16.2 Line Integrals

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Line Integrals

Similar to integrating over an interval $[a, b]$, we take a line integral by integrating over a curve C. First, define a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

This curve C can also be written as

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

Definition - Line Integral along a Plane Curve

If f is defined on a smooth curve C as given above, then the **line integral of f along C** is

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Rewritten as

$$\boxed{\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt}$$

To parametrize a line segment, it's useful to remember the vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leq t \leq 1$$

Line Integrals in Space

Definition - Line Integral along a Space Curve

Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leq t \leq b$$

or by a vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C**

$$\int_C f(x,y,z)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

Rewritten as

$$\boxed{\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt}$$

We can write it in an even more compact form

$$\int_C f(x,y,z)ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

- Note that $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt = |\mathbf{r}'(t)|dt$

Line Integrals of Vector Fields

Definition - Line Integral along of Vector Fields

Let $\mathbf{F}$ be a continuous vector field definied on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds}$$

- Note that $\mathbf{T}ds = d\mathbf{r} = \mathbf{r}'(t)dt$