A ceer that is uniformly effectively inseparable but not uniformly finitely pre-complete

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Table of contents

Ceers

Definition

A computably enumerable equivalence relation (ceer) is a c.e. subset of ω^2 that is an equivalence relation.

Example

- $\mathrm{Id} = \{(x, x) : x \in \omega)\}$. $[x]_{\mathrm{Id}} = \{x\}$.
- $\mathrm{Id}_1 = \omega^2$. $[x]_{\mathrm{Id}_1} = \omega$.
- $\{(\langle\varphi\rangle, \langle\psi\rangle) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$

Table of contents

u.e.i ceers

Definition

two disjoint c.e. sets A and B are:

- *inseparable* if there is no computable set C such that $A \subseteq C$ and $B \subseteq \overline{C}$
- *effectively inseparable* if there is a total computable function f such that for all $e, i \in \omega$ if $A \subseteq W_e$, $B \subseteq W_i$ and $W_e \cap W_i = \emptyset$ then $f(e, i) \notin W_e \cup W_i$.

Definition (Bernardi '81)

A ceer is *uniformly effectively inseparable* (u.e.i.) if it is not Id_1 and there if there is a total computable function f such that for all x, y, e, i if $[x] \subseteq W_e$, $[y] \subseteq W_i$ and $W_e \cap W_i = \emptyset$ then $f(x, y, e, i) \notin W_e \cup W_i$.

u.e.i. examples

Example

- The sets $A = \{e : \varphi_e(e) \downarrow = 0\}$ and $B = \{e : \varphi_e(e) \downarrow = 1\}$ are effectively inseparable.
  Let $f(e, i) = j$ where φ_j is the function on input j runs $\varphi_e(j)$ and $\varphi_i(j)$ and outputs 1 if $\varphi_e(j)$ converges first and 0 if $\varphi_i(j)$ converges first.
- Fix an effective enumeration of the halting set $(n_e)_e$. The relation $\{(e, i) : \varphi_{n_e}(n_e) = \varphi_{n_i}(n_i)\}$ is u.e.i.
- Id is not u.e.i.

u.f.p. ceers

Definition (Mal'tsev '63, Montagna '82)

A ceer $R \neq \mathrm{Id}_1$ is

- *precomplete* if there is a total computable function f such that for all e, i if $\varphi_e(i)\downarrow$ then $\varphi_e(i) R f(e,i)$
- A ceer is *uniformly finitely precomplete* (u.f.p.)if there is a total computable function f such that for all $D \subseteq_{\text{fin}} \omega, e, i \in \omega$ if $\varphi_e(i) \downarrow \in [D]_R$ then $f(D, e, i) R \varphi_e(i)$.

We call these functions f above *totalizers*.

u.f.p. examples

Example

- $\{(\langle\varphi\rangle, \langle\psi\rangle) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$ is u.f.p. but not precomplete.
- $\{(\langle\varphi\rangle_1, \langle\psi\rangle_1) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$ is precomplete where $\langle\varphi\rangle_1$ is an apposite Gödel coding for Σ_1 formulas.

Implications and separations

Theorem

precomplete $\implies$ *u.f.p.* $\implies$ *u.e.i.*

Theorem (Montagna)

There is a ceer that is u.f.p. but not precomplete.

Theorem (J-G)

There is a ceer that is u.e.i. but not u.f.p.

Table of contents

Proof outline

We will build a ceer R and u.e.i. witness p using finite injury.

Requirements

To ensure that R is u.e.i. we have:

$$\mathcal{S}_{x,y,e,i} : [x]_R \subseteq W_e \wedge [y]_R \subseteq W_i \wedge W_e \cap W_i = \emptyset \implies p(x,y,e,i) \notin W_e \cup W_i$$

To ensure that R is not u.f..p. we have

$$\mathcal{P}_n : \exists m,k,e \forall c[\varphi_n(\{m,k\},e,e) \downarrow = c \implies \varphi_e(e) \downarrow \in \{m,k\} \wedge c \notin [m,k]_R]$$

- We explain the tool we will use to ensure classes do not collapse.
- We will give the strategy for $\mathcal{S}$ requirements (no finite injury).
- We will give the strategy for $\mathcal{P}$ requirements (finite injury).

Union find

- We build R as the transitive symmetric closure of a directed graph E that is a partial function, i.e. $xEy \wedge xEz \implies y = z$.
- At each stage s of our construction E will be finite. This means for each class $[x]$ in R_s there is exactly one $y \in [x]$ such that y has outdegree 0. We call y the *representative* of x at stage s, denoted $\mathrm{rep}_s(x)$.

Lemma

If $x \neq y$ and x and y are representatives at all stages s then $\neg(xRy)$.

Definition

A requirement $\mathcal{R}$ is said to have *control* of a representative x if it is the only requirement that is allowed to add an edge from x.

The strategy for $\mathcal{S}_{x,y,e,i}$

1. We start by picking a witness $w = p(x, y, e, i)$ that has not been used in the construction so far.
2. If at any stage t we have not given away our witness w and see $w \in W_{e,t} \cup W_{i,t}$, then if $w \in W_e$ we set $wE_{t+1}y$ and if $w \in W_i$ then set $wE_{t+1}x$.

Note

This is enough to ensure that $\mathcal{S}_{x,y,e,i}$ is satisfied. Since $\mathcal{S}_{x,y,e,i}$ has control of w and equivalence classes merging does no cause it, this requirement cannot be injured. However, there is another way to satisfy $\mathcal{S}_{x,y,e,i}$.

3. If, at any stage s, we see xR_sy then this $\mathcal{S}_{x,y,e,i}$ is considered satisfied. We no longer need the witness w so we can give control of w to a $\mathcal{P}$ requirement.

The strategy for $\mathcal{P}_n$

1. When this requirement is initialized we pick two witnesses m, k and a code e such that $\varphi_e(e)$ runs this construction and outputs a value of our choosing at some possible future stage.
2. Next, we wait until a future stage s where we see $\varphi_n(\{m, k\}, e, e) \downarrow = c$ for some c.

Goal

We now want to gain control of $\mathrm{rep}_s(c)$ while maintain control of one of m, k.

The new directed graph

We consider a new graph: the vertices are the representatives at stage s, and for each active $\mathcal{S}$ requirement there is an edge from its witness w to $\mathrm{rep}_s(x)$ and an edge to $\mathrm{rep}_s(y)$.

The strategy for $\mathcal{P}_n$

The new directed graph

We consider a new graph: the vertices are the representatives at stage s, and for each active $\mathcal{S}$ requirement there is an edge from its witness w to $\mathrm{rep}_s(x)$ and an edge to $\mathrm{rep}_s(y)$.

- We consider the set of vertices reachable from $\mathrm{rep}_s(c)$.
- If this set omits m then we set $\varphi_e(e) = m$ and injure all lower priority $\mathcal{P}$ requirements gain control of there witnesses. We consider $\mathcal{P}_n$ satisfied unless injured later.
- At each future state the representative of c with be one of the following:
  - a witness for some $\mathcal{S}$ requirement.
  - a number under the control of a higher priority $\mathcal{P}$ requirement.
  - a number other than m under the control of $\mathcal{P}_n$.

  In all three cases we have that $\neg(cRm)$.

The strategy for $\mathcal{P}_n$

The new directed graph

We consider a new graph: the vertices are the representatives at stage s, and for each active $\mathcal{S}$ requirement there is an edge from its witness w to $\mathrm{rep}_s(x)$ and an edge to $\mathrm{rep}_s(y)$.

- If m is reachable from $\mathrm{rep}_s(c)$ in the graph then we consider a path $\mathrm{rep}(c) = w_0, w_1, \ldots, m$.
- Since the only vertices with edges out of them are witness for $\mathcal{S}$ requirements we know that $w_0, \ldots, w_{n-1}$ are all witnesses for $\mathcal{S}$ requirements.
- Let $x_0, \ldots, x_{n-1}$ being the other vertices such that there is an edge from w_j to x_j.
- We now perform a series of collapses adding E-edges $mE_{s+1}x_{n-1}, \ldots, w_1 E_{s+1} x_0$.
- The first collapse allows us to gain control of the witness w_{n-1} which allows us to perform the next collapse, and so on.

The strategy for $\mathcal{P}_n$

- We have now lost control of $\mathrm{rep}_s(m)$, but have gained control of $w_0 = \mathrm{rep}_s c$ as its requirement no longer needs it.
- We set $\varphi_e(e) = k$.
- Observe that $w_0 \neq k$ and we have control of their equivalence classes.
- Thus $\mathcal{P}_n$ is satisfied.

Thank You