DATA 605 - Discussion 4

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Page 443, Exercise C26

Verify that the function below is a linear transformation.

$T = P_2 \rightarrow \mathbb{C}^2, \quad T(a + bx + cx^2) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$

The function must have the following two properties to be a linear transformation:

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) \quad \forall \ \mathbf{u}_1, \mathbf{u}_2 \in U$
2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u}) \quad \forall \ \mathbf{u} \in U$ and $\forall \ \alpha \in \mathbb{C}$

Condition 1

We can picky dummy variables for $\mathbf{v}$, e.g. let $\mathbf{v} = d + ex + fx^2$.

$T\Big[(a + bx + cx^2) + (d + ex + fx^2)\Big] = T\Big[(a + d) + (bx + ex) + (cx^2 + fx^2)\Big]$.

Factor out the x : $T\Big[(a + d) + x(b + e) + x^2(c + f)\Big]$.

We can now check if our transformation will equal $\begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$.

$\begin{bmatrix} 2(a + d) - (b + e) \\ (b + e) + (c + f) \end{bmatrix} = \begin{bmatrix} 2a + 2d - b + e \\ (b + e) + (c + f) \end{bmatrix}$.

$\begin{bmatrix} (2a - b) + (2d - e) \\ (b + e) + (c + f) \end{bmatrix}$.

Since it's associative: $\begin{bmatrix} 2a - b \\ b + c \end{bmatrix} + \begin{bmatrix} 2d - e \\ e + f \end{bmatrix}$.

$= T(\mathbf{u}) + T(\mathbf{v})$.

Condition 2

$T\Big[\alpha(a + bx + cx^2)\Big] = T\Big[(\alpha a) + (\alpha bx) + (\alpha cx^2)\Big]$.

Factor out the $x : T\Big[(\alpha a) + x(\alpha b) + x^2(\alpha cx)\Big]$.

Following the same procedure as in the first part: $\begin{bmatrix} 2(\alpha a) - (\alpha b) \\ (\alpha b) + (\alpha c) \end{bmatrix}$.

Factor our α : $\begin{bmatrix} \alpha(2a - b) \\ \alpha(b + c) \end{bmatrix} = \alpha \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$.

$= \alpha T(\mathbf{u})$.

Since both conditions are satisfied, we can conclude that this function is a linear transformation.