16.6 - Parametric Surfaces and Their Areas

Pages: 1111 - 1122

Joshua D. Ingram

Parametric Surfaces

We can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. Suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector valued function defined on a region D in the uv-plane. So x, y, and z are functions of two variables u and v with domain D. The set of all points (x,y,z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u,v) varies throughout D, is called a **parametric surface** S. The above equations are called **parametric equations**.

- Whenever we trace out the parametric surface, we may do this by holding u (or v) constant, say $u = u_0$ (or $v = v_0$), and tracing the curve given by $\mathbf{r}(u_0, v)$ (or $\mathbf{r}(u, v_0)$). These curves are called **grid curves**.

Surfaces of Revolution

Surfaces of revolution can be represented parametrically. Consider a surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$, and let θ be the angle of rotation. If (x,y,z) is a point on S then

$$x = x \qquad y = f(x)cos(\theta) \qquad z = f(x)sin(\theta)$$

We take x and θ as parameters and regard the above equations as parametric equations of S. The parameter domain is given by $a \leq x \leq b, \ 0 \leq \theta \leq 2\pi$.

Tangent Planes

Given a parametric surface S traced out by a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

We can find its **tangent plane** at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

By setting u constant at $u = u_0$, $\mathbf{r}(u_0, v)$ becomes a vector function of a single parameter and defines a grid curve C_1 on S. The tangent vector to C_1 at P_0 is obtained by

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

Similarly, by setting v constant at $v = v_0$, $\mathbf{r}(u, v{+}0)$ becomes a vector function of a single parameter and defines a grid curve C_2 on S. The tangent vector to C_2 at P_0 is obtained by

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v \neq 0$, then the surface S is **smooth** and the **tangent plane** to this surface at a point P_0 contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ where $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ is its normal vector.

Surface Area

Definition - Surface Area of a Parametric Surface

If a smooth parametric surface S is given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \quad (u,v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

Surface Area of the Graph of a Function

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. Thus the parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

then

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Thus we get

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}$$

and we can then define the surface area formula as

$$A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\, dA$$