VICTORIA UNIVERSITY OF WELLINGTON
Te Whare Wānanga o te Ūpoko o te Ika a Māui

School of Mathematics and Statistics
Te Kura Mātai Tatauranga

PO Box 600
Wellington
New Zealand

Tel: +64 4 463 5341
Fax: +64 4 463 5045
Internet: sms-office@vuw.ac.nz

Infinitary Logic and the Harrison Linear Order

Josiah Jacobsen-Grocott

Supervisor: Adam Day

Submitted in partial fulfilment of the requirements for
MATH488 Project.

Abstract

Infinitary logic is an extension of first order logic that allows infinite conjunction and disjunction. This project looks at the Harrison linear order, using tools from infinitary logic and computability theory. The Harrison linear order is a computable linear order with no hyperarithmetic infinite descending sequence. We use computable infinite formulae and the Kreisel-Barwise compactness theorem to show that the Harrison linear order exists. We consider the Scott rank of models and use infinitary logic to give a bound on the Scott rank of computable models. We show that the Harrison linear order is an example of a model that meets this bound.

Contents

Chapter 1

Introduction

In this report, we look at infinitary logic. Infinitary logic is like first order logic, but with infinite conjunction and disjunction now allowed. The report assumes that the reader is familiar with first order logic, ordinals and some computability theory.

Chapter 2 introduces infinitary logic and gives an example of one of the differences between first order logic and infinitary logic in that the compactness theorem fails for infinitary logic. We go on to define the Scott rank of a model.

To define the Scott rank, we first define an equivalence relation $\sim_\alpha$ between tuples of models for all the ordinals. $(\mathcal{M}, \bar{a}) \sim_0 (\mathcal{N}, \bar{b})$ if $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{b})$ for all atomic formulae $\phi(\bar{v})$. If α is a limit ordinal, then $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$ if $(\mathcal{M}, \bar{a}) \sim_\beta (\mathcal{N}, \bar{b})$ for all $\beta < \alpha$. If $\alpha = \beta + 1$ then $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$ if for each $c \in M$ there is a $d \in N$ such that $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{N}, \bar{b}d)$ and for each $d \in N$ there is a $c \in M$ such that $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{N}, \bar{b}d)$.

The Scott rank of a model $\mathcal{M}$ is defined to be the least ordinal α such that if two tuples from M are equivalent at the α level, then they are equivalent at all levels. We prove that the Scott rank exists and is less than $|M|^+$.

Next we use infinitary logic to construct formulae $\phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{v})$ such that if $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{b})$ then $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$. Chapter 2 finishes by using these formulae to prove Scott's isomorphism theorem (Theorem 2.5), which says that we can describe a countable model up to isomorphism with a single countable infinitary sentence.

In Chapter 3, we cover the computability theory used in later chapters. We define Kleene's $\mathcal{O}$ and some useful functions relating to $\mathcal{O}$. We can think of Kleene's $\mathcal{O}$ as a set of notations for ordinals with an inductive definition. 0 is a notation for $\emptyset$. If a is a notation for α, then 2^a is a notation for $\alpha + 1$. If φ_e is a total computable function who's image is an increasing sequence of notations for ordinals, then $3 \cdot 5^e$ is a notation for the suprimum of the set of ordinals corresponding to the notations in the image of φ_e.

We prove that every ordinal isomorphic to a computable well ordering of $\mathbb{N}$ has a notation, and we define ω_1^{ck} to be the smallest ordinal without a notation in $\mathcal{O}$.

We go on to define Π^1_1, Σ^1_1 and Δ^1_1 sets. We show that the Π^1_1 subsets of $\mathbb{N}$ are precisely the sets that can be defined by a predicate that begins with a forall qualification over subsets of $\mathbb{N}$ and is followed by an arithmetic predicate. For Σ^1_1 sets, the predicate starts with an existential quantifier over subsets of $\mathbb{N}$, and Δ^1_1 sets are defined to be sets that are both Π^1_1 and Σ^1_1.

We show that $\mathcal{O}$ is Π^1_1 but not Σ^1_1 and conclude the chapter with Σ^1_1 bounding, which says that every Σ^1_1, and hence also computable, subset of $\mathcal{O}$ there is a computable ordinal bounding the size of the ordinals with notations in the subset of $\mathcal{O}$. The content of Chapter 3 closely follows Chapter 1 of [1].

In Chapter 4, we define computable and Δ^1_1 infinitary formulae and computable models.

We define the rank of an infinitary formula inductively to be 1 plus the suprimum of the ranks of proper subformulae. We prove that computable formulae have rank less than ω_1^{ck} and that their truth values for some computable model can be computed relative to some computable jump.

While the compactness theorem fails for infinitary logic, there is a variation called Kreisel-Barwise compactness that states that if every Δ_1^1 subset of a Π_1^1 set of Δ_1^1 sentences has a computable model, then the Π_1^1 set has a computable model. We use this theorem to construct the Harrison linear order, a computable linear order that is not a well order but has no Δ_1^1 infinite descending sequence. We prove that any computable linear order with this property must be isomorphic to $\omega_1^{ck} + \mathbb{Q} \times \omega_1^{ck} + \alpha$ for some computable ordinal α.

Chapter 5 links things back to the Scott rank. We give an alternative definition of Scott rank. The new definition can give different values than the previous definition, but the value of the Scott rank given by the previous definition can be calculated from the value in the new definition. Using this new definition, we show that the Scott rank of the Harrison linear order is greater than ω_1^{ck}. The rest of Chapter 5 is devoted to showing that the Scott rank of any computable model is less than or equal to $\omega_1^{ck} + 1$. This forces the Scott rank of the Harrison linear order to be $\omega_1^{ck} + 1$, and the Harrison linear order is an example of a model that meets the bound of Scott rank of computable models.

Chapter 2

Scott's Isomorphism Theorem

This chapter introduces infinite formulae and Scott rank and gives some theorems relating to them. We will use the notation $\overline{a} \in M^n$ to denote a sequence $a_0, \ldots, a_{n-1} \in M$, and if ϕ is a formula with free variables $v_0, \ldots, v_{n-1}$, then $\phi(\overline{a})$ denotes ϕ with v_i interpreted as a_i for each i.

2.1 Infinite Formulae

If $\mathcal{L}$ is a language, then the set of infinite formulae $\mathcal{L}_{\kappa,\omega}$ is defined inductively as for normal $\mathcal{L}$-formulae with two extra inductive cases. If X is a set of $\mathcal{L}_{\kappa,\omega}$-formulae and $|X| < \kappa$, then

$$\bigwedge_{\phi \in X} \phi$$

and

$$\bigvee_{\phi \in X} \phi$$

are $\mathcal{L}_{\kappa,\omega}$-formulae.

Satisfaction of $\mathcal{L}_{\kappa,\omega}$-formulae is defined in the normal inductive way with the two extra cases.

$$\mathcal{M} \models \bigwedge_{\phi \in X} \phi$$

if $\mathcal{M} \models \phi$ for all $\phi \in X$, and

$$\mathcal{M} \models \bigvee_{\phi \in X} \phi$$

if $\mathcal{M} \models \phi$ for some $\phi \in X$.

We say that a formula is an $\mathcal{L}_{\infty,\omega}$-formula if it is an $\mathcal{L}_{\kappa,\omega}$-formula for some cardinal κ.

Example 2.1. An example of an $\mathcal{L}_{\omega_1,\omega}$-sentence. For each n, let

$$\phi_n = \exists x_0, \ldots, x_n \forall y \bigvee_{i \leq n} y = x_i.$$

Basically, ϕ_n is a first order sentence saying that there are no more than $n+1$ elements. So the $\mathcal{L}_{\omega_1,\omega}$-sentence

$$\phi = \bigvee_{n \in \mathbb{N}} \phi_n$$

says that there are finitely many elements. From this it is possible to see that the compactness theorem fails for infinitary logic. If we let $T = \{\phi\} \cup \{\neg\phi_n : n \in \mathbb{N}\}$ then T is certainly not satisfiable, but T is finitely satisfiable, as for any finite $T_0 \subseteq T$ there is a largest n such that $\neg\psi_n \in T_0$. If a model $\mathcal{M}$ has $n+2$ many elements, then $\mathcal{M} \models T_0$.

In first order logic, if two models are isomorphic, then they are elementarily equivalent. The next theorem extends this to $\mathcal{L}_{\infty,\omega}$.

Theorem 2.2. *If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic $\mathcal{L}$-structures, then they model the same $\mathcal{L}_{\infty,\omega}$-sentences.*

Proof. Let $f : M \to N$ be an isomorphism. We will now use induction on formula complexity. If $\phi(\overline{v})$ is atomic and $\overline{a} \in M^n$ then $\mathcal{M} \models \phi(\overline{a}) \iff \mathcal{N} \models \phi(f(\overline{a}))$. Suppose $\phi(\overline{v}) = \neg\psi(\overline{v})$. By induction hypothesis $\mathcal{M} \models \psi(\overline{a}) \iff \mathcal{N} \models \psi(f(\overline{a}))$. So $\mathcal{M} \models \phi(\overline{a}) \iff \mathcal{N} \models \phi(f(\overline{a}))$. Suppose $\phi(\overline{v}) = \exists x\psi(\overline{v}x)$. Suppose $\mathcal{M} \models \phi(\overline{a})$. Then $\mathcal{M} \models \psi(\overline{a}c)$ for some $c \in M$. So $\mathcal{N} \models \psi(f(\overline{a}c))$ which means $\mathcal{N} \models \phi(f(\overline{a}))$. For the other direction, suppose $\mathcal{N} \models \phi(f(\overline{a}))$. Then $\mathcal{N} \models \psi(f(\overline{a})d)$ for some $d \in N$. So $\mathcal{M} \models \psi(\overline{a}f^{-1}(d))$ which means $\mathcal{M} \models \phi(\overline{a})$.

Finally if

$$\phi(\overline{v}) = \bigwedge_{\psi\in X} \psi(\overline{v})$$

then $\mathcal{M} \models \phi(\overline{a}) \iff \mathcal{M} \models \psi(\overline{a})$ for all $\psi \in X$ and by induction hypothesis $\mathcal{M} \models \psi(\overline{a}) \iff \mathcal{N} \models \psi(f(\overline{a}))$. Also $\mathcal{N} \models \phi(f(\overline{a})) \iff \mathcal{N} \models \psi(f(\overline{a}))$ for all $\psi \in X$. Which when put together gives $\mathcal{M} \models \phi(\overline{a}) \iff \mathcal{N} \models \phi(f(\overline{a}))$.

Since we have proved this for a complete set of logical connectives, we are done. □

2.2 $\sim_\alpha$ Equivalence Relation

In this section, we lay some ground work for Scott's isomorphism theorem (Theorem 2.5). We look at an equivalence relation $\sim_\alpha$ between tuples of elements of models. The idea is that if two tuples are equivalent at the α level, then we can define a partial isomorphism between the two models mapping one tuple to the other, and α tells us to what degree we can extended this partial isomorphism

For each ordinal α and natural number n, we define an equivalence relation $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ where $\overline{a} \in M^n$ and $\overline{b} \in N^n$. We define this inductively over the ordinals. First $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if $\mathcal{M} \models \phi(\overline{a}) \iff \mathcal{N} \models \phi(\overline{b})$ for all atomic formulae $\phi(\overline{v})$. If α is a limit ordinal, then $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{N}, \overline{b})$ for all $\beta < \alpha$. If $\alpha = \beta + 1$ then $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if for each $c \in M$ there is a $d \in N$ such that $(\mathcal{M}, \overline{a}c) \sim_\beta (\mathcal{N}, \overline{b}d)$ and for each $d \in N$ there is a $c \in M$ such that $(\mathcal{M}, \overline{a}c) \sim_\beta (\mathcal{N}, \overline{b}d)$.

From this definition, it is possible to see that if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ then $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{N}, \overline{b})$ for all $\beta < \alpha$. This means that if $\overline{a} \in M^n$ and $\overline{b} \in N^n$ and $(\mathcal{M}, \overline{a}) \nsim_\alpha (\mathcal{N}, \overline{b})$ for some α, then there is a least α with this property. Since α is least, for all $\beta < \alpha$, $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{N}, \overline{b})$. This is the definition of $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ at a limit stage, so α is not a limit ordinal.

Using induction, we can see that if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$, then there is a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ that maps $\overline{a}$ to $\overline{b}$, and we can ensure that there are at least n many other elements in the partial isomorphism for any natural number $n \leq \alpha$. The next theorem says that there is an α for which we can make this partial isomorphism a proper isomorphism.

Theorem 2.3. *Let $\mathcal{M}$ be a model with infinitely many elements. Define $\Gamma_\alpha = \{(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in M^n$ and $(\mathcal{M}, \overline{a}) \nsim_\alpha (\mathcal{M}, \overline{b})\}$ then there is an $\alpha < |M|^+$ such that $\Gamma_\beta = \Gamma_\alpha$ for all $\beta > \alpha$.*

Proof. This proof follows the one in [2].

Notice that if $\alpha < \beta$ then $\Gamma_\alpha \subseteq \Gamma_\beta$. First we show that if $\Gamma_\alpha = \Gamma_{\alpha+1}$ then $\Gamma_\beta = \Gamma_\alpha$ for all $\beta > \alpha$. We do this using induction. Suppose $\Gamma_\alpha = \Gamma_\beta$. Let $\bar{a}, \bar{b} \in M^n$. If $(\mathcal{M}, \bar{a}) \sim_\beta (\mathcal{M}, \bar{b})$ then $(\mathcal{M}, \bar{a}) \sim_{\alpha+1} (\mathcal{M}, \bar{b})$. So if $c \in M$ then there is $d, e \in M$ such that $(\mathcal{M}, \bar{a}c) \sim_\alpha (\mathcal{M}, \bar{b}d)$ and $(\mathcal{M}, \bar{a}e) \sim_\alpha (\mathcal{M}, \bar{b}c)$. Since $\Gamma_\alpha = \Gamma_\beta$, $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{M}, \bar{b}d)$ and $(\mathcal{M}, \bar{a}e) \sim_\beta (\mathcal{M}, \bar{b}c)$. Therefore $(\mathcal{M}, \bar{a}) \sim_{\beta+1} (\mathcal{M}, \bar{b})$, so $\Gamma_\alpha = \Gamma_{\beta+1}$.

Suppose β is a limit ordinal and for all $\alpha \leq \gamma < \beta$, $\Gamma_\gamma = \Gamma_\alpha$. Then if $(\bar{a}, \bar{b}) \notin \Gamma_\alpha$ then $(\bar{a}, \bar{b}) \notin \Gamma_\gamma$ for all $\gamma < \beta$. So $(\mathcal{M}, \bar{a}) \sim_\gamma (\mathcal{M}, \bar{b})$ for all $\gamma < \beta$. Which means $(\mathcal{M}, \bar{a}) \sim_\beta (\mathcal{M}, \bar{b})$ and $(\bar{a}, \bar{b}) \notin \Gamma_\beta$. Therefore $\Gamma_\alpha \supseteq \Gamma_\beta$, so $\Gamma_\alpha = \Gamma_\beta$.

Now all we need to do is show that for some $\alpha < |M|^+$, $\Gamma_\alpha = \Gamma_{\alpha+1}$. Suppose this is not the case. Then we will construct a map $f : |M|^+ \to M^{<\omega} \times M^{<\omega}$. For each $\alpha < |M|^+$ make $f(\alpha) \in \Gamma_{\alpha+1} \setminus \Gamma_\alpha$. This is well defined since if $\alpha < \beta < |M|^+$ then $\Gamma_\alpha \subsetneq \Gamma_\beta$. What is more, f is injective. This is a contradiction as $|M^{<\omega} \times M^{<\omega}| = |M| < |M|^+$. $\square$

The smallest α satisfying Theorem 2.3 for a model $\mathcal{M}$ is called the Scott rank of $\mathcal{M}$ and is denoted $\mathrm{sr}(\mathcal{M})$. Later we will use the fact that $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$ implies $(\mathcal{M}, \bar{a}) \sim_{\alpha+1} (\mathcal{N}, \bar{b})$ when $\alpha = \mathrm{sr}(\mathcal{M})$ to produce an isomorphism from $\mathcal{M}$ to $\mathcal{N}$ that maps $\bar{a}$ to $\bar{b}$.

We now want to construct formulae $\phi^{\mathcal{M}}_{\bar{a},\alpha}$ so that if $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{b})$ then $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$. We define the formulae $\phi^{\mathcal{M}}_{\bar{a},\alpha}$ inductively.

$$\phi^{\mathcal{M}}_{\bar{a},0}(\bar{v}) = \bigwedge_{\psi \in X} \psi(\bar{v})$$

where $X = \{\psi : \psi \text{ is atomic and } \mathcal{M} \models \psi(\bar{a})\} \cup \{\neg\psi : \psi \text{ is atomic and } \mathcal{M} \not\models \psi(\bar{a})\}$. If α is a limit ordinal, then

$$\phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{v}) = \bigwedge_{\beta < \alpha} \phi^{\mathcal{M}}_{\bar{a},\beta}(\bar{v}).$$

If $\alpha = \beta + 1$ then

$$\phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{v}) = \bigwedge_{c \in M} \exists d \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{v}d) \wedge \forall d \bigvee_{c \in M} \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{v}d).$$

Theorem 2.4. *$\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{b})$ if and only if $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$.*

Proof. This proof follows the one in [2].

We will use induction on the ordinals. $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},0}(\bar{b})$ if and only if $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{b})$ for all atomic formulae. Which is the definition of $(\mathcal{M}, \bar{a}) \sim_0 (\mathcal{N}, \bar{b})$. If α is a limit ordinal, then $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{b})$ if and only if $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\beta}(\bar{b})$ for all $\beta < \alpha$. Which by induction hypothesis is if and only if $(\mathcal{M}, \bar{a}) \sim_\beta (\mathcal{N}, \bar{b})$ for all $\beta < \alpha$. Which is by definition if and only if $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$.

Now we do the case where $\alpha = \beta + 1$. First suppose that $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{b})$. So

$$\mathcal{N} \models \bigwedge_{c \in M} \exists d \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{b}d).$$

So for all $c \in M$ there is $d \in N$ such that $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{b}d)$. Which means $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{N}, \bar{b}d)$. We also have

$$\mathcal{N} \models \forall d \bigvee_{c \in M} \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{b}d).$$

So for all $d \in N$ there is $c \in M$ such that $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{b}d)$. Which means $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{N}, \bar{b}d)$. So by definition $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$.

Now for the other direction. Suppose $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$. Then for all $c \in M$ there is $d \in N$ such that $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{N}, \bar{b}d)$. So

$$\mathcal{N} \models \bigwedge_{c \in M} \exists d \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{b}d).$$

Also, for all $d \in N$ there is $c \in M$ such that $(\mathcal{M}, \bar{a}c) \sim_\beta (\mathcal{N}, \bar{b}d)$. So

$$\mathcal{N} \models \forall d \bigvee_{c \in M} \phi^{\mathcal{M}}_{\bar{a}c,\beta}(\bar{b}d).$$

So $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{b})$. $\square$

2.3 Scott's Isomorphism Theorem

Theorem 2.5 (Scott's isomorphism theorem)**.** *Let $\mathcal{M}$ be a countable $\mathcal{L}$-structure. Then there is an $\mathcal{L}_{\omega_1,\omega}$ sentence $\Phi^{\mathcal{M}}$ such that if $\mathcal{N}$ is a countable $\mathcal{L}$-structure, then $\mathcal{N} \models \Phi^{\mathcal{M}}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

If $\mathcal{M}$ is an $\mathcal{L}$-structure and the α is the Scott rank of $\mathcal{M}$ then we define $\Phi^{\mathcal{M}}$ to be the sentence

$$\phi^{\mathcal{M}}_{\emptyset,\alpha} \wedge \bigwedge_{n=0}^{\infty} \bigwedge_{\bar{a} \in M^n} \forall \bar{v}[\phi^{\mathcal{M}}_{\bar{a},\alpha}(\bar{v}) \rightarrow \phi^{\mathcal{M}}_{\bar{a},\alpha+1}(\bar{v})].$$

Proof of 2.5. This proof closely follows the one in [2].

Suppose $\mathcal{N}$ is a countable $\mathcal{L}$-structure and $\mathcal{N} \models \Phi^{\mathcal{M}}$. Let $m_0, m_1, \ldots$ list the elements of M and $n_0, n_1, \ldots$ list the elements of N. We will define functions $f_0, f_1, \ldots$ inductively ensuring that for all n $(\mathcal{M}, dom(f_n)) \sim_\alpha (\mathcal{N}, Im(f_n))$ and that $f_n \subseteq f_{n+1}$. So each f_n will be a partial isomorphism. We want $f = \bigcup_{n=0}^{\infty} f_n$ to be an isomorphism so it will need to be a bijection. To ensure this, at odd stages of our induction process we will add m_i into the domain and at even stages we will add n_i to the image.

Let $f_0 : \emptyset \rightarrow N$. Since $\mathcal{N} \models \phi^{\mathcal{M}}_{\emptyset,\alpha}$ we have $\mathcal{M} \sim_\alpha \mathcal{N}$.

At stage $n = 2i + 1$, if $m_i \in dom(f_{n-1})$, then $f_n = f_{n-1}$. Otherwise, let $\bar{a} = dom(f_{n-1})$ and $\bar{b} = Im(f_{n-1})$. We have $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{b})$. Which by Theorem 2.4 means $\mathcal{N} \models \phi^{\mathcal{M}}_{\bar{b},\alpha}$. So, as $\mathcal{N} \models \Phi^{\mathcal{M}}$ we have $(\mathcal{M}, \bar{a}) \sim_{\alpha+1} (\mathcal{N}, \bar{b})$. So there is $d \in N$ such that $(\mathcal{M}, \bar{a}m_i) \sim_\alpha (\mathcal{N}, \bar{b}d)$. Define f_n to be the extension of f_{n-1} that maps m_i to d.

At stage $n = 2i$ we do a similar thing for n_i. If $n_i \in Im(f_{n-1})$ then $f_n = f_{n-1}$. Otherwise, let $\bar{a}, \bar{b}$ be defined as before. By the same argument as above, $(\mathcal{M}, \bar{a}) \sim_{\alpha+1} (\mathcal{N}, \bar{b})$. So there is $c \in M$ such that $(\mathcal{M}, \bar{a}c) \sim_\alpha (\mathcal{N}, \bar{b}n_i)$. Define f_n to be the extension of f_{n-1} that maps c to n_i.

If $f = \bigcup_{n=0}^{\infty} f_n$ then f is a bijection from M to N. All that is left is to check that f is an elementary embedding. If $\bar{a} \in M^n$ then for some f_n, $\bar{a} \in dom(f_n)$. $(\mathcal{M}, dom(f_n)) \sim_\alpha (\mathcal{N}, Im(f_n))$ means that $(\mathcal{M}, dom(f_n)) \sim_0 (\mathcal{N}, Im(f_n))$. Which means $(\mathcal{M}, \bar{a}) \sim_0 (\mathcal{N}, f(\bar{a}))$.

Using this property we can see that if R is an n-ary relation symbol then $\mathcal{M} \models R(\bar{a}) \iff \mathcal{N} \models R(f(\bar{a}))$. Similarly if g is an n-ary function symbol then $\mathcal{M} \models g(\bar{a}) = b \iff \mathcal{N} \models g(f(\bar{a})) = f(b)$. Finally if c is a constant symbol then $\mathcal{M} \models a = c \iff \mathcal{N} \models f(a) = c$. So the constant, function and relation symbols have the same interpretation. So f is an isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Now for the other direction. From Theorem 2.2, we know that if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, then they model the same sentences, so all we have to do is show that $\mathcal{M} \models \Phi^{\mathcal{M}}$. $\mathcal{M} \sim_\alpha \mathcal{M}$ so $\mathcal{M} \models \phi^{\mathcal{M}}_{\emptyset,\alpha}$. Let $\bar{a}, \bar{b} \in M^n$. Suppose $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{M}, \bar{b})$. Then as

α is the Scott rank of $\mathcal{M}$, $(\mathcal{M},\overline{a}) \sim_{\alpha+1} (\mathcal{M},\overline{b})$. Which means that for all $\overline{a},\overline{b} \in M^n$, $\mathcal{M} \models \phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b}) \rightarrow \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$. So

$$\mathcal{M} \models \bigwedge_{n=0}^{\infty} \bigwedge_{\overline{a}\in M^n} \forall\overline{v}[\phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \rightarrow \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v})].$$

So $\mathcal{M} \models \Phi^{\mathcal{M}}$. $\square$

Chapter 3

Some Computability Theory

This chapter builds up the computability theory necessary to analyse the formulae and models we look at in the next chapter. In this chapter we use the *s-m-n* and recursion theorems without going into the details of what is going on. If $f(x_0, \ldots, x_{n-1})$ is a computable function then we will use $f(x_0, \ldots, x_{m-1})$ for $m < n$ to denote the total computable function such that $\varphi_{f(x_0,\ldots,x_{m-1})}(x_m \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1})$.

3.1 Kleene's $\mathcal{O}$

Kleene's $\mathcal{O}$ is a subset of $\mathbb{N}$ that we define inductively along with a partial order $<_\mathcal{O}$.

1. $0 \in \mathcal{O}$.
2. If $n \in \mathcal{O}$ then $2^n \in \mathcal{O}$ and $n <_\mathcal{O} 2^n$.
3. If φ_e is a computable function and for all $n \in \mathbb{N}$ $\varphi_e(n) <_\mathcal{O} \varphi_e(n+1)$ then $3 \cdot 5^e \in \mathcal{O}$ and for all $n \in \mathbb{N}$, we have $\varphi_e(n) <_\mathcal{O} 3 \cdot 5^e$.

We can think of $\mathcal{O}$ and $<_\mathcal{O}$ as something like a well order. Each element has a successor and we have a way of introducing larger elements when we have a computable increasing sequence. The problem is with condition 3. We have infinitely many numbers for the same computable function, so we introduce infinitely incomparable elements.

Theorem 3.1. *$\mathcal{O}$ has no infinitely descending sequence.*

Theorem 3.2. *$\{n : n <_\mathcal{O} x\}$ is computably enumerable for any $x \in \mathcal{O}$.*

Proof. This proof is base on a proof in [1].

For this proof we will use the recursion theorem. Define the following computable function $f(x, n)$.

1. If $x = n$ then halt and output 0.
2. If $x = 2^y$ then run $f(y, n)$ and if it halts, output 1.
3. If $x = 3 \cdot 5^e$ the set $t := 0$ and loop the following:
   (a) increment t.
   (b) run $f(\varphi_e(s), n)$ for t many steps for $s := 0, \ldots, t$. If one of them halts, output 1.

So $\{n : n <_\mathcal{O} x\}$ will be the domain of the function $g(n)$ that halts if $f(x, n) \downarrow= 1$ and diverges otherwise. □

This means that $<_\mathcal{O}$ restricted to the set $\{n : n <_\mathcal{O} x\}$ is c.e. since for any $m, i \in \{n : n <_\mathcal{O} x\}$ either $i = m$, $i \in \{n : n <_\mathcal{O} m\}$ or $m \in \{n : n <_\mathcal{O} i\}$, which are all c.e. statements to check. Note also that this is done in a uniform way, so there is a computable function q such that for all $x \in \mathcal{O}$, $q(x)$ is a code for the restriction of $<_\mathcal{O}$ to the set $\{n : n <_\mathcal{O} x\}$.

$\{n : n <_\mathcal{O} x\}$ is well ordered and c.e. so it defines a computable well ordering. For some element $a \in \mathcal{O}$, if α is the order type of $\{n : n <_\mathcal{O} a\}$ then we call a a notation for α.

We define a map $||$ from notations to ordinals where if a is a notation for α then $|a| = \alpha$. We define $\omega_1^{ck} = \sup\{|a| : a \in \mathcal{O}\}$. So ω_1^{ck} is larger than any ordinal we have a notation for. Theorem 3.4 shows that ω_1^{ck} is larger than the order type of any computable well order of $\mathbb{N}$. Later we will extend this to show that all Σ_1^1 well orders are smaller than ω_1^{ck}.

We can define a computable function $+_\mathcal{O}$ such that if $a, b \in \mathcal{O}$ then $a +_\mathcal{O} b \in \mathcal{O}$ and $|a| + |b| = |a +_\mathcal{O} b|$. We define $+_\mathcal{O}$ recursively.

$$a +_\mathcal{O} b = \begin{cases} a & b = 0 \\ 2^{a +_\mathcal{O} n} & b = 2^n \\ 3 \cdot 5^{h(a,e)} & b = 3 \cdot 5^e. \end{cases}$$

Here h is defined in parallel with $+_\mathcal{O}$ and is the function such that $\varphi_{h(a,e)}(n) = a +_\mathcal{O} \varphi_e(n)$.

We use $+_\mathcal{O}$ to prove the following theorem.

Theorem 3.3. *There is a computable function g such that if $dom(\varphi_e) \subseteq \mathcal{O}$ then $|g(e)| > |a|$ for all $a \in dom(\varphi_e)$.*

Proof. This proof closely follows the one in [1].

We make $g(e) = 3 \cdot 5^i$ where $\varphi_i(n)$ sums up the first n elements of $dom(\varphi_e)$. To do this we define $h(e, n)$ to output 0 if $n = 0$, otherwise $h(e, n-1) +_\mathcal{O} m_0 +_\mathcal{O} \ldots +_\mathcal{O} m_{n-1} +_\mathcal{O} 1$, where $m_i = i$ if $\varphi_e(i)$ halts after n many steps and $m_i = 0$ otherwise. We define $g(e) = 3 \cdot 5^{h(e)}$.

Now we need to show that $|g(e)| > |a|$ for all $a \in dom(\varphi_e)$. From the definition, $h(e, n) <_\mathcal{O} h(e, n+1)$, so $g(e) \in \mathcal{O}$. For any $a \in dom(\varphi_e)$ there is n such that $\varphi_e(a)$ halts after n many steps, so $|h(e, \max\{a, n\})| \geq |a|$. Therefore $|g(e)| > |a|$.

$\square$

We define the binary relation $R_e(x, y)$ to hold if $\varphi_e(x, y) \downarrow$. We call a relation R_e well founded if there is no sequence $(x_n)_{n=0}^\infty$ such that for all n $R_e(x_{n+1}, x_n)$. This means that R_e must be irreflexive and antisymmetric, but not necessarily transitive or have an opinion on every pair. So well orders are well founded, but being well founded does not make a relation a well order.

Theorem 3.4. *There is a computable function f such that R_e is well founded on $\mathbb{N}$ if and only if $f(e) \in \mathcal{O}$. If R_e is well founded then $|f(e)|$ is greater than or equal to the order type of R_e.*

Proof. This proof closely follows the proof in [1].

We will define f by mapping initial segments of R_e to subsets of $\mathcal{O}$, using recursion with f, and then use g from Theorem 3.3 to take the sum of the heights of these subsets.

Define $h(e, n, x, y)$ by

$$h(e, n, x, y) \downarrow \iff \varphi_e(x, y) \downarrow \wedge \varphi_e(x, n) \downarrow \wedge \varphi_e(y, n) \downarrow.$$

So h is computable. We can think of $h(e, n)$ as being a restriction of R_e to the numbers R_e below n.

We define functions $t(e, n)$ and f at the same time.

$$t(e, n) \downarrow \iff \exists x (f(h(e, x)) = n) \wedge \exists x, y (R_e(x, y))$$

and $f(e) = g(t(e))$ where g is from Theorem 3.3. So t halts on the domain of the function $f(h(e,n))$, which should be a subset of $\mathcal{O}$ that includes notations for ordinals larger than every initial segment of R_e. If $R_e = \emptyset$ though, then $t(e)$ diverges everywhere.

Now we need to show that f is indeed the desired function. If $R_e = \emptyset$ then $f(e) = g(t(e)) = 0 \in \mathcal{O}$. Now suppose R_e is well founded and non empty. Then we can use transfinite induction with respect to R_e. Base case: $h(e,0) = \emptyset$, and so $f(h(e,0)) = 0$ and $|0| = |\emptyset|$. Inductive step: assume $f(h(e,n)) \in \mathcal{O}$ and $|f(h(e,n))| \geq |R_h(e,n)|$ for all n R_e below m. Then $dom(t(h(e,m))) \subseteq \mathcal{O}$. So $f(h(e,m)) \in \mathcal{O}$ and $|f(h(e,n))| < |f(h(e,m))|$ for all n R_e below m. So, as $|R_h(e,m)| \leq \sup\{|R_h(e,n)| : R_e(n,m) \wedge n \neq m\}$, we have $|f(h(e,n))| \geq |R_h(e,n)|$.

Now we suppose $f(e) \in \mathcal{O}$. We need to show that R_e is well founded. We will now use transfinite induction with respect to $<_\mathcal{O}$. Base case: $f(e) = 0$ then $R_e = \emptyset$ so it is well founded. Since $|f(h(e,n))| < |f(e)|$ for all n by induction hypothesis, every initial segment of R_e is well founded. So R_e is well founded. □

Since all computable well orders are well founded, a consequence of Theorem 3.4 is that ω_1^{ck} is the smallest ordinal larger than the order type of any computable well order. We call the ordinals smaller than ω_1^{ck} the computable ordinals.

We can use computable ordinals in relation with iterated Turing jumps. For a computable ordinal $\alpha = |a|$ we define $0^\alpha \subseteq \mathbb{N} \times \mathbb{N}$ using transfinite induction. When α is a successor, we want 0^α to contain the jump of $0^{\alpha-1}$, and when α is a limit ordinal we want $0^\aleph$ to contain 0^β for all $\beta < \alpha$. To be precise we will define 0^a for some $a \in \mathcal{O}$ and interpret 0^α as 0^a for some $|a| = \alpha$. So we define $0^0 = \emptyset$. If $a = 2^b$ then $0^a = 0^b \cup \{(a,n) : \varphi_n^{0^b}(n)\downarrow\}$. If $a = 3 \cdot 5^e$ then $0^a = \bigcup_{n\in\mathbb{N}} 0^{\varphi_e(n)}$.

We will use 0^a along with a so that we can iterate all $b <_\mathcal{O}$. By definition 0^a can compute the halting problem for 0^b for all $b <_\mathcal{O}$. So 0^a has the desired properties.

3.2 Δ_1^1 Functions

We say a function $A \subseteq \mathbb{N}$ is Σ_1^1 if

$$x \in A \iff \exists B \forall n \varphi_e(B,n,x)$$

where $B \subseteq \mathbb{N}$, $n \in \mathbb{N}$ and φ_e is some total computable function. Similarly we say A is Π_1^1 if

$$x \in A \iff \forall B \exists n \varphi_e(B,n,x).$$

If A is both Σ_1^1 and Π_1^1, then we say A is Δ_1^1. It is a theorem that the Δ_1^1 sets are precisely the hyperarithmetic sets. These definitions can be extended to the case where $A \subseteq \mathcal{P}(\mathbb{N})$. From this definition it is clear that the complement of a Σ_1^1 set is Π_1^1 and the converse.

Consider a predicate of the form $\forall m \exists B \forall n \varphi_e(B,n,m,x)$. If this predicate is true for some x, then for all m let B_m be a set that makes $\forall n \varphi_e(B_m,n,m,x)$ true. If we let $B = \{(m,n) : n \in B_m\}$, then, for an appropriate modification of φ_e to φ_i, B is a set that makes $\forall n, m \varphi_i(B,n,m,x)$ true.

On the other hand, if some B makes $\forall n, m \varphi_i(B,n,m,x)$, then by letting $B_m = \{n : (m,n) \in B\}$ this makes $\forall n \varphi_e(B_m,n,m,x)$ true. So, by combining n and m, we can rewrite the original predicate in the form $\exists B \forall n \varphi_j(B,n,x)$. So we have turned the original predicate into a Σ_1^1 predicate.

For a predicate of the form $\exists m(p(m,x))$ where $p(m,x)$ is a Π_1^1 predicate, we can take its negation $\forall m \neg p(m,x)$ and turn that into a Σ_1^1 predicate as before. If we take the negation again, we get a Π_1^1 predicate that is equivalent to the original predicate.

Now consider a predicate of the form $\exists m \exists B \forall n \varphi_e(B, n, m, x)$. Define $\varphi_i(B, n, x, l)$ by finding the least m such that $m \geq l$ or $m \notin B$ and running $\varphi_e(B, n, m, x)$, except whenever φ_e asks if $y \in B$, instead ask if $y + m \in B$. This means that the original predicate is equivalent to $\exists B \forall n \forall l \varphi_i(B, n, x, l)$, which we can turn into a Σ^1_1 predicate.

Similarly to before, if a predicate is of the form $\exists m(p(m, x))$ where $p(m, x)$ is a Π^1_1 predicate, we can take its negation, apply the above trick, and take it's negation again to get a Π^1_1 predicate. This allows us to prove the following nice theorem.

Theorem 3.5. *Every arithmetic set is a Δ^1_1 set.*

Proof. We will use induction on the number of quantifiers in an arithmetic predicate. Base case: no quantifiers, we have $p(x)$ for some total computable p. This is trivially Σ^1_1 and Π^1_1. For the inductive step, we can reduce the predicate to one of the forms that we just looked at, using the inductive hypothesis. As shown above, these can each be turned into Σ^1_1 and Π^1_1 predicates. $\square$

We can use this proof to allow us to let φ_e be arithmetic in the definition of Σ^1_1 and Π^1_1. By applying Theorem 3.5, we can get predicates that look like $\exists A \exists B \forall n \varphi_i(A, B, n, x, y)$ and $\forall A \forall B \exists n \varphi_i(A, B, n, x, y)$. By adjusting φ_i to look at a paired set, we can turn these into Σ^1_1 and Π^1_1 predicates.

A nice consequence of this is that we can think of the set quantifiers in the definition of Σ^1_1 and Π^1_1 as quantifiers of functions. We can call a set $X \subseteq \mathbb{N} \times \mathbb{N}$ a function if $\forall x \exists y \forall w((x, y) \in X \wedge (w = y \vee (x, w) \notin X))$. So if $(x, y) \in X$ then x is mapped to y. Since it is arithmetic to check if X is a function, and computable to find what x is mapped to for some set X, and we can code any function as a set, we can think of Σ^1_1 and Π^1_1 predicates as quantifying over functions if we wish.

Similarly, we can think of a function as being Σ^1_1 or Π^1_1 if the set that codes the function is Σ^1_1 or Π^1_1.

Theorem 3.6. *Every Π^1_1 or Σ^1_1 function on $\mathbb{N}$ is Δ^1_1.*

Proof. This proof closely follows the proof in [1].

Consider a Π^1_1 (Σ^1_1) function $f : \mathbb{N} \to \mathbb{N}$. We have

$$(x, y) \in f \iff p(x, y)$$

for some Π^1_1 (Σ^1_1) predicate p. Since f is a function, we also have

$$(x, y) \notin f \iff \exists w p(x, w) \wedge y \neq w)$$

which says $(x, y) \notin f$ if and only if f maps x to something else. As we have shown before, this can be turned into a Π^1_1 (Σ^1_1) predicate so the complement of f is Π^1_1 (Σ^1_1). So f must be Σ^1_1 (Π^1_1). Therefore f is Δ^1_1. $\square$

Having a numbering of the computable functions is very useful. It would be nice to have something similar for the Δ^1_1 functions and Π^1_1 and Σ^1_1 sets. To get a numbering for the Π^1_1 and Σ^1_1 sets, we can think of the eth set as being defined by the predicate $\exists B \forall n \varphi_e(B, n, x) \downarrow= 1$ or $\forall B \exists n \varphi_e(B, n, x) \downarrow= 1$. Since checking if a computable function halts is arithmetic, these predicates define Π^1_1 and Σ^1_1 sets. If we look at the original definition of Π^1_1 and Σ^1_1 sets, we see that all Π^1_1 and Σ^1_1 sets have predicates of this form.

To get a numbering of the Δ^1_1 functions is harder, but it is possible to find a Π^1_1 numbering, but we will not prove that such a numbering exists.

Towards finding a Π^1_1 numbering, notice that, since all Π^1_1 functions are Δ^1_1, we only need to consider Π^1_1 predicates. Let $p(x, y)$ be a Π^1_1 predicate, and consider the predicate

$q(x, y) = \forall w \exists z p(w, z) \wedge p(x, y)$. From a code for p, we can compute a code for q. The set X defined by q is either the empty set, or for all x there is y such that $(x, y) \in X$ and X is also defined by p.

Consider the predicate $H(e) = \exists x \forall B \exists n \varphi_e(B, n, x) \downarrow = 1$. $H(e)$ is Π^1_1 and tells us if the eth Π^1_1 set is empty or not. So using H we can get a Pi^1_1 enumeration of the Π^1_1 sets that contain a function. This numbering contains all the δ^1_1 functions. The following theorem, which we will not prove, shows that all of the sets in this enumeration contain a Δ^1_1 function.

Theorem 3.7 (Π^1_1 uniformization). *If X is a Pi^1_1 set of pairs, and for all x there is y such that $(x, y) \in X$, then there is a Π^1_1 function $f \subseteq X$.*

3.3 Bounding of Σ^1_1 sets

In this section we will work towards showing that every Σ^1_1 well order is smaller than ω^{ck}_1.

Consider a Π^1_1 predicate $\forall B \exists n \varphi_e(B, n, x)$. φ_e can only ask finitely many questions about elements of B. Let $B_{<m}$ denote the sequence $a_0 \ldots a_{m-1}$ where a_i is 1 if $i \in B$ and 0 otherwise. So $B_{<m}$ encodes the initial segment of B of length m. Let $\varphi_i(B_{<m}, x)$ be the function that runs $\varphi_e(\{a : a \in B \wedge a < m\}, n, x)$ for all $n < m$ and outputs if one of $\varphi_e(\{a : a \in B \wedge a < m\}, n, x)$ does. Then

$$\forall B \exists n \varphi_e(B, n, x) \iff \forall B \exists n \varphi_i(B_{<n}, x).$$

For a total computable $\varphi_i(\overline{a}, x)$ where $\overline{a} \in 2^{<\omega}$, we define $S_i(x)$ to be the set $\{\overline{a} \in 2^{<\omega} : \varphi_i(\overline{a}, x) = 0\}$. Since φ_i is total, $S_i(x)$ is a computable set and what is more we can compute a code for $S_i(x)$ from i and x. We define $<$ on $S_i(x)$ by $\overline{b} < \overline{a}$ if $\overline{a}$ is an initial segment of $\overline{b}$. Note $<$ is a computable relation on $S_i(x)$.

Theorem 3.8. *$(S_i(x), <)$ is well founded if and only if $\forall B \exists n \varphi_i(B_{<n}, x)$ holds.*

Proof. This proof follows the one in [1].

$(S_i(x), <)$ is not well founded if and only if there is $\overline{a}_0 > \overline{a}_1 > \ldots$ an infinite descending sequence in $S_i(x)$. Let B be the set described by $\lim_{n \to \infty} \overline{a}_n$. Then $\forall n \neg \varphi_i(B_{<n}, x)$; so $\forall B \exists n \varphi_i(B_{<n}, x)$ does not hold.

On the other hand, if $\forall B \exists n \varphi_i(B_{<n}, x)$ does not hold, then $\exists B \forall n \neg \varphi_i(B_{<n}, x)$; so $B_{<1} < B_{<2} < \ldots$ is an infinite descending sequence in $S_i(x)$. $\square$

Now we have the tools to prove the following theorem.

Theorem 3.9. *If $A \subseteq \mathbb{N}$ is Π^1_1, then $B \leq_m \mathcal{O}$.*

Proof. This proof follows the one in [1].

If A is Π^1_1, then for some total computable φ_i

$$x \in A \iff \forall B \exists n \varphi_i(B_{<n}, x) \iff (S_i(x), <) \text{ is well founded.}$$

Since $S_i(x)$ and $<$ are computable from x and i, there is a total computable function t such that $R_{t(x)}(\overline{a}, \overline{b})$ if and only if $\overline{a}, \overline{b} \in S_i(x)$ and $\overline{a} < \overline{b}$. Let f be the function from Theorem 3.4. Then $(S_i(x), <)$ is well founded if and only if $f(t(x)) \in \mathcal{O}$. So $f \circ t$ is a many-one reduction from A to $\mathcal{O}$. $\square$

Now we work towards showing that $\mathcal{O}$ is Π^1_1.

Theorem 3.10. *If $A \subseteq \mathcal{P}(\mathbb{N})$ is Σ^1_1, then $\cap A$ is Π^1_1.*

Proof. This proof is based on the one in [1].

So for some φ_e
$$X \in A \iff \exists B \forall n \varphi_e(B, n, X).$$

Let $B = \cap A$ then
$$y \in B \iff \forall X[\exists B \forall n \varphi_e(B, n, X) \to y \in X] \iff \forall X \forall B \exists n(\neg\varphi_e(B, n, X) \vee y \in X).$$

The last predicate can be turned into a Π^1_1 predicate so $\cap A$ is Π^1_1. □

We now use Theorem 3.10 to prove the following.

Theorem 3.11. *$\mathcal{O}$ is Π^1_1.*

Proof. This proof follows the proof in [1].

We will define a Σ^1_1 predicate for sets containing $<_{\mathcal{O}}$ and then use Theorem 3.10.

Consider the following predicate $p(X)$.

$$\begin{aligned}
&\langle 0, 1\rangle \in X \\
&\wedge \forall n, m, l[(\langle n, m\rangle \in X \wedge \langle m, l\rangle \in X) \to \langle n, l\rangle \in X] \\
&\wedge \forall n, m[\langle n, m\rangle \in X \to \langle m, 2^m\rangle \in X] \\
&\wedge \forall e[\forall n(\varphi_e(n) \downarrow \wedge \langle \varphi_e(n), \varphi_e(n+1)\rangle \in X) \to \forall n(\langle \varphi_e(n), 3 \cdot 5^e\rangle \in X)]
\end{aligned}$$

$p(X)$ is the inductive definition of $<_{\mathcal{O}}$ so $p(<_{\mathcal{O}})$ holds, and for any set X that satisfies $p(X)$ it must be that $<_{\mathcal{O}} \subseteq X$. Therefore $<_{\mathcal{O}} = \cap\{X : p(X)\}$. p is arithmetic so it is Σ^1_1. So $<_{\mathcal{O}}$ is Π^1_1, and hence so is $\mathcal{O}$. □

Theorem 3.12. *$\mathcal{O}$ is not Σ^1_1.*

Proof. This proof is based on one in [1].

To show this we will show that there is a Π^1_1 set that is not Σ^1_1. Since every Π^1_1 is reducible to $\mathcal{O}$ if $\mathcal{O}$ was Σ^1_1 then every Π^1_1 set would be Σ^1_1.

Define the Π predicate $Q(x)$ by $\forall B \exists n \varphi_x(B, n, x) \downarrow = 1$. Suppose $\neg Q(x)$ is Π^1_1. Then for some e
$$\neg Q(x) \iff \forall B \exists n \varphi_e(B, n, x) \downarrow = 1.$$

But then $\neg Q(e) \iff Q(e)$, a contradiction. Therefore $Q(x)$ is not Σ^1_1. □

Theorem 3.13 (Σ^1_1 bounding)**.** *If $A \subseteq \mathcal{O}$ is Σ^1_1, then there is $b \in \mathcal{O}$ such that $|b| \geq |a|$ for all $a \in A$.*

Proof. This proof closely follows the one in [1].

Since $\mathcal{O}$ is Π^1_1, let t be as in the proof of Theorem 3.9 so that $x \in \mathcal{O}$ if and only if $R_{t(x)}$ is well founded. Consider the predicate $Q(x)$ defined by

$$\exists z[z \in A \wedge \exists f \forall n, m(R_{t(x)}(n, m) \to \langle f(n), f(m)\rangle \in \{\langle a, b\rangle : a <_{\mathcal{O}} b <_{\mathcal{O}} z\})].$$

Since A is Σ^1_1, and the conjunction of Σ^1_1 predicates is Σ^1_1, $Q(x)$ is Σ^1_1. Note, asking if $\langle f(n), f(m)\rangle \in \{\langle a, b\rangle : a <_{\mathcal{O}} b <_{\mathcal{O}} z\}$ makes sense and is c.e. since $z \in A$ means $z \in \mathcal{O}$. We can think of f as being an homomorphism from $R_{t(x)}$ to $\{\langle a, b\rangle : a <_{\mathcal{O}} b <_{\mathcal{O}} z\}$.

If $R_{t(x)}$ is not well founded, then certainly $Q(x)$ should be false. Suppose $R_{t(x)}$ is well founded. Suppose that there is no bound $|b|$ bound on A. Then since $R_{t(x)}$ is a well founded computable relation, there is $z \in A$ such that $|R_{t(c)}| < |z|$. So $Q(x) \iff x \in \mathcal{O}$, so $\mathcal{O}$ is Σ^1_1, a contradiction. Therefore there is $b \in \mathcal{O}$ such that $|b| \geq |a|$ for all $a \in A$. □

Chapter 4

Computable Models

We call a model computable if we can label the elements as natural numbers and all the interpretations of function and relation symbols are computable. For example, the standard model of Peano arithmetic is computable.

4.1 Computable Formulae

We define the rank of an infinitary formula $r(\phi)$ inductively. If ϕ is atomic then $r(\phi) = 0$. If $\phi = \neg\psi$ then $r(\phi) = r(\psi)$. If $\phi = \exists x\psi(x)$ or $\phi = \forall x\psi(x)$ then $r(\phi) = r(\psi) + 1$. If

$$\phi = \bigwedge_{\psi \in X} \psi$$

or

$$\phi = \bigvee_{\psi \in X} \psi$$

then $r(\phi) = \sup\{r(\psi) : \psi \in X\} + 1$.

From this, one can see that if ϕ is an $\mathcal{L}_{\omega_1,\omega}$-formula, then $r(\phi) < \omega_1$, and that $r(\phi)$ is not a limit ordinal.

Now we describe a computable way of representing $\mathcal{L}_{\omega_1,\omega}$-formulae. We give codes to all the function, relation, variable and constant symbols, as well as to quantifiers and logical connectives. To represent an atomic formula, we just concatenate the codes for the symbols to get a code for the formula.

To code a more general formula we use the code for some computable binary relation R on $\mathbb{N} \times \mathbb{N}$, which codes a tree. If nRm then n is a child of m. To code a formula, we pair this tree with a root node and a computable function that maps nodes of the tree to codes for local symbols.

In order to be valid, the tree must have no infinite descending path, leaf nodes must map to atomic formula, quantifier codes must be paired with a variable being quantified over, and quantifier and negation nodes must have only one child. Note that checking if a number is a valid code for a formula is not computable.

We can extend this idea to Δ^1_1 formulae by allowing the codes for the map and the tree to be for Δ^1_1 functions.

Because there are only countably many codes for formulae, not all $\mathcal{L}_{\omega_1,\omega}$-formulae are computable.

Theorem 4.1. *If ϕ is a computable formula, then $r(\phi) < \omega_1^{CK}$.*

Proof. Suppose there is a computable ϕ such that $r(\phi) \geq \omega_1^{ck}$. Then we can use ϕ to construct a computable well ordering of the codes for subformulae of ϕ. To compare two different subformulae, ψ_1 and ψ_2, we enumerate all subformulae of ϕ until we have found both ψ_1 and ψ_2, keeping track of the paths to ψ_1 and ψ_2 from the route. If one is a subformula of the other, then it is smaller. Otherwise we look at the first nodes, n_{ψ_1} and n_{ψ_2}, for which the paths to ψ_1 and ψ_2 differ. If $n_{\psi_1} < n_{\psi_2}$, then ψ_1 is smaller. Otherwise ϕ_2 is smaller.

It can been seen using induction that the rank of a subformula is less than or equal to the ordinal corresponding to it in this well ordering. Therefore we have a computable well ordering of rank greater than or equal to ω_1^{ck}, a contradiction. □

The well order defined above can be turned into a computable well order of $\mathbb{N}$ by mapping $n \in \mathbb{N}$ to the nth unique subformula of ϕ in some fixed enumeration of all the subformula of ϕ. Note that the method used to construct this well order is computable. We will use this technique again later.

Theorem 4.2. *There is a computable function $f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ such that if $r(\phi) < \alpha$ then*

$$f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha) = \begin{cases} 1 & \mathcal{M} \models \phi(\overline{a}) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We will define two functions $f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ and $v(\phi, \alpha, 0^\alpha)$ together. f will be the function we want, and v is a total computable function used by f, which tries to compute $r(\phi) < \alpha$.

When $r(\phi) \geq \alpha$, f will attempt to compute a truth value for $\phi(\overline{a})$ and will halt if it finds a truth value. When $r(\phi) \leq \alpha$ this allows f to determine the truth value based on whether or not f halted with smaller oracles. For our construction to work, it must be the case that f never gives a wrong answer to the truth value of a formula, so that whenever f halts with a smaller oracle we can use use the result for larger oracles.

We will define f and v using induction on formula complexity. Base case: ϕ is atomic. f will use $\mathcal{M}$ to compute a truth value of $\phi(\overline{a})$ and output that. v will output true unless $\alpha = 0$, in which case v always outputs 0 no matter what ϕ is.

If $\phi = \neg\psi$, then for f run $f(\psi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ and if it halts output the opposite of that output. v just outputs $v(\psi, \alpha, 0^\alpha)$.

Next case: $\phi = \bigwedge_{\psi \in X} \psi$. If $\alpha = 0$ then v outputs 1. We define $v(\phi, \gamma, 0^\gamma)$ to halt and output 1 if there is $\beta < \gamma$ such that $v(\phi, \beta, 0^\beta) = 0$. $v(\phi, \gamma+1, 0^{\gamma+1})$ will output 1 if $v(\phi, \gamma, 0^\gamma)$ halts; otherwise it will enumerate all $\psi \in X$ until $v(\psi, \gamma, 0^\gamma)$ diverges and then output 0. From this definition we can see that $v(\phi, \gamma+1, 0^{\gamma+1})$ diverges if and only if $r(\phi) = \gamma + 1$.

If $\alpha = \beta + 1$ is an ordinary successor ordinal, then $v(\phi, \alpha, 0^\alpha)$ will ask if the function $h(\beta, 0^\beta)$ halts, where $h(\beta, 0^\beta)$ is the function that enumerates all $\psi \in X$ until $v(\psi, \beta, 0^\beta) = 1$. If $h(\beta, 0^\beta)$ halts, then some $\psi \in X$ has $r(\psi) \geq \alpha$ so $r(\phi) > \alpha$. So v will output 0 if $h(\beta, 0^\beta)$ halts and 1 otherwise.

For f we first define $f(\phi, \overline{a}, \mathcal{M}, \gamma, 0^\gamma)$ for the case when γ is a limit ordinal. $f(\phi, \overline{a}, \mathcal{M}, \gamma, 0^\gamma)$ will enumerate all $\beta < \gamma$ and $\psi \in X$ until either $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ halts, in which case it will output $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$, or $f(\psi, \overline{a}, \mathcal{M}, \beta, 0^\beta) \downarrow= 0$, in which case it will output 0. So by an induction argument it should be that $f(\phi, \overline{a}, \mathcal{M}, \gamma, 0^\gamma)$ halts and computes the correct output if $\gamma > r(\phi)$.

For general $\alpha = \beta + 1$, $f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ will ask if $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ halts. If it does halt, then output $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$. If $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ does not halt and $v(\phi, \alpha, 0^\alpha) = 0$ then, much like in the limit case above, $f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ will enumerate all $\psi \in X$ and $t < \omega$ until $f(\psi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha) \downarrow= 0$ in t many steps and then output 0. If $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ diverges and

$v(\phi, \alpha, 0^\alpha) = 1$ then, since $r(\psi) < \beta$ for all $\psi \in X$, it must be the case that $\mathcal{M} \models \psi(\overline{a})$ for all $\psi \in X$, so $\mathcal{M} \models \phi(\overline{a})$. So f will output true in this case.

Note in the above description of f that when α is the successor of a limit ordinal, f assumes that $v(\phi, \alpha, 0^\alpha) = 0$.

The cases for $\phi(\overline{a}) = \exists x \psi(\overline{a}x)$ are very similar. Instead of enumerating subformulae, we enumerate $b \in M$ and ask questions about $\psi(\overline{a}b)$. Note that $r(\phi)$ cannot be the successor of a limit ordinal, so we do not have to deal with that case. We define $v(\phi, \alpha, 0^\alpha) = v(\psi, \beta, 0^\beta)$.

$f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ will first ask if $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ halts. If it does halt, then f will output $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$. If $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ does not halt and $r(\phi) > \alpha$, then enumerate all $b \in M$ and $t < \omega$ until $f(\psi, \overline{a}b, \mathcal{M}, \alpha, 0^\alpha) \downarrow = 1$ in t many steps and then output 1. If $f(\phi, \overline{a}, \mathcal{M}, \beta, 0^\beta)$ diverges and $v(\phi, \alpha, 0^\alpha) = 1$ then, since $r(\psi) < \beta$, it must be the case that $\mathcal{M} \not\models \psi(\overline{a}b)$ for all $b \in M$, so $\mathcal{M} \not\models \phi(\overline{a})$. So f will output false in this case.

Using transfinite induction we can see that f and v have the desired property that whenever they halt they give the correct output. Since v is total whenever α is not a limit ordinal or the successor of a limit ordinal, we can see that $f(\phi, \overline{a}, \mathcal{M}, \alpha, 0^\alpha)$ will halt if $r(\phi) < \alpha$. So f is the desired function. □

While the compactness theorem fails for infinitary logic there is a variation which does hold for hyperarithmetic theories. We will use the following theorem without proof.

Theorem 4.3 (Kreisel-Barwise Compactness). *If T is a Π^1_1 $\mathcal{L}_{\omega_1,\omega}$-theory and every Δ^1_1 subset of T has a (computable) model, then T has a (computable) model.*

4.2 The Harrison Linear Order

In this section we will fix our language to be $\mathcal{L} = \{<, c_0, c_1, \ldots\}$ with the single binary relation $<$ and countably many constant symbols.

Consider the formula $\varphi_\alpha(x)$ which, when taken with the linear order axioms, states that $\{y : y < x\}$ has order type α. We can define $\varphi_\alpha(x)$ inductively. $\varphi_0(x) = \forall y \neg(y < x)$. For $\alpha > 0$

$$\varphi_\alpha(x) = \forall y(y < x \rightarrow \bigvee_{\beta<\alpha} \varphi_\beta(y)) \wedge \bigwedge_{\beta<\alpha} \exists y(y < x \wedge \varphi_\beta(y)).$$

For a computable ordinal α it can be shown, using transfinite induction on α, that $\varphi_\alpha(x)$ is a computable formula and that $r(\varphi_\alpha) \geq \alpha$.

Consider the following theory T.

- $<$ is a linear order.
- $\forall x \bigvee_{i=0}^{\infty} c_i = x$.
- The sentence $\exists x \varphi_\alpha(x)$ for every $\alpha < \omega_1^{ck}$.
- For each hyperarithmetic function f, the sentence $\bigvee_{i=0}^{\infty} c_{f(i+1)} \geq c_{f(i)}$.

Theorem 4.4. *T has a computable model.*

Proof. We will now use theorem 4.3. First we need to show that T is Π^1_1. The first three types of sentences are all computable and the fourth type is computable given the function f, so is hyperarithmetic. To show T is Π^1_1, we now need to show how to enumerate all the sentences in a Π^1_1 way. Since ω_1^{ck} is Π^1_1 and there is a Π^1_1 numbering of all the hyperarithmetic functions, T is Π^1_1.

Note the Π^1_1 numbering of the Π^1_1 sets containing Δ^1_1 functions we describe in Section 3.2 would be good enough here, since if $X \supseteq f$ for some Δ^1_1 function f, then

$$\bigvee_{i=0}^{\infty} c_{f(i+1)} \geq c_{f(i)} \implies \bigvee_{(i,n),(i+1,m)\in X} c_m \geq c_n.$$

Now consider any Δ^1_1 $T_0 \subseteq T$. We can turn T_0 in to a Δ^1_1 set of well orders, W_0, by taking the well order described in the proof of Theorem 4.1 for each sentence in T_0. By Σ^1_1 bounding (Theorem 3.13), there is an $\alpha < \omega_1^{ck}$ which is of greater rank than each of the well orders in W_0. So $\alpha > r(\phi)$ for any $\phi \in T_0$.

Since $r(\varphi_\beta(x)) \geq \beta$ for all β, if $\beta \geq \alpha$ then $\exists x \varphi_\beta(x) \notin T_0$. This means that a computable model of α will model T_0. Since $\alpha < \omega_1^{ck}$ there is such a model.

So by Theorem 4.3, there is a computable model of T. □

What does a computable model of T look like? It must be a linear order and since it models $\varphi_\alpha(a)$ for some a for each $\alpha < \omega_1^{ck}$, it must contain an initial segment which is isomorphic to ω_1^{ck}. So since it is computable it cannot be well founded. The last set of sentences means that it contains no Δ^1_1 infinite descending sequence. The following theorem will characterise the structure of a model of T.

Theorem 4.5. *If $\prec$ is an ill-founded linear order on $\mathbb{N}$ with no hyperarithmetic infinite descending sequence, then $(\mathbb{N}, \prec)$ is isomorphic to $\omega_1^{ck} + \mathbb{Q} \times \omega_1^{ck} + \alpha$ for some $\alpha < \omega_1^{ck}$.*

Proof. This proof closely follows the one in [3].

Consider the equivalence relation $\sim$ where if $a \prec b$ then $a \sim b$ if $\{x : a \prec x \prec b\}$ is well ordered. Since it is Π^1_1 to check if a set is well ordered, $\sim$ is a Π^1_1 equivalence relation. So the equivalence class $[a]$ is Π^1_1.

Suppose $[a]$ has no least element. The set $A = \{(x, y) \in [a] \times [a] : y \prec x\}$ is Π^1_1. By Theorem 3.7 we can get a Π^1_1-uniformization $B \subseteq A$. Pick $x_0 \in [a]$. Let x_{n+1} be the unique number such that $(x_n, x_{n+1}) \in B$. Since there is no least element, $x_0, x_1, \ldots$ is an infinite descending sequence. Since B is Π^1_1, the function f that maps n to x_n is Π^1_1 by the predicate

$$(n, x_n) \in f \iff \exists \sigma[|\sigma| = n \wedge \sigma(0) = x_0 \wedge \sigma(n) = x_n \wedge \forall n < |\sigma| - 1((\sigma(n), \sigma(n+1)) \in B)].$$

By Theorem 3.6, f is Δ^1_1, so we have a hyperarithmetic infinite descending sequence, a contradiction. Therefore each $[a]$ has a least element and as a consequence is well ordered.

This means, since $\prec$ is not well ordered, that there are at least two equivalence classes. If $a \prec b$ and for all $a \prec x \prec b$, $x \in [a]$ or $x \in [b]$, then since $[a] + [b]$ is a well ordering, $[a] = [b]$. So the equivalence classes are dense, possibly with end points.

If $\prec$ has no least element, then we can easily find a computable infinite descending sequence by picking $x_0 = 0$, and x_{n+1} to be the first $y \in \mathbb{N}$ such that $y \prec x_n$. So there is a least element and a least equivalence class.

Call this initial segment I. Next we show the order type of I is ω_1^{ck}. Suppose the order type of I is $\alpha < \omega_1^{ck}$. Then let R be a computable well ordering of $\mathbb{N}$ of rank α.

$$x \in I \iff \exists f \forall y, z(z \prec y \prec x \to R(f(z), f(y))).$$

So I is Σ^1_1 and hence Δ^1_1. Now we can construct a hyperarithmetic infinite descending sequence. Take $x_0 \notin I$. To find x_{n+1}, search for $y \in \mathbb{N}$ such that $y \prec x_n \wedge y \notin I$. So the order type of I is $\alpha \geq \omega_1^{ck}$.

If I is of order type $> \omega_1^{ck}$, then we can construct a computable well ordering of order type ω_1^{ck}. So the order type of I is ω_1^{ck}.

Consider $[a]$ not the greatest equivalence class. If we restrict the domain of $\prec$ to the set $\{x : [a] \preceq x\}$, we have a computable linear ordering with no hyperarithmetic infinite descending sequence. Since the equivalence classes are dense, it must still be ill-founded. So we can use all the information we have proven so far. Since $[a]$ is the initial segment of this linear order, the order type of $[a]$ is ω_1^{ck}.

Now all that is left is to show that the order type of the greatest equivalence class, if it exists, is $< \omega_1^{ck}$. Let $[b]$ be the greatest equivalence class. Since the set $\{x : [b] \preceq x\} = [b]$ is computable, the order type of $[b]$ must be $< \omega_1^{ck}$.

So $(\mathbb{N}, \prec) \cong \omega_1^{ck} + \mathbb{Q} \times \omega_1^{ck} + \alpha$ for some $\alpha < \omega_1^{ck}$.

$\square$

If $\mathcal{H}$ is a model of T, it has the form $\omega_1^{ck} + \mathbb{Q} \times \omega_1^{ck} + \alpha$. Any initial segment of $\mathcal{H}$ that contains ω_1^{ck} will also be a model of T. So there is a model of T isomorphic to $\omega_1^{ck} + \mathbb{Q} \times \omega_1^{ck} + \alpha$ for every $\alpha < \omega_1^{ck}$. In particular, the Harison linear order $\omega_1^{ck} + \mathbb{Q} \times \omega_1^{ck}$.

Chapter 5

Scott Rank of $\mathcal{H}$

5.1 Scott Rank of Tuples

For a model $\mathcal{M}$ and $\overline{a} \in M^n$, we define the Scott rank of $\overline{a}$, $r(\overline{a})$ to be the least ordinal α such that if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b})$ for some $b \in M^n$, then $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ for all β.

We can use this to give an alternate definition of Scott rank. We define $\mathrm{SR}(\mathcal{M}) = \sup\{r(\overline{a}) + 1 : \overline{a} \in M^{<\omega}\}$. The following theorem describes the relationship between $\mathrm{SR}(\mathcal{M})$ and $\mathrm{sr}(\mathcal{M})$.

Theorem 5.1. $\mathrm{SR}(\mathcal{M}) = \mathrm{sr}(\mathcal{M})$ *if* $\mathrm{SR}(\mathcal{M})$ *is a limit ordinal.* $\mathrm{SR}(\mathcal{M}) = \mathrm{sr}(\mathcal{M}) + 1$ *otherwise.*

Proof. If $(\mathcal{M}, \overline{a}) \sim_{\mathrm{sr}(\mathcal{M})} (\mathcal{M}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ for all β. So $r(\overline{a}) \leq \mathrm{sr}(\mathcal{M})$. On the other hand, if $\Gamma_\alpha \neq \Gamma_{\alpha+1}$, there is $\overline{a}, \overline{b} \in M^n$ such that $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b})$, but $(\mathcal{M}, \overline{a}) \nsim_{\alpha+1} (\mathcal{M}, \overline{b})$. So $r(\overline{a}) > \alpha$. When we put these together, we get that $\mathrm{sr}(\mathcal{M}) = \sup\{r(\overline{a}) : \overline{a} \in M^{<\omega}\}$.

The result follows from the definition of $\mathrm{SR}(\mathcal{M})$. □

Recall $\phi^{\mathcal{M}}_{\overline{a},\alpha}$ from Chapter 2. The next theorem shows that for computable models and ordinals, $\phi^{\mathcal{M}}_{\overline{a},\alpha}$ is computable. As we will see, this is not enough to ensure that the Scott Sentence is computable.

Theorem 5.2. *If* $\alpha < \omega_1^{ck}$ *and* $\mathcal{M}$ *is a computable model and* $\overline{a} \in M$*, then the formula* $\phi^{\mathcal{M}}_{\overline{a},\alpha}$ *is computable.*

Proof. This proof is based on one in [3].

We will use induction. Base case: $\phi^{\mathcal{M}}_{\overline{a},0}$ is the conjunction of all atomic formula (or their negation) $\psi(\overline{x})$ for which $\mathcal{M} \models \psi(\overline{a})$. Since we can enumerate all atomic formula, and $\mathcal{M}$ is computable, we can enumerate all the formulae in the conjunction. So $\phi^{\mathcal{M}}_{\overline{a},0}$ is computable.

Now we construct a function $f(\mathcal{M}, \overline{a}, \alpha)$ such that $f(\mathcal{M}, \overline{a}, \alpha)$ is a code for $\phi^{\mathcal{M}}_{\overline{a},\alpha}$.

If $\alpha = 0$, then as above we can output a code for $\phi^{\mathcal{M}}_{\overline{a},0}$. If α is a limit ordinal, then a computable tree for $\phi^{\mathcal{M}}_{\overline{a},\alpha}$ starts with a conjunction node. We enumerate notations b such that $|b| < \alpha$, and we calculate $e = f(\mathcal{M}, \overline{a}, b)$. We then modify the tree in e so that if n is a node in e it is replaced by p_b^{n+1}. Then we say the root of that tree is a child of our root.

If $\alpha = \beta + 1$, then

$$\phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{c \in M} \exists d \phi^{\mathcal{M}}_{\overline{a}c,\beta}(\overline{v}d) \wedge \forall d \bigvee_{c \in M} \phi^{\mathcal{M}}_{\overline{a}c,\beta}(\overline{v}d).$$

We can enumerate all $c \in M$ and calculate $e = f(\mathcal{M}, \overline{a}c, \beta)$. As we did above, we can relabel the nodes of e using something like p_c^{n+1} and put the roots at the appropriate places in the new tree. □

So each $\phi^{\mathcal{M}}_{\bar{a},\alpha}$ is computable. What is more, given a notation for α we can compute a code for $\phi^{\mathcal{M}}_{\bar{a},\alpha}$. Note also that $r(\phi^{\mathcal{M}}_{\bar{a},\alpha}) \geq \alpha$. We will use this in the next theorem.

Theorem 5.3. $\mathrm{SR}(\mathcal{H}) \geq \omega_1^{ck} + 1$.

Proof. This proof closely follows the one in [3].

Let $a \in \mathbb{N}$ such that a is in the part of $\mathcal{H}$ that is isomorphic to $(q, \beta) \in \mathbb{Q} \times \omega_1^{ck}$. Suppose $r(a) = \alpha < \omega_1^{ck}$. Consider the set

$$I = \{x : \mathcal{H} \models \forall y(\phi^{\mathcal{H}}_{a,\alpha}(y) \rightarrow x \prec y)\}.$$

a is an element of the form $(q, \beta) \in \mathbb{Q} \times \omega_1^{ck}$. Since any translation of $\mathbb{Q}$ is an isomorphism of $(\mathbb{Q}, <)$, there is an isomorphism of $\mathcal{H}$ that maps a to something of the form (p, β) for every $p \in \mathbb{Q}$. This means that for any b not in the well ordered initial segment of $\mathcal{H}$, there is some $y \prec b$ such that $\mathcal{H} \models \phi^{\mathcal{H}}_{a,\alpha}(y)$. So $b \notin I$.

Every b in the well ordered initial segment of $\mathcal{H}$ has that $\mathcal{H} \models \varphi_\alpha(b)$ for some $\alpha < \omega_1^{ck}$. However, if b is not in the well ordered initial segment, then the set $\{x : x \prec b\}$ is not well ordered, and so $\mathcal{H} \not\models \varphi_\alpha(b)$ for any $\alpha < \omega_1^{ck}$.

So no isomorphism of $\mathcal{H}$ can map a to something in the well ordered initial segment. Which means that I is the well ordered initial segment of $\mathcal{H}$. But $\forall y(\phi^{\mathcal{H}}_{a,\alpha}(y) \rightarrow x \prec y)$ is a computable formula, so by theorem 4.2 its true value is computable from some 0^α, $\alpha < \omega_1^{ck}$. Which means that I is hyperarithmetic, a contradiction. Therefore $r(a) \geq \omega_1^{ck}$. Which means $\mathrm{SR}(\mathcal{H}) \geq \omega_1^{ck} + 1$.

□

5.2 Scott Rank of Computable Models

We define the set WO^* to be the set of computable linear orders, R, of $\mathbb{N}$ such that

1. 0 is the least element.
2. If x is not maximal with respect to R, then there is a y such that xRy, and if zRy then $z = x$ or zRx.

We call the y satisfying condition 2 above the successor of x and denote y as $s_R(x)$.

We can create a first order formula that says $y = s_R(x)$ and use it build a theory T such that $(\mathbb{N}, R) \models T$ if and only if $R \in \mathrm{WO}^*$. This means that WO^* and s_R are arithmetic.

We define an R-analysis of models $\mathcal{M}$ and $\mathcal{N}$ with domain $\mathbb{N}$ to be a subset of $\mathbb{N} \times \bigcup_{n\in\mathbb{N}}(\mathbb{N}^n \times \mathbb{N}^n)\}$, z, such that

1. $(0, \bar{a}, \bar{b}) \in z$ if and only if $(\mathcal{M}, \bar{a}) \sim_0 (\mathcal{N}, \bar{b})$.
2. If mRn and $(n, \bar{a}, \bar{b}) \in z$, then $(m, \bar{a}, \bar{b}) \in z$.
3. $(s_R(n), \bar{a}, \bar{b}) \in z$ if and only if for each $c \in \mathbb{N}$ there is $d \in \mathbb{N}$ such that $(n, \bar{a}c, \bar{b}d) \in z$ and for each $d \in \mathbb{N}$ there is $c \in \mathbb{N}$ such that $(n, \bar{a}c, \bar{b}d) \in z$.
4. If n is a limit in R and for all mRn $(m, \bar{a}, \bar{b}) \in z$, then $(n, \bar{a}, \bar{b}) \in z$.

Since all the conditions for z to be an R-analysis of $\mathcal{M}$ and $\mathcal{N}$ can be stated as first order sentences, we can see that if $\mathcal{M}$ and $\mathcal{N}$ are computable, then

$$\{z : z \text{ is an } R\text{-analysis of } \mathcal{M} \text{ and } \mathcal{N}\}$$

is arithmetic.

Suppose R is a well order of order type α, and z is an R-analysis of $\mathcal{M}$ and $\mathcal{N}$. Suppose β is the order type of $\{m : mRn\}$. Then we can use induction with respect to R to show that

$$(n, \overline{a}, \overline{b}) \in z \iff (\mathcal{M}, \overline{a}) \sim_{\beta} (\mathcal{N}, \overline{b}).$$

This means that given R, $\mathcal{M}$, $\mathcal{N}$, we can use the above statement to define an R-analysis of $\mathcal{M}$ and $\mathcal{N}$.

The next theorem looks at the case where R is not well ordered.

Theorem 5.4. *If $R \in \mathrm{WO}^*$ and $(x_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ is a sequence such that $x_{n+1} R x_n$, then if z is an R-analysis of $\mathcal{M}$ and $\mathcal{N}$ and $(n_0, \overline{a}, \overline{b}) \in z$, there is an isomorphism of $\mathcal{M}$ and $\mathcal{N}$ that maps $\overline{a}$ to $\overline{b}$.*

Proof. We will do a back and forth construction, like in the proof of Theorem 2.5.

We will define functions $f_0, f_1, \ldots$ inductively, ensuring that for all n $(x_n, dom(f_n), Im(f_n)) \in z$ and that $f_n \subseteq f_{n+1}$.

f_0 is defined by mapping $\overline{a}$ to $\overline{b}$.

At stage $n = 2i + 1$, if $i \in dom(f_{n-1})$, then $f_n = f_{n-1}$. Otherwise, let $\overline{c} = dom(f_{n-1})$ and $\overline{d} = Im(f_{n-1})$. Since $(x_{n-1}, \overline{c}, \overline{d})) \in z$ and $x_n R x_{n-1}$ there is $e \in N$ such that $(x_n, \overline{c}i, \overline{d}e)) \in z$. So extend f_{n-1} to f_n by mapping m_i to e.

At stage $n = 2i$ we do a similar thing to ensure i is in the image of f_n and $(x_n, dom(f_n), Im(f_n)) \in z$. So $f = \bigcup_{n=1}^{\infty} f_n$ is a bijection of $\mathbb{N}$. If $\overline{c} \in \mathbb{N}^m$ then $c \subseteq dom(f_n)$ for some n, so $(x_n, dom(f_n), Im(f_n)) \in z$. Which means $(\mathcal{M}, dom(f_n)) \sim_0 (\mathcal{N}, Im(f_n))$. So every relation that holds for $\overline{c}$ in $\mathcal{M}$ holds for $f(\overline{c})$ in $\mathcal{N}$. Similarly for functions and constants. Therefore f is an isomorphism. □

We now use R-analyses to prove the following theorem.

Theorem 5.5. *If $\mathcal{M}$ is a computable model, then $\mathrm{SR}(\mathcal{M}) \leq \omega_1^{ck} + 1$.*

Proof. This proof closely follows the one in [3].

Take $\overline{a}, \overline{b} \in \mathbb{N}^n$ such that there is no automorphism of $\mathcal{M}$ that maps $\overline{a}$ to $\overline{b}$. Consider the set

$$S = \{R \in \mathrm{WO}^* : \exists z (z \text{ is an } R\text{-analysis of } \mathcal{M} \wedge \exists n (n \text{ is an } R\text{-maximum} \wedge (n, \overline{a}, \overline{b}) \in z))\}.$$

S is a Σ_1^1 set because the statements "z is an R-analysis of $\mathcal{M}$", "n is an R-maximum" and "$(n, \overline{a}, \overline{b}) \in z$" are arithmetic.

If $R \in S$ and R has an infinite descending sequence, then there is an automorphism of $\mathcal{M}$ that maps $\overline{a}$ to $\overline{b}$, a contradiction. So S is an arithmetic set of computable well orders. So by Σ_1^1 bounding, there is an ordinal $\alpha < \omega_1^{ck}$ such that for all $R \in S$ the order type of R is $< \alpha$.

Suppose $(\mathcal{M}, \overline{a}) \sim_{\alpha} (\mathcal{M}, \overline{b})$. Let R be a computable well order of rank α and n the R-maximum. So there is an R-analysis of $\mathcal{M}$ z and $(n, \overline{a}, \overline{b}) \in z$. A contradiction, therefore $(\mathcal{M}, \overline{a}) \nsim_{\alpha} (\mathcal{M}, \overline{b})$.

This means that $r(\overline{a}) \leq \omega_1^{ck}$. So $\mathrm{SR}(\mathcal{M}) \leq \omega_1^{ck} + 1$. □

So the Scott rank of $\mathcal{H}$ is $\omega_1^{ck} + 1$.

Bibliography

[1] G. E. Sacks, *Higher recursion theory*. Springer, 1990.

[2] D. Marker, *Model theory: an introduction*. Springer, 2011.

[3] D. Marker, *Lectures on Infinitary Model Theory*, vol. 46 of *Lecture Notes in Logic*. Cambridge University Press, 2016.