1.) $f(x) = \tan(x)$, $f(\frac{\pi}{4}) = 1$ and $f(\frac{3\pi}{4}) = -1$. Why does this not contradict Bolzano's Theorem?:
Our function $\tan(x)$ is one that is unbounded and thus, we would not find $x \in [\pi/4, 3\pi/4]$ for $\tan(x) = 0$.

Proof:

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ For $\sin(x)$: $\frac{1}{2} \leq \sin(x) \leq 1$ for $x \in [\pi/4, 3\pi/4]$
As for $\cos(x)$: $-\frac{\sqrt{2}}{2} \leq \cos(x) \leq \frac{\sqrt{2}}{2}$ for $x \in [\pi/4, 3\pi/4]$

$\tan(x)$ cannot be continuous for $\tan(x) = 0$ for $x \in [\pi/4, 3\pi/4]$ given that the numerator is never 0 but strictly positive. For it to be continuous $\tan(x) = 0$ needs to exist to which the numerator $= 0$.

Therefore, it does not contradict Bolzano's theorem where no points are present for $\tan(x) = 0$ for $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$.

2.) Continuous function, $f(x)$ is periodic with period $= 2$ s.t: $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.
Show $\exists\, c \in [0,1]$ s.t: $f(c) = f(c+1)$.

Assume that $f(c) - f(c+1) = a$ where $a \in \mathbb{R}/0$.
then: $f(c+1) - f(c+2) = -a$.

Define function $G(c) = f(c) - f(c+1)$
Since we see $\left.\begin{matrix} G(c) = a \text{ and} \\ G(c+1) = -a \end{matrix}\right\}$ between $[0,1]$ there must exist m such that $G(m+c) = 0$ with $m \in [c, c+1]$, to follow Bolzano's property.

3.) Let $f: \mathbb{R} \to \mathbb{R}$ be a function s.t: $|f(x) - f(y)| \leq 2|x-y|$, $\forall x, y \in \mathbb{R}$.
Prove f is uniformly continuous on $\mathbb{R}$.

$|f(x) - f(y)| = |f(y) - f(x)| \leq 2|y-x|$. We can have $\varepsilon = 2|y-x|$.
$\frac{\varepsilon}{2} = |y-x|$.

We can set $\delta = \frac{\varepsilon}{2}$ to make $|x-y| < \delta$, resulting $|f(y) - f(x)| \leq \varepsilon$, $\forall x, y \in \mathbb{R}$, making uniform continuity.

4.) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and periodic with $T > 0$ s.t. $f(x+T) = f(x)$ $\forall x \in \mathbb{R}$.
Prove that f is uniformly continuous:

We know from theorem earlier that a continuous function bounded by $[a,b]$ as domain is also uniformly continuous. Thus with $a \in \mathbb{R}$, we can say that from $[a, a+T]$ that f is uniformly continuous.

Claim: We state that $[a, \infty)$ is uniformly continuous

Base cases: f on $[a, a+T]$ and $[a, a+2T]$ are uniformly continuous as they are fixed on inclusive bounds from theorem in class. // Periodicity property will be used later.

n^{th} case: f on $[a, a+nT]$ where $n \in \mathbb{N}$ is uniformly continuous as claim.

$n+1^{th}$ case: f on $[a, a+(n+1)T]$ can be decomposed as: $[a, a+nT]$ and $[a+nT, a+(n+1)T]$

We know that points from $[a+(n-1)T, a+nT]$ are uniformly continuous from n^{th} case and property of uniform continuity can be mapped to $[a+nT, a+(n+1)T]$ by shifting T:

$\forall \varepsilon > 0, \exists \delta > 0$ $|f(x) - f(y)| < \varepsilon$, $|x-y| < \delta$ where $x, y \in [a+(n-1)T, a+nT]$

$\downarrow$

$\forall \varepsilon > 0, \exists \delta > 0$ $|f(x+T) - f(y+T)| < \varepsilon$, $|x+T-(y+T)| < \delta$ where $x, y \in [a+(n-1)T, a+nT]$.
replace interval with $[a+nT, a+(n+1)T]$.

Additionally x, y between $a+(n-1)T$ and $a+(n+1)T$ are still uniformly convergent since they can be similarly mapped from points: $[a, a+2T]$.

Thus, f on $[a, a+nT]$ is uniformly continuous as $n \to \infty$.

WLO, f on $[a-nT, a]$ is also uniformly continuous as $n \to \infty$. Thus, f is uniformly continuous on $\mathbb{R}$