V$  be a 4-dimensional  $L$ -vector space, fix a basis of  $V$ , let  $h$  be the hermitian or skew-hermitian form



on  $V$  with matrix

$$H = \begin{pmatrix} & I_2 \\ \varepsilon I_2 & \end{pmatrix} = \begin{pmatrix} & 1 & \\ \varepsilon 1 & & 1 \\ & \varepsilon 1 & \end{pmatrix}$$

where  $\varepsilon = 1$  in the hermitian case and  $\varepsilon = -1$  for skew-hermitian. (It turns out that many things are the same regardless of  $\varepsilon$ , though some minor things do depend on  $\varepsilon$ .) Note that  $h$  has Witt index 2. The associated special unitary group is

$$\mathrm{SU}_{4,2}(L, h) = \{X \in \mathrm{SL}_4(L) : X^* H X = H\}$$

The 4 marks the fact that  $\dim_L V = 4$ , and the 2 records the Witt index of  $h$ .

**Exercise 6.1.1.** Show that if  $t \in \mathrm{SU}_{4,2}(L, h)$  is a diagonal matrix, then  $t$  has the form

$$t = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \bar{t}_1^{-1} & \\ & & & \bar{t}_2^{-1} \end{pmatrix} \quad t_1, t_2 \in L^\times$$

**Definition 6.1.2.** The subgroup of all diagonal matrices in  $\mathrm{SU}_{4,2}(L, h)$  is called  $T(k)$ . In the general theory of algebraic groups,  $T$  is called a “maximal torus.”

**Definition 6.1.3.** Rather than a maximal torus, what we need is actually something called a “maximal  $k$ -split torus.” For that role, we will need the following subgroup of  $T(k)$ , which we will call  $S(k)$ .

$$S(k) = \left\{ \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & s_1^{-1} & \\ & & & s_2^{-1} \end{pmatrix} : s_1, s_2 \in k^\times \right\}$$

One way to think about this is that  $S(k)$  is the set of fixed points of  $T(k)$  under entry-wise conjugation.  $S(k)$  will play the role that the diagonal subgroup played for  $\mathrm{SL}_n$ .

**Definition 6.1.4.** For  $i = 1, \dots, 4$  let  $\alpha_i : S(k) \rightarrow k^\times$  be the map which picks off the  $i$ th diagonal entry.

**Definition 6.1.5.** The **character group** of  $S$ , denoted  $X^*(S)$ , is the group of group homomorphisms  $S(k) \rightarrow k^\times$ . Elements of  $X^*(S)$  are called **characters**. The group operation in  $X^*(S)$  looks like

$$(\alpha_i + \alpha_j)(s) = \alpha_i(s) \cdot \alpha_j(s)$$

for any  $1 \leq i, j \leq 4$  and  $s \in S(k)$ . The  $\cdot$  on the right hand side is multiplication in  $k$ .

**Lemma 6.1.6.**  $X^*(S)$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  with basis  $\{\alpha_1, \alpha_2\}$ .

**Remark 6.1.7.** A calculation which we will do in much more generality later shows that the root system of  $\mathrm{SU}_{4,2}(L, h)$  (with respect to  $S$ ) is the following set of 8 characters.

$$\Phi_k = \{\pm 2\alpha_1, \pm 2\alpha_2, \pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)\}$$

This is the root system of type  $C_2$ . Roots of the form  $\pm 2\alpha_i$  are long, and roots of the form  $\pm\alpha_i \pm \alpha_j$  with  $i \neq j$  are short.

### 6.1.2 $\mathrm{SU}_{n,q}(L, h)$ with $n = 2q$ , general quasisplit case

Again, let  $L/k$  be a quadratic extension and  $\sigma(x) = \bar{x}$  be the conjugation operation on  $L$ . Let  $q \geq 1$  and let  $n = 2q$  and let  $V$  be an  $n$ -dimensional  $L$ -vector space, fix a basis of  $V$ , and let  $h$  be the nondegenerate hermitian or skew-hermitian form on  $V$  with matrix

$$H = \begin{pmatrix} & I_q \\ \varepsilon I_q & \end{pmatrix}$$

Note that  $h$  has Witt index  $q$ , half the dimension of  $V$ . (Witt index can never be more than half the dimension, so this is the “maximal Witt index” case.) The associated special unitary group to  $h$  (and  $H$ ) is

$$\mathrm{SU}_{n,q}(L, h) = \mathrm{SU}_{2q,q}(L, h) = \{X \in \mathrm{SL}_{2q}(L) : X^* H X = H\}$$

**Exercise 6.1.8.** Generalizing exercise 6.1.1, show that if  $t \in \mathrm{SU}_{2q,q}(L, h)$  is a diagonal matrix, then  $t$  has the form

$$t = \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_q & & \\ & & \bar{t}_1^{-1} & & \\ & & & \ddots & \\ & & & & \bar{t}_q^{-1} \end{pmatrix} \quad t_1, \dots, t_q \in L^\times$$

**Definition 6.1.9.** The set  $T(k)$  of all diagonal matrices in  $\mathrm{SU}_{2q,q}(L, h)$  is a maximal torus. As in the  $(n, q) = (4, 2)$  case, we are more interested in the maximal split torus  $S(k)$  described below.

$$S(k) = \left\{ \begin{pmatrix} s_1 & & & & \\ & \ddots & & & \\ & & s_q & & \\ & & s_1^{-1} & & \\ & & & \ddots & \\ & & & & s_q^{-1} \end{pmatrix} : s_i \in k^\times \right\}$$

**Definition 6.1.10.** For  $i = 1, 2, \dots, n$  let  $\alpha_i : S(k) \rightarrow k^\times$  be the map which picks off the  $i$ th diagonal entry.

**Definition 6.1.11.** The **character group** of  $S$ , denoted  $X^*(S)$ , is the group of group homomorphisms  $S(k) \rightarrow k^\times$ .

**Lemma 6.1.12.**  $X^*(S)$  is isomorphic to  $\mathbb{Z} \oplus \dots \oplus \mathbb{Z} = \mathbb{Z}^q$  with basis  $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ .

**Remark 6.1.13.** A calculation which we will do in much more generality later shows that the root system of  $\mathrm{SU}_{2q,q}(L, h)$  (with respect to  $S$ ) is the root system of type  $C_q$ .

$$\Phi_k = \{\pm 2\alpha_i\} \cup \{\pm\alpha_i \pm \alpha_j : i \neq j\} \quad 1 \leq i, j, \leq q$$

**Exercise 6.1.14.** How big is the set  $\Phi_k$ ?

### 6.1.3 $\mathrm{SU}_{5,2}(L, h)$ minimal non-quasisplit case

The case  $(n, q) = (5, 2)$  is the smallest non-quasisplit special unitary group of interest. Now  $V$  is a 5-dimensional  $L$ -vector space, and  $h$  is the hermitian or skew-hermitian form on  $V$  with matrix

$$H = \begin{pmatrix} & I_2 & \\ \varepsilon I_2 & & \\ & & c_1 \end{pmatrix} = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ \varepsilon 1 & & & & \\ & \varepsilon 1 & & & \\ & & & & c_1 \end{pmatrix}$$

Note that  $c_1 \in k$  in the hermitian case and  $c_1 = k(\sqrt{d})$  in the skew-hermitian case, and  $c \neq 0$  in either case. A general diagonal matrix in  $\mathrm{SU}_{5,2}(L, h)$  has the form

$$t = \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & \bar{t}_1^{-1} & & \\ & & & \bar{t}_2^{-1} & \\ & & & & 1 \end{pmatrix} \quad t_1, t_2 \in L^\times$$

Let  $T(k)$  be the subgroup of such matrices. The maximal  $k$ -split torus we need is the subgroup  $S(k) \subset T(k)$  where the diagonal entries lie in  $k$ .

$$S(k) = \left\{ \begin{pmatrix} s_1 & & & & \\ & s_2 & & & \\ & & s_1^{-1} & & \\ & & & s_2^{-1} & \\ & & & & 1 \end{pmatrix} : s_1, s_2 \in k^\times \right\}$$

For  $i = 1, \dots, 4$  let  $\alpha_i : S(k) \rightarrow k^\times$  be the map which picks off the  $i$ th diagonal entry. The character group of  $S$  is the group of group homomorphisms  $S(k) \rightarrow k^\times$ , and it is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  with basis  $\{\alpha_1, \alpha_2\}$ . The root system of  $\mathrm{SU}_{5,2}(L, h)$  (with respect to  $S$ ) is the following set of 12 characters.

$$\Phi_k = \{\pm\alpha_1, \pm\alpha_2, \pm 2\alpha_1, \pm 2\alpha_2, \pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)\}$$

This is the root system of type  $\mathrm{BC}_2$ . This root system is called “non-reduced” because it contains roots which are proportional but not negatives, e.g.  $\alpha_1$  and  $2\alpha_1$ . Depending on where you look, this does not meet the definition of a root system. However, we do consider it a root system here, and the traditional definition is a “reduced” root system. Note that  $\mathrm{BC}_q$  is actually the only non-reduced root system, so expanding the definition in this way adds only one case.

### 6.1.4 $\mathrm{SU}_{n,q}(L, h)$ general case

Let  $L/k = k(\sqrt{d})/k$  be a quadratic extension with conjugation operation  $\sigma(x) = \bar{x}$ , and let  $V$  be an  $n$ -dimensional  $L$ -vector space. Fix a basis of  $V$ , and let  $h$  an nondegenerate isotropic

(meaning  $q \geq 1$ ) hermitian or skew-hermitian form on  $V$  with Witt index  $q$ , with associated (skew-)hermitian matrix  $H$ . Then the associated special unitary group is

$$\mathrm{SU}_{n,q}(L, h) = \{X \in \mathrm{SL}_n(L) : X^* H X = H\}$$

Note that we are no longer assuming that  $h$  has maximal Witt index. Instead, we just know that  $1 \leq q \leq \frac{n}{2}$ . Our goal is to generalize everything from section 6.1.2 to this more general situation, and have Matlab code to describe and verify all of it.

We choose a basis of  $V$  in which the matrix  $H$  is

$$H = \begin{pmatrix} & I_q \\ \varepsilon I_q & \\ & C \end{pmatrix}$$

with  $C$  diagonal. When  $H$  has this form, the full diagonal subgroup of  $\mathrm{SU}_{n,q}(L, h)$  is

$$T(k) = \left\{ \begin{pmatrix} T_1 & & & \\ & \bar{T}_1^{-1} & & \\ & & \pm 1 & \\ & & & \ddots \\ & & & & \pm 1 \end{pmatrix} \right\}$$

where  $T_1$  is an arbitrary  $q \times q$  invertible diagonal matrix with entries in  $L$ , and there are  $n - 2q$  independent  $\pm 1$  along the diagonal after  $\bar{T}_1^{-1}$ . But the maximal  $k$ -split torus subgroup that we need is

$$S(k) = \left\{ \begin{pmatrix} S_1 & & \\ & S_1^{-1} & \\ & & I_{n-2q} \end{pmatrix} = \begin{pmatrix} s_1 & & & & & \\ & \ddots & & & & \\ & & s_q & & & \\ & & & s_1^{-1} & & \\ & & & & \ddots & \\ & & & & & s_q^{-1} \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} : s_i \in k^\times \right\}$$

A major additional complication in the case  $n > 2q$  is that the root system  $\Phi_k = \Phi(\mathrm{SU}_{n,q}(L, h), S)$  is no longer “reduced,” because it contains some scalar multiples of its elements other than  $\pm 1$ . Thankfully, it only includes scalar multiples of  $\pm 1, \pm 2, \pm \frac{1}{2}$ . The root system is called  $\mathrm{BC}_q$ , which is

$$\Phi_k = \{\pm \alpha_i, \pm 2\alpha_i : 1 \leq i \leq q\} \cup \{\pm \alpha_i \pm \alpha_j : 1 \leq i, j \leq q, i \neq j\}$$

The Lie algebra of  $\mathrm{SU}_{n,q}(L, h)$  is

$$\mathfrak{su}_{n,q}(L, h) = \{X \in \mathfrak{sl}_n(L) : X^* H + H X = 0\}$$

We will work out what a general element of  $\mathfrak{su}_{n,q}(L, h)$  looks like. Let  $X \in \mathfrak{su}_{n,q}(L, h)$ , and write it in block form

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

with the block sizes corresponding to those in  $H$ . That is,  $X_{11}, X_{12}, X_{21}, X_{22}$  are all  $q \times q$ ,  $X_{13}$  and  $X_{23}$  have  $q$  rows and  $(n - 2q)$  columns, while  $X_{31}$  and  $X_{32}$  have  $q$  columns and  $(n - 2q)$  rows, and  $X_{33}$  is  $(n - 2q) \times (n - 2q)$ . Then  $X^*H + HX$  becomes

$$\begin{aligned} X^*H + HX &= \begin{pmatrix} X_{11}^* & X_{21}^* & X_{31}^* \\ X_{12}^* & X_{22}^* & X_{32}^* \\ X_{13}^* & X_{23}^* & X_{33}^* \end{pmatrix} \begin{pmatrix} I_q & & \\ \varepsilon I_q & & \\ & & C \end{pmatrix} + \begin{pmatrix} & I_q & \\ \varepsilon I_q & & \\ & & C \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon X_{21}^* & X_{11}^* & X_{31}^* C \\ \varepsilon X_{22}^* & X_{12}^* & X_{32}^* C \\ \varepsilon X_{23}^* & X_{13}^* & X_{33}^* C \end{pmatrix} + \begin{pmatrix} X_{21} & X_{22} & X_{23} \\ \varepsilon X_{11} & \varepsilon X_{12} & \varepsilon X_{13} \\ CX_{31} & CX_{32} & CX_{33} \end{pmatrix} \end{aligned}$$

so for  $X$  to belong to  $\mathfrak{su}_{n,q}$  we get the following nine equations, with some redundancies.

$$\begin{aligned} \varepsilon X_{21}^* + X_{21} &= 0 \\ X_{11}^* + X_{22} &= 0 \\ X_{31}^* C + X_{23} &= 0 \\ \varepsilon X_{22}^* + \varepsilon X_{11} &= 0 \\ X_{12}^* + \varepsilon X_{12} &= 0 \\ X_{32}^* C + \varepsilon X_{13} &= 0 \\ \varepsilon X_{23}^* + CX_{31} &= 0 \\ X_{13}^* + CX_{32} &= 0 \\ X_{33}^* C + CX_{33} &= 0 \end{aligned}$$

Removing redundancies and rearranging, combined with the facts that  $C$  is invertible, diagonal,  $\overline{C} = \varepsilon C$ , we get the following conditions for  $X \in \mathfrak{su}_{n,q}$ .

$$\begin{aligned} X_{22} &= -X_{11}^* \\ X_{12} &= -\varepsilon X_{12}^* \\ X_{21} &= -\varepsilon X_{21}^* \\ X_{13} &= -X_{32}^* \overline{C} = -\varepsilon X_{32}^* C \\ X_{23} &= -X_{31}^* C = -\varepsilon X_{31}^* \overline{C} \\ X_{33} &= C^{-1} X_{33}^* C \end{aligned}$$

So an arbitrary element of  $\mathfrak{su}_{n,q}$  looks like

$$X = \begin{pmatrix} X_{11} & X_{12} & -\varepsilon X_{32}^* C \\ X_{21} & -X_{11}^* & -X_{31}^* C \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

with  $X_{12}$  and  $X_{21}$  hermitian or skew-hermitian depending on  $\varepsilon$ ,  $X_{33} = C^{-1}X_{33}^*C$ , and  $X_{11}, X_{31}, X_{32}$  arbitrary. In particular, the diagonal entries of  $X_{12}$  and  $X_{21}$  are either in  $k$  or “purely imaginary” depending on  $\varepsilon$ .

Let  $X \in \mathfrak{su}_{n,q}(L, h)$  be a generic element of the Lie algebra and let  $s \in S(k)$  be a generic torus element. We want to work out the root spaces in  $\mathfrak{su}_{n,q}$  by comparing  $sXs^{-1}$  and  $\alpha(s)X$ .

$$sXs^{-1} = \begin{pmatrix} s_1 & & & & & \\ & \ddots & & & & \\ & & s_q & & & \\ & & & s_1^{-1} & & \\ & & & & \ddots & \\ & & & & & s_q^{-1} \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} (x_{ij}) \begin{pmatrix} s_1^{-1} & & & & & \\ & \ddots & & & & \\ & & s_q^{-1} & & & \\ & & & s_1 & & \\ & & & & \ddots & \\ & & & & & s_q \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

where each  $X_{ij}$  is a block matrix as described below.

$$X_{11} = (s_i s_j^{-1} x_{ij}) = \begin{pmatrix} x_{11} & \cdots & s_1 s_q^{-1} x_{1q} \\ \vdots & \ddots & \vdots \\ s_q s_1^{-1} x_{q1} & \cdots & x_{qq} \end{pmatrix}$$

$$X_{12} = (s_i s_j x_{i,q+j}) = \begin{pmatrix} s_1^2 x_{1,q+1} & \cdots & s_1 s_q x_{1,2q} \\ \vdots & \ddots & \vdots \\ s_q s_1 x_{q,q+1} & \cdots & s_q^2 x_{q,2q} \end{pmatrix}$$

$$X_{13} = (s_i x_{i,2q+j}) = \begin{pmatrix} s_1 x_{1,2q+1} & \cdots & s_1 x_{1n} \\ \vdots & \ddots & \vdots \\ s_q x_{q,2q+1} & \cdots & s_q x_{qn} \end{pmatrix}$$

$$X_{21} = (s_i^{-1} s_j^{-1} x_{q+i,j}) = \begin{pmatrix} s_1^{-2} x_{q+1,1} & \cdots & (s_1 s_q)^{-1} x_{q+1,q} \\ \vdots & \ddots & \vdots \\ (s_q s_1)^{-1} x_{2q,1} & \cdots & s_q^{-2} x_{2q,q} \end{pmatrix}$$

$$\begin{aligned}
X_{22} &= (s_i^{-1} s_j x_{q+i, q+j}) = \begin{pmatrix} x_{q+1, q+1} & \cdots & s_{q+1}^{-1} s_q x_{q+1, 2q} \\ \vdots & \ddots & \vdots \\ s_q^{-1} s_1 x_{2q, q+1} & \cdots & x_{2q, 2q} \end{pmatrix} \\
X_{23} &= (s_i^{-1} x_{q+i, 2q+j}) = \begin{pmatrix} s_1^{-1} x_{q+1, 2q+1} & \cdots & s_1^{-1} x_{q+1, n} \\ \vdots & \ddots & \vdots \\ s_q^{-1} x_{2q, 2q+1} & \cdots & s_q^{-1} x_{2q, n} \end{pmatrix} \\
X_{31} &= (s_j^{-1} x_{2q+i, j}) = \begin{pmatrix} s_1^{-1} x_{2q+1, 1} & \cdots & s_q^{-1} x_{2q+1, q} \\ \vdots & \ddots & \vdots \\ s_1^{-1} x_{n, 1} & \cdots & s_q^{-1} x_{n, q} \end{pmatrix} \\
X_{32} &= (s_j x_{2q+i, q+j}) = \begin{pmatrix} s_1 x_{2q+1, q+1} & \cdots & s_q x_{2q+1, 2q} \\ \vdots & \ddots & \vdots \\ s_1 x_{n, q+1} & \cdots & s_q x_{n, 2q} \end{pmatrix} \\
X_{33} &= (x_{2q+i, 2q+j}) = \begin{pmatrix} x_{2q+1, 2q+1} & \cdots & x_{2q+1, n} \\ \vdots & \ddots & \vdots \\ x_{n, 2q+1} & \cdots & x_{nn} \end{pmatrix}
\end{aligned}$$

By inspecting the above matrices, we can see that the roots are

$$\Phi = \{\pm\alpha_i, \pm 2\alpha_i : 1 \leq i \leq q\} \cup \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq q, i \neq j\}$$

which is the non-reduced root system of type  $\text{BC}_q$ . The root spaces are described in the tables below. In the table, we assume  $i \neq j$ , and that all dimensions are over  $k$ .

Root	Blocks	Dimension of root space
$+\alpha_i$	$X_{13}, X_{32}$	$2(n-2q)$
$-\alpha_i$	$X_{31}, X_{23}$	$2(n-2q)$
$+2\alpha_i$	$X_{12}$	1
$-2\alpha_i$	$X_{21}$	1
$\alpha_i + \alpha_j$	$X_{12}$	2
$-(\alpha_i + \alpha_j)$	$X_{21}$	2
$\pm(\alpha_i - \alpha_j)$	$X_{11}, X_{22}$	2

There are three root lengths: short, medium, and long. The root spaces for long roots are easiest to describe: they lie along the diagonal of the  $X_{12}$  and  $X_{21}$  blocks, each root space occupying a single matrix entry. Since the  $X_{12}$  and  $X_{21}$  blocks each satisfy  $X = -\varepsilon X^*$ , these diagonal entries satisfy  $x = -\varepsilon \bar{x}$ . If  $\varepsilon = -1$ , this means the diagonal entries in question lie in  $k$ , while if  $\varepsilon = 1$  they lie in  $k\sqrt{d}$ , the “purely imaginary” elements of the quadratic extension  $k(\sqrt{d})/k$ . To denote this, let

$$p_\varepsilon = \begin{cases} 1 & \varepsilon = -1 \\ \sqrt{d} & \varepsilon = 1 \end{cases} \quad k_\varepsilon = kp_\varepsilon = \begin{cases} k & \varepsilon = -1 \\ k\sqrt{d} & \varepsilon = 1 \end{cases}$$

Then we can describe the root spaces as

$$\begin{aligned}
\mathfrak{g}_{2\alpha_i} &= \{E_{i, i+q}(x) : x \in k_\varepsilon\} = \{E_{i, i+q}(p_\varepsilon v) : v \in k\} \\
\mathfrak{g}_{-2\alpha_i} &= \{E_{i+q, i}(x) : x \in k_\varepsilon\} = \{E_{i+q, i}(p_\varepsilon v) : v \in k\}
\end{aligned}$$

Consequently, the root space maps and root subgroup maps are simple to describe.

$$\begin{aligned}
\text{Lie}X_{2\alpha_i} : k &\rightarrow \mathfrak{su}_{n,q} & \text{Lie}X_{2\alpha_i}(v) &= E_{i,q+i}(p_\varepsilon v) \\
\text{Lie}X_{-2\alpha_i} : k &\rightarrow \mathfrak{su}_{n,q} & \text{Lie}X_{-2\alpha_i}(v) &= E_{q+i,i}(p_\varepsilon v) \\
X_{2\alpha_i} : k &\rightarrow \text{SU}_{n,q} & X_{2\alpha_i}(v) &= 1 + E_{i,q+i}(p_\varepsilon v) \\
X_{-2\alpha_i} : k &\rightarrow \text{SU}_{n,q} & X_{-2\alpha_i}(v) &= 1 + E_{q+i,i}(p_\varepsilon v)
\end{aligned}$$

Next, we describe the root spaces and root subgroup maps for medium length roots, those of the form  $\pm\alpha_i \pm \alpha_j$  with  $i \neq j$ . Their root spaces live in the  $X_{11}, X_{12}, X_{21}$ , and  $X_{22}$  blocks. Since  $X_{12}$  and  $X_{21}$  are hermitian or skew-hermitian, the root spaces for  $\pm(\alpha_i + \alpha_j)$  occur in transpose-pairs of entries. Since  $X_{22} = -X_{11}^*$ , the root spaces for  $\alpha_i - \alpha_j$  occur in pairs of entries, one from  $X_{11}$  and the accompanying entry of  $X_{22}$  determined by that entry.

Root	Block	Specific entries
$\alpha_i - \alpha_j$	$X_{11}, X_{22}$	$x_{ij}, x_{q+j,q+i}$
$\alpha_i + \alpha_j$	$X_{12}$	$x_{i,q+j}, x_{j,q+i}$
$-\alpha_i - \alpha_j$	$X_{21}$	$x_{q+i,j}, x_{q+j,i}$

For each of these roots, the specific listed entries of an arbitrary element  $X = (x_{ij}) \in \mathfrak{su}_{n,q}$  can be anything, besides their pairwise dependency. So each root space has dimension 1 over  $L$ , hence dimension 2 over  $k$ .

$$\begin{aligned}
\text{Lie}X_{\alpha_i - \alpha_j} : L &\rightarrow \mathfrak{su}_{n,q} & \text{Lie}X_{\alpha_i - \alpha_j}(v) &= E_{ij}(v) - E_{q+j,q+i}(\bar{v}) \\
\text{Lie}X_{\alpha_i + \alpha_j} : L &\rightarrow \mathfrak{su}_{n,q} & \text{Lie}X_{\alpha_i + \alpha_j}(v) &= E_{i',q+j'}(v) - E_{j',q+i'}(\varepsilon \bar{v}) \\
\text{Lie}X_{-\alpha_i - \alpha_j} : L &\rightarrow \mathfrak{su}_{n,q} & \text{Lie}X_{-\alpha_i - \alpha_j}(v) &= E_{q+j',i'}(v) - E_{q+i',j'}(\varepsilon \bar{v})
\end{aligned}$$

where  $i' = \min(i, j)$  and  $j' = \max(i, j)$ . The conventions on  $\text{Lie}X_{\pm(\alpha_i + \alpha_j)}$  are chosen so that

$$\text{Lie}X_{-\alpha_i - \alpha_j}(v) = \text{Lie}X_{\alpha_i + \alpha_j}(v)^T$$

Finally, we describe the root spaces and root subgroup maps for short roots, those of the form  $\pm\alpha_i$ . For  $+\alpha_i$ , the root space lives in the  $X_{13}$  and  $X_{32}$  blocks, while for  $-\alpha_i$  it lives in the  $X_{23}$  and  $X_{31}$  blocks. The root space takes up a whole row or column of each block. Since  $X_{13} = -\varepsilon X_{32}^* C$ , the entries of a row in  $X_{13}$  are determined by the entries of a corresponding column in  $X_{32}$ . There are  $n - 2q$  of these entries, which can be any element of  $L$ , so the root space associated to  $\pm\alpha_i$  has dimension  $2(n - 2q)$  over  $k$ .

$$\mathfrak{g}_{\alpha_i} = \left\{ \begin{pmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & X_{32} & 0 \end{pmatrix} \right\}$$

where

$$X_{32} = \begin{pmatrix} 0 & v_1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & v_{n-2q} & 0 \end{pmatrix} \quad X_{13} = -\varepsilon X_{32}^* C = \begin{pmatrix} 0 & \cdots & 0 \\ -\varepsilon c_1 \bar{v}_1 & \cdots & -\varepsilon c_{n-2q} \bar{v}_n \\ 0 & \cdots & 0 \end{pmatrix}$$



with the column vector  $(v_1, \dots, v_{n-2q})$  occurring as the  $i$ th column of  $X_{32}$  and the  $i$ th row of  $X_{13}$ , and the elements  $v_1, \dots, v_{n-2q}$  can be anything from  $L$ . Similarly,

$$\mathfrak{g}_{-\alpha_i} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & X_{23} \\ X_{31} & 0 & 0 \end{pmatrix} \right\}$$

where

$$X_{31} = \begin{pmatrix} 0 & v_1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & v_{n-2q} & 0 \end{pmatrix} \quad X_{23} = -X_{31}^* C = \begin{pmatrix} 0 & \dots & 0 \\ -c_1 \bar{v}_1 & \dots & -c_{n-2q} \bar{v}_n \\ 0 & \dots & 0 \end{pmatrix}$$

with  $(v_1, \dots, v_{n-2q})$  being the  $i$ th column of  $X_{31}$  and corresponding to the  $i$ th row of  $X_{23}$ .

Root	Block	Specific entries	Range
$\alpha_i$	$X_{32}$	$x_{2q+j, q+i}$	$1 \leq j \leq n-2q$
	$X_{13}$	$x_{i, 2q+j}$	$1 \leq j \leq n-2q$
$-\alpha_i$	$X_{31}$	$x_{2q+j, i}$	$1 \leq j \leq n-2q$
	$X_{23}$	$x_{q+i, 2q+j}$	$1 \leq j \leq n-2q$

## 6.2 Pinning special unitary groups

### 6.2.1 $\mathrm{SU}_{4,2}(L, h)$ minimal quasisplit case

We describe a pinning of  $\mathrm{SU}_{4,2}(L, h)$ . Specializing the calculation of root spaces for  $\mathrm{SU}_{n,q}(L, h)$  from the previous section, the root system is type  $\mathbf{C}_2$ .

$$\Phi_k = \{\pm 2\alpha_1, \pm 2\alpha_2, \pm \alpha_1 \pm \alpha_2\}$$

Recall the notation

$$p_\varepsilon = \begin{cases} 1 & \varepsilon = -1 \\ \sqrt{d} & \varepsilon = 1 \end{cases} \quad k_\varepsilon = kp_\varepsilon = \begin{cases} k & \varepsilon = -1 \\ k\sqrt{d} & \varepsilon = 1 \end{cases}$$

The pinning maps for short roots are

$$\begin{aligned} X_{2\alpha_i} : k &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{2\alpha_i}(v) &= 1 + E_{i, q+i}(p_\varepsilon v) \\ X_{-2\alpha_i} : k &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{-2\alpha_i}(v) &= 1 + E_{q+i, i}(p_\varepsilon v) \end{aligned}$$

$$\begin{aligned} X_{2\alpha_1}(v) &= \begin{pmatrix} 1 & & p_\varepsilon v & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} & X_{-2\alpha_1}(v) &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ p_\varepsilon v & & 1 & \\ & & & 1 \end{pmatrix} \\ X_{2\alpha_2}(v) &= \begin{pmatrix} 1 & & & \\ & 1 & & p_\varepsilon v \\ & & 1 & \\ & & & 1 \end{pmatrix} & X_{-2\alpha_2}(v) &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p_\varepsilon v & & & 1 \end{pmatrix} \end{aligned}$$

The pinning maps for medium roots are

$$\begin{aligned} X_{\alpha_i - \alpha_j} : L &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{\alpha_i - \alpha_j}(v) &= 1 + E_{ij}(v) - E_{q+j, q+i}(\bar{v}) \\ X_{\alpha_i + \alpha_j} : L &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{\alpha_i + \alpha_j}(v) &= 1 + E_{i', q+j'}(v) - \varepsilon E_{j', q+i'}(\bar{v}) \\ X_{-(\alpha_i + \alpha_j)} : L &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{-(\alpha_i + \alpha_j)}(v) &= 1 + E_{q+j', i'}(v) - \varepsilon E_{q+i', j'}(\bar{v}) \end{aligned}$$

where  $i' = \min(i, j)$  and  $j' = \max(i, j)$ . Explicitly, these look like

$$\begin{aligned} X_{\alpha_1 - \alpha_2}(v) &= \begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{v} & 1 \end{pmatrix} & X_{\alpha_2 - \alpha_1}(v) &= \begin{pmatrix} 1 & & & \\ v & 1 & & \\ & & 1 & -\bar{v} \\ & & & 1 \end{pmatrix} \\ X_{\alpha_1 + \alpha_2}(v) &= \begin{pmatrix} 1 & & & v \\ & 1 & -\varepsilon \bar{v} & \\ & & 1 & \\ & & & 1 \end{pmatrix} & X_{-\alpha_1 - \alpha_2}(v) &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ -\varepsilon \bar{v} & 1 & & \\ v & & & 1 \end{pmatrix} \end{aligned}$$

**Exercise 6.2.1.** Let  $\alpha = 2\alpha_1$  and  $\beta = -\alpha_1 + \alpha_2$ . We will verify the commutator formula in this case.

- (a) Verify that  $\alpha + \beta$  and  $\alpha + 2\beta$  are in  $\Phi_k$ , but that these are the only positive integer linear combinations of the form  $i\alpha + j\beta$  which are in  $\Phi_k$ . In other words, on the right hand side of the commutator formula, we get something like

$$\prod_{\substack{i, j \geq 1 \\ i\alpha + j\beta \in \Phi_k}} X_{i\alpha + j\beta} \left( N_{ij}^{\alpha\beta}(u, v) \right) = X_{\alpha + \beta} (?) \cdot X_{\alpha + 2\beta} (?)$$

- (b) Use the existing Matlab code for  $\mathrm{SU}_{4,2}(L, h)$  to calculate the left hand side of the commutator formula in this case:

$$\left[ X_{2\alpha_1}(u), X_{-\alpha_1 + \alpha_2}(v) \right] = ?$$

- (c) Using your answer for part (b), work out what the inputs to  $X_{\alpha + \beta}$  and  $X_{\alpha + 2\beta}$  should be in the formula in part (a). In other words, determine the two expressions below.

$$\begin{aligned} N_{11}^{2\alpha_1, -\alpha_1 + \alpha_2}(u, v) &= ? \\ N_{12}^{2\alpha_1, -\alpha_1 + \alpha_2}(u, v) &= ? \end{aligned}$$

### 6.2.2 $\mathrm{SU}_{n,q}(L, h)$ with $n = 2q$ , general quasisplit case

We obtain pinning maps for a general quasisplit special unitary group by taking  $n = 2q$  in the calculations of section 6.1.4, which allows us to ignore the short roots. The pinning maps for long roots are

$$\begin{aligned} X_{2\alpha_i} : k &\rightarrow \mathrm{SU}_{n,q} & X_{2\alpha_i}(v) &= 1 + E_{i, q+i}(p_\varepsilon v) \\ X_{-2\alpha_i} : k &\rightarrow \mathrm{SU}_{n,q} & X_{-2\alpha_i}(v) &= 1 + E_{q+i, i}(p_\varepsilon v) \end{aligned}$$

and for medium roots they are

$$\begin{aligned}
X_{\alpha_i - \alpha_j} : L &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{\alpha_i - \alpha_j}(v) &= 1 + E_{ij}(v) - E_{q+j, q+i}(\bar{v}) \\
X_{\alpha_i + \alpha_j} : L &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{\alpha_i + \alpha_j}(v) &= 1 + E_{i', q+j'}(v) - \varepsilon E_{j', q+i'}(\bar{v}) \\
X_{-(\alpha_i + \alpha_j)} : L &\rightarrow \mathrm{SU}_{n,q}(L, h) & X_{-(\alpha_i + \alpha_j)}(v) &= 1 + E_{q+j', i'}(v) - \varepsilon E_{q+i', j'}(\bar{v})
\end{aligned}$$

where  $i' = \min(i, j)$  and  $j' = \max(i, j)$ .

### 6.2.3 $\mathrm{SU}_{5,2}(L, h)$ minimal non-quasisplit case

We describe the pinning maps for  $\mathrm{SU}_{5,2}(L, h)$ , the minimal non-quasisplit case. The pinning maps for long and medium roots are the same as in the  $\mathrm{SU}_{4,2}(L, h)$  case, with an extra row and column of zeros. For long roots these are

$$\begin{aligned}
X_{2\alpha_1}(v) &= \begin{pmatrix} 1 & p_\varepsilon v & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} & X_{-2\alpha_1}(v) &= \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ p_\varepsilon v & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \\
X_{2\alpha_2}(v) &= \begin{pmatrix} 1 & & & & \\ & 1 & p_\varepsilon v & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} & X_{-2\alpha_2}(v) &= \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ p_\varepsilon v & & & 1 & \\ & & & & 1 \end{pmatrix}
\end{aligned}$$

And for medium roots the pinning maps are

$$\begin{aligned}
X_{\alpha_1 - \alpha_2}(v) &= \begin{pmatrix} 1 & v & & & \\ & 1 & & & \\ & & 1 & & \\ & & -\bar{v} & 1 & \\ & & & & 1 \end{pmatrix} & X_{\alpha_2 - \alpha_1}(v) &= \begin{pmatrix} 1 & & & & \\ v & 1 & & & \\ & & 1 & -\bar{v} & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \\
X_{\alpha_1 + \alpha_2}(v) &= \begin{pmatrix} 1 & & & v & \\ & 1 & -\varepsilon \bar{v} & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} & X_{-\alpha_1 - \alpha_2}(v) &= \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -\varepsilon \bar{v} & 1 & \\ v & & & & 1 \\ & & & & 1 \end{pmatrix}
\end{aligned}$$

The new thing that happens in the  $(n, q) = (5, 2)$  case is the appearance of short roots  $\pm\alpha_1, \pm\alpha_2$ . The associated root spaces in the Lie algebra are

$$\begin{aligned}\mathfrak{g}_{\alpha_1} &= \begin{pmatrix} 0 & & & -\varepsilon c_1 \bar{v}_1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \\ & v_1 & & 0 \end{pmatrix} & \mathfrak{g}_{\alpha_2} &= \begin{pmatrix} 0 & & & -\varepsilon c_1 \bar{v}_1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \\ & & v_1 & 0 \end{pmatrix} \\ \mathfrak{g}_{-\alpha_1} &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -\varepsilon \bar{v}_1 \\ & & & 0 \\ v_1 & & & 0 \end{pmatrix} & \mathfrak{g}_{-\alpha_2} &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -\varepsilon \bar{v}_1 \\ & & & 0 \\ & v_1 & & 0 \end{pmatrix}\end{aligned}$$

#### 6.2.4 $\mathrm{SU}_{6,2}(L, h)$ next non-quasisplit case

The  $(n, q) = (5, 2)$  case isn't necessarily that easy to generalize from, but the  $(n, q) = (6, 2)$  case is a bit easier to see the patterns with. As in the  $(5, 2)$  case, the root spaces and root subgroups for long and medium roots are essentially the same as in the quasisplit  $(4, 2)$  case, just with more rows and columns of zeros padded on the ends. So we just depict the root spaces for short roots in the  $(6, 2)$  case.

$$\begin{aligned}\mathfrak{g}_{\alpha_1} &= \begin{pmatrix} 0 & & & -\varepsilon c_1 \bar{v}_1 & -\varepsilon c_2 v_2 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & v_1 & & 0 & \\ & v_2 & & & 0 \end{pmatrix} & \mathfrak{g}_{\alpha_2} &= \begin{pmatrix} 0 & & & -\varepsilon c_1 \bar{v}_1 & -\varepsilon c_2 \bar{v}_2 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & v_1 & & 0 & \\ & v_2 & & & 0 \end{pmatrix} \\ \mathfrak{g}_{-\alpha_1} &= \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & -\varepsilon \bar{v}_1 & -\varepsilon \bar{v}_2 \\ & & & 0 & \\ v_1 & & & 0 & \\ v_2 & & & & 0 \end{pmatrix} & \mathfrak{g}_{-\alpha_2} &= \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -\varepsilon \bar{v}_1 & -\varepsilon \bar{v}_2 \\ & v_1 & & 0 & \\ & v_2 & & & 0 \end{pmatrix}\end{aligned}$$

#### 6.2.5 $\mathrm{SU}_{n,q}(L, h)$ general case

Let  $L/k$  be a quadratic extension with conjugation operation  $\sigma(x) = \bar{x}$ , and let  $V$  be an  $n$ -dimensional  $L$ -vector space. Fix a basis of  $V$ , and let  $h$  an arbitrary hermitian form on  $V$ , with associated hermitian matrix  $H$  with Witt index  $q$ . Then the associated special unitary group is

$$\mathrm{SU}_{n,q}(L, h) = \{X \in \mathrm{SL}_n(L) : X^* H X = H\}$$

Note that we are no longer assuming that  $h$  has maximal Witt index. Instead, let  $q$  be the Witt index of  $h$ , and we just know that  $1 \leq q \leq \frac{n}{2}$ . Our goal is to generalize everything from

section 6.2.2 to this more general situation, and have Matlab code to describe and verify all of it.

We want to know what the maps  $\text{LieX}(n, q, \alpha, u)$  should look like. To figure this out, we need to do the following calculations: let  $X \in \mathfrak{g} = \mathfrak{su}_{n,q}(L, h)$  be an arbitrary element of the Lie algebra, so  $X^*H + HX = 0$ . Let  $s \in S(k)$  be an arbitrary element of the torus subgroup. The torus  $S$  acts on the Lie algebra by conjugation.

$$S(k) \times \mathfrak{g} \rightarrow \mathfrak{g} \quad (s, X) \mapsto sXs^{-1}$$

For a character  $\alpha \in \text{Hom}(S(k), k^\times)$ , there is an associated subspace  $\mathfrak{g}_\alpha$ , defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : sXs^{-1} = \alpha(s)X, \forall s \in S(k)\}$$

Conceptually,  $\mathfrak{g}_\alpha$  is the set of “simultaneous eigenvectors” for all for  $S$  (viewing elements of  $S$  as linear transformations of  $\mathfrak{g}$  via the conjugation action). A **root** is a character  $\alpha$  such that the subspace  $\mathfrak{g}_\alpha$  is nonzero.

**Exercise 6.2.2.** Determine the roots and root spaces for the group  $\text{SU}_{6,2}(L, h)$ . (The root system should be type  $\text{BC}_2$ .)

**Exercise 6.2.3.** Determine the roots and root spaces for the group  $\text{SU}_{n,q}(L, h)$  where  $h$  is the hermitian form Witt index  $q$  and with matrix

$$H = \begin{pmatrix} & I_q \\ I_q & \\ & & C \end{pmatrix}$$

where  $C$  is diagonal.

## 6.3 Coding specifications

**Name:** Tr

**Inputs:** Number or symbolic variable  $u$

**Outputs:** Number/symbolic variable

**Description:** For a quadratic extension  $L/k$  with conjugation operation  $\sigma(x) = \bar{x}$ , the trace of  $u \in L$  is

$$\text{Tr}(u) = u + \bar{u}$$

This function is just allows us to write the formulas for things like  $N_{ij}^{\alpha\beta}(u, v)$  slightly more compactly.

**Name:** LieX\_SU

**Inputs:** Positive integer  $n$ , the root system  $\text{BC}_q$  as an instance of the RootSystem class, the form matrix  $H$ , a root  $\alpha$ , and a vector  $u$

**Outputs:** Symbolic matrix

**Description:** Given a vector  $u$  of length equal to the dimension of the root space  $\mathfrak{g}_\alpha$ , output a matrix with  $u$  coefficients in the  $\mathfrak{g}_\alpha$  root space. There are essentially three cases here:  $\alpha$  is short, medium, or long. When  $\alpha$  is long, the root space has dimension 1, so  $u$  is just a scalar.

**Name:** X\_SU

**Inputs:** Positive integer  $n$ , the root system  $BC_q$  (or  $C_q$  in the quasisplit case), the form matrix  $H$ , a root  $\alpha$ , and a vector  $u$

**Outputs:** Symbolic matrix

**Description:** Matrix exponential of  $\text{LieX\_SU}$  of the same inputs.

**Name:** W\_SU

**Inputs:** Positive integer  $n$ , the root system  $BC_q$  (or  $C_q$  in the quasisplit case), the form matrix  $H$ , a root  $\alpha$ , and a vector  $u$

**Outputs:** Symbolic matrix

**Description:** In the quasisplit case  $n = 2q$ ,  $w_\alpha(u)$  is described by the formula

$$w_\alpha(u) = X_\alpha(u) \cdot X_{-\alpha}(-u^{-1}) \cdot X_\alpha(u)$$

where  $u$  can be either an element of  $k$  or  $L$ . Note that because of the  $u^{-1}$  appearing in the middle factor,  $u$  must be nonzero in order for  $w_\alpha(u)$  to be defined.

Hopefully (probably),  $w_\alpha(u)$  is very similar to this in the general case, at least for medium and long roots. However, for short roots, the root space has potentially large dimension and so this is likely to be more complicated. That general case is not yet figured out/implemented in Matlab yet.

**Name:** CommutatorCoefficientSU

**Inputs:** Positive integer  $n$ , the root system  $BC_q$  (or  $C_q$  in the quasisplit case), two roots  $\alpha, \beta$ , two positive integers  $i, j$ , and two symbolic variables/scalars  $u, v$ ,

**Outputs:** Vector

**Description:** For the case  $n = 2q$ , some of these are described in Lemma 2.2.4 of <https://users.math.msu.edu/users/ruiterj2/math/Documents/Publications/Thesis%202022-05-09.pdf>. That was for  $H$  in a slightly different form, but it should not be a big difference. The bigger issue is that Lemma 2.2.4 ignored the case of two short roots adding up to a third short root which occurs even in the quasisplit case  $(n, q) = (6, 3)$ . Matlab code doing this is still in progress.

**Name:** WeylGroupConjugationCoefficientSU

**Inputs:** Positive integer  $n$ , the root system  $BC_q$  (or  $C_q$  in the quasisplit case), the form

matrix  $H$ , two roots  $\alpha, \beta$ , and a vector  $u$

**Outputs:** Vector

**Description:** This coefficient is  $\varphi_{\alpha\beta}(v)$  appearing in the formula

$$w_\alpha(u) \cdot X_\beta(v) \cdot w_\alpha(u)^{-1} = X_{\sigma_\alpha(\beta)}(\varphi_{\alpha\beta}(v))$$

In the quasisplit case, this always takes the form  $\varphi_{\alpha\beta}(v) = \pm v_{\pm 1}$ , but I don't know if that will carry over to the non-quasisplit case. This is not yet implemented in Matlab.

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