1.) Prove that for any $\varepsilon > 0$ there exists $N > 0$ s.t:

$$(L-\varepsilon)(y_{n+1}-y_N) < x_{n+1} - x_N < (L+\varepsilon)(y_{n+1}-y_N) \text{ for } \forall n > N.$$

$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L$. This means $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \geq N'$ $\left|\frac{x_{n+1}-x_n}{y_{n+1}-y_n} - L\right| < \varepsilon$. $N' : N+1$

By Abs val def: $-\varepsilon < \frac{x_{n+1}-L}{y_{n+1}-y_n} < \varepsilon$.

Hint: $x_{n+1} - x_{N'} = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \dots + (x_{N'+1} - x_{N'})$

Add them up

$$\begin{cases} (L-\varepsilon)(y_{n+1}-y_n) < x_{n+1}-x_n < (L+\varepsilon)(y_{n+1}-y_n) \\ (L-\varepsilon)(y_{n+1}-y_n) < x_n - x_{n-1} < (L+\varepsilon)(y_n-y_{n-1}) \\ \vdots \\ (L-\varepsilon)(y_{N'+1}-y_{N'}) < x_{N'+1}-x_{N'} < (L+\varepsilon)(y_{N'+1}-y_{N'}) \end{cases}$$

$(L-\varepsilon)(y_{n+1}-y_{N'}) < x_{n+1}-x_{N'} < (L+\varepsilon)(y_{n+1}-y_{N'})$ for $\forall \varepsilon > 0, \exists N > 0, \forall n \geq N'$; which also implies $\forall n > N$.

- It's important to note the inequalities don't change since $(y_n)_n$ is an increasing sequence.

2.) Let $(x_n)_n$ and $(y_n)_n$ be 2 sequences of real #s, s.t. $(y_n)_n$ is increasing and $\lim_{n\to\infty} y_n = \infty$.

$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L$, for some $L \in \mathbb{R}$. Prove that $\lim_{n\to\infty} \frac{x_n}{y_n} = L$

if $\lim_{n\to\infty} \frac{x_n}{y_n} = L$, $\forall \varepsilon > 0, \exists N > 0$, where $\forall n \geq N$: $\left|\frac{x_n}{y_n} - L\right| < \varepsilon$

This means: $L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon$.

$$(L-\varepsilon)y_n < x_n < (L+\varepsilon)y_n.$$

For $n+1^{th}$ case: $\left|\frac{x_{n+1}}{y_{n+1}} - L\right| < \varepsilon$:

$$L - \varepsilon < \frac{x_{n+1}}{y_{n+1}} < L + \varepsilon.$$

$$(L-\varepsilon)y_{n+1} < x_{n+1} < (L+\varepsilon)y_{n+1}$$

Combine by subtract: $(L-\varepsilon)y_{n+1} - (L-\varepsilon)y_n < x_{n+1} - x_n < (L+\varepsilon)y_{n+1} - (L+\varepsilon)y_n$.

$$(L-\varepsilon)(y_{n+1}-y_n) < x_{n+1}-x_n < (L+\varepsilon)(y_{n+1}-y_n)$$

This equation satisfies the earlier condition as to how: $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L$, thus making $\lim_{n\to\infty} \frac{x_n}{y_n} = L$ TRUE.

3.) if $\lim_{n\to\infty} a_n = L$ then: $\lim_{n\to\infty} \frac{a_1+a_2+\dots+a_n}{n}$.

From def of limits, we have: $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N$: $\boxed{-\varepsilon + L < a_n < L+\varepsilon}$

Then since N is fixed: $|a_1+a_2+\dots+a_{N-1}| < D$ as we're adding finite sums finite times where $D \in \mathbb{R}$.

$$\lim_{n\to\infty} \frac{a_1+a_2+\dots+a_n}{n} = \lim_{n\to\infty}\left(\frac{a_1+a_2+\dots+a_{N-1}}{n} + \frac{a_N+a_{N+1}+\dots+a_n}{n}\right)$$

$$= \lim_{n\to\infty} \frac{a_1+a_2+\dots+a_{N-1}}{n} + \lim_{n\to\infty} \frac{a_N+a_{N+1}+\dots+a_n}{n}$$

$$\lim_{n\to\infty} -\frac{D}{n} < \lim_{n\to\infty} \frac{a_1+a_2+\dots+a_{N-1}}{n} < \lim_{n\to\infty} \frac{D}{n}.$$

$$\lim_{n\to\infty} \left|\frac{D}{n} - 0\right| < \varepsilon, \Rightarrow -\varepsilon < \frac{D}{n} < \varepsilon \Rightarrow -\varepsilon > \frac{n}{D} > \varepsilon, \quad n > D\varepsilon, \quad N = D\varepsilon$$

$$\lim_{n\to\infty} \left|\frac{a_N+a_{N+1}+\dots+a_n}{n} - L\right| < \varepsilon \Rightarrow L-\varepsilon < \frac{a_N+a_{N+1}+\dots+a_n}{n} < L+\varepsilon.$$

$$n(L-\varepsilon) < a_N+a_{N+1}+\dots+a_n < n(L+\varepsilon)$$

$$(L-\varepsilon) < \frac{a_N+a_{N+1}+\dots+a_n}{n} < L+\varepsilon$$

interval is 2ε

We know that's true because:

$L-\varepsilon < a_N < L+\varepsilon$

$L-\varepsilon < a_{N+1} < L+\varepsilon$

$\vdots$

$L-\varepsilon < a_n < L+\varepsilon$

add

$$= (n-(N+1))(L-\varepsilon) < a_N+a_{N+1}+\dots+a_n < (n-(N-1))\cdot(L+\varepsilon)$$

$$= (n-N+1)L - (n-N+1)\varepsilon < a_N+\dots+a_n < (n-N+1)L + (n-N+1)\varepsilon$$

$$= L-\varepsilon < \frac{a_N+\dots+a_n}{n-N+1} < L+\varepsilon.$$

$$= \frac{(n-N+1)}{n}\cdot(L-\varepsilon) < \frac{a_N+\dots+a_n}{n} < \frac{(n-N+1)}{n}(L+\varepsilon)$$

$(n-N+1)\varepsilon + (n-N+1)\varepsilon = \frac{2n-2N+2}{n}$ as interval length

interval length is $\frac{2n-2N+2}{n}\varepsilon \leq 2\varepsilon$ for $N \geq 1$, where the sequence starts at 1. Thus, $(L-\varepsilon) < \frac{a_N+\dots+a_n}{n} < L+\varepsilon$.

$$\lim_{n\to\infty}\left(\frac{D}{n} + \frac{a_N+a_{N+1}+\dots+a_n}{n}\right) = \lim_{n\to\infty} \frac{D}{n} + \lim_{n\to\infty} \frac{a_N+a_{N+1}+\dots+a_n}{n}$$

$$= 0 + L$$

$$= L$$

4) Sequence $a_1 = 1$ and $a_{n+1} = 3 - \frac{1}{a_n}$, we assert that it's positive and monotonically increasing:

Base Case: $a_1 = 1$, $a_2 = 3 - 1 = 2$
Assume for n^{th} case: $a_n \geq a_{n-1}$ for $n > 1$ and $a_n \in \mathbb{R}^+$.
$n+1^{th}$ Case: $a_{n+1} = 3 - \frac{1}{a_n}$.

Since $a_n \geq a_{n-1}$

$$\frac{1}{a_n} \leq \frac{1}{a_{n-1}}$$
$$-\frac{1}{a_n} \geq -\frac{1}{a_{n-1}}$$

$\Rightarrow$ Thus, $3 - \frac{1}{a_n} \geq 3 - \frac{1}{a_{n-1}}$
$a_{n+1} \geq a_n$, we satisfied monotonocity.

Bounded: We claim that $n \to \infty$ $a_n \leq 3$, no matter what.

Base Case: $a_1 = 1$, and $a_1 \leq 3$ by default.
n^{th} Case: Assume $a_n \leq 3$, and $a_n \geq 1$. Since it's increasing from 1.

$n+1^{th}$ Case: $a_{n+1} = 3 - \frac{1}{a_n}$ and we know that

$$1 \leq a_n \leq 3$$
$$1 \geq \frac{1}{a_n} \geq \frac{1}{3}$$
$$-1 \leq -\frac{1}{a_n} \leq -\frac{1}{3}$$
$$2 \leq 3 - \frac{1}{a_n} \leq 2\tfrac{2}{3} < 3,$$

Therefore 3 is the upper bound and thus, $(a_n)_n$ sequence is bounded

$a_n \leq a_{n+1} \to a_n \leq \left(3 - \frac{1}{a_n}\right)$
$a_n^2 \leq 3a_n - 1$

$a_n^2 - 3a_n + 1 \leq 0$
$\downarrow$
$$\frac{3 \pm \sqrt{9-4}}{2} \Rightarrow \boxed{\frac{3 \pm \sqrt{5}}{2}}$$

We assert that $\lim_{n \to \infty} 3 - \frac{1}{a_{n-1}} = \frac{3+\sqrt{5}}{2}$

$\lim_{n \to \infty} a_n + \lim_{n \to \infty} a_{n-1} = 2L$, by def of limit laws.

$$L + \left(3 - \frac{1}{L}\right) = 2L, \quad L^2 + 3L - 1 = 2L^2$$
$$-L^2 + 3L - 1 = 0$$
$$L^2 - 3L + 1 = 0 \Rightarrow \frac{3+\sqrt{5}}{2} \text{ as the limit}$$