

# Kernel Learning And Its Application In Nonlinear Support Vector Machines

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## 1 Introduction

- Linearly separable data classes
- Hyperplane classifiers - Solving an optimization problem

## 2 Linear SVMs

- Maximum margin separator
- Limitations

## 3 Nonlinear SVMs

- The kernel trick
- Solving nonlinear separable classification problems

# Linearly separable data classes

First, let's consider a given data set  $\mathcal{X}$  of labeled points (inputs) with individual labels  $y_i \in \{-1, 1\}$ , e.g.  $(x_1, y_1), \dots, (x_m, y_m) \in \mathcal{X} \times \{-1, 1\}$ .

Our goal is to implement a classification method, which is able to classify new and unlabeled data points with the right or "best" label.

- Plot of data points -

# Linearly separable data classes

In machine learning, a well established classification method are the so called **Support Vector Machines** (SVM). Developed by Vladimir Vapnik and his coworkers in the 1990s, SVMs are still a relevant topic and an even more powerful tool for **classification** and **regression**.

# Hyperplane classifiers

The underlying learning algorithm of SVMs yields to find a hyperplane in some dot product space  $\mathcal{H}$ , which separates the data. A hyperplane of the form

$$\langle w, x \rangle + b = 0 \quad (1)$$

where  $w \in \mathcal{H}$ ,  $b \in \mathbb{R}$  shall be considered [p.11 Kernel Learning Book].  
Furthermore decision functions

$$f(x) = \text{sgn}(\langle w, x \rangle + b) \quad (2)$$

can be used as similarity measures for unlabeled data.

# Hyperplane classifiers - A constrained optimization problem

The optimal hyperplane can be calculated by finding the normal vector that leads to the largest margin. Thus we need to solve the optimization problem

$$\begin{aligned} \min_{w \in \mathcal{H}, b \in \mathbb{R}} \quad & \tau(w) = \frac{1}{2} \|w\|^2 \\ \text{subject to} \quad & y_i (\langle w, x \rangle + b) \geq 1 \quad \forall i = 1, \dots, m. \end{aligned} \tag{3}$$

The constraints in (3) ensure that  $f(x_i)$  will be  $+1$  for  $y_i = +1$  and  $-1$  for  $y_i = -1$ . The  $\geq 1$  on the right hand side of the constraints effectively fixes the scaling of  $w$ . This leads to the maximum margin hyperplane. A detailed explanation can be found in [Chap 7, Kernel Learning Book].

# Hyperplane classifiers - Lagrangian

The constrained optimization problem in (3) can be re-written using the method of Lagrange multipliers. This leads to the Lagrangian

$$L(w, b, \alpha) = \frac{1}{2} \|w\|^2 - \sum_{i=1}^m \alpha_i (y_i (\langle w, x \rangle + b) - 1) \quad (4)$$

subject to  $\alpha_i \geq 0 \ \forall i = 1, \dots, m$ . Here,  $\alpha_i$  are the Lagrange multipliers. The Lagrangian  $L$  has to be minimized with respect to the primal variables  $w$  and  $b$  and maximized with respect to the dual variables  $\alpha_i$  (in other words, a saddle point has to be found).

# Hyperplane classifiers - KKT conditions

The Karush-Kuhn-Tucker (KKT) complementarity conditions of optimization theory state, that at the saddle point, the derivatives of  $L$  with respect to the primal variables must vanish, since

$$\frac{\partial}{\partial b} L(w, b, \alpha) = 0 \text{ and } \frac{\partial}{\partial w} L(w, b, \alpha) = 0 \quad (5)$$

leads to

$$\sum_{i=1}^m \alpha_i y_i = 0 \text{ and } w = \sum_{i=1}^m \alpha_i y_i x_i. \quad (6)$$

The solution vector  $w$  thus has an expansion in terms of a subset of the training patterns, namely those patterns with non-zero  $\alpha_i$ , called Support Vectors (SVs).



# Hyperplane classifiers - Dual optimization problem

We can again re-write our optimization problem by substituting (6) into the Lagrangian (4) to eliminate the primal variables. This yields the dual optimization problem, which is usually solved in practice

$$\begin{aligned} \max_{\alpha \in \mathbb{R}^m} \quad & W(\alpha) = \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i,j=1}^m \alpha_i \alpha_j y_i y_j \langle x_i, x_j \rangle \\ \text{subject to} \quad & \alpha_i \geq 0 \quad \forall i = 1, \dots, m \text{ and } \sum_{i=1}^m \alpha_i y_i = 0. \end{aligned} \tag{7}$$

# Hyperplane classifiers - Dual optimization problem

Finally, the decision function can be re-written using (6) as

$$f(x) = \operatorname{sgn} \left( \sum_{i=1}^m \alpha_i y_i \langle x, x_i \rangle + b \right), \quad (8)$$

where  $b$  can be computed by exploiting  $\alpha_i [y_i (\langle x_i, w \rangle + b) - 1] = 0$ , which follows from the KKT conditions.

# Maximum margin separator

We now have all the theoretical background to go back to our initial classification problem. We can implement a SVM as an maximum margin separator for the given data set  $\mathcal{X}$ .

- Python code, plots, etc. -

# Limitations

Let's consider the following data set.

- Plot of nonlinear separable data -

# Paragraphs of Text

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## Heading

- 1 Statement
- 2 Explanation
- 3 Example

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# Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption

# Theorem

Theorem (Mass–energy equivalence)

$$E = mc^2$$

## Example (Theorem Slide Code)

```
\begin{frame}  
\frametitle{Theorem}  
\begin{theorem}[Mass--energy equivalence]  
$E = mc^2$  
\end{theorem}  
\end{frame}
```

# Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2012].



John Smith (2012)

Title of the publication

*Journal Name* 12(3), 45 – 678.

# The End