
ENEL420 Formula Sheet

The bitchiest of all the exams

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October 31, 2020

1 Interpolation

Interpolation, convolution with a kernel function:

$$f(k_0) = \sum_k f(k) \frac{K(k_0 - k)}{\text{convolution kernel}} \quad (1)$$

Where $f(k_0)$ is the interpolated signal and $f(k)$ is the data

1.1 Kernels

Nearest Neighbour:

$$K(x) = \text{rect}(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Linear:

$$K(x) = \text{tri}(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Sinc:

$$K(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad (4)$$

2 Aliasing

General Case to avoid aliasing:

$$\frac{2f_u}{n} < f_s < \frac{2f_l}{n-1} \quad (5)$$

Where,

$$1 \leq n \leq I_{MLT} \left(\frac{f_u}{B} \right) \quad (6)$$

I_{MLT} is Maximum integer less than

3 Z-Transform

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \quad (7)$$

Or,

$$F(s) = \sum_{n=0}^{\infty} f(nT)e^{-nTs} \quad (8)$$

4 Quantisation Noise

RMS quantisation noise:

$$\sigma = \frac{\Delta}{\sqrt{12}} = \frac{\Delta}{2\sqrt{3}} \quad (9)$$

Noise Power:

$$\text{NP} \propto \Delta^2 \quad (10)$$

Therefore is noise power is decreased by $\frac{1}{2}$,

$$10\log_{10}\left(\frac{1}{2}\right) = 6\text{dB} \quad (11)$$

5 Filters

Direct realisation:

$$H(z) = \frac{\sum_{k=0}^K b_k z^{-k}}{1 + \sum_{m=1}^M a_m z^{-m}} \quad (12)$$

Linear Phase, if a filter has linear phase it can be expressed in the form:

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)} \quad (13)$$

Where,

$$\theta(\omega) = -(\alpha\omega + \beta) \quad (14)$$

Windowing the impulse response is defined by the inverse Fourier Transform,

$$h_D(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_D(\omega)e^{j\omega n} d\omega \quad (15)$$

non-recursive FIR Filters,

$$y(n) = b_0x(n) + b_1x(n-1) \quad (16)$$

Recursive,

$$y(n) = ay(n-1) + bx(n) \quad (17)$$

So,

$$H(z) = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{\frac{j2\pi kn}{N}} z^{-n} \quad (18)$$

6 Detection Theory

Likelihood Ratio,

$$\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)} \quad (19)$$

6.1 Likelihood ratio test

Choose null hypothesis if,

$$\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)} > \gamma \quad (20)$$

Choose alternative hypothesis if,

$$\frac{p(\mathbf{x}|H_0)}{p(\mathbf{x}|H_1)} \leq \gamma \quad (21)$$

6.2 Log Likelihood

Choose null hypothesis if,

$$-x[0] + \frac{1}{2} > 0 \leftrightarrow x[0] < \frac{1}{2} \quad (22)$$

Choose alternative hypothesis if,

$$-x[0] + \frac{1}{2} \leq 0 \leftrightarrow x[0] \geq \frac{1}{2} \quad (23)$$

Solve for γ using,

$$P_{FA} = \int_{\mathbf{x}: L(\mathbf{x}) < \gamma} p(\mathbf{x}|H_0) d\mathbf{x} \quad (24)$$

6.3 Missed Detection

The probability of missed detection is,

$$P_M = p(x[0] \leq \frac{1}{2} | H_1) = p(x[0] \leq \frac{1}{2} | u = 1) \quad (25)$$

6.4 Detection Error Rate

$$P_{FA}P(H_0) + P_MP(H_1) \quad (26)$$

Where P_{FA} is the probability of falsely selecting the alternative hypothesis and P_M is the probability of falsely selecting the null hypothesis.

6.5 Detection Threshold

To find the detection threshold use the following,

$$P_{FA} = \int_{\mathbf{x}} 1(\log(L(\mathbf{x})) - c < \gamma) \cdot p(\log(L(\mathbf{x})) - c | H_0) \, d\mathbf{x} \quad (27)$$

6.6 NP Detector of Known Signal in WGN

The signal likelihood under $w[n] \sim N(0, \sigma^2)$ (Normal distribution) for the null hypothesis,

$$p(\mathbf{x} | H_0) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x[n]^2}{2\sigma^2}\right) \quad (28)$$

For the alternative hypothesis,

$$p(x | H_1) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x[n] - s[n])^2}{2\sigma^2}\right) \quad (29)$$

Therefore the Log-Likelihood ratio test is,

$$\log(L(\mathbf{x})) = \log\left(\frac{p(\mathbf{x} | H_0)}{p(\mathbf{x} | H_1)}\right) = \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (s[n]^2 - 2s[n]x[n]) \quad (30)$$

6.7 Multivariate Gaussian PDF

Null Hypothesis,

$$P(\mathbf{x}|H_0) = \frac{1}{(\sqrt{2\pi}\sigma)^N} \exp\left(-\frac{1}{2\sigma^2} \mathbf{x}^T \mathbf{x}\right) (\delta = \sigma^2 \mathbf{I}) \quad (31)$$

Alternative Hypothesis,

$$P(\mathbf{x}|H_1) = \frac{1}{|2\pi(\mathbf{C} + \sigma^2 \mathbf{I})|} \exp\left(-\frac{1}{2} \mathbf{x}^T (\mathbf{C} + \sigma^2 \mathbf{I})^{-1} \mathbf{x}\right) (\Delta = \mathbf{C} + \sigma^2 \mathbf{I}) \quad (32)$$

LLR,

$$\log(L(\mathbf{x})) \propto -\mathbf{x}^T \mathbf{C} (\mathbf{C} + \sigma^2 \mathbf{I})^{-1} \mathbf{x} \quad (33)$$

Simplified,

$$\sum_{n=0}^{N=1} \frac{\delta_n}{\delta_n + \sigma^2} y[n]^2 \quad (34)$$

6.8 Mixture Model

$$p(\mathbf{x}|\theta) = \sum_{k=1}^K w_k p_k(\mathbf{x}|\theta) \quad (35)$$

Where, $p_k(\mathbf{x}|\theta)$ is the k^{th} base distribution and w_k is the mixing weight of $p_k(\mathbf{x}|\theta)$
Where,

$$w_k \geq 0 \quad \text{and} \quad \sum_{k=1}^K w_k = 1 \quad (36)$$

7 Fourier Transform

7.1 Properties

Shift Theorem

$$x(t - t_0) \leftrightarrow X(f) \exp(-j2\pi f t_0) \quad (37)$$

Scaling

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right) \quad (38)$$

Convolution

$$x(t) \otimes y(t) \leftrightarrow X(f)Y(f) \quad (39)$$

For a real signal

$$X(-f) = X^*(f) \quad (40)$$

DFT

$$X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk/N) \quad (41)$$

IDFT

$$x(n) = \sum_{k=0}^{N-1} X(k) \exp(j2\pi nk/N) \quad (42)$$

Parseval's Theorem

$$\sum_{n=0}^{N-1} [f(n)]^2 = \sum_{k=0}^{N-1} |F(k)|^2 \quad (43)$$

2D Fourier Transform can be expressed as a weighted sum of harmonics,

$$\exp(j2\pi(ux + vy)) \quad (44)$$

7.2 2D FT

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-j2\pi(ux + vy)) dx dy \quad (45)$$

2D DFT,

$$F(p, q) = \frac{1}{N} \sum_{m=0}^{N-1} \left(\frac{1}{N} \sum_{n=0}^{N-1} f(m, n) \exp(-j2\pi qn/N) \right) \exp(-j2\pi pm/N) \quad (46)$$

Rotation of an image rotating by α ,

$$F_{\alpha}(\rho; \phi) = \int_0^{2\pi} \int_0^{\infty} f(r; \theta) \exp(-j2\pi \rho r \cos(\theta - \phi)) r dr d\theta \quad (47)$$

So,

$$F_{\alpha}(\rho; \phi) = F(\rho; \phi - \alpha) \quad (48)$$

8 Projection

$$P(\theta, x') = \int_{-\infty}^{\infty} f(x', y') dy' \quad (49)$$