1.) $\exists$ a sequence $(x_n)_n$ of points in S so that $\lim_{n\to\infty} x_n = \sup S$.

S is a set meaning it contains that are not duplicates of each other. And since it's bounded by lower and upper, $\exists$ bound $D>0$ s.t: $-D \le x \le D$, $\forall x \in S$.

Thus, we can organize each elements where $\forall n \in \mathbb{N}$, $x_n < x_{n+1}$ // Strictly increasing.
Thus, we construct a set that is bonded and monotonically increasing.

Therefore, there's a supremum through MCT. Thus, looking at def of lim and sup:

$\forall \varepsilon>0, \exists N>0, \forall n>N$: $\quad\quad$ $\forall \varepsilon>0, \exists n \in \mathbb{N}$ s.t:

$$|x_n - L| < \varepsilon \quad\quad\quad x_n \le R \le x_n + \varepsilon,\ |R - x_n| \le \varepsilon$$

The limit is the supremum as it can fit within ε where n can be any value greater than N.

2.) For a sequence $(a_n)_n$, prove that its subsequences $(a_{2n})_n$, $(a_{2n+1})_n$, and $(a_{3n})_n$ are each convergent. Then $(a_n)_n$ is convergent.

$$\boxed{\begin{array}{l}\text{both } (a_{2n})_n \text{ and } (a_{2n+1})_n \text{ could converge diff.}\\ (a_{3n})_n \text{ consists of odd and even terms.}\end{array}}$$

Let's choose $N^* = \max\{N_{2n}, N_{2n+1}, N_{3n}\}$
for $2n$, $2n+1$, let's use k as index where: $2k > N^*$, $2k+1 > N^*$, $3n > N^*$, for $k, n \in \mathbb{N}$:

$\forall \varepsilon>0$ by def:

$$|a_{2k} - L_1| < \varepsilon,\ |a_{2k+1} - L_2| < \varepsilon,\ |a_{3n} - L_3| < \varepsilon$$

if we have $3n$ as evens: $|a_{2k} - L_3| < \varepsilon$
$|a_{2k+6} - L_3| < \varepsilon$
$\vdots$
$|a_{2k+6p} - L_3| < \varepsilon$

if we have $3n$ as odds: $|a_{2k+1} - L_3| < \varepsilon$ $\quad$ for $p \in \mathbb{N}$.
$|a_{2k+7} - L_3| < \varepsilon$
$\vdots$
$|a_{2k+1+6p} - L_3| < \varepsilon$

- For even terms of $3n$, they converge to L_3 with ε error which happens to converge to L_1.
  $L_1 = L_3$ by limit uniqueness
- For odd terms of $3n$, they converge to L_3 with ε error which also happens to L_2.
  $L_2 = L_3$ by limit uniqueness.

Then $L_1 = L_2 = L_3$, therefore with $L_1 = L_2$, $(a_n)_n$ converges to L. $(L = L_1 = L_2 = L_3)$

3.) $|x_{n+1} - x_n| < \frac{1}{2^n}$ for $\forall n \geq 1, n \in \mathbb{N}$. Prove that $(x_n)_n$ is a Cauchy sequence.

Def: $(x_n)_n$ is Cauchy if $\forall \varepsilon > 0, \exists N > 0$ s.t. $|x_n - x_m| < \varepsilon, \forall n,m > N$.

$$|x_{n+k} - x_{n+(k-1)} + x_{n+(k-1)} - x_{n+(k-2)} + \ldots + x_{n+1} - x_n| \leq |x_{n+k} - x_{n+k-1}| + |x_{n+k-1} - x_{n+k-2}| + \ldots + |x_{n+1} - x_n|$$
$$\leq \frac{1}{2^{n+k-1}} + \frac{1}{2^{n+k-2}} + \ldots + \frac{1}{2^n}$$
$$\leq \frac{1}{2^n}\left(\frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \ldots + 1\right)$$
$$< \frac{1}{2^n} \cdot 2$$
$$< \frac{1}{2^{n-1}},$$

$$\frac{1}{2^{n-1}} < \varepsilon \Rightarrow 2^{n-1} > \frac{1}{\varepsilon} \Rightarrow n-1 > \log_2\left(\frac{1}{\varepsilon}\right), \quad n > 1 + \log_2\left(\frac{1}{\varepsilon}\right)$$

$$\boxed{N := 1 + \log_2\left(\frac{1}{\varepsilon}\right)}$$

Thus, $\forall n,m > 1 + \log_2\left(\frac{1}{\varepsilon}\right), \forall \varepsilon > 0$, $|x_m - x_n|$ is Cauchy.

4.) Prove that for any $a \in \mathbb{R}$, the sequence:

$$x_n = \frac{\sin a}{2} + \frac{\sin(2a)}{2^2} + \ldots + \frac{\sin(na)}{2^n}$$

We have:

$$x_{n+1} = \frac{\sin a}{2} + \frac{\sin(2a)}{2^2} + \ldots + \frac{\sin(na)}{2^n} + \frac{\sin((n+1)a)}{2^{n+1}} \quad \text{by def.}$$

$$x_{n+1} - x_n = \frac{\sin((n+1)a)}{2^{n+1}}$$ as $n \to \infty$ $-1 \leq \sin(na) \leq 1$. As both upper and lower bounds

Therefore using Inequality bounds:

$$\frac{-1}{2^{n+1}} \leq x_{n+1} - x_n \leq \frac{1}{2^{n+1}}$$
$$|x_{n+1} - x_n| \leq \frac{1}{2^{n+1}}.$$

From earlier, we saw that if $|x_{n+1} - x_n| < \frac{1}{2^n}$, the sequence $(x_n)_n$ is a Cauchy Sequence.
And such a Cauchy sequence does have a converging limit.

Therefore, since $\frac{1}{2^{n+1}} < \frac{1}{2^n}$, $(x_n)_n$ is a Cauchy sequence and has a converging limit.