ENEL420 Formula Sheet

The bitchiest of all the exams

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1 Interpolation

Interpolation, convolution with a kernel function:

$$f(k_0) = \sum_k f(k) \underbrace{K(k_0 - k)}_{\text{convolution kernel}} \tag{1}$$

Where $f(k_0)$ is the interpolated signal and $f(k)$ is the data

1.1 Kernels

Nearest Neighbour:

$$K(x) = \text{rect}(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Linear:

$$K(x) = \text{tri}(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Sinc:

$$K(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{4}$$

2 Aliasing

General Case to avoid aliasing:

$$\frac{2f_u}{n} < f_s < \frac{2f_l}{n-1} \tag{5}$$

Where,

$$1 \leq n \leq I_{MLT}\left(\frac{f_u}{B}\right) \tag{6}$$

I_{MLT} is Maximum integer less than

3 Z-Transform

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} \tag{7}$$

Or,

$$F(s) = \sum_{n=0}^{\infty} f(nT) e^{-nTs} \tag{8}$$

4 Quantisation Noise

RMS quantisation noise:

$$\sigma = \frac{\Delta}{\sqrt{12}} = \frac{\Delta}{2\sqrt{3}} \tag{9}$$

Noise Power:

$$\text{NP} \propto \Delta^2 \tag{10}$$

Therefore is noise power is decreased by $\frac{1}{2}$,

$$10\log_{10}\left(\frac{1}{2}^2\right) = 6\text{dB} \tag{11}$$

5 Filters

Direct realisation:

$$H(z) = \frac{\sum_{k=0}^{K} b_k z^{-k}}{1 + \sum_{m=1}^{M} a_m z^{-m}} \tag{12}$$

Linear Phase, if a filter has linear phase it can be expressed in the form:

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)} \tag{13}$$

Where,

$$\theta(\omega) = -(\alpha\omega + \beta) \tag{14}$$

Windowing the impulse response is defined by the inverse Fourier Transform,

$$h_D(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_D(\omega) e^{j\omega n} \quad d\omega \tag{15}$$

non-recursive FIR Filters,

$$y(n) = b_0 x(n) + b_1 x(n-1) \tag{16}$$

Recursive,

$$y(n) = ay(n-1) + bx(n) \tag{17}$$

So,

$$H(z) = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{\frac{j2\pi kn}{N}} z^{-n} \tag{18}$$

6 Detection Theory

Likelihood Ratio,

$$\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)} \tag{19}$$

6.1 Likelihood ratio test

Choose null hypothesis if,

$$\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)} > \gamma \tag{20}$$

Choose alternitive hypothesis if,

$$\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)} \leq \gamma \tag{21}$$

6.2 Log Likelihood

Choose null hypothesis if,

$$-x[0] + \frac{1}{2} > 0 \leftrightarrow x[0] < \frac{1}{2} \tag{22}$$

Choose alternitive hypothesis if,

$$-x[0] + \frac{1}{2} \leq 0 \leftrightarrow x[0] \geq \frac{1}{2} \tag{23}$$

Solve for γ using,

$$P_{FA} = \int_{\mathbf{x}:L(\mathbf{x})<\gamma} p(\mathbf{x}|H_0) d\mathbf{x} \tag{24}$$

6.3 Missed Detection

The probability of missed detection is,

$$P_M = p(x[0] \leq \frac{1}{2}|H_1) = p(x[0] \leq \frac{1}{2}|u = 1) \tag{25}$$

6.4 Detection Error Rate

$$P_{FA}P(H_0) + P_M P(H_1) \tag{26}$$

Where P_{FA} is the probability of falsely selecting the alternative hypothesis and P_M is the probability of falsely selecting the null hypothesis.

6.5 Detection Threshold

To find the detection threshold use the following,

$$P_{FA} = \int_{\mathbf{x}} 1(\log(L(\mathbf{x})) - c < \gamma) \cdot p(\log(L(\mathbf{x})) - c|H_0) \quad d\mathbf{x} \tag{27}$$

6.6 NP Detector of Known Signal in WGN

The signal likelihood under $w[n]$ $N(0, \sigma^2)$ (Normal distribution) for the null hypothesis,

$$p(\mathbf{x}|H_0) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x[n]^2}{2\sigma^2}\right) \tag{28}$$

For the alternative hypothesis,

$$p(x|H_1) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x[n] - s[n])^2}{2\sigma^2}\right) \tag{29}$$

Therefore the Log-Likelihood ratio test is,

$$\log(L(\mathbf{x})) = log\left(\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)}\right) = \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (s[n]^2 - 2s[n]x[n]) \tag{30}$$

6.7 Multivariate Gaussian PDF

Null Hypothesis,

$$P(\mathbf{x}|H_0) = \frac{\mathbf{1}}{(\sqrt{2\pi}\sigma)^N} \exp\left(-\frac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right) (\delta = \sigma^2\mathbf{I}) \tag{31}$$

Alternative Hypothesis,

$$P(\mathbf{x}|H_1) = \frac{\mathbf{1}}{|2\pi(\mathbf{C} + \sigma^2\mathbf{I})|} exp\left(-\frac{1}{2}\mathbf{x}^T(\mathbf{C} + \sigma^2\mathbf{I})^{-1}\mathbf{x}\right) (\Delta = \mathbf{C} + \sigma^2\mathbf{I}) \tag{32}$$

LLR,

$$\log(L(\mathbf{x})) \propto -\mathbf{x}^T\mathbf{C}(\mathbf{C} + \sigma^2\mathbf{I})^{-1}\mathbf{x} \tag{33}$$

Simplified,

$$\sum_{n=0}^{N=1} \frac{\delta_n}{\delta_n + \sigma^2} y[n]^2 \tag{34}$$

6.8 Mixture Model

$$p(\mathbf{x}|\theta) = \sum_{k=1}^{K} w_k p_k(\mathbf{x}|\theta) \tag{35}$$

Where, $p_k(\mathbf{x}|\theta)$ is the k^{th} base distribution and is the mixing weight of $p_k(\mathbf{x}|\theta)$
Where,

$$w_k \geq 0 \quad \text{and} \quad \sum_{k=1}^{K} w_k = 1 \tag{36}$$

7 Fourier Transform

7.1 Properties

Shift Theorem

$$x(t - t_0) \leftrightarrow X(f)\exp(-j2\pi f t_0) \tag{37}$$

Scaling

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right) \tag{38}$$

Convolution

$$x(t) \otimes y(t) \leftrightarrow X(f)Y(f) \tag{39}$$

For a real signal

$$X(-f) = X^*(f) \tag{40}$$

DFT

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-j2\pi nk/N) \tag{41}$$

IDFT

$$x(n) = \sum_{k=0}^{N-1} X(k)\exp(j2\pi nk/N) \tag{42}$$

Parseval's Theorem

$$\sum_{n=0}^{N-1}[f(n)]^2 = \sum_{k=0}^{N-1}|F(k)|^2 \tag{43}$$

2D Fourier Transform can be expressed as a weighted sum of harmonics,

$$\exp(j2\pi(ux + vy)) \tag{44}$$

7.2 2D FT

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-j2\pi(ux + vy))dxdy \tag{45}$$

2D DFT,

$$F(p, q) = \frac{1}{N}\sum_{m=0}^{N-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} f(m, n)\exp(-j2\pi qn/N)\right)\exp(-j2\pi pm/N) \tag{46}$$

Rotation of an image rotating by α,

$$F_\alpha(\rho;\phi) = \int_0^{2\pi}\int_0^{\infty} f(r;\theta)\exp(-j2\pi\rho r\cos(\theta - \phi))rdrd\theta \tag{47}$$

So,

$$F_\alpha(\rho;\phi) = F(\rho;\phi - \alpha) \tag{48}$$

8 Projection

$$P(\theta, x') = \int_{-\infty}^{\infty} f(x', y')dy' \tag{49}$$

8.1 Wiener filtering

$$W(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \Phi(u,v)} \tag{50}$$

Where $\Phi(u,v)$ is an estimate of the spatial frequency dependent noise-to-signal power

The estimate of the image spectrum is given by,

$$\hat{F}(u,v) = G(u,v)W(u,v) \tag{51}$$

If $H(u,v)$ is large,

$$\hat{F}(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v)} \approx F(u,v) \tag{52}$$

However when $H(u,v)$ is small,

$$\hat{F}(u,v) = G(u,v)\left(\frac{H^*(u,v)}{|H(u,v)|^2 + \Phi}\right) \approx G(u,v)\frac{H^*(u,v)}{\Phi} \approx 0 \tag{53}$$

9 Diffraction

The phase difference is,

$$\phi = (d_1 - d_2)\left(\frac{2\pi}{\lambda}\right) = (\mathbf{x} \cdot \hat{\mathbf{k_0}} - \mathbf{x} \cdot \hat{\mathbf{k}})\left(\frac{2\pi}{\lambda}\right) \tag{54}$$

So,

$$\phi = -\mathbf{x} \cdot (\mathbf{k} - \mathbf{k_0}) \tag{55}$$

Therefore the total scattering is,

$$\psi(\mathbf{k}) = \int f(\mathbf{x})\exp(i(\mathbf{k} - \mathbf{k_0}) \cdot \mathbf{x})d\mathbf{x} \tag{56}$$

The scattering vector is defined as,

$$\mathbf{u} = (\hat{\mathbf{k}} - \hat{\mathbf{k_0}})/\lambda \tag{57}$$

Therefore the scattered field in terms of $\mathbf{u}$ is,

$$F(\mathbf{u}) = \int f(\mathbf{x}) \exp(i2\pi\mathbf{u}\cdot\mathbf{x}) d\mathbf{x} \tag{58}$$

Therefore by taking the inverse Fourier transform the object can be reconstructed,

$$f(\mathbf{x}) = \int F(\mathbf{u}) \exp(-i2\pi\mathbf{u}\cdot\mathbf{x}) d\mathbf{u} \tag{59}$$