Abstract homomorphisms of non-quasi-split special unitary groups

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Contents

1 Generalizing the construction of Kassabov and Sapir

1.1 Original construction

Example 1. Let k be a field. For $1 \leq i \neq j \leq 3$, let $e_{ij} : k \to \mathrm{SL}_3(k)$ be the map which takes $u \in k$ to the transvection matrix $e_{ij}(u)$. For example,

$$e_{13} : k \to \mathrm{SL}_3(k) \qquad e_{13}(u) = \begin{pmatrix} 1 & & u \\ & 1 & \\ & & 1 \end{pmatrix}$$

Note that e_{ij} is a group homomorphism from $(k, +)$ into $\mathrm{SL}_3(k)$, i.e. $e_{ij}(u) \cdot e_{ij}(v) = e_{ij}(u + v)$. Let $U_{ij} = e_{ij}(k)$, so e_{ij} gives an isomorphism between $(k, +)$ and U_{ij}. Let

$$w_{12} = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \qquad w_{23} = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}$$

and note that

$$w_{12} \cdot e_{13}(u) \cdot w_{12}^{-1} = e_{23}(u) \tag{1}$$
$$w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1} = e_{12}(u) \tag{2}$$

Let K be an algebraically closed field, and let

$$\rho : \mathrm{SL}_3(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism (not necessarily well-behaved with respect to any kind of geometric structure). Let $U = e_{13}(k)$, so e_{13} is an isomorphism between $(k, +)$ and U.

Let $V = \rho(U)$ and let $A = \overline{V}$ be the Zariski closure of V. Since V is a subgroup of $\mathrm{GL}_m(K)$, so is A. That is, A is closed under matrix multiplication. We define another binary operation on A as follows:

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a, b) = \left[\rho(w_{23}) \cdot a \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot b \cdot \rho(w_{12}^{-1})\right]$$

A calculation involving the relations (1), (2), and the commutator relation

$$\left[e_{12}(u), e_{23}(v)\right] = e_{13}(uv)$$

shows that

$$\mu_\rho\Big(\rho\big(e_{13}(u)\big), \rho\big(e_{13}(v)\big)\Big) = e_{13}(uv)$$

From this, it follows that μ_ρ maps the subset $V \times V$ (a dense subset of $A \times A$) to V. Since μ_ρ is continuous, it follows that μ_ρ maps $A \times A$ to A.

1.2 Replace SL_3 with arbitrary group

Definition 1. Let k be a field and K be an algebraically closed field. Let G be an algebraic k-group, and let

$$\rho : G(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_{12}, e_{13}, e_{23} : k \to G(k)$ such that

$$\Big[e_{12}(u), e_{23}(v)\Big] = e_{13}(uv)$$

and suppose have elements $w_{12}, w_{23} \in G(k)$ such that

$$w_{12} \cdot e_{13}(u) \cdot w_{12}^{-1} = e_{23}(u)$$
$$w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1} = e_{12}(u)$$

Let $U = e_{13}(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a, b) = \Big[\rho(w_{23}) \cdot a \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot b \cdot \rho(w_{12}^{-1})\Big]$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b, and some fixed elements $\rho(w_{23})$ and $\rho(w_{12})$ in $\mathrm{GL}_m(K)$.

Lemma 1. $\mu_\rho\Big(\rho \circ e_{13}(u), \rho \circ e_{13}(v)\Big) = \rho \circ e_{13}(uv)$

Proof.

$$\begin{aligned}
\mu_\rho\Big(\rho \circ e_{13}(u), \rho \circ e_{13}(v)\Big) &= \Big[\rho(w_{23}) \cdot \rho(e_{13}(u)) \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot \rho(e_{13}(v)) \cdot \rho(w_{12}^{-1})\Big] \\
&= \rho\Big[w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1}, w_{12} \cdot e_{13}(v) \cdot w_{12}^{-1}\Big] \\
&= \rho\Big[e_{12}(u), e_{23}(v)\Big] \\
&= \rho \circ e_{13}(uv)
\end{aligned}$$

□

Corollary 1. *$\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$*

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A. □

Proposition 1. *A is a commutative unital algebraic ring under the two binary operations of matrix multiplication and μ_ρ.*

Proof. Immediate from Lemma 3.2 from *Linear representations of Chevalley groups over commutative rings.* □

1.3 Replace some notation

Definition 2. Let k be a field and K be an algebraically closed field. Let G be an algebraic k-group, and let

$$\rho : G(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_\alpha, e_\beta, e_\gamma : k \to G(k)$ such that

$$\Big[e_\alpha(u), e_\beta(v)\Big] = e_\gamma(uv)$$

and suppose have elements $w_{\gamma\to\beta}, w_{\gamma\to\alpha} \in G(k)$ such that

$$w_{\gamma\to\beta} \cdot e_\gamma(u) \cdot w_{\gamma\to\beta}^{-1} = e_\beta(u)$$
$$w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} = e_\alpha(u)$$

Let $U = e_\gamma(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a, b) = \Big[\rho(w_{\gamma\to\alpha}) \cdot a \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot b \cdot \rho(w_{\gamma\to\beta}^{-1})\Big]$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b, and some fixed elements $\rho(w_{\gamma\to\alpha})$ and $\rho(w_{\gamma\to\beta})$ in $\mathrm{GL}_m(K)$.

Lemma 2. $\mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) = \rho \circ e_\gamma(uv)$

Proof.

$$\begin{aligned}
\mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) &= \Big[\rho(w_{\gamma\to\alpha}) \cdot \rho \circ e_\gamma(u) \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot \rho \circ e_\gamma(v) \cdot \rho(w_{\gamma\to\beta}^{-1})\Big] \\
&= \rho\Big[w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1}, w_{\gamma\to\beta} \cdot e_\gamma(v) \cdot w_{\gamma\to\beta}^{-1}\Big] \\
&= \rho\Big[e_\alpha(u), e_\beta(v)\Big] \\
&= \rho \circ e_\gamma(uv)
\end{aligned}$$

□

Corollary 2. *$\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$*

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A. □

Proposition 2. *A is a commutative unital algebraic ring under the two binary operations of matrix multiplication and μ_ρ.*

Proof. Immediate from Lemma 3.2 from *Linear representations of Chevalley groups over commutative rings.* □

Example 2. Let $G = \mathrm{SL}_n$ and let $e_\alpha = e_{12}$ and $e_\beta = e_{23}$ and $e_\gamma = e_{13}$. Then we have

$$[e_\alpha(u), e_\beta(v)] = e_\gamma(uv)$$

Then with

$$w_{\gamma \to \beta} = w_{12}$$
$$w_{\gamma \to \alpha} = w_{23}$$

we have

$$w_{\gamma \to \beta} \cdot e_\gamma(u) \cdot w_{\gamma \to \beta}^{-1} = e_\beta(u)$$
$$w_{\gamma \to \alpha} \cdot e_\gamma(u) \cdot w_{\gamma \to \alpha}^{-1} = e_\alpha(u)$$

(a) Suppose $k = K = K^{\mathrm{alg}}$ and ρ is the inclusion $\mathrm{SL}_n(k) \hookrightarrow \mathrm{GL}_n(k)$. Then $U = V = e_{13}(k)$, and V is already closed so $A = V = U$ (technically U is a subset of a different group than $V = A$, but whatever). As an abelian group (under the operation of matrix multiplication), A is isomorphic to $(k, +)$. Since ρ is just inclusion, Lemma 3 says that

$$\mu_\rho\Big(e_\gamma(u), e_\gamma(v)\Big) = e_\gamma(uv)$$

i.e. e_γ is a ring isomorphism $(k, +, \cdot) \to (A, \cdot, \mu_\rho)$.

(b) Suppose $k = K = \mathbb{C}$ and $\rho : \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is the composition of inclusion with entrywise complex conjugation. As in the previous example, $A = V = U$. Also as above, A is isomorphic to $(k, +)$ as an abelian group under matrix multiplication. Now Lemma 3 tells us

$$\mu_\rho\Big(e_\gamma(\overline{u}), e_\gamma(\overline{v})\Big) = e_\gamma(\overline{uv})$$

i.e. the map

$$f : \mathbb{C} \to A \qquad f(u) = e_\gamma(\overline{u})$$

is a ring isomorphism.

(c) Let L/k be a Galois extension, and let $K = L^{\mathrm{alg}} = k^{\mathrm{alg}}$ be the algebraic closure. Fix an embedding $L \hookrightarrow K$. Let $\sigma \in \mathrm{Gal}(L/k)$, and let $\rho : \mathrm{SL}_n(L) \to \mathrm{GL}_n(K)$ be the composition of inclusion and entrywise application of σ. Then again $A = V = U$, though technically $U = e_{13}(L) \subseteq \mathrm{SL}_n(L)$, and $A = V = e_{13}(L) \subseteq \mathrm{GL}_n(K)$. As abelian groups, $(L, +) \cong (A, \cdot)$. Lemma 3 says

$$\mu_\rho\Big(e_\gamma(\sigma u), e_\gamma(\sigma v)\Big) = e_\gamma\Big(\sigma(uv)\Big)$$

so the map

$$f : L \to A \qquad f(u) = e_\gamma(\sigma u)$$

gives a ring isomorphism $(L, +, \cdot) \cong (A, \cdot, \mu_\rho)$.

1.4 Generalize commutator relation

Definition 3. Let k be a field and K be an algebraically closed field. Let G be an algebraic k-group, and let

$$\rho : G(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_\alpha, e_\beta, e_\gamma : k \to G(k)$ such that

$$\Big[e_\alpha(u), e_\beta(v)\Big] = e_\gamma\Big(N_{\alpha\beta}(u,v)\Big)$$

and suppose we have an element $h_\gamma \in G(k)$ such that

$$h_\gamma \cdot e_\gamma\Big(N_{\alpha\beta}(u,v)\Big) \cdot h_\gamma^{-1} = uv$$

and suppose we have elements $w_{\gamma\to\beta}, w_{\gamma\to\alpha} \in G(k)$ such that

$$\begin{aligned} w_{\gamma\to\beta} \cdot e_\gamma(u) \cdot w_{\gamma\to\beta}^{-1} &= e_\beta(u) \\ w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} &= e_\alpha(u) \end{aligned}$$

Let $U = e_\gamma(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a,b) = \rho(h_\gamma) \cdot \Big[\rho(w_{\gamma\to\alpha}) \cdot a \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot b \cdot \rho(w_{\gamma\to\beta}^{-1})\Big] \cdot \rho(h_\gamma)^{-1}$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b, and some fixed elements $\rho(w_{\gamma\to\alpha})$ and $\rho(w_{\gamma\to\beta})$ in $\mathrm{GL}_m(K)$.

Lemma 3. $\mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) = \rho \circ e_\gamma(uv)$

Proof.

$$\begin{aligned} \mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) &= \rho(h_\gamma) \cdot \Big[\rho(w_{\gamma\to\alpha}) \cdot \rho \circ e_\gamma(u) \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot \rho \circ e_\gamma(v) \cdot \rho(w_{\gamma\to\beta}^{-1})\Big] \cdot \rho(h_\gamma)^{-1} \\ &= \rho\Big(h_\gamma \cdot \Big[w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1}, w_{\gamma\to\beta} \cdot e_\gamma(v) \cdot w_{\gamma\to\beta}^{-1}\Big] \cdot h_\gamma^{-1}\Big) \\ &= \rho\Big(h_\gamma \cdot \big[e_\alpha(u), e_\beta(v)\big] \cdot h_\gamma^{-1}\Big) \\ &= \rho\Big(h_\gamma \cdot e_\gamma\big(N_{\alpha\beta}(u,v)\big) \cdot h_\gamma^{-1}\Big) \\ &= \rho\Big(e_\gamma(uv)\Big) \\ &= \rho \circ e_\gamma(uv) \end{aligned}$$

□

Corollary 3. *$\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$*

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A. □

Proposition 3. *A is a commutative unital algebraic ring under the two binary operations of matrix multiplication and μ_ρ.*

Proof. Immediate from Lemma 3.2 from *Linear representations of Chevalley groups over commutative rings.* □