Topological classification of classes of enumeration degrees

Josiah Jacobsen-Grocott*

December 7, 2022

Abstract

A point in a represented second-countable T_0 space can be identified with the set of basic open sets containing that point. By representing a point by an enumeration of the indices of the basic open sets containing that point we can consider the enumeration degrees of the points in a second-countable T_0 space. For example, the ω-product of the Sierpiński space is universal for second-countable T_0 spaces and gives us all enumeration degrees and the Hilbert cube gives us all continuous degrees.

Kihara, Ng, and Pauly [1] have studied various classes that arise from different spaces. They show that any enumeration degree is contained in a class arising from some decidable, submetrizable space, and that no T_1-space contains all enumeration degrees. We call a class of degrees a $\mathcal{T}$ class if it comes from a $\mathcal{T}$ space. So Kihara, Ng, and Pauly show that $\mathcal{D}_e$ is not T_1. Similarly they separate T_2 classes from T_1 classes and $T_{2.5}$ classes from T_2 classes by showing that no T_2 class contains all the cylinder-cototal degrees and no $T_{2.5}$ class contains all degrees arising from $(\mathbb{N}_{\mathrm{rp}})^\omega$. We answer several questions posed in their article: we extend their results to show that the cylinder-cototal degrees are T_2-quasi-minimal and the $(\mathbb{N}_{\mathrm{rp}})^\omega$ degrees are $T_{2.5}$ quasi-minimal. We then give separations of the $T_{2.5}$ classes from the submetrizable classes using the Arens co-d-CEA degrees and the Roy halfgraph above degrees.

1 Introduction

In this paper we study connections between topology and enumeration reducibility. Enumeration reducibility, introduced by Friedberg and Rogers [2], is a relation defined on sets of natural numbers. We say that a set A is enumeration reducible to a set B (written $A \leq_e B$) if every enumeration of B can uniformly compute an enumeration of A. Enumeration reducibility is a preorder and gives rise to the enumeration degrees $\mathcal{D}_e$.

The Turing degrees properly embed into the enumeration degrees via the map induced by $A \mapsto A \oplus A^c$. The degrees in the image of this embedding are known as the total degrees. The total degrees can also be defined in terms of enumeration reducibility: a degree is total if it contains a set A such that $A^c \leq_e A$. The degrees of sets A with the property that $A \leq_e A^c$ are known as the cototal degrees [3]. Every total degree is cototal, but there are cototal degrees which are not total.

A subclass of the cototal degrees introduced by Miller [4] is the continuous degrees. These are taken by looking at points in a computably represented metric space and defining a reducibility

*Author partially supported by NSF Grant No. DMS-2053848

between them. Kihara and Pauly [5] extended this idea to define degrees of points in arbitrary second-countable topological spaces, using the notion of a countably based space.

Definition 1.1. A cb_0 space $\mathcal{X}$ is a second countable T_0 space given with a listing of a basis $(\beta_e)_{e\in\omega}$. Given a cb_0 space $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ and a point $x \in X$ the coded neighborhood filter of x is $\mathrm{NBase}_{\mathcal{X}}(x) = \{e \in \omega : x \in \beta_e\}$. We define the degrees of a space $\mathcal{X}$ to be $\mathcal{D}_{\mathcal{X}} = \{\mathbf{a} \in \mathcal{D}_e : \exists x \in X[\mathrm{NBase}(x) \in \mathbf{a}]\}$.

This definition of the degree of a point is not the same as the definition Miller [4] used, but Kihara and Pauly [5] observed that it coincides on computably represented metric spaces if we use as our basis the set of balls with rational radius and rational center.

Miller [4] showed that the continuous degrees are the degrees of a universal computably represented metric space, namely Hilbert's cube $[0,1]^\omega$. For enumeration degrees Kihara and Pauly showed that $\mathcal{D}_e$ is the class of degrees of a universal second-countable T_0 space, namely the ω-product of Sierpiński space $\mathbb{S}^\omega$ where $\mathbb{S} = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$. For a point $x \in \mathbb{S}^\omega$ we have that $\mathrm{NBase}(x) \equiv_e \{n : x(n) = 1\}$.

From the definition, every cb_0 space $\mathcal{X}$ gives a class of enumeration degrees $\mathcal{D}_{\mathcal{X}}$. From the universality of $\mathbb{S}^\omega$ we have that every class $\mathcal{C}$ of enumeration degrees is $\mathcal{D}_{\mathcal{X}}$ for some cb_0 space $\mathcal{X}$, namely $\mathcal{X} = \{x \in \mathbb{S}^\omega : \deg(\mathrm{NBase}(x)) \in \mathcal{C}\}$. So the study of subclasses of the enumeration degrees is the study of cb_0 spaces.

Kihara, Ng and Pauly [1] looked at many cb_0 spaces from classical topology to expand the zoo of enumeration degrees. They discovered some new classes as well as spaces that give rise to some previously studied classes. Some new classes that are of particular interest in this paper are the cocylinder degrees, the doubled co-CEA degrees, the Arens co-d-CEA degrees and the Roy halfgraph above degrees. We also look at the degrees of the relatively prime integer topology, $\mathbb{N}_{\mathrm{rp}}$. We will define these classes in the sections that first make use of them.

Kihara, Ng and Pauly [1] also looked at topological separation axioms and how they interact with classes of enumeration degrees. The separation axioms that we explore are as follows.

Definition 1.2. A topological space is considered

- T_0 (Kolmogorov) if for any $x \neq y$ there is an open set U such that either $x \in U, y \notin U$ or $x \notin U, y \in U$. In other words, points can be distinguished by the topology.
- T_1 (Fréchet) if for any $x \neq y$ there are open U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Equivalently if $\{x\}$ is closed for any x.
- T_2 (Hausdorff) if for any $x \neq y$ there are disjoint open U, V such that $x \in U, y \in V$.
- $T_{2.5}$ (Urysohn) if for any $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.
- *Submetrizable* if there is a coarser topology on the space that is metrizable. In other words, if $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ is submetrizable then there is a collection of $\mathcal{X}$-open sets $(\alpha_e)_{e\in\omega}$ such that $(X, (\alpha_e)_{e\in\omega})$ is metrizable.

We have the following series of implications:

$$\text{metrizable} \implies \text{submetrizable} \implies T_{2.5} \implies T_2 \implies T_1 \implies T_0$$

It is well known that this hierarchy is strict for second countable spaces.

Kihara, Ng and Pauly [1] showed that for any enumeration degree $\mathbf{a}$ there is a decidable, submetrizable cb_0 space $\mathcal{X}$ such that $\mathbf{a} \in \mathcal{D}_{\mathcal{X}}$. Similarly there is a (non-decidable) metric space $\mathcal{Y}$ such that $\mathbf{a} \in \mathcal{D}_{\mathcal{Y}}$. So even if we require the topology to be decidable the non-metrizable separation axioms give us all enumeration degrees. Other effective restrictions yield new classes, for example decidable, computably regular T_2 spaces give subclasses of the continuous degrees [6, 7] and effectively G_δ spaces give subclasses of the cototal degrees [1]. It is an open question what the class of degrees that can arise from a decidable, metrizable cb_0 space is, although from Theorem 8.14 we know that this is strictly larger than the class of continuous degrees.

These non-metrizable separation axioms may not give us new classes of degrees, but we can use the separation axioms to classify classes of degrees.

Definition 1.3. Given a collection of cb_0 spaces $\mathcal{T}$ we say that a class $\mathcal{C}$ of enumeration degrees is $\mathcal{T}$ if there is some $\mathcal{X} \in \mathcal{T}$ such that $\mathcal{D}_{\mathcal{X}} = \mathcal{C}$.

We have the same implications of the separation axioms, but because multiple different cb_0 spaces may give rise to the same class of degrees it is not clear that these implications are strict. In fact, Kihara, Ng and Pauly [1] considered another separation axiom, the notion of a T_D space. For second countable spaces T_D lies strictly between T_0 and T_1. However they show that for any T_D cb_0 space $\mathcal{X}$ there is a T_1 space $\mathcal{Y}$ such that $\mathcal{D}_{\mathcal{X}} = \mathcal{D}_{\mathcal{Y}}$, so this is a case of two topological separation axioms that are distinct for second countable spaces, but not for classes of enumeration degrees.

Kihara, Ng and Pauly [1] gave some separations for this classification of classes of degrees. They showed that $\mathcal{D}_e$ is T_0 but not T_1, that cylinder-cototal degrees are T_1 but not T_2, and that $\mathcal{D}_{\mathbb{N}^\omega_{\mathrm{rp}}}$ is T_2 but not $T_{2.5}$. They did not show $T_{2.5}$ and submetrizable are different notions for classes of degrees and asked as a question if there is a $T_{2.5}$ class that is not submetrizable. They also showed that the degrees of the Gandy-Harrington topology do not arise from any metrizable space, giving a separation between submetrizable and metrizable for classes of degrees.

Kihara, Ng and Pauly [1] suggested some candidates for classes that could be $T_{2.5}$ but not submetrizable. They introduced the Arens co-d-CEA degrees and the Roy halfgraph above degrees. Both classes arise from spaces that are $T_{2.5}$ but not submetrizable. Another candidate class introduced by Kihara, Ng and Pauly was the doubled co-d-CEA degrees. This class contains both the Arens co-d-CEA degrees and the Roy halfgraph degrees and arises from a space that is T_2 but not $T_{2.5}$. Kihara, Ng and Pauly asked if the doubled co-d-CEA degrees are a $T_{2.5}$ class or not.

In Section 5 we answer this question and show that the doubled co-d-CEA degrees are not $T_{2.5}$, giving a new example of a class that is T_2 but not $T_{2.5}$. As a result of this we have that the class of degrees that are Arens co-d-CEA or Roy halfgraph is a strict subset of the doubled co-d-CEA degrees. This separation is of interest in its own right because the doubled co-d-CEA degrees arise from a quasi-Polish space and the previous separation uses $\mathbb{N}^\omega_{\mathrm{rp}}$ which is not quasi-Polish.

In Section 6 we prove that the Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable answering the question of Kihara, Ng and Pauly about the distinction between $T_{2.5}$ and submetrizable. In the proof of these results we introduce a general method that could be used to get similar results. We also introduce the notion of a space being effectively submetrizable as part of the general method.

In Section 7 we look at the relationship between the Arens co-d-CEA degrees and the Roy halfgraph degrees and prove that neither of class is contained in the other. This answers another question of Kihara, Ng and Pauly [1].

Kihara, Ng and Pauly [1] extend the notion of quasi-minimal to give a strong way in which a class of enumeration degrees can be not $\mathcal{T}$. An enumeration degree $\mathbf{a}$ is quasi-minimal if it is not above any nonzero total degrees. The following extends this idea to general classes.

Definition 1.4. For a cb_0 space $\mathcal{X}$ we say that a degree $\mathbf{a} \in \mathcal{D}_e$ is $\mathcal{X}$ quasi-minimal if $\mathbf{a} \notin \mathcal{D}_{\mathcal{X}}$ and for all $\mathbf{b} \in \mathcal{D}_{\mathcal{X}}$ if $\mathbf{b} \leq \mathbf{a}$ then $\mathbf{b} = \mathbf{0}$.

For a class $\mathcal{C} \subseteq \mathcal{D}_e$ and a set of cb_0 spaces $\mathcal{T}$, we say that $\mathcal{C}$ is $\mathcal{T}$ quasi-minimal if for every $\mathcal{X} \in \mathcal{T}$ the is $\mathbf{a} \in \mathcal{C}$ such that $\mathbf{a}$ is $\mathcal{X}$ quasi-minimal.

Kihara, Ng and Pauly show that $\mathcal{D}_e$ is T_1-quasi-minimal and as a result any class of enumeration degrees that is downwards dense in the enumeration degrees (that is for each $\mathbf{a} >_e \mathbf{0}$ there is a $\mathbf{b}$ in our class such that $\mathbf{0} <_e \mathbf{b} \leq_e \mathbf{a}$), like the class of semicomputable degrees (introduced by Jockusch [8] and proven to be downwards dense by Kihara, Ng and Pauly [1]) and the class of enumeration-1-generic degrees (introduced by Badillo and Harris [9] and proven to be downwards dense by Badillo, Liliana, Harris and Soskova [10]), is also T_1-quasi-minimal. They also prove that the telegraph-cototal degrees, a T_1 class containing the doubled co-d-CEA degrees, are quasi-minimal for countable disjoint unions of effective T_2 spaces. A question they ask is if there is a quasi-minimal separation of T_2 and $T_{2.5}$.

In section 3 we modify the proof that the cylinder-cototal degrees are not T_2 to show that they are T_2-quasi-minimal. In section 4 we answer the above question by modifying the proof that $\mathcal{D}_{\mathbb{N}^{\omega}_{\mathrm{rp}}}$ is not $T_{2.5}$ to show that it is $T_{2.5}$ quasi-minimal.

An open question we ask is if there is a quasi-minimal separation of $T_{2.5}$ from submetrizable. We also ask if there is a quasi-minimal separation of submetrizable from metrizable. Miller [4] showed that no continuous degree is quasi-minimal. So a class is quasi-minimal for computable metric spaces if it contains a quasi-minimal degree. The situation becomes more complex when we consider non-decidable spaces as any enumeration degree can be coded into the presentation of a basis of a metrizable space.

In Section 8 we look at metrizable cb_0 spaces that have different bases than the basis of rational radius balls centered at rational points. We Show that there is a metrizable cb_0 space $\mathcal{X}$ such that $\mathcal{D}_{\mathcal{X}}$ contains all quasi-minimal doubled co-d-CEA degrees. As a result the doubled co-d-CEA degrees are not metrizable quasi-minimal. Hence neither are the Arens co-d-CEA or Roy halfgraph degrees. So a different $T_{2.5}$ space is needed if one wants to get a quasi-minimal separation of $T_{2.5}$ from submetrizable. In this section we also asked about the degrees of points in decidable, metrizable cb_0 spaces. We show that there is a decidable, metrizable cb_0 space whose degrees contain a quasi-minimal degree. Hence the class of degrees of points in decidable, metrizable cb_0 spaces is larger than the continuous degrees. We leave open the question of whether there is any degree that is not the degree of a point in a decidable, metrizable cb_0 space.

2 Preliminaries

In this sections we go over some background notions related to cb_0 spaces and enumeration reducibility that are needed for this paper. More specific definitions, for instance the definition of a particular class of degrees, will be given in the relevant section. The content of this section gives some background and should apply to the whole paper.

2.1 Enumeration and Medvedev degrees

Recall that $A \leq_e B$ if there is a uniform way of computing an enumeration of A from any enumeration of B. Another way of defining enumeration reducibility is in terms of enumeration operators. Given an effective listing $(W_e)_{e\in\omega}$ of c.e. sets we define the enumerations operators $(\Psi_e)_{e\in\omega}$ as follows:

$$\Psi_e(B) = \{n : \exists\langle n, u\rangle \in W_e[D_u \subseteq A\}$$

We have that $A \leq_e B$ if and only if there is some e such that $A = \Psi_e(B)$. We will use $\Psi_{e,s}$ to denote the enumeration operator corresponding to $W_{e,s}$. Here $(W_{e,s})_{s\in\omega}$ is the effective approximation of W_e with increasing finite sets. We have that $(\Psi_{e,s}(B))_{s\in\omega}$ is an approximation of $\Psi_e(B)$ with finite sets. The operators Ψ_e are monotonic in that $\Psi_e(A) \subseteq \Psi_e(B)$ if $A \subseteq B$ and correspond to continuous functions on $\mathbb{S}^\omega$.

Medvedev reducibility [11] is defined on subsets of ω^ω. For $P, Q \subseteq \omega^\omega$ we say $Q \leq_M P$ if there is a Turing operator Φ such that for every $p \in P$ we have that $\Phi(p) \in Q$. If we let E_A be the set of all enumerations of A then we have $A \leq_e B$ if an only if $E_A \leq_M E_B$.

2.2 Represented spaces

Represented spaces are a way of defining notions of computably for arbitrary spaces, using the notions of compatibility on ω^ω. These are a key tool in the study computable analysis [12].

Definition 2.1. A represented space $\mathcal{X}$ is a set X and a partial surjection $\delta :\subseteq \omega^\omega \to X$. Given a represented space $\mathcal{X} = (X, \delta)$ and a point $x \in X$ we say that a point $p \in \omega^\omega$ is a δ-*name* for x if $\delta(p) = x$. We define $\mathrm{Name}_{\mathcal{X}}(x) = \{p : \delta(p) = x\}$.

For an example one nonstandard representation of ω^ω is to say that p is a δ-name for $f \in \omega^\omega$ if $\mathrm{range}(p) = \{\langle n, m\rangle : f(n) \neq m\}$. In this example it is possible that a name for a point f does not compute f, even though it describes f uniquely.

For arbitrary represented spaces $\mathcal{X}, \mathcal{Y}$ Kihara and Pauly [5] define a reducibility notion of points $\leq_{\mathbf{T}}$, by $x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y} \iff \mathrm{Name}_{\mathcal{X}}(x) \leq_M \mathrm{Name}_{\mathcal{Y}}(y)$. For arbitrary represented spaces, the degrees of points in that space are Medvedev degrees. They study the degree spectra of represented spaces in [5].

For cb_0 spaces $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ there is a natural representation δ given by $\delta(p) = x$ if p is an enumeration of $\mathrm{NBase}(x)$. This is well defined because $\mathcal{X}$ is a T_0 space: if $x \neq y \in \mathcal{X}$ then without loss of generality there is some open U such that $x \in U, y \notin U$. So there is an e such that $x \in \beta_e \subseteq U$ and $y \notin \beta_e$ and hence $\mathrm{NBase}(x) \neq \mathrm{NBase}(y)$. The embedding of $\mathcal{D}_e$ into $\mathcal{D}_M$ means that cb_0 spaces $\mathcal{X}, \mathcal{Y}$ we have $x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y}$ if and only if $\mathrm{NBase}(x) \leq_e \mathrm{NBase}(y)$.

The example of a represented space above comes from a cb_0 space, the ω product of the cofinite topology, $(\omega_{\mathrm{cof}})^\omega$. Here a subbasis can be given as $\beta_{\langle n,m\rangle} = \{f : f(n) \neq m\}$ and the representation we used above was $\delta(p) = f$ if $\mathrm{range}(p) = \mathrm{NBase}_{(\omega_{\mathrm{cof}})^\omega}(f)$. For a cb_0 space we technically need a basis rather than a subbasis, however a subbasis can be turned to into a basis by taking finite intersections. Since $\{e : x \in \beta_e\} \equiv_e \{\sigma \in \omega^{<\omega} : x \in \beta_{\sigma(0)} \cap \cdots \cap \beta_{\sigma(|\sigma|-1)}\}$ using a subbasis rather than a basis will not change the degree of a point. In this paper we will sometimes specify and work with a cb_0 space in terms of a subbasis rather than a basis.

Our remark above about a $(\omega_{\mathrm{cof}})^\omega$-name for f not necessarily computing f can be stated as saying that while $f : (\omega_{\mathrm{cof}})^\omega \leq_{\mathbf{T}} f : \omega^\omega$ there are f such that $f : \omega^\omega \not\leq_{\mathbf{T}} f : (\omega_{\mathrm{cof}})^\omega$. The degrees of $(\omega_{\mathrm{cof}})^\omega$ are the degrees of complements of graphs of total functions. This class is know as graph cototal degrees [13, 3]. This class is known to be a proper subclass of the cototal degrees [3] and to contain non-total degrees.

As a tool in Sections 3, 4 and 5 we will make use of multi-representations [14]. The difference between a multi-representation and a single valued representation is that in a multi-representation

$\delta :\subseteq \omega^\omega \rightrightarrows X$ is a multi function. So a point $p \in \omega^\omega$ may be a name for more than one distinct point in X.

2.3 Computability of spaces and functions

For represented spaces $\mathcal{X}, \mathcal{Y}$ and a partial function $f :\subseteq \mathcal{X} \to \mathcal{Y}$ we say that a partial function $F :\subseteq \omega^\omega \to \omega^\omega$ is a realizer if $f(\delta_{\mathcal{X}}(p)) = \delta_{\mathcal{Y}}(F(p))$ for every $p \in \mathrm{dom}(f \circ \delta_{\mathcal{X}})$. We say that a function $f :\subseteq \mathcal{X} \to \mathcal{Y}$ is computable if it has a computable realizer.

For topological spaces there are several different notions of a computable representation. Since we are interested in the degrees of a space, we want a notion of compatibility that prevents coding non-computable information directly into the basis.

Definition 2.2. For a cb_0 space $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ we say $\mathcal{X}$ is *decidable* if the subset relation between positive Boolean combinations of $\emptyset$ and $(\beta_e)_{e\in\omega}$ is computable. We say $\mathcal{X}$ is *strongly decidable* if the subset relation between positive Boolean combinations of $\emptyset$, $(\beta_e)_{e\in\omega}$ and $(\overline{\beta_e})_{e\in\omega}$ is computable.

Many natural spaces are strongly decidable, for instance $2^\omega, \omega^\omega, \mathbb{S}^\omega, [0,1]^\omega$ and $(\omega_{\mathrm{cof}})^\omega$. In section 6 we will introduce another notion of compatibility for spaces—that of being effectively submetrizable.

3 The cylinder-cototal degrees are T_2-quasi-minimal

The cylinder-cototal degrees were introduced by Kihara, Ng and Pauly [1]. They are defined to be the degrees of the the cocylinder space $\omega^\omega_{\mathrm{co}} = (\omega^\omega, (\beta_e)_{e\in\omega})$ where $\beta_e = \{x \in \omega^\omega : \sigma_e \not\prec x\}$ for an effective enumeration $(\sigma_e)_{e\in\omega}$ of $\omega^{<\omega}$. This is a coarser topology than the usual one on ω^ω because $\beta_e = \bigcup\{[\sigma] : \sigma \in \omega^{|\sigma_e|}, \sigma \neq \sigma_e\}$ is open under the usual topology. The space is T_1 but not T_2. Kihara, Ng and Pauly [1] prove that the cylinder-cototal degrees are a subclass of the graph cototal degrees by embedding $\omega^\omega_{\mathrm{co}}$ into $(\omega_{\mathrm{cof}})^\omega$, the space of the graph cototal degrees. One of the reasons the cylinder-cototal degrees are interesting is that they give us a separation of T_1 and T_2 for classes of degrees.

Theorem 3.1 (Kihara, Ng, and Pauly [1])**.** *The cylinder-cototal degrees are not T_2.*

By modifying the proof of the above theorem we are able to turn this into a quasi-minimal separation.

Theorem 3.2. *The cylinder-cototal degrees are T_2-quasi-minimal.*

To prove their result, Kihara, Ng and Pauly prove two important lemmas involving Hausdorff spaces and network representations of spaces. To state these lemmas we need to introduce the terminology used.

Definition 3.3. Given a topological space $\mathcal{X}$, a point $x \in X$ and a collection $\mathcal{N} \subseteq \mathcal{P}(X)$ have the following.

- $\mathcal{N}$ is a *network* at x if for each open $U \ni x$ there is $N \in \mathcal{N}$ such that $x \in N \subseteq U$.
- $\mathcal{N}$ is a *strict network* at x it is a network at x and $x \in N$ for each $N \in \mathcal{N}$.

- $\mathcal{N}$ is a *cs-network* for $\mathcal{X}$ if for any convergent sequence $(x_n)_{n\in\omega}$ and open $U \ni \lim_n x_n$ there is $N \in \mathcal{N}$ and $m \in \omega$ such that $\{x_n : n > m\} \subseteq N \subseteq U$.

Given a space $\mathcal{X}$ and a countable cs-network $\mathcal{N} \subseteq \mathcal{P}(X)$, the representation of $\mathcal{X}$ from $\mathcal{N}$ is $\delta_{\mathcal{N}}$ where $\delta_{\mathcal{N}}(p) = x$ if $\{N_{p(e)} : e \in \omega\}$ is a strict network at x.

For some simple examples of cs-networks consider the following. If $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ is a cb_0 space then $(\beta_e)_{e\in\omega}$ is a cs-network. If $\mathcal{X}$ is regular then $(\overline{\beta_e})_{e\in\omega}$ is a cs-network.

A network representation does not necessarily give us a class of enumeration degrees like a cb_0 space does, but Kihara, Ng and Pauly [1] make the following observation that connects points in network representations with enumeration degrees.

Observation 3.4. *If $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ is a cb_0 space and $\mathcal{Y} = (Y, \mathcal{N})$ is a space with countable cs-network, then $y : \mathcal{Y} \leq_{\mathbf{T}} x : \mathcal{X}$ if and only if there is $J \leq_e \mathrm{NBase}_{\mathcal{X}}(x)$ such that $\{N_e : e \in J\}$ is a strict network at y.*

To prove Theorem 3.1 Kihara, Ng and Pauly [1] consider a different type of representation, they call the closure representation.

Definition 3.5. Given a space $\mathcal{X} = (X, \mathcal{N})$ with a countable cs-network $\mathcal{N}$, the *closure representation* of $\mathcal{X}$ is $\overline{\delta_{\mathcal{N}}}$ where $\overline{\delta_{\mathcal{N}}}(p) = x$ if $\{N_{p(e)} : e \in \omega\}$ is a network at x and $x \in \overline{N_{p(e)}}$ for all $e \in \omega$. For a point $x \in \mathcal{X}$ we say that x is *nearly computable* if there is a computable $\overline{\delta_{\mathcal{N}}}$ name for x. For a space $\mathcal{Y}$ and point $y \in \mathcal{Y}$ we say that y is *nearly $\mathcal{X}$-quasi-minimal* if

$$\forall x \in \mathcal{X}[x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y} \implies x \text{ is nearly computable}]$$

The main difference between the closure representation and the network representation is that the closure representation includes more names for a point x. So being nearly computable is a weaker notion that being computable as a point x may have a computable $\overline{\delta_{\mathcal{N}}}$ name, but no computable $\delta_{\mathcal{N}}$ name.

In general, these representations are not single valued; that is, a name $p \in \omega^\omega$ may be a name for multiple distinct points. The following observation by Kihara, Ng and Pauly [1] gives a condition for the closure representation to be single valued.

Observation 3.6. *If $\mathcal{X}$ is a T_2 space and $\mathcal{N}$ is a cs-network for $\mathcal{X}$ then $\overline{\delta_{\mathcal{N}}}$ is a single valued representation of $\mathcal{X}$.*

From this observation we can conclude that if $\mathcal{X}$ is a T_2 space and $\mathcal{N}$ is a countable network for $\mathcal{X}$ then there are only countably many points with a computable name as each computable $p \in \omega^\omega$ represents at most one point and there are only countably many computable $p \in \omega^\omega$.

Recall that a function $f \in \omega^\omega$ is *A-computably dominated* if there is an A-computable function g such that $f(n) \leq g(n)$ for all n. Given a space $\mathcal{X} = (X, \mathcal{N})$ the *disjointness diagram* of $\mathcal{X}$ is the set $\{(e, i) : N_e \cap N_i = \emptyset\}$. Now we can state the main lemma used to prove Theorem 3.1.

Lemma 3.7 (Kihara, Ng, Pauly [1])**.** *Let $f \in \omega^\omega$ be a function that is not C'-computably dominated. Then for any second countable $\mathcal{X} = (X, \mathcal{N})$ with C-c.e. disjointness diagram we have that $f : \omega^\omega_{\mathrm{co}}$ (viewing f as a point in $\omega^\omega_{\mathrm{co}}$) is nearly $\mathcal{X}$-quasi-minimal.*

Since we are modifying Kihara, Ng and Pauly [1]'s proof of Lemma 3.1, we will give it here.

Proof of Theorem 3.1. Let $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ be a T_2 cb_0 space. Let C be an oracle such that the disjointness diagram of $(\beta_e)_{e\in\omega}$ is C-c.e. By Lemma 3.7, if f is not C'-computably dominated, then $f : \omega^\omega_{\mathrm{co}}$ is nearly $\mathcal{X}$-quasi-minimal. Since X is T_2, there are only countably many points in $\mathcal{X}$ that are nearly C-computable by Observation 3.6. However, there are uncountably many functions which are not C'-computably dominated. Thus, one can choose a function which is not $\mathbf{T}$-equivalent to any nearly computable point in $\mathcal{X}$. $\square$

The above proof is close to a quasi-minimal separation. If $f \in \omega^\omega_{\mathrm{co}}$ is not C'-computably dominated then there are only countably many degrees in $\mathcal{D}_\mathcal{X}$ that might be below $\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)$. We now use forcing to avoid computing any of these degrees.

Lemma 3.8. *Given a set A and countable collection of non-c.e. sets $(C_i)_{i\in\omega}$ there is a function f such that f is not A-computably dominated and for each i we have $C_i \not\leq_e \mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)$.*

Proof. We construct f in stages with $f = \cup f_s$.

At stage $s = 2n$ we consider the nth A-partial computable function φ^A_n. If $\varphi^A_n(|f_s|)\downarrow$ then set $f_{s+1} = f_s \cup \{(|f_s|, \varphi^A_n(|f_s|) + 1)\}$, otherwise $f_{s+1} = f_s$. At stage $s = 2\langle e, i\rangle + 1$ we ask if there is $\sigma \succ f_s$ such that $\Psi_e(\{\tau : \tau \perp \sigma\}) \not\subseteq C_i$ then take $f_{s+1} = \sigma$ otherwise set $f_{s+1} = f_s$.

The even stages give us that f is not A-computably dominated. So we now only need to show f satisfies the other condition. Suppose that $C_i = \Psi_e(\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f))$. Let $s = 2\langle e, i\rangle + 1$. If there was $\sigma \succ f_s$ such that $\Psi_e(\{\tau : \tau \perp \sigma\}) \not\subseteq X_i$ then we would have $f \succ \sigma$ for one such σ and $\Psi_e(\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)) \not\subseteq C_i$. So there is no such σ and thus $C_i = \Psi_e(\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)) \subseteq \Psi_e(\{\tau : \tau \not\succeq f_s\}) \subseteq C_i$. Hence C_i is c.e., a contradiction. $\square$

Using this we can modify the proof Theorem 3.1 to get a quasi-minimal separation.

Proof of Theorem 3.2. Let $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ be a T_2 cb_0 space. Let C be an oracle such that the disjointness diagram of $(\beta_e)_{e\in\omega}$ is C-c.e. Let $(C_i)_{i\in\omega}$ be a listing of sets whose degrees are those of the non-computable nearly C-computable points in $\mathcal{X}$. By Observation 3.6 we know that we can find such a listing. From Lemma 3.8 we can find an f that is not C'-computably dominated and has $C_i \not\leq_e \mathrm{NBase}_{\mathrm{co}}(f)$ for any i. By the proof of Theorem 3.1 we have that f is nearly $\mathcal{X}$-quasi-minimal. Since f is not $\emptyset'$-computably dominated we have that $\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)$ is not c.e. and hence f is $\mathcal{X}$-quasi-minimal $\square$

4 A $T_{2.5}$-quasi-minimal class

In this section we look at the relatively prime integer topology. This topology is defined as follows. Let $\mathbb{Z}_+$ be the set of positive integers. The basic open sets in this topology are $\{a + b\mathbb{Z} : \gcd(a, b) = 1\}$. We write $\mathbb{N}_{\mathrm{rp}} = (\mathbb{Z}_+, \{a + b\mathbb{Z} : \gcd(a, b) = 1\})$ for the cb_0 space. It is known that $\mathbb{N}_{\mathrm{rp}}$ is second countable, T_2 and not $T_{2.5}$ [15].

Proposition 4.1. *$\mathbb{N}_{\mathrm{rp}}$ is strongly decidable, and the sets $a + b\mathbb{Z}, \overline{a + b\mathbb{Z}}$ are uniformly computable.*

Proof. Given a finite collection of basic open sets $(a_0 + b_0\mathbb{Z}), \ldots, (a_{n-1}, b_{n-1}\mathbb{Z})$ let $b_n = \prod_{i<n} b_i$. For each $k < n$ we can uniformly compute $c_{k,0} < \cdots < c_{k,n_k-1} < b_n$ such that $a_k + b_k\mathbb{Z} = \bigcup_{i<n_k} c_{k,i} + b_n\mathbb{Z}$. So we have that the subset relationships between Boolean combinations of $((a_0 + b_0\mathbb{Z}), \ldots (a_{n-1}, b_{n-1}\mathbb{Z}), \emptyset)$ is the same as the subset relationships between Boolean combinations of $(\{c_{0,i} : i < n_0\}, \ldots \{c_{n-1,i} : i < n_{n-1}\}, \emptyset)$ and is hence decidable uniformly in

$a_0, \dots, a_{n-1}, b_0 \dots, b_{n-1}$. Hence the subset relationships between Boolean combinations of $(\{a+b\mathbb{Z} : \gcd(a,b) = 1\}, \emptyset)$ is decidable.

To see that $\mathbb{N}_{\text{rp}}$ is strongly decidable, we first show that for any a, b, c, d (no relatively prime assumptions) $(a + b\mathbb{Z}) \cap (c + d\mathbb{Z}) \neq \emptyset$ if and only if $a \equiv c \mod \gcd(b, d)$. If $a \equiv c \mod \gcd(b, d)$ then $a = c + k\gcd(b, d)$ for some k, so $a = c + k(nd - mb)$ for some m, n. So $a + kmb = c + knd \in (a + b\mathbb{Z}) \cap (c + d\mathbb{Z})$. On the other hand $a + bm \equiv a \mod \gcd(b, d)$ and $c + dn \equiv c \mod \gcd(b, d)$ so if $a \not\equiv c \mod \gcd(b, d)$ then $(a + b\mathbb{Z}) \cap (c + d\mathbb{Z}) = \emptyset$.

So $(a + b\mathbb{Z}) \cap (c + d\mathbb{Z}) \neq \emptyset$ if and only if $(a + b\mathbb{Z}) \cap (c + \gcd(b, c)\mathbb{Z}) \neq \emptyset$. Let $F = \{0 \leq c < b : (\forall d \mid b)[\gcd(c, d) > 1 \vee (c + d\mathbb{Z}) \cap (a + b\mathbb{Z}) \neq \emptyset\}$. Then we have that $\overline{a + b\mathbb{Z}} = \bigcup_{c \in F} c + b\mathbb{Z}$. So we get that $\overline{a + b\mathbb{Z}}$ is uniformly computable and using the same argument as we did to show $\mathbb{N}_{\text{rp}}$ is decidable we can show that $\mathbb{N}_{\text{rp}}$ is strongly decidable. $\square$

Kihara, Ng and Pauly[1] proved that the notions of T_2 and $T_{2.5}$ are distinct for classes of enumeration degrees.

Theorem 4.2. *$\mathcal{D}_{(\mathbb{N}_{\text{rp}})^\omega}$ is not $T_{2.5}$.*

In this section we modify the proof of the above theorem show that there are T_2 classes of degrees that are $T_{2.5}$-quasi-minimal.

Theorem 4.3. *$\mathcal{D}_{(\mathbb{N}_{\text{rp}})^\omega}$ is $T_{2.5}$-quasi-minimal.*

The idea behind the proof of Theorem 4.2 is similar to that or Theorem 3.1, but instead of looking at the closure representation, Kihara, Ng and Pauly introduce a new representation.

Definition 4.4. Given a space $\mathcal{X} = (X, \mathcal{N})$ with countable cs-network, we define the representation $\widetilde{\delta_\mathcal{N}}$ where $\widetilde{\delta_\mathcal{N}}(p) = x$ if $\{N_{p(e)} : e \in \omega\}$ is a network at x and $\overline{N_{p(e)}} \cap \overline{N_{p(i)}} \neq \emptyset$ for all $e, i \in \omega$. For a point $x \in \mathcal{X}$ we say that x is *$\widetilde{*}$-nearly computable* if the is a computable $\widetilde{\delta_\mathcal{N}}$ name for x. For a space $\mathcal{Y}$ and point $y \in \mathcal{Y}$ we say that y is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal if

$$\forall x \in \mathcal{X}[x : \mathcal{X} \leq_\mathbf{T} y : \mathcal{Y} \implies x \text{ is } \widetilde{*}\text{-nearly computable}]$$

A $\overline{\delta_\mathcal{N}}$-name for a point x is a $\widetilde{\delta_\mathcal{N}}$-name for x, but a $\widetilde{\delta_\mathcal{N}}$-name for a point x may not be $\overline{\delta_\mathcal{N}}$-name. As we will see $\widetilde{\delta_\mathcal{N}}$ is not nessisarily single valued on T_2 spaces, but Kihara, Ng and Pauly observed that it is single valued on $T_{2.5}$ spaces.

Observation 4.5. *If $\mathcal{X}$ is a $T_{2.5}$ space and $\mathcal{N}$ is a cs-network for $\mathcal{X}$ then $\widetilde{\delta_\mathcal{N}}$ is a single valued representation of $\mathcal{X}$.*

Rather than working directly with $\mathbb{N}_{\text{rp}}$ Kihara, Ng and Pauly use the fact that $\mathbb{N}_{\text{rp}}$ is countable and nowhere $T_{2.5}$ and prove results about an arbitrary countable nowhere $T_{2.5}$ space $\mathcal{H}$. We say $\mathcal{H}$ is nowhere $T_{2.5}$ if for all open $U, V \subseteq \mathcal{H}$ we have that $\overline{U} \cap \overline{V} \neq \emptyset$. It is known that $\mathbb{N}_{\text{rp}}$ is nowhere $T_{2.5}$ [15]. If $\mathcal{H} = (\omega, (H_e)_{e\in\omega})$ is nowhere $T_{2.5}$ then a *witness for being nowhere $T_{2.5}$* is a set $\Lambda \subseteq \omega^3$ such that $H_e, H_d \neq \emptyset$ then the set $\Lambda_{e,d} = \{(e, d, n) \in \Lambda\}$ is nonempty and $\Lambda_{e,d} \subseteq \overline{H_e} \cap \overline{H_d}$.

Lemma 4.6 (Kihara, Ng, Pauly). *Let $\mathcal{H} = (\omega, (H_e)_{e\in\omega})$ be a represented, second countable space with c.e. witness for being nowhere $T_{2.5}$ and let $x \in \omega^\omega$ be 1-C-generic. Then for any space $\mathcal{Y} = (Y, \mathcal{N})$ which is strongly decidable relative to C and $\mathcal{N}$ is a cs-network, we have $x : \mathcal{H}^\omega$ is $\widetilde{*}$-nearly $\mathcal{Y}$-quasi-minimal.*

In the case of $\mathbb{N}_{\text{rp}}$ Proposition 4.1 tells us that there is a computable witness that $\mathbb{N}_{\text{rp}}$ is nowhere $T_{2.5}$ namely $\Lambda = \{(n, \langle a, b\rangle, \langle c, d\rangle) : n \in \overline{a + b\mathbb{Z}} \cap \overline{c + d\mathbb{Z}}\}$. So the lemma can be applied here.

Proof of Theorem 4.2. Let $\mathcal{X} = (X, \mathcal{N})$ be a $T_{2.5}$ space. Let C be an oracle such that $\mathcal{X}$ is strongly decidable relative to C. By Lemma 4.6, for any 1-C-generic point $x \in \omega^\omega$ we have that $x : (\mathbb{N}_{\text{rp}})^\omega$ is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal. By Observation 4.5 since $\mathcal{X}$ is a $T_{2.5}$-space, there are only countably many points in $\mathcal{X}$ that are $\widetilde{*}$-nearly computable. However, there are uncountably many points in ω^ω which are 1-C-generic. Thus, one can choose such a point which is not $\equiv_{\mathbf{T}}$-equivalent (in terms of $(\mathbb{N}_{\text{rp}})^\omega$) to any $\widetilde{*}$-nearly computable points in $\mathcal{X}$. $\square$

In the above proof we use a counting argument to separate the two classes. Using forcing we can get a stronger result of $\mathcal{X}$-quasi-minimality.

Lemma 4.7. *Given a countable* cb_0 *space* $\mathcal{H} = (\omega, (H_e)_{e\in\omega})$*, a countable collection of non-c.e. sets* $(X_i)_{i\in\omega}$ *and set* C*, there is a 1-*C*-generic set* $x \in \omega^\omega$ *such that* $\mathbf{0} <_e \text{NBase}_{\mathcal{H}^\omega}(x)$ *and* $X_i \not\leq_e \text{graph}(x)$ *for each* i*.*

Proof. We will use forcing to construct x in stages with $x = \bigcup_s x_s$. We fix $H_e \neq \omega, \emptyset$ and points $a \in H_e, b \notin H_e$.

At stage $s = 3n$ let W_n be the nth c.e. set. If $\langle |x_s|, e\rangle \in W_n$ then set $x_{s+1} = x_s{}^\frown b$ otherwise $x_{s+1} = x_s{}^\frown a$. This ensures that $W_n \neq \text{NBase}_{\mathcal{H}^\omega}(x)$.

At stage $s = 3n + 1$ let V_n be the nth C-c.e. subset of $\omega^{<\omega}$. If there is $\sigma \in V_n$ such that $x_s \prec \sigma$ then set $x_{s+1} = \sigma$ otherwise set $x_{s+1} = x_s$.

At stage $s = 3\langle e, i\rangle + 2$ let Ψ_e be the eth enumeration operator. Ask if there is a number $n \notin X_i$ and $\sigma \succ x_s$ such that $n \in \Psi_e(\text{graph}(\sigma))$. If yes then set $x_{s+1} = \sigma$, otherwise $x_{s+1} = x_s$.

At stages $3n$ we ensured that $\text{NBase}_{\mathcal{H}^\omega}(x)$ is not c.e. and at stages $3n + 1$ we ensure that x is 1-C-generic. Now we need to show that $X_i \not\leq_e x$ for any i. Suppose that $X_i = \Psi_e(\text{graph}(x))$. Then let $s = 3\langle e, i\rangle+2$. If there is an extension $\sigma \succ x_s$ that $\Psi_e(\text{graph}(\sigma)) \nsubseteq X_i$ then we would have $x \succ \sigma$ for some such σ and $\Psi_e(\text{graph}(x)) \supseteq \Psi_e(\text{graph}(\sigma)) \nsubseteq X_i$, so it must be that $\Psi_e(\text{graph}(\sigma)) \subseteq X_i$ for all $\sigma \succ x_s$. So we have $X_i = \Psi_e(x) \subseteq \{n : \exists\sigma \succ x_s, n \in \Psi_e(\text{graph}(\sigma))\} \subseteq X_i$. So we have that X_i is c.e., a contradiction. $\square$

Now we can replace the counting argument used in the proof of Theorem 4.2 to get the quasi-minimal separation.

Proof of Theorem 4.3. Let $\mathcal{X} = (X, (\beta_e)_e)$ be a $T_{2.5}$ space. Let C be an oracle such that $\mathcal{X}$ is strongly decidable relative to C. Let $(X_i)_{i\in\omega}$ be a list of non-c.e. sets with enumeration degrees those of the non-computable $\widetilde{*}$-nearly computable points in $\mathcal{X}$. Since the open sets of $\mathbb{N}_{\text{rp}}$ are uniformly computable we have that $x : (\mathbb{N}_{\text{rp}})^\omega \leq_{\mathbf{T}} x : \omega^\omega$ for all x, so by Lemma 4.7 there is a 1-C-generic x such that $x : (\mathbb{N}_{\text{rp}})^\omega$ is non-computable and $z : \mathcal{X} \leq_{\mathbf{T}} x : (\mathbb{N}_{\text{rp}})^\omega$ means that $z : \mathcal{X}$ is computable or $z : \mathcal{X}$ is not $\widetilde{*}$-nearly-computable. But from Lemma 4.6 we have that $x : (\mathbb{N}_{\text{rp}})^\omega$ is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal, so $x : (\mathbb{N}_{\text{rp}})^\omega$ is $\mathcal{X}$-quasi-minimal. $\square$

5 The doubled co-d-CEA degrees

In this section we show that the doubled co-d-CEA degrees are not $T_{2.5}$. Kihara, Ng and Pauly [1] introduced the doubled co-d-CEA degrees as the degrees of points in the product of the double origin topology $\mathcal{DO}^\omega$. Rather than working directly with this topology it is easier for us to work with the characterization in terms of sets that they came up with.

Definition 5.1. A set $X = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$ is doubled co-d-CEA if $(A \cup B)^c, P, N$ are Y-c.e. and A, B, P, N are disjoint. A degree is doubled co-d-CEA if it contains a doubled co-d-CEA set.

Since the doubled origin space is T_2 the doubled co-d-CEA degrees are a T_2 class. Question 4 of the open questions asked by Kihara, Ng and Pauly [1] is if these degrees are a proper T_2 class. We show that they are a proper T_2 class.

Theorem 5.2. *The doubled co-d-CEA degrees are not $T_{2.5}$.*

Proof. We will make use of the $\widetilde{*}$-name concept from section 4 again in this proof.

Consider some $T_{2.5}$ cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$. Let Y be such that $\mathcal{X}$ is strongly Y-decidable. We will build coinfinite Y-c.e. sets $P, N \subseteq C$ such that for any partition $A \sqcup B = C^c$ we have that the doubled co-d-CEA degree $Y \oplus Y^c \oplus (A \cup P) \oplus (B \oplus N)$ does not enumerate the name of any non $\widetilde{*}$-nearly-Y'-computable $x \in X$. By doing this we will have constructed a size continuum class of doubled co-d-CEA degrees, only countably many of which can contain the name of a point in $\mathcal{X}$. Hence we will have shown that $\mathcal{D}_{\mathcal{DO}^\omega} \not\subseteq \mathcal{D}_{\mathcal{X}}$.

Let $\mathcal{Q} = \{(C_q, P_q, N_q) : P_q \cap N_q = \emptyset \wedge P_q, N_q \subseteq C_q \subseteq_{\text{fin}} \omega\}$. For $p, q \in \mathcal{Q}$, $u \in \omega$ and $a \subseteq u$ we define the following:

- $q \preceq p$ if $C_q \supseteq C_p, P_q \supseteq P_p, N_q \supseteq N_p$.
- $q \preceq_u p$ (q extends p above u) if $q \preceq p$ and $C_q{\restriction}u = C_p{\restriction}u, P_q{\restriction}u = P_p{\restriction}u, N_q{\restriction}u = N_p{\restriction}u$.
- $a \lhd_u p$ (a is a p-compatible choice of $A{\restriction}u$) if $a \subseteq u \setminus C_p$.
- $p(a, u) = Y \oplus Y^c \oplus (a \cup P_p) \oplus (u \setminus (C_p \cup a) \cup N_p)$

Note that if $a \lhd_u p$ and $q \preceq_u p$ then $a \lhd_u q$ and $p(a, u) \subseteq q(a, v)$ for any $v \geq u$. This does not necessarily hold if $q \preceq p$.

We will build a Y-computable sequence $q_0 \succeq q_1 \succeq \dots$ and have $C = \bigcup_s C_{q_s}$, $P = \bigcup_s P_{q_s}$ and $N = \bigcup_s N_{q_s}$. This will ensure that C, P, N are Y-c.e. and $P \sqcup N \subseteq C$. The requirements $\mathcal{R}_e$ are that for any partition $A \sqcup B = C^c$ we have that if $\Psi_e(Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)) = \mathrm{NBase}(x)$ for some $x \in X$ then x is $\widetilde{*}$-nearly-Y'-computable.

The strategy for $\mathcal{R}_e$ works as follows. Each requirement has a restriction u and when it sets q_{s+1} it needs to ensure that $q_{s+1} \preceq_u q_s$. If at stage s the value of u is undefined then let $u = \max(C_{q_s}) + 2$. If at some later stage t a higher priority requirement acts and we have $q_t \not\preceq_u q_s$ then we consider $\mathcal{R}_e$ injured and u to be undefined. $\mathcal{R}_e$ needs to be able to handle any partition of $u \setminus C$ so for each $a \lhd_u q_s$ we create a new subrequirement $\mathcal{R}_e^a$. If $\mathcal{R}_e$ is injured then we remove these subrequirments. $\mathcal{R}_e^a$ is satisfied if $\mathcal{R}_e$ is satisfied for each partition $A \sqcup B = C^c$ with $A{\restriction}u = a$.

The idea of the strategy for $\mathcal{R}_e^a$ is as follows. We consider the potential point x which is named by Ψ_e of some partition extending a. We first wait until a stage where we see a way to force $x \in \beta_{n_0}$ and a separate way to force $x \in \beta_{n_1}$ for some n_0, n_1 with $\overline{\beta_{n_0}} \cap \overline{\beta_{n_1}} = \emptyset$. We then put everything above u into C and injure lower priority requirements so that we always have the option to put x in β_{n_0} or β_{n_1} without changing other facts about x. The next step is to wait until we see way to put $x \in \beta_m$ for some β_m disjoint from β_{n_i} for some i. We then put x in both β_m and β_{n_i}. If we get past the waiting step then we will be able to ensure that there is no potential point x and satisfy the requirement that way. If we are stuck at the waiting step forever then we will show that x is close to computable.

The details of the strategy for $\mathcal{R}_e^a$ use states w, c, d and work as follows.

- State w: we wait until a stage r where we see some $p_0, p_1 \preceq_u q_r$ with $n_0 \in \Psi_e(p_0(a,u))$ and $n_1 \in \Psi_e(p_1(a,u))$ such that $\overline{\beta_{n_0}} \cap \overline{\beta_{n_1}} = \emptyset$. Then we set $q_{r+1} = (C_q \cup [u,r), P_{q_r}, N_{q_r})$ and injure all lower priority requirements.

- State c: we wait until a stage $v > r$ where we see some $q \preceq_r q_v$ with $m \in \Psi_e(q(a,u))$ such that $\beta_{n_0} \cap \beta_m = \emptyset$ or $\beta_{n_1} \cap \beta_m = \emptyset$. In the first case we set $q_{v+1} = (C_q, P_q \cup P_{p_0}, N_q \cup N_{p_0})$ and in the second case we set $q_{r+1} = (C_q, P_q \cup P_{p_1}, N_q \cup N_{p_1})$. All lower priority requirements are injured, along with $\mathcal{R}_e^b$ requirements that are in state c. This requirement moves to state d.

- State d: the requirement is considered finished and cannot be injured by fellow $\mathcal{R}_e^b$ requirements.

This completes the construction of C, P, N. Now we move onto the verification.

Claim 5.2.1. *Each requirement is injured only finitely often.*

Proof. If a requirement $\mathcal{R}_e$ is never injured after stage s then it acts only once more to split into the $\mathcal{R}_e^a$ requirements. Suppose that an $\mathcal{R}_e^a$ is never injured by higher priority requirements after stage s.

If $\mathcal{R}_e^a$ is in state c then either $\mathcal{R}_e^a$ is injured by an $\mathcal{R}_e^b$ requirement and moves back to state w or it acts once and moves to state d, where it never acts again and can no longer be injured by other $\mathcal{R}_e^b$ requirements. So each $\mathcal{R}_e^a$ requirement acts finitely often from state c.

Since each $\mathcal{R}_e^b$ requirement can act only finitely often in state c, and there are only finitely many of these requirements, we can let $t > s$ be a stage after which all $\mathcal{R}_e^b$ will not act in state c. If $\mathcal{R}_e^a$ is in c or d then it will never again act. If $\mathcal{R}_e^a$ is in state w then $\mathcal{R}_e^a$ will not be injured at any later stage and will act at most once more to move into state c. □

Claim 5.2.2. *C^c is infinite.*

Proof. Let s be the last stage where $\mathcal{R}_e$ was injured. We have that if u is the restriction chosen by $\mathcal{R}_e$ at stage $t > s$ then $e \leq |C^c \restriction u|$. This follows from induction and the fact that $\max(C_{q_t}) + 1 < u$ means that $\max(C_{q_t}) + 1 \notin C_{q_j}$ for any $j \geq t$. □

Claim 5.2.3. *Each $\mathcal{R}_e$ is satisfied.*

Proof. Consider some partition $A \sqcup B = C^c$. Let $Q = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$ and fix e. We will show that $\mathcal{R}_e$ is satisfied for Q. Let s be the last stage where a subrequirement $\mathcal{R}_e^b$ changes its state. There is some subrequirement $\mathcal{R}_e^a$ such that $Q \restriction u = (q_s(a,u)) \restriction u$. Note that for any t we have $q_t(a,u) \subseteq Q$. Let l be the last state that $\mathcal{R}_e^a$ is in. We will look at the three cases.

- $l = d$: when we entered state d at stage s we ensured that there was $m, n \in \Psi_e(Q)$ such that $\beta_m \cap \beta_n = \emptyset$. So $\Psi_e(Q)$ is not the $\mathcal{X}$-name of a point in $\mathcal{X}$.

- $l = c$: Let $r > s$ be the stage when $\mathcal{R}_e^a$ moves to state c for the last time. Suppose that $\mathrm{NBase}(x) = (\Psi_e(Q))$ for some $x \in X$. Then since $\overline{\beta_{n_0}}$ and $\overline{\beta_{n_1}}$ are disjoint there must be $m \in \mathrm{NBase}(x)$ and $i \in 2$ such $\beta_m \cap \beta_{n_i} = \emptyset$. Since $m \in \Psi_e(Q)$ there are some finite $D \subseteq A \cup P, E \subseteq B \cup N$ and $t > s$ such that $m \in \Psi_{e,t}(Y \oplus Y^c \oplus D \oplus E)$. Consider $q = (C_{q_t} \cup D \setminus u \cup E \setminus u, P_{q_t} \cup D \setminus u, N_{q_t} \cup E \setminus u)$. We have that $m \in \Psi_{e,t}(q(a,u))$ and $q \preceq_r q_t$ since $[u,r) \subseteq C$. Thus at stage t we could have used q to enter state d, a contradiction.

- $\underline{l = w}$ Suppose that $\text{NBase}(x) = \Psi_e(Q)$ for some $x \in X$. We will show that x is $\widetilde{*}$-nearly-Y'-computable. Consider the set

  $$J = \{n : n \in \Psi_{e,t}(Y \oplus Y^c \oplus (D \cup P) \oplus (E \cup N)) \text{ for some } t > s, D \sqcup E \subseteq_{\text{fin}} C^c \text{ with } a = D{\restriction}u\}$$

  Note that $J \leq_e Y'$ because C, P, N are Y-c.e. We claim that J is a $\widetilde{\delta_\beta}$-name for x. Suppose not. Then since $J \supseteq \text{NBase}(x)$ there must be some $n_0, n_1 \in J$ such that $\overline{\beta_{n_0}} \cap \overline{\beta_{n_1}} = \emptyset$. Since $n_i \in J$ there is $t_i > s, D \sqcup E \subseteq_{\text{fin}} C^c$ with $a = D{\restriction}u$ such that $n_i \in \Psi_{e,t_i}(Y \oplus Y^c \oplus (D \cup P_{q_t}) \oplus (E \cup N_{q_t}))$. Consider $p_i = (C_{q_t} \cup D \setminus u \cup E \setminus u, P_{q_t} \cup D \setminus u, N_{q_t} \cup E \setminus u)$. We have that $p_i \preceq_u q_t$ and $n_i \in \Psi_{e,t}(p_i(a, u))$. Then at stage $\max(t_0, t_1)$ we would have used p_0, p_1 to move to state c, a contradiction.

  So J is a $\widetilde{\delta_\beta}$-name for x and hence x is $\widetilde{*}$-nearly-Y'-computable.

$\square$

Since the requirements are satisfied the construction works and we have a class of continuum many doubled co-d-CEA degrees $\mathcal{C}$ such that $\mathcal{C} \cap \mathcal{D}_\mathcal{X}$ is a countable set, and hence $\mathcal{D}_{\mathcal{DO}^\omega} \not\subseteq \mathcal{D}_\mathcal{X}$. Since $\mathcal{X}$ was an arbitrary $T_{2.5}$ cb_0 space we have that the doubled co-d-CEA degrees are not $T_{2.5}$. $\square$

6 Separating $T_{2.5}$ classes from submetrizable classes

In this section we give our main result: there are $T_{2.5}$ classes of degrees that are not submetrizable. We do this for two example classes, the Arens co-d-CEA degrees and the Roy halfgraph above degrees. Kihara, Ng and Pauly [1] show that both these classes are $T_{2.5}$ as they arise from the decidable $T_{2.5}$ spaces $\mathcal{QA}^\omega$ and $\mathcal{QR}^\omega$ respectively. We will give formal definitions of these classes later in this section. The definitions of the spaces $\mathcal{QA}^\omega$ and $\mathcal{QR}^\omega$ can be found in [1]. For now we will go over the general method that is used to prove these separations.

6.1 General method

A submetrizable space arises by adding open sets some underlying metric space, however a cb_0 submetrizable space does not tell us which sets are open under the metric. We would like an effective way to find a name for a point with respect to the underlying metric space from a name for a point with respect to the submetrizable space. We can do this with many natural examples, but it is not always possible. This motivates the following definition.

Definition 6.1. We call a submetrizable cb_0 space $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ *effectively submetrizable* if there is a continuous, injective function $f : \mathcal{X} \to [0,1]^\omega$ such that $\text{NBase}_{[0,1]^\omega}(f(x)) \leq_e \text{NBase}_\mathcal{X}(x)$.

If $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ is a computable metric space and $\mathcal{Y} = (X, (\beta_e)_{e\in\omega} \sqcup (\alpha_e)_{e\in\omega})$ is submetrizable then $\mathcal{Y}$ is effectively submetrizable. If one looks at the examples of submetrizable cb_0 spaces in [1] they will see that these spaces are all effectively submetrizable, even the non-decidable cb_0 spaces like the Gandy-Harrington topology. Thus every enumeration degree is an $\mathcal{X}$-degree for some decidable, effectively submetrizable cb_0 space $\mathcal{X}$.

Since $[0,1]^\omega$ is universal for second countable metrizable spaces for any submetrizable cb_0 space $\mathcal{X}$ there is an oracle Y such that $\mathcal{X}$ is Y-effectively submetrizable.

Miller [4] proved that there is no quasi-minimal continuous degree. So if x is a point in an effectively submetrizable cb_0 space $\mathcal{X}$ and f is a witness that $\mathcal{X}$ is effectively submetrizable, then $\mathrm{NBase}_{\mathcal{X}}(x)$ is quasi-minimal implies that $\mathrm{NBase}_{[0,1]^\omega}(f(x))$ is c.e. since we have $\mathrm{NBase}_{[0,1]^\omega}(f(x)) \leq_e \mathrm{NBase}_{\mathcal{X}}(x)$, so we can conclude that $\mathcal{X}$ has only countably many quasi-minimal degrees.

The above is one way of showing a class is not effectively submetrizable, but it does not help us in the case of the Arens co-d-CEA and Roy halfgraph degrees as we do not know if they contain uncountably many quasi-minimal degrees. By looking more closely at the total degrees below a continuous degree we come up with our method of separation.

Definition 6.2. A countable class $\mathcal{S} \subseteq 2^\omega$ is a *Scott set* if it is closed under join, Turing reducability and for any $X \in \mathcal{S}$ and nonempty $\Pi^0_1(X)$ class G there is $Y \in \mathcal{S} \cap G$. The collection $\{\deg_T(X) : X \in \mathcal{S}\}$ is called a *Scott ideal.*

So every Scott ideal contains PA degrees. In fact, for any $Y \in \mathcal{S}$ we have that $\mathcal{S}$ contains a set that is PA relative to Y.

Theorem 6.3 (J. Miller [4])**.** *If* $\mathbf{v}$ *is a non-total continuous degree then the set* $\{\mathbf{b} <_e \mathbf{v} : \mathbf{b}$ *is total*$\}$ *is a Scott ideal. Notably, there is total* $\mathbf{b} <_e \mathbf{v}$ *such that* $\mathbf{b}$ *is a PA degree.*

Now we have the tools we need to prove the following

Lemma 6.4. *If* $\mathcal{C}$ *is an uncountable class of enumeration degrees and* $\mathcal{B}$ *is a countable class of non PA total degrees such that for any* $\mathbf{a} \in \mathcal{C}$ *we have* $\{\mathbf{b} \in \mathcal{D}_T : \mathbf{b} \leq_e \mathbf{a}\} \subseteq \mathcal{B}$*, then* $\mathcal{C} \nsubseteq \mathcal{D}_{\mathcal{X}}$ *for any effectively submetrizable* cb_0 *space* $\mathcal{X}$*.*

Proof. Take $\mathcal{C}$ as in the statement of the theorem. Let $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ be an effectively submetrizable cb_0 space, with witness f. We will show that $\mathcal{C} \cap \mathcal{D}_{\mathcal{X}}$ is countable.

Consider some $x \in X$. Suppose that $\mathrm{NBase}_{\mathcal{X}}(x) \in \mathbf{a}$ for some $\mathbf{a} \in \mathcal{C}$. So we have that $\mathrm{NBase}_{[0,1]^\omega}(f(x)) \leq_e \mathbf{a}$. Since $\mathbf{a}$ does not bound any PA degrees we have that $\mathrm{NBase}_{[0,1]^\omega}(f(x))$ has total degree. So we have that that $\deg_e(\mathrm{NBase}_{[0,1]^\omega}(f(x))) \in \mathcal{B}$. Since f is injective, $\mathrm{NBase}_{[0,1]^\omega}(f(x))$ uniquely determines x and there are only countably many $x \in \mathcal{X}$ such that $\deg_e(\mathrm{NBase}_{\mathcal{X}}(x)) \in \mathcal{C}$. So $\mathcal{C} \cap \mathcal{D}_{\mathcal{X}}$ is countable, and hence $\mathcal{C} \nsubseteq \mathcal{D}_{\mathcal{X}}$. □

Now we relativize to get the result for arbitrary submetrizable spaces.

Theorem 6.5. *Suppose that for each* $Y \subseteq \omega$*, we have an uncountable class of enumeration degrees* $\mathcal{C}^Y$ *and* $\mathcal{B}^Y$ *a countable class of non* Y*-PA total degrees, such that for any* $\mathbf{a} \in \mathcal{C}$ *we have* $Y \oplus Y^c \leq_e \mathbf{a}$ *and* $\{\mathbf{b} \in \mathcal{D}_T : \mathbf{b} \leq_e \mathbf{a}\} \subseteq \mathcal{B}^Y$*, then* $\bigcup_Y \mathcal{C}^Y \nsubseteq \mathcal{D}_{\mathcal{X}}$ *for any submetrizable* cb_0 *space* $\mathcal{X}$*.*

Proof. Let $\mathcal{X} = (X, (\beta_e)_{e\in\omega})$ be a submetrizable space. Let $f : X \to [0,1]^\omega$ be a continuous injection. Let $Y = \{\langle n, m\rangle : \beta_n \subseteq f^{-1}[\alpha_m]\}$ where $(\alpha_e)_{e\in\omega}$ is the standard basis on $[0,1]^\omega$. So for any $x \in \mathcal{X}$ we have that $\mathrm{NBase}_{[0,1]^\omega}(x) \leq_e Y \oplus Y^c \oplus \mathrm{NBase}_{\mathcal{X}}(x)$. We will show that the $\mathcal{C}^Y \nsubseteq \mathcal{D}_{\mathcal{X}}$.

Consider some $x \in X$. Suppose that $\mathrm{NBase}_{\mathcal{X}}(x) \in \mathbf{a}$ for some $\mathbf{a} \in \mathcal{C}^Y$. So we have that $Y \oplus Y^c \leq_e \mathrm{NBase}_{\mathcal{X}}(x)$ and hence $\mathrm{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c \leq_e \mathbf{a}$. If $\mathrm{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$ is not a total degree, then since it is a continuous degree and $Y \oplus Y^c \leq_e \mathrm{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$, by Theorem 6.3 there is some total degree $\mathbf{b} \leq \mathrm{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$ such that $\mathbf{b}$ is PA relative to Y. So it must be that $\mathrm{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$ is total and hence $\deg_e(\mathrm{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c) \in \mathcal{B}^Y$. Since $\mathrm{NBase}_{[0,1]^\omega}(x)$ uniquely determines x and the downward closure of $\mathcal{B}^Y$ is countable there are only countably many $x \in \mathcal{X}$ such that $\deg_e(\mathrm{NBase}_{\mathcal{X}}(x)) \in \mathcal{C}^Y$. So $\mathcal{C}^Y \cap \mathcal{D}_{\mathcal{X}}$ is countable and hence $\mathcal{C}^Y \nsubseteq \mathcal{D}_{\mathcal{X}}$. □

Now that we have a method to prove classes are not submetrizable, we can use it on some classes to get new separations.

6.2 Arens co-d-CEA degrees

Now we show that the Arena co-d-CEA degrees are not submetrizable. The Arens co-d-CEA degrees were introduced in [1] and are the degrees of points in a $T_{2.5}$ space. By proving that the Arens co-d-CEA degrees are not submetrizable we prove that the notions of $T_{2.5}$ and submetrizable are distinct for classes of enumeration degrees.

Definition 6.6. A degree is Arens co-d-CEA if it contains a set of the form:

$$Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$$

Where $(A_0 \cup A_1)^c, N, P_0, P_1, M$ are Y-c.e. A_0, A_1, N are disjoint, $P_0, P_1, M \subseteq N$ are pairwise disjoint, and there is a partition $N_0 \sqcup N_1 = N$ such that N_0, N_1 are Y-c.e. and $P_0 \subseteq N_0, P_1 \subseteq N_1$.

When referring to an Arens co-d-CEA set (or subset) we will often use the notation $L \oplus R \oplus Z$ to keep the track of the different columns of $(A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$. By definition L, R, Z are disjoint and the $L \oplus R$ part is a doubled co-d-CEA set by itself. The Z part is of co-d-c.e. degree and keeps track of numbers that have been added to $(A_0 \cup A_1)^c$ but not to $P_0 \cup P_1$. In order for some number in $(A_0 \cup A_1)^c$ to not show up in Z it must appear in N. This means the number is in N_0 or N_1, so if the number is also in $P_0 \cup P_1$ then we know which of P_0, P_1 contains it. The set M gives us a way of adding numbers in N back into Z.

We will introduce some notation that will help us keep track of the different c.e. sets when constructing an Arens co-d-CEA degree, or class of degrees. Let

$$Q = \{(C^q, P^q = P^q_0 \sqcup P^q_1, N^q = N^q_0 \sqcup N^q_1, M^q) : P^q_i \subseteq N^q_i, M^q \subseteq N^q \subseteq C^q, M^q \cap P^q = \emptyset\}$$

Here C^q is meant to represent the c.e. set $(A_0 \cup A_1)^c$. For $p, q \in Q$, $u \in \omega$ and $a = a_0 \sqcup a_1 \subseteq u$ we define the following:

- $q \preceq p$ if $C^q \supseteq C^p, P^q_i \supseteq P^p_i, N^q_i \supseteq N^p_i, M^q \supseteq M^p$.
- $q \preceq_u p$ (q extends p above u) if $q \preceq p$ and $C^q{\restriction}u = C^p{\restriction}u, P^q{\restriction}u = P^p{\restriction}u, N^q{\restriction}u = N^p{\restriction}u, M^q{\restriction}u = M^p{\restriction}u$.
- $a \lhd_u q$ (a is a q compatible choice of $A{\restriction}u$) if $a = u \setminus C^q$.
- $q(a) = (a_0 \cup P^q_0) \oplus (a_1 \cup P^q_1) \oplus (C^q \setminus N^q \cup M^q)$.
- q is considered *u-robust* if $C^q \setminus N^q \subseteq u$ and $C^q \setminus u$ is an interval.

Note that we can have $q \preceq p$ but $p(a) \nsubseteq q(a)$. This is why we need the notion of a condition being u-robust. Note that for a u-robust p if $a \lhd_u p$ and $q \preceq_u p$ then $a \lhd_u q$ and $p(a) \subseteq q(a)$. Note also that for every condition $p \in Q$ and $u \in \omega$ there is $q \preceq_u p$ such that $p(a) = q(a)$ and q is u-robust.

Theorem 6.7. *The Arens co-d-CEA degrees are not submetrizable.*

Proof. We use Lemma 6.5 and show there are c.e. sets $C, P = P_0 \sqcup P_1, N = N_0 \sqcup N_1, M$ such that C^c is infinite and for any partition $A_0 \sqcup A_1 = C^c$ we have that $X = (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus (C \setminus N \cup M)$ has Arens co-d-CEA degree and if any $f \leq_e X$ is the graph of a total function, then $f \leq_T \mathbf{0}'$ and $\deg_T(f)$ is not PA.

Fix a nonempty Π^0_1 class G where each $x \in G$ has PA degree, and fix a computable tree T with $[T] = G$.

We will build a computable sequence $q_0 \succeq q_1 \succeq \dots$ of $q_i \in Q$ and have $C = \bigcup_s C^{q_s}$, $P = \bigcup_s P^{q_s}$, $N = \bigcup_s N_{q_s}$ and $M = \bigcup_s M^{q_s}$. This will ensure that C, P, N are c.e. and will produce a Arens co-d-CEA set for any partition of C^c. The requirements $\mathcal{R}_e$ are that for any partition $A_0 \sqcup A_1 = C^c$ we have that if $f = \Psi_e((A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus (C \setminus N \cup M))$ is the graph of a total function then $f \leq_T \emptyset'$ and $f \notin G$.

The strategy for $\mathcal{R}_e$ works as follows. If at stage s, $\mathcal{R}_e$ is initialized then let $u_e = \max(C^{q_s}) + 2$. If at some later stage t we have $q_t \not\preceq_{u_e} q_s$ then we consider $\mathcal{R}_e$ injured and will reinitialize it. For each $a \lhd_{u_e} q_s$ we create a new subrequirement $\mathcal{R}_e^a$. If $\mathcal{R}_e$ is injured then we remove these subrequirments. $\mathcal{R}_e^a$ is satisfied if $\mathcal{R}_e$ is satisfied for each partition $A_0 \sqcup A_1 = C^c$ with $A_i \restriction u = a_i$. We add an interval consisting of the $\mathcal{R}_e^a$ to the order of requirements placing the interval just below $\mathcal{R}_e$ in priority. Initially the restriction u_e^a given to each $\mathcal{R}_e^a$ is u_e but this may increase if $\mathcal{R}_e^e$ is injured by a higher priority $\mathcal{R}_e^b$. When $\mathcal{R}_e^a$ is injured, we set $u_e^a = \max(C^{q_s}) + 1$.

The strategy for each $\mathcal{R}_e^a$ has states g, w, c, n, d. Initially they are in state g. Let $u = u_e^a$, the actions for each state are as follows.

- State g: we start with $n = 0$ and $\sigma_0 = \emptyset$. If at some stage s we see some $x \in \omega$ and u-robust $q \preceq_u q_s$ such that $\langle n, x\rangle \in \Psi_{e,s}(q(a))$ then we set $q_{s+1} = q$ and injure all lower priority requirements. If $\sigma_n{}^\frown x \notin T$ then we go to state w otherwise we remain in state g and set $n = n+1, \sigma_{n+1} = \sigma_n{}^\frown x$.

- State w: we wait until at some stage s we see $m, x_0, x_1 \in \omega$ and a u-robust pair $r_0, r_1 \preceq_u q_s$ such that $\langle m, x_i\rangle \in \Psi_{e,s}(r_i(a))$ and $x_0 \neq x_1$. We set $q_{s+1} = (C^{q_s} \cup C^{r_0} \cup C^{r_1}, P^{q_s}, N^{q_s}, M^{q_s})$. Let $v = \max(C^{q_{s+1}}) + 1$. All lower priority requirements are injured and we enter state c.

- State c: we wait until we see a stage t such that for some v-robust $p \preceq_v q_t$ we have $\langle m, x_2\rangle \in \Psi_{e,t}(p(a))$ for some x_2. Pick i such that $x_i \neq x_2$. Set $q_{t+1} = (C^p, P^p, N^p \cup N^{r_i}, M^p)$ (N^{r_i} and N^p do not conflict because $p \preceq_v q_{s+1}$). Note that now we have $q_{t+1}(a)\restriction s \subseteq r_i(a), p(a)$. Set $o = \max(C^{q_{t+1}}) + 1$ and enter state c.

- State n: we wait until we see a stage ℓ such that for some o-robust $h \preceq_o q_t$ we have $\langle m, y\rangle \in \Psi_{e,t}(v(a))$ for some y. If $y \neq x_i$ then set $q_{\ell+1} = (C^h, P^h \cup P^{r_i}, N^h, M^h \cup M^{r_i})$ and move into state d. Otherwise $y \neq x_2$ and we set $q_{\ell+1} = (C^h, P^h, N^h, M^h \cup v \setminus (u \cup P^h))$

- State d: in this state $\mathcal{R}_e^a$ is considered satisfied.

This completes the construction of C, P, N, M. Now we move onto the verification.

Claim 6.7.1. *Each requirement is injured only finitely often.*

Proof. If a requirement $\mathcal{R}_e$ is never injured after stage s then it acts only once more to split into the $\mathcal{R}_e^a$ requirements.

Suppose that an $\mathcal{R}_e^a$ is never injured by higher priority requirements after stage s. The first case we need to deal with is if $\mathcal{R}_e^a$ remains in state g and injures lower priority requirements infinitely often. Each time it acts n increases, so we have that $f = \bigcup_n \sigma_n \in 2^\omega$ and f is computable. Since every $\sigma \prec f$ is in T we have that $f \in G$, but this is a contradiction as no PA degree is computable. So $\mathcal{R}_e^a$ acts only finitely often in state g.

In states w, c, n the requirement acts at most once, so there is a stage after which $\mathcal{R}_e^a$ stops injuring lower priority requirements. □

For each requirement $\mathcal{R}_e$ let u_e be the restriction that is given to $\mathcal{R}_e$ after the last time it is injured.

Claim 6.7.2. $C^c = \{u_e - 1 : e \in \omega\}$

Proof. Because the restriction u_e is defined to be $\max(C^{q_s}) + 2$ we have that $u_e - 1 \notin C^{q_s}$. Since lower priority requirements only work above u_e we have that $u_e - 1 \notin C$. On the other hand, whenever a subrequirment $\mathcal{R}_e^a$ acts it makes sure that $[u_e^a, \max(C^{q_s})] \subseteq C^{q_s}$. So we have that $[u_e^a, u_{e+1} - 1) \subseteq C$. □

Claim 6.7.3. *Each $\mathcal{R}_e$ is satisfied.*

Proof. Consider some partition $A \sqcup B = C^c$. Let $X = (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus (C \setminus N \cup M)$ and fix e. We will show that $\mathcal{R}_e$ is satisfied for X. Let s be the last stage where a subrequirement $\mathcal{R}_e^b$ changes its state. There is some subrequirement $\mathcal{R}_e^a$ such that $X \upharpoonright u_e^a = q_s(a) \upharpoonright u_e^a$. Note that for any t we have $q_t(a) \subseteq X$. Let l be the last state that $\mathcal{R}_e^a$ is in and $u = u_e^a$. We will look at the four cases.

- $\underline{l = d}$: when we entered state d at stage t we ensured that $\langle m, y\rangle, \langle m, x_i\rangle \in \Psi_{e,t}(q_{t+1}(a)) \subseteq \Psi_e(X)$ with $y \neq x_i$, so $\Psi_e(X)$ is not the graph of a total function.

- $\underline{l = g}$: consider the last value n takes. We know from Claim 6.9.1 that $\mathcal{R}_e^a$ acts finitely often, so n is finite. Suppose that $n \in \mathrm{dom}(\Psi_e(X))$. Then there are some finite $L \oplus R \oplus Z \subseteq X$ and $t > s$ such that $n \in \mathrm{dom}(\Psi_{e,t}(L \oplus R \oplus Z))$. Consider

  $$q = (C^{q_t} \cup L \setminus u \cup R \setminus u \cup Z \setminus u, (P_0^{q_t} \cup L \setminus u) \sqcup (P_1^{q_t} \cup R \setminus u), N^{q_t} \cup L \setminus u \cup R \setminus u \cup Z \setminus u, M^{q_t} \cup Z \setminus u)$$

  We have that $q \prec_u q_t$ and $n \in \mathrm{dom}(\Psi_{e,t}(q(a)))$. So at stage t we would have set $q_{t+1} = q$, a contradiction. So $n \notin \mathrm{dom}(\Psi_e(X))$ and thus $\Psi_e(X)$ is not a total function.

- $\underline{l = c}$: Since $\mathcal{R}_e^a$ is never injured after stage s we have a fixed collection m, r_0, r_1 such that $r_i \not\preceq q_t$ for any t. Furthermore, because of the restriction imposed by $\mathcal{R}_e^a$ we have that for all $t > s$ we have $N^{r_i} \setminus u \subseteq C^{q_t} \setminus N^{q_t} \subseteq [u, v)$. Suppose that $m \in \mathrm{dom}(\Psi_e(X))$. Then, like in the previous case, there are some finite $L \oplus R \oplus Z \subseteq X$ and $t > s$ such that $m \in \mathrm{dom}\Psi_{e,t}(L \oplus R \oplus Z)$. Let $z = \max(L \cup R \cup Z)$. Consider

  $$q = (C^{q_t} \cup [v, z], (P_0^{q_t} \cup L \setminus v) \sqcup (P_1^{q_t} \cup R \setminus v), N^{q_t} \cup [v, z], M^{q_t} \cup Z \setminus v)$$

  We have that $q \preceq_v q_t$ is v-robust and $m \in \mathrm{dom}(\Psi_{e,t}(q(a)))$. Thus at stage t we could have used q to enter state n, a contradiction. So $m \notin \mathrm{dom}(\Psi_e(X))$, and thus $\Psi_e(X)$ is not a total function.

- $\underline{l = n}$: Since $\mathcal{R}_e^a$ is never injured after stage s we have a fixed collection m, r_i, p such that $r_i, p \not\preceq q_t$ for any t. Like before, we have that for all $t > s$ we have $N^p \setminus v \subseteq C^{q_t} \setminus N^{q_t} \subseteq [v, o)$. Suppose that $m \in \mathrm{dom}(\Psi_e(X))$. Then, like in the previous case, there are some finite $L \oplus R \oplus Z \subseteq X$ and $t > s$ such that $m \in \mathrm{dom}\Psi_{e,t}(L \oplus R \oplus Z)$. Let $z = \max(L \cup R \cup Z)$ Consider

  $$q = (C^{q_t} \cup [o, z], (P_0^{q_t} \cup L \setminus v) \sqcup (P_1^{q_t} \cup R \setminus v), N^{q_t} \cup [o, z], M^{q_t} \cup Z \setminus v)$$

  We have that $q \preceq_o q_t$ is o-robust and $m \in \mathrm{dom}(\Psi_{e,t}(q(a)))$. Thus at stage t we could have used q to enter state n, a contradiction. So $m \notin \mathrm{dom}(\Psi_e(X))$, and thus $\Psi_e(X)$ is not a total function.

- $\underline{l = w}$ Suppose that $f = \Psi_e(X)$ is a total function. We will show that $f \notin G$ and that $f \leq_T 0'$. Since we left state g there is some n such that $f\restriction n+1 \notin T$, so $f \notin G$. To compute $f(m)$ from $\mathbf{0}'$ search for a stage $t > s$ and u-robust $r \preceq_u q_t$ such that $m \in \mathrm{dom}(\Psi_{e,t}(r(a)))$ and $r \not\leq q_k$ for all k. $\mathbf{0}'$ can carry out this search, and since $m \in \mathrm{dom}(\Psi_e(X))$ the search will halt. We claim that $f(m) = \Psi_e(r(a))(m)$. Suppose not. Since $m \in \mathrm{dom}(\Psi_e(X))$ there are some finite $L \oplus R \oplus Z \subseteq X$ and $k > t$ such that $m \in \mathrm{dom}(\Psi_{e,k}(L \oplus R \oplus Z))$. Let $z = \max(L \cup R \cup Z)$ Consider

$$q = (C^{q_t} \cup [u, z], (P_0^{q_t} \cup L \setminus u) \sqcup (P_1^{q_t} \cup R \setminus u), N^{q_t} \cup [u, z], M^{q_t} \cup Z \setminus u)$$

We have that q is u-robust and $q \not\leq q_k$ for all k. But we also have $r \not\leq q_k$ for all k So at some stage j we would have entered state c using robust extensions of q and r along with m, $f(m)$ and $\Psi_e(r(a))(m)$. This is a contradiction, so $f(m) = \Psi_e(r(a))(m)$ and hence $f \leq_T 0'$.

□

We can relativize this construction to get a relativizable class as in the statement of Theorem 6.5, and then apply the theorem to get that the Arens co-d-CEA degrees are not submetrizable. □

6.3 Roy halfgraph degrees

Now we give another example of a class that is $T_{2.5}$ by not submetrizable. The Roy halfgraph degrees were introduced in [1] and are defined as follows.

Definition 6.8. Define $\tilde{\omega} = \omega \cup \{-1, \infty\}$ For a function $f : \omega \to \tilde{\omega}$ we define

$$\mathrm{HalfGraph}(f) = \{\langle n, m\rangle \in \omega : f(n) \in \omega \wedge f(n) = 2m\} \oplus \{\langle n, m\rangle \in \omega : f(n) \in \omega \wedge f(n) \geq 2m\}$$

$$\mathrm{HalfGraph}^+(f) = \{\langle n, m\rangle \in \omega : f(n) \leq 2m\} \oplus \{\langle n, m\rangle \in \omega : f(n) \geq 2m\}$$

We say that f is Y*-computably dominated* if there is a Y-partial computable φ such that for all n we have $f(n) \in \omega \implies \varphi(n)\downarrow \geq f(n)$.

We say a degree $\mathbf{d}$ is *Roy halfgraph above* if it contains a set of the form

$$Y \oplus Y^c \oplus \mathrm{HalfGraph}^+(f)$$

where f is Y-computably dominated and $\mathrm{HalfGraph}(f)$ is Y c.e.

Kihara, Ng and Pauly [1] show that Roy halfgraph graph degrees are the degrees of the product space of Roy's lattice space $\mathcal{QR}^\omega$, a $T_{2.5}$ space which is not submetrizable. Now we prove that this class of degrees do not arise from any submetrizable cb_0 space.

Theorem 6.9. *The Roy halfgraph degrees are not submetrizable.*

Proof. Given a partial function $\psi :\subseteq \omega \to \omega$ we define a *Roy extension of* ψ to be a total function $f : \omega \to \tilde{\omega}$ such that $f^{-1}(\omega) = \mathrm{dom}(\psi)$.

We will build a partial function $\psi :\subseteq \omega \to \omega$ such that $\mathrm{HalfGraph}(\psi)$ is c.e., ψ is computably dominated, $\mathrm{dom}(\psi)$ is coinfinite and for any Roy extension f of ψ we have if $h \leq_e \mathrm{HalfGraph}^+(f)$ then $h \leq_T \mathbf{0}'$ and $\deg_T(h)$ is not PA.

Now we consider what enumeration operators on the halfgraph of a function look like. If we see $2\langle n, m\rangle \in \mathrm{HalfGraph}^+(f)$ then we know that $f(n) \leq 2m$ and if we see $2\langle n, m\rangle+1 \in \mathrm{HalfGraph}^+(f)$

then we know that $f(n) \geq 2m$. So an enumeration of $\mathrm{HalfGraph}^+(f)$ can be viewed as a refinement of even ended intervals containing f. This is the idea behind the following notation.

Let I be the set of all closed intervals in $\tilde{\omega}$ with end points in $\{2n : n \in \omega\} \cup \{-1, \infty\}$. Let $(\alpha_v)_{v\in\omega}$ be an effective listing of all functions $\alpha : \omega \to I$ such that $\alpha(n) = \tilde{\omega}$ for all but finitely many n. For $f : \omega \to \tilde{\omega}$ define $\Phi_e(f) = \{n : \exists\langle x, v\rangle \in W_e[\forall n(f(n) \in \alpha_v(n))]\}$. For any f we have that $X \leq_e \mathrm{HalfGraph}^+(f)$ if and only if $X = \Phi_e(f)$ for some e. For $f :\subseteq \omega \to \tilde{\omega}$, note that if $n \in \Phi_e(f)$ via α then there is $g : \omega \to \omega$ such that $f(m) = g(m)$ for all $m \in f^{-1}(\omega)$ and $n \in \Phi_e(g)$ via α.

We will use a finite injury construction to create partial functions ψ, φ such that $\mathrm{HalfGraph}(\psi)$ is c.e., φ is partial computable and φ dominates ψ.

We will build ψ and φ in stages and use the following notation. Let $Q = \{(q, p) : q, p :\subseteq \omega \to \omega \wedge \mathrm{dom}(q) = \mathrm{dom}(p) \subseteq_{\text{fin}} \omega \wedge \forall n \in \mathrm{dom}(q)[q(n) \leq p(n)]\}$. For $(q_0, p_0), (q_1, p_1) \in Q$, $u \in \omega$ and $a : u \to \tilde{\omega}$ we define the following:

- $(q_0, p_0) \preceq (q_1, p_1)$ if $\mathrm{HalfGraph}(q_0) \supseteq \mathrm{HalfGraph}(q_1)$ and $p_0 \supseteq p_1$.
- $(q_0, p_0) \preceq_u (q_1, p_1)$ ((q_0, p_0) extends (q_1, p_1) above u) if $(q_0, p_0) \preceq (q_1, p_1)$, $q_0\restriction u = q_1\restriction u$ and $\mathrm{dom}(q_0) \setminus u$ is an interval.
- $a \lhd_u (q, p)$ if $a : u \to \tilde{\omega}$ and $\mathrm{HalfGraph}(a) = \mathrm{HalfGraph}(q\restriction u)$.

Note that if $a \lhd_u (q_1, p_1)$ and $(q_0, p_0) \preceq_u (q_1, p_1)$ then $a \lhd_u (q_0, p_0)$ and $\Phi_e(a \cup q_1) \subseteq \Phi_e(a \cup q_0)$. Note that if $(q_0, p_0) \preceq (q_1, p_1)$ and $q_1(n) = 2m$ then $q_0(n) = 2m$, but if $q_1(n) = 2m + 1$ then $q_0(n)$ can be any number in $[2m, p_1(n)]$.

For $\alpha : \omega \to I$ and $a \lhd_u (q, p)$ we say $a, (q, p) \Vdash_u f \in \alpha$ if for all n we have that if $\alpha(n) \neq \tilde{\omega}$ then $(a \cup q)(n) \in \alpha(n)$ and in addition if $n \geq u$ then $q(n) \in 2\mathbb{Z}$. We say $a, (q, p) \Vdash_u f \notin \alpha$ if for all $(q_0, p_0) \preceq_u (q, p)$ we have $a, (q_0, p_0) \nVdash_u f \in \alpha$. Note that because even values of q cannot change in extensions if $(q_0, p_0) \preceq_u (q_1, p_1)$ and $a, (q_1, p_1) \Vdash_u f \in \alpha$ then $a, (q_0, p_0) \Vdash_u f \in \alpha$.

We will build a computable sequence $(q_0, p_0) \succeq (q_1, p_1) \succeq \ldots$ and have $\psi = \lim_n q_n$ with $\varphi = \bigcup_n p_n$. This will ensure that $\mathrm{HalfGraph}(\psi)$ is c.e. and ψ is computably dominated by φ. Fix a Π^0_1 class G such that for each $x \in G$ $\deg_T(x)$ is PA and a computable tree T such that $[T] = G$. The requirements $\mathcal{R}_e$ are that for any Roy extension f of ψ we have that if $h = \Phi_e(f)$ is the graph of a total function then $h \leq_T \emptyset'$ and $h \notin G$.

The strategy for $\mathcal{R}_e$ works as follows. If at stage s $\mathcal{R}_e$ is initialized then let $u = \max(\mathrm{dom}(q_s))+2$. If at some later stage t we have $(q_t, p_t) \npreceq_u (q_s, p_s)$ then we consider $\mathcal{R}_e$ injured and will reinitialize it. For each $a \lhd_u (q_s, p_s)$ we create a new subrequirement $\mathcal{R}^a_e$. We put an order on these subrequirements and add them to the list of all requirements as an interval just below $\mathcal{R}_e$ in priority. If $\mathcal{R}_e$ is injured then we remove these subrequirements. $\mathcal{R}^a_e$ is satisfied if $\mathcal{R}_e$ is satisfied for each Roy extension f of ψ with $a \subseteq f$.

Each $\mathcal{R}^a_e$ has states g, w, c, d and a bound u_a. Initially they are in state g and have $u_a = u$. Each time $\mathcal{R}^a_e$ is injured by a higher priority requirement we set $u_a = \max(\mathrm{dom}(q_s)) + 1$ and return to state g. When an $\mathcal{R}^a_e$ sets q_s we will have that $[u_a, \max(\mathrm{dom}(q_s))] \subseteq \mathrm{dom}(q_s)$ so that lower priority $\mathcal{R}^b_e$ can work with a higher restriction than u_a without the need to split into more subrequirements. The actions for each state are as follows.

- State g: we start with $n = 0$ and $\sigma_0 = \emptyset$. If at some stage s we see some x, v such that $\langle\langle n, x\rangle, v\rangle \in W_{e,s}$ and we see $(q_{s+1}, p_{s+1}) \preceq_{u_a} (q_s, p_s)$ such that $a, (q_{s+1}, p_{s+1}) \Vdash_{u_a} f \in \alpha_v$. We injure all lower priority requirements. If $\sigma_n{}^\frown x \notin T$ then we go to state w otherwise we remain in state g and set $n = n + 1, \sigma_{n+1} = \sigma_n{}^\frown x$.

- State w: we wait until at some stage s we see $m, x_0, x_1, v_0, v_1 \in \omega$ such that $\langle\langle m, x_i\rangle, v_i\rangle \in W_{e,s}$ and $a, (q_s, p_s) \nVdash_{u_a} f \notin \alpha_{v_i}$ for $i \in 2$. Chose a condition $(q,p) \preceq_{u_a} q_s$ such that for all $n \geq u_a$ the following hold.

  1. If $\alpha_{v_0}(n) \cap \alpha_{v_1}(n) \neq \emptyset$ then $q(n) \in \alpha_{v_0}(n) \cap \alpha_{v_1}(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_{v_i}(n)) < \min(\alpha_{v_{1-i}}(n))$ then $q(n) = \max(\alpha_{v_i}(n)) + 1$ and $p(n) \geq \min(\alpha_{v_{1-i}})$

  So we get that $q(n) \in \alpha_{v_0}(n) \iff q(n) \in \alpha_{v_1}(n)$. Set $(q_{s+1}, p_{s+1}) = (q, p)$. If $a, (q,p) \Vdash_{u_a} f \in \alpha_{v_0}, \alpha_{v_1}$ then move to state d. Otherwise move to state c. Let $r = \max(\mathrm{dom}(q) + 1$. Note that $a, (q_s, p_s) \nVdash_{u_a} f \notin \alpha_{v_i}$ but $a, (q,p) \Vdash_r f \notin \alpha_{v_i}$.

- State c: we wait until we at some stage t we see a pair $x, v \in \omega$ such that $\langle\langle m, x\rangle, v\rangle \in W_{e,t}$ and $a, (q_t, p_t) \nVdash_r f \notin \alpha_v$. Pick i such that $x_i \neq x$. Choose a condition $(q,p) \preceq_{u_a} (q_s, p_s)$ such that the following all hold.

  1. If $\alpha_{v_i}(n) \cap \alpha_v(n) \neq \emptyset$ then $q(n) \in \alpha_{v_i}(n) \cap \alpha_v(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_v(n)) < \min(\alpha_{v_i}(n))$ then $q(n) = \max(\alpha_v(n)) + 1$.

  We set $(q_{t+1}, p_{t+1}) = (q, p)$. If $a, (q,p) \Vdash_{u_a} f \in \alpha_v, \alpha_{v_i}$ then we will move to state d. Otherwise we will remain in state c. If we remain, then note that there is n such that $q(n) = \max(\alpha_v(n)) + 1$. Since $\max(\alpha_{v_{1-i}}) < q_t(n) \in \alpha_v(n)$ we now have $a, (q,p) \Vdash_{u_a} f \notin \alpha_{v_{1-i}}$. We redefine $x_{1-i} = x, v_{1-i} = v, r = \max(\mathrm{dom}(q) + 1$.

- State d: in this state $\mathcal{R}_e^a$ is considered satisfied.

This completes the construction of ψ and φ. Now we move onto the verification.

Claim 6.9.1. *Each requirement is injured only finitely often.*

Proof. If a requirement $\mathcal{R}_e$ is never injured after stage s then it acts only once more to split into the $\mathcal{R}_e^a$ requirements. Suppose that $\mathcal{R}_e^a$ is never injured after stage s. Then there is a stage after which the state of $\mathcal{R}_e^a$ remains the same. In states w and d it is clear $\mathcal{R}_e^a$ can act at most once. We now look at the other two states.

First, suppose that $\mathcal{R}_e^a$ remains in state g and injures lower priority requirements infinitely often. Each time it acts n increases, so we have that $h = \bigcup_n \sigma_n \in 2^\omega$ and h is computable. Since every $\sigma \prec h$ is in T we have that $h \in G$, but this is a contradiction as no PA degree is computable.

Second, suppose that $\mathcal{R}_e^a$ remains in state c and injures lower priority requirements infinitely often. Let $n_0 \ldots n_{k-1}$ be the values where $q_s(n) \notin \alpha_{v_i}(n)$ for $i \in 2$. Each time $\mathcal{R}_e^a$ acts the collection $n_0 \ldots n_{k-1}$ either stays the same or decreases. So there is a stage after which this collection is fixed. Consider one of these n. At each stage t where $\mathcal{R}_e^a$ acts we have that $q_t(n) \in \alpha_v(n)$ so $q_{t+1}(n) > q_t(n)$. This can only happen finitely often, a contradiction. □

Claim 6.9.2. $\mathrm{dom}(\psi)$ *is coinfinite.*

Proof. Let s be the last stage where $\mathcal{R}_e$ was injured. We have that if u is the restriction chosen by $\mathcal{R}_e$ at stage $t > s$, then $e = |\mathrm{dom}(\psi)^c \restriction u|$. This follows by induction and from the fact that $\max(\mathrm{dom}(q_t)) + 1 < u$ means that $\max(\mathrm{dom}(q_t)) + 1 \notin \mathrm{dom}(q_j)$ for any $j \geq t$. □

Claim 6.9.3. *Each $\mathcal{R}_e$ is satisfied.*

Proof. Consider some Roy extension f of ψ. Fix e. We will show that $\mathcal{R}_e$ is satisfied for f. There is some subrequirement $\mathcal{R}_e^a$ such that $f{\restriction}u_a = a \cup \psi{\restriction}u_a$. Note that for any t we have $\Phi_e(a \cup q_t) \subseteq \Phi_e(f)$. Let s be a stage such that $\mathcal{R}_e^a$ is never injured after this stage. Let l be the last state that $\mathcal{R}_e^a$ is in. We will look at the four cases.

- $\underline{l = d}$: when we entered state d at stage we ensured that $\langle m, x_0\rangle, \langle m, x_1\rangle \in \Phi_{e,t}(a \cup q_{t+1}) \subseteq \Phi_e(f)$ for $x_0 \neq x_1$ so $\Phi_e(f)$ is not a function.

- $\underline{l = g}$: consider the last values n takes. We know from Claim 6.9.1 that $\mathcal{R}_e^a$ acts finitely often, so n is finite. Suppose that $n \in \mathrm{dom}(\Phi_e(f))$. Then there are some $x, v \in \omega$ and $t > s$ such that $\langle\langle n, x\rangle, v\rangle \in W_{e,t}$ and $f(m) \in \alpha_v(m)$ for all m. But then there is $(q, p) \preceq_{u_a} (q_t, p_t)$ such that $a \cup q = f{\restriction}\{n : \alpha_v(n) \neq \tilde{\omega}\}$. Define $q'(n) = q(n) - 1$ if $q(n)$ is odd and $n \geq u_a$, and $q'(n) = q(\alpha_v)$ otherwise. We have $(q', p) \preceq_{u_a} (q_t, p_t)$ and $a, (q', p) \Vdash f \in \alpha_v$. Thus the requirement would be able to act again using (q', p), a contradiction.

- $\underline{l = c}$: Let $t - 1$ be the last stage when $\mathcal{R}_e^a$ acts. So from stage t onward we have fixed m, r and $(q_k, p_k) \preceq_r (q_t, p_t)$ for all $k \geq t$. Suppose that $m \in \mathrm{dom}(\Phi_e(f))$ then like in the case above there is some stage k and $(q, p) \preceq_r (q_k, p_k)$ such that $m \in \mathrm{dom}(\Phi_{e,k}(a \cup q))$. But then $\mathcal{R}_e^a$ would have acted again at stage k, a contradiction.

- $\underline{l = w}$: Suppose that $h = \Phi_e(f)$ is a total function. We will show that $h \notin G$ and that $h \leq_T 0'$. Since we left state g there is some n such that $h{\restriction}n + 1 \notin T$, so $h \notin G$. To compute $h(m)$ search for a stage $t > s$ and $(q, p) \preceq_{u_a} q_t$ such that $m \in \mathrm{dom}(\Phi_{e,t}(a \cup q))$ and $\psi{\restriction}\mathrm{dom}(q) = q$. $\mathbf{0'}$ can carry out this search since ψ is $\mathbf{0'}$ computable, and $m \in \mathrm{dom}(\Phi_e(f))$ so the search will halt. We claim that $f(m) = \Phi_e(a \cup q)$.

  Suppose not. Let α_v be the witness that $m \in \mathrm{dom}(\Phi_{e,t}(a \cup q))$. Since $m \in \mathrm{dom}(\Phi_e(f))$ there is $j \geq t$ and $(q_1, p_1) \preceq_{u_a} (q_j, p_j)$ such that $\langle m, f(m)\rangle \in \Phi_{e,j}(a \cup q_1)$ via some witness α_{v_1} and $\psi{\restriction}\mathrm{dom}(q_1) = q_1$. So for all k we have that $(q_k, p_k) \nVdash f \notin \alpha_v$,$(q_k, p_k) \nVdash f \notin \alpha_{v_1}$. But we would have used α_v and α_{v_1} to enter state c, a contradiction. So $f(m) = \Phi_e(a \cup q)$ and hence $f \leq_T 0'$.

$\square$

By meeting all $\mathcal{R}_e$ requirements we have ensured ψ has the desired properties. By relativizing this construction we can find classes of Roy halfgraph degrees $\mathcal{C}^Y$ as in the statement of theorem 6.5. So the Roy halfgraph degrees are not submetrizable. $\square$

7 Arens co-d-CEA degrees and Roy halfgraph degrees above

We have seen that the Arens co-d-CEA degrees and the Roy halfgraph above degrees are both examples of classes which are $T_{2.5}$ but not submetrizable. In this section we look at the relationship between these classes. Kihara, Ng and Pauly [1] showed that both of these classes contain the co-d-CEA degrees, and that the Roy halfgraph degrees are a subclass of the doubled co-d-CEA degrees. We show that the Arens co-d-CEA degrees are also a subclass of the doubled co-d-CEA degrees.

Proposition 7.1. *Every Arens co-d-CEA degree is doubled co-d-CEA*

Proof. Consider an Arens co-d-CEA set $Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$. We have that $Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1)$ is doubled co-d-CEA by definition. Consider the set $(A_0 \cup A_1 \cup N)^c \cup M = (A_0 \cup A_1)^c \cap N^c \cup M = (A_0 \cup A_1)^c \cap (N^c \cup M)$. Since $(A_0 \cup A_1)^c$ is Y-c.e. let f be a Y computable enumeration of $(A_0 \cup A_1)^c$. Then we have $Y \oplus Y^c \oplus (A_0 \cup A_1)^c \cap (N^c \cup M) \equiv_e Y \oplus Y^c \oplus \{n : f(n) \in N^c\} \cup \{n : f(n) \in M\}$. The right hand side is a co-d-CEA and hence doubled co-d-CEA. The join of two doubled co-d-CEA degrees is doubled co-d-CEA so we have that $Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$ has doubled co-d-CEA degree. □

Now we know from Theorem 5.2 that these classes are both a proper subclasses of the doubled co-d-CEA degrees and since the co-d-CEA degrees are a submetrizable class [1] we know that they both properly contain the co-d-CEA degrees. The next question to ask is if these two classes are distinct from each other. We now give separations that show neither class is contained in the other. The proofs below make use of some of the notation and ideas from the proofs of Theorems 6.7 and 6.9

Theorem 7.2. *There is a Roy halfgraph degree that is not an Arens co-d-CEA degree.*

Proof. Our proof is a finite injury construction. Since we are building a single Roy halfgraph degree we will use an extension of the partial order from 6.9. Let

$$Q = \{(q,p) : q : \omega \to \tilde{\omega} \wedge |q^{-1}[\omega \cup \{\infty\}]| < \omega \wedge p : q^{-1}[\omega] \to \omega \wedge \forall n \in \mathrm{dom}(p)[q(n) \leq p(n)]\}$$

The difference from before is that now q is total and its range is no longer restricted to ω. For $(q_0, p_0), (q_1, p_1) \in Q$ and $u \in \omega$ we define the following:

- $(q_0, p_0) \preceq (q_1, p_1)$ if $\mathrm{HalfGraph}(q_0) \supseteq \mathrm{HalfGraph}(q_1)$ and $p_0 \supseteq p_1$.
- $(q_0, p_0) \preceq_u (q_1, p_1)$ ((q_0, p_0) extends (q_1, p_1) above u) if $(q_0, p_0) \preceq (q_1, p_1)$, $q_0 \restriction u = q_1 \restriction u$.

We will again make use of the enumeration of sequences of even ended intervals $(\alpha_v)_{v \in \omega}$ and the operators $(\Phi_e)_{e \in \omega}$. This time we define $\Vdash_u$ a little differently. For $\alpha : \omega \to I$ we say $(q,p) \Vdash_u f \in \alpha$ if for all n we have $f(n) \in \alpha(n)$ and for all $n \geq u$ we have $f(n) \in 2\mathbb{Z}$ or $\alpha(n) = \tilde{\omega}$. We say $(q,p) \Vdash_u f \notin \alpha$ if for all $(q_0, p_0) \preceq_u (q,p)$ we have $(q_0, p_0) \nVdash_u f \in \alpha$. Note that because even values of q cannot change in extensions if $(q_0, p_0) \preceq_u (q_1, p_1)$ and $(q_1, p_1) \Vdash_u f \in \alpha$ then $(q_0, p_0) \Vdash_u f \in \alpha$.

We will build a computable sequence $(q_0, p_0) \succeq (q_1, p_1) \succeq \dots$ such that $f = \lim_s q_s$ is well defined. We will have that $\mathrm{HalfGraph}^+(f)$ has Roy halfgraph degree. Note that if q_0 and q_1 differ only in that $q_0(n) = -1$ and $q_1(n) = \infty$ then for an appropriate p we have $(q_0, p) \preceq (q_1, p) \preceq (q_0, p)$ so $\preceq$ is not a partial order. The reason f will be well defined is that requirements put up restrictions, so for each u there is a large enough s such that $(q_s, p_s) \succeq_u (q_{s+1}, p_{s+1}) \succeq_u \dots$ and so $f \restriction u = q_s \restriction u$.

The requirements will be $\mathcal{R}_{e,i,j,N,P,M,A}$ where $e, i, j \in \omega$, A, N, P, M are enumeration operators such that given some total set Y they produce Y-c.e. sets $A^Y, N^Y = N_0^Y \sqcup N_1^Y, P^Y = P_0^Y \sqcup P_1^Y, M^Y$ with $P_0 \subseteq N_0^Y \subseteq A^Y, P_1^Y \subseteq N_1^Y \subseteq A^Y$, and $M^Y \subseteq N_Y \setminus P^Y$. The intuition for $\mathcal{R}_{e,i,j,N,P,M,A}$ is that if $Y = \Phi_i(f)$ is the graph of a total function and $Y \oplus \Phi_e(f)$ is an Arens co-d-CEA set of the form $Y \oplus (A_0 \cup P_0^Y) \oplus (A_1 \cup P_1^Y) \oplus ((A_0 \cup A_1 \cup N^Y)^c \cup M^Y)$ for some $A_0 \sqcup A_1 = (A^Y)^c$ then we need to have $\Psi_j(Y \oplus \Phi_e(f)) \neq \mathrm{HalfGraph}^+(f)$. So we say $\mathcal{R}_{e,i,j,N,P,M,A}$ is satisfied if one of the following conditions holds.

1. $Y = \Phi_i(X)$ is not the graph of a total function.

2. $\Phi_e(X) \neq (A_0 \cup P_0^Y) \oplus (A_1 \cup P_1^Y) \oplus ((A_0 \cup A_1 \cup N^Y)^c \cup M^Y)$ for any partition $A_0 \sqcup A_1 = (A^Y)^c$.

3. $\Psi_j(Y \oplus \Phi_e(f)) \neq \text{HalfGraph}^+(f)$.

Now we consider the strategies for $\mathcal{R}_{e,i,j,A,N,P,M}$. Each $\mathcal{R}_{e,i,j,A,N,P,M}$ will be given some restriction u by higher priority requirements and must ensure that $(q_{s+1}, p_{s+1}) \preceq_u (q_s, p_s)$ whenever it sets (q_{s+1}, p_s). Let $Y_s = \Phi_{i,s}(q_s)$. There are several strategies for $\mathcal{R}_{e,i,j,A,N,P,M}$. First we try to meet condition 2 directly. If we ever see some $(q, p) \preceq_u (q_s, p_s)$ such that for $L \oplus R \oplus Z = \Phi_{e,s}(q)$ we have L, R, Z are not disjoint or one of the pairs $(P_0^Y \cup M^Y, R), (P_1^Y \cup M^Y, L), (P^Y, Z)$ is not disjoint, then we set $(q_{s+1}, p_{s+1}) = (q, p)$ and injure lower priority requirements by forcing their restriction to be larger than the use of $\Phi_{e,s}, \Phi_{i,s}$. At each stage where $\mathcal{R}_{e,i,j,A,N,P,M}$ is active we will run this strategy then move on to the next strategy.

For the second strategy we try to meet condition 1. We do this, by running a simplified version of the strategy we used for Theorem 6.9 for $\mathcal{R}_i^a$ to try to make $\Psi_i(X)$ not the graph of a total function. This time we start in state w.

- <u>State w</u>: we wait until at some stage s we see $m, x_0, x_1, v_0, v_1 \in \omega$ such that $\langle\langle m, x_i\rangle, v_i\rangle \in W_{e,s}$ and $(q_s, p_s) \nVdash_u f \notin \alpha_{v_i}$ for $i \in 2$. Chose a condition $(q, p) \preceq_u q_s$ such that for all $n \geq u$ the following hold.

  1. If $\alpha_{v_0}(n) \cap \alpha_{v_1}(n) \neq \emptyset$ then $q(n) \in \alpha_{v_0}(n) \cap \alpha_{v_1}(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_{v_i}(n)) < \min(\alpha_{v_{1-i}}(n))$ then $q(n) = \max(\alpha_{v_i}(n)) + 1$ and $p(n) \geq \min(\alpha_{v_{1-i}})$.

  So we get that $q(n) \in \alpha_{v_0}(n) \iff q(n) \in \alpha_{v_1}(n)$. Set $(q_{s+1}, p_{s+1}) = (q, p)$. If $(q, p) \Vdash_u f \in \alpha_{v_0}, \alpha_{v_1}$ then move to state d. Otherwise move to state c. Let $r = \max(\text{dom}(q) + 1$. Note that $(q_s, p_s) \nVdash_u f \notin \alpha_{v_i}$ but $(q, p) \Vdash_r f \notin \alpha_{v_i}$.

- <u>State c</u>: we wait until we at some stage t we see a pair $x, v \in \omega$ such that $\langle\langle m, x\rangle, v\rangle \in W_{e,t}$ and $(q_t, p_t) \nVdash_r f \notin \alpha_v$. Pick i such that $x_i \neq x$. Choose a condition $(q, p) \preceq_u (q_s, p_s)$ such that the following all hold.

  1. If $\alpha_{v_i}(n) \cap \alpha_v(n) \neq \emptyset$ then $q(n) \in \alpha_{v_i}(n) \cap \alpha_v(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_v(n)) < \min(\alpha_{v_i}(n))$ then $q(n) = \max(\alpha_v(n)) + 1$.

  We set $(q_{t+1}, p_{t+1}) = (q, p)$. If $a, (q, p) \Vdash_u f \in \alpha_v, \alpha_{v_i}$ then we will move to state d. Otherwise we will remain in state c. If we remain, then note that there is an n such that $q(n) = \max(\alpha_v(n)) + 1$. Since $\max(\alpha_{v_{1-i}}) < q_t(n) \in \alpha_v(n)$ we now have $(q, p) \Vdash_u f \notin \alpha_{v_{1-i}}$. We redefine $x_{1-i} = x, v_{1-i} = v, r = \max(\text{dom}(q) + 1$.

- <u>State d</u>: in this state $\mathcal{R}_{e,i,j,A,N,P,M}$ is considered satisfied via case 1.

If the strategy finishes then we get that Y is multivalued, and thus not the graph of a function. If the strategy remains in state c then $m \notin \text{dom}(Y)$ so Y is not total. In both these cases $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied.

If the strategy never leaves state w then it is possible that Y is the graph of a total function. However if this is the case then we have the following important observation: if Y is the graph of a total function and $(r, p) \not\preceq_u (q_t, p_t)$ for all t then it must be that $\Phi_i(r) \subseteq Y$ as otherwise we would have used (r, p) to move into state c at some point. We make repeated use of this in the third strategy to ensure that Y is consistent at every stage where we act.

The third strategy tries to meet condition 3. Define $L_s \oplus R_s \oplus Z_s = \Phi_{e,s}(q_s)$ and $D_s = L_s \cup R_s \cup Z_s$. For $(q,p) \in Q$ and $v \in \omega$ we say that $(q,p) \to_v f(n) \leq 2m$ (implies $f(n) \leq 2m$) if $2\langle n,m\rangle \in \Psi_j(Y \oplus \Phi_e(q\restriction v)$ and $(q,p) \to_v f(n) \geq 2m$ if $2\langle n,m\rangle + 1 \in \Psi_j(Y \oplus \Phi_e(q\restriction v)$.

When $\mathcal{R}_{e,i,j,A,N,P,M}$ is initialized we pick a witness $x > u$ and start with $q_s(x) = -1$. The steps for this strategy are as follows.

1. If we ever see a stage s and $v \in \omega$ where we see $(q_s, p_s) \to_s f(x) \leq 0$ then we injure all lower priority requirements with restriction s and set $q_{s+1}(x) = \infty$. If we are waiting forever at this step then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 3.

2. Next we wait until we see a stage $t > s$ where $Y_s \subseteq Y_t$, $D_s \subseteq D_t \cup N^{Y_t}$ and $(q_t, p_t) \to_t f(x) \geq 2$. Then injure all lower priority requirements with restriction t and set $q_{t+1}(x) = -1$ and $D = D_s \cup D_t$.

   Note that for any Arens co-d-CEA set $Y \oplus L \oplus R \oplus Z$ we have that $L \cup R \cup Z \cup N = \omega$. So if we wait forever to see $D_s \subseteq D_t \cup N^{Y_t}$ then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 2.

   If we have $L_s \oplus R_s \oplus Z_s \subseteq L_t \oplus R_t \oplus Z_t$ then the strategy is finished—this is because if $\mathcal{R}_{e,i,j,A,N,P,M}$ is not injured after stage t then we will have we have $(q_t, p_t) \to_t f(n) \leq 0 \wedge f(n) \geq 2$ meeting condition 3. Otherwise we move on to the next step.

3. We wait until a stage $l > t$ where $Y_t \subseteq Y_l$ and $L_l \oplus R_l \oplus Z_l$ looks like a subset of an Arens co-d-CEA set on D, more precisely, we must have $D \cap A^{Y_l} \cap (L_t \cup R_t) \subseteq P^{Y_l}$, $D \cap Z_t \subseteq A^{Y_l}$ and $D \cap N^{Y_l} \cap Z_t \subseteq M^{Y_l}$. Then set $q_{l+1}(x) = \infty$, and injure lower priority requirements with restriction l.

   For any Arens co-d-CEA set $Y \oplus L \oplus R \oplus Z$ we must have $(A_0 \cup A_1)^c \cap (L \cup R) \subseteq P$, $Z \subseteq (A_0 \cup A_1)^c$ and $N \cap Z \subseteq M$. So if we wait forever at this step, then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 2.

4. We wait until a stage $r > l$ where we have $Y_l \subseteq Y_r$ and where $L_s \oplus R_s \oplus Z_s$ looks like an Arens co-d-CEA set on D. Then set $q_{r+1}(x) = -1$, and injure lower priority requirements with restriction r.

5. We repeat steps 4 and 3 until we have $l < r$ such that $A^{Y_l}, P^{Y_l}, N^{Y_l}, M^{Y_l}$ agrees with $A^{Y_r}, P^{Y_r}, N^{Y_r}, M^{Y_r}$ on D. This must happen eventually as D is finite and $A^{Y_l}, P^{Y_l}, N^{Y_l}, M^{Y_l}$ can only increase each time we repeat these steps.

   Since the witness of $L_r \oplus R_r \oplus Z_r \subseteq \Phi_{e,t}(q_t)$ only uses finitely many axioms there is a large enough n such that for any axiom (z, α) used we have if $2n \in \alpha(x)$ then $\infty \in \alpha(x)$. Set $p_{r+1}(x) = 2n + 1$, $q_{r+1}(x) = 1$. The next step will be repeated several times; we start with $i = 0$, $s_0 = l$ and $r_0 = r$.

6. We wait for a stage $k > r_i$ where $Y_{r_i} \subseteq Y_k$ and $L_k \oplus R_k \oplus Z_k$ looks like an Arens co-d-CEA set on $D_{s_i} \cup D_r$, more precisely $P_0^{Y_k} \cap (D_{s_i} \cup D_r) \subseteq L_k$, $P_1^{Y_k} \cap (D_{s_i} \cup D_r) \subseteq R_k$, $M^{Y_k} \cap (D_{s_i} \cup D_r) \subseteq Z_k$, $Z_k \cap (D_{s_i} \cup D_r) \subseteq A^{Y_k}$ and $D_{s_i} \cup D_r \subseteq D_k \cup N^{Y_k}$. Again, if we are waiting forever at this step, then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 2 or condition 1.

   We now have several cases to consider.

   (a) If for we have $L_{s_i} \oplus R_{s_i} \oplus Z_{s_i} \subseteq L_k \oplus R_k \oplus Z_k$ then $(q_k, p_k) \to_k f(x) \leq 2i$ but $q(k) > 2i$, so we injure all lower priority requirements with restriction k and set $q_{k+1} = q_k$. We have now met condition 3.

(b) If we have $L_{s_i} \nsubseteq L_k \cup N_0^{Y_k}$, $R_{s_i} \nsubseteq R_k \cup N_1^{Y_k}$ or $Z_{s_i} \nsubseteq Z_k \cup N^{Y_k}$, then set $q_{k+1}(x) = 2i$ and injure all lower priority requirements with restriction k. In this case we are meeting condition 2 directly with the first strategy, as either $(L_{s_i} \oplus R_{s_i} \oplus Z_{s_i}) \cup (L_k \oplus R_k \oplus Z_k)$ is not the join of three disjoint sets or N^{Y_k} does not match with $L_{s_i} \oplus R_{s_i} \oplus Z_{s_i}$.

(c) Otherwise set $i = i+1$, $s_i = k$, $q_{k+1}(x) = 2i+1$ and repeat this step.

Now we verify this part of the construction. Consider a requirement $\mathcal{R} = \mathcal{R}_{e,i,j,A,N,P,M}$. Let s be a stage after which $\mathcal{R}$ is never injured. We need to show that $\mathcal{R}$ acts only finitely often and is satisfied. If $\mathcal{R}$ acts at stage $t > s$ via the fist strategy then it never acts again and has ensured that $\Phi_i(f) \oplus \Phi_e(f)$ is not an Arens co-d-CEA set with A, N, P, M, so $\mathcal{R}$ is satisfied via condition 2.

If at some point the second strategy acts then by the same sort of verification used in the proof of Theorem 6.9 we can see that $\mathcal{R}$ acts only finitely often and is satisfied via condition 1.

Now we must consider the third strategy. If we end up waiting forever at any of the steps then $\mathcal{R}$ acts only finitely often and by looking at each step and the current state of $q(x)$ one can see that $\mathcal{R}$ must be satisfied by one of the three conditions. Similarly if the strategy finishes then $\mathcal{R}$ never acts again and is satisfied directly.

Suppose towards a contradiction that $\mathcal{R}$ acts infinitely often. Then it must be that we are in case (c) of step 6 for all values of $i \in \omega$. Since $Y_{s_i} \subseteq Y_{s_{i+1}}$ we have that $N^{Y_{s_i}} \subseteq N^{Y_{s_{i+1}}}$. Since case (b) does not apply we have that $L_{s_0} \subseteq L_{s_1} \cup N_0^{Y_{s_1}} \subseteq L_{s_2} \cup N_0^{Y_{s_2}} \subseteq \dots$ and $R_{s_0} \subseteq R_{s_1} \cup N_1^{Y_{s_1}} \subseteq R_{s_2} \cup N_1^{Y_{s_2}} \subseteq \dots$ Now consider when $i = n$. This means that $L_r \oplus R_r \oplus Z_r \subseteq L_{s_i} \oplus R_{s_i} \oplus Z_{s_i}$. However, since we finished repeating steps 3 and 4 we have that $Z_l \cap D = Z_r \cap D$ and since $L_r \oplus R_r \oplus Z_r$ and $L_l \oplus R_l \oplus Z_l$ disagree somewhere on D there must be $k \in L_r \cap R_l$ or $k \in L_l \cap R_r$. But then the first strategy would act, a contradiction. This also means that the largest i we can get is $i = n$ so $q_{s_i}(x)$ will not exceed $p_{s_i}(x)$.

So $\mathcal{R}$ acts only finitely often and is satisfied. So $\mathrm{HalfGraph}^+(f)$ is not of Arens co-d-CEA degree.

□

Now we proof the other direction.

Theorem 7.3. *There is an Arens co-d-c.e. degree that is not a Roy halfgraph above degree.*

Proof. Note that this proof is very similar to the proof of Theorem 7.2 above and shares a lot of the same structure and ideas.

We will use a modified set of conditions from the set Q from the proof of Theorem 6.7 to construct a specific Arens co-d-c.e. set. Let $P = \{(a,q) : q \in Q, a \subseteq_{\text{fin}} C^q\}$. For $(a,q),(b,p) \in P$ we define the following new notions:

- $(a,q) \preceq (b,p)$ if $q \preceq p$.
- $(a,q) \preceq_u (b,p)$ if $q \preceq_u p$ and $a \restriction u = b \restriction u$.
- $q(a) = (a \cup P_0^q) \oplus ((C^q)^c \setminus a \cup P_0^q) \oplus (C^q \setminus N^q \cup M^q)$.

We will build a computable sequence $(a_0,q_0) \succeq (a_1,q_1) \succeq \dots$ and have $C = \bigcup_s C^{q_s}$, $P = \bigcup_s P^{q_s}$, $N = \bigcup_s N^{q_s}$,$M = \bigcup_s M^{q_s}$ and ensure that $A_0 = \lim_n a_s$ is a well defined. We will define $A_1 = C^c \setminus A_0$ so A_0, A_1 is partition of C^c. This will ensure that C, P, N, M are c.e. and that $X = L \oplus R \oplus Z := (A_0 \cup P_0) \oplus (A_1 \cup P_1) \cup (C \setminus N \cup M)$ is an Arens co-d-c.e. set.

The requirements are $\mathcal{R}_{e,i,j,H}$ where H is an c.e. operator. We think of $\Psi_i(X) \oplus \Psi_e(X)$ as being a Roy halfgraph set where $\Psi_i(X)$ is the total part Y and $\Psi_e(X)$ is $\text{HalfGraph}_+(f)$ for some $f : \omega \to \tilde{\omega}$ with $H^Y = \text{HalfGraph}(f)$. If this really is the case, then we want $\Psi_j(\Psi_i(X) \oplus \Psi_e(X)) \neq X$. We say that $\mathcal{R}_{e,i,j,H}$ is satisfied if one of the following holds.

1. $Y = \Psi_i(X)$ is not the graph of a total function.

2. There is no $f : \omega \to \tilde{\omega}$ such that $\Psi_e(X) = \text{HalfGraph}^+(f)$ and $H^Y = \text{HalfGraph}(f)$.

3. $X \neq \Psi_j(Y \oplus \Psi_e(X))$.

Fix a requirement $\mathcal{R} = \mathcal{R}_{e,i,j,H}$ and let u be the restriction given to $\mathcal{R}$ by higher priority requirements. We will now give a strategy for $\mathcal{R}$.

Like before, the first strategy for $\mathcal{R}$ is to attempt to make Y not the graph of a total function. Again, For this we use the strategy from the proof of Theorem 6.7. This time we only need states w, c, n, d and not g.

- State w: we wait until at some stage s we see $m, x_0, x_1 \in \omega$ and a pair $(b_0, r_0), (b_1, r_1) \preceq_u (a_s, q_s)$ such that $\langle m, x_k \rangle \in \Psi_{i,s}(r_k(b_k))$ and $x_0 \neq x_1$. Let v bound the use of $\langle m, x_k \rangle \in \Psi_{i,s}(r_k(b_k))$. Without loss of generality we can assume $[u, v] \subseteq C^{r_k} \subseteq [0, v]$ as we can find some $(b, r) \preceq_u (a_s, q_s)$ with this property that has $r(b){\upharpoonright}v = r_k(b_r){\upharpoonright}v$ by putting to $[u, v] \setminus C^{r_k}$ into P^r. Note this means we are assuming $b_0 = b_1 = a_s \upharpoonright u$. We set $q_{s+1} = (C^{q_s} \cup C^{r_0} \cup C^{r_1}, P^{q_s}, N^{q_s}, M^{q_s})$ and $a_{s+1} = a_s \setminus C^{q_{s+1}}$. All lower priority requirements are injured with restriction v and we enter state c.

- State c: we wait until we see a stage t such that for some $(b, p) \preceq_s (a_t, q_t)$ we have $\langle m, x_2 \rangle \in \Psi_{i,t}(p(b))$ for some x_2. Pick k such that $x_k \neq x_2$. Set $q_{t+1} = (C^p, P^p, N^p \cup N^{r_k}, M^p)$ (We know N^{r_k} and N^p do not conflict because $p \preceq_v q_{s+1}$ and v bounds N^{r_k}) and set $a_{t+1} = b$. Note that now we have $q_{t+1}(b){\upharpoonright}v \subseteq r_k(b), p(b)$. Let o bound $C^{q_{t+1}}$ and the use of $\langle m, x_2 \rangle \in \Psi_{i,t}(p(b))$. We injure all lower priority requirements with restriction o and enter state c.

- State n: we wait until we see a stage ℓ such that for some $(a, h) \preceq_o (a_\ell, q_\ell)$ we have $\langle m, y \rangle \in \Psi_{i,\ell}(h(a))$ for some y. If $y \neq x_k$ then set $q_{\ell+1} = (C^h, P^h \cup P^{r_k}, N^h, M^h \cup M^{r_k})$ and move into state d. Otherwise $y \neq x_2$ and we set $q_{\ell+1} = (C^h, P^h, N^h, M^h \cup v \setminus (u \cup P^h))$ and enter state d. In either case all lower priority requirements are injured with the use of $\langle m, y \rangle \in \Psi_{i,\ell}(v(a))$.

- State d: in this state $\mathcal{R}$ is considered satisfied.

If this strategy ends in state d then we have ensured that Y is multivalued. If it ends in state c or n then we have that $m \notin \text{dom}(Y)$. In either case $\mathcal{R}$ is satisfied via condition 1.

If we remain forever in state w then it might be Y is the graph of a total function. However if this is the case then we again have the following important observation: if Y is the graph of a total function and $(a, q) \not\preceq_u (a_t, q_t)$ for all t then it must be that $\Psi_i(q(a)) \subseteq Y$ as otherwise we would have used (a, q) to move into state c at some point. We make repeated use of this in the third strategy to ensure that Y is consistent at every stage where we act. While this first strategy is waiting in state w we will enact the second strategy.

Before we describe the second strategy we introduce some notation. We define $Y_s = \Psi_i(q_s(a_s))$ and $H_s = H^{Y_s}$. Since it is easier to think of a Roy halfgraph set in terms of the function f rather than the formal definition we will think of elements of $\Psi_e(q(a))$ as putting restrictions on f. We say that $q(a) \to f(n) \geq 2m$ if $2\langle n, m \rangle \in \Psi_e(q(a))$ and $q(a) \to f(n) \leq 2m$ if $2\langle n, m \rangle + 1 \in \Psi_e(q(a))$.

For an interval $[x, y] \in I$ we define $q(a) \to f(n) \in [x, y]$ if $q(a) \to x \leq f(n) \leq y$ (I is the set of even ended intervals from the proof of Theorem 6.9). For a sequence of intervals $\alpha : \omega \to I$ we say $q(a) \to f \in \alpha$ if for all n we have $q(a) \to f(n) \in \alpha(n)$. In a similar way we define $H_s \to f(n) \geq 2m$ and $H_s \to f(n) = 2m$ (We cannot define $H_s \to f(n) \leq 2m$ as the HalfGraph(f) does not give upper bounds on $f(n)$ unless $f(n)$ is even).

For facts going the other way we say $Y_s, \alpha \to x \in L$ if $\{x\} \oplus \emptyset \oplus \emptyset \subseteq \Psi_j(Y_s \oplus (\{\langle n, m\rangle : 2m \leq \min(\alpha(n))\} \oplus \{\langle n, m\rangle : 2m \geq \max(\alpha(n))\}))$. Similarly we define $Y_s, \alpha \to x \in R$ and $Y_s, \alpha \to x \in Z$.

The steps of the second strategy for $\mathcal{R}$ are as follows:

1. Pick some $x \notin C^{q_s} \cup u$ and set $a_{s+1} = a_s \cup \{x\}, q_{s+1} = q_s$.

2. Wait for a stage s where we see some α_0 such that $Y_s, \alpha_0 \to x \in L$ and $q_s(a_s) \to f \in \alpha_0$. Set $a_{s+1} = a_s \setminus \{x\}, q_{s+1} = q_s$. Injure all lower priority requirements with the use of $Y_s, \alpha_0 \to x \in L$ and $q_s(a_s) \to f \in \alpha_0$.

3. Like in the previous step wait for a stage t such that $Y_s \subseteq Y_t$ and we see some α_1 such that $Y_t, \alpha_1 \to x \in R$ and $q_t(a_t) \to f \in \alpha_1$. Set $a_{s+1} = a_s \cup \{x\}, q_{t+1} = q_t$. Injure all lower priority requirements with the use of $Y_t, \alpha_1 \to x \in R$ and $q_t(a_t) \to f \in \alpha_1$.

4. Let $k = \max\{m : \alpha_i(m) \neq \tilde{\omega}\} + 1$. Now we wait until a stage $s_1 > t$ such that $Y_t \subseteq Y_{s_1}$ and H_{s_1}, f are well behaved up to k. More precisely, for each $n < k$ we want the following:

   - $H_{s_1} \to f(n) = 2m \iff q_{s_1}(a_{s_1}) \to f(n) = 2m$.
   - $H_{s_1} \to f(n) \geq 2m \wedge H_{s_1} \nrightarrow f(n) \geq 2m + 2 \iff q_{s_1}(a_{s_1}) \to 2m \leq f(n) \leq 2m + 2$.
   - $H_{s_1} \nrightarrow f(n) \geq 0 \implies q_{s_1}(a_{s_1}) \to f(n) \leq 0 \vee q_{s_1}(a_{s_1}) \to f(n) \geq \max(8, \min(\alpha_0(n)), \min(\alpha_1(n)))$.

   We use 8 here because the interval $[0, 8]$ can be divided into 4 even ended sub intervals. Set $a_{s_1+1} = a_{s_1} \setminus \{x\}, q_{s_1+1} = q_{s_1}$. Injure all lower priority requirements with the use that witnesses H_{s_1}, f are well behaved.

5. Again search for a $t_1 > s_1$ such that $Y_{s_1} \subseteq Y_{t_1}$ and H_{t_1}, f are well behaved up to k. Set $a_{t_1+1} = a^{t_1} \cup \{x\}, q_{t_1+1} = q_{t_1}$. Injure all lower priority requirements with the use that witnesses H_{t_1}, f are well behaved.

6. Repeat the previous two steps until we have $s_i < t_i$ such that H_{s_i} and H_{t_i} agree upto k. Since k is finite we only have to do this finitely many times. If we have that $q_{t_i}(a_{t_i}) \to f \in \alpha_0 \cap \alpha_1$ then the strategy is finished and we have satisfied condition 3.

   If this is not the case, then there is some $n < k$ such that $H_{s_i}, H_{t_i} \nrightarrow f(n) \geq 0$ and we have $q_{s_i}(a_{s_i}) \to f(n) \leq 0$ and $q_{t_i}(a_{t_i}) \to f(n) \geq 8$ or vica versa. Now set $a^{t_i+1} = a^{t_i} \setminus \{x\}, C^{q_{t_i+1}} = C^{q_{t_i}} \cup \{x\}$ and leave the rest of q_{t_i+1} unchanged from q_{t_i}. Injure all lower priority requirements.

7. Wait until a stage $r > t_i$ when we see $Y_{t_i} \subseteq Y_r$ and β such that $Y_r, \beta \to x \in Z$ and $q_r(a_r) \to f \in \beta$. Furthermore we want that f and H_r are well behaved upto $\max\{m : \beta(m) \neq \tilde{\omega}\} + 1$.

   Since H_r, f are well behaved on k we have that $q_r(a_r) \to f(n) \geq 4$ or $q_r(a_r) \to f(n) \leq 4$ for the n from the previous step. We will assume that $q_r(a_r) \to f(n) \geq 4$ and $q_{s_i}(a_{s_i}) \to f(n) \leq 0$ and give the steps for this case. The cases where $q_r(a_r) \to f(n) \leq 4$ or $q_{t_i}(a_{t_i}) \to f(n) \leq 0$ are similar.

   In this case we add x to $N_0^{q_{r+1}}$, injure lower priority requirements and proceed to the next step.

8. We now wait for a stage $v > r$ where we see one of $q_v(a_v) \to f(n) \geq 2$ or $q_v(a_v) \to f(n) \leq 2$. If $q_v(a_v) \to f(n) \geq 2$ then we add x to $P_0^{q_v+1}$. If $q_v(a_v) \to f(n) \leq 2$ then we add x to M^{q_v+1}. In either case we have ensured $q_v(a_v) \to f(n) \in \emptyset$ so we have met condition 2.

If the strategy finishes, then as explained we meet either 2 or 3. If the strategy waits forever at one of the steps then $\mathcal{R}$ is also met. If we are waiting because we never see a $t > s$ where $Y_s \subseteq Y_t$ then either Y is not total or we have an opportunity to use the tools from the proof of Theorem 6.7. Either way we meet condition 1.

If we are waiting forever at a step for some other reason then one of the other conditions will be met. In the case of steps 2 and 3 we meet condition 3 and in the case of steps 4,5, 6 and 8 we meet condition 2. For step 7 will be condition 3 if we never see $Y_r, \beta \to x \in Z$ and condition 2 if we never see that H_r, f are well behaved.

□

8 Metrizable classes and degrees

8.1 The doubled co-d-c.e. degrees

In Section 5.2 we showed that the doubled co-d-CEA degrees are not $T_{2.5}$ and in Section 6 we show that the Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable. A natural question to ask is if these results can be improved to quasi-minimal separations. In this section we give a negative answer to that question. This result and others come from the following theorem.

Theorem 8.1. *For each Y there is a metrizable* cb$_0$ *space $\mathcal{X}_Y$ such that*

$$\mathcal{D}_{\mathcal{X}_Y} = \{\mathbf{a} : \mathbf{a} \text{ is doubled co-d-c.e. in } Y\}$$

In fact $\mathcal{X}_Y$ is homeomorphic to $\omega \times 2^\omega$.

Proof. Fix a total set Y and Y-c.e. sets C, P, N such that $P \cap N = \emptyset$ and $P, N \subseteq C$. Let $X = \{f : C^c \cup P \cup N \to 2 : f[P] = \{0\}, f[N] = \{1\}\}$. We give a subbasis $(\beta_n)_{n\in\omega}$ of $\mathcal{X}$ as:

- $f \in \beta_{3n}$ if $f(n) = 0$,
- $f \in \beta_{3n+1}$ if $f(n) = 1$ and
- $f \in \beta_{3n+2}$ if $n \in Y$.

If C^c is infinite then $\mathcal{X} \cong 2^\omega$ otherwise X is finite. We have that $\mathcal{D}_{\mathcal{X}} = \{\mathbf{a} : \exists A \sqcup B = C^c[Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N) \in \mathbf{a}]\}$. By taking the disjoint union over all Y-c.e. sets C, P, N where C^c is infinite we get the desired $\mathcal{X}_Y$. □

Corollary 8.2. *There is a second countable metric space $\mathcal{X}$ such that for all computably submetrizable spaces $\mathcal{Y}$ we have $\mathcal{D}_{\mathcal{X}} \nsubseteq \mathcal{D}_{\mathcal{Y}}$.*

Corollary 8.3. *There is a second countable metric space $\mathcal{X}$ such that the quasi-minimal degrees in $\mathcal{D}_{\mathcal{X}}$ are the exactly the quasi-minimal doubled co-d-CEA degrees.*

Corollary 8.2 tells that a lot of complexity can be coded into a non-computable basis of a metrizable space, but it does not tell much about submetrizable classes that are not metrizable. We know there are effectively submetrizable classes that are not metrizable, for instance the co-d-CEA degrees. This follows from the fact that there are quasi-minimal co-d-CEA degrees relative to any oracle. However, since the co-d-CEA degrees are contained in the doubled co-d-CEA degrees Corollary 8.3 tells us that this is not a quasi-minimal separation. In fact every effectively submetrizable space $\mathcal{X}$ can have only countably many quasi-minimal degrees since points representing these degrees get mapped to computable points under the continuous injection $f : \mathcal{X} \to [0,1]^\omega$. Because it is possible to encode any countable set of degrees into the basis of a metric space we have the following result.

Proposition 8.4. *There is no effectively submetrizable class of degrees $\mathcal{C}$ that is metrizable quasi-minimal.*

However, if we drop the effective requirement then this becomes and open question.

Question 8.5. Is the a submetrizable class of degrees $\mathcal{C}$ that is metrizable quasi-minimal?

Corollary 8.3 means that there is no hope to separate the classes of degrees of a $T_{2.5}$ space from the classes of degrees of an arbitrary second countable submetrizable space using the notion of $\mathcal{T}$ quasi-minimal with the given examples of $T_{2.5}$ spaces. If there is such a separation we will need to look at new spaces.

Question 8.6. Is there a (decidable) $T_{2.5}$ class of degrees that is submetrizable quasi-minimal?

While the doubled co-d-CEA degrees are not metrizable quasi-minimal it may still be able to get quasi-minimal separations by adding a computability constraint. In this vain we have the following questions.

Question 8.7. For every decidable $T_{2.5}$ space $\mathcal{X}$ is there a doubled co-d-CEA degree that is $\mathcal{X}$ quasi-minimal?

Question 8.8. For every effectively submetrizable space $\mathcal{X}$ is there a Arens co-d-CEA degree or Roy halfgraph degree that is $\mathcal{X}$ quasi-minimal?

Since an effectively submetrizable space can have at most countably many quasi-minimal degrees and any countable collection of enumeration degrees can be encoded into an effectively submetrizable space the previous question is equivalent to the following.

Question 8.9. Are there uncountably many quasi-minimal Arens co-d-CEA degrees or quasi-minimal Roy halfgraph degrees?

Now we explore the spaces from Theorem 8.1 a little more. By combining them all together in the right way we can get a new cb_0 space that represents all doubled co-d-CEA degrees.

Definition 8.10. We define the doubled co-d-CEA space ($\mathcal{DCD}$) as follows. For each triple of c.e. functionals C, P, N with $P^Y, N^Y \subseteq C^Y, P^Y \cap N^Y = \emptyset$ define the set $\mathcal{X}_{C,P,N} = \{(Y, f) : Y \in 2^\omega, f : (C^Y)^c \cup P^Y \cup N^Y, f[P^Y] = 1, f[N^Y] = 0\}$. The subbasis $(\beta_e)_{e\in\omega}$ of $\mathcal{X}_{C,P,N}$ is coded by pairs $\langle\sigma, m\rangle$, $\sigma \in 2^{<\omega}, m \in \omega$ with $\beta_{\langle\sigma,2n\rangle} = \{(Y, f) : \sigma \prec Y, (n, 0) \in f\}$ and $\beta_{\langle\sigma,2n+1\rangle} = \{(Y, f) : \sigma \prec Y, (n, 1) \in f\}$.

Let $\Gamma_e = (\Gamma_{e,2}, \Gamma_{e,1}, \Gamma_{e,0})$ be a effective listing of all valid triples of functionals. We define $\mathcal{DCD}$ to be $\bigsqcup_e \mathcal{X}_{\Gamma_e}$. The subbasis of $\mathcal{DCD}$ is given by $\beta_{\langle e,\sigma,m\rangle}$ where $\beta_{\langle e,\sigma,m\rangle}$ is the open set $\beta_{\langle\sigma,m\rangle}$ in X_{Γ_e}.

Theorem 8.11. *$\mathcal{D}_{\mathcal{DCD}}$ is the class of doubled co-d-CEA degrees.*

Proof. Consider a point $(Y, f) \in \mathcal{X}_{\Gamma_e}$. Let $(C, P, N) = \Gamma_e$. We have that

$$\mathrm{NBase}_{\mathcal{DCD}}(Y, f) = \{\langle e, \sigma, 2n\rangle : \sigma \prec Y, f(n) = 0\} \cup \{\langle e, \sigma, 2n+1\rangle : \sigma \prec Y, f(n) = 1\}$$

Let $A = f^{-1}[\{1\}] \setminus C^Y, B = f^{-1}[\{0\}] \setminus C^Y$. Let $X = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$. So we have that X is doubled co-d-CEA and $\mathrm{NBase}_{\mathcal{DCD}}(Y, f) \equiv_e X$.

Now consider some doubled co-d-CEA set $X = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$. Let e be such that $\Gamma_{e,2}(Y) = (A \cup B)^c, \Gamma_{e,1}(Y) = P, \Gamma_{e,0}(Y) = N$. Let $f : A \cup B \cup P \cup N \to 2$ be given by $f(n) = 1$ if $n \in A \cup P$, $f(n) = 0$ if $n \in B \cup N$. Then we have that $(Y, f) \in \mathcal{X}_{\Gamma_e}$. Then just like above we have that $\mathrm{NBase}_{\mathcal{DCD}}(Y, f) \equiv_e X$. $\square$

So we have a different space that represents the doubled co-d-CEA degrees. It is less natural than the double origin topology, but more explicitly represents these degrees.

Theorem 8.12. *$\mathcal{DCD}$ is $T_2 \setminus T_{2.5}$.*

Proof. Since $\mathcal{DCD}$ gives us the doubled co-d-CEA degrees we know that it cannot be $T_{2.5}$. So now we need to show that the space is T_2.

Fix and e and consider $\mathcal{X}_{C,P,N}$ for $(C, P, N) = \Gamma_e$. Consider two distinct points $(Y_0, f_0), (Y_1, f_1) \in \mathcal{X}_{C,P,N}$. If $Y_0 \neq Y_1$ then there is $\sigma_0 \prec Y_0$ and $\sigma_1 \prec Y_1$ such that $\sigma_0 \perp \sigma_1$. Consider the open sets $V_0 = \bigcup_{m\in\omega} \beta_{\langle\sigma_0,m\rangle}$ and $V_1 = \bigcup_{m\in\omega} \beta_{\langle\sigma_1,m\rangle}$. Since $\sigma_i \prec Y_i$ we have $Y_i \in V_i$. For any $(Y, f) \in V_0$, $\sigma_0 \prec Y$ so $\sigma_1 \not\prec Y$. Hence $(Y, f) \notin V_1$, so V_0 and V_1 are disjoint.

Now suppose that $Y_0 = Y_1$. So $f_0 \neq f_1$ and there is n such that $f_0(n) \neq f_1(n)$. So we have $(Y_i, f_i) \in \beta_{\langle\langle\rangle,2n+f_i(n)\rangle}$. If $(Y, f) \in \beta_{\langle\langle\rangle,2n+f_0(n)\rangle}$ then $f(n) = f_0(n) \neq f_1(n)$ so $(Y, f) \notin \beta_{\langle\langle\rangle,2n+f_1(n)\rangle}$. Hence $\beta_{\langle\langle\rangle,2n+f_0(n)\rangle}$ and $\beta_{\langle\langle\rangle,2n+f_0(n)\rangle}$ are disjoint.

$\square$

8.2 Decidable, metrizable degrees

We know that any enumeration degree can be realized in a decidable, submetrizable cb_0 space. A natural question to ask is the following.

Question 8.13. What is the class of degrees $\mathbf{a}$ such that $\mathbf{a} \in \mathcal{D}_{\mathcal{X}}$ for some decidable, metrizable cb_0 space $\mathcal{X}$?

We know that this class includes all continuous degrees, since $[0, 1]^\omega$ is decidable with the usual basis. Theorem 8.14 below show that this class contains a quasi-minimal, and hence not continuous, degree, so this class is larger than that of the continuous degrees. It remains open whether there are any enumeration degrees that do not belong to this class.

Theorem 8.14. *There is a decidable, metrizable* cb_0 *space $\mathcal{X}$ such that $\mathcal{D}_X$ contains a quasi-minimal degree.*

Proof. The metric space $\mathcal{X} = (X, d)$ we will construct will be $X = \omega \times \omega \cup \{\infty\}$ with the metric given by $d((a, n), (b, m)) = 2^{-\min(n,m)}$ if $(a, n) \neq (a, m)$ and $d((a, n), \infty) = 2^{-n}$. So $\omega \times \omega$ has the discrete topology and ∞ is the limit of all sequences where the second coordinate is increasing. The basis we will use is given by $\beta_{2\langle a,n\rangle} = \{(a, n)\}$ and

$$\beta_{2n+1} = \{(a, m) : m \geq n \wedge (\forall k \leq n)(p_k \nmid a \vee n \notin B_k \vee m \in B_n)\} \cup \{\infty : B_n \text{ is cofinite}\}$$

where $(p_n)_{n\in\omega}$ is the sequence of primes and $(B_n)_{n\in\omega}$ are uniformly computable sets that we will build by finite injury. There will be infinitely many n such that B_n is cofinite, hence for each n there is $m > n$ such that $\infty \in \beta_{2m+1} \subseteq B(\infty, 2^{-n})$. So $(\beta_n)_{n\in\omega}$ is a basis of $\mathcal{X}$.

To ensure that $(X, (\beta_e)_{e\in\omega})$ is decidable we will ensure that $n \in B_n \subseteq \omega \setminus n$ and we will ensure for all $m < n$ if $n \in B_m$ then $B_n \subseteq B_m$. This means that if $n \in B_m$ then $\beta_{2n+1} \subseteq \beta_{2m+1}$. To show that the $\subseteq$ relation on positive Boolean combinations is computable it is enough to look at questions of the form $\bigcap_{i<k} \beta_{e_i} \subseteq \bigcup_{j<k'} \beta_{d_j}$. If some e_i is even then, since the B_n are uniformly computable we can answer the question computably. So we can assume that all e_i are odd. Since $|\bigcap_{i<k} \beta_{e_i}|$ is either 0 or ω we can assume all d_j are odd. Let $e = \min\{e_i : i < k\}$. Let $e_i = 2r_i + 1$ and $d_j = 2v_j + 1$. If there is i, j such that $r_i \in B_{v_j}$ then it is true that $\bigcap_{i<k} \beta_{e_i} \subseteq \bigcup_{j<k'} \beta_{d_j}$ as $\beta_{e_i} \subseteq \beta_{d_j}$. If this is not the case then let $r = \max\{r_i : i < k\}$ and consider $p = \prod_{j:v_j \leq r} p_j$. Since $r \notin B_{v_j}$ for any j we have that $(p, r) \notin \beta_{d_j}$. On the other hand since $r_i \notin B_{v_j}$ for any i, j we have that $(p, r) \in \beta_{e_i}$ for each $i < k$.

Now we move on to the construction of $(B_n)_{n\in\omega}$. We define $A_s = \{n : s \in B_n\}$, which means that $B_n = \{s : n \in A_s\}$. We will build A_s in stages and have $A = \lim_s A_s$. Note that $\emptyset \oplus A = \mathrm{NBase}(\infty)$ so we want to ensure that A is quasi-minimal. The requirements are $\mathcal{N}_e : A \neq W_e$ and $\mathcal{R}_e : \Psi_e(A)$ is the graph of a total function $\implies \Psi_e(A) \leq_e \emptyset$. Each requirement will be given a restriction $u \leq s$ by higher priority requirements and will not be allowed to change the value of $A\restriction u$.

The strategy for an $\mathcal{N}_e$ requirement is as follows. If $\mathcal{N}_e$ is initialized at stage s then we will use s as a witness and give lower priority requirements restriction $s + 1$ and will not change A. By definition we must have $s \in A_s$. We wait until a stage $t > s$ where we see $s \in W_e$. Then we set $A_{t+1} = (A_s \setminus \{s\}) \cup \{t + 1\}$ and give all lower priority requirements restriction t. The strategy is finished after this step.

The strategy for an $\mathcal{R}_e$ requirement is as follows. If $\mathcal{R}_e$ is initialized at stage s then we give lower priority requirements restriction s at move wait until we a stage t where we see a pair $\langle x, y\rangle, \langle x, z\rangle \in \Psi_{e,t}(A_s \cup [s, t])$ with $y \neq z$. When we see such a pair we set $A_{t+1} = A_s \cup [s, t + 1]$ and give all lower priority requirements restriction t. The strategy is now finished. Since $A\restriction s$ has not changed between stages s and t defining A_{t+1} as above will not violate the requirement that $B_n \subseteq B_m$ if $n \in B_m$.

The $\mathcal{N}_e$ requirement ensures that $A \neq W_e$. The $\mathcal{R}_e$ requirement ensures that if $\Psi_e(A)$ is the graph of a total function then $\Psi_e(A) = \Psi_e(A_s \cup [s, \infty))$ for some s and is hence computable. So A has quasi-minimal degree and hence is $\mathcal{D}_{(X,(\beta_e)_{e\in\omega})} = \{\mathbf{0}, \deg_e(A)\}$ contains a quasi-minimal degree. □

References

[1] Takayuki Kihara, Keng Meng Ng, and Arno Pauly. Enumeration degrees and non-metrizable topology, 2019.

[2] Richard M. Friedberg and Hartley Rogers, Jr. Reducibility and completeness for sets of integers. *Z. Math. Logik Grundlagen Math.*, 5:117–125, 1959.

[3] Uri Andrews, Hristo A. Ganchev, Rutger Kuyper, Steffen Lempp, Joseph S. Miller, Alexandra A. Soskova, and Mariya I. Soskova. On cototality and the skip operator in the enumeration degrees. *Trans. Amer. Math. Soc.*, 372(3):1631–1670, 2019.

[4] Joseph S. Miller. Degrees of unsolvability of continuous functions. *J. Symbolic Logic*, 69(2):555–584, 2004.

[5] Takayuki Kihara and Arno Pauly. Point degree spectra of represented spaces. *Forum Math. Sigma*, 10:Paper No. e31, 27, 2022.

[6] Tanja Grubba, Matthias Schröder, and Klaus Weihrauch. Computable metrization. *MLQ Math. Log. Q.*, 53(4-5):381–395, 2007.

[7] Klaus Weihrauch. Computably regular topological spaces. *Log. Methods Comput. Sci.*, 9(3):3:5, 24, 2013.

[8] Carl G. Jockusch, Jr. Semirecursive sets and positive reducibility. *Trans. Amer. Math. Soc.*, 131:420–436, 1968.

[9] Liliana Badillo and Charles M. Harris. An application of 1-genericity in the Π^0_2 enumeration degrees. In *Theory and applications of models of computation*, volume 7287 of *Lecture Notes in Computer Science*, pages 604–620. Springer, Heidelberg, 2012.

[10] Liliana Badillo, Charles M. Harris, and Mariya I. Soskova. Enumeration 1-genericity in the local enumeration degrees. *Notre Dame J. Form. Log.*, 59(4):461–489, 2018.

[11] Yu. T. Medvedev. Degrees of difficulty of the mass problem. *Dokl. Akad. Nauk SSSR (N.S.)*, 104:501–504, 1955.

[12] Klaus Weihrauch. *Computable analysis*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000. An introduction.

[13] Boris Ya. Solon. Co-total enumeration degrees. In *CiE*, volume 3988 of *Lecture Notes in Computer Science*, pages 538–545. Springer, 2006.

[14] Matthias Schröder. Effectivity in spaces with admissible multirepresentations. volume 48, pages 78–90. 2002. Dagstuhl Seminar on Computability and Complexity in Analysis, 2001.

[15] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in topology*. Springer-Verlag, New York-Heidelberg, second edition, 1978.