16.5 Curl and Divergence

Pages: 1103 - 1111

Joshua D. Ingram

Curl

Definition - Curl

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{curl\ \mathbf{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i}\ + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j}\ + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}}$$

To aid in remembering the formula above, know that

$$\nabla = \frac{\partial}{\partial x}\mathbf{i}\ +\ \frac{\partial}{\partial y}\mathbf{j}\ +\ \frac{\partial}{\partial z}\mathbf{k}$$

This allows us to write

$$curl\ \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Which is equal to the boxed equation above.

Theorem 16.5.3

If f is a function of three variables that has continuous second-order partial derivatives, then

$$curl(\nabla f) = \mathbf{0}$$

Theorem 16.5.4

If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and *curl* $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

- *curl* is associated with rotations, so if we think about particles in a fluid near a point (x,y,z) that rotate about an axis, the *curl* $\mathbf{F}(x, y, z)$ points in the direction of that axis and its length measures how quickly the particles move about that axis. (see page 1106)
- If *curl* $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P.

Divergence

Definition - Divergence

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the **divergence of F** is the function of three variables defined by

$$\boxed{div\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

In terms of ∇, the divergence of $\mathbf{F}$ can be written as

$$div\ \mathbf{F} = \nabla \cdot \mathbf{F}$$

- Note that *curl* $\mathbf{F}$ is a vector field but *div* $\mathbf{F}$ is a scalar field.

Theorem 16.5.11

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$div\ curl\ \mathbf{F} = 0$$

- If $\mathbf{F}$(x,y,z) is the velocity of a fluid, then *div* $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x,y,z).
- If *div* $\mathbf{F}(x, y, z) = 0$, then $\mathbf{F}$ is said to be **incompressible**.
- $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace Operator**.

Vector Forms of Green's Theorem

Given the curl and divergence operators, we can rewrite Green's Theorem as follows

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (curl\ \mathbf{F}) \cdot \mathbf{k}\ dA}$$