Homework 4 (Due: October 16th, 2021)

Math-M-Addicts Group A+

2021-2022

Problem 1. (Answer only) Let A be the set of all integers that have even number of ones in their binary representation. Find the 2021-st smallest element in A.

Problem 2. (Answer only) ABC is a right triangle with $|AB| = 5$, $|AC| = 4$ and $|BC| = 3$. Point D on hypotenuse AB is chosen in such a way that the in-radius of triangle ACD is equal to the radius of the excircle of triangle BCD touching side BD. Find those radii.

Problem 3. Real numbers x, y and z satisfy

$$x+y+z=5$$

$$xy+yz+zx=8$$

Prove that $1 \leq x \leq \frac{7}{3}$.

Problem 4. Give an example of 4 green and 4 yellow points on a plane such that for any 3 points of one color there exists point of the other color so that they are 4 vertices of parallelogram.

Problem 5. The sum of 9 distinct real numbers is positive. Prove that there are at least 28 ways to choose 3 of the numbers, so that the sum of the three is positive.

Problem 6. In triangle ABC, $|AB| = \frac{|AC|+|BC|}{2}$. Let I and O be the incenter and the circumcenter of ABC.

a) Prove that $\angle OIC = 90^\circ$.

b) Suppose E and F be the midpoints of AC and BC, and CI meets AB at D. Prove that I is the circumcenter of DEF.

Problem 7. Suppose in a sequence of real numbers $0 < x_1 < 1$ and for any integer $n > 0$

$$x_{n+1} = x_n + \frac{x_n^2}{n(n+1)}$$

Prove that x_n is a bounded sequence.

Problem 8. ⧫ True or False: one can choose 2021 consecutive positive integers so that no chosen number is divisible by the sum of its decimal digits.