The failure of Selman's Theorem for hyperenumeration reducibility

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Enumeration degrees

Definition

For two sets $A, B \subseteq \omega$ we say that $A \leq_e B$ if there is a c.e. set W such that:

$$x \in A \iff \exists \langle x, u \rangle \in W[D_u \subseteq B]$$

where $(D_u)_{u \in \omega}$ is listing of all finite sets by strong indices.

- From an effective listing of c.e. sets $(W_e)_{e \in \omega}$ we obtain an effective listing of enumeration operators $(\Psi_e)_{e \in \omega}$. Defined by $A = \Psi_e(B)$ if $A \leq_e B$ via W_e.
- $\leq_e$ is a preorder and, like with Turing reducibility and the Turing degrees, we get the enumeration degrees $\mathcal{D}_e$.

Total and cototal sets

Definition

We say that a set A is *total* if $\overline{A} \leq_e A$. We say that A is cototal if $A \leq_e \overline{A}$. A degree is *total* (*cototal*) if it contains a total (cototal) set.

- If A is total then $B \leq_e A$ if and only if B is c.e. in A.
- For any set A we have that $A \oplus \overline{A}$ is both total and cototal.
- The Turing degrees embed onto the total degrees via the map induced by $A \mapsto A \oplus \overline{A}$.
- The cototal degrees are a proper subclass of the enumeration degrees and the total degrees are a proper subclass of the cototal degrees.

Selman's Theorem

As we have seen, we can define Turing reducibility in terms of enumeration reducibility. Selman's theorem gives us a way of defining enumeration reducibility in terms of Turing reducibility.

Theorem (Selman's Theorem)

$A \leq_e B$ if and only if for all X if $B \leq_e X \oplus \overline{X}$ then $A \leq_e X \oplus \overline{X}$.

There is another way to define enumeration reducibility in terms of enumerations. We have that $A \leq_e B$ if every enumeration of B uniformly computes an enumeration of A. Here an enumeration of A is a total, onto function $f : \omega \to A$. In this context, Selman's theorem shows that we can drop the uniformity in the definition..

Proof of Selman's Theorem

Proof.

Suppose that $B \nleq_e A$. We will use forcing to build a enumeration f of A that is not above B. At stage s given initial segment $\sigma_s \in \omega^{<\omega}$ we ask if there is $\tau \succeq \sigma_s$ and $n \notin B$ such that $n \in \Psi_s(\tau)$ and $\mathrm{range}(\tau) \subseteq A$. If there is such a τ then we set $\sigma_{s+1} = \tau$. If there is no such τ then let $k = \min(A \setminus \mathrm{range}(\sigma_s))$ and set $\sigma_{s+1} = \sigma_s{}^\frown k$.

By construction we have that $f = \bigcup_s \sigma_s$ is an enumeration of A. Now suppose towards a contradiction that $B = \Psi_e(f)$ for some e. Then at stage e we must not have found any τ. So for all $\tau \succ \sigma_e$ with $\mathrm{range}(\tau) \subseteq A$ we have that $\Psi_e(\tau) \subseteq B$. So as $B = \Psi_e(f)$ we have:

$$n \in B \iff \exists \tau \succ \sigma_e[\mathrm{range}(\tau) \subseteq A]$$

Hence $B \leq_e A$. □

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Hyperenumeration reducibility

- Now we define hyperenumeration reducibility as introduced by Sanchis in 1978.

Definition

We say that $A \leq_{he} B$ if there is a c.e. set W such that

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u \rangle \in W \wedge D_u \subseteq B]$$

- Like with enumeration reducibility this is a preorder and the equivalence classes give us the hyperenumeration degrees $\mathcal{D}_{he}$.
- From an effective listing of c.e. sets $(W_e)_{e \in \omega}$ we obtain an effective listing of hyperenumeration operators $(\Gamma_e)_{e \in \omega}$.

Hypertotal degrees.

Definition

We say that a set A is *hypertotal* if $\overline{A} \leq_{he} A$. We say that A is *hypercototal* if $A \leq_{he} \overline{A}$. A degree (in either $\mathcal{D}_e$ or $\mathcal{D}_{he}$) is *hypertotal* (*hypercototal*) if it contains a hypertotal (hypercototal) set.

We have a similar relationship between the hypertotal degrees and the hyperarithmetic degrees as the relationship between the total and Turing degrees.

From the definition of $\leq_{he}$ we have that if $A \leq_{he} B$ then A is Π^1_1 in B. It is not hard to show that if A is Π^1_1 in B then $A \leq_{he} B \oplus \overline{B}$. So $A \leq_h B \iff A \oplus \overline{A} \leq_{he} B \oplus \overline{B}$. The hyperarithmetic degrees embed onto the total degrees via the map induced by $A \mapsto A \oplus \overline{A}$.

Theorem (Sanchis)

There is a hyperenumeration degree that is not hypertotal.

Relating $\leq_e$ and $\leq_{he}$

Sanchis proved an interesting result about the relationship between enumeration reducibility and hyperenumeration reducibility.

Theorem (Sanchis)

If $A \leq_e B$ then $A \leq_{he} B$ and $\overline{A} \leq_{he} \overline{B}$.

This means that if f is an enumeration of A then $A \oplus \overline{A} \leq_{he} f$. So when working with hyperenumeration redicibility we want a new notion of a hyperenumeration.

Recall the definition of $A = \Gamma_e(B)$.

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u\rangle \in W_e \wedge D_u \subseteq B]$$

Now consider the tree $S_e \subseteq \omega^{<\omega}$ defined by

$$n^\frown x \notin S_e \iff \exists y \preceq x, u \leq |x|[\langle n, y, u\rangle \in W_{e,|x|} \wedge D_u \subseteq B]$$

We have that $S_e \leq_T B$ and $\overline{S_e} \leq_e B$. Define $S_{e,n} = \{x : n^\frown x \in S_e\}$. We have that

$$n \in A \iff S_{e,n} \text{ is well founded}$$

So $A \leq_{he} \overline{S_e}$. We call a tree which hyperenumerates A in the way that S_e does a hyperenumeration of A.

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E-pointed trees in Cantor space

Definition

A tree T is *e-pointed* if for every path $P \in [T]$ we have that T is c.e. in P. We say T is *uniformly* e-pointed if there is a single operator Ψ_e such that for all paths $P \in [T]$ we have $T = \Psi_e(P)$.

McCarthy studied e-pointed trees in Cantor space and was able to characterize their enumeration degrees.

Theorem (McCarthy)

If $T \subseteq 2^{<\omega}$ is uniformly e-pointed then T is cototal. Furthermore for a degree $\mathbf{a} \in \mathcal{D}_e$ *the following are equivalent:*

- $\mathbf{a}$ *is cototal.*
- $\mathbf{a}$ *contains an e-pointed tree $T \subseteq 2^{<\omega}$.*
- $\mathbf{a}$ *contains a uniformly e-pointed tree $T \subseteq 2^{<\omega}$ with no dead ends.*

E-pointed trees in Baire space with dead ends

In Baire space we have the following characterization in terms of hypertotal sets.

Theorem (Goh, J-G, Miller, Soskova)

If $T \subseteq \omega^{<\omega}$ is uniformly e-pointed then T is hypercototal. Furthermore for a degree $\mathrm{a} \in \mathcal{D}_e$ (or $\mathcal{D}_{he}$) the following are equivalent:

- a *is hypercototal.*
- a *contains an e-pointed tree $T \subseteq \omega^{<\omega}$.*
- a *contains a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$.*

E-pointed trees in Baire space without dead ends

When we consider only e-pointed trees that do not have dead ends then things become more complex

Theorem (Goh, J-G, Miller, Soskova)

There is an arithmetic set that is not enumeration equivalent to any e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends.

Theorem (Goh, J-G, Miller, Soskova)

There is a uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends that is not of cototal enumeration degree.

Question

Is there an e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends that is not enumeration equivalent to any uniformly e-pointed tree $T \subseteq \omega^{<\omega}$ without dead ends.

Connection to Selman's theorem

Theorem (J-G)

There is a uniformly e-pointed tree with no dead ends that is not hypertotal.

This leads us to a contradition of Selman's theorem.

Corollary

There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.

Connection to Selman's theorem

Corollary

There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.

Proof.

We will have $A = T$ and $B = \overline{T}$ where T is a uniformly e-pointed tree with no dead ends that is not hypertotal. Suppose that T is Π^1_1 in X. Since T has no dead ends there must be a path $P \in [T]$ such that $P \leq_h X$. So $T \leq_e P$ and by previous lemma we have $\overline{T} \leq_{he} \overline{P} \leq_h X$ So we get that $\overline{T} \leq_{he} X \oplus \overline{X}$. □

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Admissible sets

The usual definition of a Π^1_1 set of natural numbers is a set of the form $m \in X \iff \forall f \in \omega^\omega \exists n[R(f, n, m)]$ where R is a computable relation. However admissibility gives us another definition in terms of $L_{\omega_1^{CK}}$ that is useful.

Definition

A set M is *admissible* is it is transitive, closed under union, pairing and Cartesian product as well as satisfying the following to properties:

Δ_1-comprehension: for every Δ_1 definable class $A \subseteq M$ and set $a \in M$ the set $A \cap a \in M$.

Σ_1-collection: for every Σ_1 definable class relation $R \subseteq M^2$ and set $a \in M$ such that $a \subseteq \mathrm{dom}(R)$ there is $b \in M$ such that $a = R^{-1}[b]$.

Admissible sets

- The smallest admissible set is HF the collection of hereditarily finite sets. Looking at the Δ_1 and Σ_1 subsets of HF is one notion of computability. We have that the Δ_1 subsets of HF are computable sets and the Σ_1 subsets of HF are c.e. sets.
- We generalize this to an arbitrary admissible set M by calling a set $A \subseteq M$ M-computable if it is a Δ_1 subset of M and M-c.e. if it is a Σ_1 subset of M.
- The smallest admissible set containing ω as an element is $L_{\omega_1^{CK}}$. We have that the $L_{\omega_1^{CK}}$-c.e. subsets of ω are precisely the Π^1_1 sets.

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The forcing partial order

Let $\{T_\sigma : \sigma \in \omega^{<\omega}\}$ be an effective listing of all finite trees in $\omega^{<\omega}$ where for each $\sigma \in \omega^{<\omega}$ sequence $T_{\sigma^\frown 0}, T_{\sigma^\frown 1}, \ldots$ lists each finite tree that contains T_σ infinitely often.

Definition

A condition p is a pair $(T^p, L^p : T^p \times T^p \to \omega_1^{CK}) \in L_{\omega_1^{CK}}$ such that:

1. $T^p \subseteq \omega^{<\omega}$ is a well founded tree.
2. For each $\sigma \in T^p$ we have that $T_\sigma \subseteq T^p$.
3. $L^p(\sigma, \tau) = 0$ if and only if $\sigma \in T_\tau$.
4. If $\rho \prec \tau$ then $L^p(\sigma, \tau) = 0$ or $L^p(\sigma, \tau) < L^p(\sigma, \rho)$.
5. For each $\tau \in T^p$ and $n < \omega$ the set $\{\sigma : L^p(\sigma, \tau) \leq n\}$ is finite.

For two conditions p and q we say $p \leq q$ if $T^q \preceq T^p$ and $L^q \subseteq L^p$.

Tools

Lemma

The set of conditions is $L_{\omega_1^{CK}}$-c.e. and the relation $\leq$ on conditions is $L_{\omega_1^{CK}}$-computable.

Lemma

Let $A \subseteq \omega^{<\omega}$ be a set such that for all $\sigma^\frown i \in A$ we have $\sigma \in T^p$ and $\{\tau : L^p(\tau, \sigma) \leq 1\} \subseteq T_{\sigma^\frown i} \subseteq T^P \cup A$. For such an A we can define a condition $q = p[A]$ with $T^q = T^p \cup A\}$ such that q is a valid condition. If we also have that $T^p \preceq T^p \cup A$ then $q \leq p$.

Corollary

If $\mathcal{G}$ is a sufficiently generic filter then $T^{\mathcal{G}}$ is a uniformly e-pointed tree with no dead ends.

The forcing relation

Definition

For a condition p we define $S_e^p \subseteq \omega^\omega$ to be the tree where

$$n^\frown x \notin S_e^p \iff \exists y \prec x, u \leq |x|[\langle n, y, u\rangle \in W_{e,|x|} \wedge D_u \subseteq T^p]$$

For a filter $\mathcal{G}$ we define $S_e^{\mathcal{G}} \bigcap_{p\in\mathcal{G}} S_e^p$.
We define $p \Vdash \mathrm{rank}(S_{e,x}^{\mathcal{G}}) \leq \alpha$ if $\mathrm{rank}(S_{e,x}^p) \leq \alpha$.

So by definition of Γ_e we have $\Gamma_e(T^{\mathcal{G}}) = \{n : S_{e,n}^{\mathcal{G}}$ is well founded$\}$.
From this definition it is clear that if $p \Vdash \mathrm{rank}(S_{e,x}^{\mathcal{G}}) \leq \alpha$ then for any $\mathcal{G} \ni p$ we have that $\mathrm{rank}(S_{e,x}^{\mathcal{G}}) \leq \alpha$.

Lemmas

Lemma

Fix a condition p. Suppose that for each $i \in \omega, r \leq p$ there is $q \leq r$ such that $q \Vdash \mathrm{rank}(S^{\mathcal{G}}_{e,x^\frown i}) \leq \beta$ for some $\beta < \omega_1^{CK}$ then there is $\hat{p} \leq p$ and $\alpha < \omega_1^{CK}$ such that $\hat{p} \Vdash \mathrm{rank}(S^{\mathcal{G}}_{e,x}) \leq \alpha$.

Lemma

If for all $q \leq p$ and $\alpha < \omega_1^{CK}$ we have $q \nVdash \mathrm{rank}(S^{\mathcal{G}}_{e,x}) \leq \alpha$ then $p \Vdash S^{\mathcal{G}}_{e,x}$ is ill founded. Formally, for all sufficiently generic filters $\mathcal{G} \ni p$ we have that $S^{\mathcal{G}}_{e,x}$ contains an infinite path.

Main result

Theorem (J-G)

There is a uniformly e-pointed tree in $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.

Proof.

We say $p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$ if there is $\sigma \in T^p$ and $\alpha < \omega_1^{CK}$ such that $p \Vdash \mathrm{rank}(S^{\mathcal{G}}_{e,\langle\sigma\rangle}) \leq \alpha$, or if there is $\sigma \notin T^p$ such that the initial segment of σ in T^p is not a leaf and $p \Vdash S^{\mathcal{G}}_{e,\langle\sigma\rangle}$ is ill founded. To show that $T^{\mathcal{G}}$ is not hypertotal it is enough for us the show that the sets $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ are dense for each e.

Main result

Proof continued.

Suppose towards a contradiction, that $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ is not dense. Let p be such that for all $q \leq p$ we have $q \nVdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$.
Consider some leaf $\sigma \in T^p$ and let i, j be such that $T_{\sigma^\frown i} = T_{\sigma^\frown j} = \{\rho : L^p(\rho, \sigma) \leq 1\}$. Now consider $q = p[\{\sigma^\frown i\}]$; this is well defined by previous lemma. By assumption on p we have that $q \nVdash S^{\mathcal{G}}_{e,\langle\sigma^\frown j\rangle}$ is ill founded, so by previous lemma there is $r \leq q, \alpha < \omega_1^{CK}$ such that $r \Vdash \operatorname{rank}(S^{\mathcal{G}}_{e,\langle\sigma^\frown j\rangle}) \leq \alpha$. Now consider $r' = r[\{\sigma^\frown j\}]$. Since $\sigma^\frown i \in T^r$ we have $\{\rho : L^r(\rho, \sigma) \leq 1\} \subseteq T_{\sigma^\frown i} = T_{\sigma^\frown j}$ and thus the condition r' is a valid condition. Since $r \leq p$ and σ is a leaf in T^p we have that $r' \leq p$. But we have $S^r_e \supseteq S^{r'}_e$ so $r' \Vdash \operatorname{rank}(S^{\mathcal{G}}_{e,\langle\sigma^\frown j\rangle}) \leq \alpha$ a contradiction. So we have that the set $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ is dense.
So for sufficiently generic $\mathcal{G}$ we have that $T^{\mathcal{G}}$ is uniformly e-pointed without dead ends and for all e we have $\overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$, and thus $\overline{T^{\mathcal{G}}} \not\leq_{he} T^{G}$. $\square$

Thank You