# 16.7 - Surface Integrals

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Joshua D. Ingram

## Parametric Surfaces

Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D$$

### Definition - Riemann Sum Definition of a Surface Integral of a Parametric Surface

We assume that the parameter domain D is a rectangle and we divide it into subrectangles $R_{ij}$ with dimensions $\Delta u$ and $\Delta v$. The surface S is then divided into corresponding patches $S_{ij}$. We then evaluate $f$ at a point $P_{ij}^*$ in each patch, multiply by the area $\Delta S_{ij}$ of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\ \Delta S_{ij}$$

We can then take the limit as the number of patches increases and we define the **surface integral of f over the surface S** as

$$\iint_S f(x,y,z)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\ \Delta S_{ij}$$

### Definition - Surface Integral of a Parametric Surface

We can approximate the patch area $\Delta S_{ij}$ by the area of an approximating parallelogram in the tangent plan. We use

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\ \Delta u\ \Delta v$$

Where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of $S_{ij}$. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown that

$$\boxed{\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|dA}$$

### Mass and Center of Mass

If a thin sheet has the shape of a surface S and the density at the point (x,y,z) is $\rho(x,y,z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x,y,z)d\mathbf{S}$$

Its **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)d\mathbf{S} \qquad \bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)d\mathbf{S} \qquad \bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)d\mathbf{S}$$

## Graphs of Functions

Any surface S with equation $z = g(x,y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x,y)$$

Thus

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}$$

where

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}$$

Therefore,

$$\boxed{\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}dA}$$

## Oriented Surfaces

To define surface integrals of vector fields, we must rule out *nonorientable* surfaces like the Mobius strip since it only has one side. We only consider *orientable* (two-sided) surfaces moving forward.

### Definition - Orientable Surface

We first start with a surface S that has a tangent plane at every point (x,y,z) on S. There are then two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = \mathbf{1}_1$ at every point.

If it is possible to choose a unit normal vector at every point (x,y,z) so that $\mathbf{n}$ varies continuously over S, then S is called an **orientable surface** and the given choice of $\mathbf{n}$ provides S with an **orientation**.

For a surface $z = g(x,y)$ given as the graph of $g$, we can use the following equation to associate with the surface a natural orientation given by the unit normal vector

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}}$$

Since the $\mathbf{k}$ component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u,v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by **-n**.

## Surface Integrals of Vector Fields

### Definition - Surface Integral of F over S

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS}$$

This integral is also called the **flux** of **F** across S.

If S is given by a vector function $\mathbf{r}(u, v)$, then

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA}$$

In the case of a surface S given by a graph $z = g(x, y)$, we can think of $x$ and $y$ as parameters and then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R) dA$$