16.3 The Fundamental Theorem for Line Integrals

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Fundamental Theorem of Line Integrals

Theorem 16.3.1 - Fundamental Theorem of Calculus

$$\int_a^b F'(x)dx = F(b) - F(a)$$

If we think of the gradient vector ∇f as a sort of derivative of f, then we can regard the following as the "fundamental theoreom of line integrals:"

Theorem 16.3.2 - Fundamental Theorem of Line Integrals*

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves, or **paths**, that have the same initial point A and terminal point B. One implication of *Theorem 16.3.2* is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenver ∇f is continuous.

- The line integral of a *conservative* vector field depends only on the initial point and the terminal point of the curve.

- If $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \nabla f \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and terminal points.
- A curve is called **closed** if its terminal point coincides with its initial point, $\mathbf{r}(b) = \mathbf{r}(a)$

Theorem 16.3.3 - Independence of Path

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem 16.3.4 - Independence of Path and Conservative Vector Fields

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem 16.3.5*

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

- A **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints.
- A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D.

Theorem 16.3.6*

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout D}$$

Then $\mathbf{F}$ is conservative.