1.a) Prove $\lim_{x\to 2}(4x^2-x+2)=16$. Formal def: $\forall \varepsilon>0, \exists \delta>0$ s.t. when $|x-2|<\delta$, then we have $|4x^2-x+2-16|<\varepsilon$ in turn.

Roots of $4x^2-x-14$:

$$\frac{1\pm\sqrt{1-4(-14\cdot 4)}}{8}=\frac{1\pm\sqrt{1+224}}{8}=\frac{1\pm 15}{8}, \text{ Roots: } 2, -1.75$$

Factor: $|4x^2-x-14|=|4(x-2)(x+1.75)|<4|x-2||x+1.75|=\varepsilon$

$4\cdot\delta\cdot|x+1.75|=\varepsilon$, We know that $1<x<3$.

$4\cdot\delta\cdot 4.75=\varepsilon$ $2.75<x<4.75$

$\delta=\varepsilon/19$

if we have $\delta=\varepsilon/19$, then in turn with $|x-2|<\delta$ we can have $|4x^2-x+2-16|<\varepsilon, \forall\varepsilon>0$.

1.b) Prove $\lim_{x\to 1}\frac{x^2+1}{x+1}=1$: We assert that $\left|\frac{x^2+1}{x+1}-1\right|<\varepsilon$ where $|x-1|<\delta, \forall\varepsilon>0$ to which $\exists\delta>0$.

and $\left|\frac{x^2+1}{x+1}-\frac{(x+1)}{x+1}\right| \Rightarrow \left|\frac{x^2-x}{x+1}\right|$, we know that $\frac{1}{2}<x<\frac{3}{2}$ as our focus. $\frac{3}{2}<x+1<\frac{5}{2}$, as a relation.

$$\left|\frac{x^2-x}{x+1}\right|<\left|\frac{x(x-1)}{3/2}\right|=\frac{2}{3}\cdot|x|\cdot|x-1|<\frac{2}{3}\cdot\frac{3}{2}\cdot\delta=\varepsilon, \quad \delta=\varepsilon.$$

For $\forall\varepsilon>0, \exists\delta>0$ s.t: $\left|\frac{x^2+1}{x+1}-1\right|<\varepsilon$ and $|x-1|<\delta$.

2.a) $a=5x$ and $\frac{a}{5}=x$. $b=3x$ and $\frac{b}{3}=x$

$$\lim_{x\to 0}\frac{\sin(5x)-\sin(3x)}{x}=\lim_{x\to 0}\frac{\sin(5x)}{x}-\lim_{x\to 0}\frac{\sin(3x)}{x}$$
$$=\lim_{a\to 0}5\frac{\sin(a)}{a}-3\lim_{b\to 0}\frac{\sin(b)}{b}$$
$$=5-3$$
$$=2$$

$$\lim_{x\to 0}\frac{\sin(5x)-\sin(3x)}{x}=2$$

2b) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$

$$=\frac{x^2+x-x}{x(x^2+x)}=\frac{x^2}{x^3+x^2}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\frac{1}{x^2+1}=1$$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)=1$$

2c) $\lim_{x\to -2}\frac{x^3+8}{x+2}$

$$\frac{x^3+8}{x+2}=x^2-2x+4, \quad \text{then } (-2)^2-2(-2)+4=12$$

$$\lim_{x\to -2}\frac{x^3+8}{x+2}=12$$

3.) $f(x) = \begin{cases} cx^2+2x, & \text{if } x<2 \\ x^3-cx & \text{if } x\geq 2 \end{cases}$ Determine c:

- $c(2)^2+2(2) = 4c+4$
- $(2)^3-2c = 8-2c$

$4c+4 = 8-2c$

$-4 = -6c$, $c = \frac{2}{3}$

For f to be continuous; $f(x) = \begin{cases} \frac{2}{3}x^2+2x & \text{if } x<2 \\ x^3-\frac{2}{3}x & \text{if } x\geq 2. \end{cases}$

We say that f is continuous at 2 if: $\lim_{x\to 2} f(x) = f(2)$.

Show that $\lim_{x\to 2^-} f(x) = \frac{20}{3}$:

$\exists \delta_1 > 0$ s.t. $\forall \varepsilon > 0$ we have $\left|\frac{2}{3}x^2+2x-\frac{20}{3}\right| < \varepsilon$

$|2x^2+6x-20| < 3\varepsilon$.

$2-\delta < x < 2$.

$-2+\delta_1 > x > -2$

$\delta_1 > x+2$ $(3x+6)^2 < 9\delta_1^2$

$|2x^2+6x-20| < |(3x+6)^2| < 9\delta_1^2$

$9\delta_1^2 = \varepsilon$ $\boxed{\delta_1 = \sqrt{\varepsilon}/3}$

$\delta_1^2 = \varepsilon/9$

Show that $\lim_{x\to 2^+} f(x) = \frac{20}{3}$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3-\frac{2}{3}x = (2)^3-\frac{2}{3}(2)$

$= 8-\frac{4}{3}$

$= \frac{20}{3}$.

Thus, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \frac{20}{3}$.

Proof that $\lim_{x\to 2^+} f(x) = \frac{20}{3}$:

$\left|x^3-\frac{2}{3}x-\frac{20}{3}\right| = \left|x^3-8-\frac{2}{3}(x-2)\right| \leq |x^3-8| + \left|\frac{2}{3}(x-2)\right|$,

$\leq \delta + \frac{2}{3}\delta = \varepsilon$

$\frac{5}{3}\delta = \varepsilon$, $\delta = \frac{3}{5}\varepsilon$

$\boxed{\text{we know: } 2<x<2+\delta \quad x-2<\delta.}$

4.) $f(x) = \begin{cases} 1+2x^2 & \text{if } x \text{ is rational} \\ 1+x^4 & \text{if } x \text{ is irrational.} \end{cases}$

→ From both sides, positive and negatives, there are rational and irrational numbers:

We claim that $\lim_{x\to 0} 1+2x^2 = 1$ and $\lim_{x\to 0} 1+x^4 = 1$ with ε,δ def.

↳ $\forall \varepsilon > 0, \exists \delta_1 > 0$ s.t: $|1+2x^2-1| < \varepsilon$ and $|x| < \delta_1$

with $-\delta_1 < x < \delta_1$

$$|2x^2| < 2\delta_1^2 = \varepsilon$$
$$\delta_1^2 = \varepsilon/2$$
$$\delta_1 = \sqrt{\varepsilon/2}$$

We claim within $(-\delta_1, \delta_1)$ there exists rational numbers

Claim: There's none: Proof by contradiction:
Through Archimedean prop we have $N \in \mathbb{N}$ s.t: $N \cdot \delta_1 > 1$. Then we have $\delta_1 > 1/N$, $1/N$ is a rational number. Thus, within $(-\delta, \delta)$ there do exist rational #s and $\lim_{x\to 0} 1+2x^2$ for when x is rational gets satisfied.

$\forall \varepsilon > 0, \exists \delta_2 > 0$ s.t: $|1+x^4-1| < \varepsilon$ and $|x| < \delta_2$

$$-\delta_2 < x < \delta_2$$
$$|x^4| < \varepsilon \text{ and } -\delta_2 < x < \delta_2$$
$$-\delta_2^4 < x^4 < \delta_2^4$$
$$|x^4| < \delta_2^4 = \varepsilon, \quad \delta_2 = \varepsilon^{1/4}$$

We also claim within $(-\delta_2, \delta_2)$ there exists irrational numbers.

Claim: There's none: Proof by contradiction:
Through Archimedean prop, we have $N \in \mathbb{N}$ s.t: $N \cdot \delta_2 > \sqrt{2}$. Then we have $\delta_2 > \frac{\sqrt{2}}{N}$. $\frac{\sqrt{2}}{N}$ is irrational and bounded within δ_2.
Thus within $(-\delta_2, \delta_2)$, there do exist irrational #s and $\lim_{x\to 0} 1+x^4 = 1$ for when x is irrational gets satisfied.

For completeness chose $\delta^* = \min\{\delta_1, \delta_2\}$ and you'll be able to see the limit converge to 1 irrespective to whether our x is rational or irrational number.