Strong Minimal Pairs in the Enumeration Degrees

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Abstract

We prove that there are strong minimal pairs in the enumeration degrees and that the degrees of the left and right sides of strong minimal pairs include Σ^0_2 degrees, although it is unknown if there is a strong minimal pair in the Σ^0_2 enumeration degrees. We define a stronger type of minimal pair we call a strong super minimal pair, and show that there are none of these in the enumeration degrees, answering a question of Lempp, Slaman and Soskova [1]. We leave open the question of the existence of a super minimal pair in the enumeration degrees.

1 Introduction

Enumeration reducibility, the relation $\leq_e$ on 2^ω, can be defined as $A \leq_e B$ if given any enumeration of B we can compute an enumeration of A. Like Turing reducibility $\leq_e$ is a pre-order and we define $\mathcal{D}_e$ to be the partial order of $\equiv_e$ equivalence classes. $\mathcal{D}_e$ has a join given by the usual operation $\deg_e(A) \vee \deg_e(B) = \deg_e(\{2x : x \in A\} \cup \{2x+1 : x \in B\})$ and a least element $\mathbf{0} = \deg_e(\emptyset)$ the degree of c.e. sets. So $\mathcal{D}_e$ is an upper semilattice.

Like in the case of Turing reducibility and Turing functionals there is a more useful characterization of enumeration reducibility using enumeration operators. Let $(W_e)_e$ be a uniformly c.e. sequence of all c.e. sets and $(D_u)_u$ be a computable listing of all finite sets. We define $\Psi_e(A) = \{x : \exists u[\langle x, u\rangle \in W_e \wedge D_u \subseteq A]\}$. We get that $A \leq_e B$ if and only if there is an e such that $A = \Psi_e(B)$. Selman [2] proved that this is equivalent to the previous definition.

Two points to note about the enumeration operators that are different from the Turing case is that $\Psi_e(A)(n)$ is always defined and Ψ_e is monotonic: if $A \subseteq B$ then $\Psi_e(A) \subseteq \Psi_e(B)$.

The Turing degrees properly embed into the enumeration degrees via the map induced by $A \mapsto A \oplus \overline{A}$. This means that every countable partial order can be embedded in the enumeration degrees (Lagemann [3]). However, structurally the enumeration degrees are very different from the Turing degrees. Gutteridge [4] proved that the enumeration degrees are downward dense. Gutteridge's proof does not relativize though, and later Cooper [5] showed that there are empty intervals in the enumeration degrees.

Now we look at minimal pairs.

Definition 1.1. In an upper semilattice with least element $\mathbf{0}$ a pair $\mathbf{a}, \mathbf{b}$ is a:

- *minimal pair* if $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

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- *strong minimal pair* if it is a minimal pair, and for all $\mathbf{x}$ such that $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ we have $\mathbf{x} \vee \mathbf{b} = \mathbf{a} \vee \mathbf{b}$.
- *super minimal pair* if both $\mathbf{a}, \mathbf{b}$ and $\mathbf{b}, \mathbf{a}$ are strong minimal pairs.
- *strong super minimal pair* if it is a minimal pair, and for all $\mathbf{x}, \mathbf{y}$ such that $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ and $\mathbf{0} < \mathbf{y} \leq \mathbf{b}$ we have $\mathbf{x} \vee \mathbf{y} = \mathbf{a} \vee \mathbf{b}$.

It is clear that if $\mathbf{a}$ and $\mathbf{b}$ are distinct minimal degrees then $\mathbf{a}$ and $\mathbf{b}$ form a strong super minimal pair, so the question of the existence of these types of minimal pairs is only of interest in upper semilattices with downward density, like the enumeration degrees, the enumeration degrees below $\mathbf{0}'$ ($\mathcal{D}_e(\leq \mathbf{0}')$) and the c.e. Turing degrees. In the case of the c.e. degrees recent work by Cai, Liu, Liu, Peng and Yang [6] proves that there are no strong minimal pairs.

In the case of the enumeration degrees, in Section 4 we prove that there does exist a strong minimal pair. The existence of a strong minimal pair in the enumeration degrees was know earlier, however this proof has not been published before, so we cover it in Section 3. The construction in section 3 produces a strong minimal pair A, B where A is Π^0_2 and $B = \emptyset'$. The proof can be generalized to show that there are continuum many strong minimal pairs, however it only gives examples of strong minimal pairs where $B \geq_e \emptyset'$. The construction of a strong minimal pair in Section 4 is more direct than the one in Section 3 and can also be generalized to give continuum many strong minimal pairs. In Section 5 we modify the construction from Section 4 to build a strong minimal pair A, B where A is Σ^0_2 and B is Π^0_2. While both sides of a strong minimal pair can be Σ^0_2, it is open whether or not there is a strong minimal pair in $\mathcal{D}_e(\leq \mathbf{0}')$.

We prove in Section 2 that there is no strong super minimal pair in the enumeration degrees. The question of the existence of a super minimal pair in the enumeration degrees remains open.

The statements of the existence or not of these types of minimal pairs are all part of the two quantifier theory of the degree structure. Lempp, Slaman and Soskova [1] proved that the three quantifier theory of D_e is undecidable, and Kent [8] proved that the three quantifier theory of $\mathcal{D}_e(\leq \mathbf{0}')$ is undecidable. The single quantifier theory of both of these structures is known to be decidable thanks to results from Lagemann [3]. However, it is not known whether or not the two quantifier theory of the enumeration degrees is decidable. Showing that the two quantifier theory of a degree structure $\mathcal{D}$ is decidable is equivalent to finding a effective way to answer the following question.

Question 1.2 (Generalized extension of embeddings)**.** Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_0, \ldots, \mathcal{Q}_{k-1}$ is it true that every embedding of P into $\mathcal{D}$ can be extended to $\mathcal{Q}_i$ for some $i < k$?

The case where $k = 1$ is known as the extension of embeddings problem for $\mathcal{D}$. For the enumeration degrees, Lempp, Slaman and Soskova [1] show that the extension of embeddings problem is decidable, but the decidability of the generalized form is still open.

The questions about what types of minimal pairs exist in the enumeration degrees are directly relevant to Question 1.2 as they can be rephrased as the existence of embeddings of the diamond that do not allow certain extensions. The lack of strong super minimal pairs is the reason for the failure of an attempt to build an algorithm to decide the two quantifier theory of the enumeration degrees suggested by Lempp, Slaman and Soskova [1].

2 No Strong Super Minimal Pairs

We prove that there are no strong super minimal pairs in the enumeration degrees. This proof is similar to Gutteridge's proof of downwards density [4], and makes use of the Gutteridge operator Θ. Gutteridge's proof is split into two cases: one where $\mathbf{a}$ is Δ_2^0 and one where $\mathbf{a}$ is not Δ_2^0. Similarly our proof is split into two cases. For the first case we have the following lemma.

Lemma 2.1 (M. Soskova)**.** *If A is Δ_2^0 then A, B is not a strong minimal pair in $\mathcal{D}_e$ for any B.*

The proof relies on some results about Kalimullin pairs [7], defined below.

Definition 2.2. A and B are a Kalimullin pair ($\mathcal{K}$-pair) if there is a c.e. set $W \subseteq \omega^2$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is called *trivial* if one of A, B is c.e.

We use the following two facts about $\mathcal{K}$-pairs.

Theorem 2.3 (The minimal pair $\mathcal{K}$-property, Kalimullin [7])**.** *A, B are a $\mathcal{K}$-pair if and only if for all $X \subseteq \omega$, $A \oplus X$ and $B \oplus X$ form a minimal pair relative to X. i.e. $Y \leq_e A \oplus X, Y \leq_e B \oplus X \implies Y \leq_e X$.*

Theorem 2.4 (Kalimullin [7])**.** *Every nonzero Δ_2^0 degree computes a nontrivial $\mathcal{K}$-pair.*

Proof of Lemma 2.1. Suppose that A is Δ_2^0 and A, B form a minimal pair. Then by Theorem 2.4 let $X, Y \leq_e A$ be a nontrivial $\mathcal{K}$-pair. Then consider $X \oplus B$ and $Y \oplus B$. If $X \oplus B \equiv_e Y \oplus B \equiv_e A \oplus B$ then by Theorem 2.3 $A \leq_e B$ a contradiction. But by assumption X, Y are both non-c.e. and bounded by A, so A, B is not a strong minimal pair. $\square$

For the second case of his proof Gutteridge constructed an operator Θ, now known as the Gutteridge operator. Gutteridge constructed Θ so that the following would hold:

$$\text{If } A \text{ is not } \Delta_2^0 \text{ then } \emptyset <_e \Theta(A) <_e A. \tag{1}$$

Our proof below relies on the particular form of Θ, not just the fact that (1) holds, so we remind the reader of this.

The construction of Θ uses a c.e. set B with the property that each column $B^{[k]} = \{x : \langle k, x\rangle \in B\}$ is finite and an initial segment of ω, that is $x + 1 \in B^{[k]} \implies x \in B^{[k]}$. We also have $B^{Int} = \{\langle k, x\rangle : \langle k, x+1\rangle \in B\}$ which is also c.e. Let $n_k = |B^{[k]}| - 1$. $\Theta(A)$ is defined to be the set $B^{Int} \cup \{\langle k, n_k\rangle : k \in A\}$.

From this we can see the following.

Lemma 2.5. $\Theta(A \cup C) = \Theta(A) \cup \Theta(C)$.

Proof.

$$\begin{aligned}
\Theta(A \cup C) &= B^{Int} \cup \{\langle k, n_k\rangle : k \in A \cup C\} \\
&= B^{Int} \cup \{\langle k, n_k\rangle : k \in A\} \cup B^{Int} \cup \{\langle k, n_k\rangle : k \in C\} \\
&= \Theta(A) \cup \Theta(C)
\end{aligned}$$

$\square$

Using this we can prove the following lemma.

Lemma 2.6. *If A and C are not Δ^0_2 then there are X, Y such that $\emptyset <_e X \leq_e A$, $\emptyset <_e Y \leq_e C$, and $X \oplus Y <_e A \oplus C$.*

Proof. Take $X = \Theta(A \oplus \emptyset), Y = \Theta(\emptyset \oplus C)$. Since $A \oplus \emptyset \equiv_m A$, by (1) we have that $0 <_e X <_e A$ as desired. Similarly $0 <_e Y <_e C$. By Lemma 2.5 we have that $X \cup Y = \Theta(A \oplus C)$.

Next we show that $X \oplus Y \equiv_e X \cup Y$. It is clear that $X \cup Y \leq_e X \oplus Y$, so we need to consider the other direction. We have that

$$\begin{aligned} X \oplus Y =& \{2x : x \in X\} \cup \{2x+1 : x \in Y\} \\ =& B^{Int} \oplus B^{Int} \cup \{2\langle k, n_k\rangle : k \in A \oplus \emptyset\} \cup \{2\langle k, n_k\rangle + 1 : k \in \emptyset \oplus C\} \\ =& B^{Int} \oplus B^{Int} \cup \{2\langle k, n\rangle : \langle k, n\rangle \in X \cup Y, k \text{ is even}\} \cup \\ & \{2\langle k, n\rangle + 1 : \langle k, n\rangle \in X \cup Y, k \text{ is odd}\} \end{aligned}$$

From this we see that $X \oplus Y \leq_e X \cup Y$, as $B^{Int} \oplus B^{Int}$ is c.e. So, as $A \oplus C$ is not Δ^0_2, by (1) we have that $\emptyset <_e X \oplus Y \equiv_e \Theta(A \oplus C) <_e A \oplus C$. $\square$

Putting both lemmas together we get the following theorem.

Theorem 2.7. *There are no strong super minimal pairs in the enumeration degrees.*

3 A Strong Minimal Pair

In this section we use $\mathcal{K}$-pairs to show that there is a strong minimal pair in the enumeration degrees. We would like to thank the anonymous referee to an earlier version of this paper for pointing out this proof.

For this proof we need some more facts about $\mathcal{K}$-pairs.

Theorem 3.1 (The ideal $\mathcal{K}$-property, Kalimullin [7]). *For sets $A, B, C \subseteq \omega$ we have the following:*

1. *If A, B are a $\mathcal{K}$-pair and $C \leq_e B$ then A, C is a $\mathcal{K}$-pair.*
2. *If both A, B and A, C are $\mathcal{K}$-pairs then $A, B \oplus C$ is a $\mathcal{K}$-pair.*

We will only need 1 for out purposes.

Theorem 3.2 (The main $\mathcal{K}$-property, Kalimullin [7]). *If A, B are a nontrivial $\mathcal{K}$-pair then:*

1. *$A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.*
2. *$\overline{A} \leq_e B \oplus \emptyset'$ and $\overline{B} \leq_e A \oplus \emptyset'$.*

Now we look at a method of building $\mathcal{K}$-pairs. For a set X we define $L_X = \{\sigma \in 2^{<\omega} : \sigma \leq_{\text{lex}} X\}$. We have that $L_X \leq_e X$ since if $\sigma \leq_{\text{lex}} D \subseteq X$ then $\sigma \leq_{\text{lex}} X$. We define $R_X = \overline{L_X} = \{\sigma \in 2^{<\omega} : \sigma >_{\text{lex}} X\} \leq_e \overline{X}$. L_X, R_X form a $\mathcal{K}$-pair with witness $W = \{\langle \sigma, \tau\rangle : \sigma <_{\text{lex}} \tau\}$. Since $\rho \prec X \iff \exists \sigma \in L_X, \tau \in R_X[\rho \preceq \sigma, \tau]$ we have that $X \oplus \overline{X} \leq_e L_X \oplus R_X$. Now we have the tools needed to prove the existence of a strong minimal pair.

Theorem 3.3. *There is a strong minimal pair A, B in the enumeration degrees. Furthermore A can be Π^0_2 and B can be Π^0_1.*

Proof. Consider some non-low Δ^0_2 set Y (For example $Y = \emptyset'$). Consider $X = K_Y = \bigoplus_e \Psi_e(Y)$. We will prove that $A = R_X$ and $B = \emptyset'$ is a strong minimal pair. Since Y is Δ^0_2 we have $L_X \leq_e K_Y \leq_e Y \leq_e \emptyset'$. Since Y is not low $Y' \not\leq_e \emptyset'$ and $Y' = K_Y \oplus \overline{K_Y} \equiv_e L_X \oplus R_X$, we must have $R_X \not\leq_e \emptyset'$. Consider some non-c.e. $C \leq_e A$. By the ideal $\mathcal{K}$-property we have that C, L_X is a $\mathcal{K}$-pair. So we have that $A = \overline{L_X} \leq_e C \oplus \emptyset'$ by the main $\mathcal{K}$-property since neither L_X or C is c.e. So A, B is a strong minimal pair.

Since Y is Δ^0_2 we have that K_Y and L_X are Σ^0_2, so $A = \overline{L_X}$ is Π^0_2. $\emptyset' \equiv_e \overline{K_\emptyset}$ which is Π^0_1. □

Now we know there are strong minimal pairs, a question we can ask is how many such pairs are there? If A, B is a strong minimal pair and $C \subseteq \omega$ is such that $B \leq_e C$ and $A \not\leq_e C$ then A, C is a strong minimal pair since $A \leq_e X \oplus B \leq_e X \oplus C$ for any X such that $\emptyset <_e X \leq_e A$. If $A \not\leq_e B$ then there are $2^{\aleph_0}$ many $C \geq_e B$ such that $A \not\leq_e C$ so the existence of one strong minimal pair tells us there are $2^{\aleph_0}$ many strong minimal pairs in the enumeration degrees.

If we restrict our attention to the left side A of a strong minimal pair A, B then we can observe that if $C \leq_e A$ is not c.e. then C, B is also a minimal pair and for any non-c.e. $X \leq_e C$ we have $C \leq_e A \leq X \oplus B$. So the existence of a strong minimal pair A, B tells us that there are at least $\aleph_0$ many degrees $\mathbf{a}$ such that $\mathbf{a}$ is the left side of a strong minimal pair.

In the proof of Theorem 3.3 we chose the set X so that R_x would be Π^0_2 and not below $\emptyset'$. If we take any X such that $X \not\leq_e \emptyset'$ then either $L_x \not\leq \emptyset'$ or $R_X \not\leq \emptyset'$. Thus, by the same argument from Theorem 3.3, either $L_x, \emptyset'$ of $R_x, \emptyset'$ is a strong minimal pair. Since there are $2^{\aleph_0}$ many $X \subseteq \omega$ but only countably many of them are below $\emptyset'$ there are $2^{\aleph_0}$ many left sides of a strong minimal pair.

4 Forcing Construction of a Strong Minimal Pair

In this section we give a new proof of the existence of a strong minimal pair in the enumeration degrees. This proof is longer than the one given in Section 3, but it is a more direct construction, and can be modified more easily.

Theorem 4.1. *There is a strong minimal pairs A, B in the enumeration degrees.*

Proof. The first step is to consider the requirements. We have:

$$\mathcal{R}_e : \ \exists \Gamma[\Gamma(\Psi_e(A) \oplus B) = A] \vee \Psi_e(A) \text{ is c.e.}$$

and

$$\mathcal{N}_e : \ \Psi_e(B) \neq A$$

Satisfying $\mathcal{N}_e$ gives us that $A \not\leq_e B$ and satisfying $\mathcal{R}_e$ gives us that for all degrees x such that $\mathbf{0} < \mathbf{x} < \deg_e(A)$ we have $\mathbf{x} \vee \deg_e(B) = \deg_e(A) \vee \deg_e B$, so notably $\mathbf{x} \not< \deg_e(B)$. If $\mathbf{0} < \mathbf{y} \leq \deg_e(B)$ then $\mathbf{y} \not\leq \deg_e(A)$ as otherwise $\deg_e(A) \leq \mathbf{y} \vee \deg_e(B) = \deg_e(B)$ contradicting an $\mathcal{N}_e$ requirement. By density there is an $\mathbf{x}$ such that $\mathbf{0} < \mathbf{x} < \deg_e(A)$ so we will have that $B \notin \mathbf{0}$ and hence a strong minimal pair.

The Γ that we will use to satisfy $\mathcal{R}_e$ will have a very specific form and will in fact be chosen ahead of time. We define

$$\Gamma_e = \{\langle a, p\rangle : \exists v[D_p = D_v \oplus \{\langle e, a, v\rangle\}]\}$$

The intuitive idea is that we will enumerate $\langle e, a, v\rangle \in B$ to code the fact that $D_v \subseteq \Psi_e(A) \implies a \in A$. In other words, $B^{[e]}$ will look like an enumeration operator that computes A from $\Psi_e(A)$.

We will do two rounds of forcing to construct A and B. The first round will produce a pair $A(X), B(X)$ satisfying all $\mathcal{R}_e$ requirements for each $X \in 2^\omega$. Then we will force along 2^ω to find an X such that $A(X), B(X)$ satisfies all $\mathcal{N}_e$ requirements.

Definition 4.2. The forcing partial $\mathbb{P} = (P, \leq)$ we will use will be defined as follows. A condition $p \in P$ will consist of a disjoint pair of computable sets (A_p, C_p) with $A_p \cup C_p$ coinfinite. We say that $p \leq q$ if $A_p \supseteq A_q$ and $C_p \supseteq C_q$.

If $\mathcal{G}$ is a generic filter on $\mathbb{P}$ then we define $A_\mathcal{G} = \bigcup_{p \in G} A_p$. So we can think of p as determining a subset of $A_\mathcal{G}$ and a subset of $\overline{A_\mathcal{G}}$. The definition of $B_\mathcal{G}$ is more complex, and it will look at $p \notin \mathcal{G}$. We will give this definition later.

Definition 4.3. For $p \in P$ and $e, n \in \omega$ we say $p \Vdash n \in \Psi_e(A)$ if $n \in \Psi_e(A_p)$ and $p \Vdash n \notin \Psi_e(A)$ if $n \notin \Psi_e(\overline{C_p})$. We say n *is determined for* e *by* p and write $p \Vdash \Psi_e(A)(n)$ if either $p \Vdash n \in \Psi_e(A)$ or $p \Vdash n \notin \Psi_e(A)$. We say $p \Vdash \Psi_e(A)$ is c.e. if for all n we have $p \Vdash \Psi_e(A)(n)$.

It is clear from the definition that if $p \in \mathcal{G}$ and $p \Vdash n \in \Psi_e(A)$ then $n \in \Psi_e(A_\mathcal{G})$; similarly if $p \Vdash n \notin \Psi_e(A)$ then $n \notin \Psi_e(A_\mathcal{G})$. If $p \Vdash \Psi_e(A)$ is c.e. then, as each n is determined for e by p, we have $\Psi_e(A_\mathcal{G}) = \Psi_e(A_p) = \Psi_e(\overline{C_p})$ which is c.e. for any $\mathcal{G} \ni p$.

Lemma 4.4. *For every $p \in P, e \in \omega$ we have either*

1. *There is $q \leq p$ such that $q \Vdash \Psi_e(A)$ is c.e.*
2. *There is $n \in \omega$ and $F \subseteq_{\text{fin}} \overline{A_p \cup C_p}$ such that $(A_p \cup F, C_p) \Vdash n \in \Psi_e(A)$ and $(A_p, C_p \cup F) \Vdash n \notin \Psi_e(A)$.*

Proof. Suppose we are given $p \in P$ and case 2 fails. We will show that case 1 holds for $q = (A_q, C_p)$ where A_q is built as follows. We have requirements

$$\mathcal{P}_n : \ n \in \Psi_e(\overline{C_p}) \iff n \in \Psi_e(A_q)$$

Along with the requirement that $A_q \cup C_p$ is coinfinite and A_q and C_p are disjoint. We build sequences $A_p = A_0 \subseteq A_1 \subseteq \dots$ and $m_0 < m_1 < \dots$ with $\{m_t : t \in \omega\}$ disjoint from all A_s. A requirement $\mathcal{P}_n$ is unmet at stage s if $n \notin \Psi_{e,s}(A_s)$ and a requirement needs attention at stage s if it is unmet and there is some pair $\langle n, u\rangle \in \Psi_{e,s}$ such that $D_u \subseteq \overline{C_p}$

We start with $m_0 = \max(A_p \cup C_p) + 1$ and $A_0 = A_p$. At stage s let $\mathcal{P}_n$ be the highest priority requirement that needs attention (if there is no such requirement then let $A_{s+1} = A_s, m_{s+1} = m_s + 1$). So there is a pair $\langle n, u\rangle \in \Psi_{e,s}$ such that $D_u \subseteq \overline{C_p}$. Wait until we see a possibly new pair $\langle n, v\rangle \in \Psi_e$ such that $D_v \subseteq \overline{C_p}$ and $\min(D_v \setminus A_s) > m_s$, then define $A_{s+1} = A_s \cup D_v$ and $m_{s+1} = \min(\overline{A_{s+1} \cup C_p} \setminus (m_s + 1))$. By assumption this search will always terminate eventually as otherwise $F = ((m_s + 1) \cup D_u) \setminus (A_p \cup C_p)$ will have $(A_p \cup F, C_p) \Vdash n \in \Psi_e(A)$ and $(A_p, C_p \cup F) \Vdash n \notin \Psi_e(A)$, a contradiction.

So the sequence $(m_s)_s$ is computable and increasing so the set $A_q = \overline{C_p} \setminus \{m_s : s \in \omega\}$ is computable and has that $A_q \cup C_p$ is coinfinite, A_q and C_p are disjoint and $A_p \subseteq A_q$. If a requirement $\mathcal{P}_n$ ever needs attention then it is met no more than n stages later and $n \in \Psi_e(A_q)$. On the other hand if $\mathcal{P}_n$ never needs attention then $n \notin \Psi_e(\overline{C_p})$. So every requirement is satisfied and $q \Vdash \Psi_e(A)$ is c.e. as desired. $\square$

The key point of Lemma 4.4 is that if we cannot find a p that forces $\Psi_e(A)$ is c.e. and satisfy $\mathcal{R}_e$ that way, then we can always find n that is not determined for e. We will use this to satisfy $\mathcal{R}_e$ using Γ_e while maintaining the choice of whether $n \in \Psi_e(A)$ or not.

Next we will use this to build an embedding $H : 2^{<\omega} \to P$ and a function $S : 2^{<\omega} \to \{f :\subseteq \omega \to \omega : f \text{ is finite}\}$. The idea is that for $X \in 2^\omega$ we will have $A(X) = \bigcup_{\sigma \prec X} A_{H(\sigma)}$ and if $\Psi_e(A(X))$ is not c.e. then for all $\sigma \prec X$, $|\sigma| > e \implies [S(\sigma)(e) \in \Psi_e(A(X)) \leftrightarrow \sigma^\frown 1 \prec X]$.

Construction of H and S. We define H, S as follows. At each stage of the construction we will start considering a new $\mathcal{R}_e$ requirement. When we can force that $\Psi_e(A)$ is c.e. we will do so immediately. For other requirements, case 2 of Lemma 4.4 will always apply. These requirements will be considered active and will need to be handled at each step. To help us keep track what requirements are active for a given σ we use a function $Z : 2^{<\omega} \to \{F \subseteq_{\text{fin}} \omega\}$. We start with $H(\emptyset) = (\emptyset, \emptyset), Z(\emptyset) = \emptyset$.

Given a node σ and $H(\sigma), Z(\sigma)$ we ask if there is $p \leq H(\sigma)$ such that $p \Vdash \Psi_{|\sigma|}(A)$ is c.e. If yes, then we can satisfy $\mathcal{R}_{|\sigma|}$ by making sure we choose extensions of p for $H(\sigma^\frown j)$. Otherwise we redefine $Z(\sigma) := Z(\sigma) \cup \{|\sigma|\}$ so that $\mathcal{R}_{|\sigma|}$ is now active and set $p = H(\sigma)$.

Let $0 = e_0, \ldots, e_{k-1}$ list $Z(\sigma)$. For each $i < k$ define n_i, F_i to be a pair satisfying case 2 of Lemma 4.4 for p and e_i. By assumption of $e_i \in Z(\sigma)$ case 1 has failed so case 2 applies. Define $F = \bigcup_{i<k} F_i$, $S(\sigma)(e_i) = n_i$, $Z(\sigma^\frown j) = Z(\sigma)$.

Finally define $H(\sigma^\frown 1) = (A_p \cup F, C_p)$ and $H(\sigma^\frown 0) = (A_p, C_p \cup F)$. Clearly $p \geq H(\sigma^\frown 1), H(\sigma^\frown 0)$ and for each $i < k$, $H(\sigma^\frown 1) \Vdash n_i \in \Psi_{e_i}(A)$ and $H(\sigma^\frown 0) \Vdash n_i \notin \Psi_{e_i}(A)$.

End of Construction.

For $X \in 2^\omega$ we define $A(X) = \bigcup_{\sigma \prec X} A_{H(\sigma)}$ and

$$B(X) = \{\langle e, a, v\rangle : (\exists \sigma \geq_{\text{lex}} X \upharpoonright |\sigma|) \\ [a \in A_{H(\sigma)} \wedge e \in \operatorname{dom}(S(\sigma)) \wedge D_v = \{S(\tau)(e) : \tau \prec \sigma, |\tau| \geq e, \sigma(|\tau|) = 1\}]\}$$

Let us try to understand the definition of $B(X)$. From the definition of Γ_e we want that if $\langle e, a, v\rangle \in B(X)$ then $D_v \subseteq \Psi_e(A(X)) \to a \in A(X)$. We also want that if $\Psi_e(A(X))$ is not c.e. then for all $a \in A(X)$ there exists a v such that $\langle e, a, v\rangle \in B(X) \wedge D_v \subseteq \Psi_e(A(X))$. If $\sigma \prec X$ then $A_{H(\sigma)} \subseteq A(X)$ and $\{S(\tau)(e) : \tau \prec \sigma, |\tau| \geq e, \sigma(|\tau|) = 1\}]\} \subseteq \Psi_e(A(X))$ so that is where the conditions on v and a come from. The reason we need to add axioms for $\sigma \geq_{\text{lex}} X \upharpoonright |\sigma|$ instead of just $\sigma \prec X$ is that the latter is too restrictive and will not allow to meet the $\mathcal{N}_e$ requirements in the next stage.

Lemma 4.5. *If $X \in 2^\omega$ then the pair $A(X), B(X)$ satisfies $\mathcal{R}_e$ for each e.*

Proof. Case 1: $e \notin \operatorname{dom}(S(\sigma))$ for any $\sigma \prec X$. Then by construction, for $\sigma = X \upharpoonright (e+1)$ we have $H(\sigma) \Vdash \Psi_e(A)$ is c.e. So as $A_{H(\sigma)} \subseteq A(X) \subseteq \overline{C_{H(\sigma)}}$ we have $\Psi_e(A(X))$ is c.e.

Case 2: we assume $e \in \operatorname{dom}(S(\sigma))$ for all $\sigma \prec X$ with $|\sigma| \geq e$. By induction along $\sigma \prec X$ we can see that $A_{H(\sigma)} \subseteq A(X) \subseteq \overline{C_{H(\sigma)}}$. So by construction we have that $A_{H(\sigma^\frown 1)} \setminus A_{H(\sigma^\frown 0)} \subseteq A(X) \iff S(\sigma)(e) \in \Psi_e(A(X))$, for all $\sigma \prec X$, $|\sigma| \geq e$. Given $\sigma \prec X$, $|\sigma| \geq e$, $a \in A_{H(\sigma)}$ and $D_v = \{S(\tau)(e) : \tau \prec \sigma, |\tau| \geq e, \sigma(|\tau|) = 1\}$ we have that $D_v \subseteq \Psi_e(A(X))$ and $\langle e, a, v\rangle \in B(X)$ so by definition of Γ_e, $a \in \Gamma_e(\Psi_e(A(X)) \oplus B(X))$. Therefore $A(X) \subseteq \Gamma_e(\Psi_e(A(X)) \oplus B(X))$.

On the other hand, if $a \in \Gamma_e(\Psi_e(A(X)) \oplus B(X))$ then by definition of Γ_e and $B(X)$, there is some $\sigma \geq_{\text{lex}} X \upharpoonright |\sigma|$ such that for $D_v = \{S(\tau)(e) : \tau \prec \sigma \wedge \sigma(|\tau|) = 1\}$ We have $D_v \subseteq \Psi_e(H(X))$ and $a \in A_{H(\sigma)}$. If $\sigma >_{\text{lex}} X \upharpoonright |\sigma|$ then let n be the first place that they differ. So $\sigma(n) = 1$

and hence $S(\sigma\upharpoonright n)(e) \in D_v$ but $H(X\upharpoonright(n+1)) \Vdash S(\sigma\upharpoonright n)(e) \notin \Psi_e(A)$. So a was not put in $\Gamma_e(\Psi_e(H_e(X)) \oplus B(X))$ by $\langle e, a, v\rangle$, a contradiction. So $\sigma \prec X$ and hence $a \in A_{H(\sigma)} \subseteq A(X)$. Therefore $A(X) = \Gamma_e(\Psi_e(A(X)) \oplus B(X))$ $\square$

Now all that is left is to diagonalize and satisfy all $\mathcal{N}_e$ requirements.

Construction of X**.** We pick a path, $X \in 2^\omega$, satisfying one $\mathcal{N}_e$ requirement at a time. We start with $\sigma_0 = \emptyset$. Suppose at stage $s+1$ we are given σ_s. Let $Y_s = \sigma_s{}^\frown 1^\frown 0^\omega$. To satisfy $\mathcal{N}_s$ ask if $A_{H(\sigma^\frown 1)} \subseteq \Psi_s(B(Y_s))$. If yes then $\sigma_{s+1} = \sigma_s{}^\frown 0$ otherwise $\sigma_{s+1} = \sigma_s{}^\frown 1$. Let $X = \cup_s \sigma_s$.

End of Construction.

Lemma 4.6. *If X is defined as above then $A(X)$ and $B(X)$ satisfy $\mathcal{N}_e$ for each e.*

Proof. If $X \succ \sigma_s{}^\frown 0$ then $A_{H(\sigma^\frown 1)} \subseteq \Psi_s(B(Y_s))$ and by definition of B, $B(Y_s) \subseteq B(X)$ so $A_{H(\sigma^\frown 1)} \subseteq \Psi_s(B(X))$ but $A_{H(\sigma^\frown 1)} \nsubseteq A(X)$ as $\sigma^\frown 0 \prec X$, so $\mathcal{N}_e$ is met. On the other hand if $X \succ \sigma_s{}^\frown 1$, then $A_{H(\sigma^\frown 1)} \nsubseteq \Psi_s(B(Y_s))$ and $B(X) \subseteq B(Y_s)$, so $A_{H(\sigma^\frown 1)} \nsubseteq \Psi_s(B(X))$, but $A_{H(\sigma^\frown 1)} \subseteq H_e(X)$. So $\mathcal{N}_s$ is satisfied. $\square$

So $(A(X), B(X))$ satisfy all the requirements and form a strong minimal pair. $\square$

An immediate corollary of this proof is that there are continuum many strong minimal pairs in the enumeration degrees.

Corollary 4.7. *For ever set $Y \in 2^\omega$ there is a strong minimal pair A_Y, B_Y such that if $Y \neq Z$ then $A_Y \neq A_Z, B_Y \neq B_Z$.*

Proof. Fix Y we build X as in the construction of X from the proof of Theorem 4.1 but on even stages $2s$ we set $X(2s) = Y(s)$ and on odd stages $2s+1$ we chose $X(2s+1)$ to satisfy $\mathcal{N}_s$ as before. $\square$

It is also interesting to note the reduction $A \oplus B \leq_e \Psi_e(A) \oplus B$ is uniform in e with $\Gamma_e(\Psi_e(A) \oplus B) = \emptyset$ if $\Psi_e(A)$ is c.e. Furthermore B can enumerate the set $\{e : 0 <_e \Psi_e(A)\}$ by looking at which columns of B are nonempty.

The forcing conditions are symmetric. By applying the same forcing steps to $\overline{A} = \bigcup_{p \in H(x)} C_p$ that we apply to A we can make it that both A, B and $\overline{A}, B$ are strong minimal pairs (we can also construct examples like this with the $\mathcal{K}$-pair construction). Note that $A \oplus \overline{A}, B$ will not be a strong minimal pair as $L_A, R_A \leq_e A \oplus \overline{A}$ and Lemma 2.1 says the left side of a strong minimal pair cannot bound a non trivial $\mathcal{K}$-pair.

If we wanted to modify the construction to get a super minimal pair we would quickly run into problems. The design of B is very precise and if we add some point $\langle e, a, v\rangle$ to B at some stage where we have ensured $A_s \subseteq A$, then it could be that already $D_v \subseteq \Psi_e(A_s)$. So we would have to ensure that $a \in A$, but then because we want $\Gamma_i(\Psi_i(B) \oplus A) = B$ we are in the reverse situation and may need to add things to B. This could go on indefinitely and end up making A and B cofinite or require us to add numbers to A or B that we have ensured are not in A or B and break a negative condition. We could try increasing C_p so that this case cannot happen, but the set $\{\langle e, a, v\rangle : p \Vdash D_v \subseteq \Psi_e(A)\}$ is not computable so we would be using a new partial order and Lemma 4.4 no longer holds.

5 Complexity of a Strong Minimal Pair

Now we look at what oracle is needed to carry out the construction of the Section 4. To work out if case 1 of Lemma 4.4 can be applied for a given condition p and number e we ask if there exists $q \leq p$ such that $\Psi_e(A_q) = \Psi_e(\overline{C_q})$. Since P is the set of pairs of disjoint computable sets, we can encode it as a Π^0_2 set of natural numbers. Similarly asking if $\Psi_e(A_q) = \Psi_e(\overline{C_q})$ is a Π^0_2 question. So asking if case 1 of Lemma 4.4 can be applied is a Σ^0_3 question. Asking if a pair n, F witnesses case 2 of Lemma 4.4 holding is something $\mathbf{0}'$ can answer, so is not going to add to the complexity of the construction. Hence H and S are Δ^0_4.

$A, B \leq_T H \oplus S \oplus X$ so we need to work out the complexity of X. To construct X we ask question of the form "is $A_{H(\sigma\hat{\ }1)} \subseteq \Psi_e(B(\sigma\hat{\ }1\hat{\ }0^\omega))$?" which is $\Pi^0_2(H \oplus S)$. So X is Δ^0_6. When the answer was yes, B increased in size. Therefore A is Δ^0_6 and B is Π^0_5.

Clearly there are some minor modifications that would reduce the complexity. We make some more serious changes to get the following result.

Theorem 5.1. *There is a strong minimal pair A, B in the enumeration degrees such that A is Σ^0_2 and B is Π^0_2 and quazi-minimal.*

A set B is quazi-minimal if every function f such graph$(f) \leq_e B$ is a computable function. In other words the only degree below $\deg_e(B)$ that is the image of a Turing degree is $\mathbf{0}_e$.

Proof. This is a finite injury argument. The idea is that we run the construction using $0'$ as an oracle, but rather than building a whole tree we only build nodes along what we believe to be on the true path (on X). $0'$ will often be wrong about what is the true path and this is where the injury comes in.

We use a restricted set of forcing conditions, $Q = \{p \in P : A_p, C_p \text{ are finite}\}$. In the proof of for Lemma 4.4 the q we build to meet case 1 was in fact infinite, so to ensure we satisfy $\mathcal{R}_e$ when case 2 does not apply we will make A enumeration 1-generic.

Definition 5.2 ([9])**.** A set A is *enumeration 1-generic* if for every W_e either there is $u \in W_e$ such that $D_u \subseteq A$ or there is $F \subseteq_{\text{fin}} \overline{A}$ such that for all $u \in W_e$, $D_u \cap F \neq \emptyset$.

For $q \in Q$ we say $q \Vdash \Psi_e(A)$ is c.e. if for all enumeration 1-generic $A \supseteq A_q$ with $\overline{A} \supseteq C_q$ we have that $\Psi_e(A) = \Psi_e(\overline{C_q})$. We have a new version of Lemma 4.4 that applies to Q.

Lemma 5.3. *For every $q \in Q, e \in \omega$ we have either*

1. *$q \Vdash \Psi_e(A)$ is c.e.*

2. *There is $n \in \omega$, $F \subseteq_{\text{fin}} \overline{A_q \cup C_q}$ such that $(A_q \cup F, C_q) \Vdash n \in \Psi_e(A)$ and $(A_q, C_q \cup F) \Vdash n \notin \Psi_e(A)$.*

Proof. Consider a pair $q \in Q, e \in \omega$, and suppose that case 2 does not hold. Let G be an enumeration 1-generic such that $A_q \subseteq G \subseteq \overline{C_q}$. Then suppose that there is n such that $n \in \Psi_e(\overline{C_q})$ but $n \notin \Psi_e(G)$. Then consider the c.e. set $W = \{u : \langle n, u\rangle \in \Psi_e\}$. Since G is enumeration 1-generic and there is no $u \in W$ such that $D_u \subseteq G$ we have that there is $E \subseteq_{\text{fin}} \overline{G}$ such that for all $u \in W$ we have $D_u \cap E \neq \emptyset$.

Pick $v \in W$ such that $D_v \subseteq \overline{C_q}$ (Since $n \in \Psi_e(\overline{C_q})$ there must be some v). Now consider $F = (D_v \setminus A_q) \cup (E \setminus C_q)$. $D_v \subseteq A_q \cup F$ so $(A_q \cup F, C_q) \Vdash n \in \Psi_e(G)$. On the other hand $E \subseteq C_q \cup F$ so for each $u \in W$, $D_u \nsubseteq \overline{C_q \cup F}$, and thus $(A_q, C_q \cup F) \Vdash n \notin \Psi_e(G)$, a contradiction. So it must be that $\Psi_e(G) \supseteq \Psi_e(\overline{C_q})$. We already have $\Psi_e(G) \subseteq \Psi_e(\overline{C_q})$ as $G \subseteq \overline{C_q}$, so $\Psi_e(G) = \Psi_e(\overline{C_q})$. Since G was arbitrary, we have $q \Vdash \Psi_e(A)$ is c.e. $\square$

The Γ_e we will use this time is a little different, only needing one witness from $\Psi_e(A)$:

$$\Gamma_e = \{\langle a, p\rangle : \exists m[D_p = \{m\} \oplus \{\langle e, a, m\rangle\}]\}$$

Since we are making B a Π^0_2 set we will start with all axioms in B and remove broken axioms as we go.

Construction of Σ^0_2 A and Π^0_2 B. At each stage of the construction we will have a tuple

$$(\sigma_s \in 2^{<\omega}, n_s = |\sigma_s|, H_s : n_s + 1 \to Q, (F_{n,s})_{n<n_s}, S_s :\subseteq \omega \times n_s \to \omega, B_s)$$

with $H = \lim_s H_s$, $S = \lim_s S_s$, $X = \lim_s \sigma_s$, $B = \bigcap B_s$, $F_n = \bigcup_s F_{n,s}$, $A = \bigcup_n A_{H(n)} = \bigcup_s A_{H_s(n_s)}$ and $\overline{A} = \bigcup_n C_{H(n)} = \bigcup\{F_n : X(n) = 0\}$. We will have $H_s(n+1) < H_s(n)$ and $F_{n,s} \subseteq A_{H_s(n+1)}$ if $\sigma(n)_s = 1$ and $C_{H_s(n)} = \bigcup\{F_k : \sigma_s(k) = 0, k < n\}$.

The requirements we will use are slightly different than in the Section 4. We will break each $\mathcal{R}_e$ requirement into ω many requirements $\mathcal{R}_{e,n}$ for $n \geq e$.

$$\mathcal{R}_{e,n} :\ A_{H(n)} \cup F_n \subseteq \Gamma_e(\Psi_e(A_{H(n)} \cup F_n) \oplus B)) \subseteq \overline{C_{H(n)}}$$

This means that if every $\mathcal{R}_{e,n}$ requirement is satisfied (and X contains infinitely many 1's) then $\Gamma_e(\Psi_e(A) \oplus B) = A$. If some $\mathcal{R}_{e,n}$ cannot be satisfied then by Lemma 5.3 we will have $H(n) \Vdash \Psi(A)$ is c.e. The $\mathcal{N}_e$ requirements will not change.

$$\mathcal{N}_i :\ \Psi_i(B) \neq A$$

And we have new requirements to make sure that A is enumeration 1-generic.

$$\mathcal{E}_i :\ \exists u \in W_i[D_u \subseteq A_{H(i)}] \vee \forall u \in W_i[D_u \cap C_{H(i)} \neq \emptyset]$$

And requirements to make sure that B is quazi-minimal.

$$\mathcal{Q}_e :\ \Psi_e(B) \neq \text{graph}(f) \text{ for any non-computable } f$$

The priority of the requirements is given by $\mathcal{R}_{0,n} < \cdots < \mathcal{R}_{n,n} < \mathcal{N}_n < \mathcal{E}_n < \mathcal{Q}_n < \mathcal{R}_{0,n+1}$.

A requirement $\mathcal{R}_{e,n}$ requires attention at stage $s+1$ if it has not been satisfied, $\sigma_s(n) = 0$ and there is $m, u < s$ that shows case 2 holds for $(A_{H_s(n_s)}, C_{H_s(n)})$ with m and D_u (not $(A_{H_s(n)}, C_{H_s(n)})$). $\emptyset'$ can answer this question.

We say that a requirement $\mathcal{N}_i$ requires attention at stage $s+1$ if it has not been initialized and $i < |\sigma_s|$. There are two cases to how $\mathcal{N}_i$ will act depending on if $F_{i,s} \nsubseteq \Psi_e(B_s)$ or if $F_{i,s} \subseteq \Psi_e(D)$ for some finite $D \subseteq B_s$. An $\mathcal{E}_i$ requirement needs attention at stage $s+1$ if $n_s = i$. It is only when satisfying these $\mathcal{E}_i$ requirements that we will increase n_s, so every $\mathcal{E}_i$ requirements will require attention at some stage. We say that a requirement $\mathcal{Q}_i$ requires attention if it has not been initialized and $i < |\sigma_s|$ and there is x, y, z such that $z \neq y$ and $\langle x, y\rangle, \langle x, z\rangle \in \Psi_i(B_s)$.

Assume at stage s we have $(n_s, \sigma_s, H_s, (F_{n,s})_{n<n_s}, S_s, A_s B_s)$. At stage $s+1$ Consider the highest priority requirement that requires attention. All lower priority requirements will be considered unsatisfied.

- *Case one: the requirement is $\mathcal{R}_{e,n}$.* By assumption we have $m, u < s+1$ that shows e is in case 2 for $(A_{H_s(n_s)}, C_{H_s(n)})$. We set
  - $\sigma_{s+1} = \sigma_s \restriction (n+1)$.

- $n_{s+1} = n + 1$.
- $F_{n,s+1} = F_{n,s} \cup D_u \cup C_{H_s(n_s)} \setminus C_{H_s(n)}$.
- $F_{k,s+1} = F_{k,s}$ for $k < n$.
- $H_{s+1} = H_s \restriction (n+1) \cup \{(n+1, (A_{H_s(n_s)}, C_{H_s(n)} \cup F_{n,s+1}))\}$ if $\sigma(n) = 0$.
- $H_{s+1} = H_s \restriction (n+1) \cup \{(n+1, (A_{H_s(n_s)} \cup F_{n,s+1}, C_{H_s(n)}))\}$ if $\sigma(n) = 1$.
- $S_{s+1} = S_s \restriction (\omega \times n + 1) \cup ((e, n), m)$.

The reason we add all the extra elements to $F_{n,s+1}$ is because D_u may have contained some of them and we need to respect the axioms on modified parts of B. For each $a \in F_{n,s+1}$ and i, k such that $k = S_{s+1}(i, n)$ we ask if $\{v : \langle i, a, v\rangle \in B_s\} = \omega$. If yes then we define $\{v : \langle i, a, v\rangle \in B_{s+1}\} = \{k\}$. Intuitively this change to B means that $a \in \Gamma_i(\Psi_i(A) \oplus B)$ if and only if $k \in \Psi_i(A)$.

- *Case two: the requirement is $\mathcal{N}_i$ and $F_{i,s} \nsubseteq \Psi_i(B_s)$.* We have that $i < n_s$, $\sigma_s(i) = 0$. Since $F_{i,s} \nsubseteq \Psi_i(B_s)$ we will redefine $\sigma_{s+1}(i)$ to be 1 and add $F_{i,s}$ to A.
  - $\sigma_{s+1} = (\sigma_s \restriction i)^\frown 1$.
  - $n_{s+1} = i + 1$.
  - $F_{k,s+1} = F_{k,s}$ for $k \leq i$.
  - $H_{s+1} = H_s \restriction n_{s+1} \cup \{(n_{s+1}, (A_{H_s(n_s)} \cup F_{i,s} \cup C_{H_s(n_s)} \setminus C_{H_s(i)}, C_{H_s(i)}))\}$.
  - $S_{s+1} = S_s \restriction (\omega \times n_{s+1})$ ‘and $B_{s+1} = B_s$.
- *Case three: the requirement is $\mathcal{N}_i$ and $F_{i,s} \subseteq \Psi_i(B_s)$.* We have that $i < n_s$, $\sigma_s(i) = 0$ and there is finite $D \subseteq B$ such that $F_{i,s} \subseteq \Psi_i(D)$. We will add elements to A to ensure that D will remain a subset of B. Let $P = (\{a : \exists e, m[\langle e, a, m\rangle \in D]\} \cup C_{H_s(n_s)}) \setminus C_{H_s(i+1)}$. We set
  - $\sigma_{s+1} = (\sigma_s \restriction i + 1)$.
  - $n_{s+1} = i + 1$.
  - $F_{k,s+1} = F_{k,s}$ for $k \leq i$.
  - $H_{s+1} = H_s \restriction n_{s+1} \cup \{(n_{s+1}, (A_{H_s(n_s)} \cup P, C_{H_s(i+1)}))\}$.
  - $S_{s+1} = S_s \restriction (\omega \times n_{s+1})$ ‘and $B_{s+1} = B_s$.
- *Case four: the requirement is $\mathcal{Q}_i$ and $\langle x, y\rangle, \langle x, z\rangle \in \Psi_i(B_s)$.* We have that $i < n_s$ and a finite $D \subseteq B$ such that $\langle x, y\rangle, \langle x, z\rangle \in \Psi_i(D)$. We will add elements to A to ensure that D will remain a subset of B. Let $P = (\{a : \exists e, m[\langle e, a, m\rangle \in D]\} \cup C_{H_s(n_s)}) \setminus C_{H_s(i+1)}$. We set
  - $\sigma_{s+1} = (\sigma_s \restriction i + 1)$.
  - $n_{s+1} = i + 1$.
  - $F_{k,s+1} = F_{k,s}$ for $k \leq i$.
  - $H_{s+1} = H_s \restriction n_{s+1} \cup \{(n_{s+1}, (A_{H_s(n_s)} \cup P, C_{H_s(i+1)}))\}$.
  - $S_{s+1} = S_s \restriction (\omega \times n_{s+1})$ ‘and $B_{s+1} = B_s$.
- *Case five: the requirement is $\mathcal{E}_i$.* Note that the other two cases do not increase n_s. This is where we do so. Ask if there is $u \in W_i$ such that $p = (A_{H_s(n_s)} \cup D_u, C_{H_s(n_s)}) \in Q$. If not then set $p = H_s(n_s)$. Now take the least $m \in \overline{A_p \cup C_p}$ and set

- $\sigma_{s+1} = \sigma_s{}^\frown 0$.
- $n_{s+1} = n_s + 1$.
- $F_{n_s,s+1} = \{m\}$.
- $F_{k,s+1} = F_{k,s}$ for $k < n_s$.
- $H_{s+1} = H_s \restriction n_s \cup \{(n_s, p), (n_{s+1}, (A_p \cup F_{n_s,s+1}, C_p))\}$.
- $S_{s+1} = S_s$ and $B_{s+1} = B_s$.

End of Construction.

Now we move on to the verification.

Lemma 5.4. *A is Σ^0_s and B is Π^0_2.*

Proof. The construction only removes points from B so B is $\emptyset'$-co-c.e. At each stage we have that $A_{H_s(n_s)} \subseteq A_{H_{s+1}(n_{s+1})}$ so $\bigcup_s A_{H_s(n_s)}$ is $\emptyset'$-c.e. At each stage the only value of H_s that changes is the final one, so for each n there is s such that $H(n) = H_s(n_s)$. Hence $A = \bigcup_n A_{H(n)} = \bigcup_s A_{H_s(n_s)}$ is Σ^0_2. □

Lemma 5.5. *A is enumeration 1-generic.*

Proof. Consider a requirement $\mathcal{E}_i$. Consider the last stage s where $n_s = i$. Then at stage $s+1$ we looked for $u \in W_i$ such that $D_u \cap C_{H_{s+1}(i)} = \emptyset$. If there was such a u then we set $D_u \subseteq A_{H_{s+1}(i)}$, and if not then, $D_u \cap C_{H_{s+1}(i)} \neq \emptyset$ for all $u \in W_i$. Since no higher priority requirements act after stage s we have $n_t \geq i$ for all $t > s$ and $H(i) = H_{s+1}(i)$. Thus $\mathcal{E}_i$ is satisfied. □

Lemma 5.6. *B is quazi-minimal.*

Proof. Consider a requirement $\mathcal{Q}_i$. Suppose that $\text{graph}(f) = \Psi_i(B)$. It is sufficient for us to show that f is computable. Let s be stage such that $\mathcal{Q}_i$ is not injured at any stage $t \geq s$. We claim that $\text{graph}(f) = \Psi_i(B_s)$. Since $B \subseteq B_s$ we have $\text{graph}(f) \subseteq \Psi_i(B_s)$ so if $\text{graph}(f) \neq \Psi_i(B_s)$ then there is $x, y \neq z$ such that $\langle x, y\rangle, \langle x, z\rangle \in \Psi(B_s)$. This means that we would have acted with some finite $D \subseteq B_s$ and P at some stage $\leq s$ according to the strategy for $\mathcal{Q}_i$.

Since $D \not\subseteq B$ some $\mathcal{R}_{e,n}$ requirement removed an axiom $\langle e, a, m\rangle \in D$ from B at some stage $t > s$. Since $\mathcal{Q}_i$ is not injured after stage s we must have $n > i$ and the strategy for $\mathcal{R}_{e,n}$ put $a \in F_{n,t}$. So $a \notin A_{H_t(n)} \cup C_{H_t(n)} \supseteq P \cup C_{H_s(i+1)} \supseteq \{a : \exists e, m[\langle e, a, m\rangle \in D]\}$, a contradiction. □

Lemma 5.7. *A, B satisfies $\mathcal{R}_e$ and $\mathcal{N}_i$ for all $e, i \in \omega$.*

Proof. Consider an $\mathcal{N}_i$ requirement. Let s be the last stage where $\mathcal{N}_i$ is injured. So $F_i = F_{i,s}$ as only higher priority requirements can change $F_{i,s}$. If $F_i \subseteq \Psi_i(B)$ then $F_i \subseteq \Psi_i(B_t)$ for all t. So when $\mathcal{N}_i$ acted for the last time we must have set $\sigma_t(i) = 0$. Since no lower priority requirements will change $\sigma_t(i)$ we have $X(i) = 0$ and $F_i \subseteq \overline{A}$.

If $F_i \not\subseteq \Psi_e(B)$ then suppose that when $\mathcal{N}_i$ acted for the last time it set $\sigma_t(i) = 0$. At this stage we had $D \subseteq B_t$ so there must have been a later stage k where $D \not\subseteq B_k$. If we remove an axiom $\langle e, a, m\rangle$ from B then we have $a \in F_{n,k}$ for some $n \leq n_k$. But because lower priority requirements do not remove elements from $H_k(i+1)$ we have that $a \notin A_{H_k(i+1)} \cup C_{H_k(i+1)} \supseteq P \cup C_{H_t(i+1)} \supseteq \{a : \exists e, m[\langle e, a, m\rangle \in D]\}$, a contradiction. So $F_i \subseteq A$. Hence each $\mathcal{N}_i$ is satisfied.

Consider an $\mathcal{R}_e$ requirement. We have two cases to deal with here. First, suppose that all $\mathcal{R}_{e,n}$ sub-requirements are satisfied. Now we show that $\Gamma_e(\Psi_e(A) \oplus B) = A$. Consider some $a \in A$. If

$\{m : \langle e,a,m\rangle \in B\} = \omega$ then we must have $a \in \Gamma_e(\Psi_e(A) \oplus B$. If $\{m : \langle e,a,m\rangle \in B\} \neq \omega$ then when we removed the missing elements at some stage s, we ensured that there is m such that $\langle e,a,m\rangle \in B$ and $m, F_{k,s}$ satisfied case 2 of Lemma 5.3 for e and $(A_{H_s(n_s)}, C_{H_s(k)})$ and $a \in F_{k,s}$. Since $a \in A$ it must be that $A_{H_s(k)} \cup F_{k,s} \subseteq A$; this is because when we change $F_{k,s}$ at some later stage t we ensure that $F_{k,s}$ is a subset of one of $F_{n,t}, A_{H_t(n))}, C_{H_t(n)}$ for some n. So we have that $m \in \Psi_e(A)$ and hence $a \in \Gamma_e(\Psi_e(A) \oplus B)$.

Now consider $a \notin A$. Then there must be a least stage s with $a \in F_{k,s}$ for some k. Since $a \notin A$ there is some n such that $a \in F_n \subseteq \overline{A}$. Since $\mathcal{R}_{e,n}$ is satisfied and $X(n) = 0$ there is a stage t and m, u, n' such that $a \in D_u \subseteq F_n$ and m, D_u satisfied case 2 of Lemma 5.3 for e and $(A_{H_t(n_t)}, C_{H_t(n')})$. Note it is possible that t is smaller that the stabilizing stage of $F_{n,s}$ and that $n' \neq n$ but this does not matter. We would have used m, D_u to satisfy $\mathcal{R}_{e,n'}$ and have $\{m\} = \{v : \langle e,a,v\rangle \in B\}$. Since $a \notin A$ it must be that $C_{H_t(n')} \subseteq \overline{A}$ as every time we add a part of $C_{H_s(n)}$ to A we make sure $F_{n',s} \subseteq A$. Since $m \notin \Psi_e(\overline{C_{H_t(n')} \cup D_u})$ we have $m \notin \Psi_e(A)$ and so $a \notin \Gamma_e(\Psi_e(A) \oplus B)$.

Now suppose there is some $\mathcal{R}_{e,n}$ that is never satisfied. Then we argue in a similar vain to Lemma 5.3 that $\Psi_e(A) = \Psi_e(\overline{C_{H(n)}})$. Suppose not. Then there is $m \in \Psi_e(\overline{C_{H(n)}}) \setminus \Psi_e(A)$. Consider the c.e. set $\{u : D_u \subseteq \overline{C_{H(n)}}, \langle m,u\rangle \in W_e\}$. Since $m \notin \Psi_e(A)$ and A is enumeration-1-generic there must be finite $F \subseteq \overline{A} \setminus C_{H(n)}$ such that $m \in \Psi_e(A \cup F)$ but $m \notin \Psi_e(\overline{C_{H(n)} \cup F})$. But then we would have used m, F to satisfy $\mathcal{R}_{e,n}$ at some sufficiently large stage. □

This completes the proof. □

Is it possible for A or B to have lower complexity? Lemma 2.1 tells us that A cannot be Δ^0_2 or, in fact, be above any non c.e. Δ^0_2 set. So we have shown that Σ^0_2 is a strict lower bound on the complexity of A. As for B, we know from Theorem 3.3 that there B can have complexity Π^0_1. B cannot have lower complexity because it cannot be c.e. We have shown that both sides of a strong minimal pair can be Σ^0_2 but we do not know if this can happen at the same time.

Question 5.8 (Open)**.** Is there a strong minimal pair in $\mathcal{D}_e(\leq \mathbf{0}')$?

We leave open the questions about super minimal pairs.

Question 5.9 (Open)**.** Is there a super minimal pair in the enumeration degrees?

Question 5.10 (Open)**.** Is there a super minimal pair in $\mathcal{D}_e(\leq \mathbf{0}')$?

Before we can hope to find an algorithm that decides the two quantifier theory of $\mathcal{D}_e$ or $\mathcal{D}_e(\leq \mathbf{0}')$, we need be able to find to find answers to the questions above.

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