Classification of classes of enumeration degrees of non-metrizable spaces by topological separation axioms.

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Enumeration degrees

- We say $A \leq_e B$ if every enumeration of B (uniformly) computes an enumeration of A.
- Like in the case of Turing degrees this gives us a pre-order and we define $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$. $\mathcal{D}_e$ is the set of $\equiv_e$ equivalence classes.
- The Turing degrees properly embed into the enumeration degrees via the map induced by the map $A \mapsto \mathrm{graph}(A)$. The degrees in the image of this map are called the *total* degrees. Degrees which are not above any non-zero total degree are called *quasi-minimal*.
- A subclass we will see later is the *graph cototal* degrees, which are the degrees of the complements of total functions.

Degrees of points in a space

The continuous degrees, introduced by Miller, are another subclass of the enumeration degrees that arise from a reduction on points in computable metric spaces. Kihara and Pauly extend this idea to general topological spaces as follows.

Definition

- A cb_0 space $\mathcal{X}$ is a second countable $\mathcal{T}_0$ space given with a listing of a basis $(\beta_e)_e$.
- Given a cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ and a point $x \in X$ the name of x, $\mathrm{Name}_{\mathcal{X}}(x) = \{e \in \omega : x \in \beta_e\}$.
- We define the degrees of a space $\mathcal{X}$ to be $\mathcal{D}_{\mathcal{X}} = \{\mathrm{a} \in \mathcal{D}_e : \exists x \in X[\mathrm{Name}(x) \in \mathrm{a}]\}$.

Example spaces

- The product of Sierpiński space $\mathbb{S}^\omega$ where $\mathbb{S} = \{0, 1\}$ with open sets $\{\emptyset, \{1\}, \mathbb{S}\}$, is universal for second countable T_0 spaces. $D_{\mathbb{S}^\omega} = D_e$.
- Cantor space 2^ω gives the total degrees.
- $(\omega_{\mathrm{cof}})^\omega$ the product of the cofinite topology, gives cototal degrees.
- Hilbert's cube $[0, 1]^\omega$, gives us the continuous degrees.

Separation axioms

Definition

A topological space is considered

- T_0 if for any $x \neq y$ the is an open set U such that either $x \in U, y \notin U$ or $x \notin U, y \in U$.
- T_1 if $\{x\}$ is closed for any x.
- T_2 (Hausdorff) if for any $x \neq y$ there are disjoint open U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$.
- $T_{2.5}$ if for any $x \neq y$ there are disjoint closed neighborhoods C, D such that $x \in C, y \notin C$ and $x \notin D, y \in D$.
- *Submetrizable* if its topology comes from taking a metric space and adding open sets.

We have the following series in implications: metrizable $\implies$ submetrizable $\implies T_{2.5} \implies T_2 \implies T_1 \implies T_0$

More examples

- Sierpiński space is $T_0 \setminus T_1$.
- $(\omega_{\mathrm{cof}})^\omega$ is $T_1 \setminus T_2$.
- The cylinder cototal degrees, a subclass of the graph cototal degrees, come from the $T_1 \setminus T_2$ space $(\omega^\omega)_{\mathrm{co}}$.
- The Golomb space $\mathbb{N}_{\mathrm{rp}}$ is a $T_2 \setminus T_{2.5}$ space.
- The doubled co-d-CEA degrees come from a $T_2 \setminus T_{2.5}$ space.
- The Roy halfgraph degrees and the Arens co-d-CEA degrees are both subclasses of the doubled co-d-CEA degrees that come from spaces that are $T_{2.5}$ but not submetrizable.
- The Gandy-Harrington topology is a submetrizable space that is not metrizable.

Definition

A set is *doubled co-d-CEA* if it is of the form $\mathrm{graph}(Y) \oplus (A \cup N) \oplus (B \cup P)$ where $N, P, (A \cup B)^c$ are Y-c.e. and A, B, N, P are disjoint.

Separating classes with separation axioms

We have that if $\mathcal{X}$ is a computable metric space, then $\mathcal{D}_{\mathcal{X}}$ is a subclass of the continuous degrees. However this is the only case where a separation axiom gives us a nontrivial class of enumeration degrees.

Theorem (Kihara, Ng, Pauly)

For every degree $\mathbf{a} \in \mathcal{D}_e$ *there is a computable submetrizable space* $\mathcal{X}$ *such that such that* $\mathbf{a} \in \mathcal{D}_{\mathcal{X}}$.

So the submetrizable degrees are the same as the T_0 degrees. However we can still make separations at the level of classes.

Separating classes with separation axioms

Definition

For a cb_0 space $\mathcal{X}$ we say that a degree $\mathrm{a} \in \mathcal{D}_e$ is $\mathcal{X}$ quasi-minimal if $\mathrm{a} \notin \mathcal{D}_{\mathcal{X}}$ and for all $\mathrm{b} \in \mathcal{D}_{\mathcal{X}}$ if $\mathrm{b} \leq_e \mathrm{a}$ then $\mathrm{b} = 0$.

So, since $\mathcal{D}_{2^\omega}$ is the total degrees, 2^ω-quasi-minimal and quasi-minimal mean the same thing.

Definition

For class $\mathcal{C} \subseteq \mathcal{D}_e$ and a set of cb_0 spaces $\mathcal{T}$, we say that $\mathcal{C}$ is $\mathcal{T}$-quasi-minimal (not $\mathcal{T}$) if for every $\mathcal{X} \in \mathcal{T}$ the is $\mathrm{a} \in \mathcal{C}$ such that a is $\mathcal{X}$-quasi-minimal ($\mathrm{a} \notin \mathcal{D}_{\mathcal{X}}$).

Note that there is an $\mathcal{X}$-quasi-minimal degree if and only if $\mathcal{D}_e$ is $\{X\}$-quasi-minimal. Clearly if $\mathcal{C}$ is $\mathcal{T}$-quasi-minimal then $\mathcal{C}$ is not $\mathcal{T}$.

Known separations

Recall that the product of Golomb space $\mathbb{N}_{\mathrm{rp}}^{\omega}$ is T_2 space, and recall that $(\omega^\omega)_{\mathrm{co}}$ is T_1.

Theorem (Kihara, Ng, Pauly)

- *$D_{\mathbb{S}^\omega}$ is T_1-quasi-minimal.*
- *The cylinder cototal degrees $\mathcal{D}_{(\omega^\omega)_{\mathrm{co}}}$ are not T_2.*
- *$D_{\mathbb{N}_{\mathrm{rp}}^{\omega}}$ is not $T_{2.5}$.*

The proofs of the last two results use a counting argument in the final step. By replacing the final step with a forcing construction the last two points can be strengthened to.

Theorem (J-C)

- *The cylinder cototal degrees are T_2-quasi-minimal.*
- *$D_{\mathbb{N}_{\mathrm{rp}}^{\omega}}$ is $T_{2.5}$-quasi-minimal.*

Separation of $T_{2.5}$ and Submetrizable

Recall that the Arens co-d-CEA degrees and Roy halfgraph degrees both come from $T_{2.5}$ spaces.

Theorem (J-C)

The Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable.

Since the doubled co-d-CEA degrees included the Arens co-d-CEA degrees and Roy halfgraph degrees, A corollary is that the doubled co-d-CEA degrees are not submetrizable. It is unknown if the doubled co-d-CEA degrees are $T_{2.5}$ or not, but we do have the following.

Theorem (J-C)

The Arens co-d-CEA degrees are a strict subset of the doubled co-d-CEA degrees.

Doubled co-d-CEA separation

We will give a sketch of the proof that the doubled co-d-CEA degrees are not submetrizable, since it has the same structure, but is less technical. The key idea is to use the

Proof part 1.

First we use finite injury to build c.e. sets $N, P \subseteq C$ with $N \cap P = \emptyset$ such that for any partition $A \sqcup B = C^c$ we have that $(A \cup N) \oplus (B \cup P)$ is not PA and does not compute any non Δ^0_2 total degree. This gives us a class we will call $\mathcal{C}$ of continuum many doubled co-d-CEA degrees that do not bound a Scott ideal or any non Δ^0_2 total degree.

Doubled co-d-CEA separation

Proof part 2.

Next we consider some arbitrary computable metric space $\mathcal{X} = (X, (\alpha_e)_e)$ and submetrizable extension $\mathcal{Y} = (X, (\alpha_e)_e \cup (\beta_i)_i)$. Fix a degree $\mathbf{a} \in \mathcal{C}$. Suppose that for some point $x \in X$ we have that $\mathrm{Name}_{\mathcal{Y}}(x) \in \mathbf{a}$ then $\mathrm{Name}_{\mathcal{X}}(x) \leq_e \mathbf{a}$. Since $\mathbf{a}$ does not bound a Scott ideal, $\mathrm{Name}_{\mathcal{X}}(x)$ must have total degree (by theorem of Miller). Hence $\mathrm{Name}_{\mathcal{X}}(x) \leq_e 0'$. So there are only countably many $x \in X$ such that $\deg(\mathrm{Name}_{\mathcal{Y}}(x)) \in \mathcal{C}$, so $\mathcal{C} \nsubseteq \mathcal{D}_{\mathcal{Y}}$. The result for non computable submetrizable spaces is done by relativization. □

This part of the proof is the same as with Arens co-d-CEA and Roy halfgraph degrees.

Thank you

Thank You