Classification of classes of enumeration degrees of non-metrizable spaces by topological separation axioms.

Josiah Jacobsen-Grocott

University of Wisconsin—Madison
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Enumeration degrees

- We say $A \leq_e B$ if every enumeration of B (uniformly) computes an enumeration of A.
- Like Turing riducibility this is a pre-order and we define $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$. $\mathcal{D}_e$ is the set of $\equiv_e$ equivalence classes.
- The Turing degrees properly embed into the enumeration degrees the map induced by $A \mapsto \mathrm{graph}(A)$. The degrees in the image of this map are called the *total* degrees. There are degrees which are not above any non-zero total degree. These are called *quasi-minimal*.

Degrees of points in a space

The continuous degrees, introduced by Miller, are another subclass of the enumeration degrees that arise from a reduction on points in computable metric spaces. Kihara and Pauly extend this idea to general topological spaces as follows.

Definition

- A cb_0 space $\mathcal{X}$ is a second countable $\mathcal{T}_0$ space given with a listing of a basis $(\beta_e)_e$.
- Given a cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ and a point $x \in X$ the name of x, $\mathrm{Name}_{\mathcal{X}}(x) = \{e \in \omega : x \in \beta_e\}$.
- We define the degrees of a space $\mathcal{X}$ to be $\mathcal{D}_{\mathcal{X}} = \{\mathbf{a} \in \mathcal{D}_e : \exists x \in X[\mathrm{Name}(x) \in \mathbf{a}]\}$.

Example spaces

- The product of the Sierpiński space $\mathbb{S}^\omega$ where $\mathbb{S} = \{0, 1\}$ with open sets $\{\emptyset, \{1\}, \mathbb{S}\}$, is universal for second countable T_0 spaces. We have that $\mathcal{D}_{\mathbb{S}^\omega} = \mathcal{D}_e$. This follow from the fact that for any $x \in \mathbb{S}^\omega$ we have $\mathrm{Name}_{\mathbb{S}^\omega}(x) \equiv_e \{n : x(n) = 1\}$. This means that any class of enumeration degrees is $\mathcal{D}_\mathcal{X}$ for some $\mathcal{X} \subseteq \mathbb{S}^\omega$.
- Cantor space 2^ω gives the total degrees.
- Hilbert's cube $[0, 1]^\omega$ is universal for second countable metric spaces, and gives us the continuous degrees.

Motivation

- Kihara, Ng and Pauly look at many different spaces from topology and discover many new classes of enumeration degrees.
- A second part of their work is to establish a classification and hierarchy of classes of degrees by looking at what types of spaces a particular class of degrees could arise from.

Separation axioms

Definition

A topological space is considered

- T_0 if for any $x \neq y$ there is an open set U such that either $x \in U, y \notin U$ or $x \notin U, y \in U$.
- T_1 if $\{x\}$ is closed for any x.
- T_2 (Hausdorff) if for any $x \neq y$ there are disjoint open U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$.
- $T_{2.5}$ if for any $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.
- *Submetrizable* if its topology comes from taking a metric space and adding open sets.

Separating degrees with separation axioms

- We have the following series in implications:
  metrizable $\implies$ submetrizable $\implies$ $T_{2.5} \implies T_2 \implies T_1 \implies T_0$.
  It is well known that this hierarchy is strict for second countable spaces.
- One question is if the separation axioms give rise to different classes of degrees. For instance we could define the T_1 degrees to be the set the $\{\mathbf{a} : \exists \mathcal{X} \in T_1[\mathbf{a} \in \mathcal{D}_{\mathcal{X}}]\}$.
- Kihara and Pauly show that the continuous degrees are. $\bigcup \mathcal{D}_{\mathcal{X}}$ where $\mathcal{X}$ is a computable metric space.

Theorem (Kihara, Ng, Pauly)

For every degree $\mathbf{a} \in \mathcal{D}_e$ *there is a computable submetrizable space* $\mathcal{X}$ *such that such that* $\mathbf{a} \in \mathcal{D}_{\mathcal{X}}$.

- So the submetrizable degrees are the same as the T_0 degrees and hence the same as the T_1 degrees, T_2 degrees and $T_{2.5}$ degrees.

Separating classes with separation axioms

The separation axioms may not give us new classes of degrees, but they can still be used to categorize classes of degrees.

Definition

Given a collection of cb_0 spaces $\mathcal{T}$ we say that a class $\mathcal{C}$ of enumeration degrees is $\mathcal{T}$ if there is some $\mathcal{X} \in \mathcal{T}$ such that $\mathcal{D}_{\mathcal{X}} = \mathcal{C}$.

So any $\mathcal{C} \subseteq \mathcal{D}_e$ is T_0 and the continuous degrees and total degrees are both computably metrizable. This leads to the following question.

Question

Is the separation hierarchy T_0, T_1, T_2, $T_{2,5}$, submetrizable, metrizable a strict hierarchy on classes of degrees?

Known separations

The Golomb space $\mathbb{N}_{\mathrm{rp}} = (\mathbb{Z} \setminus \{0\}, (a + b\mathbb{Z} : \gcd(a, b) = 1))$ and its product $\mathbb{N}_{\mathrm{rp}}^{\omega}$ is a know $T_2 \setminus T_{2.5}$ space. The cocylinder topology $(\omega^{\omega})_{\mathrm{co}} = (\omega^{\omega}, (\omega^{\omega} \setminus [\sigma])_{\sigma \in \omega^{<\omega}})$ is a $T_1 \setminus T_2$ space the degrees of which are know as the cylinder cototal degrees.

Theorem (Kihara, Ng, Pauly)

- $\mathcal{D}_{\mathbb{S}^{\omega}}$ *is* $T_0 \setminus T_1$*.*
- *The cylinder cototal degrees are* $T_1 \setminus T_2$*.*
- $\mathcal{D}_{\mathbb{N}_{\mathrm{rp}}^{\omega}}$ *is* $T_2 \setminus T_{2.5}$*.*
- *There is a computably submetrizable space* $\mathcal{X}$ *such that* $\mathcal{D}_{\mathcal{X}}$ *is not computably metrizable.*

Question (Kihara, Ng, Pauly)

Is there a $T_{2.5}$ class of degrees that is not submetrizable?

Separation of $T_{2.5}$ and Submetrizable

The Arens co-d-CEA degrees and Roy halfgraph degrees were introduced by Kihara, Ng and Pauly. Both come from non submetrizable, computable $T_{2.5}$ spaces and are subclasses of the doubled co-d-CEA degrees, a class that comes from a $T_2 \setminus T_{2.5}$ space.

Theorem (J-G)

The Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable.

A corollary is that the doubled co-d-CEA degrees are not submetrizable. It is unknown if the doubled co-d-CEA degrees are $T_{2.5}$ or not, but we do have the following.

Theorem (J-G)

The Arens co-d-CEA degrees do not contain the doubled co-d-CEA degrees.

Doubled co-d-CEA separation

We will give a sketch of the proof that the doubled co-d-CEA degrees are not submetrizable, since it has the same structure, but is less technical. Fist we give the definition of doubled co-d-CEA.

Definition

A set is *doubled co-d-CEA* if it is of the form $\mathrm{graph}(Y) \oplus (A \cup N) \oplus (B \cup P)$ where $N, P, (A \cup B)^c$ are Y-c.e. and A, B, N, P are disjoint.

Proof part 1.

First we use finite injury to build c.e. sets $N, P \subseteq C$ with $N \cap P = \emptyset$ such that for any partition $A \sqcup B = C^c$ we have that $(A \cup N) \oplus (B \cup P)$ is not PA and does not compute any non Δ^0_2 total degree. This gives us a class we will call $\mathcal{C}$ of continuum many doubled co-d-CEA degrees that do not bound a Scott ideal or any non Δ^0_2 total degree.

Doubled co-d-CEA separation

Proof part 2.

Next we consider some arbitrary computable metric space $\mathcal{X} = (X, (\alpha_e)_e)$ and submetrizable extension $\mathcal{Y} = (X, (\alpha_e)_e \cup (\beta_i)_i)$. Fix a degree $\mathbf{a} \in \mathcal{C}$. Suppose that for some point $x \in X$ we have that $\mathrm{Name}_{\mathcal{Y}}(x) \in \mathbf{a}$ then $\mathrm{Name}_{\mathcal{X}}(x) \leq_e \mathbf{a}$. Since $\mathbf{a}$ does not bound a Scott ideal, $\mathrm{Name}_{\mathcal{X}}(x)$ must have total degree (by a theorem of Miller). Hence $\mathrm{Name}_{\mathcal{X}}(x) \leq_e 0'$. So there are only countably many $x \in X$ such that $\deg(\mathrm{Name}_{\mathcal{Y}}(x)) \in \mathcal{C}$, so $\mathcal{C} \nsubseteq \mathcal{D}_{\mathcal{Y}}$.

The result for non computable submetrizable spaces is done by relativization. □

This part of the proof is the same as with Arens co-d-CEA and Roy halfgraph degrees.

Reverse separation for metrizable and submetrizable

The submetrizable space we looked at in the previous proof had a particular form. This prompted the following definition.

Definition

A space $\mathcal{Y} = (Y, (\beta_i)_i)$ is *effectively submetrizable* if there is a computable $(\alpha_e)_e \subseteq (\beta_i)_i$ such that $\mathcal{X} = (Y, (\alpha_e)_e)$ is a computable metric space.

We still have that the effectively submetrizable degrees are all enumeration degrees. However we have the following result.

Theorem (J-G)

There is a second countable metric space $\mathcal{X}$ such that $\mathcal{D}_{\mathcal{X}}$ is not effectively submetrizable.

Quasi-minimal

Definition

For a cb_0 space $\mathcal{X}$ we say that a degree $\mathsf{a} \in \mathcal{D}_e$ is $\mathcal{X}$ quasi-minimal if $\mathsf{a} \notin \mathcal{D}_{\mathcal{X}}$ and for all $\mathsf{b} \in \mathcal{D}_{\mathcal{X}}$ if $\mathsf{b} \leq_e \mathsf{a}$ then $\mathsf{b} = 0$.

So, since $\mathcal{D}_{2^\omega}$ is the total degrees, 2^ω-quasi-minimal and quasi-minimal mean the same thing.

Definition

For class $\mathcal{C} \subseteq \mathcal{D}_e$ and a set of cb_0 spaces $\mathcal{T}$, we say that $\mathcal{C}$ is $\mathcal{T}$-quasi-minimal if for every $\mathcal{X} \in \mathcal{T}$ the is $\mathsf{a} \in \mathcal{C}$ such that a is $\mathcal{X}$-quasi-minimal.

Clearly if $\mathcal{C}$ is $\mathcal{T}$-quasi-minimal then $\mathcal{C}$ is not $\mathcal{T}$.

Quasi-minimal results

Kihara, Ng and Pauly showed that $\mathcal{D}_e$ is T_1-quasi-minimal and give several other Quasi-minimal results. Recall that the cylinder cototal degrees are $T_1 \setminus T_2$ and that $D_{\mathbb{N}^{\omega}_{\mathrm{rp}}}$ is $\mathcal{T}_2 \setminus T_{2.5}$. The proofs of these two results use a counting argument in the final step. By replacing the final step with a forcing construction they can be strengthened to.

Theorem (J-G)

- *The cylinder cototal degrees are T_2-quasi-minimal.*
- $D_{\mathbb{N}^{\omega}_{\mathrm{rp}}}$ *is $T_{2.5}$-quasi-minimal.*

The proof of our separation of metric classes and effectively submetrizable classes also gives us the following.

Theorem

The doubled co-d-CEA degrees are not metric-quasi-minimal.

Thank You