A ceer that is uniformly effectively inseparable but not uniformly finitely precomplete

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Ceers

Definition

A computably enumerable equivalence relation (ceer) is a c.e. subset of ω^2 that is an equivalence relation.

Example

- $\mathrm{Id} = \{(x, x) : x \in \omega)\}$. $[x]_{\mathrm{Id}} = \{x\}$.
- $\mathrm{Id}_1 = \omega^2$. $[x]_{\mathrm{Id}_1} = \omega$.
- $\{(\langle\varphi\rangle, \langle\psi\rangle) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$

Reducibility on ceers

Definition

A We say a ceer A is *reducible* to B ($A \leq B$) if there is a total computable function f such that $xAy \iff f(x)Bf(y)$.

Note that $A \leq B$ implies that $A \leq_m B$ (reducing pairs to pairs) but the converse does not hold.

Universal ceers

Definition

We say a ceer R is *universal* if for all ceers A, we have $A \leq R$.

Example

- $R = \bigoplus_e R_e$ is universal, where R_e is the transitive symmetric closure of the eth c.e. subset of ω^2.
- $\{(\langle\varphi\rangle, \langle\psi\rangle) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$ is universal.
- Id is not universal.
- $\{(e, i) : \varphi_e(e)\downarrow \wedge \varphi_i(i)\downarrow\}$ is not universal.

u.f.p. ceers

Definition (Mal'tsev '63, Montagna '82)

A ceer $R \neq \mathrm{Id}_1$ is

- *precomplete* if there is a total computable function f such that for all e, i if $\varphi_e(i)\downarrow$ then $\varphi_e(i) R f(e,i)$
- A ceer is *uniformly finitely precomplete* (u.f.p.)if there is a total computable function f such that for all $D \subseteq_{\mathrm{fin}} \omega, e, i \in \omega$ if $\varphi_e(i)\downarrow \in [D]_R$ then $f(D,e,i) R \varphi_e(i)$.

We call these functions f above *totalizers*.

u.f.p. examples

Example

- $\{(\langle\varphi\rangle, \langle\psi\rangle) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$ is u.f.p. but not precomplete.
- $\{(\langle\varphi\rangle_1, \langle\psi\rangle_1) : \mathrm{PA} \vdash \varphi \leftrightarrow \psi\}$ is precomplete where $\langle\varphi\rangle_1$ is an appropriate Gödel coding for Σ_1 formulas.

u.e.i ceers

Definition

two disjoint c.e. sets A and B are:

- *inseparable* if there is no computable set C such that $A \subseteq C$ and $B \subseteq \overline{C}$
- *effectively inseparable* if there is a total computable function f such that for all $e, i \in \omega$ if $A \subseteq W_e$, $B \subseteq W_i$ and $W_e \cap W_i = \emptyset$ then $f(e, i) \notin W_e \cup W_i$.

Definition (Bernardi '81)

A ceer is *uniformly effectively inseparable* (u.e.i.) if it is not Id_1 and there if there is a total computable function f such that for all x, y, e, i if $[x] \subseteq W_e$, $[y] \subseteq W_i$ and $W_e \cap W_i = \emptyset$ then $f(x, y, e, i) \notin W_e \cup W_i$.

u.e.i. examples

Theorem (Andrews, Badaev, and Sorbi)

Every u.e.i. ceer is universal.

Example

- The sets $A = \{e : \varphi_e(e) \downarrow= 0\}$ and $B = \{e : \varphi_e(e) \downarrow= 1\}$ are effectively inseparable.
  Let $f(e, i) = j$ where φ_j is the function on input j runs $\varphi_e(j)$ and $\varphi_i(j)$ and outputs 1 if $\varphi_e(j)$ converges first and 0 if $\varphi_i(j)$ converges first.
- Fix an effective enumeration of the halting set $(n_e)_e$. The relation $\{(e, i) : \varphi_{n_e}(n_e) = \varphi_{n_i}(n_i)\}$ is u.e.i.
- Id is not u.e.i.

Implications and separations

Theorem

precomplete $\implies$ *u.f.p.* $\implies$ *u.e.i.*

Theorem (Montagna)

There is a ceer that is u.f.p. but not precomplete.

Theorem (J-G)

There is a ceer that is u.e.i. but not u.f.p.

Proof outline

We will build a ceer R and u.e.i. witness p using finite injury.

Requirements

To ensure that R is u.e.i. we have:

$$\mathcal{S}_{x,y,e,i} : [x]_R \subseteq W_e \wedge [y]_R \subseteq W_i \wedge W_e \cap W_i = \emptyset \implies p(x, y, e, i) \notin W_e \cup W_i$$

To ensure that R is not u.f..p. we have

$$\mathcal{P}_n : \exists m, k, e \forall c [\varphi_n(\{m, k\}, e, e) \downarrow = c \implies \varphi_e(e) \downarrow \in \{m, k\} \wedge c \notin [m, k]_R]$$

Union find

- We build R as the transitive symmetric closure of a directed graph E that is a partial function, i.e. $xEy \wedge xEz \implies y = z$.
- At each stage s of our construction E will be finite. This means for each class $[x]$ in R_s there is exactly one $y \in [x]$ such that y has outdegree 0. We call y the *representative* of x at stage s, denoted $\mathrm{rep}_s(x)$.

Lemma

If $x \neq y$ and x and y are representatives at all stages s then $\neg(xRy)$.

Definition

A requirement $\mathcal{R}$ is said to have *control* of a representative x if it is the only requirement that is allowed to add an edge from x.

Thank you

Thank You