Homework 3 (Due: October 9th, 2021)

Math-M-Addicts Group A+

2021-2022

Problem 1. Let $x = \dfrac{\sum\limits_{n=1}^{44} \cos n^\circ}{\sum\limits_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

Problem 2. (Answer only) Five distinct integers are chosen from set $\{1, 2, ..., 90\}$ are chosen at random and ordered $V < W < X < Y < Z$. Find $\lfloor 10E(Y) \rfloor$.

Problem 3. Set A consists of 20 distinct numbers. Any 11-element subset of A has two elements that add up to 5. Find, with proof, the sum of all elements of A.

Problem 4. Some parliament set up several committees, so that any committee has at least two members, and every two committees have at least one person in common. Prove that you can give all member of the parliament a red, white or blue scarf, so each committee has at least two colors of scarves among their members.

Problem 5. Prove that any polynomial $P(x)$ with real coefficients can be represented as a difference of polynomials $P(x) = Q(x) - R(x)$, where both Q and R are increasing functions of x.

Problem 6. Let n be a positive integer, such that both $3n + 1$ and $4n + 1$ are perfect squares. Prove that n is a multiple of 7.

Problem 7. Let ω_1, ω_2 and ω_3 be three circles, none of which is inside, intersects or touches any other. For a point P outside of all circles, define A_i and B_i to be the points where tangents from P touch ω_i ($i = 1, 2, 3$). Point P is called nice if A_1B_1, A_2B_2 and A_3B_3 share a point. Prove that all nice points, if they exist at all, lie on the same circle.

Problem 8. ♦ The first quadrant is partitioned into unit squares. In the initial configuration there is just one marble in the left-hand lower corner (see Figure 1). A valid operation consists of replacing any existing marble by two marbles, one above and one to the right of it provide both of those squares are empty. That is, no square can ever be occupied by 2 marbles. Is it possible to make the 6-square area colored red in Figure 2 free of marbles?
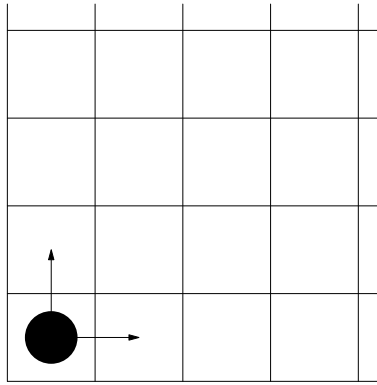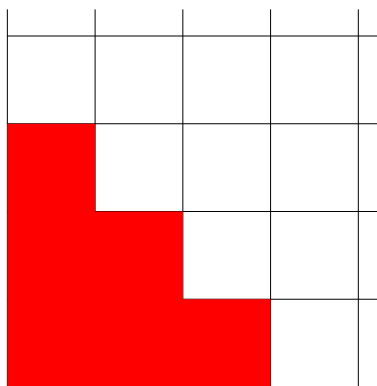
Figure 1: Problem 8: initial configuration.

Figure 2: Problem 8: figure to free.