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COSC 4780 HW #7 Problem 1.1

6.2 a) Prove: $[\![C_1; (C_2; C_3): comm]\!] = [\![(C_1; C_2); C_3: comm]\!]$

Proof:

By the inductive hypothesis,
$[\![C_1: comm]\!]$, $[\![C_2: comm]\!]$ and $[\![C_3: comm]\!]$ are all elements of $store \rightarrow store_\perp$.
For every $s \in store$, we have
$[\![C_2; C_3: comm]\!](s) = [\![C_3: comm]\!]([\![C_2: comm]\!](s))$.
By the inductive hypothesis for $[\![C_2: comm]\!]$,
$[\![C_2: comm]\!](s) = s_1 \in store_\perp$.
If $s_1 = \perp$, then $[\![C_3: comm]\!](\perp) = \perp$, which is an element of $store_\perp$. On the other hand, if $s_1 \in store$, then the inductive hypothesis on $[\![C_3: comm]\!]$ lets us conclude that: $[\![C_3: comm]\!](s_1) \in store_\perp$.
Hence, $[\![C_2; C_3: comm]\!] \in store \longrightarrow store_\perp = [\![comm]\!]$.
If we let C_A be an arbitrary command such that:
$[\![C_A: comm]\!]$ represents $[\![C_2; C_3: comm]\!]$, the we can repeat the aforementioned steps to show that:
$[\![C_1; C_A: comm]\!] \rightarrow [\![comm]\!]$ meaning $[\![C_1; (C_2; C_3): comm]\!]$ holds.
$\because$ $[\![C_1: comm]\!]$, $[\![C_2: comm]\!]$ and $[\![C_3: comm]\!]$ are all elements of $store \longrightarrow store_\perp$, we can apply the same argument to show that $[\![C_1; C_2: comm]\!] \rightarrow [\![comm]\!]$ and consequently,
$[\![(C_1; C_2); C_3: comm]\!] \longrightarrow [\![comm]\!]$.
$[\![C_1; (C_2; C_3): comm]\!] = [\![comm]\!]$ and $[\![(C_1; C_2); C_3: comm]\!] = [\![comm]\!]$
$\therefore [\![C_1; (C_2; C_3): comm]\!] = [\![(C_1; C_2); C_3: comm]\!]$. ∎

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6.2 c) Prove:

$[\![$while E do C_1 od; C_2 : comm$]\!]$ = $[\![$while E do C od; if E then C_1 else C_2 fi : comm$]\!]$

Proof:

$[\![$while E do C_1 od; C_2 : comm$]\!]$ simplifies to:

$[\![$while E do C_1 od : comm$]\!]$; $[\![C_2$: comm$]\!]$

By the inductive hypothesis,

$[\![E : \text{boolexp}]\!] \in \text{Store} \to \text{Bool}$ and $[\![C_1 : \text{comm}]\!] \in \text{Store} \to \text{Store}_\perp$.

Let s be an arbitrary store such that:

$[\![$while E do C_1 od : comm$]\!](s) = w(s)$, where $w(s) = s'$

iff there is some $k \geq 0$ such that $w_k(s) = s'$.

By the least element principle, we assume that such a k exists such that $\exists k : \mathbb{N}.\ w_k(s) = (s')$.

Because there is a least $k \in \mathbb{N}$ such that $w_k(s) = (s')$, then we know $[\![E : \text{boolexp}]\!](s')$ must return a boolean given $[\![E : \text{boolexp}]\!] \in \text{Store} \to \text{Bool}$.

There are two return cases: True or False

True:

Because k is the minimum k (i.e. $k = 1$), then if we return true:

$w_k(s) = w_1(s') = w_0([\![C_1 : \text{comm}]\!](s')) = \perp$. Thus, $w_k(s) = \perp$

$\perp$ is an element of $\text{Store}_\perp$ meaning:

$[\![$while E do C_1 od : comm$]\!]$ holds

because $[\![C_1 : \text{comm}]\!] \in \text{Store} \to \text{Store}_\perp$.

False:

In the case that $[\![E : \text{boolexp}]\!](s')$ returns false, then it would not evaluate, leaving us with $[\![C_2 : \text{comm}]\!]$

looking at $[\![$while E do C od ; if E then C_1 else C_2 fi : comm$]\!]$

if E evaluates to false then we are also left with $[\![C_2$: comm$]\!]$

meaning there is equivalence between the two sides.

$\therefore$ in both cases we know that:

$[\![$while E do C_1 od; C_2 : comm$]\!]$ = $[\![$while E do C od ; if E then C_1 else C_2 fi : comm$]\!]$. ∎