Selman's theorem for hyperenumeration reducibility

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Abstract

Hyperenumeration reducibility was first introduced by Sanchis [1]. The relationship between hyperenumeration and hyperarithmetic reducibility shares many parallels with the relationship between enumeration and Turning reducibility. We ask if this relationship can be pushed to prove and analog of Selman's Theorem for hyperenumeration reducibility. By studying e-pointed trees in Baire space we are able to get a counter example. An e-pointed tree T is a tree with no dead ends and the property that every path in T enumerates T. We prove that if T is an e-pointed tree then for all X if T is Π^1_1 in X then $\overline{T}$ is Π^1_1 in X. We build an e-pointed tree T such that $\overline{T}$ is not hyperenumeration reducible to T.

1 Introduction

Enumeration reducibility ($\leq_e$) is a reducibility that captures the notion of how difficult it is to enumerate a given set of numbers. There are several definition, but the one we find most useful in this paper is the given by Friedberg and Rogers [2] when they introduced the reducibility.

Definition 1.1. For sets $A, B \subseteq \omega$ we say that $A \leq B$ if there is a c.e. set of axioms W such that:

$$n \in A \iff \exists u[\langle n, u\rangle \in W \wedge D_u \subseteq B]$$

Here $(D_u)_{u\in\omega}$ is the collection of all finite sets given by strong indexes.

One useful property of this definition is that it gives us a collection of enumeration operators $(\Psi_e)_{e\in\omega}$. We define $A = \Psi_e(B)$ if $A \leq_e B$ via the eth c.e. set W_e. Enumeration reducibility is a reducibility on the positive information about a set. This can be seen by the fact that if $A \subseteq B$ then $\Psi_e(A) \subseteq \Psi_e(B)$.

Enumeration reducibility is a pre-order and the equivalence classes form an upper semi-lattice $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets and joins given by the usual operation. There is also an enumeration jump given by $A \mapsto \bigoplus_{e\in\omega} \Psi_e(A) \oplus \bigoplus_{e\in\omega} \overline{\Psi_e(A)}$. Like with the Turing jump we have that $A <_e A'$.

One aspect of enumeration reducibility that has been well studied is its relationship with the Turning reducibility. The Turning degrees embed into the enumeration degrees via the map induced by $A \mapsto A \oplus \overline{A}$. This follows from the fact that $A \oplus \overline{A} \leq_e B \oplus \overline{B} \iff A \leq_T B$. This embedding is known to be a proper embedding [3], and the Turing and enumeration jump coincide on these degrees. The image of the Turing degrees is know as the total degrees.

Definition 1.2. We say that a set A is *total* if $\overline{A} \leq_e A$. We say that A is cototal if $A \leq_e \overline{A}$. A degree is *total* (*cototal*) if it contains a total (cototal) set.

It is known that the total degrees are a proper subclass of the cototal degrees and that the cototal degrees are a proper subclass of all enumeration degrees [4]. It is known that both the jump [5] and the total degrees [6] are definable within the structure $(\mathcal{D}_e, \leq_e)$.

While there are many similarities between these classes of degrees they are structurally different. One notable difference is the fact that, while there are minimal Turing degrees, Gutteridge [7] proved that the enumeration degrees are downwards dense.

We have seen that Turing reducibility can be defined in terms of enumeration reducibility. An important early result of Selman [8] shows how to define enumeration reducibility in terms of Turing reducibility.

Theorem 1.3 (Selman's Theorem). *$A \leq_e B$ if and only if, for all X if $B \leq_e X \oplus \overline{X}$ then $A \leq_e X \oplus \overline{X}$.*

This theorem states that an enumeration degree is uniquely determined by total degrees above it. This means that the total degrees form an automorphism base for the enumeration degrees.

Sanchis [1] introduced the notion of hyperenumeration reducibility $\leq_{he}$, an analogue of enumeration reducibility relating to hyperarithmetic reducibility rather than Turing reducibility.

Definition 1.4 (Sanchis 1978 [1]). We say that $A \leq_{he} B$ if there is a c.e. set W such that

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u\rangle \in W \wedge D_u \subseteq B]$$

Sanchis proved that $\leq_{he}$ is a pre-order giving rise to the hyperenumeration degrees $\mathcal{D}_{he}$ and proved that the map $A \mapsto A \oplus \overline{A}$ induces an embedding of the hyperarithmetic degrees into the hyperenumeration degrees. The main result of Sanchis' paper was proving that there is a hyperenumeration degree that is not a hyperarithmetic degree.

A couple of other useful results Sanchis proved are the fact that we can allow W in the definition of $\leq_{he}$ to be Π^1_1 and the fact that if $A \leq_e B$ then $A \leq_{he} B$ and $\overline{A} \leq_{he} \overline{B}$.

In this paper we look at a couple of the aspects of the relationships between Turing reducibility and enumeration reducibility and see if they also hold for the relationship between hyperarithmetic reducibility and hyperenumeration reducibility. In section 3 we show that Selman's theorem fails for hyperenumeration reducibility: we show there are sets $A <_{he} B$ such that the $\{X : A \leq_{he} X \oplus \overline{X}\} = \{X : B \leq_{he} X \oplus \overline{X}\}$. The proof of this works by constructing a uniformly e-pointed tree without dead ends that is not of hypertotal degree. A A set is hypertotal if $\overline{A} \leq_{he} A$ and hypercototal if $A \leq_{he} \overline{A}$.

E-pointed trees were introduced by McCarthy [9] and were used to characterize the cototal enumeration degrees.

Definition 1.5. A tree T is *e-pointed* if for every path $P \in [T]$ we have that T is c.e. in P. We say T is *uniformly* e-pointed if there is a single operator Ψ_e such that for all paths $P \in [T]$ we have $T = \Psi_e(P)$.

McCarthy proved that the degree of a uniformly e-pointed tree on $2^{<\omega}$ is a cototal set, and characterized the cototal degrees as the degrees of uniformly e-pointed trees on $2^{<\omega}$ without dead ends and as the degrees of general e-pointed trees on $2^{<\omega}$.

Goh, Jacobsen-Grocott, Miller and Soskova [10] have studied e-pointed on $\omega^{<\omega}$. They found some interesting connections to hyperenumeration reducibility and the notion of hypercototality. They proved that every uniformly e-pointed tree is hypercototal and enumeration degrees of these trees are the same as the degrees of hypercototal sets and the same as the degrees of general e-pointed trees.

They found that when considering e-pointed trees on $\omega^{<\omega}$ without dead ends things become different. They proved there is an arithmetic set that is not enumeration equivalent to any e-pointed tree without a dead end. They also proved that there is a uniformly e-pointed tree without dead ends is not of cototal enumeration degree.

In this section 3 we prove a stronger separation

Theorem 1.6. *There is a uniformly e-pointed tree $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.*

This has some interesting corollaries, one of which is the failure of Selman's theorem for hyperenumeration reducibility.

Corollary 1.7. *There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.*

This is one way in which in which hyperenumeration reducibility is different from enumeration reducibility. In section 4 we prove that, like the enumeration degrees, the hyperenumeration degrees are downwards dense, giving another example of how these degree structures are similar. We prove this by adapting Gutteridge's original proof, but in section 4 we describe a problem that arises when trying to do priority constructions for hyperenumeration reducibility and give a method that can be used to solve this problem in cases like downwards density.

In section 5 we look at some other natural reducibilities that could be considered hyperarithmetic analogues of enumeration reducibility, and we consider their relationship to hyperarithmetic and enumeration reducibility.

2 Preliminaries

Some basic points of notation. We use n, m, i, j, k for natural numbers. We use α, β, γ for ordinals. We use $\sigma, \tau, \rho, \upsilon, x, y, z$ to represent strings of natural numbers. $\langle \sigma \rangle$ corresponds to the Gödel number of the string σ. We use T and S to refer to trees in $\omega^{<\omega}$. We will give a brief overview of some of the tools of higher computability theory that we will use in this paper. A more in depth introduction to higher computability can be found in Sacks' book [11].

2.1 Admissible sets and higher computability theory

The usual definition of a Π^1_1 set of natural numbers is a set of the form $m \in X \iff \forall f \in \omega^\omega \exists n[R(f, n, m)]$ where R is a computable relation. However admissibility gives us another definition in terms of $L_{\omega_1^{CK}}$ that is useful.

Definition 2.1. A set M is *admissible* is it is transitive, closed under union, pairing and Cartesian product as well as satisfying the following to properties:

Δ_1-comprehension: for every Δ_1 definable class $A \subseteq M$ and set $a \in M$ the set $A \cap a \in M$.

Σ_1-collection: for every Σ_1 definable class relation $R \subseteq M^2$ and set $a \in M$ such that $a \subseteq \mathrm{dom}(R)$ there is $b \in M$ such that $a = R^{-1}[b]$.

The smallest admissible set is HF the collection of hereditarily finite sets. Looking at the Δ_1 and Σ_1 subsets of HF is one notion of computability. We have that the Δ_1 subsets of HF are computable sets and the Σ_1 subsets of HF are the c.e. sets. We generalize this to an arbitrary admissible set M by calling a set $A \subseteq M$ M-computable if it is a Δ_1 subset of M and M-c.e. if it is a Σ_1 subset of M.

The smallest admissible set containing ω is $L_{\omega_1^{CK}}$. We have that the $L_{\omega_1^{CK}}$-c.e. subsets of ω are precisely the Π^1_1 sets. This means that the $L_{\omega_1^{CK}}$-computable subsets of ω sets are the hyperarithmetic sets. Note that Δ_1-comprehension means that the hyperarithmetic sets are precisely $\mathcal{P}(\omega) \cap L_{\omega_1^{CK}}$.

These results about Π^1_1 and hyperarithmetic sets can be relativized for some set X. We define L_X to be the smallest admissible set containing X. We have that $A \subseteq \omega$ is Π^1_1 in X if and only if it is L_X-c.e. and hyperarithmetic in X if and only if $A \in L_X$. Note that while we have $\mathrm{ORD}^{L_X} = \omega_1^X$ and $L_{\omega_1^X} \subseteq L_X$ it is only sometimes the case that $L_X = L_{\omega_1^X}$.

2.2 Hyperenumeration reducibility

Since this paper is focused on hyperenumeration reducibility we will use this section to discuss some concepts that will be useful for the rest of the paper.

We defined enumeration reducibility in terms of operators $(\Psi_e)_{e\in\omega}$. Sanchis' definition of hyperenumeration reducibility gives us hyperenumeration operators $(\Gamma_e)_{e\in\omega}$. We say $m \in \Gamma_e(A) \iff \forall f \in \omega^\omega \exists u, n[m^\frown\langle f \upharpoonright n, u\rangle \in W_e \wedge D_u \subseteq A]$.

We examine the relationship between Γ_e and Ψ_e. Both use the same set W_e in there definition. Consider the tree $S^A_e = \{x \in \omega^{<\omega} : \exists z \preceq x, u \in \omega[\langle x, u\rangle \in W_e \wedge D_u \subseteq A]\}$. From the definition of Γ_e we have that $n \in \Gamma_e(A)$ if and only if S^A_e does not have and infinite path starting with n. We have that $S^A_e \leq_T A$ and $\overline{S^A_e} \leq_e A$. In our construction in section 3 we will use these trees S^A_e in our forcing notion. While the notation is a little different, this is similar to how Sanchis proofed the existence of a non hypertotal degree. S^A_e inspires us to come up with the notion of a hyperenumeration of a set.

Definition 2.2. We say that a tree S is a *hyperenumeration* of a set B if $B = \{n : \exists f \in [S](f(0) = n\}$.

From this we have that $B \leq_{he} \overline{S}$ via the same operator for every hyperenumeration S of B. By coding a set X into a layer of S^X_e we have that for every X such that B is Π^1_1 in X there is a hyperenumeration S of B such that $S \equiv_T X$. So the hyperenumerations of B characterize the total *he*-degrees above $\deg_{he}(B)$ much like how the enumerations of B characterize the *e*-degrees above the $\deg_e(B)$.

2.3 Some facts about trees

We will deal a lot with trees in this paper so it is useful to have to operations on trees. For a tree $S \subseteq \omega^{<\omega}$ and string x we define $\mathrm{Ext}(S, x)$ to be the subtree $\{y : x^\frown y \in S\}$. A relation on trees that we will use is $\preceq$. We say $T \preceq S$ if S is an end extension of T. That is, $T \subseteq S$ and for all $\sigma \in S$ the longest initial segment of σ that is in T is a leaf in T.

Now we define $\mathrm{rank}(S)$ for a well founded tree S using transfinite recursion. We define $\mathrm{rank}(\emptyset) = 0$. Given a tree S we define $\mathrm{rank}(S) = \sup_{i \in S} \mathrm{rank}(\mathrm{Ext}(S, i)) + 1$.

As it turns out, this function rank is in fact $L_{\omega_1^{CK}}$-partial computable, i.e. its graph is $L_{\omega_1^{CK}}$-c.e. To help the reader feel more familiar with computability on $L_{\omega_1^{CK}}$ we include a sketch of the proof of this fact.

For a tree $T \in L_{\omega_1^{CK}}$ and function $f \in L_{\omega_1^{CK}}$ we say that f is a rank function on T if $\mathrm{dom}(f) = T$, $\mathrm{range}(f) \subseteq \omega_1^{CK}$, for each leaf $x \in T$ we have $f(x) = 1$ and for each non leaf $y \in T$ we have that $f(y) = \sup_{y^\frown i \in T} f(y^\frown i) + 1$. Since the quantifies are all bounded it is $L_{\omega_1^{CK}}$-computable to check if f is a rank function on T. If f is a rank function on T then f is unique and $f(\emptyset) = \mathrm{rank}(T)$. So we can define rank by $\mathrm{rank}(T) = \alpha$ if there is a rank function f on T such that $f(\emptyset) = \alpha$ or $\alpha = 0$ and $T = \emptyset$. So we now have a Σ_1 definition of rank, the only problem is that its domain may not consist of all well founded trees $T \in L_{\omega_1^{CK}}$.

To prove that the domain of rank is all well founded trees in $L_{\omega_1^{CK}}$ we use induction on the true rank of T. Suppose all trees of rank less than T are in the domain of rank. Then for each $i \in T$ there is a rank function f_i for $\mathrm{Ext}(T, i)$. Since the map, $S \mapsto f$ where f is the rank function on S, is $L_{\omega_1^{CK}}$-c.e, Σ_1-collection tells us that the map $i \mapsto f_i$ is in $L_{\omega_1^{CK}}$. So we can build a rank function $f \in L_{\omega_1^{CK}}$ on T by $f(i^\frown x) = f_i(x)$ and $f(\emptyset) = \sup_{i \in T} f_i(\emptyset) + 1$.

One nice result of this is that if a tree $T \in L_{\omega_1^{CK}}$ is well founded, then it has rank $< \omega_1^{CK}$ and the set of all well founded trees in $L_{\omega_1^{CK}}$ is $L_{\omega_1^{CK}}$-c.e. This could also be seen by observing that trees in $L_{\omega_1^{CK}}$ are Δ_0 definable and so hyperarithmetic.

3 A uniformly e-pointed tree in ω^ω without dead ends that is not of hyper total degree

In this section we prove the following theorem.

Theorem 1.6. *There is a uniformly e-pointed tree $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.*

3.1 The forcing partial order

To build this we will need a new set of forcing conditions similar to those used in the construction of a uniformly e-pointed tree without dead ends that is not of introenumerable degree. So let $\{T_\sigma : \sigma \in \omega^{<\omega}\}$ be an effective listing of all finite trees in $\omega^{<\omega}$ where for each $\sigma \in \omega^{<\omega}$ sequence $T_{\sigma^\frown 0}, T_{\sigma^\frown 1}, \ldots$ lists each finite tree that contains T_σ infinitely often. We will need a labeling that is allowed to use any ordinal below ω_1^{CK}. From now on a condition is some $p = (T^p, L^p : T^p \times T^p \to \omega_1^{CK}) \in L_{\omega_1^{CK}}$ where the following hold:

1. $T^p \subseteq \omega^{<\omega}$ is a well founded tree.
2. For each $\sigma \in T^p$ we have that $T_\sigma \subseteq T^p$.
3. $L^p(\sigma, \tau) = 0$ if and only if $\sigma \in T_\tau$.
4. If $\rho \prec \tau$ then $L^p(\sigma, \tau) = 0$ or $L^p(\sigma, \tau) < L^p(\sigma, \rho)$.
5. For each $\tau \in T^p$ and $n < \omega$ the set $\{\sigma : L^p(\sigma, \tau) \leq n\}$ is finite.

For two conditions p and q we say $p \leq q$ if $T^q \preceq T^p$ and $L^q \subseteq L^p$. For a filter $\mathcal{G}$ we define $T^{\mathcal{G}} = \bigcup_{p \in \mathcal{G}} T^p$. The fact that we must have $T^q \preceq T^q$ means that if $p \in \mathcal{G}$, σ is not a leaf in T^p and $\sigma^\frown i \notin T^p$ then $\sigma^\frown i \notin T^{\mathcal{G}}$. So we have a way of forcing strings into the complement of $T^{\mathcal{G}}$.

Proposition 3.1. *The set of conditions is $L_{\omega_1^{CK}}$-c.e. and the relation $\leq$ on conditions is $L_{\omega_1^{CK}}$-computable.*

Proof. Properties 2—5 are all straightforwardly Δ_1 conditions. To check if a tree T is well founded we ask if there is a rank function $f \in L$ such that $f(\sigma) = \text{rank}(\text{Ext}(T, \sigma))$, so a Σ_1 question. So property 1 is a Σ_1 condition. Hence the set of valid conditions is $L_{\omega_1^{CK}}$-c.e.

To check if $q \leq p$ is clearly Δ_1 so $\leq$ is an $L_{\omega_1^{CK}}$-computable relation with $L_{\omega_1^{CK}}$-c.e. domain. $\square$

Proposition 3.2. *For a condition p we have $L^p(\sigma, \tau) \geq \text{rank}(\{\rho : \tau^\frown\rho \in T^p, \sigma \notin T_{\tau^\frown\rho}\})$ for all $\sigma, \tau \in T^p$.*

Proof. We will use induction on $L^p(\sigma, \tau)$. Base case, $L^p(\sigma, \tau) = 0$. Then $\sigma \in T_\tau$ so $\{\rho : \tau^\frown\rho \in T^p, \sigma \notin T_{\tau^\frown\rho}\} = \emptyset$ and $\text{rank}(\emptyset) = 0$. Now suppose the proposition holds for all $\beta < \alpha$ and $L^p(\sigma, \tau) = \alpha$. Then we have for each $\tau^\frown i \in T^p$ we have $L^p(\sigma, \tau^\frown i) \geq \text{rank}(\{\rho : \tau^\frown i^\frown\rho \in T^p, \sigma \notin T_{\tau^\frown i^\frown\rho}\})$ by induction hypothesis. By property 4 and definition of rank we have $L^p(\sigma, \tau) \geq \sup_{\tau^\frown i \in T^p} L^p(\sigma, \tau^\frown i) + 1 \geq \sup_{\tau^\frown i \in T^p} \text{rank}(\{\rho : \tau^\frown i^\frown\rho \in T^p, \sigma \notin T_{\tau^\frown i^\frown\rho}\}) + 1 = \text{rank}(\{\rho : \tau^\frown\rho \in T^p, \sigma \notin T_{\tau^\frown\rho}\})$. $\square$

In order for this forcing notion to have non trivial generics we need a way to extend conditions. Fix a condition p. Let $A \subseteq \omega^{<\omega}$ be a set such that for all $\sigma^\frown i \in A$ we have $\sigma \in T^p$ and $\{\tau : L^p(\tau, \sigma) \leq 1\} \subseteq T_{\sigma^\frown i} \subseteq T^P \cup A$. For such an A we can define $q = p[A]$ by $T^q = T^p \cup A\}$ and L^q given by

$$L^q(\sigma, \tau) = \begin{cases} L^p(\sigma, \tau) & \sigma, \tau \in T^p \\ \langle\sigma\rangle + \text{rank}(\text{Ext}(T^q, \tau)) & \sigma \in A, \sigma \notin T_\tau \\ 0 & \sigma \in T_\tau \\ L^p(\sigma, \rho) - 1 & \rho^\frown i = \tau \in A, \sigma \notin T_\tau, L^p(\sigma, \rho) < \omega \\ \langle\sigma\rangle & \text{otherwise} \end{cases}$$

Lemma 3.3. *If A meets the requirement of the definition then $p[A]$ is a valid condition. If we also have that $T^p \preceq T^p \cup A$ then $p[A] \leq p$.*

Proof. We show that $q = p[A]$ is well defined. Our requirement for A ensures that 1 and 2 hold. For 3—5, since $L^p = L^q \restriction T^p \times T^p$ the only way we can run into a problem is when considering $\sigma^\frown i \in A$. If $\rho \prec \tau \in T^q$ then by definition $L^q(\sigma^\frown i, \rho) > L^q(\sigma^\frown i, \tau)$. If $\rho \prec \sigma^\frown i$ then $L^p(\tau, \rho) \geq L^p(\tau, \sigma)$. If $0 < L^p(\tau, \sigma) < \omega$ then $L^q(\tau, \sigma^\frown i) = L^p(\tau, \sigma) - 1 < L^p(\tau, \sigma)$. If $L^p(\tau, \sigma) \geq \omega$ then $L^q(\tau, \sigma^\frown i) < \omega$. So 4 holds.

Fix n and τ and consider the set $\{\rho : L^q(\rho, \tau) \leq n\}$. If $\tau \in T^p$ then we have added at most n many elements to the set, so it is still finite. If $\tau = \sigma^\frown i \in A$ and ρ is in this set then either ρ belongs to the finite set $\{\rho : L^p(\rho, \sigma) \leq n + 1\}$ or $\langle\rho\rangle \leq n$. So there are only finitely many ρ that can be in $\{\rho : L^q(\rho, \tau) \leq n\}$. So 5 holds.

Now consider the set $\{\rho : L^q(\rho, \tau) = 0\}$. If $\tau \in T^p$ then $L^q(\sigma, \tau) \geq 1$ for each $\sigma \in A$, so we have $\{\rho : L^q(\rho, \tau) = 0\} = \{\rho : L^p(\rho, \tau) = 0\} = T_\tau$. If $\tau \in A$ then by definition of L^q we have $\rho \in T_\tau$ if and only if $L^q(\rho, \tau) = 0$. So 3 holds.

Since $L^p \subseteq L^q$ if $T^p \preceq T^p \cup A = T^q$ then $p[A] \leq p$. $\square$

Corollary 3.4. *If $\mathcal{G}$ is a sufficiently generic filter then $T^{\mathcal{G}}$ is a uniformly e-pointed tree with no dead ends.*

Proof. First we show that for each condition p and $\sigma \in T^p$ the set $\{q \leq p : \sigma \text{ is not a dead end}\}$ is dense below p. If σ is a dead end in T^p then enumeration of $(T_\sigma)_{\sigma\in\omega^\omega}$ gives us an i such that $T_{\sigma^\frown i} = \{\rho : L^p(\rho,\sigma) \leq 1\}$. Thus we can take $p[\{\sigma^\frown i\}] < p$ where σ is no longer a dead end. So $T^{\mathcal{G}}$ does not have any dead ends.

To show $T^{\mathcal{G}}$ is uniformly e-pointed consider some path $P \in [T^{\mathcal{G}}]$. We will show that $T^G = \bigcup_{\sigma\prec P} T_\sigma$. If $\sigma \in T^{\mathcal{G}}$ then $\sigma \in T^p$ for some $p \in G$. So by property 2 we have that $T_\sigma \subseteq T^p \subseteq T^{\mathcal{G}}$. On the other hand if $\sigma \in T^{\mathcal{G}}$ then consider a sequence $p_0 > p_1 > \cdots \subseteq G$ with $P \restriction n \in T^{p_n}$. Now consider the sequence $(L^{p_n}(\sigma, P\restriction n))_{n\in\omega}$. Since $L^{p_n} \subseteq L^{p_{n+1}}$ property 4 means that this is a decreasing sequence. Since ω_1^{CK} is a well order there is n such that $L^{p_n}(\sigma, P\restriction n) = 0$. So we have that $\sigma \in T_{P\restriction n}$. Hence $T^G = \bigcup_{\sigma\prec P} T_\sigma$. □

3.2 The forcing relation

Now that we have a forcing partial order and some useful operations on conditions, we will talk about forcing with conditions. We define $S_e^p \subseteq \omega^\omega$ to be the tree where $x \notin S_e^p \iff \exists y \prec x[y \in \Psi_e(T^p)]$. For a filter $\mathcal{G}$ we define $S_e^{\mathcal{G}} \bigcap_{p\in\mathcal{G}} S_e^p$. So $x \notin S_e^{\mathcal{G}} \iff \exists y \prec x[y \in \Psi_e(T^{\mathcal{G}})]$. By definition of Γ_e we have that $\Gamma_e(T^{\mathcal{G}}) = \{n : \mathrm{Ext}(S_e^{\mathcal{G}}, n) \text{ is well founded}\}$.

We define $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$ if $\mathrm{rank}(\mathrm{Ext}(S_e^p, x)) \leq \alpha$. From this definition it is clear that if $p \Vdash \mathrm{rank}(S_e^{\mathcal{G}} \restriction x) \leq \alpha$ then for any $\mathcal{G} \ni p$ we have that $\mathrm{rank}(S_e^{\mathcal{G}} \restriction x) \leq \alpha$. We now work towards proving the opposite direction.

Lemma 3.5. *Fix a condition p. Suppose that for each $i \in \omega, r \leq p$ there is $q \leq r$ such that $q \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x^\frown i)) \leq \beta$ for some $\beta < \omega_1^{CK}$ then there is $\hat{p} \leq p$ and $\alpha < \omega_1^{CK}$ such that $\hat{p} \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$.*

Proof. The function $(q, e) \mapsto S_e^q$ is $L_{\omega_1^{CK}}$-partial computable so by composition, the map $(q, e, x) \mapsto \mathrm{rank}(\mathrm{Ext}(S_e^q, x))$ is also $L_{\omega_1^{CK}}$-partial computable. So the set

$$C = \{(i, r, q, \beta) : q \leq r \land \mathrm{rank}(\mathrm{Ext}(S_e^q x^\frown i)) \leq \beta\}$$

is $L_{\omega_1^{CK}}$-c.e.

For each i we will define a condition r_i as follows. For each leaf $\sigma \in T^p$ let k_σ be the ith number such that $T_{\sigma^\frown k_\sigma} = \{\tau : L^p(\tau, \sigma) \leq 1\}$. Now we define $A_i = \{\sigma^\frown k_\sigma : \sigma \text{ is a leaf in } T^p\}$ and define $r_i = p[A_i]$. The definition of r_i only involves computable operations so the map $i \mapsto r_i$ is $L_{\omega_1^{CK}}$-computable and since $\omega \in L_{\omega_1^{CK}}$ the set $\{(i, r_i)\} \in L_{\omega_1^{CK}}$ by Σ_1-collection. Using Σ_1-collection again, this time with the set C, we get that there is a function $f \in L_{\omega_1^{CK}}$ such that $f(i) = (q_i, \beta_i)$ for some $q_i \leq r_i$ and β_i such that $q_i \Vdash S_e^{\mathcal{G}} \restriction x^\frown i \leq \beta_i$. Let $\alpha = \sup_i\{\beta_i : i \in \omega\}$. Since $f \in L_{\omega_1^{CK}}$, $\alpha < \omega_1^{CK}$.

To build $\hat{p}$ let and $T^{\hat{p}} = \bigcup_{i\in\omega} T^{q_i}$. Since $f \in L_{\omega_1^{CK}}$ we have that $T^{\hat{p}} \in L_{\omega_1^{CK}}$. $T^{\hat{p}}$ will satisfy property 1 because the sets $T^{q_i} \setminus T^p$ are disjoint and so $T^{\hat{p}}$ is well founded.

We define $L^{\hat{p}}$ using the following tools. For $\tau \in T^{\hat{p}}$ let τ_p the longest initial segment of τ that is in T^p. For $\sigma, \tau \in T^{\hat{p}}$ let $\mathrm{rank}(\sigma, \tau) = \mathrm{rank}(\{\rho : \tau^\frown\rho \in T^{\hat{p}}, \sigma \notin T_{\tau^\frown\sigma}\})$. Note that both of these operations are $L_{\omega_1^{CK}}$-computable. Define

$$L^{\hat{p}}(\sigma, \tau) = \begin{cases} L^p(\sigma, \tau) & \sigma, \tau \in T^p \\ 0 & \sigma \in T_\tau \\ L^p(\sigma, \tau_p) - |\tau| + |\tau_p| & \sigma \in T^p \setminus T_\tau, \tau \notin T^p, L^p(\sigma, \tau_p) < \omega \\ \langle\sigma\rangle + \mathrm{rank}(\sigma, \tau) & \text{otherwise} \end{cases}$$

Now we prove that $\hat{p}$ is a valid condition. Since it is built in an effective way out of $L_{\omega_1^{CK}}$-computable functions $L^{\hat{p}}$ is $L_{\omega_1^{CK}}$-computable. Since $\mathrm{dom}(L^{\hat{p}}) \in L_{\omega_1^{CK}}$ we have that $L^{\hat{p}} \in L_{\omega_1^{CK}}$. So we have that $\hat{p} \in L_{\omega_1^{CK}}$.

Now we show that $\hat{p}$ has the properties of a condition. Property 2 is straightforward. Property 3 follows from the definition of $L^{\hat{p}}$ and the fact that it held for each q_i.

For property 4 consider $\sigma, \rho \prec \tau$, and suppose that $L^{\hat{p}}(\sigma, \rho) > 0$. We look at several cases.

- $\sigma, \tau \in T^p$. Then $\rho \in T^p$ so by 4 for p we have $L^{\hat{p}}(\sigma, \tau) = L^p(\sigma, \tau) < L^p(\sigma, \rho) = L^{\hat{p}}(\sigma, \rho)$.
- $\sigma \in T_\tau$. Then $L^{\hat{p}}(\sigma, \tau) = 0 < L^{\hat{p}}(\sigma, \rho)$.
- $\sigma \in T^p \setminus T_\tau, \tau \notin T^p, L^p(\sigma, \tau_p) < \omega$. We have two subcases: if $\rho \notin T^p$ then $\tau_p = \rho_p$ so $L^{\hat{p}}(\sigma, \tau) = L^p(\sigma, \tau_p) - |\tau| + |\tau_p| < L^p(\sigma, \tau_p) - |\rho| + |\rho_p| = L^{\hat{p}}(\sigma, \rho)$. If $\rho \in T^p$ then $\rho \preceq \tau_p$ so $L^{\hat{p}}(\sigma, \tau) = L^p(\sigma, \tau_p) - |\tau| + |\tau_p| < L^p(\sigma, \tau_p) \leq L^p(\sigma, \rho) = L^{\hat{p}}(\sigma, \rho)$.
- Otherwise $L^{\hat{p}}(\sigma, \tau) = \langle \sigma \rangle + \mathrm{rank}(\sigma, \tau)$. If $\rho \notin T^p$ or $\sigma \notin T^p$ then $L^{\hat{p}}(\sigma, \rho) = \langle \sigma \rangle + \mathrm{rank}(\sigma, \rho) > \langle \sigma \rangle + \mathrm{rank}(\sigma, \tau)$ as $\rho \prec \tau$. If $\rho, \sigma \in T^p$ then consider i such that $\tau \in T^{q_i}$. By Proposition 3.2 we have that $\mathrm{rank}(\sigma, \tau) \leq L^{q_i}(\sigma, \tau) < L^{q_i}(\sigma, \rho) = L^p(\sigma, \rho) = L^{\hat{p}}(\sigma, \rho)$.

For property 5 fix τ and n. Suppose that $L^{\hat{p}}(\sigma, \tau) \leq n$. Then one of the following is true: $L^p(\sigma, \tau) \leq n$ or $\sigma \in T_\tau$ or $L^p(\sigma, \tau_p) - |\tau| + |\tau_p| \leq n$ or $\langle \sigma \rangle + \mathrm{rank}(\sigma, \tau) \leq n$. So σ is a member of the finite set $\{\sigma : L^p(\sigma, \tau) \leq n\} \cup T_\tau \cup \{\sigma : L^p(\sigma, \tau_p) \leq n + |\tau|\} \cup \{\sigma : \langle \sigma \rangle \leq n\}$. Hence the set $\{\sigma : L^{\hat{p}}(\sigma, \tau) \leq n\}$ is finite.

So we have shown that $\hat{p}$ is a valid condition. Since $T^p \preceq T^{\hat{p}}$ and $L^p \subseteq L^{\hat{p}}$ we have $\hat{p} \leq p$. Consider $\mathrm{Ext}(S_e^{\hat{p}}, x)$. By definition of $T^{\hat{p}}$ we have that $S_e^{\hat{p}} \subseteq S_e^{q_i}$ for each $i \in \omega$, so $\mathrm{rank}(\mathrm{Ext}(S_e^{\hat{p}}, x^\frown i)) \leq \mathrm{rank}(\mathrm{Ext}(S_e^{q_i}, x^\frown i)) < \alpha$. Thus we have $\mathrm{rank}(\mathrm{Ext}(S_e^{\hat{p}}, x)) \leq \alpha$ as desired. $\square$

Now we use this lemma to show that if a condition p cannot be extended to some q that forces $S_e^{\mathcal{G}}$ to have computable rank then p in fact forces $S_e^{\mathcal{G}}$ to be ill founded. We say $p \Vdash \mathrm{Ext}(S_e^{\mathcal{G}}, x)$ is ill founded if for all sufficiently generic filters $\mathcal{G} \ni p$ we have that $\mathrm{Ext}(S_e^{\mathcal{G}}, x)$ contains an infinite path.

Lemma 3.6. *If for all $q \leq p$ and $\alpha < \omega_1^{CK}$ we have $q \nVdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$ then $p \Vdash \mathrm{Ext}(S_e^{\mathcal{G}}, x)$ is ill founded.*

Proof. We define $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ if $\forall q \leq p, \alpha < \omega_1^{CK}[q \nVdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha]$.To prove this lemma, we first prove the simpler statement: if $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ then there is $q \leq p, i \in \omega$ such that $q \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty$.

Suppose this statement fails for some p and x. Then $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$, so we have that $\forall q \leq p, \alpha < \omega_1^{CK}[q \nVdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha]$, and there is no $q \leq p, i \in \omega$ such that $q \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty$ so we have $\forall q \leq p, i \in \omega[\exists r \leq q, \alpha < \omega_1^{CK}(r \Vdash (\mathrm{Ext}(S_e^{\mathcal{G}}, x^\frown i)) \leq \alpha)]$. So by Lemma 3.5 there is $\hat{p} \leq p$ such that $\hat{p} \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$. This contradicts the fact that $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$, so the statement holds.

Now we use this to prove the lemma. Since the set $\{q \leq p\} \subseteq \{q \leq r\}$ for $r \leq p$ we have that if $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ then $r \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ for all $r \leq p$. So if $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ then the set $\{q \leq p : \exists i \in \omega[q \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty]\}$ is dense above p. So if $p \in \mathcal{G}$ for some sufficiently generic $\mathcal{G}$ then there is $q \in \mathcal{G}$ and $i \in \omega$ such that $q \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty$. By repeating this argument we can build a sequence $X \in \omega^\omega$ such that for all $y \prec X$ there is $q \in \mathcal{G}$ such that $q \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, y)) = \infty$. We have that $X \in S_e^{\mathcal{G}}$ as otherwise there would be some $r \in \mathcal{G}$ and $y \prec X$ such that $r \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, y)) = 0$, a contradiction of $\mathcal{G}$ being a filter and $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, x)) = \infty$. $\square$

Now we have all the tools needed to prove the main result of this section.

Theorem 1.6. *There is a uniformly e-pointed tree $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.*

Proof. We show that for a sufficiently generic $\mathcal{G}$ we have that $T^{\mathcal{G}}$ is not hypertotal. We say $p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$ if there is $\sigma \in T^p$ and $\alpha < \omega_1^{CK}$ such that $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)) \leq \alpha$, or if there is $\sigma \notin T^p$ such that the initial segment of σ in T^p is not a leaf and $p \Vdash \mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)$ is ill founded. To show that $T^{\mathcal{G}}$ is not hypertotal it is enough for us the show that the sets $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ are dense for each e. To see this consider the two cases. If $p \in \mathcal{G}$ and there is $\sigma \in T^p$ and $\alpha < \omega_1^{CK}$ such that $p \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)) \leq \alpha$ then we have that $\mathrm{Ext}(S_e^p, \langle\sigma\rangle)$ is well founded and so $\mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle) \subseteq \mathrm{Ext}(S_e^p, \langle\sigma\rangle)$ is also well founded so $\sigma \in T^{\mathcal{G}} \cap \Gamma_e(T^{\mathcal{G}})$. On the other hand if there is $\sigma \notin T^p$ such that the initial segment of σ in T^p is not a leaf and $p \Vdash \mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)$ is ill founded, then by definition $p \in \mathcal{G}$ means that $\mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)$ is ill founded, so $\sigma \notin \Gamma_e(T^{\mathcal{G}})$. Since the initial segment of σ in T^p is not a leaf, no $q \leq p$ has $\sigma \in T^q$ so $\sigma \notin T^{\mathcal{G}}$.

Suppose towards a contradiction, that $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ is not dense. Let p be such that for all $q \leq p$ we have $q \nVdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$. Consider some leaf $\sigma \in T^p$ and let i, j be such that $T_{\sigma^\frown i} = T_{\sigma^\frown j} = \{\rho : L^p(\rho, \sigma) \leq 1\}$. Now consider $q = p[\{\sigma^\frown i\}]$; this is well defined by Lemma 3.3. By assumption on p we have that $q \nVdash \mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma^\frown j\rangle)$ is ill founded, so by Lemma 3.6 there is $r \leq q, \alpha < \omega_1^{CK}$ such that $r \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma^\frown j\rangle)) \leq \alpha$. Now consider $r' = r[\{\sigma^\frown j\}]$. Since $\sigma^\frown i \in T^r$ we have $\{\rho : L^r(\rho, \sigma) \leq 1\} \subseteq T_{\sigma^\frown i} = T_{\sigma^\frown j}$ and thus the condition r' is a valid condition. Since $r \leq p$ and σ is a leaf in T^p we have that $r' \leq p$. But we have $S_e^r \supseteq S_e^{r'}$ so $r' \Vdash \mathrm{rank}(\mathrm{Ext}(S_e^{\mathcal{G}}, \langle\sigma^\frown j\rangle)) \leq \alpha$ a contradiction. So we have that the set $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ is dense.

So for sufficiently generic $\mathcal{G}$ we have that $T^{\mathcal{G}}$ is uniformly e-pointed without dead ends and for all e we have $\overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$, and thus $\overline{T^{\mathcal{G}}} \nleq_{he} T^{G}$. ◻

4 Downwards density

4.1 The hyper Gutteridge operator

Gutteridge [7] proved the downwards density of the non-Δ_2^0 enumeration degrees using an operator Θ with the properties that if $\Psi_e(\Theta(A)) = A$ then A is c.e. and if $\Theta(A)$ is c.e. then A is Δ_2^0. Here we will take Gutteridge's construction and run it in $L_{\omega_1^{CK}}$ to produce a hyperenumeration operator Λ with similar properties. Thus we get the following result

Theorem 4.1. *If $A \subseteq \omega$ and $A \nleq_{he} \overline{\mathcal{O}}$ then there is $C \subseteq \omega$ such that $\emptyset <_{he} C <_{he} A$.*

Proof. Recall the definition of Θ: there is a c.e. set $B = \bigoplus_{k\in\omega} n_k$ which is the join of ω many initial segments of ω. Θ is defined by $\Theta(A) = B \cup \{(k, n_k) : k \in A\}$. B is built using finite injury to ensure that if $\Psi_e(\Theta(A)) = A$ then A is c.e. If $\Theta(A)$ is c.e. then $A \leq_e \overline{B}$ and so A is Δ_2^0. Hence for any non-Δ_2^0 set A we have that $\emptyset <_e \Theta(A) <_e A$.

To ensure that if $\Psi_e(\Theta(A)) = A$ then A is c.e. B has the property that for any $D \subseteq n \geq e$ we have that $n \in \Psi_e(\Theta(D)) \iff n \in \Psi_e(\Theta(D \cup (\omega \setminus n)))$. So if $\Psi_e(\Theta(A)) = A$ then

$$D \preceq A \iff D\restriction e \prec A\restriction e \wedge \forall n \in D \setminus e[n \in \Psi_e(\Theta(D\restriction n)]$$

We will use this idea to build Λ.

Before we start building Λ we need to set up some notation. For an $L_{\omega_1^{CK}}$-c.e. set A, given by formula $\exists y\varphi(x,y)$ where φ is Δ_0, we define $A_\alpha = \{x \in L_\alpha : L_\alpha \models \exists y\varphi(x,y)\}$. Since φ is Δ_0 we have that $A = \bigcup_{\alpha<\omega_1^{CK}} A_\alpha$. In this manner we can think of $L_{\omega_1^{CK}}$-c.e. sets as being enumerated over ordinal stages. Using this, for a set $B \in L_{\omega_1^{CK}}$ and ordinal $\alpha < \omega_1^{CK}$ we can define $\Gamma_{e,\alpha}(B) \in L_{\omega_1^{CK}}$ and get an $L_{\omega_1^{CK}}$-computable map $(B,e,\alpha) \mapsto \Gamma_{e,\alpha}(B)$. This is the hyperenumeration analogue of $\Psi_{e,s}(D)$. We can define $\Gamma_{e,\alpha}(B)$ more explicitly as $\Gamma_{e,\alpha}(B) = \{n : \text{rank}(S_{e,n}(B)) \leq \alpha\}$.

In the enumeration case, it is clear that $\Psi_e(W) = \bigcup_{s\in\omega} \Psi_{e,s}(W_s)$ for a c.e. set W, but this is not so clear for an $L_{\omega_1^{CK}}$-c.e. set A. Γ_e is monotonic, so we have that $\bigcup_{\alpha\in\omega_1^{CK}} \Gamma_{e,\alpha}(A_\alpha) \subseteq \Gamma_e(A)$. The other direction is needed for our construction, so we will prove it here using the rank of nodes in $S_e(A)$.

Claim 4.1.1. $\bigcup_{\alpha\in\omega_1^{CK}} \Gamma_{e,\alpha}(A_\alpha) = \Gamma_e(A)$ *For any* $L_{\omega_1^{CK}}$*-c.e.* A.

Proof. Consider some node $x \in S_e(A)$ with ordinal rank. We will use induction the rank of x and Σ_1^1 bounding to prove that there is $\alpha < \omega_1^{CK}$ such that x has rank $< \alpha$ in $S_e(A_\alpha)$. Base case if x is a leaf then by definition of Γ_e there is a finite $D_u \subseteq A$ such that $(x,u) \in W_e$. There is $\alpha < \omega_1^{CK}$ such that $D_u \subseteq A_\alpha$ so x is a leaf in $S_e(A_\alpha)$.

For the next step, suppose by the inductive hypothesis that for each $i \in \omega$ there is a least $\alpha_i < \omega_1^{CK}$ such that $x^\frown i$ has rank $< \alpha_i$ in $S_e(A_{\alpha_i})$. Consider the map $i \mapsto \alpha_i$. This is $L_{\omega_1^{CK}}$-computable and hence by Σ_1-collection there is a $\beta < \omega_1^{CK}$ such that $\alpha_i < \beta$ for all i. So it must be that x has rank $\leq \beta$ in $S_e(A_\beta)$.

So we have that if $n \in \Gamma_e(A)$ then there is $\alpha < \omega_1^{CK}$ such that $\text{rank}(S_{e,n}(A_\alpha)) < \alpha$ hence we have that $n \in \Gamma_{e,\alpha}(A_\alpha)$. $\square$

For Our construction of Λ we will modify Gutteridge's proof. We will build an $L_{\omega_1^{CK}}$-c.e. set $B = \bigoplus_{k\in\omega} n_k$ and define $\Lambda(A) = B \cup \{(k, n_k) : k \in A\}$. We will build B using sages in ω_1^{CK} and satisfy the following requirements for $D \subseteq m \geq e$:

$$\mathcal{R}_{e,m,D} : m \in \Gamma_e(B \cup \{(k,n_k) : k \in D\}) \iff m \in \Gamma_e(B \cup \{(k,n_k) : k \in D \vee k \geq m\})$$

We chose an ordering of requirements so that $\mathcal{R}_{e,m,D}$ is higher priority than $\mathcal{R}_{i,m+1,E}$. Note that this means the priority of our requirements has order type ω.

Now we can move onto the construction. A requirement $\mathcal{R}_{e,n,D}$ requires attention at stage α if there $n \notin \Gamma_{e,\alpha}(B_\alpha \cup \{(k, B_\alpha^{[k]}) : k \in D\})$ and there is $B \in L_\alpha$ such that $B_\alpha \subseteq B$ and $B_\alpha^{[k]} = B^{[k]}$ for $k < n$, B is the join of initial segments of ω and $n \in \Gamma_{e,\alpha}(B \cup \{(k, B^{[k]}) : k \in D\})$.

At stage α we consider the highest priority requirement that requires attention with some witness B. We then define $B_{\alpha+1} = B$. This completes the construction.

By the monotonicity of the Γ_e each requirement will need to act at most once, and that means that each column of B is finite. Now suppose that $\Gamma_e(\Lambda(A)) = A$ for some A and e. It is enough for us to show that A is $L_{\omega_1^{CK}}$-c.e. We claim that $A = \cup\{D : D\restriction e = A\restriction e \wedge \exists\alpha\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D\restriction n))]\}$. If D is such that $\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D\restriction n))]$ then by induction on $n \geq e$ we can see that $D \subseteq A$. So what we need to prove is that all $n \in A$ are contained in some such D. Fix $n \in A \setminus e$ and consider $D = A\restriction n$. We have is $\Lambda(D \cup (\omega \setminus n))$ is $L_{\omega_1^{CK}}$-c.e. Since $n \in A$ by Claim 4.1.1 there is a stage $\alpha < \omega_1^{CK}$ such that $n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D \cup (\omega \setminus n)))$. So at stage α or earlier the requirement $R_{e,n,D}$ will have acted and we have $n \in \Gamma_{e,\alpha+1}(\Lambda_{\alpha+1}(D))$. We can assume by induction that D has the property $\exists\alpha\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D\restriction n))]$. We have now proven that $D \cup \{n\} = A\restriction n+1$ also has this property, thus by induction $A = \cup\{D : D\restriction e = A\restriction e \wedge \exists\alpha\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D\restriction n))]\}$. Hence A is $L_{\omega_1^{CK}}$-c.e. $\square$

4.2 Downwards density below $\overline{\mathcal{O}}$

We have proven Downwards density for most degrees in $\mathcal{D}_{he}$, but the proof may not work when a degree is below $\overline{\mathcal{O}}$. If we look at some of the proofs of downwards density for the degrees below $\mathbf{0}'_e$ and try to translate them, then we have a problem. They are finite injury constructions and rely on the following property of enumeration operators: $\Psi_e(A) = \bigcup_{D \subseteq_{\text{fin}} A} \Psi_e(D)$. This property does not hold for hyperenumeration operators. In fact there are many sets A and operators e such that $\Gamma_e(A) \neq \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$. For example, if A is the graph of a non-hyperarithmetic function and Γ_e is such that $0 \in \Gamma_e(B)$ if and only if B contains the graph of some function. None of the hyperarithmetic subsets of A will contain a function, but A does contain a function, so $0 \in \Gamma_e(A) \setminus \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$.

The reason we did not have this problem when adapting the Gutteridge operator was because of the special way that it was constructed. First, it is important to note that for Π^1_1 sets A we do have $\Gamma_e(A) = \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$ by Claim 4.1.1. This property also holds for sets X of the form $X = \Lambda(A)$. To see this fix some Γ_e. For $n \geq e$ we have $n \in \Gamma_e(X) \iff n \in \Gamma_e(\Lambda(A \upharpoonright n))$. Here $\Lambda(A \upharpoonright n)$ is a Π^1_1 set, if $n \in \Gamma_e(X)$ then there is some hyperarithmetic $H \subseteq \Lambda(A \upharpoonright n) \subseteq X$ with have $n \in \Gamma_e(H)$. The result for $n < e$ comes from some coding of indices of reductions.

We will make use of this idea in the proof of the following:

Theorem 4.2. *If A is not Π^1_1 and $\Lambda(A)$ is Π^1_1 then there are $X <_{he} A$ such that $X >_e \mathbf{0}$.*

Proof. Let $(A_s)_{s<\omega_1^{CK}}$ be an $L_{\omega_1^{CK}}$-computable approximation to A. We know there must be one since $\Lambda(A)$ is Π^1_1. We will build $L_{\omega_1^{CK}}$-c.e. operator Ψ, and define $X = \Psi(A)$. There are two types of requirements we need to satisfy

$$\mathcal{A}_e : X \neq V_e \text{ where } V_e \text{ is the } e\text{th } \Pi^1_1 \text{ set}$$

that will ensure that $\mathbf{0} <_e X$, and

$$\mathcal{B}_e : \Gamma_e(X) \neq A$$

that will ensure that $A \not\leq X$. The ordering of requirements is $A_0 < B_0 < A_1 < \ldots$. We will build Ψ and X in ω_1^{CK} many stages. We will want to be able to add infinitely much to columns of X so we will build X as a subset of $\omega_1^{CK^2}$. By fixing an $L_{\omega_1^{CK}}$-computable injection from ω_1^{CK} to ω (for instance the well founded part of a Harrison order)we can turn X into a subset of ω.

We will use and an infinite injury construction here, putting the requirements on a tree of strategies. The outcome of node σ on the tree will be a set $\hat{A} \in L_{\omega_1^{CK}}$ that represents strategy σ's guess at A at this stage of the construction. Strategies $\sigma^\frown\hat{A}$ and below will only add axioms of the form (n, H) to Ψ for sets $H \supseteq \hat{A}$. This way their work will not interfere with strategies who think $\hat{A} \nsubseteq A$. Each strategy σ can put up a restriction $u < \omega_1^{CK}$ and require that strategies to the right of them on the tree only put things in columns $\geq u$ of X. For σ we consider its restriction u to be the sup of all the restrictions put up by nodes to the left of or above σ. The ordering of outcomes is $\hat{A} < \hat{B}$ if $\hat{A} \supseteq \hat{B}$. This is only a partial order sets, but we will see in the construction that we only use a limited collection of outcomes for each strategy, and this collection will be linearly ordered. We will argue that the left most path visited cofinally often is correct in its guesses about A and along this path all requirements are met. When a strategy has outcome $\hat{A}$ to the left of a previous outcome $\hat{B}$, it adds $(n, \hat{A})$ to Ψ for all n added by strategies below $\hat{B}$ to ensure that they cannot axioms they added cannot interfere with the strategies below $\hat{A}$.

Strategies: The strategy for a node σ of requirement $\mathcal{B}_e$ is to find a witness m where $m \in A \setminus \Gamma_e(X)$. When this strategy is initialized at some stage s it is given outcome $\hat{A}$ from it parent node and restriction u, the sup of the restrictions put up by nodes to the left of σ. σ has one variable m_s that it keeps track of. When initialized, we start with $m_{s+1} = 0$. At a limit stage s, we define $m_s = \liminf_{t<s} m_t$ if that exists, otherwise $m_s = 0$. We also have $X_s = \Psi_s(\hat{A})$ which is σ's guess at X at stage s. In the verification, we will prove that X_s eventually agrees with X on the first u many columns, that $m = \lim_{s<\omega_1^{CK}} m_s$ exists, and that $A(m) \neq \Gamma_e(X)(m)$.

Given m_s and X_s at stage s, the strategy asks if $m_s \notin \Gamma_{e,s}(X_s)$ and there is some $H \in L_s$ such that $H \subseteq (\omega_1^{CK} \setminus u) \times \omega_1^{CK}$ and $m_s \in \Gamma_{e,s}(X_s \cup H)$. If yes, then we put $m_s \in X$: we add axioms $(\langle \alpha, \beta \rangle, \hat{A})$ to Ψ_{s+1} for each $\langle \alpha, \beta \rangle \in X_s \cup H$ with $\alpha \geq u$. This will injure all lower priority $\mathcal{A}$ requirements. If no and $m_s \notin (A_s \cup \hat{A}) \triangle \Gamma_{e,s}(X_s)$, then we need to pick a new witness: set m_{s+1} to be the least m in $(A_s \cup \hat{A}) \triangle \Gamma_{e,s}(X_s)$. Otherwise $m_{s+1} = m_s$. No matter what, the outcome of σ is always $\hat{A}$, and σ does not put up any restriction.

The strategy for a node τ of requirement $\mathcal{A}_e$ is as follows. We will try to build an $L_{\omega_1^{CK}}$-c.e. aproximation $(P_s)_s$ to A by encoding parts of A into column u of X. The aproximation to A will eventually fail, as A is not $L_{\omega_1^{CK}}$-c.e., and we will use this point of difference to ensure that so that $X \neq V_e$. To this end we will build a sequence of coding points $(n_\beta, m_\beta)_{\beta<\alpha_s}$. The only changes we will make to this sequence is add a new element to the sequence or remove the last element (if there is one). So at limit stages s we can have $\alpha_s = \liminf_{t<s} \alpha_t$ and that will ensure that all the coding points are well defined at stage s. The idea with the coding points is that for all but the top one we have ensured that $\langle u, n_\beta \rangle \in V_e$ and that $\langle u, n_\beta \in X$ only if $m_\beta \in A$. Our aproximation P_s will be $\hat{A} \cup \{m_\beta : \langle u, n_\beta \rangle \in V_{e,s}\}$. Since $V_{e,s}$ is increasing P_S is be increasing as long as we do not remove coding points (n_β, m_β) after putting $m_\beta \in P$.

We are trying to make $X \neq V_e$, so when we notice $m_\beta \notin A_s$ it appears that we have succeeded and do not need to do anything. The problem with this, is that once we have $\alpha_s > \omega$ there are infinitely many m_β so we may see $m_\beta \notin A_t$ for a different β at each stage $t > s$ but have $m_\beta \in A$ for all $\beta < \alpha_s$. We could avoid this problem if we knew that A_s stabilized on hyperarithmetic sets, but we only know that it stabilizes on finite sets. This is enough for us, but we will have to keep track of the β where $m_\beta \notin A_s$, and sometimes our outcome will need to include numbers that are not in A_s.

To this end the τ will keep track of a finite sequence of victories $\hat{\beta}_s = \beta_0 > \beta_1 > \cdots > \beta_{k-1}$ with the property that $m_{\beta_0} < \cdots < m_{\beta_{k-1}}$ are numbers we thought were out of A at the previous stage. When τ is initialized we start with $\hat{\beta} = \emptyset$ and at limit stages s we define $\hat{\beta}_s$ be the longest sequence $\hat{\beta}$ such that $\hat{\beta}_i = \lim_{t<s} \hat{\beta}_{t,i}$. At a stage s the requirement updates the victories as the first step. We consider β such that m_β is the smallest $m \in \overline{A_s}$ with the property for all $i < k$, $m_{\beta_i} \leq m \implies \beta_i > \beta$. If there is such a β then we add it to our sequence of victories for $\hat{\beta}_{s+1}$ and remove all victories $\beta_i < \beta$. Next we remove invalid victories: if there is any $i < k$ such that $m_{\beta_i} \in A_s$ then we remove that victory and all victories for $j \geq i$. The reason we also remove larger victories is to deal with the fact that A_s only stabilizes on finite sets. In the verification we will prove that an initial segment of $\hat{\beta}_s$ stabilizes, and to ensure that that initial segment is β_s cofinally often, we need to remove smaller β_i whenever we see a change.

If the sequence of victories has become empty, then we think $P_s \subseteq A$ so we need to consider adding a coding point. If α_s is a successor, then we consider the highest coding point (n, m). If $\langle u, n \rangle \in X \setminus V_e$ then we can satisfy the $\mathcal{A}_e$ without any victories. So if $\langle u, n \rangle \notin V_{e,s}$ and $m \notin A_s$, then we add the axiom $(\langle u, n \rangle, P_s)$ to Ψ_{s+1} to keep $\langle u, n \rangle \in X$. This will, however invalidate the coding point, meaning we will have to remove it and try again if we ever see $\langle u, n \rangle$ enter V_e. If

$\langle u, n\rangle \in V_e$ then it is time to add a new coding point. If (n, m) has been invalidated, the remove it from the sequence of coding points first. If (n, m) has not been invalidated then we add m to P_{s+1}. To picking a new coding point we chose n_{α_s} to be the least unused number in column u and m_{α_s} to be the least member of $A_s \setminus P_s$ if there is one. We then add the axiom $(\langle u, n_{\alpha_s}\rangle, P_s \cup \{m_{\alpha_s}\})$ to Ψ_{s+1}. If $A_s \subseteq P_s$ then we cannot add a new coding point yet.

If there are no victories and α_s is a limit or 0, then we proceed to add a new coding point as above.

Finally we come to defining the outcome and restriction of τ. We always impose restriction $u+1$ on lower priority requirements so they cannot interfere with our coding points in column u. To define the outcome, we use the sequence of victories. If there are no victories this means we have outcome P_s since it looks like $P_s \subseteq A$. If there are victories, then we consider the least victory β_{k-1}. Since it looks like $m_{\beta_{k-1}} \notin A$ but $m_\beta \in A$ for all $\beta < \beta_{k-1}$ we give outcome $\hat{A} \cup \{m_\beta : \beta < \beta_{k-1}\}$. Note that, as promised above, the collection of outcomes is linearly ordered.

Verification: We will use induction to argue that for each node σ on the true path, the following hold

1. There is a left most outcome $\hat{A}$ that is visited cofinally often.
2. $\hat{A} \subseteq A$.
3. For all $\hat{B} < \hat{A}$ that were outcomes of σ we have $\hat{B} \not\subseteq A$.
4. $\Psi(\hat{A})^{[u]} = \Psi(A)^{[u]}$.
5. σ stops adding axioms to Ψ after some stage.
6. The requirement for σ is satisfied.

We start with the case where σ is a strategy for a $\mathcal{B}_e$ requirement. Let s be a stage after which no node to the left of σ is visited and no node above σ adds axioms to Ψ. In this case σ only has one outcome $\hat{A}$, and since it only adds axioms using $\hat{A}$, 1. through 4. hold. So we just need to check that the requirement was met and stops adding axioms. Consider the set $W = \bigcup_{s<\omega_1^{CK}} \Gamma_{e,s}(X_s) = \lim_{s<\omega_1^{CK}} \Gamma_{e,s}(X_s)$. Since W is $L_{\omega_1^{CK}}$-c.e. there is some least $m \in W \triangle A$. So we have that $m = \lim_{s<\omega_1^{CK}} m_s$, and once this limit settles down σ puts axioms into Ψ at most once more, so 5. is satisfied.

Now to show the requirement is met. We have two cases. Case 1: suppose that $m \in W$. Then $m \notin A$ and $m \in \Gamma_e(X)$ since $X_s\Psi_s(\hat{A}) \subseteq \Psi(A) = X$, so the requirement is satisfied.

Case 2: suppose that $m \in A$. Then consider the set $X^* = \Psi(\hat{A}) \cup \{n : \exists H, \tau \prec \sigma[\tau \text{ put } (n, H) \in \Psi]\}$. Since any number put into X by a strategy to the right of σ is put into X with a subset of $\hat{A}$ when we next visit σ, and because requirements to the left of σ only add axioms (n, H) for $H \not\subseteq A$ by 3., we have that $X \subseteq X^*$ and $X^{[v]} = X^{*[v]}$ for all $v < u$. It is clear that X^* is $L_{\omega_1^{CK}}$-c.e. so if $m \in \Gamma_e(X^*)$ then by claim 4.1.1 there is $t < \omega_1^{CK}$ and hyperarithmetic $H \subseteq X^*$ such that $m \in \Gamma_{e,t}(H)$. Since X an X^* agree on the first u many columns we have that $m \in W$ as σ will have acted at some stage $\geq t$ to ensure this. But $m \in A$, a contradiction, so $m \notin \Gamma_e(X^*) \supseteq \Gamma_e(X)$. So the requirement is satisfied.

Now we consider the case where σ is a strategy for an $\mathcal{A}_e$ requirement. First we will argue that we stop adding coding locations after some stage. Consider the set $P = \bigcup_s P_s$. This is an $L_{\omega_1^{CK}}$-c.e. set, so there is some least $m \in P \triangle A$. Consider some stage s such that $A \restriction m+1 =$

$A_t\restriction m+1 = P_s\restriction m+1\triangle\{m\}$ for all $t > s$. If $m \in P$ then after stage s we will always have $m = m_{\beta_0}$ as the first victory, so σ will stop growing P and will not add any more coding locations. If $m \notin P$ then after stage s whenever we chose a new coding location (n', m') we will have $m' = m$ since m never leaves A_t this location will never be invalidated, so, since $m \notin P$, it must be than $\langle u, n'\rangle \notin V_e$ so we stop adding coding locations. In either case 5. is satisfied.

Next we argue that the sequence of victories stabilizes on an initial segment. If $P \subseteq A$ then this initial segment will be the empty set. Otherwise, observe that $P_s = \{m_\beta : \beta < \alpha_s\}$ (with the last element excluded if it has not been added). Consider the sequence $(\beta_i)_{i<k}$ defined by taking β_i is the least β such that m_β is the least element of $\{m_\beta : \beta < \beta_{i-1}\} \setminus A$ if this set is nonempty. Since α_s is well founded this sequence must be finite and have some length k. Consider a stage s after which $A_t\restriction m_{\beta_{k-1}} + 1$ and P_t have stabilized. At all stages $t > s$ where σ is visited we must have $\beta_0 > \cdots > \beta_k$ as a proper initial segment of the sequence of victories as these are all true victories and no other victories could be added for $m_\beta < m_{\beta_{k-1}}$ after stage s. So after stage s the outcome of σ will always be a subset of $\hat{A} := \hat{B} \cup \{m_\beta : \beta < \beta_{k-1}\}$ where $\hat{B}$ was the outcome of the parent of σ. We now claim that $\hat{A}$ will satisfy 1. through 3.

If the outcome is ever $\hat{C} < \hat{A}$ then it must be that $m_{\beta_{k-1}} \in \hat{C}$ so 3. is satisfied and after stage s we never have any outcome left of $\hat{A}$. Since we could not extend our sequence $(\beta + i)_{i<k}$ is must be that $\hat{A} \subseteq A$ so 2. is satisfied. This also means that any victory β added to the end of our sequence of victories must have $m_\beta \in A$. This means that β will eventually be removed from our sequence of victories when we see $m_\beta \in A_t$ for some t. We remove a victory, we also remove all victories for $m > m_\beta$, so there will be cofinally many stages where the sequence of victories is just $(\beta_i)_{i<k}$. Hence the outcome of σ will cofinally often be $\hat{A}$, satisfying 1. To see 4. recall that we a coding location (n,m) was added as stage t we used $P_t \cup \{m\}$ to put it in X. So if it was added before the coding location $(n_{\beta_{k-1}}, m_{\beta_{k-1}})$ was added then $\langle u, n\rangle \in \Psi_e(\hat{A})$ and if it was added after, then $\langle u, n\rangle \notin \Psi(A)$.

To see that the requirement for σ is satisfied, we need to look at two cases. First, if the sequence of victories was empty this meant that $P \subseteq A$ and we stopped adding coding locations because the top location (n, m) had $\langle u, n\rangle \notin V_e$ if this location was never invalidated then, $m \in A$ and $\langle u, n\rangle \in \Psi(A)$. If it was invalidated, then we added the axiom $(\langle u, n\rangle, P)$ to Ψ so $\langle u, n\rangle \in \Psi(A)$. Second, if the sequence of victories was not empty then $m_{\beta_0} \notin A$, so $\langle u, n_{\beta_0}\rangle \notin \Psi(A)$ but $m_{\beta_0} \in P$ so $\langle u, n_{\beta_0}\rangle \in V_e$.

This completes the induction. Note that condition 1. ensures that there is a true path. Since each requirement on the true path is satisfied we have that X is not $L_{\omega_1^{CK}}$-c.e. and $A \not\leq_{he} X$. The fact that $X \leq_{he} A$ follows from Proposition 5.4, which is proved in the next section.

□

5 Other reducibilities

We now look at some other reducibilities that are different from $\leq_{he}$ but could be considered notions of hyperenumeration reducibility. We show most of these reducibilities $\leq_*$ share some of the properties of $\leq_{he}$ like extending enumeration reducibility and having $A \leq_* B \oplus \overline{B} \iff A$ is Π^1_1 in B. The reducibility to consider is the notion of relatively Π^1_1.

Definition 5.1. We say that A is *relatively* Π^1_1 *in* B, $A \leq_{\Pi^1_1} B$, if whenever B is Π^1_1 in X we have that A is Π^1_1 in X.

We say that A is *uniformly relatively* Π^1_1 *in* B, $A \leq_{u\Pi^1_1} B$, if there is a computable f such that if $B = \Gamma_e(X \oplus \overline{X})$ then $A = \Gamma_{f(e)}(X \oplus \overline{X})$

We used hyperenumeration operators to define $\leq_{u\Pi^1_1}$, but it could equivalently be defined by saying there is Turing operator that turns hyperenumerations of B into hyperenumerations of A or that there is a computable function that turns Π^1_1 formulas for B into Π^1_1 formulas for A.

The fact that composition of hyperenumeration operators is uniform means that $A \leq_{he} B \implies A \leq_{u\Pi^1_1} B$ and by definition we have $A \leq_{u\Pi^1_1} B \implies A \leq_{\Pi^1_1} B$. It is natural to ask if these implications are distinct. From Theorem 1.6 we can see that $\leq_{he}$ is different from $\leq_{\Pi^1_1}$ because by definition each relatively Π^1_1 degree is uniquely determined by the total degrees above it. A closer look at the proof of Corollary 1.7 show us that $\overline{T} \leq_{u\Pi^1_1} T$ for any uniformly e-pointed without dead ends, hence $\leq_{he}$ and $\leq_{u\Pi^1_1}$ are different. The remaining possible separation is an open question.

Question 5.2. Are there sets A and B such that $A \leq_{\Pi^1_1} B$ and $A \not\leq_{u\Pi^1_1} B$.

A negative answer to the above questions could be seen as a proof of Selman's theorem for $\leq_{u\Pi^1_1}$. One approach to try to answer this question is to see if one can transform Selman's original proof to this context.

On the other hand, we observe that the uniformity of an e-pointed tree T without dead ends is important have $\overline{T} \leq_{u\Pi^1_1} T$. Perhaps there is a sufficiently generic non-uniformly e-pointed tree T without dead ends such that $\overline{T} \not\leq_{u\Pi^1_1} T$. Such a result would be interesting because it would show that there is no notion hyperenumeration operators for relatively Π^1_1.

Another way that may be natural to define hyperenumeration reducibility is by changing the nature of the set W in the usual definition of enumeration reducibility.

Definition 5.3. We say that A is *continuously higher enumeration reducible* to B, $A \leq_{che} B$ if there is a Π^1_1 set W such that $n \in A \iff \exists u[\langle n, u\rangle \in W \wedge D_u \subseteq B]$.

We say that A is ω_1^{CK}*-enumeration reducible* to B, $A \leq_{\omega_1^{CK}} B$, if there is an $L_{\omega_1^{CK}}$-c.e. set W such that $n \in A \iff \exists H[(n, H) \in W \wedge H \subseteq B]$

Both these reducibilities can be thought of as relativizations of enumeration reducibility to $L_{\omega_1^{CK}}$. In the case of continuously higher enumeration operators these are, like enumeration operators, continuous functions on $\mathcal{P}(\omega)$ with the Scott topology (Basic open sets take the form $\{X : X \supseteq D_u\}$), hence the name. This reducibility could be thought of as an enumeration analogue of continuously higher Turing reducibility.

Both of these reducibilities imply hyperenumeration reducibility. For $\leq_{che}$ this follows from the fact (Sanchis [1]) that we can replace the c.e. set in the definition of hyperenumeration reducibility with a Π^1_1 set. It takes a bit more work for ω_1^{CK}-enumeration reducibility.

Proposition 5.4. *If* $A \leq_{\omega_1^{CK}} B$ *then* $A \leq_{he} B$.

Proof. Suppose W is a $L_{\omega_1^{CK}}$-c.e. set of pairs such that $n \in A \iff \exists H[(n, H) \in W \wedge H \subseteq B]$. Since W is $L_{\omega_1^{CK}}$-c.e. there is an $L_{\omega_1^{CK}}$-computable injection $f : \omega_1^{CK} \to W$. Consider the set

$$V = \{\langle n, \sigma^\frown k, u\rangle : \exists H, i, e[(n, H) \in W, D_u = H \restriction k, |\sigma| = \langle i, e\rangle, e \in \mathcal{O}, H = \Psi_i(\emptyset^{(e)})]\}$$

Since W and $\mathcal{O}$ are $L_{\omega_1^{CK}}$-c.e., V is also $L_{\omega_1^{CK}}$-c.e. and hence Π^1_1. Now all that is needed is to check that $A \leq_{he} B$ via V. If $H \subseteq B$ and $(n, H) \in W$ then V will put $n \in A$ as every path of length

$\langle i, e\rangle + 1$ will be removed for $H = \Psi_i(\emptyset^{(e)})$. If there is no $H \subseteq B$ such that $(n, H) \in W$ then we can build a path f as follows:

$$f(\langle i, e\rangle) = \begin{cases} 0 & e \notin \mathcal{O} \vee (n, \Psi_i(\emptyset^{(e)})) \notin W \\ \text{least } k \text{ such that } \Psi_i(\emptyset^{(e)}) \upharpoonright k \not\subseteq B & \text{otherwise.} \end{cases}$$

Note for all $\sigma \prec f$ we have that $\langle n, \sigma, u\rangle \notin V$ for any u with $D_u \subseteq B$. □

So both these reducibilities imply hyperenumeration reducibility. These implications are strict. In fact, if we consider some set X with $L_{\omega_1^{CK}} \in L_X$ then anything ω_1^{CK} enumeration reducible to $X \oplus \overline{X}$ will be hyperarithmetic in X and there are sets hyperenumeration reducible to $X \oplus \overline{X}$ that are not hyperarithmetic in X, for instance $\mathcal{O}^X$.

Since these are weaker than hyperenumeration reducibility, it can be that Selman's theorem holds for these reducibilities. For continuously higher enumeration reducibility we have a proof of Selman's theorem that uses the enumeration degrees.

Theorem 5.5. *The continuously higher enumeration degrees embed as the enumeration degrees above $\mathcal{O}$ via the map $X \mapsto \mathcal{O} \oplus X$.*

Proof. For one direction, suppose that $X \oplus \mathcal{O} \leq_e Y \oplus \mathcal{O}$. Then $X \leq_e Y \oplus \mathcal{O} \leq_{che} Y$.

For the other direction, suppose that $X =\leq_{che} Y$ via the Π^1_1 set W. Let f be an m-reduction of W to $\mathcal{O}$. We define a c.e. set $W_e = \{\langle n, u\rangle : D_u = D_v \oplus D_q \wedge D_q = \{2f(\langle n, v\rangle + 1\}\}$. So we have that $n \in \Psi_e(Y \oplus \mathcal{O}) \iff \exists v[D_v \subseteq Y \wedge f(\wedge n, v\rangle) \in \mathcal{O}] \iff \exists v[D_v \subseteq Y \wedge \langle n, v\rangle \in W] \iff n \in X$. So $X \oplus \mathcal{O} \leq_e Y \oplus \mathcal{O}$.

To see that this embedding is onto, observe that ever enumeration degree above $\mathcal{O}$ contains a set of the form $X \oplus \mathcal{O}$. □

To see how this gives us Selman's theorem recall that every enumeration degree **a** above $\mathcal{O}$ is uniquely determined by the class total degrees above **a**. This means the every *che*-degree is uniquely determined by the class of degrees above it that map to a total enumeration degree. If an enumeration degree above $\mathcal{O}$ is total then it will contain a set of the form $X \oplus \overline{X} \oplus \mathcal{O}$ and be the image of a *che*-total degree.

Note that there are *che*-total degrees that get mapped to non-total e-degrees. For instance $\mathcal{O}$ is not total.

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