Math Exam Review: Answers

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1 Introduction

This document is meant to be used as tool for relearning anything you might have forgotten during the school year. This is the solution to every question in the exam review. **If there is any problem, anything you don't understand, or anything that is not clear, please contact me on Discord: JoshL#4217 or by Email: josh.liu@student.tdsb.on.ca. I will be able to update this document and improve it.**

2 Exam review questions

1. For what values of the variable(s) is each rational expression not defined?

   a)
   $$\frac{y-2}{5x(y-2)}$$

   x and y are not defined, they're variables

   b)
   $$\frac{x^2-7x+12}{x^2-x-12}$$

   x is not defined, it's a variable

2. Simplify the following and state restrictions.

   a)
   $$\begin{aligned}
   &\frac{1}{x-1}-\frac{1}{1-x} && x \neq 1\\
   &=\frac{1}{x-1}+\frac{1}{-(1-x)} && x \neq 1\\
   &=\frac{1}{x-1}+\frac{1}{x-1} && x \neq 1\\
   &=\frac{2}{x-1} && x \neq 1
   \end{aligned}$$

b)

$$
\begin{aligned}
&\frac{a-a^3+\frac{2}{a}}{\frac{8}{a}-2a^3} && a \neq 0, \frac{8}{a}-2a^3 \neq 0 \\
&= \frac{\frac{a^2-a^4+2}{a}}{\frac{8-2a^4}{a}} && a \neq 0, 8-2a^4 \neq 0 \\
&= \frac{a^2-a^4+2}{8-2a^4} && a \neq 0, a^4 \neq 4 \\
&= \frac{(2-a^2)(a^2+1)}{2(4-a^4)} && a \neq 0, \pm\sqrt{2} \\
&= \frac{(2-a^2)(a^2+1)}{2(2-a^2)(2+a^2)} && a \neq 0, \pm\sqrt{2} \\
&= \frac{a^2+1}{2(2+a^2)} && a \neq 0, \pm\sqrt{2} \\
&= \frac{a^2+1}{4+2a^2} && a \neq 0, \pm\sqrt{2}
\end{aligned}
$$

c)

$$
\begin{aligned}
&\frac{x^2+4x-5}{2x^2+11x+5} \div \frac{x^2-3x+2}{2x^2+5x+2} \\
&= \frac{(x+5)(x-1)}{(2x+1)(x+5)} \div \frac{(x-2)(x-1)}{(2x+1)(x+1)} && x \neq -\frac{1}{2}, -5, -1 \\
&= \frac{(x+5)(x-1)}{(2x+1)(x+5)} \cdot \frac{(2x+1)(x+1)}{(x-2)(x-1)} && x \neq -\frac{1}{2}, -5, -1, 2, 1 \\
&= \frac{\cancel{(x+5)}\cancel{(x-1)}}{\cancel{(2x+1)}\cancel{(x+5)}} \cdot \frac{\cancel{(2x+1)}(x+1)}{(x-2)\cancel{(x-1)}} && x \neq -\frac{1}{2}, -5, -1, 2, 1 \\
&= \frac{x+1}{x-2} && x \neq -\frac{1}{2}, -5, -1, 2, 1
\end{aligned}
$$

d)

$$
\begin{aligned}
&\frac{x^2-4x+3}{x^2-x} \\
&= \frac{(x-3)(x-1)}{x(x-1)} && x \neq 0, 1 \\
&= \frac{(x-3)\cancel{(x-1)}}{x\cancel{(x-1)}} && x \neq 0, 1 \\
&= \frac{x-3}{x} && x \neq 0, 1
\end{aligned}
$$

e)

$$
\begin{aligned}
&\frac{x^2+x-6}{x^2+5x+4} \div \frac{x^2+4x+3}{x^2+6x+8} \\
&= \frac{(x+3)(x-2)}{(x+4)(x+1)} \div \frac{(x+3)(x+1)}{(x+4)(x+2)} && x \neq -4, -1, -2 \\
&= \frac{(x+3)(x-2)}{(x+4)(x+1)} \cdot \frac{(x+4)(x+2)}{(x+3)(x+1)} && x \neq -4, -1, -2, -3, -1 \\
&= \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+4)}(x+1)} \cdot \frac{\cancel{(x+4)}(x+2)}{\cancel{(x+3)}(x+1)} && x \neq -4, -1, -2, -3, -1 \\
&= \frac{(x-2)(x+1)}{(x+1)^2} && x \neq -4, -1, -2, -3, -1 \\
&= \frac{x^2-x-2}{x^2+2x+1} && x \neq -4, -1, -2, -3, -1
\end{aligned}
$$

f)

$$
\begin{aligned}
&\frac{x-3}{x+2} - \frac{6}{x-4} && x \neq 4, -2 \\
&= \frac{(x-3)(x-4)-6(x+2)}{(x+2)(x-4)} && x \neq 4, -2 \\
&= \frac{x^2-7x+12-6x-12}{(x+2)(x-4)} && x \neq 4, -2 \\
&= \frac{x^2-13x}{(x+2)(x-4)} && x \neq 4, -2 \\
&= \frac{x^2-13x}{x^2-2x-8} && x \neq 4, -2
\end{aligned}
$$

g)

$$
\begin{aligned}
&\frac{2x^2-x-1}{3x^2+x-2} \div \frac{2x^2-5x-3}{3x^2-11x+6} \\
&= \frac{(2x-1)(x+1)}{(3x-2)(x+1)} \div \frac{(2x+1)(x-3)}{(3x-2)(x-3)} && x \neq \frac{2}{3}, -1, 3 \\
&= \frac{(2x-1)(x+1)}{(3x-2)(x+1)} \cdot \frac{(3x-2)(x-3)}{(2x+1)(x-3)} && x \neq \frac{2}{3}, -1, 3, -\frac{1}{2} \\
&= \frac{(2x-1)\cancel{(x+1)}}{\cancel{(3x-2)(x+1)}} \cdot \frac{\cancel{(3x-2)(x-3)}}{(2x+1)\cancel{(x-3)}} && x \neq \frac{2}{3}, -1, 3, -\frac{1}{2} \\
&= \frac{2x-1}{2x+1} && x \neq \frac{2}{3}, -1, 3, -\frac{1}{2}
\end{aligned}
$$

h)

$$
\begin{aligned}
&\frac{3}{x^2-3x-4}-\frac{2}{x^2+5x+4} \\
&=\frac{3}{(x-4)(x+1)}-\frac{2}{(x+4)(x+1)} && x\neq 4,-1,-4 \\
&=\frac{3(x+4)-2(x-4)}{(x-4)(x+1)(x+4)} && x\neq 4,-1,-4 \\
&=\frac{3x+12-2x+8}{(x-4)(x+1)(x+4)} && x\neq 4,-1,-4 \\
&=\frac{x+20}{(x-4)(x+1)(x+4)} && x\neq 4,-1,-4
\end{aligned}
$$

3. Factor Completely.

a)

$$
\begin{aligned}
&4ax^5-\frac{axy^4}{4} \\
&=ax(4x^4-\frac{y^4}{4}) \\
&=\frac{ax(16x^4-y^4)}{4} \\
&=\frac{ax(4x^2-y^2)(4x^2+y^2)}{4} \\
&=\frac{ax(2x-y)(2x+y)(4x^2+y^2)}{4}
\end{aligned}
$$

4. Solve by factoring

a)

$$
\begin{aligned}
2x^2-9x+10&=0 \\
(2x+5)(x+2)&=0 \\
2x+5&=0 && -x+2=0 \\
x&=-\frac{5}{2} && x=-2 \\
x&=-\frac{5}{2},-2
\end{aligned}
$$

b)

$$
\begin{aligned}
2x^2-7x&=4 \\
2x^2-7x-4&=0 \\
(2x+1)(x-4)&=0 \\
2x+1&=0 && x-4=0 \\
x&=-\frac{1}{2} && x=4 \\
x&=-\frac{1}{2},4
\end{aligned}
$$

5. Solve $5x^2 + 1 = 8x$ using the quadratic formula. Express the answer in simplest radical form.

$$5x^2 + 1 = 8x \qquad\qquad ax^2 + bc + c = 0$$

$$5x^2 - 8x + 1 = 0 \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 5 \cdot 1}}{2 \cdot 5}$$

$$x = \frac{8 \pm \sqrt{44}}{10}$$

$$x = \frac{8 \pm 4\sqrt{11}}{10}$$

$$x = \frac{8 + 4\sqrt{11}}{10} \qquad\qquad x = \frac{8 - 4\sqrt{11}}{10}$$

$$x = \frac{4 + 2\sqrt{11}}{5} \qquad\qquad x = \frac{4 - 2\sqrt{11}}{5}$$

$$x = \frac{4 + 2\sqrt{11}}{5}, \frac{4 - 2\sqrt{11}}{5}$$

6. Evaluate the expression $E(x) = \frac{x^2+5x+6}{x^2+7x+10}$ for $x = -11$

$$E(x) = \frac{x^2 + 5x + 6}{x^2 + 7x + 10}$$

$$E(x) = \frac{(x+3)(x+2)}{(x+5)(x+2)}$$

$$E(x) = \frac{x+3}{x+2}$$

$$E(-11) = \frac{-11+3}{-11+2}$$

$$E(-11) = \frac{8}{9}$$

7. Solve for x (leave in answers in the radical form if necessary)

a)

$$-2x^2 - 3x + 1 = 0$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot -2 \cdot 1}}{2 \cdot -2} \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{17}}{-4}$$

$$x = \frac{-3 \pm \sqrt{17}}{4}$$

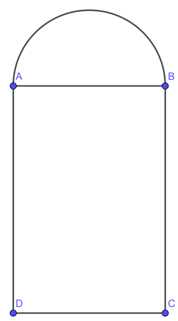
$$x = \frac{-3 + \sqrt{17}}{4}, \frac{-3 - \sqrt{17}}{4}$$

b)

$$
\begin{aligned}
6^{x-5} &= \frac{1}{36^{2x+5}} \\
6^{x-5} &= 36^{-(2x+5)} \\
6^{x-5} &= 6^{2(-2x-5)} \\
6^{x-5} &= 6^{-4x-10} \\
x-5 &= -4x-10 \\
5x &= -5 \\
x &= -1
\end{aligned}
$$

8. A rectangular window is continued at the top side with a semi-circle. The perimeter of the window is 10m. Find the dimensions of the window that will maximize the area of the window.

$$
\begin{aligned}
2(\overline{AB}+\overline{BC}) &= 10 \\
\overline{AB}+\overline{BC} &= 5
\end{aligned}
$$



Let x represent $\overline{AB}$

$$
\begin{aligned}
Area_{ABCD} &= x(10-x) \quad & Area_{\widehat{AB}} &= (\frac{x}{2})^2\pi \\
&= 10x - x^2 & &= \frac{x^2\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
Area &= 10x - x^2 + \frac{x^2\pi}{4} \\
0 &= x(10 - x + \frac{x\pi}{4}) \\
0 &= x(x(\frac{-4+\pi}{4}) - 10) \\
x &= 0, \frac{40}{\pi - 4}
\end{aligned}
$$

Maximum area is the average of the x_1, x_2, meaning maximum area is where $x = \frac{20}{\pi-4}$. No need to really worry about this, I was told it doesn't matter.

9. Determine the vertex of $y = -4x^2 - 16x + 5$ and state if it is a maximum or minimum

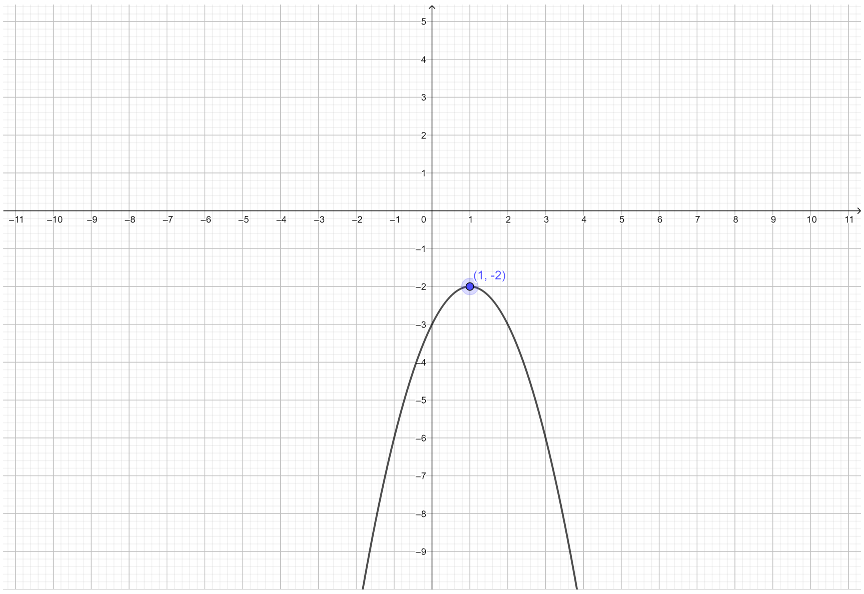
   Remember that looking of the first term (the $-4x^2$) if it's facing up (Is positive), then it is a minimum, if facing down (Is negative), the it is a maximum. In this case, $-4x^2$ is negative, meaning it was a **maximum**

10. Complete the square and represent graphically the following quadratic equation $y = -x^2 + 2x - 3$

$$
y = -(x^2 - 2x + 3)
$$

$$y = -((x-1)^2 + 3 - 1)$$
$$y = -(x-1)^2 - 2$$

Here is the graphical representation, vertex is $(1, -2)$, facing down.



11. Given that θ is an angle in standard position, $0° \leq \theta \leq 360°$, with $\cos\theta = -\frac{\sqrt{3}}{2}$. Find all possible values of θ. Remember CAST rule

$$\cos\beta = \frac{\sqrt{3}}{2}$$
$$\beta = 30°$$

$$\cos(180-\beta) = \cos(180+\beta) = -\cos\beta$$

$$180-\beta = \theta \qquad\qquad 180+\beta = \theta$$
$$180-30 = \theta \qquad\qquad 180+30 = \theta$$
$$150 = \theta \qquad\qquad 210 = \theta$$
$$\theta = 150°, 210°$$

12. In $\triangle ABC, \angle B = 107°$, $a = 18.7$, $c = 10.5$. Solve the triangle. as you're given two sides and one angle (SAS) you can use the cosine law. $c^2 = a^2 + b^2 - 2ab\cos C$ to find the side,

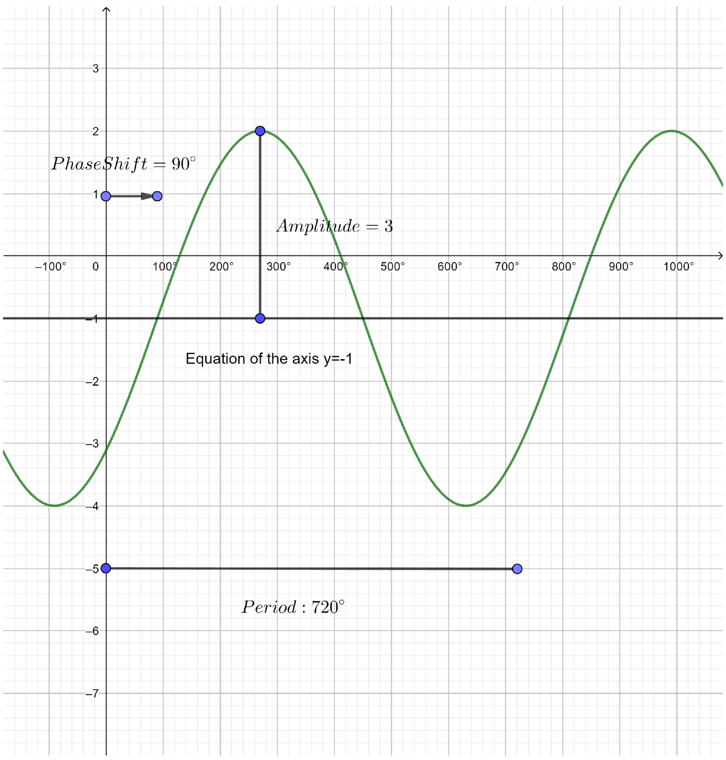
the sine law $\frac{\sin A}{a} = \frac{\sin B}{B} = \frac{\sin C}{C}$ to find the angles

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\cos B \\
b^2 &= 18.7^2 + 10.5^2 - 2 \cdot 18.7 \cdot 10.5 \cdot \cos 107^\circ \\
b &= 23.974035297475865 \\
\frac{\sin B}{b} &= \frac{\sin C}{c} \\
\frac{\sin 107^\circ}{23.984} &= \frac{\sin C}{10.5} \\
0.419 &= \sin C \\
24.8^\circ &= \angle C \\
\angle A &= 180 - 24.8 - 107 \\
\angle A &= 48.2
\end{aligned}$$

13. On graph paper provided, sketch one cycle of $y = 3\sin\frac{1}{2}(\theta - 90^\circ) - 1$

$$\begin{aligned}
&\text{Amplitude} = 3 \\
&\text{Equation of the axis} : y = -1 \\
&\text{Phase Shift} = 90^\circ right \\
&\text{Period} : \frac{360}{\frac{1}{2}} = 720^\circ
\end{aligned}$$

The graph is drawn as shown



14. Given the function $y = -4\cos 2(\theta + \frac{180}{3}) + 5$, state the amplitude, period, phase shift, and

range.

$$\begin{aligned}
&\text{Amplitude} = 4\\
&\text{Equation of the axis} : y = 5\\
&\text{Phase Shift} = \frac{180}{3} = 60° left\\
&\text{Period} : \frac{360}{2} = 180°
\end{aligned}$$

15. Solve the equation for $0 \leq \theta \leq 360°$. $\sin\theta(3 - 4\cos^2\theta) = 0$

$$\sin\theta = 0 \qquad\qquad 3 - 4\cos^2\theta = 0$$

$$\theta = 0°, 180°, 360° \qquad\qquad \cos\theta = -\frac{\sqrt{3}}{2}$$

$$\theta = 0°, 180°, 360° \qquad\qquad \theta = 180 - 30, 180 + 30$$

$$\theta = 0°, 180°, 360° \qquad\qquad \theta = 150°, 210°$$

$$\theta = 0°, 150°, 180°, 210°, 360°$$

16. Solve the equation $2\sin^2\theta - \sin\theta - 1 = 0$ for $0 \leq \theta \leq 360°$.

Let x represent $\sin\theta$

$$\begin{aligned}
2x^2 - x - 1 &= 0\\
(2x + 1)(x - 1) &= 0\\
x &= -\frac{1}{2}, 1
\end{aligned}$$

$$\sin\theta = -\frac{1}{2} \qquad\qquad \sin\theta = 1$$

$$\theta = 180 + 30, 360 - 30 \qquad\qquad \theta = 90°$$

$$\theta = 210°, 330° \qquad\qquad \theta = 90°$$

$$\theta = 90°, 210°, 330°$$

17. Question Missing there's just nothing in the paper

18. Question Missing there's just nothing in the paper

19. Prove the identity:

a) $\frac{\sin^2\theta - 1}{\cos^2\theta - 1} = \frac{1}{\sin^2\theta} - 1$

Remember that $\sin^2\theta + \cos^2\theta = 1$

$$\begin{aligned}
LS &= \frac{\sin^2\theta - 1}{\cos^2\theta - 1} & RS &= \frac{1}{\sin^2\theta} - 1\\
&= \frac{-\cos^2\theta}{-\sin^2\theta} & &= \frac{1 - \sin^2\theta}{\sin^2\theta}\\
&= \frac{\cos^2\theta}{\sin^2\theta} & &= \frac{\cos^2\theta}{\sin^2\theta}
\end{aligned}$$

$$\because LS = RS$$

$$\therefore \frac{\sin^2\theta - 1}{\cos^2\theta - 1} = \frac{1}{\sin^2\theta} - 1$$

b) $\tan\theta + \frac{\cos\theta}{1+\sin\theta} = \frac{1}{\cos\theta}$

Remember that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\begin{aligned} LS &= \tan\theta + \frac{\cos\theta}{1+\sin\theta} \\ &= \frac{\sin\theta(1+\sin\theta)+\cos\theta}{\cos\theta(1+\sin\theta)} \\ &= \frac{sin^2\theta+\cos^2\theta+\sin\theta}{\cos\theta(1+\sin\theta)} \\ &= \frac{1+\sin\theta}{\cos\theta(1+\sin\theta)} \\ &= \frac{1}{\cos\theta} \end{aligned} \qquad RS = \frac{1}{\cos\theta}$$

$$\because LS = RS$$
$$\therefore \tan\theta + \frac{\cos\theta}{1+\sin\theta} = \frac{1}{\cos\theta}$$

c) $\frac{\sin x+\tan x}{\cos x+1} = \tan x$

$$\begin{aligned} LS &= \frac{\sin x+\tan x}{\cos x+1} \\ &= \frac{\sin x + \frac{\sin x}{\cos x}}{\cos x+1} \\ &= \frac{\frac{\sin x\cos x+\sin x}{\cos x}}{\cos x+1} \\ &= \frac{\sin x\cos x+\sin x}{\cos x(\cos x+1)} \\ &= \frac{\sin x(\cos x+1)}{\cos x(\cos +1)} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned} \qquad RS = \tan x$$

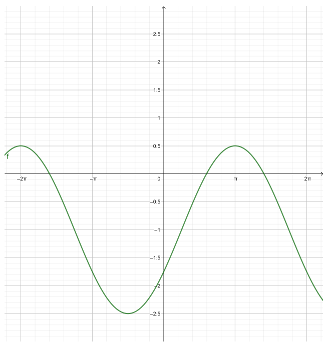
$$\because LS = RS$$
$$\therefore \frac{\sin x+\tan x}{\cos x+1} = \tan x$$

20. Complete the table

Sine Function	$y = -3\sin(2x-60)+1$	
Amplitude		2
Period		270°
Phase Shift		30° left
Vertical Shift		None
Domain		
Range		

Sine Function	$y = -3\sin(2x - 60) + 1$	$2\sin(\frac{4}{3}(x + 30))$
Amplitude	3	2
Period	180°	270°
Phase Shift	30° right	30° left
Vertical Shift	1 up	None
Domain	D=$\{x \in \mathbb{R}\}$	D=$\{x \in \mathbb{R}\}$
Range	R=$\{y \in \mathbb{R} \mid -2 \leq y \leq 4\}$	R=$\{y \in \mathbb{R} \mid -2 \leq y \leq 2\}$

21. State a possible equation for the cosine function shown.



$$Min = -2.5 \qquad Max = 0.5$$

$$\text{Amplitude} = \frac{Max - Min}{2} \qquad \text{Period} = \frac{2\pi}{3\pi}$$

$$\text{Amplitude} = \frac{0.5 + 2.5}{2} \qquad \text{Period} = \frac{2}{3}$$

$$\text{Amplitude} = 1.5$$

$$\text{Equation of the axis: } y = \frac{Max + Min}{2} \qquad \text{Phase Shift} = \pi \text{ right}$$

$$\text{Equation of the axis: } y = \frac{-2.5 + 0.5}{2}$$

$$\text{Equation of the axis: } y = -1$$

$$y = 1.5\cos(\frac{2}{3}(x - \pi)) - 1$$

22. Consider the function $y = 3\sin 2(x + 90) + 5$.
    State the Phase Shift, Period, Vertical displacement, Amplitude, Domain, Range
    a) Graph the function over two complete graphs

Phase Shift	90° left
Period	$\frac{360}{2} = 180°$
Vertical Displacement	5
Amplitude	3
Domain	$D = \{x \in \mathbb{R}\}$
Range	$R = \{y \in \mathbb{R} \mid 2 \leq y \leq 8\}$

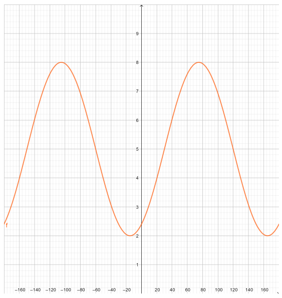
I will most likely not do the graph, if you really want me to add this, ask me on discord.

23. Graph the function $y = -3\sin(2x - 240)$ for $-180° \leq x \leq 180°$

$$y = -3\sin(2x - 240)$$

$$y = -3\sin(2(x - 120))$$

Phase Shift	$120°$ right
Period	$\frac{360}{2} = 180°$
Vertical Displacement	0
Amplitude	3
Domain	$D = \{x \in \mathbb{R}\}$
Range	$R = \{y \in \mathbb{R} \mid -3 \leq y \leq 3\}$



24. On a typical day at an ocean port, the water has a maximum depth of $18m$ at 6:00 a.m. The minimum depth of 9 m occurs 6.8 h later. Sketch one complete cycle and write an equation of the form: $h = a\cos b(t-d)+c$ to describe the relationship between the depth h of the water and the time t. *I'm assuming t is in terms of hours as before it says 6.8 h later.*

$$\begin{aligned} \text{Max} &= 18 \\ \text{Vertical Displacement} &= \frac{Max + Min}{2} \\ \text{Vertical Displacement} &= \frac{18+9}{2} \\ \text{Vertical Displacement/c} &= \frac{27}{2} \\ \text{Phase Shift} &= 6.8 \text{ right} \\ \text{Phase Shift/d} &= 6.8 \text{ right} \end{aligned}$$

$$\begin{aligned} \text{Min} &= 9 \\ \text{Amplitude} &= \frac{Max - Min}{2} \\ \text{Amplitude} &= \frac{18-9}{2} \\ \text{Amplitude/a} &= \frac{9}{2} \\ \text{Period} &= 2 * 6.8 \\ \text{Period/b} &= 13.6 \end{aligned}$$

$$h = 13.5\cos 13.6(t-6.8)+4.5$$

Again, graphs are annoying, if you want me to explain ask me on discord.

25. Given the function $f(x) = \frac{x+1}{x^2}$, determine and simplify $f(\frac{1}{x})$

$$\begin{aligned} f(\frac{1}{x}) &= \frac{\frac{1}{x}+1}{(\frac{1}{x})^2} \\ f(\frac{1}{x}) &= \frac{\frac{1+x}{x}}{\frac{1}{x^2}} \\ f(\frac{1}{x}) &= \frac{x^2(x+1)}{x} \\ f(\frac{1}{x}) &= x^2 + x \end{aligned}$$

26. Solve the triangle $\triangle ABC$ in which $AB = 23.3cm$ and $BC = 26.8cm$ and $\angle ABC = 113°$. Round the angles to the nearest degree and lengths to one decimal place. Include diagram.

Use the cosine law to find the missing side$c^2 = a^2 + b^2 - 2ab\cos C$. Then use sine law to find the missing angles $\frac{\sin A}{a} = \frac{\sin B}{B} = \frac{\sin C}{C}$.

$$c = 23.3 \qquad a = 26.8 \qquad B = 113°$$

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2 \times a \times c\cos B \\
b^2 &= 26.8^2 + 23.3^2 - 2 \times 23.3 \times 26.8\cos 113° \\
b^2 &= 1749.11 \\
b &= 41.8 \\
\frac{\sin A}{a} &= \frac{\sin B}{B} \\
\frac{\sin A}{26.8} &= \frac{\sin 113°}{41.8} \\
\sin A &= \frac{26.8 \times \sin 113°}{41.8} \\
A &= \arcsin(\frac{26.8 \times \sin 113°}{41.8}) \\
A &= 36° \\
C &= 180 - A - B \\
C &= 180 - 36 - 113 \\
C &= 31°
\end{aligned}$$

For diagram refer to this link.

27. Determine how much money is needed today in order to have \$6000 in three years at 6% compounded quarterly.

    Compound interest formula: $A = P(1 + \frac{r}{n})^{nt}$. (If you want to understand these variables search online)

$$P = 6000 \qquad r = 0.06$$
$$n = 4 \qquad t = 3$$
$$A = 6000(1 + \frac{0.06}{4})^{3*4}$$
$$A = 7173.71$$

28. Suppose you begin a saving program to have \$10 000 after 10 years. You plan to make regular deposits every month into an investment account that pays 6% compounded monthly, calculate each regular deposit.

    This uses the future value annuity formula $FV = P\frac{(1+\frac{r}{n})^{nt}-1}{\frac{r}{n}}$

$$n = 12 \qquad t = 10$$
$$r = 0.06 \qquad FV = 10000$$

$$FV = P\frac{(1+\frac{r}{n})^{nt}-1}{\frac{r}{n}}$$
$$10000 = P\frac{(1+\frac{0.06}{12})^{12\times 10}-1}{\frac{0.06}{12}}$$
$$\frac{10000}{\frac{(1+\frac{0.06}{12})^{12\times 10}-1}{\frac{0.06}{12}}} = P$$
$$61.02 = P$$

29. A penny is tossed into the air from a bridge and falls to the water below. The height of the penny h metres, relative to the water t seconds after being thrown is given by $h = -5t^2 + 10t + 20$.

   a) Determine the max height of the penny above the water.

   I am *pretty* sure that this involves completing the square. *Maybe I got the name wrong*

$$h = -5t^2 + 10t + 20$$
$$h = -5(t^2 - 2t - 4)$$
$$h = -5((t-1)^2 - 5)$$
$$h = -5(t-1)^2 + 25 \text{Vertex: } (1, 25)$$

   Maximum height is 25 metres

   b) How long does it take the penny to reach its max height?

   it takes 1 second.

   Note: The x value of the maximum is the average of the two x intercepts $\frac{x_1+x_2}{2}$. Replacing x_1, x_2 with the quadratic formula you get the x value of vertex is $-\frac{b}{a}$.

30. What transformations have been performed on $y = f(x)$ to attain $y = 3f(-x+1) - 2$?

$$y = 3f(-(x-1)) - 2$$

Vertical stretch by a factor of 3
Reflect along the y axis
Horizontal shift 1 unit to the right
Vertical shift 2 units down

31. Describe the transformations applied to $y = \sqrt{49 - x^2}$ to produce $y = -2\sqrt{49 - (x+5)^2} - 6$.

Reflect along the x axis
Vertical stretch by a factor of 2
Horizontal shift 5 units to the left
Vertical shift 6 units down

32. Given $f(x) = 4x^2 - 5$, determine

   a) $f(2)$

$$f(2) = 4 \times 2^2 - 5$$
$$f(2) = 11$$

b) $f(x) = 31$

$$\begin{aligned} 31 &= 4x^2 - 5 \\ 36 &= 4x^2 \\ x &= \pm 3 \end{aligned}$$

33. Determine the inverse of $f(x) = \frac{2}{5+x} - 1$, and state its domain.

$$\begin{aligned} f(x) &= \frac{2}{5+x} - 1 \\ x &= \frac{2}{5+f'(x)} - 1 \\ x+1 &= \frac{2}{5+f'(x)} \\ 5+f'(x) &= \frac{2}{x+1} \\ f'(x) &= \frac{2}{x+1} - 5 \\ R &= \{f(x) \in \mathbb{R} \mid f(x) \neq -1\} \\ D' &= \{x \in \mathbb{R} \mid x \neq -1\} \end{aligned}$$

34. For each case,
    - find the inverse
    - state if the inverse is a function or not

a) $f(x) = \frac{1}{2}x - 3$

$$\begin{aligned} x &= \frac{1}{2}f'(x) - 3 \\ x+3 &= \frac{1}{2}f'(x) \\ f'(x) &= 2x + 6 \end{aligned}$$

This inverse is a function

b) $f(x) = x^2 + 2$

$$\begin{aligned} x &= f'(x)^2 + 2 \\ x-2 &= f'(x)^2 \\ \pm\sqrt{x-2} &= f'(x) \end{aligned}$$

This inverse is not a function

c) $f(x) = \frac{2x}{x+1}$

$$\begin{aligned} x &= \frac{2f'(x)}{f'(x)+1} \\ xf'(x) + x &= 2f'(x) \\ f'(x)(x-2) &= -x \\ f'(x) &= \frac{x}{2-x} \end{aligned}$$

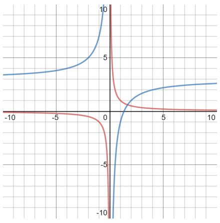
This inverse is a function

d) $f(x) = 2(x+2)^2 + 2$

$$\begin{aligned} x &= 2(f'(x)+2)^2 + 2 \\ x - 2 &= 2(f'(x)+2)^2 \\ \frac{x-2}{2} &= (f'(x)+2)^2 \\ \pm\frac{x-2}{2} - 2 &= f'(x) \\ &\text{This inverse is not a function} \end{aligned}$$

35. Given $f(x) = \frac{1}{x}$

a) Sketch the image of $y = 2f(-\frac{1}{2}x) + 3$



b) Write and simplify the equation of the image in (a).

$$\begin{aligned} y &= 2f(-\frac{1}{2}x) + 3 \\ y &= 2 \times \frac{1}{-\frac{1}{2}x} + 3 \\ y &= 2 \times \frac{2}{x} + 3 \\ y &= \frac{4}{x} + 3 \end{aligned}$$

c) State the domain and range of the image in (a).

$$\begin{aligned} D &= \{x \in \mathbb{R} \mid x \neq 0\} \\ R &= \{y \in \mathbb{R} \mid y \neq 3\} \end{aligned}$$

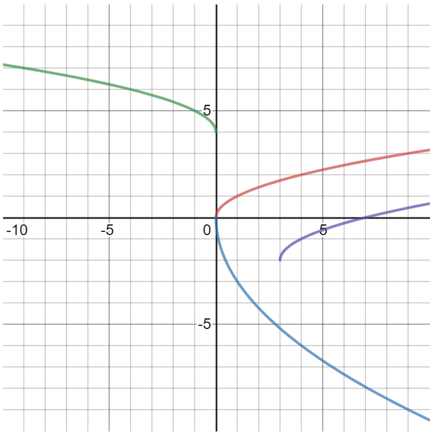
36. Sketch on following curves on the same grid.

a) $y = \sqrt{x}$

b) $y = -3\sqrt{x}$

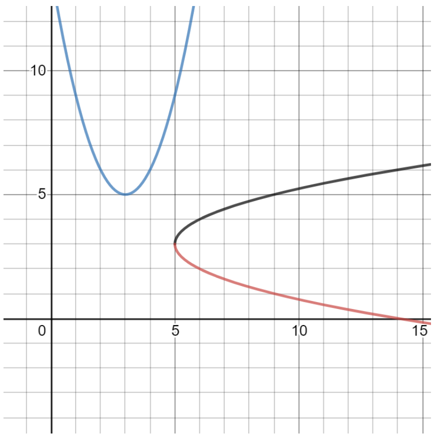
c) $y = \sqrt{-x} + 4$

d) $y = \sqrt{x-3} - 2$



37. Determine the inverse of $y = (x-3)^2 + 5$ and state its domain (the inverse) Sketch both curves on the same grid

$$\begin{aligned} y &= (x-3)^2 + 5 \\ x &= (y'-3)^2 + 5 \\ x - 5 &= (y'-3)^2 \\ \pm\sqrt{x-5} &= y' - 3 \\ 3 \pm \sqrt{x-5} &= y \\ D' &= \{x \in \mathbb{R} \mid 5 \leq x\} \end{aligned}$$



38. For each of the following sequences.
    a) Find a possible formula for t_n
    b) Use your formula to find t_{21}. Express your answer in exact values only
        i. $6, -18, 54$

        This is a geometric sequence, following the pattern $a, ar, ar^2, ar^3 \ldots$ The general/explicit formula is $a_n = a_1 r^{n-1}$ and the recursive formula is $a_n = ra_{n-1}$

$$
\begin{aligned}
r &= \frac{-18}{6} & a_1 &= 6 \\
r &= -2 a_n & &= 6 \times (-2)^{n-1} \\
a_{21} &= 6 \times (-2)^{21-1} \\
a_{21} &= 6291456
\end{aligned}
$$

ii. $\frac{4}{5}, \frac{7}{25}, \frac{10}{125}$

This is not a geometric or arithmetic sequence, but a pattern. Notice how the numerator is in increments of 3 and the denominator is multiplied by 5. Answer is $a_n = \frac{3n+1}{5^n}$. But we solve it by splitting it into two parts, the numerator and denominator.

Numerator: This is an arithmetic sequence, following the pattern $a, a+d, a+2d, a+3d\ldots$ The general/explicit formula is $a_n = a_1 + d(n-1)$ and the recursive formula is $a_n = a_{n-1} + d$

$$
\begin{aligned}
d &= 7 - 4 & a_1 &= 4 \\
d &= 3 \\
a_n &= 4 + 3(n-1)
\end{aligned}
$$

Denominator: This is a geometric sequence

$$
\begin{aligned}
r &= \frac{25}{5} & a_1 &= 5 \\
r &= 5 \\
a_n &= 5 \times 5^{n-1}
\end{aligned}
$$

Put together,

$$
\begin{aligned}
a_n &= \frac{4 + 3(n-1)}{5 \times 5^{n-1}} \\
a_n &= \frac{3n+1}{5^n} \\
a_{21} &= \frac{64}{476837158203125}
\end{aligned}
$$

iii. $25, 16, 9, 4$

Again, this is not a arithmetic or geometric sequence but a pattern.

It's pretty difficult to describe this, if you want me to go into detail, ask me on discord.

$$
\begin{aligned}
a_n &= (6-n)^2 \\
a_{21} &= (6-21)^2 \\
a_{21} &= 225
\end{aligned}
$$

39. If the terms $27, x, y, 8$ form a geometric sequence, find x and y.

$$\begin{aligned}
a_1 &= 27 \\
a_4 &= 27r^{4-1} \\
8 &= 27r^3 \\
\frac{8}{27} &= r^3 \\
\sqrt[3]{\frac{8}{27}} &= r \\
\frac{2}{3} &= r \\
x &= 27 \times \frac{2}{3} \\
x &= 18 \\
y &= 18 \times \frac{2}{3} \\
y &= 12
\end{aligned}$$

40. Determine the sum of the series $81 + 77 + 73 + \ldots + 5$.

This is an arithmetic sequence, the formula for the sum of series is $S_n = \frac{n(a_1+a_n)}{2}$

$$\begin{aligned}
d &= 77 - 81 & a_1 &= 81 \\
d &= -4 & a_n &= 5 \\
5 &= 81 - 4(n-1) \\
-76 &= -4(n-1) \\
19 &= n - 1 \\
20 &= n \\
S_n &= \frac{20(81+5)}{2} \\
S_n &= 860
\end{aligned}$$

41. Determine the value of n for which $(2^1)(2^2)(2^3)(2^4)\ldots(2^n) = 2^210$

Remembering the power rule, $a^n \times a^m = a^{n+m}$ we see the above equals $2^{1+2+3+4+\ldots+n} = 2^210$ we see the exponent is a arithmetic sequence

$$\begin{aligned}
a_1 &= 1 & a_n &= n
\end{aligned}$$

There are n terms

$$\begin{aligned}
S_n &= \frac{n(1+n)}{2} \\
210 &= \frac{n(n+1)}{2} \\
420 &= n^2 + n \\
0 &= n^2 + n - 420 \\
0 &= (n-20)(n+21) \\
n &= 20
\end{aligned}$$

42. Determine a recursion formula for the sequence $14, 10, 4, 6, -2, \ldots$

This is an arithmetic sequence

$$\begin{aligned} d &= 10 - 14 & a_1 &= 14 & (1)\\ d &= -4 & & & (2)\\ a_n &= 14 - 4(n-1) & & & (3)\\ & & & & (4) \end{aligned}$$

43. For the geometric sequence $2, 6, 18, \ldots, 39366$ what term is 39366?

This is a geometric sequence

$$\begin{aligned} r &= \frac{6}{2} & a_1 &= 2\\ r &= 3\\ a_n &= 2 \times 3^{n-1}\\ 39366 &= 2 \times 3^{n-1}\\ 19683 &= 3^{n-1}\\ 3^9 &= 3^{n-1}\\ 9 &= n - 1\\ 10 &= n \end{aligned}$$

44. Determine the sum of the series $14 + 11 + 8 + \ldots - 61$.

This is an arithmetic sequence

$$\begin{aligned} d &= 11 - 14 & a_1 &= 14\\ d &= -3\\ -61 &= 14 - 3(n-1)\\ -75 &= -3(n-1)\\ 25 &= n - 1\\ 26 &= n\\ S_n &= \frac{26(14-61)}{2}\\ S_n &= -661 \end{aligned}$$

45. A principal of \$200 is invested at the end of each year for 15 years at 5.2% compounded annually. Determine the amount after 15 years.

$$A = 200 \qquad r = 0.052 \qquad n = 1 \qquad t = 15$$

$$\begin{aligned} A &= P(1 + \frac{r}{n})^{nt}\\ &= 200(1 + \frac{0.052}{1})^{15}\\ &= 427.82 \end{aligned}$$

46. Write a formula for the nth term of each sequence $\frac{1}{2}, \frac{4}{6}, \frac{7}{18}, \frac{10}{54}, \frac{13}{162}, \ldots$

This is a similar problem to 38 ii. The numerator is an arithmetic sequence, the denominator is a geometric sequence.

Numerator:

$$\begin{aligned} d &= 4-1 & a_1 &= 1 \\ d &= 3 \\ a_n &= 1+3(n-1) \end{aligned}$$

Denominator:

$$\begin{aligned} r &= \frac{6}{2} & a_1 &= 2 \\ r &= 3 \\ a_n &= 2 \times 3^{n-1} \end{aligned}$$

Put together,

$$a_n = \frac{1+3(n-1)}{2 \times 3^{n-1}}$$

47. Given the series $3+7+11+15+19+\ldots$, determine a_{100} and S_{100}.

$$\begin{aligned} d &= 7-3 & a_1 &= 3 && (5) \\ d &= 4 &&&& (6) \\ a_n &= 3+4(n-1) &&&& (7) \\ a_{100} &= 3+4(100-1) &&&& (8) \\ a_{100} &= 399 &&&& (9) \\ S_{100} &= \frac{100(4+399)}{2} &&&& (10) \\ S_{100} &= 20150 &&&& (11) \\ &&&&& (12) \end{aligned}$$

48. Solve the exponential equation $4^{2x-3} = 8^{5x+3}$.

$$\begin{aligned} 4^{2x-3} &= 8^{5x+3} \\ (2^2)^{2x-3} &= (2^3)^{5x+3} \\ 2^{2(2x-3)} &= 2^{3(5x+3)} \\ 2(2x-3) &= 3(5x+3) \\ 4x-6 &= 15x+9 \\ -15 &= 11x \\ x &= -\frac{15}{11} \end{aligned}$$

49. Find the amount of \$1000 at 6.38% compounded quarterly for 3 years.

$$P = 1000 \qquad r = 0.0638 \qquad n = 4 \qquad t = 3$$

$$\begin{aligned} A &= 1000(1 + \frac{0.0638}{4})^{4\times 3} \\ &= 1209.12 \end{aligned}$$

#49 and #50 were both in one question in the exam sample. So #50 in the paper is #51 here.

50. How much do you have to invest each month starting today in order to become a millionaire in 40 years. The investment pas you 12% compounded monthly.

$$FV = 1000000 \qquad r = 0.12 \qquad n = 12 \qquad t = 40$$

$$\begin{aligned} FV &= P\frac{(1+\frac{r}{n})^{nt} - 1}{\frac{r}{n}} \\ 1000000 &= P\frac{(1+\frac{0.12}{12})^{12\times 40}}{\frac{0.12}{12}} \\ \frac{1000000}{\frac{(1+\frac{0.12}{12})^{12\times 40}}{\frac{0.12}{12}}} &= P \\ 84.28 &= P \end{aligned}$$

51. Write an explicit formula for the sequence determined by the recursion formula. $a_1 = -4, a_n = a_{n-1} + 6$.

$$\begin{aligned} a_1 &= -4 \qquad\qquad d = 6 \\ a_n &= -4 + 6(n-1) \end{aligned}$$

Ok, that's it. I just want to clarify that many the solutions would not get you full points on a real exam, you would need a concluding sentence, listing variables, and tables/mapping to show your steps when graphing. Here I used a_n for sequence questions, but in the exam, it is better to use t_n. I also wrote the interest formulas a bit differently as I think this is a bit more detailed. If you are a bit confused after that, ask me on discord. This exam sheet is also not complete, there are a few that this paper does not include.

3 Unused Formulas

Some Formulas that are not used, but have learned in the school year.

Sum of geometric sequence: $S_n = \frac{a(r^n-1)}{r-1}$

Present value annuity formula: $PV = P\frac{1-(1+\frac{r}{n}^{-nt})}{\frac{r}{n}}$

SOH,CAH,TOA: $\sin x = \frac{o}{a}, \cos x = \frac{a}{h}, \tan x = \frac{o}{a}$

Inverse trig: $\csc x = \frac{1}{\sin x}, \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x}$

Simple interest: $A = Prt$

Exponent rules: $(a^n)^m = a^{nm}$

Equations Used