DATA 605 - Assignment 7

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March 15, 2018

1.

Let $X_1, X_2, ..., X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let Y denote the minimum of the X_i's. Find the distribution of Y.

Solution

We first consider how many ways we can assign Y to one of the x_i variables.

Since there are n variables, with k values, there are k^n total possibilities.

The number of ways we can get $Y = 1$ is the total number of possibilities less the number of possibilities that Y is *not* 1. So, this can be done in $k^n - (k-1)^n$ ways.

Similarly, the number of ways to get $Y = 2$ would be $k^n - (k-2)^n - [k^n - (k-1)^n] = (k-1)^n - (k-2)^n$.

The pattern we get is the recurrence relation $P(Y = m) = \frac{(k-m+1)^n - (k-m)^n}{k^n}$.

2.

Your organization owns a copier (future lawyers, etc.) or MRI (future doctors). This machine has a manufacturer's expected lifetime of 10 years. This means that we expect one failure every ten years. (Include the probability statements and R Code for each part.)

a.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a geometric. (Hint: the probability is equivalent to not failing during the first 8 years.)

Solution

Let p be the probability that the machine fails, and $q = 1 - p$ the probability that it doesn't.

We're looking for the first failure (success) after 8 years, so this is a geometric distribution. Since we're only expecting one failure in 10 years, $p = 0.1,\ q = 0.9$.

$P(X > 8) = 1 - P(X \leq 8)$

$1 - (1 - q^{i+1}) = q^{i+1} = 0.9^9 = 0.3874.$

$E[X] = \frac{1}{p} = \frac{1}{0.1} = 10.$

$Var(X) = \frac{1-p}{p^2} = \frac{0.9}{(0.1)^2} = 90.$

Standard deviation $= \sqrt{Var(X)} = \sqrt{90} \approx 9.486833.$

```r
pdf <- pgeom(8, 0.1, lower.tail = F)

p <- 0.1
q <- 1 - p
ex <- p^-1
var <- q/p^2
sd <- sqrt(var)

cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard Deviation"),
            c(pdf, ex, var, sd))
    )
##
##  Probability = 0.387420
##
##  Expected Value = 10.000000
##
##  Variance = 90.000000
##
##  Standard Deviation = 9.486833
```

b.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as an exponential.

Solution

$P(X > 8) = e^{-\lambda i}$, with $\lambda = 0.1$.

$P(X > 8) = e^{-0.8} = 0.449329$.

$E[X] = \frac{1}{\lambda} = 10$.

$Var(X) = \frac{1}{\lambda^2} = \frac{1}{0.1^2} = 100$.

Standard Deviation $= \sqrt{Var(X)} = \sqrt{100} = 10$.

```r
pdf <- pexp(8, 0.1, lower.tail = F)

l <- 0.1
ex <- 1/l
var <- 1/l^2
sd <- sqrt(var)

cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard Deviation"),
            c(pdf, ex, var, sd))
    )
##
##  Probability = 0.449329
##
##  Expected Value = 10.000000
##
```

```
##  Variance = 100.000000
##
##  Standard Deviation = 10.000000
```

c.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a binomial. (Hint: 0 success in 8 years)

Solution

We're looking for 0 successes in 8 years. So, $P(0)$ with $n = 8$.

$P(0) = \binom{8}{0}(0.1)^0 \times 0.98 - 0 = (0.9)^8 = 0.4304672.$

$E[X] = np = 8 \cdot 0.1 = 0.8.$

$Var(X) = npq = 0.8 \cdot 0.9 = 0.72.$

Standard Deviation $= \sqrt{Var(X)} = \sqrt{0.72} = 0.8485281.$

```r
pdf <- pbinom(0, 8, 0.1)

n <- 8
i <- 0
p <- 0.1
q <- 0.9
ex <- n*p
var <- n*p*q
sd <- sqrt(var)

cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard Deviation"),
            c(pdf, ex, var, sd))
    )
##
##  Probability = 0.430467
##
##  Expected Value = 0.800000
##
##  Variance = 0.720000
##
##  Standard Deviation = 0.848528
```

d.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a Poisson.

Solution

Since Poisson uses averages, and we expect one failure every 10 years, we can say the average yearly failure rate is 0.1. We're looking for 0 failures in the first 8 years.

$P_i(t) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}$

$P_0(8) = \frac{(0.1 \cdot 8)^0 e^{-0.1 \cdot 8}}{0!} = e^{-0.8} = 0.449329.$

$E[X] = Var(X) = \lambda \cdot t = 0.1 \cdot 8 = 0.8.$

Standard Deviation $= \sqrt{Var(X)} = \sqrt{0.8} = 0.8944272.$

```
lambda <- 0.1
t <- 8
i <- 0
ex <- lambda*t
var <- lambda*t
sd <- sqrt(var)
pdf <- ppois(i, t*lambda)

cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard Deviation"),
            c(pdf, ex, var, sd))
    )
##
##  Probability = 0.449329
##
##  Expected Value = 0.800000
##
##  Variance = 0.800000
##
##  Standard Deviation = 0.894427
```