Abstract homomorphisms of non-quasi-split special unitary groups

Joshua Ruiter

November 5, 2024

Contents

1 Generalizing the construction of Kassabov and Sapir

1.1 Original construction

Example 1. Let k be a field. For $1 \leq i \neq j \leq 3$, let $e_{ij} : k \to \mathrm{SL}_3(k)$ be the map which takes $u \in k$ to the transvection matrix $e_{ij}(u)$. For example,

$$e_{13} : k \to \mathrm{SL}_3(k) \qquad e_{13}(u) = \begin{pmatrix} 1 & & u \\ & 1 & \\ & & 1 \end{pmatrix}$$

Note that e_{ij} is a group homomorphism from $(k, +)$ into $\mathrm{SL}_3(k)$, i.e. $e_{ij}(u) \cdot e_{ij}(v) = e_{ij}(u + v)$. Let $U_{ij} = e_{ij}(k)$, so e_{ij} gives an isomorphism between $(k, +)$ and U_{ij}. Let

$$w_{12} = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \qquad w_{23} = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}$$

and note that

$$w_{12} \cdot e_{13}(u) \cdot w_{12}^{-1} = e_{23}(u) \tag{1}$$
$$w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1} = e_{12}(u) \tag{2}$$

Let K be an algebraically closed field, and let

$$\rho : \mathrm{SL}_3(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism (not necessarily well-behaved with respect to any kind of geometric structure). Let $U = e_{13}(k)$, so e_{13} is an isomorphism between $(k, +)$ and U.

Let $V = \rho(U)$ and let $A = \overline{V}$ be the Zariski closure of V. Since V is a subgroup of $\mathrm{GL}_m(K)$, so is A. That is, A is closed under matrix multiplication. We define another binary operation on A as follows:

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a, b) = \Big[\rho(w_{23}) \cdot a \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot b \cdot \rho(w_{12}^{-1})\Big]$$

A calculation involving the relations (1), (2), and the commutator relation

$$\Big[e_{12}(u), e_{23}(v)\Big] = e_{13}(uv)$$

shows that

$$\mu_\rho\Big(\rho\big(e_{13}(u)\big), \rho\big(e_{13}(v)\big)\Big) = e_{13}(uv)$$

From this, it follows that μ_ρ maps the subset $V \times V$ (a dense subset of $A \times A$) to V. Since μ_ρ is continuous, it follows that μ_ρ maps $A \times A$ to A.

1.2 Replace SL_3 with arbitrary group

Definition 1. Let k be a field and K be an algebraically closed field. Let G be an algebraic k-group, and let

$$\rho : G(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_{12}, e_{13}, e_{23} : k \to G(k)$ such that

$$\Big[e_{12}(u), e_{23}(v)\Big] = e_{13}(uv)$$

and suppose have elements $w_{12}, w_{23} \in G(k)$ such that

$$w_{12} \cdot e_{13}(u) \cdot w_{12}^{-1} = e_{23}(u)$$
$$w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1} = e_{12}(u)$$

Let $U = e_{13}(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a, b) = \Big[\rho(w_{23}) \cdot a \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot b \cdot \rho(w_{12}^{-1})\Big]$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b, and some fixed elements $\rho(w_{23})$ and $\rho(w_{12})$ in $\mathrm{GL}_m(K)$.

Lemma 1. $\mu_\rho\Big(\rho \circ e_{13}(u), \rho \circ e_{13}(v)\Big) = \rho \circ e_{13}(uv)$

Proof.

$$\begin{aligned}
\mu_\rho\Big(\rho \circ e_{13}(u), \rho \circ e_{13}(v)\Big) &= \Big[\rho(w_{23}) \cdot \rho(e_{13}(u)) \cdot \rho(w_{23})^{-1}, \rho(w_{12}) \cdot \rho(e_{13}(v)) \cdot \rho(w_{12}^{-1})\Big] \\
&= \rho\Big[w_{23} \cdot e_{13}(u) \cdot w_{23}^{-1}, w_{12} \cdot e_{13}(v) \cdot w_{12}^{-1}\Big] \\
&= \rho\Big[e_{12}(u), e_{23}(v)\Big] \\
&= \rho \circ e_{13}(uv)
\end{aligned}$$

$\square$

Corollary 1. *$\mu_\rho(V \times V) \subseteq V$, and consequently $\mu_\rho(A \times A) \subseteq A$*

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A. $\square$

Proposition 1. *A is a commutative unital algebraic ring under the two binary operations of matrix multiplication and μ_ρ.*

Proof. Immediate from Lemma 3.2 from *Linear representations of Chevalley groups over commutative rings.* $\square$

1.3 Replace some notation

Definition 2. Let k be a field and K be an algebraically closed field. Let G be an algebraic k-group, and let

$$\rho : G(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_\alpha, e_\beta, e_\gamma : k \to G(k)$ such that

$$\Big[e_\alpha(u), e_\beta(v)\Big] = e_\gamma(uv)$$

and suppose have elements $w_{\gamma\to\beta}, w_{\gamma\to\alpha} \in G(k)$ such that

$$w_{\gamma\to\beta} \cdot e_\gamma(u) \cdot w_{\gamma\to\beta}^{-1} = e_\beta(u)$$
$$w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} = e_\alpha(u)$$

Let $U = e_\gamma(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a, b) = \Big[\rho(w_{\gamma\to\alpha}) \cdot a \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot b \cdot \rho(w_{\gamma\to\beta}^{-1})\Big]$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b, and some fixed elements $\rho(w_{\gamma\to\alpha})$ and $\rho(w_{\gamma\to\beta})$ in $\mathrm{GL}_m(K)$.

Lemma 2. $\mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) = \rho \circ e_\gamma(uv)$

Proof.

$$\begin{aligned}
\mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) &= \Big[\rho(w_{\gamma\to\alpha}) \cdot \rho \circ e_\gamma(u) \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot \rho \circ e_\gamma(v) \cdot \rho(w_{\gamma\to\beta}^{-1})\Big] \\
&= \rho\Big[w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1}, w_{\gamma\to\beta} \cdot e_\gamma(v) \cdot w_{\gamma\to\beta}^{-1}\Big] \\
&= \rho\Big[e_\alpha(u), e_\beta(v)\Big] \\
&= \rho \circ e_\gamma(uv)
\end{aligned}$$

□

Corollary 2. $\mu_\rho(V \times V) \subseteq V$, *and consequently* $\mu_\rho(A \times A) \subseteq A$

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A. □

Proposition 2. *A is a commutative unital algebraic ring under the two binary operations of matrix multiplication and μ_ρ.*

Proof. Immediate from Lemma 3.2 from *Linear representations of Chevalley groups over commutative rings.* □

Example 2. Let $G = \mathrm{SL}_n$ and let $e_\alpha = e_{12}$ and $e_\beta = e_{23}$ and $e_\gamma = e_{13}$. Then we have

$$[e_\alpha(u), e_\beta(v)] = e_\gamma(uv)$$

Then with

$$w_{\gamma\to\beta} = w_{12}$$
$$w_{\gamma\to\alpha} = w_{23}$$

we have

$$w_{\gamma\to\beta} \cdot e_\gamma(u) \cdot w_{\gamma\to\beta}^{-1} = e_\beta(u)$$
$$w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} = e_\alpha(u)$$

(a) Suppose $k = K = K^{\text{alg}}$ and ρ is the inclusion $\mathrm{SL}_n(k) \hookrightarrow \mathrm{GL}_n(k)$. Then $U = V = e_{13}(k)$, and V is already closed so $A = V = U$ (technically U is a subset of a different group than $V = A$, but whatever). As an abelian group (under the operation of matrix multiplication), A is isomorphic to $(k, +)$. Since ρ is just inclusion, Lemma 3 says that

$$\mu_\rho\Big(e_\gamma(u), e_\gamma(v)\Big) = e_\gamma(uv)$$

i.e. e_γ is a ring isomorphism $(k, +, \cdot) \to (A, \cdot, \mu_\rho)$.

(b) Suppose $k = K = \mathbb{C}$ and $\rho : \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is the composition of inclusion with entrywise complex conjugation. As in the previous example, $A = V = U$. Also as above, A is isomorphic to $(k, +)$ as an abelian group under matrix multiplication. Now Lemma 3 tells us

$$\mu_\rho\Big(e_\gamma(\overline{u}), e_\gamma(\overline{v})\Big) = e_\gamma(\overline{uv})$$

i.e. the map

$$f : \mathbb{C} \to A \qquad f(u) = e_\gamma(\overline{u})$$

is a ring isomorphism.

(c) Let k be any field, let k^{alg} be the algebraic closure, and fix an embedding $k \hookrightarrow k^{\text{alg}}$. Let $\sigma : k \to k$ be a field automorphism. Let $\rho : \mathrm{SL}_n(k) \to \mathrm{GL}_n(K)$ be the composition of inclusion and entrywise application of σ. Then again $A = V = U$, though technically $U = e_{13}(L) \subseteq \mathrm{SL}_n(k)$, and $A = V = e_{13}(k) \subseteq \mathrm{GL}_n(K)$. As abelian groups, $(k, +) \cong (A, \cdot)$. Lemma 3 says

$$\mu_\rho\Big(e_\gamma(\sigma u), e_\gamma(\sigma v)\Big) = e_\gamma\Big(\sigma(uv)\Big)$$

so the map

$$f : L \to A \qquad f(u) = e_\gamma(\sigma u)$$

gives a ring isomorphism $(k, +, \cdot) \cong (A, \cdot, \mu_\rho)$.

(d) Let k be any field, let k^{alg} be the algebraic closure, and fix an embedding $k \hookrightarrow k^{\text{alg}}$. Let $\tau : k^{\text{alg}} \to k^{\text{alg}}$ be a field automorphism. Let $\rho : \mathrm{SL}_n(k) \to \mathrm{GL}_n(K)$ be the composition of inclusion and entrywise application of σ. Then

$$U = e_\gamma(k) \subseteq \mathrm{SL}_n(k)$$
$$V = e_\gamma(\tau k) \subseteq \mathrm{GL}_n(K)$$
$$A = V$$

As above, $(k, +, \cdot) \cong (A, \cdot \mu_\rho)$ via the map $k \to A, u \mapsto e_\gamma(\tau u)$.

Example 3. Let $G = \mathrm{SL}_n$ and let $T \subset G$ be the diagonal torus. Let $\Phi = \Phi(G, T)$ be the associated root system, of type A_{n-1}. Concretely,

$$\Phi = \{\alpha_{ij} : 1 \leq i \neq j \leq n\}$$

where $\alpha_{ij} = \alpha_i - \alpha_j$ and $\alpha_i : T \to k^\times$ is the character that picks off the ith diagonal entry. Let $\alpha, \beta \in \Phi$ such that $\alpha + \beta = \gamma \in \Phi$. That is, $\alpha = \alpha_{ij}$ and $\beta = \alpha_{j\ell}$ and $\gamma = \alpha_{i\ell}$ for some three distinct i, j, ℓ. There are maps $e_\alpha, e_\beta, e_\gamma : k \to \mathrm{SL}_n(k)$ such that $e_\alpha(k)$ is the root subgroup

$$e_\alpha(k) = \left\{x \in \mathrm{SL}_n(k) : txt^{-1} = \alpha(t)x, \forall t \in T(k)\right\}$$

Concretely, $e_{\alpha_{ij}}(u)$ is the matrix with 1's on the diagonal and u in the (i, j) entry and the set of all such matrices is the root subgroup associated to the root α_{ij}. Because $\alpha + \beta = \gamma$, we have a commutator relation

$$\Big[e_\alpha(u), e_\beta(v)\Big] = e_\gamma(\pm uv)$$

If the sign is negative, just reverse the roles of α and β so that the sign is positive, so we may assume that

$$\Big[e_\alpha(u), e_\beta(v)\Big] = e_\gamma(uv)$$

Also, there are "Weyl group elements" $w_{\gamma\to\alpha}, w_{\gamma\to\beta} \in \mathrm{SL}_n(k)$ so that

$$\begin{aligned} w_{\gamma\to\beta} \cdot e_\gamma(u) \cdot w_{\gamma\to\beta}^{-1} &= e_\beta(u) \\ w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} &= e_\alpha(u) \end{aligned}$$

I put "Weyl group elements" in quotation marks because techincally the Weyl group is the quotient $N_G(T)/Z_G(T) = N_G(T)/T$, and so elements of the Weyl group are not in $G(k)$. The elements above are, literally speaking, elements of the normalizer $N_G(T)$, representing elements of the quotient which is the Weyl group.

1.4 Generalize commutator relation

Definition 3. Let k be a field and K be an algebraically closed field. Let G be an algebraic k-group, and let

$$\rho : G(k) \to \mathrm{GL}_m(K)$$

be a group homomorphism. Suppose we have functions $e_\alpha, e_\beta, e_\gamma : k \to G(k)$ such that

$$\Big[e_\alpha(u), e_\beta(v)\Big] = e_\gamma\Big(N_{\alpha\beta}(u,v)\Big)$$

and suppose we have an element $h_\gamma \in G(k)$ such that

$$h_\gamma \cdot e_\gamma\Big(N_{\alpha\beta}(u,v)\Big) \cdot h_\gamma^{-1} = e_\gamma(uv)$$

and suppose we have elements $w_{\gamma\to\beta}, w_{\gamma\to\alpha} \in G(k)$ such that

$$\begin{aligned} w_{\gamma\to\beta} \cdot e_\gamma(u) \cdot w_{\gamma\to\beta}^{-1} &= e_\beta(u) \\ w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} &= e_\alpha(u) \end{aligned}$$

Let $U = e_\gamma(k)$ and let $V = \rho(U)$ and let $A = \overline{V}$. Note that V and A are abelian subgroups of $\mathrm{GL}_m(K)$. Define

$$\mu_\rho : A \times A \to \mathrm{GL}_m(K) \qquad \mu_\rho(a,b) = \rho(h_\gamma) \cdot \Big[\rho(w_{\gamma\to\alpha}) \cdot a \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot b \cdot \rho(w_{\gamma\to\beta}^{-1})\Big] \cdot \rho(h_\gamma)^{-1}$$

Note that μ_ρ is continuous, since it is defined purely in terms of matrix multiplication and inversion of a, b, and some fixed elements $\rho(w_{\gamma\to\alpha})$ and $\rho(w_{\gamma\to\beta})$ in $\mathrm{GL}_m(K)$.

Lemma 3. $\mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) = \rho \circ e_\gamma(uv)$

Proof.

$$\begin{aligned} \mu_\rho\Big(\rho \circ e_\gamma(u), \rho \circ e_\gamma(v)\Big) &= \rho(h_\gamma) \cdot \Big[\rho(w_{\gamma\to\alpha}) \cdot \rho \circ e_\gamma(u) \cdot \rho(w_{\gamma\to\alpha})^{-1}, \rho(w_{\gamma\to\beta}) \cdot \rho \circ e_\gamma(v) \cdot \rho(w_{\gamma\to\beta}^{-1})\Big] \cdot \rho(h_\gamma)^{-1} \\ &= \rho\Big(h_\gamma \cdot \Big[w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1}, w_{\gamma\to\beta} \cdot e_\gamma(v) \cdot w_{\gamma\to\beta}^{-1}\Big] \cdot h_\gamma^{-1}\Big) \\ &= \rho\Big(h_\gamma \cdot \big[e_\alpha(u), e_\beta(v)\big] \cdot h_\gamma^{-1}\Big) \\ &= \rho\Big(h_\gamma \cdot e_\gamma\big(N_{\alpha\beta}(u,v)\big) \cdot h_\gamma^{-1}\Big) \\ &= \rho\Big(e_\gamma(uv)\Big) \\ &= \rho \circ e_\gamma(uv) \end{aligned}$$

□

Corollary 3. $\mu_\rho(V \times V) \subseteq V$*, and consequently* $\mu_\rho(A \times A) \subseteq A$

Proof. The first inclusion is immediate from Lemma 3. The second inclusion follows from this as μ_ρ is continuous and V is dense in A. □

Proposition 3. *A is a commutative unital algebraic ring under the two binary operations of matrix multiplication and μ_ρ.*

Proof. Immediate from Lemma 3.2 from *Linear representations of Chevalley groups over commutative rings.* □

Example 4. Split special orthogonal group, where h is needed INCOMPLETE

Example 5. Quasi-split special unitary group, where h is needed INCOMPLETE

In the following definition, an algebraic k-group is a functor from the category of k-algebras to the category of groups. So if G is an algebraic k-group, then G is not literally a group, but $G(k)$ is a group, called the group of k-rational points. A morphism of algebraic groups is a natural transformation $G \to H$, which comes with literal group homomorphisms $G(R) \to H(R)$ for every k-algebra R.

Definition 4. Let G be an isotropic[1] reductive algebraic k-group, and let $\mathfrak{g}$ be the Lie algebra of G. Let $S \subseteq G$ be a maximal[2] k-split torus. A **character** of S is a morphism of algebraic groups $S \to \mathbb{G}_m$. The set of characters is denoted $X(S)$. For $\alpha \in X(S)$, let

$$\mathfrak{g}_\alpha(k) = \{X \in \mathfrak{g}(k) : s \cdot X = \alpha(s)X, \forall s \in S(k)\}$$

For every $\alpha \in X(S)$, $\mathfrak{g}_\alpha(k)$ is a Lie subalgebra of $\mathfrak{g}(k)$. If k is algebraically closed and characteristic zero, then $\dim(\mathfrak{g}_\alpha(k))$ is always zero or one, but for an arbitrary field k it may have larger dimension. Let

$$\begin{aligned} d_\alpha &= \dim_k(\mathfrak{g}_\alpha(k)) \\ V_\alpha &= (\mathbb{G}_a)^{d_\alpha} \end{aligned}$$

We may think of $V_\alpha(k)$ as a k-vector space with dimension d_α. In fact, as a vector space, $V_\alpha(k) \cong \mathfrak{g}_\alpha(k)$, though we do not think of $V_\alpha(k)$ as being a subset of the Lie algebra $\mathfrak{g}(k)$ or having any kind of Lie algebra structure. When k is algebraically closed and characteristic zero, V_α is just $\{0\}$ or k, but in general it may have larger dimension.

The **relative root system of G with respect to S** is

$$\Phi = \Phi(G, S) = \{\alpha \in X(S) : \mathfrak{g}_\alpha(k) \neq 0\} = \{\alpha \in X(S) : d_\alpha > 0\}$$

The **Weyl group** of (G, S) is $W = N_G(S)/Z_G(S) = N_G(S)/S$. Usually, the functorial perspective on W is unnecessary and we just care about the group of k-points, $W(k) = N_{G(k)}(S(k))/S(k)$. Actually, most of the time we only care about the groups of k-points of all of this stuff.

Definition 5. Let Φ be a (possibly non-reduced) root system. For $\alpha, \beta \in \Phi$, let

$$(\alpha, \beta) = \{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}_{\geq 1}\}$$

Lemma 4. *Let G, k^3, S, Φ be as in Definition 4. Let $\alpha, \beta \in \Phi$ so that $\gamma = \alpha + \beta \in \Phi$ and α, β are not proportional[4]. There exist*

- *Maps $e_\alpha : V_\alpha(k) \to G(k)$ aned $e_\beta : V_\beta(k) \to G(k)$*
- *For each element of (α, β) a function $e_{i\alpha+j\beta} : k \to G(k)$*

[1] Isotropic just means it contains a torus of some positive dimension.
[2] S is maximal among k-split tori, not necessarily maximal among all tori
[3] Probably need to assume k is characteristic zero, or at least not characteristic 2 or 3.
[4] A relative root system may not be reduced, and we do not want $\alpha = \beta$ even though 2α may be a root.

- *For each element of (α, β) a function*[5] $N_{ij}^{\alpha\beta} : V_\alpha(k) \times V_\beta(k) \to V_{i\alpha+j\beta}(k)$

such that for all $u \in V_\alpha(k)$ and $v \in V_\beta(k)$ we have

$$\Big[e_\alpha(u), e_\beta(v)\Big] = \prod_{(\alpha,\beta)} e_{i\alpha+j\beta}\Big(N_{ij}^{\alpha\beta}(u,v)\Big)$$

Proof. INCOMPLETE ☐

Lemma 5. *Let G, k, S, Φ be as in Definition 4. Let $\alpha, \gamma \in \Phi$ be roots of the same length. There exist*

- *An element*[6] $w_{\gamma\to\alpha} \in G(k)$
- *A function*[7] $\varphi_{\gamma\to\alpha} : V_\gamma(k) \to V_\alpha(k)$

such that for all $u \in V_\alpha(k)$ we have

$$w_{\gamma\to\alpha} \cdot e_\gamma(u) \cdot w_{\gamma\to\alpha}^{-1} = e_\alpha\Big(\varphi_{\gamma\to\alpha}(u)\Big)$$

Proof. INCOMPLETE ☐

Lemma 6. *Let G, k, S, Φ be as in Definition 4. Let $\alpha, \beta \in \Phi$ so that $\gamma = \alpha + \beta \in \Phi$ and α, β are not proportional, and let $N_{ij}^{\alpha\beta}$ be the function from Lemma 4. For every $u \in V_\alpha(k)$ and $v \in V_\beta(k)$, there exists an element $h \in G(k)$ such that*

$$h_\gamma(u,v) \cdot \left(\prod_{(\alpha,\beta)} e_{i\alpha+j\beta}\big(N_{ij}^{\alpha\beta}(u,v)\big)\right) \cdot h_\gamma(u,v)^{-1} = INCOMPLETE$$

Proof. I'm not even sure what the right hand side of the equation above is supposed to be INCOMPLETE ☐

[5] $N_{ij}^{\alpha\beta}$ should be homogeneous of degree i in the first input and homogeneous of degree j in the second input, but I am not sure if this is actually needed in any way for the construction.

[6] The element $w_{\gamma\to\alpha}$ should be in $N_{G(k)}(S(k))$ and satisfy $w^2 = 1$, but I am not sure if this is necessary to the calculation.

[7] I am pretty sure $\varphi_{\gamma\to\alpha}$ should be an isomorphism of vector spaces.

2 Generalizing Chevalley construction

If Φ is a reduced irreducible root system, and R is a commutative ring, then the construction of Chevalley gives a procedure for obtaining a group $G(\Phi, R)$ which is the group of R-points of a functor $G(\Phi, -)$ which is a split, semisimple algebraic group. Concretely, the construction describes $G(\Phi, R)$ as a group of automorphisms of a vector space as generated by elements $x_\alpha(r)$ where $\alpha \in \Phi$ and $r \in R$.

What I want is a generalization of this which is generalized in at least two ways:

- Allow for Φ to be non-reduced
- The construction should produce non-split groups

Definition 6. Let V be a $\mathbb{Q}$-vector space with an inner product $(-,-)$. Define

$$\langle -,- \rangle : V \times V \to V \qquad \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

and note that $\langle -,- \rangle$ is only linear in the first argument. For $\alpha \in V$, the reflection across the hyperplane perpendicular to α is the map

$$\sigma_\alpha : V \to V \qquad \sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \alpha$$

A **root system** (in V) is a set $\Phi \subset V$ such that

- Φ is finite
- $0 \notin \Phi$
- For any $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$
- For any $\alpha \in \Phi$, Φ is preserved (set-wise) by the reflection σ_α. (In particular, $-\alpha \in \Phi$ because $\sigma_\alpha(\alpha) = -\alpha$.)

A root $\alpha \in \Phi$ is **multipliable** if $n\alpha \in \Phi$ for some $n > 1$. In other words, a root is not multipliable if $\mathbb{Q}\alpha \cap \Phi = \{\pm\alpha\}$. A root system is **reduced** if $\mathbb{Q}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha \in \Phi$, i.e. Φ is reduced if there are no multipliable roots.

The **rank** of Φ is the dimension of the subspace of V spanned by Φ. Φ is **reducible** if there exist subsets Φ_1, Φ_2 such that $\Phi = \Phi_1 \cup \Phi_2$ and $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1, \beta \in \Phi_2$. Φ is **irreducible** if it is not reducible.

The next lemma shows that even a multipliable root is severly limited in what multiples of it may occur in the root system.

Lemma 7. *Let α be a multipliable root of a root system Φ. Then $\mathbb{Q}\alpha \cap \Phi = \{\pm\alpha, \pm 2\alpha\}$.*

Proof. INCOMPLETE □

Definition 7. Let Φ be a root system. The **Weyl group** of Φ is the subgroup of $\mathrm{Aut}(V)$ generated by σ_α for $\alpha \in \Phi$.

$$W(\Phi) = \langle \sigma_\alpha : \alpha \in \Phi \rangle \subset \mathrm{Aut}(V)$$

Example 6. Type A, B, C, BC root systems INCOMPLETE