

Abstract homomorphisms of non-quasi-split special unitary groups

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1 Generalizing the construction of Kassabov and Sapir

1.1 Original construction

Example 1. Let k be a field. For $1 \leq i \neq j \leq 3$, let $e_{ij} : k \rightarrow \mathrm{SL}_3(k)$ be the map which takes $u \in k$ to the transvection matrix $e_{ij}(u)$. For example,

$$e_{13} : k \rightarrow \mathrm{SL}_3(k) \quad e_{13}(u) = \begin{pmatrix} 1 & & u \\ & 1 & \\ & & 1 \end{pmatrix}$$

Note that e_{ij} is a group homomorphism from $(k, +)$ into $\mathrm{SL}_3(k)$, i.e. $e_{ij}(u) \cdot e_{ij}(v) = e_{ij}(u + v)$. Let $U_{ij} = e_{ij}(k)$, so e_{ij} gives an isomorphism between $(k, +)$ and U_{ij} . Let

$$w_{12} = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \quad w_{23} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

and note that

$$w_{12} \cdot e_{13}(u) \cdot w_{12}^{-1} = e_{23}(u) \tag{1}$$

$$w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1} = e_{12}(u) \tag{2}$$

Let K be an algebraically closed field, and let

$$\rho : \mathrm{SL}_3(k) \rightarrow \mathrm{GL}_m(K)$$

be a group homomorphism (not necessarily well-behaved with respect to any kind of geometric structure). Let $U = e_{13}(k)$, so e_{13} is an isomorphism between $(k, +)$ and U .

Let $V = \rho(U)$ and let $A = \overline{V}$ be the Zariski closure of V . Since V is a subgroup of $\mathrm{GL}_m(K)$, so is A . That is, A is closed under matrix multiplication. We define another binary operation on A as follows:

$$\mu_\rho : A \times A \rightarrow \mathrm{GL}_m(K) \quad \mu_\rho(a, b) = \left[\rho(w_{23}) \cdot a \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot b \cdot \rho(w_{12}^{-1}) \right]$$

A calculation involving the relations (1), (2), and the commutator relation

$$\left[e_{12}(u), e_{23}(v) \right] = e_{13}(uv)$$

shows that

$$\mu_\rho \left(\rho(e_{13}(u)), \rho(e_{13}(v)) \right) = e_{13}(uv)$$

From this, it follows that μ_ρ maps the subset $V \times V$ (a dense subset of $A \times A$) to V . Since μ_ρ is continuous, it follows that μ_ρ maps $A \times A$ to A .

1.2 Replace SL_3 with arbitrary group

Definition 1. Let k be a field and K be an algebraically closed field. Let G be an algebraic k -group, and let

$$\rho : G(k) \rightarrow \text{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_{12}, e_{13}, e_{23} : k \rightarrow G(k)$ such that

$$\left[e_{12}(u), e_{23}(v) \right] = e_{13}(uv)$$

and suppose have elements $w_{12}, w_{23} \in G(k)$ such that

$$\begin{aligned} w_{12} \cdot e_{13}(u) \cdot w_{12}^{-1} &= e_{23}(u) \\ w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1} &= e_{12}(u) \end{aligned}$$

Let $U = e_{13}(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\text{GL}_m(K)$. Define

$$\mu_\rho : A \times A \rightarrow \text{GL}_m(K) \quad \mu_\rho(a, b) = \left[\rho(w_{23}) \cdot a \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot b \cdot \rho(w_{12}^{-1}) \right]$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b , and some fixed elements $\rho(w_{23})$ and $\rho(w_{12})$ in $\text{GL}_m(K)$.

Lemma 1. $\mu_\rho(\rho \circ e_{13}(u), \rho \circ e_{13}(v)) = \rho \circ e_{13}(uv)$

Proof.

$$\begin{aligned} \mu_\rho(\rho \circ e_{13}(u), \rho \circ e_{13}(v)) &= \left[\rho(w_{23}) \cdot \rho(e_{13}(u)) \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot \rho(e_{13}(v)) \cdot \rho(w_{12}^{-1}) \right] \\ &= \rho \left[w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1}, w_{12} \cdot e_{13}(v) \cdot w_{12}^{-1} \right] \\ &= \rho \left[e_{12}(u), e_{23}(v) \right] \\ &= \rho \circ e_{13}(uv) \end{aligned}$$

□

Corollary 1. $\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A . □

Proposition 1. A is a ring under the two binary operations of matrix multiplication and μ_ρ .

Proof. **INCOMPLETE**

□

1.3 Replace some notation

Definition 2. Let k be a field and K be an algebraically closed field. Let G be an algebraic k -group, and let

$$\rho : G(k) \rightarrow \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_\alpha, e_\beta, e_\gamma : k \rightarrow G(k)$ such that

$$[e_\alpha(u), e_\beta(v)] = e_\gamma(uv)$$

and suppose have elements $w_{\gamma \rightarrow \beta}, w_{\gamma \rightarrow \alpha} \in G(k)$ such that

$$\begin{aligned} w_{\gamma \rightarrow \beta} \cdot e_\gamma(u) \cdot w_{\gamma \rightarrow \beta}^{-1} &= e_\beta(u) \\ w_{\gamma \rightarrow \alpha} \cdot e_\gamma(u) \cdot w_{\gamma \rightarrow \alpha}^{-1} &= e_\alpha(u) \end{aligned}$$

Let $U = e_\gamma(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \rightarrow \mathrm{GL}_m(K) \quad \mu_\rho(a, b) = \left[\rho(w_{\gamma \rightarrow \alpha}) \cdot a \cdot \rho(w_{\gamma \rightarrow \alpha})^{-1}, \rho(w_{\gamma \rightarrow \beta}) \cdot b \cdot \rho(w_{\gamma \rightarrow \beta})^{-1} \right]$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b , and some fixed elements $\rho(w_{\gamma \rightarrow \alpha})$ and $\rho(w_{\gamma \rightarrow \beta})$ in $\mathrm{GL}_m(K)$.

Lemma 2. $\mu_\rho(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)) = \rho \circ e_\gamma(uv)$

Proof.

$$\begin{aligned} \mu_\rho(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)) &= \left[\rho(w_{\gamma \rightarrow \alpha}) \cdot \rho(e_\gamma(u)) \cdot \rho(w_{\gamma \rightarrow \alpha})^{-1}, \rho(w_{\gamma \rightarrow \beta}) \cdot \rho(e_\gamma(v)) \cdot \rho(w_{\gamma \rightarrow \beta})^{-1} \right] \\ &= \rho \left[w_{\gamma \rightarrow \alpha} \cdot e_\gamma(u) \cdot w_{\gamma \rightarrow \alpha}^{-1}, w_{\gamma \rightarrow \beta} \cdot e_\gamma(v) \cdot w_{\gamma \rightarrow \beta}^{-1} \right] \\ &= \rho \left[e_\alpha(u), e_\beta(v) \right] \\ &= \rho \circ e_\gamma(uv) \end{aligned}$$

□

Corollary 2. $\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A . □

Proposition 2. A is a ring under the two binary operations of matrix multiplication and μ_ρ .

Proof. **INCOMPLETE**

□

1.4 Generalize commutator relation

Definition 3. Let k be a field and K be an algebraically closed field. Let G be an algebraic k -group, and let

$$\rho : G(k) \rightarrow \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_\alpha, e_\beta, e_\gamma : k \rightarrow G(k)$ such that

$$\left[e_\alpha(u), e_\beta(v) \right] = e_\gamma \left(N_{\alpha\beta}(u, v) \right)$$

and suppose have elements $w_{\gamma \rightarrow \beta}, w_{\gamma \rightarrow \alpha} \in G(k)$ such that

$$\begin{aligned} w_{\gamma \rightarrow \beta} \cdot e_\gamma(u) \cdot w_{\gamma \rightarrow \beta}^{-1} &= e_\beta(u) \\ w_{\gamma \rightarrow \alpha} \cdot e_\gamma(u) \cdot w_{\gamma \rightarrow \alpha}^{-1} &= e_\alpha(u) \end{aligned}$$

Let $U = e_\gamma(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \rightarrow \mathrm{GL}_m(K) \quad \mu_\rho(a, b) = \textcolor{red}{INCOMPLETE}$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b , and some fixed elements $\rho(w_{\gamma \rightarrow \alpha})$ and $\rho(w_{\gamma \rightarrow \beta})$ in $\mathrm{GL}_m(K)$.

Lemma 3. $\mu_\rho \left(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v) \right) = \rho \circ e_\gamma(uv)$

Proof. INCOMPLETE □

Corollary 3. $\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A . □

Proposition 3. A is a ring under the two binary operations of matrix multiplication and μ_ρ .

Proof. INCOMPLETE □