Structural and topological aspects of the enumeration and hyperenumeration degrees

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Enumeration reducibility

Definition (Friedberg and Rogers 1959)

A set A is *enumeration reducible* to a set B ($A \leq_e B$) if there is a program that transforms any enumeration of B into an enumeration of A.

In practice, we use that $A \leq_e B$ if and only if there is a c.e. set of axioms W such that

$$x \in A \iff \exists \langle x, u \rangle \in W [D_u \subseteq B]$$

where $(D_u)_{u \in \omega}$ is a listing of all finite sets by strong indices.

Example

$K \leq_e A$ for any A since K is c.e.
$\overline{K} \not\leq_e K$ since $\overline{K}$ is not c.e.

Degree structure and operators

- Like with Turing reducibility $\leq_T$ we have that $\leq_e$ is a pre-order and taking equivalences classes gives us a degree structure $\mathcal{D}_e$.
- The lowest element of $\mathcal{D}_e$ is 0_e which is the equivalence class of all c.e. sets.
- From an effective listing of c.e. sets $(W_e)_{e\in\omega}$ we obtain an effective listing of enumeration operators $(\Psi_e)_{e\in\omega}$, defined by $A = \Psi_e(B)$ if $A \leq_e B$ via the set of axioms W_e.
- Unlike with Turing operators $\Psi_e(A)$ is always a set. We also have that these operators are monotonic: if $B \subseteq A$ then $\Psi_e(B) \subseteq \Psi_e(A)$.
- Gutteridge showed that the enumeration degrees are downward dense.

Total and cototal sets

Definition

We say that a set A is *total* if $\overline{A} \leq_e A$. We say that A is cototal if $A \leq_e \overline{A}$. A degree is *total* (*cototal*) if it contains a total (cototal) set.

- If A is total then $B \leq_e A$ if and only if B is c.e. in A.
- For any set A we have that $A \oplus \overline{A}$ is both total and cototal.
- The Turing degrees embed as the total degrees via the map induced by $A \mapsto A \oplus \overline{A}$.
- So $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.
- The cototal degrees are a proper subclass of the enumeration degrees and the total degrees are a proper subclass of the cototal degrees.

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Degrees of points in a space

The continuous degrees, introduced by Miller, are another subclass of the enumeration degrees that arise from a reduction on points in computable metric spaces. Kihara and Pauly extend this idea to general topological spaces as follows:

Definition

- A cb_0 space $\mathcal{X}$ is a second countable $\mathcal{T}_0$ space given with a listing of a basis $(\beta_e)_e$.
- Given a cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ and a point $x \in X$ the name of x is $\mathrm{NBase}_{\mathcal{X}}(x) = \{e \in \omega : x \in \beta_e\}$.
- We define the degrees of a space $\mathcal{X}$ to be $\mathcal{D}_{\mathcal{X}} = \{\mathrm{a} \in \mathcal{D}_e : \exists x \in X[\mathrm{NBase}(x) \in \mathrm{a}]\}$.

Example spaces

Example

- The product of the Sierpiński space $\mathbb{S}^\omega$ where $\mathbb{S} = \{0, 1\}$ with open sets $\{\emptyset, \{1\}, \mathbb{S}\}$, is universal for second countable T_0 spaces. We have that $\mathcal{D}_{\mathbb{S}^\omega} = \mathcal{D}_e$. This follow from the fact that for any $x \in \mathbb{S}^\omega$ we have $\mathrm{NBase}_{\mathbb{S}^\omega}(x) \equiv_e \{n : x(n) = 1\}$. This means that any class of enumeration degrees is $\mathcal{D}_{\mathcal{X}}$ for some $\mathcal{X} \subseteq \mathbb{S}^\omega$.
- Cantor space 2^ω gives the total degrees.
- Hilbert's cube $[0, 1]^\omega$ is universal for second countable metric spaces, and gives us the continuous degrees.

Motivation

- Kihara, Ng and Pauly look at many different spaces from topology and discover many new classes of enumeration degrees.
- A second part of their work is to establish a classification and hierarchy of classes of degrees by looking at what types of spaces a particular class of degrees could arise from.

Separation axioms

Definition

A topological space is considered

- T_0 if for any $x \neq y$ there is an open set U such that either $x \in U, y \notin U$ or $x \notin U, y \in U$.
- T_1 if $\{x\}$ is closed for any x.
- T_2 (Hausdorff) if for any $x \neq y$ there are disjoint open U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$.
- $T_{2.5}$ if for any $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.
- *Submetrizable* if its topology comes from taking a metric space and adding open sets.

Separating degrees with separation axioms

- We have the following series in implications:
metrizable $\implies$ submetrizable $\implies$ $T_{2.5}$ $\implies$ T_2 $\implies$ T_1 $\implies$ T_0.
It is well known that this hierarchy is strict for second countable spaces.
- One question is if the separation axioms give rise to different classes of degrees. For instance we could define the T_1 degrees to be the set the $\{\mathrm{a} : \exists \mathcal{X} \in T_1[\mathrm{a} \in \mathcal{D}_{\mathcal{X}}]\}$.

Theorem (Kihara, Ng, Pauly)

For every degree $\mathrm{a} \in \mathcal{D}_e$ *there is a decideable submetrizable space* $\mathcal{X}$ *such that such that* $\mathrm{a} \in \mathcal{D}_{\mathcal{X}}$.

- So the submetrizable degrees are the same as the T_0 degrees and hence the same as the T_1 degrees, T_2 degrees and $T_{2.5}$ degrees.

Separating classes with separation axioms

The separation axioms may not give us new classes of degrees, but they can still be used to categorize classes of degrees.

Definition

Given a collection of cb_0 spaces $\mathcal{T}$ we say that a class $\mathcal{C}$ of enumeration degrees is $\mathcal{T}$ if there is some $\mathcal{X} \in \mathcal{T}$ such that $\mathcal{D}_{\mathcal{X}} = \mathcal{C}$.

So any $\mathcal{C} \subseteq \mathcal{D}_e$ is T_0 and the continuous degrees and total degrees are both computably metrizable. This leads to the following question.

Question

Is the separation hierarchy T_0, T_1, T_2, $T_{2,5}$, submetrizable, metrizable a strict hierarchy on classes of degrees?

Known separations

The Golomb space $\mathbb{N}_{\mathrm{rp}} = (\mathbb{Z}^+, (a + b\mathbb{Z} : \gcd(a, b) = 1))$ and its product $\mathbb{N}_{\mathrm{rp}}^\omega$ is a known $T_2 \setminus T_{2.5}$ space. The cocylinder topology $\omega_{\mathrm{co}}^\omega = (\omega^\omega, (\omega^\omega \setminus [\sigma])_{\sigma \in \omega^{<\omega}})$ is a $T_1 \setminus T_2$ space the degrees of which are known as the cylinder cototal degrees.

Theorem (Kihara, Ng, Pauly)

- *$\mathcal{D}_{\mathbb{S}^\omega}$ is $T_0 \setminus T_1$.*
- *The cylinder cototal degrees are $T_1 \setminus T_2$.*
- *$\mathcal{D}_{\mathbb{N}_{\mathrm{rp}}^\omega}$ is $T_2 \setminus T_{2.5}$.*
- *There is a decidable, submetrizable space $\mathcal{X}$ such that $\mathcal{D}_{\mathcal{X}}$ is not metrizable.*

Question (Kihara, Ng, Pauly)

Is there a $T_{2.5}$ class of degrees that is not submetrizable?

Separation of $T_{2.5}$ and Submetrizable

The Arens co-d-CEA degrees and Roy halfgraph degrees were introduced by Kihara, Ng and Pauly. Both come from non submetrizable, decidable $T_{2.5}$ spaces and are subclasses of the doubled co-d-CEA degrees, a class that comes from a decidable $T_2 \setminus T_{2.5}$ space.

Theorem (J-G)

The Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable.

A corollary is that the doubled co-d-CEA degrees are not submetrizable. In fact the doubled co-d-CEA degrees give us another separation of T_2 classes from $T_{2.5}$ classes.

Theorem (J-G)

The doubled co-d-CEA degrees are not $T_{2.5}$.

Quasi-minimal

Definition

For a cb_0 space $\mathcal{X}$ we say that a degree $\mathbf{a} \in \mathcal{D}_e$ is $\mathcal{X}$-quasi-minimal if $\mathbf{a} \notin \mathcal{D}_{\mathcal{X}}$ and for all $\mathbf{b} \in \mathcal{D}_{\mathcal{X}}$ if $\mathbf{b} \leq_e \mathbf{a}$ then $\mathbf{b} = \mathbf{0}$.

So, since $\mathcal{D}_{2^\omega}$ is the total degrees, 2^ω-quasi-minimal and quasi-minimal mean the same thing.

Definition

For class $\mathcal{C} \subseteq \mathcal{D}_e$ and a set of cb_0 spaces $\mathcal{T}$, we say that $\mathcal{C}$ is $\mathcal{T}$-quasi-minimal if for every $\mathcal{X} \in \mathcal{T}$ the is $\mathbf{a} \in \mathcal{C}$ such that $\mathbf{a}$ is $\mathcal{X}$-quasi-minimal.

If $\mathcal{C}$ is $\mathcal{T}$-quasi-minimal then $\mathcal{C}$ is not $\mathcal{T}$.

Quasi-minimal results

Kihara, Ng and Pauly showed that $\mathcal{D}_e$ is T_1-quasi-minimal and give several other quasi-minimal results. Recall that the cylinder cototal degrees are $T_1 \setminus T_2$ and that $D_{\mathbb{N}^\omega_{\mathrm{rp}}}$ is $\mathcal{T}_2 \setminus T_{2.5}$. By modifying the proofs of these two results I was able to get the following.

Theorem (J-G)

- *The cylinder cototal degrees are T_2-quasi-minimal.*
- *$D_{\mathbb{N}^\omega_{\mathrm{rp}}}$ is $T_{2.5}$-quasi-minimal.*

Not quasi-minimal results

Theorem

There is a (non-decidable) metrizable space $\mathcal{DCD}_0$ such that $\mathcal{D}_{\mathcal{DCD}_0}$ contains all quasi-minimal doubled co-d-CEA degrees.

$\mathcal{DCD}_0$ is an example of a metrizable class that is not effectively submetrizable.

Corollary

- *The doubled co-d-CEA degrees, and hence also the Arens co-d-CEA degrees and Roy halfgraph degrees, are not metrizable-quasi-minimal.*
- *There is a metrizable class of degrees that is not effectively submetrizable.*
- *There is no effectively submetrizable class of degrees that is metrizable-quasi-minimal.*

Deciable, metrizable degrees

Any enumeration degree can arise from a decidable submetrizable space or a non-decidable metrizable space.

Question

What are the degrees of decideable, metrizable spaces.

This class will include all continuous degrees but,

Theorem (J-G)

There is a decideable metrizable cb_0-space $\mathcal{X}$ such that $\mathcal{D}_{\mathcal{X}}$ contains a quasi-minimal degree.

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Hyperenumeration reducibility

Definition (Sanchis 1978)

We say that $A \leq_{he} B$ if there is a c.e. set W such that

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u\rangle \in W \wedge D_u \subseteq B]$$

- Like with enumeration reducibility this is a preorder and the equivalence classes give us the hyperenumeration degrees $\mathcal{D}_{he}$.
- From an effective listing of c.e. sets $(W_e)_{e\in\omega}$ we obtain an effective listing of hyperenumeration operators $(\Gamma_e)_{e\in\omega}$.
- Sanchis proved, if $A \leq_e B$ then $A \leq_{he} B$ and $\overline{A} \leq_{he} \overline{B}$.

Hypertotal degrees.

Definition

We say that a set A is *hypertotal* if $\overline{A} \leq_{he} A$. We say that A is *hypercototal* if $A \leq_{he} \overline{A}$. A degree (in either $\mathcal{D}_e$ or $\mathcal{D}_{he}$) is *hypertotal* (*hypercototal*) if it contains a hypertotal (hypercototal) set.

- If $A \leq_{he} B$ then A is Π^1_1 in B.
- If A is Π^1_1 in B then $A \leq_{he} B \oplus \overline{B}$.
- $A \leq_h B \iff A \oplus \overline{A} \leq_{he} B \oplus \overline{B}$.
- The hyperarithmetic degrees embed onto the hypertotal degrees via the map induced by $A \mapsto A \oplus \overline{A}$.

Theorem (Sanchis)

There is a hyperenumeration degree that is not hypertotal.

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E-pointed trees in Cantor space

Definition (Montalbán)

A tree T is *e-pointed* if for every path $P \in [T]$ we have that T is c.e. in P. We say T is *uniformly* e-pointed if there is a single operator Ψ_e such that for all paths $P \in [T]$ we have $T = \Psi_e(P)$.

McCarthy studied e-pointed trees in Cantor space and was able to characterize their enumeration degrees.

Theorem (McCarthy)

For a degree $\mathbf{a} \in \mathcal{D}_e$ *the following are equivalent:*

- $\mathbf{a}$ *is cototal.*
- $\mathbf{a}$ *contains an e-pointed tree* $T \subseteq 2^{<\omega}$*.*
- $\mathbf{a}$ *contains a uniformly e-pointed tree* $T \subseteq 2^{<\omega}$ *with no dead ends.*

E-pointed trees in Baire space with dead ends

In Baire space we have the following characterization in terms of hypercototal sets.

Theorem (Goh, J-G, Miller, Soskova)

For a degree $\mathbf{a} \in \mathcal{D}_e$ *(or* $\mathcal{D}_{he}$*) the following are equivalent:*

- $\mathbf{a}$ *is hypercototal.*
- $\mathbf{a}$ *contains an e-pointed tree* $T \subseteq \omega^{<\omega}$ *with dead ends.*
- $\mathbf{a}$ *contains a uniformly e-pointed tree* $T \subseteq \omega^{<\omega}$ *with dead ends.*

E-pointed trees in Baire space without dead ends

When we consider only e-pointed trees that do not have dead ends then things become more complex

Theorem (Goh, J-G, Miller, Soskova)

There is an arithmetic set that is not enumeration equivalent to any e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends.

Theorem (Goh, J-G, Miller, Soskova)

There is a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends that is not of cototal enumeration degree.

Question

Is there an e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends that is not enumeration equivalent to any uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends.

Topological classification

Theorem (J-G)

All these classes are T_1 but not T_2

Proof.

They all contain the cototal degrees so are not T_2. The hypercototal degrees are the degrees of a T_1 space.
Consider the space:
$\mathcal{X} = \{F \subseteq \omega^\omega : F = [T]$ for some uniformly e-pointed tree via $\Psi\}$ with basis given by $\alpha_\sigma = \{F \in X : [\sigma] \cap F \neq \emptyset\}$. The degrees of $\mathcal{X}$ give us all uniformly e-pointed trees via Ψ. $\square$

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Selman's Theorem

Selman's theorem gives us a way of defining enumeration reducibility in terms of total degrees.

Theorem (Selman's Theorem)

$A \leq_e B$ if and only if, for all X if $B \leq_e X \oplus \overline{X}$ then $A \leq_e X \oplus \overline{X}$.

From the original definition of enumeration reducibility. We have that $A \leq_e B$ if every enumeration of B uniformly computes an enumeration of A. In this context, Selman's theorem shows that we can drop the uniformity in the definition.

Connection to e-pointed trees

Theorem (J-G)

There is a uniformly e-pointed tree with no dead ends that is not hypertotal.

This shows that the analogue of Selman's theorem fails for hyperenumeration reducibility.

Corollary

There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.

Proof idea.

Let $A = T$ and $B = \overline{T}$ for a non hypertotal uniformly e-pointed tree T without dead ends. $\square$

The Gutteridge operator

Theorem (Gutteridge '71)

For every $\mathbf{a} \neq \mathbf{0}_e$ *there is* $\mathbf{b} \in \mathcal{D}_e$ *such that* $\mathbf{0} < \mathbf{b} < \mathbf{a}$.

As part of his proof, Gutteridge constructed an enumeration operator Θ with the following properties:

1. If A is not c.e. then $\Theta(A) <_e A$.
2. If $\Theta(A)$ is c.e. then A is Δ^0_2.

The hyper Gutteridge operator

Theorem (J-G)

There is a hyperenumeration operator Λ such that for all A:

1. *If A is not Π^1_1 then $\Lambda(A) <_{he} A$.*
2. *If $\Lambda(A)$ is Π^1_1 then $A \leq_{he} \overline{\mathcal{O}}$.*

Downward density below $\overline{\mathcal{O}}$

Theorem (J-G)

For every X such that $\emptyset <_{he} X \leq_{he} \overline{\mathcal{O}}$ there is Y such that $\emptyset <_{he} Y <_{he} X$.

Difficulty with injury arguments

For an enumeration operator we have that $\Psi_e(A) = \bigcup_{D \subseteq_{\text{fin}} A} \Psi_e(D)$. For a hyper enumeration operator it may be that $\Gamma_e(A) \neq \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$.

Questions

Question

We proved that notion of hyperenumeration reducibility in terms of operators does not match up with a definition in terms of hyperenumerations, but is possible to define a different reducibility in terms of hyperenumerations. Does a version of Selman's theorem hold for this reducibility?

Question

Are the hypertotal degrees definable in $\mathcal{D}_{he}$? How complex is the theory of $\mathcal{D}_{he}$? Are the hypertotal degrees an automorphism base?

Thank You