In the following, the potentials f will be of polynomial form with only even powers appearing, $f(t) = \sum_{i=1}^{n} b_i x^{2i}$, with $b_i \in \mathbb{R}$, in connection with the similar form of truncations of expansions of p-frame potentials. One rewriting of the polynomial potential minimization problem on the unit circle involves rewriting the integral as a sum of integrals obtained by expanding the potential $f(t) = \sum_{i=1}^{n} c_i T_i(t)$ in Chebyshev polynomials after replacing $\langle x, y\rangle$ with $\cos\theta_{x,y}$. This replaces the original problem

$$\min_{\mu \in \mathcal{P}(\mathbb{T})} \int_{\mathbb{T}} \int_{\mathbb{T}} f(\langle x, y\rangle) d\mu(x) d\mu(y)$$

with its equivalent

$$\min_{\nu \in \mathcal{P}([-\pi,\pi])} \sum_{i=1}^{n} c_n \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos n(x-y) d\nu(x) d\nu(y) = \min_{\nu \in \mathcal{P}([-\pi,\pi])} \sum_{i=1}^{n} c_n \sum_{k} a_k \left[\int_{-\pi}^{\pi} g_k(t) d\nu(t) \right]^2$$

where $g_k(t)$ are selected from the functions $\sin kt$ and $\cos kt$ which arise from repeated use of the addition formula in the earlier expression.

Now, probability measures on the unit circle are in one-to-one correspondences with certain Toeplitz matrices, as detailed in the following classical result.

Theorem 0.1 (Theorem 1.4, Shohat, Tamarkin, pg. 7)**.** *A necessary condition that the trigonometric moment problem*

$$\mu_n = \int_0^{2\pi} e^{in\theta} d\mu, \;\; n = 0, \pm 1, \pm 2, ..., \mu_{-n} = \overline{\mu_n}$$

have a solution is that all Toeplitz forms

$$\sum_{j,l=0}^{n} \mu_{j-l} x_j \overline{x_l} \geq 0, \;\; n = 0, 1, 2, ...$$

This theorem allows for another rewriting of the above problem, namely to,

$$P^* = \min_{\nu} P(\nu_i, a_i, c_i) \text{ s.t. } A_\nu = \begin{pmatrix} 1 & \nu_1 & \nu_2 & \cdots & \nu_n \\ \nu_{-1} & 1 & \nu_1 & \cdots & \nu_{n-1} \\ \nu_{-2} & \nu_{-1} & 1 & \cdots & \nu_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_{-n} & \nu_{-(n-1)} & \nu_{-(n-2)} & \cdots & 1 \end{pmatrix} \succeq 0.$$

Noting that the function being minimized is a polynomial in ν_i while the constraints are semi-algebraic, it is natural to guess that for nice, rational coefficients a_i and c_i the optimal value is algebraic. The following argument verifies this (based off an argument used by Fickus, Jasper, and Mixon to answer a similar question for optimal real projective packings).

Proposition 0.2. *For rational coefficients, P^*, the optimal value to the polynomial potential minimization problem is algebraic.*

Proof. Let $S = \{\{\nu_i\}_{i=-n}^{n}, \mid A_\nu \succeq 0, \; \nu_0 = 1\}$ denote the first moments for a probability measure supported on the circle. S is a semi-algebraic set as is $T = \{(\nu, x) \mid \nu \in S, \; x \geq P_{a,c}(\nu)\}$, for a, c rational coefficients. The Tarski-Seidenberg principle implies that the projection $proj_x(T)$, as the projection of a semi-algebraic set is also semi-algebraic. Thus $P^* = \min(proj_x T)$ is algebraic. □

~~Actually, it is not necessary to use this type of argument when at least one of the values of the (real) coefficients in the corresponding orthogonal polynomials (Gegenbauer or Chebyshev) is negative. The existence of a discrete measure in this case allows to settle the question of algebraicity of the optimal value for spheres as well.~~

~~Recall that polynomial energy integrals on spheres are given in the form~~

Set

$$S = \{Q \in \mathbb{R}^{N\times N},\ Q \succeq 0,\ \mathrm{diag}(Q) = 1,\ \text{and } \mathrm{rank}(Q) \leq d+1\}$$

Then measures ν of support at most N with equal masses correspond to matrices $\nu = Q \in S$ (by abuse of notation) so that

$$I_F(\nu) = \frac{1}{N^2}\sum_{i,j=1}^{N} F(Q_{i,j}) = \frac{1}{N^2}\sum_{i,j=1}^{N}\sum_{k=0}^{m}\alpha_k (Q_{i,j})^k.$$

These simple observations allow to for a similar result as the first proposition above.

Proposition 0.3. *For polynomial potentials $F(t) = \sum_{k=0}^{m}\alpha_k t^k$, such that α_k are rational, Q^*, the optimal value to the energy minimization problem for N equally weighted points on the sphere $\mathbb{S}^d$, is algebraic.*

Proof. S is a semi-algebraic set and so $T = \{(Q,x) \mid Q \in S,\ \nu = (Q,\omega), x \geq I_F(\nu) = \frac{1}{N^2}\sum_{i,j}\alpha_k Q_{i,j}^k\}$ is semi-algebraic as α_k are rational. Again, the Tarski-Seidenberg principle implies that the projection $proj_x(T)$ is also semi-algebraic. Thus $\min(proj_x T)$ is algebraic. Since a discrete measure attains the minimal value, this value is optimal over probability measures on the sphere and $Q^* = \min(proj_x T)$ so that Q^* is algebraic. $\square$