Probability: Chapter 2 - Axioms of Probability

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2.2 Sample Space and Events

Sample Space - the set of all possible outcomes in an experiment

- Denoted $S = \{outcome_1, outcome_2 ...\}$

Event - any subset of the sample space

Union of Events - the event that contains any outcomes in all the "unioned" events *(e.g. blue OR black)*

- Denoted $E \cup F$
- Denoted EF

Intersection of Events - the event that contains any outcomes that occur in both events

- Denoted $E \cap F$
- An event E that is a subset of an event F is denoted:
  - $E \subset F$ *"E subset of f"*
  - $F \supset E$ *"F superset of E"*

These follow rules several familiar rules:

- ***Commutative Laws***
  - $E \cup F = F \cup E$
  - $EF = FE$
- ***Associative Laws***
  - $(E \cup F) \cup G = E \cup (F \cup G)$
  - $(EF)G = E(FG)$
- ***Distributive Laws***
  - $(E \cup F)G = EG \cup FG$
  - $EF \cup G = (E \cup G)(F \cup G)$

DeMorgan's Laws

$$\boxed{(\cup_{i=1}^{n} E_i)^c = \cap_{i=1}^{n} E_i^c}$$

$$\boxed{(\cap_{i=1}^{n} E_i)^c = \cup_{i=1}^{n} E_i^c}$$

2.3 Axioms of Probability

We can define the probability of an event in terms of its long-run *relative frequency*

$$\boxed{P(E) = \lim_{n\to\infty} \frac{n(E)}{n}}$$

Three Axioms of Probability

Axiom 1

$$0 \leq P(E) \leq 1$$

Axiom 2

$$P(S) = 1$$

Axiom 3

For any sequence of *mutually exlcusive* events $E_1, E_2, ...$

$$P(\cup_{i=1}^{n}) = \sum_{i=1}^{\infty} P_(E_i)$$

2.4 Some Simple Propositions

Proposition 4.1

$$P(E^c) = 1 - P(E)$$

Proposition 4.2

$$\text{If } E \subset F, \text{ then } P(E) \leq P(F)$$

Proposition 4.3

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Proposition 4.4

$$P(E_1 \cup ... \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2})... + ...(-1)^{r+1} \sum_{i_1<...<i_r} P(E_{i_1}E_{i_2}...E_{i_r}) + ... + (-1)^{n+1} P(E_1...E_n)$$

2.5 Sample Spaces Having Equally Likely Outcomes

For a sample space of a finite set, $S = \{1, 2, ..., N\}$, if all single outcomes are equally then

$$P(\{1\}) = P(\{2\}) = ... = P(\{N\})$$

Which implies

$$P(\{i\}) = \frac{1}{N},\ i = 1, 2, ..., N$$

For any event,

$$P(\{E\}) = \frac{\text{number of outcomes in E}}{\text{number of outcomes in S}}$$

2.6 Probability as a Continuous Set Function

A sequence of events $\{E_n, n \geq 1\}$ is increasing if

$$E_1 \subset E_2 \subset ... \subset E_n \subset E_{n+1} \subset ...$$

A sequence of events $\{E_n, n \geq 1\}$ is decreasing if

$$E_1 \supset E_2 \supset ... \supset E_n \supset E_{n+1} \supset ...$$

For an increasing sequence of events, we define a new event by

$$\lim_{n \to \infty} E_n = \cup_{i=1}^{\infty} E_i$$

For decreasing sequence of events, we define a new event by

$$\lim_{n \to \infty} E_n = \cap_{i=1}^{\infty} E_i$$

Propisition 6.1

If $\{E_n, n \geq 1\}$ is either an increasing or decreasing sequence of events, then

$$\lim_{n \to \infty} P(E_n) = P(\lim_{n \to \infty} E_n)$$