[Carillo, Figalli, Pattachini for manifolds - theorem, corollary, and proof sketch from July 2019, later adapted for https://arxiv.org/abs/2003.01597]

Theorem

Suppose we have a smooth Riemannian manifold Ω. There's a function F that takes numbers from 0 to ∞, and suppose it's decreasing near zero. Specifically, as we get close to zero, it behaves like

$$F(t) = -(C + \text{small stuff})\, t^{\alpha}$$

for some positive constant C and an exponent $\alpha > 2$. Now, suppose we have a local minimizer $\widetilde{\mu}$ for a functional I_F. If there's a point y in Ω where the curvature is not too negative nearby, then y can't be a limit point of the support of $\widetilde{\mu}$.

Proof sketch

Following Carillo, Figalli, Patacchini's argument in a slightly different setting: we argue by contradiction. Assume that the support of our minimizer $\widetilde{\mu}$ includes a sequence of points $\{y_k\}$ that gets closer and closer to y. We're also assuming that the curvature around y is better than $-1/K^2$.

For simplicity, let's make things neat: assume all these points and the neighborhood U_y where y is located are within a certain radius that makes things easier to work with (injectivity radius). We can map everything around y into a flat space $\mathbb{R}^d$ using the exponential map.

Now, we can pick a subsequence of $\{y_k\}$ so that:

1. The directions of the vectors from y to y_k converge to a specific direction n_y.
2. The distances from y to y_k are getting smaller in a nice way.

Next, consider a hyperbolic plane Ω' with constant curvature that's worse (more negative) than the curvature of Ω around y. There's a nice mapping between points in Ω and Ω' that preserves distances along certain paths.

Now, let's take two points y_k and y_l (with $l > k$) and look at the distances between them. Using some geometry (the law of cosines), we can set up an equation relating the distances between these points.

For large enough k and l, we'll show that the distance between y_k and y_l is less than the distance from y to y_k. This will imply that the geodesic from y to y_k is the longest side in the triangle formed with points y, y_k, and y_l.

Using properties of hyperbolic geometry and some theorems, we find out that if the angle adjacent to this longest side is too small, we get a contradiction with our earlier assumptions about directions and distances.

By applying some lemmas about measures and angles in small triangles, we derive a relationship between the distances that leads to a contradiction. The idea is that if y_k and y_l were in the support of our measure $\widetilde{\mu}$, it would mean that the distances wouldn't work out as expected.

Thus, we conclude that our original assumption must be wrong, meaning that y can't be a limit point of the support of $\widetilde{\mu}$. This completes the proof sketch.

Corollary

For any closed compact smooth manifold Ω, the local minimizers of the functional

$$\iint_{\Omega} F(\rho(x,y))\, d\mu(x) d\mu(y)$$

are discrete in the d_∞ topology. This happens because the sectional curvature is nice and bounded below on such manifolds.