Math-M-Addicts. Group A+. Homework 1.

Due 09/25/2021.

Problem 1 (Just The Answer – numeric answer is sufficient) A 15×15 table is filled with numbers from 1 to 225, so that each number occurs exactly once. Also numbers increase in each row from left to right, and in each column from top to bottom. How many possibilities are there for the number at the intersection of the 5th column from the left, and 5th row from the top?

Problem 2 (JTA) An infinite sequence is obtained by adding the corresponding terms of two geometric progressions. This sequence starts with 1, 2, 11, 50. Find the next term.

Problem 3 Suppose $a \neq 0$ is a real number, and x, y, z satisfy

$$\begin{cases} x + y + z = a \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} \end{cases}$$

Prove that at least one of x, y and z is equal to a.

Problem 4. Anthony and Bob broke into a Dunkin Donuts store and found 199 donuts there. They started playing the following game. If there were currently n donuts in the store, a player could eat either 1 or $\lfloor \frac{n}{2} \rfloor$ donuts. Anthony goes first. Whoever eats the last donut loses. Who would win with best play? Note: if there is only one donut left, a player is required to eat it and lose.

Problem 5 The alphabet of Twin Letter Tribe has just two letters. Each word in their language consists of 7 letters with any two words differing in at least three places.

a) Prove that the language has no more than 16 words.

b) Can it actually have 16 words?

Problem 6 Let AA_1, BB_1 and CC_1 be altitudes of acute scalene triangle ABC, and let H be its orthocenter. Let D be the reflection of point C with respect to line A_1B_1. Prove that the circumcircle of C_1DH passes through the circumcenter of ABC.

Problem 7 Prove that there exist infinite many positive integer n such that $n^2 + 1$ divides $n!$.

Black Diamond Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(i) $f(1) = 1$

(ii) $f(x + y) = f(x) + f(y)$ for any real x and y

(iii) $f(x)f(\frac{1}{x}) = 1$ for all real $x \neq 0$.

Prove that $f(x) = x$ for all $x \in \mathbb{R}$.