The failure of Selman's Theorem for hyperenumeration reducibility

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Enumeration degrees

Definition

For two sets $A, B \subseteq \omega$ we say that $A \leq_e B$ if there is a c.e. set W such that:

$$x \in A \iff \exists \langle x, u \rangle \in W[D_u \subseteq B]$$

where $(D_u)_{u \in \omega}$ is listing of all finite sets by strong indices.

- From an effective listing of c.e. sets $(W_e)_{e \in \omega}$ we obtain an effective listing of enumeration operators $(\Psi_e)_{e \in \omega}$. Defined by $A = \Psi_e(B)$ if $A \leq_e B$ via W_e.
- $\leq_e$ is a preorder on sets of natural numbers and, like with Turing reducibility and the Turing degrees, the equivalence classes give us the enumeration degrees $\mathcal{D}_e$.

Total and cototal sets

Definition

We say that a set A is *total* if $\overline{A} \leq_e A$. We say that A is cototal if $A \leq_e \overline{A}$. A degree is *total* (*cototal*) if it contains a total (cototal) set.

- If A is total then $B \leq_e A$ if and only if B is c.e. in A.
- For any set A we have that $A \oplus \overline{A}$ is both total and cototal.
- The Turing degrees embed into the enumeration degrees as the total degrees via the map induced by $A \mapsto A \oplus \overline{A}$.
- The cototal degrees are a proper subclass of the enumeration degrees and the total degrees are a proper subclass of the cototal degrees.

Selman's Theorem

As we have seen, we can define Turing reducibility in terms of enumeration reducibility. Selman's theorem gives us a way of defining enumeration reducibility in terms of Turing reducibility.

Theorem (Selman's Theorem)

$A \leq_e B$ if and only if for all X if $B \leq_e X \oplus \overline{X}$ then $A \leq_e X \oplus \overline{X}$.

There is another way to define enumeration reducibility in terms of enumerations. We have that $A \leq_e B$ if every enumeration of B uniformly computes an enumeration of A. Here an enumeration of A is a total, onto function $f : \omega \to A$. In this context, Selman's theorem shows that we can drop the uniformity in the definition..

Hyperenumeration reducibility

- Now we define hyperenumeration reducibility as introduced by Sanchis in 1978.

Definition

We say that $A \leq_{he} B$ if there is a c.e. set W such that

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u\rangle \in W \wedge D_u \subseteq B]$$

- Like with enumeration reducibility this is a preorder and the equivalence classes give us the hyperenumeration degrees $\mathcal{D}_{he}$.
- From an effective listing of c.e. sets $(W_e)_{e\in\omega}$ we obtain an effective listing of hyperenumeration operators $(\Gamma_e)_{e\in\omega}$.

Hypertotal degrees.

Definition

We say that a set A is *hypertotal* if $\overline{A} \leq_{he} A$. We say that A is *hypercototal* if $A \leq_{he} \overline{A}$. A degree (in either $\mathcal{D}_e$ or $\mathcal{D}_{he}$) is *hypertotal* (*hypercototal*) if it contains a hypertotal (hypercototal) set.

Hyperenumeration reducibility and the hypertotal sets have some analogies with enumeration reducibility and total sets.

- We have that if $A \leq_{he} B \oplus \overline{B}$ if and only if A is Π^1_1 in B.
- $A \leq_h B$ if and only if $A \oplus \overline{A} \leq_{he} B \oplus \overline{B}$.
- The hyperarithmetic degrees embed in $\mathcal{D}_{he}$ as the hypertotal degrees via the map induced by $A \mapsto A \oplus \overline{A}$.

Theorem (Sanchis)

There is a hyperenumeration degree that is not hypertotal.

Relating $\leq_e$ and $\leq_{he}$

Sanchis proved an interesting result about the relationship between enumeration reducibility and hyperenumeration reducibility.

Theorem (Sanchis)

If $A \leq_e B$ then $A \leq_{he} B$ and $\overline{A} \leq_{he} \overline{B}$.

This means that if f is an enumeration of A then $A \oplus \overline{A} \leq_{he} f$. So when working with hyperenumeration redicibility we want a new notion of a hyperenumeration.

Hyperenumerations

Recall the definition of $A = \Gamma_e(B)$.

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u\rangle \in W_e \wedge D_u \subseteq B]$$

Now consider the tree $S_e \subseteq \omega^{<\omega}$ defined by

$$n^\frown x \notin S_e \iff \exists y \preceq x, u \leq |x|[\langle n, y, u\rangle \in W_{e,|x|} \wedge D_u \subseteq B]$$

We have that $\overline{S_e} \leq_e B$. Define $S_{e,n} = \{x : n^\frown x \in S_e\}$. We have that

$$n \in A \iff S_{e,n} \text{ is well founded}$$

So $A \leq_{he} \overline{S_e}$. We call a tree which hyperenumerates A in the way that S_e does a *hyperenumeration* of A.

E-pointed trees in Cantor space

Definition

A tree T is *e-pointed* if for every path $P \in [T]$ we have that T is c.e. in P. We say T is *uniformly* e-pointed if there is a single operator Ψ_e such that for all paths $P \in [T]$ we have $T = \Psi_e(P)$.

McCarthy studied e-pointed trees in Cantor space and was able to characterize their enumeration degrees.

Theorem (McCarthy)

If $T \subseteq 2^{<\omega}$ is uniformly e-pointed then T is cototal. Furthermore for a degree $\mathrm{a} \in \mathcal{D}_e$ the following are equivalent:

- a *is cototal.*
- a *contains an e-pointed tree $T \subseteq 2^{<\omega}$.*
- a *contains a uniformly e-pointed tree $T \subseteq 2^{<\omega}$ with no dead ends.*

E-pointed trees with dead ends

In Baire space we have the following characterization in terms of hypertotal sets.

Theorem (Goh, J-G, Miller, Soskova)

If $T \subseteq \omega^{<\omega}$ is uniformly e-pointed then T is hypercototal. Furthermore for a degree $\mathrm{a} \in \mathcal{D}_e$ *(or* $\mathcal{D}_{he}$*) the following are equivalent:*

- a *is hypercototal.*
- a *contains an e-pointed tree* $T \subseteq \omega^{<\omega}$.
- a *contains a uniformly e-pointed tree* $T \subseteq \omega^{<\omega}$.

E-pointed trees in Baire space without dead ends

When we consider only e-pointed trees that do not have dead ends then things become more complex

Theorem (Goh, J-G, Miller, Soskova)

There is an arithmetic set that is not enumeration equivalent to any e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends.

Theorem (Goh, J-G, Miller, Soskova)

There is a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends that is not of cototal enumeration degree.

Question

Is there an e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends that is not enumeration equivalent to any uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends.

Connection to Selman's theorem

Theorem (J-G)

There is a uniformly e-pointed tree with no dead ends that is not hypertotal.

This leads us to a contradition of Selman's theorem.

Corollary

There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.

Connection to Selman's theorem

Corollary

There are sets A, B such that $B \nleq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.

Proof.

We will have $A = T$ and $B = \overline{T}$ where T is a uniformly e-pointed tree with no dead ends that is not hypertotal. Suppose that T is Π^1_1 in X. Since T has no dead ends there must be a path $P \in [T]$ such that $P \leq_h X$. So $T \leq_e P$ and by previous lemma we have $\overline{T} \leq_{he} \overline{P} \leq_h X$ So we get that $\overline{T} \leq_{he} X \oplus \overline{X}$. □

Thank you

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